Appendix to Lecture 5. Proofs

I now show that the limit appearing in the definition of the Riemann integral \( \int_{a}^{b} f(y) \, dy \) exists, when \( f \) is a continuous function. First of all, I need a definition and a lemma.

**Definition:** If \( f : A \to B \), where \( A \) is a subset of \( \mathbb{R}^n \) and \( B \) is a subset of \( \mathbb{R}^k \), then \( f \) is uniformly continuous, if for every \( \varepsilon > 0 \), there is a \( \delta > 0 \) such that \( ||f(x) - f(y)|| < \varepsilon \) whenever \( ||x - y|| < \delta \).

**Lemma 5.17:** Let \( f : A \to B \), where \( A \) and \( B \) are subsets of \( \mathbb{R}^n \) and \( \mathbb{R}^k \), respectively. If \( A \) is compact and \( f \) is continuous, then \( f \) is uniformly continuous.

That is, every continuous function is uniformly continuous on a compact set.

**Proof:** Let \( \varepsilon > 0 \). Since \( f \) is continuous, for every \( x \in A \), there is a \( \delta > 0 \) such that if \( ||x - y|| < 2\delta \), then \( ||f(x) - f(y)|| < \varepsilon/2 \). Then \( \{ B_{\delta}(x) \mid x \in A \} \) forms an open cover of \( A \), where \( B_{\delta}(x) = \{ y \in A \mid ||y - x|| < \delta \} \). Since \( A \) is compact, the Heine-Borel theorem implies that there is a finite subcover \( B_{\delta}(x(1)), B_{\delta}(x(2)), \ldots, B_{\delta}(x(M)) \). Let \( \delta = \min(\delta_x, \delta_{x(1)}, \ldots, \delta_{x(M)}) \). Suppose that \( x \) and \( y \) in \( A \) are such that \( ||x - y|| < \delta \). Then \( x \in B_{\delta}(x(m)) \), for some \( m \), so that \( ||x - x(m)|| < \delta \). Since \( ||x - y|| < \delta \), \( ||y - x(m)|| \leq ||y - x|| + ||x - x(m)|| < 2\delta \), and hence \( ||f(y) - f(x(m))|| < \varepsilon/2 \). Similarly, since \( ||x - x(m)|| < \delta \), it follows that \( ||f(x) - f(x(m))|| < \varepsilon/2 \). Hence \( ||f(y) - f(x)|| \leq ||f(y) - f(x(m))|| + ||f(x(m)) - f(x)|| < \varepsilon/2 + \varepsilon/2 = \varepsilon \). 

**Theorem 5.18:** If \( f : [a, b] \to \mathbb{R} \) is continuous, where \( a < b \), then
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \frac{b-a}{N} \left( a + \frac{n}{N} (b-a) \right) \text{ exists.}
\]

**Proof:** It is sufficient to show that the sequence in \( N \), \( \frac{1}{N} \sum_{n=0}^{N-1} \frac{b-a}{N} \left( a + \frac{n}{N} (b-a) \right) \), is Cauchy. Let \( \varepsilon > 0 \). Since \( f \) is continuous and the interval \([a, b]\) is compact, \( f \) is uniformly continuous by the previous lemma. Let \( \delta > 0 \) be such that \( ||f(x) - f(y)|| < \frac{\varepsilon}{2(b-a)} \), if \( ||x - y|| < \delta \). Let \( K \) be so large that \( \frac{b-a}{K} < \delta \). I show that if \( M > K \), then \( ||y - x|| < \delta \).
\[
\sum_{n=0}^{K-1} \frac{b - a}{K} f\left(a + \frac{n}{K} (b - a)\right) - \sum_{m=0}^{M-1} \frac{b - a}{M} f\left(a + \frac{m}{M} (b - a)\right) \leq \varepsilon,
\]

so that the sequence \(\sum_{n=0}^{N} \frac{b - a}{N} f\left(a + \frac{n}{N} (b - a)\right)\) is Cauchy.

The numbers \(a + \frac{n}{K} (b - a)\), for \(n = 0, 1, \ldots, K\), and \(a + \frac{m}{M} (b - a)\), for \(m = 0, 1, \ldots, M\), form \(Q\) distinct numbers, where \(Q \leq K + M\). Let these numbers be \(t_0, t_1, \ldots, t_q\), where \(a = t_0 < t_1 < \ldots < t_q = b\). Let

\[
I = \sum_{q=0}^{Q-1} (t_{q+1} - t_q) f(t_q).
\]

It is sufficient to show that \(I - \sum_{n=0}^{K-1} \frac{b - a}{K} f\left(a + \frac{n}{K} (b - a)\right) < \varepsilon/2\), if \(K\) exceeds \(\frac{b - a}{\delta}\), since if we replace \(K\) by \(M\), where \(M > K\), we have the inequality \(I - \sum_{m=0}^{M-1} \frac{b - a}{M} f\left(a + \frac{m}{M} (b - a)\right) < \varepsilon/2\) as well. For \(n = 0, \ldots, K-1, a + \frac{n(b - a)}{K} = t_{q(n)}\), for some \(q(n)\), where \(q(n) < q(n+1)\), for all \(n\). Notice that if \(q\) is a positive integer such that \(q(n) \leq q \leq q(n+1)\), then

\[
\left| a + \frac{n}{K} (b - a) - t_q \right| \leq \frac{b - a}{K} < \delta,
\]

so that

\[
\left| f(t_q) - f\left(a + \frac{n}{K} (b - a)\right) \right| \leq \frac{\varepsilon}{2(b - a)}.
\]

This inequality explains the last inequality in the following series of equations and inequalities.

\[
\left| I - \sum_{n=0}^{K-1} \frac{b - a}{K} f\left(a + \frac{n}{K} (b - a)\right) \right|
\]

\[
= \sum_{q=0}^{Q-1} f(t_q) (t_{q+1} - t_q) - \sum_{n=0}^{K-1} \frac{b - a}{K} f\left(a + \frac{n}{K} (b - a)\right)
\]

\[
= \sum_{n=0}^{K-1} \left[ \frac{b - a}{K} f\left(a + \frac{n}{K} (b - a)\right) - \sum_{q=q(n)}^{q(n+1)-1} f(t_q) (t_{q+1} - t_q) \right]
\]
\[
= \sum_{n=0}^{K-1} \left[ f(t_{q(n)}) \sum_{q+q(n)}^{q(n+1)-1} (t_{q+1} - t_q) - \sum_{q=q(n)}^{q(n+1)-1} f(t_q) (t_{q+1} - t_q) \right]
\]
\[
\leq \sum_{n=0}^{K-1} \sum_{q=q(n)}^{q(n+1)-1} \left| f(t_{q(n)}) - f(t_q) \right| (t_{q+1} - t_q)
\]
\[
< \sum_{n=0}^{K-1} \frac{\varepsilon}{2(b-a)} \frac{b-a}{K} = \frac{\varepsilon}{2}.
\]

Use has been made of the facts that \( t_{q(n)} = a + \frac{n}{K} (b-a) \) and \( \sum_{q=q(n)}^{q(n+1)} (t_{q+1} - t_q) = \frac{b-a}{K} \).

I now prove the fundamental theorem of calculus, theorem 5.15. First I require a lemma.

**Lemma 5.19**: If \( f: [a, b] \rightarrow \mathbb{R} \) is continuous, where \( a < b \), then

\[
\left| \int_a^b f(y) \, dy \right| \leq \int_a^b |f(y)| \, dy.
\]

**Proof**: Since

\[
\int_a^b f(y) \, dy = \lim_{N \to \infty} \sum_{n=0}^{N-1} \frac{b-a}{N} f\left(a + \frac{n}{N} (b-a)\right),
\]

it follows that

\[
\left| \int_a^b f(y) \, dy \right| = \left| \lim_{N \to \infty} \sum_{n=0}^{N-1} \frac{b-a}{N} f\left(a + \frac{n}{N} (b-a)\right) \right| = \lim_{N \to \infty} \left| \sum_{n=0}^{N-1} \frac{b-a}{N} f\left(a + \frac{n}{N} (b-a)\right) \right|
\]
\[
\leq \lim_{N \to \infty} \sum_{n=0}^{N-1} \frac{b-a}{N} \left| f\left(a + \frac{n}{N} (b-a)\right) \right| = \int_a^b |f(y)| \, dy,
\]

where the second equation applies because the absolute value is a continuous function, the inequality follows from the triangle inequality for the absolute value, and the last equation is the definition of the integral of the continuous function \(|f(y)|\).

**Proof of theorem 5.15**: Let \( c \in (a, b) \) and let \( \varepsilon \) be a positive number. Since \( f \) is continuous, there is a positive number \( \delta \) such that \( |f(x) - f(c)| < \varepsilon \), if \( |x - c| < \delta \). Lemma 5.19 implies that if \( c < x < c + \delta \), then
\[ \left| \int_{c}^{x} (f(y) - f(c)) \, dy \right| \leq \int_{c}^{x} |f(y) - f(c)| \, dy \leq \int_{c}^{x} \, dy = |x - c|. \]

Therefore if \( c < x < c + \delta \), then
\[ \left| \int_{a}^{c} f(y) \, dy - \int_{a}^{c} f(y) \, dy - f(c) (x - c) \right| = \left| \int_{a}^{x} f(y) \, dy - f(c) (x - c) \right| \leq \varepsilon |x - c|. \]

Similarly, if \( c - \delta < x < c \), then
\[ \left| \int_{a}^{c} f(y) \, dy - \int_{a}^{c} f(y) \, dy - f(c) (x - c) \right| = \left| \int_{x}^{c} (-f(y) + f(c) (c - x)) \, dy \right| \leq \varepsilon |x - c|. \]

Therefore if \( |x - c| < \delta \), then
\[ \left| \int_{a}^{c} f(y) \, dy - \int_{a}^{c} f(y) \, dy - f(c) (x - c) \right| \leq \varepsilon |x - c|, \]

so that
\[ \frac{dF(c)}{dx} = \left. \frac{d}{dx} \int_{a}^{x} f(y) \, dy \right|_{x=a} = f(c), \]

by the definition of the derivative. Since differentiable functions are continuous, \( F(x) \) is continuous in \( x \), for \( c \in (a, b) \).

It remains to be shown that \( F \) is continuous at \( a \) and \( b \). It should be clear from the definition of the Riemann integral that
\[ F(a) = \int_{a}^{a} f(y) \, dy = 0. \]

Let \( \varepsilon \) be a positive number. Since \( f \) is continuous, there is a positive number \( \delta \) such that \( \delta \leq \varepsilon \) and \( |f(x) - f(a)| < \varepsilon \), if \( x - a < \delta \) and \( |f(x) - f(b)| < \varepsilon \), if \( b - x < \delta \). If \( x - a < \delta \), then
\[ |F(x) - F(a)| = \left| \int_{a}^{x} f(y) \, dy \right| \leq \int_{a}^{x} |f(y)| \, dy \leq \int_{a}^{x} (f(a) + \varepsilon) \, dy \leq (x - a)(f(a) + \varepsilon) \]
\[ \leq \varepsilon (f(a) + \varepsilon), \]

where the last inequality follows because \( x - a < \delta \leq \varepsilon \). Since \( \varepsilon (f(a) + \varepsilon) \) may be made arbitrarily small, \( F \) is continuous at \( a \).
Similarly if $b - x < \delta$, then

$$\left| F(b) - F(x) \right| = \left| \int_x^b f(y) \, dy \right| \leq \int_x^b |f(y)| \, dy \leq \int_x^b |f(b) + \varepsilon| \, dy \leq (b - x)(f(b) + \varepsilon)$$

$$\leq \varepsilon(f(b) + \varepsilon),$$

so that $F$ is continuous at $b$. \qed