1) A consumer has an income of 162 and buys quantities of three commodities, goods 1, 2, and 3. The price of good 1 is 1, that of good 2 is 2, and that of good 3 is 3. The consumer’s utility function is

\[ \ln x_1 + \ln x_2 + \ln x_3, \]

where \( x_n \) is the quantity purchased of good \( n \), for \( n = 1, 2, \) and \( 3 \). The consumer spends his or her income so as to maximize utility.

a) Write down a formal description of the consumer’s constrained optimization problem.
b) Write down the Lagrangian function for this problem.
c) Solve the problem.
d) What is the consumer’s marginal utility of wealth at the optimum?

Suppose now that goods 2 and 3 are rationed so that the consumer can buy no more than 4 units of good 2 and 6 units of good 3. The consumer is allotted 4 units of rationing tickets for good 2 and 6 units of rationing tickets for good 3. The consumer must both pay for these goods at their regular prices and hand in rationing tickets in order to obtain them.

e) Write down a formal description of the consumer’s constrained optimization problem with these two additional constraints.
f) Write down the Lagrangian function for this new problem.
g) Solve the new problem.
h) What is the consumer’s marginal utility, \( \lambda \), of wealth at the optimum?
i) What is the consumer’s marginal utility, \( \gamma_2 \), of having more rationing tickets for good 2 at the optimum? What is the consumer’s marginal utility, \( \gamma_3 \), of having more rationing tickets for good 3 at the optimum?

Suppose there is black market for rationing tickets and that \( p_2 \) and \( p_3 \) are the respective prices of rationing tickets for goods 2 and 3.

j) What are the smallest values of \( p_2 \) and \( p_3 \) such that the consumer buys no extra rationing tickets?
k) The price \( p_2 \) is a function of \( \lambda \) and \( \gamma_2 \) and \( p_3 \) is a function of \( \lambda \) and \( \gamma_3 \). What are these functions?

2) Consider the production function

\[ y = x_1^{\alpha} x_2^{1/2}, \]
where \( y \) is the quantity of output and \( x_1 \) and \( x_2 \) are the quantities of inputs 1 and 2, respectively. Let \( p \) be the price of the output and let \( q_1 \) and \( q_2 \) be the prices of inputs 1 and 2, respectively. Assume that \( 0 < \alpha < 1/2 \). Maximum profit is

\[
\pi(p, q_1, q_2, \alpha) = \max_{x_1 \geq 0, x_2 \geq 0} \left[ px_1 x_2^{1/2} - q_1 x_1 - q_2 x_2 \right].
\]

Find a formula for

\[
\frac{\partial \pi(p, q_1, q_2, \alpha)}{\partial \alpha}
\]

at \( \alpha = 1/4 \) in terms of \( q_1, q_2, \) and \( p \).