1) Suppose that a consumer buys two commodities and that the consumer's relative interest in
the first good increases with its relative price. More specifically, assume that the utility
function of the consumer is
\[ u(x_1, x_2, p_1, p_2) = p_1^{-1} \ln(x_1) + \ln(x_2), \]
where \( x_1 \) and \( x_2 \) are the quantities consumed of goods 1 and 2, respectively, and \( p_1 \) and \( p_2 \) are the
respective prices of goods 1 and 2. Let \( w \) be the consumer's wealth, where \( w > 0 \). The
consumer's maximization problem is
\[
\max_{x_1 \geq 0, x_2 \geq 0} u(x_1, x_2, p_1, p_2)
\quad \text{s.t.} \quad p_1 x_1 + p_2 x_2 \leq w.
\]
Let \( V(p_1, p_2, w) = \max_{x_1 \geq 0, x_2 \geq 0} \{ u(x_1, x_2, p_1, p_2) \mid p_1 x_1 + p_2 x_2 \leq w \} \).
Compute \( V(1, 2, 10) \).

2) A linear program is a problem of the form
\[
\max_{x_1 \geq 0, \ldots, x_n \geq 0} \sum_{n=1}^{N} c_n x_n
\quad \text{s.t.} \quad \sum_{k=1}^{N} b_{kn} x_n \leq a_k, \text{ for } k = 1, \ldots, K.
\]
This problem may be written in abbreviated form as
\[
\max_{x \in \mathbb{R}^N} c x
\quad \text{s.t.} \quad B x \leq a,
\]
where \( c \) is an \( N \)-vector, \( B \) is an \( K \times N \) matrix, \( a \) is a \( K \)-vector, and \( \mathbb{R}^N = \{ x \in \mathbb{R}^N \mid x \geq 0, \text{ for all } n \} \).
This problem is called the primal problem. The \( N \)-vector \( x \) is said to be feasible for the primal
problem if \( B x \leq a \). Suppose that \( a > 0 \), for all \( k \), and that the primal problem has a non-zero
solution \( \bar{x} \).

a) Use the Kuhn-Tucker theorem to show that there is a non-negative \( K \)-vector \( \bar{y} \) such that,
for all \( k \), \( \bar{y}_k = 0 \), if \( \sum_{n=1}^{N} b_{kn} \bar{x}_n < a_k \), and \( \bar{x} \) solves the problem.
\[
\max [c.x - y^T B x] = \max_{x \in \mathbb{R}^N} \sum_{n=1}^{N} \left( c.x - \sum_{k=1}^{K} y^T b_{k} x \right).
\]

b) Show that, for all \( n \), \( \sum_{k=1}^{K} y_{n} b_{k} \geq c \) and \( x = 0 \), if \( \sum_{k=1}^{K} y_{n} b_{k} > c \).

c) Show that for some \( n \), \( \sum_{k=1}^{K} y_{n} b_{k} = c \).

The dual linear program is

\[
\min \sum_{k=1}^{K} a_{k} y_{k}
\]

\[ y_{k} \in \mathbb{R}^K \]

s.t. \( \sum_{k=1}^{K} b_{k} y_{k} \geq c \),

which may be written as

\[
\min_{y \in \mathbb{R}^K} a.y
\]

s.t. \( y^T B \geq c^T \).

A \( K \)-vector \( y \) is feasible for the dual problem if \( y^T B \geq c^T \).

d) Show that if \( x \) is feasible for the primal problem and \( y \) is feasible for the dual problem, then \( c.x \leq y^T B x \leq a.y \).

e) Show that if \( x \) solves the primal problem and \( y \) is as in question 1, then \( c.x = a.y \).

f) Show that if \( x \) solves the primal problem and \( y \) is as in question 1, then \( y \) solves the dual problem.

3) Consider the problem

\[
\max \quad 2x_{1} + x_{2}
\]

s.t. \( x_{1} + 3x_{2} \leq 19 \)

\[ x_{1} \geq 0, x_{2} \geq 0 \]

\[
3x_{1} + x_{2} \leq 11.
\]

a) Draw the constraint set.

b) Solve this problem.
c) Define the dual problem.

d) Solve the dual problem.