Math Camp
Homework #10

There is a fish farm in which fish are started in period 1 and may be harvested in each of $T > 1$ successive periods, $t = 1, \ldots, T$. The question is how much to harvest in each period. The stock of fish at the beginning of period $t$ is the non-negative number $s_t$. Take the initial stock $s_1$ to be a given positive number. The decision variable is the quantity of fish, $c_t$, caught in period $t$, for $t = 1, \ldots, T$. The variable $c_t$ may be any number, negative or positive, where a negative value of $c_t$ corresponds to adding more fish of the same age as those already in a pond. The stock of fish at the end of period $t$ is $s_t - c_t$. The stock at the end of period $t$ is $s_t - c_t$, and the stock at the beginning of period $t + 1$ is

$$s_{t+1} = 2a(s_t - c_t) - a(s_t - c_t)^2,$$

where, for all $t < T$, $a$ and $s_t$ are positive constants. The price of fish at time $t$ is $p_t$, where $p_t > 0$, for all $t$. Assume that the farm has no costs other than buying fish to grow. That is, they need not be fed. Then the profits of the farm are

$$-p_t s_t + \sum_{t=1}^{T} p_t c_t.$$

Let $V(s)$ be the maximum total profits of the farm from time $t$ until time $T$, when the stock of fish at the beginning of period $t$ is $s_t$. Let $c_t(s_t)$ be the optimal catch of fish in period $t$ as a function of the stock of fish at the beginning of the period. Recall that the catch may be negative.

a) Describe formally the maximization problem defining $V_t$.

b) Describe formally the Bellman equation for the value functions.

c) Show that for every period $t$,

$$c_t(s_t) = \gamma_t + s_t$$

and

$$V(s_t) = p_t s_t + \Gamma_t,$$

where $\gamma_t$ and $\Gamma_t$ are constants.