Multiple Equilibria and Sunspots in a Security Market with Investment Restrictions

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I. Motivation and Main Results

A. **History of the project**

B. **Multiple Equilibria/Sunspot Equilibria: applied theory (Finance – asset pricing) vs theory (Economics – financial equilibrium)**

C. **Main Results about the Structure of FE in the Simplest Example: Multiple Equilibria/Sunspot Equilibria**

II. A Model of Financial Equilibrium with Investment Restrictions

- Two specializations of FE:

  (a) "Trees:" stocks with the properties that
  (i) the return from stock *g* is in terms of good *g*, and
  (ii) endowments consist wholly of stocks (and possibly bonds, also having good-specific returns)
  (b) "Logs:" log-linear expected utility

- FE is described by the equations representing household optimization (first-order conditions, budget constraints, and investment restrictions) and market clearing conditions

- Glossary

Glossary	Applied Theory*	Theory**
time	$t = 0, 1\ (t = 0 \equiv \omega = 0)$	
uncertainty	$\omega = u, d\ (\text{or } u, m, d)$	$\omega = 1, 2\ (\text{or } 1, 2, 3)$
probabilities	$\pi(\omega)$	
goods / households	$g = \ , * \ /\ i = 1, 2$	$g = 1, 2\ /\ h = 1, 2$
consumption	$c_i^g(\omega)$	$c_h^g(\omega)$
investment (portfolio)	$\theta_i\ (\alpha_i, \theta_i)$	$s_h\ (b_h, s_h)$
spot goods prices	$p^g(\omega)$	
stock prices	S^g	q^g
Lagrange multipliers	$\phi_i(\omega), \psi$	$\lambda_h(\omega), \mu$
stochastic weights	$\lambda_i(\omega) \sim 1/\phi_i(\omega)$	$\eta_h(\omega) \sim \beta_h/\lambda_h(\omega)$

* Basak, Cass, and Pavlova, Multiple equilibria in a security market with investment restrictions, preprint, December, 2004.
** Cass and Pavlova, On trees and logs, **JET**, 116 (2004), 41-83.

II.A. Extended Form Equations

To begin with we ignore the investment restrictions, and assume only intrinsic uncertainty.

The first use of "Trees" and "Logs" – converting units of good *g* into units of good *g* **per dividend of stock *g***

$$c_h^g(\omega) \longrightarrow \frac{c_h^g(\omega)}{\delta^g(\omega)},$$

where $\delta^g(\omega)$ is the dividend of stock *g* at spot ω

II.A. Extended Form Equations (cont'd.)

- Problem of the Household

Every household solves the problem:

$$Max_{(c_h, s_h^1)} \sum_g \alpha_h^{0,g} log c_h^g(0) + \sum_{\omega>0} \pi(\omega)\beta_h \sum_g \alpha_h^{1,g} log c_h^g(\omega)$$

subject to — with multipliers,

$$p(0)(c_h(0) - s_h^0) + q(s_h^1 - s_h^0) \leq 0 \qquad \lambda_h(0)$$

$$p(\omega)(c_h(\omega) - s_h^1) \leq 0\ ,\ \text{all } \omega > 0 \qquad \lambda_h(\omega)$$

$$c_h \gg 0$$

- Market clearing conditions for spot goods and stocks

⋮

II.A. Extended Form Equations (cont'd.)

$$D_{c_h^{0,g}} u_h(c_h^0, c_h^1) = \frac{\alpha_h^{0,g}}{c_h^g(0)} = \lambda_h(0)p^g(0), \text{ all } g, h$$

$$D_{c_h^{1,g}(\omega)} u_h(c_h^0, c_h^1) = \frac{\pi(\omega)\beta_h\alpha_h^{1,g}}{c_h^g(\omega)} = \lambda_h(\omega)p^g(\omega), \text{ all } g, h, \omega > 0$$

$\vdots$

$$\textstyle\sum_h c_h^g(\omega) = 1, \text{ all } g, \omega$$

$\vdots$

where $(c_h^0, c_h^1) = (c_h(0), (c_h(\omega), \omega > 0))$

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II.B. Reduced Form Equations

The second use of "Trees" and "Logs" – reducing the number of variables and equations

We can use the previous equations to simplify the system to consist of just the spot goods price equations, no-arbitrage conditions, budget constraints, and stock market clearing conditions. At this point, it is convenient to replace the Lagrange multipliers $\lambda_h(\omega)$ by the stochastic weights $\eta_h(\omega) \equiv \beta_h/\lambda_h(\omega)$.

After some manipulation, this yields:

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II.B. Reduced Form Equations (cont'd.)

good eqs.:

$$p(0) = \textstyle\sum_h (\alpha_h^0/\beta_h)\eta_h(0) \quad \text{SGP's}$$

$$p(\omega) = \pi(\omega)\textstyle\sum_h \alpha_h^1\eta_h(\omega), \ \omega > 0$$

$$q = \eta_h(0)\textstyle\sum_{\omega>0}[1/\eta_h(\omega)]p(\omega), \ h < H \quad \text{NAC's}$$

$$(1 + 1/\beta_h) - \textstyle\sum_{\omega>0}[1/\eta_h(\omega)]p(\omega)s_h^0 = 0, \ h < H \quad \text{BC's}$$

bad eqs.:

$$\pi(\omega)\eta_h(\omega) - p(\omega)s_h^1 = 0, \ \omega > 0$$

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II.B. Reduced Form Equations (cont'd.)

Note that

(1) By the analogue of Walras' law, Mr. H's budget constraints are redundant, and

(2) For the purposes of analysis, the variables and equations can be reduced even further by normalizing prices, say, by setting $\eta_1(\omega) = 1$, all ω, and then (i) substituting for spot goods prices, and (ii) using Ms. 1's NAC's, substituting for stock prices in the remaining NAC's. This leaves only $\eta_h(\omega)$, ω, $h > 1$, and s_h^1, $h < H$, as variables.

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III. The Leading Example: 2 states, goods, and households

Notice specially that, since there are equal number of states and stocks, the stock market is potentially complete. For simplicity (and without any loss of generality), we take

$\alpha_h^0 = \alpha_h^1 = \alpha_h = (a_h, 1 - a_h)$ with $0 < a_h < 1$, and

$\beta_h = \beta > 0$, $h = 1, 2$,

and relabel $\eta_2(\omega) \equiv \eta(\omega)$, all ω.

We establish, first, a multiplicity of equilibria, then second (based on the same technique), an abundance of sunspot equilibria – some of which are especially interesting from an "applied" perspective.

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III.A. A Specific Investment Restriction

Suppose that Mr. 2 faces the additional constraint

$$q^1 s_2^{1,2} \geq \gamma W_2^0 \equiv \gamma(q^1 s_2^{0,1} + q^2 s_2^{0,2})$$

or $s_2^{1,2} \geq \gamma(q s_2^{0,1} + s_2^{0,2})$

with $\gamma > 0$, (now) relative price $q = q^1/q^2$, and associated multiplier $\mu \geq 0$.

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III.A. A Specific Investment Restriction (cont'd)

Adding the constraint entails two modifications of the reduced form equations: First, the term μ is added to the right hand side of Mr. 2's NAC for stock 2, and second, the complementary slackness condition

$$\min\{s_2^{1,2} - \gamma(qs_2^{0,1} + s_2^{0,2}), \mu\} = 0$$

is added to his first-order conditions.

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III.A. A Specific Investment Restriction (cont'd)

This leads to our first main result, based on distinguishing the second period budget constraints from the rest of the reduced form equations, **the singular equations** (TSE) from **the regular equations** (TRE) (this labeling reflects the fact that the reduced form equations have an especially critical point – which will be formally described if time permits).

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III.A. A Specific Investment Restriction (cont'd)

Proposition 1. Assume that $a_2 < a_1$, $a_2 s_1^{0,1} + (1-a_2)s_1^{0,2} \neq 0$, and $A \equiv \frac{1-a_1 s_1^{0,1} + (1-a_1)s_1^{0,2}}{a_2 s_1^{0,1} + (1-a_2)s_1^{0,2}} > 0$.

Then, there are two cases to consider.

Case 1. $\mu = 0$

TRE have a unique solution with $\eta(\omega) = \eta^* = A$, all ω, so that $p(\omega)$ is colinear to $p(1)$, all ω. In particular, this implies that TSE reduce to a single equation which has a one-dimensional continuum of solutions s_1^{1*} s.t.

$(a_1 + a_2\eta^*, (1-a_1) + (1-a_2)\eta^*)s_1^{1*} = 0$.

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III.A. A Specific Investment Restriction (cont'd)

Proposition 1. (cont'd)

Case 2. $\mu > 0$

There is an open interval $(\underline{\gamma}, \overline{\gamma})$ with $0 < \underline{\gamma} < \overline{\gamma}$ s.t., for $\gamma \in (\underline{\gamma}, \overline{\gamma})$, TRE have exactly two distinct solutions with, for example, $\underline{\eta}(d)^* < \eta^* < \overline{\eta}(d)^*$.

(Here and after, ***it is convenient*** to utilize the so-called applied theory notation $\omega = 0, u, d$ [sic]).

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III.A. A Specific Investment Restriction (cont'd)

Moreover, TSE have a unique solution independent of the stochastic weights

$$s_1^{1*} = \left(\frac{1-a_2}{a_1-a_2}, \frac{-a_2}{a_1-a_2}\right). //$$

We emphasize that this example is very robust. The results we describe obtain on an open set of all the parameters of the model, including γ .

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III.B. An Abstract Investment Restriction

For this analysis, we ignore the investment restriction itself and elaborate the analysis of local perturbations of TRE (still including μ as a variable) around the "stationary" solution $\eta(\omega) = \eta*$, all ω, in order to derive properties of their global solutions. The basic rationale is that if $\mu > 0$, then we can "tailor" many investment restrictions which depend on endogenous variables (when they are specified in parametric form). In fact, this can be done for the specific investment restriction introduced in the preceding section – as illustrated in Fig. 1 below.

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III.B. An Abstract Investment Restriction (cont'd)

Let $\Phi(\xi,\theta)=0$ represent TRE, where

$\xi=(\eta(0),\eta(u),\mu)$ (the "dependent" variables),

$\theta=\eta(d)$ (the "independent" variable),

$\Xi=\mathbb{R}^3_{++}\cup\{\eta^*,\eta^*,0\}$, and

$\Theta=\{\theta\in\mathbb{R}_{++}\text{: there is }\xi\in\Xi\text{ s.t. }\Phi(\xi,\theta)=0\}$.

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III.B. An Abstract Investment Restriction (cont'd)

Proposition 2. Under the same hypotheses as in Proposition 1, there is a C^∞ mapping $\varphi:\Theta\to\Xi$ s.t. , for $\theta\in\Theta$,

(i) $\xi\in\Xi$ and $\Phi(\xi,\theta)=0 \Leftrightarrow \xi=\varphi(\theta)$ and

(ii) $\theta\neq\eta^*$ and $\xi=\varphi(\theta) \Rightarrow \mu>0$.

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III.B. An Abstract Investment Restriction (cont'd)

Remark. In fact, $D\mu|_{\eta(d)=\eta^*}=0$ while $D^2\mu|_{\eta(d)=\eta^*}>0$, and $D\eta(u)|_{\eta(d)=\eta^*}=-\pi(d)/\pi(u)$ (so that for the variable $y=\pi(u)\eta(u)+\pi(d)\eta(d)$, we have $Dy|_{\eta(d)=\eta^*}=0$ while $D^2y|_{\eta(d)=\eta^*}>0$).

Applying Proposition 2 for the specific investment restriction introduced earlier yields Proposition 1, as illustrated in Figure 1.

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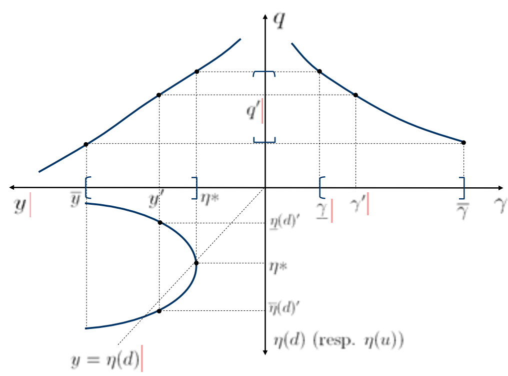
Figure 1. Relating the local and global results



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III.B. An Abstract Investment Restriction (cont'd)

A very important aspect of both Propositions is that if $\eta(d)^*\neq\eta(u)^*$ (which must be the case when $\eta(d)^*\neq\eta^*$), then

$$\text{Rank}\begin{pmatrix}p(u)^*\\p(d)^*\end{pmatrix}=2 \quad\text{and}$$

$$\begin{pmatrix}p(u)^*\\p(d)^*\end{pmatrix}^{-1} \text{ is independent of } \eta(\omega)^*,\ \omega>0.$$

Both this critical property of TSE and the local analysis of TRE appear to be quite generalizable.

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III.B. An Abstract Investment Restriction (cont'd)

Interpretation of the Propositions Concerning Multiple Equilibria. There are two types of equilibria, Pareto efficient (E-type) and Pareto inefficient (I-type). For the E-type, there is a continuum of equilibria, but in which spot goods prices and allocations are identical. Moreover, the asset market is incomplete. For the I-type, there are exactly two distinct equilibria, but in which the portfolio strategies are identical (in a T-period environment with T>2, this means that investors follow a buy-and-hold strategy). Moreover, the asset market is complete.

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III.C. Sunspot Equilibria

Suppose now that, in addition to intrinsic uncertainty $\omega = u, d$, there is also extrinsic uncertainty $\sigma = g, b$, so that overall uncertainty is $\pi(\sigma, \omega) = \pi(\sigma)\pi(\omega)$, with $\sigma = g, b$, $\omega = u, d$. Consider the extreme case where $\eta(g, u) = \eta(g, d) = \eta^*$. Then Proposition 2 obtains as stated. More generally, for $(\eta(g, u), \eta(g, d), \eta(b, d)) \in \Theta$ (the "independent" variables now), there will be a 2-dimensional manifold of sunspot equilibria, while the nonsunspot equilibria correspond to the restriction $\eta(g, \omega) = \eta(b, \omega)$, $\omega = u, d$.

III.C. Sunspot Equilibria (cont'd)

Interpretation of the Proposition 2 (taken as) Concerning Sunspot Equilibria. Say that a "sunspot realization σ matters" if, conditional on its realization, spot good prices do not smooth consumption, i.e., $\eta(\sigma, u) \neq \eta(\sigma, d)$. (Note that if a sunspot realization doesn't matter [resp. does matter] then the corresponding conditional allocation is [resp. isn't] Pareto optimal). There are sunspot equilibria in which sunspots don't matter when $\sigma = g$, but do matter when $\sigma = b$.

IV. Some Extensions

These are more or less in order of decrease in our level of understanding. Their common feature is that each appears amenable to local analysis.

- **3 intrinsic states of the world and 2 goods**

  ⋮
- **T >2 Periods**

  ⋮
- **More than 2 (intrinsic or extrinsic) states and 2 goods**

  ⋮

V. Further Research

- More than 2 households
- Robustness (more than simply parametric):

  Beyond the forest (aka trees)

  Beyond the mill (aka logs) – the most problematic, but also the most interesting – intuition is from the textbook example illustrated in Figure 2.

Figure 2. Robustness of utility functions

The Walrasian model with G=H=2, log-linear utility functions, $a_h log c_h^1 + (1 - a_h) log c_h^2$ with $a_2 < a_1$, and total endowments (1,1)