# Games with Incomplete Awareness

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**Abstract**

A new game form termed games with incomplete awareness is defined. This game form captures unawareness as to other players' actions, as well as unawareness of the existence of some players. It also captures interactive unawareness: the awareness of players as to what other players are aware of, and so on. The Nash equilibrium solution is extended to this game form and it is shown that every game with incomplete awareness possesses an extended Nash equilibrium. **JEL Classification: C72,D81,D82.**

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# 1 Introduction

While the modeling tools for analyzing various aspects of strategic interactions have been continuously evolving and successfully implemented, it is evident that many economic interactions are governed by forces that are not fully captured by our existing tools. One such aspect of interactive decision making is the presence of unawareness and its strategic implications. In this paper we develop and study a modeling tool that allows us to capture unawareness in an interactive strategic setting, namely, games with incomplete awareness.

We interpret unawareness as the inability to reason. In the context of strategic interactions the relevant fundamental objects are actions and players. Hence, unawareness is restricted to these objects. A player – Alice – can be unaware of some of the actions available to another player – Bob. These actions are out of her scope of reasoning. But subject to what she is aware of, Alice is assumed to be just like any other player in a regular game. Things become more complicated when we consider that Alice may actually be aware of Bob's actions but Bob may be unaware that Alice is aware of his actions. It is Bob that has a limited view of the world, in the sense that he does not fully appreciate Alice's comprehension of the strategic situation.

For a concrete example, consider the following game with incomplete awareness. We begin with the game depicted in (1) below. This game represents all the actions available to every player and the payoffs associated with each action profile. Assume that Alice and Bob are both aware of all the actions available in the game. We also assume that they are commonly aware of each other's existence. However, Alice is unaware that Bob is aware of *all* her actions. She is only aware that he is aware of the actions $\{a_1, a_2, b_1, b_2, b_3\}$, i.e. she is unaware that he is aware of her third action. We assume that Bob is fully aware of the extent to which Alice is aware of his awareness, i.e. Bob is aware that Alice is aware that Bob is aware of $\{a_1, a_2, b_1, b_2, b_3\}$. We also assume that Bob is aware that Alice is aware of the whole action set $\{a_1, a_2, a_3, b_1, b_2, b_3\}$. Turning to higher levels of iterative awareness; since Alice is only aware of Bob being aware of $\{a_1, a_2, b_1, b_2, b_3\}$, she cannot be aware that he is aware that she is aware of anything beyond this set, otherwise, she would be aware that he is able to reason about her reasoning about the additional action $a_3$, so he must be able to reason about $a_3$ as far as Alice can deduce which contradicts our assumption that Alice is unaware that Bob is aware of $a_3$. In particular, any higher order iteration of awareness of Alice and Bob which is not considered above is assumed to be associated with the set $\{a_1, a_2, b_1, b_2, b_3\}$.

| | Bob | | | |
|---|---|---|---|---|
| | | $b_1$ | $b_2$ | $b_3$ |
| Alice | $a_1$ | 0,2 | 3,3 | 0,2 |
| | $a_2$ | 2,2 | 2,1 | 2,1 |
| | $a_3$ | 1,0 | 4,0 | 0,1 |

(1)

In this example we already have a glimpse at some of the assumptions about interactive awareness that will be used in defining games with incomplete awareness. In this example we made explicit the assumption that every player is aware of her own actions and that there is common awareness of this fact, and that if Alice is aware that Bob is aware of something then she must be aware of it as well.

The game depicted in (1) has a unique Nash equilibrium $(a_2, b_1)$ obtained by eliminating strictly dominated strategies. However, while both players are aware that this is the game being played, we assumed that Alice is only aware that Bob is aware of two of her actions. In other words, Alice is unaware that Bob is aware of $a_3$. While Alice and Bob both view the game as in (1) Alice perceives that Bob finds the game being played as depicted in (2).

| | Bob | | | |
|---|---|---|---|---|
| | | $b_1$ | $b_2$ | $b_3$ |
| Alice | $a_1$ | 0,2 | 3,3 | 0,2 |
| | $a_2$ | 2,2 | 2,1 | 2,1 |

(2)

Since Alice finds that, as far as Bob is aware, he finds that *she* perceives the game as in (2), and so on for every higher order awareness, Alice finds that Bob views the game as a standard normal form game with complete awareness. Taking Nash equilibria as the solution concept for normal form games, Alice may deduce that Bob plays according to the Nash equilibrium $(a_1, b_2$ of the game in (2) which is also the Pareto dominant outcome of this normal form game. Alice, who sees herself as being more aware than Bob, will be inclined to choose $a_3$ which is her best response to $b_2$. Bob can make the exact same deduction that we, as modelers, just made, since he is aware of all the actions and is fully aware of what Alice is aware that he is aware of. Hence, Bob will play his best response to $a_3$. We will have that Alice chooses $a_3$ and Bob chooses $b_3$ as a result of this higher order unawareness. We end up with the worst possible payoff for Alice and a low payoff for Bob although both are aware of the full extent of the game and both act rationally given their perceived view of the game.

A game with incomplete awareness is defined as a normal form game and an unawareness construction, were an unawareness construction describes the actions and players that each player is aware of, the actions and players that each player is aware that each other player is aware of and so on. This definition has the property that if we only consider what a given player – Alice – is aware of, including what she is aware of higher orders of awareness of players, we find that she is aware of a game with incomplete awareness. Furthermore, every player is aware that other players are viewing the game as some game with incomplete awareness and this holds for higher orders of iterated awareness.

Since every high order iteration of awareness yields a game with incomplete awareness we can define the extension of Nash equilibria to games with incomplete awareness. We assign a mixed strategy of the player corresponding to a given high order awareness. For example, if we consider Alice's awareness of Bob's awareness, we are considering how Alice views Bob's view of the game and associate a mixed strategy of Bob for this specific view. Obviously, this strategy needs to be supported in the set of actions that Alice is aware of. The assignment of mixed strategies to each iterated level of awareness is said to constitute an extended Nash equilibria if at every awareness iteration the strategy assigned is a best response to the strategies assigned for other players. Specifically, Alice is playing a best response to the strategy assigned to Bob in the game that Alice is aware that Bob is aware of. The strategy that Bob is assigned in the game that Alice is aware that Bob is aware of, is a best response to the strategy assigned to Alice in the game that Alice is aware that Bob is aware that Alice is aware of, and so on.

The term extended Nash equilibrium is justified by showing that its definition implies that if at some point the high order awareness leads to a game with complete awareness – a normal form game – the strategies assigned by an extended Nash equilibrium constitute a Nash equilibrium of this normal form game. This follows from the assumption that the extended solution must assign identical strategies to different high order uncertainties that have identical views of the game. Furthermore, since we show that there is exactly one possible game with incomplete awareness imbedded at a high order of awareness of any given game with incomplete awareness, the definition of our solution implies that the same Nash equilibrium is played in all the manifestations of this embedded normal form game.

The intuition behind the extended Nash equilibrium is straightforward. With every view of the game being played a player is playing a best response to how she finds that others are playing according to the extended solution, in the sense that she considers the game restricted to what she is aware that they are aware of. Furthermore, if two different perspectives (iterations of awareness) coincide in their description of the game as well as awareness of others about the game and so on, then the same strategic behavior is assumed for these two viewpoints. In the game depicted in (1) we can associate the action $b_2$ with the game that Alice is aware that Bob is aware of, and the action $a_1$ with the game that Alice is aware that Bob is aware that Alice is aware of, and so on for every level of awareness. Hence we associate the Nash equilibrium $(a_1, b_2)$ for the game in (2). Once we chose this equilibrium for the game that Alice is aware that Bob is aware of, the definition implies that Alice must play the unique best response $a_3$. Since how Bob views that Alice views the game and higher orders of awareness coincides with how Alice actually views the game, we have that Bob must play a best response to the same strategy $a_3$ and hence we must have we also have that Bob chooses $b_3$ as suggested above.

We show that not all games with incomplete awareness have that all higher orders of awareness terminate with a game of complete awareness. The question then arises whether an arbitrary game with incomplete awareness posses an extended Nash equilibrium. Our main result states that they do. Hence, the extended Nash equilibrium solution is non-empty.

The work is the product of two strands of research. A wide variety of economic interactions are modeled with boundedly rational players, usually in an ad hoc manner that considers very simple situations of unawareness. For example, Merton (1987) is a typical example where investors are fully rational but aware only of a subset of the firms they can invest in.[1] These type of models are prevalent in the behavioral literature pointing to unawareness as one possible source of deviation from theories of rational decision makers. On the other hand, recent advancements in the foundational frameworks for interactive unawareness provide the basis for the applicable construction provided in this paper. The combination of the demand for a strategic framework for solving interactive economic situations with unawareness and the supply of abstract methods of representing interactive unawareness lead to the results presented here.

We are motivated by applications of models of unawareness as achieved by Modica, Rustichini and Tallon (1998) and more recently by Kawamura (2004) who applied the foundational work of Modica and Rustichini (1994,1999) on modeling single person unawareness to a general equilibrium model with unawareness. In order to enable an interactive setting we refer to the foundations laid by Feinberg (2004c), Heifetz, Meier and Schipper (2004) and Li (2004) for interactive unawareness[2]. Our interpretation of unawareness and the properties of higher order unawareness build directly on Feinberg (2004c) which provides an epistemic justification for the interactive unawareness construction used in this paper. Games with incomplete

[1]In particular, there is no reasoning or consideration of the unawareness of others.

[2]For epistemic foundations of unawareness see also Fagin and Halpern (1987), Fagin, Halpern, Moses and Vardi (1995) and Halpern (2001).

awareness can be seen as semantic models for interactive unawareness in a strategic setting. This follows the semantic approach of Heifetz, Meier and Schipper (2004). Heifetz, Meier and Schipper (2004) provide a general class of models for interactive unawareness. Their approach allows for unawareness of abstract events and does not utilize specific features of games.[3] The work of Li (2004) provides a set theoretic representation of unawareness. While there are more substantial differences in how high order unawareness is captured, Li's expression of unawareness as projections into a subspace of a product space, is similar to our restriction of awareness in a game to a subset of actions[4]. We also note the work of Ewerhart (2001) where interactive unawareness is modeled in relation to the agreeing to disagree theorem.

A different problem of limited comprehension of players in games was confronted by Crawford and Haller (1990) and Blume (2000). They considered players cognition to be limited in the sense that they posses private languages representing the naming of actions. Considering a repeated symmetric coordination game the players may be unable to distinguish some of the actions and hence they may be forced to treat them equally strategically. Blume and Gneezy (2000) study players with possibly different languages for the symmetric coordination games and also test this model experimentally[5]. They consider optimal attainable strategies which are best responses to how a player perceives the strategies of other players' types that have a coarser language. We adopt a conceptually similar approach to the justification of our extended Nash equilibrium solution, i.e. playing a best response to how one boundedly perceives what others are playing.

In Section 2 we define games with incomplete awareness. In Section 3 we define the extended Nash equilibrium solution for games with incomplete awareness and show its non-emptiness as well as additional properties of games with incomplete awareness. Section 4 concludes with a discussion.

# 2 Games with Incomplete Awareness

A game with incomplete awareness is defined as a normal form game $\Gamma$ and an unawareness construction that describes which actions and players each player in $\Gamma$ is aware of, which actions and players each player is aware that other players are aware of, and so on.

Consider a normal form game $\Gamma = (I, \prod_{i\in I} A_i, u_i : \prod_{j\in I} A_j \to R)$. The unawareness construction we use is based on the general definition introduced in Feinberg (2004a) applied

[3]This leads to different interpretations of high order awareness in the two approaches.

[4]Li's main concern is a unified partition-like framework for representing unawareness and the product representation is mostly used for expository reasons.

[5]See also the experimental work by Blume and Gneezy (2004).

to the set of players $I$ and the set of actions $\alpha = \bigcup_{i \in I} A_i$ in the given game. An unawareness construction associates with every $n$-tuple of players, denoted $\theta = (i_1, ..., i_n)$, a set of players $I_\theta \subseteq I$ and a set of actions $\alpha_\theta \subseteq \alpha$. The interpretation of $I_\theta$ and $\alpha_\theta$ is the set of players and actions that $i_1$ is aware that $i_2$ is aware that ... that $i_n$ is aware of.

We denote $\Theta = \bigcup\limits_{n=0}^{\infty} (I)^n$ and let $I_{\{\emptyset\}} = I$ and $\alpha_{\{\emptyset\}} = \alpha$. We assume that $I_\theta \neq \emptyset$ if and only if $\alpha_\theta \neq \emptyset$ and we denote $\bar{\Theta} = \{\theta \in \Theta | I_\theta \neq \emptyset \text{ and } \alpha_\theta \neq \emptyset\}$. We refer to $\theta \in \bar{\Theta}$ as an instance of higher order awareness, as a state of awareness or as iterated awareness. It will be convenient to denote $\theta \succ \bar{\theta}$ whenever $\theta = (i_1, ..., i_n)$ and $\bar{\theta} = (i_{k_1}, i_{k_2}, ..., i_{k_m})$ with $1 \leq k_1 < k_2 < ... < k_m \leq n$ and $m < n$. The order $\succ$ is a partial transitive order on finite words $\theta$ comprised of letters from the finite alphabet $I$. The weak order $\succeq$ also allows for the two words to coincide.

We postulate the following consistency conditions for higher order of unawareness:

1. For every $\theta = (i_1, ..., i_n) \in \Theta$, if $\theta \succeq \bar{\theta}$ we have that $I_\theta \subset I_{\bar{\theta}}$ and $\alpha_\theta \subset \alpha_{\bar{\theta}}$.

   This condition states that whenever Alice is aware that Bob is aware of an action or a player then Alice must also be aware of that action or player. It also requires that if Alice is aware that Bob is aware of something then Bob is indeed aware of it and that there is higher order awareness of these two properties. These conditions correspond to our particular interpretation of awareness and high order awareness. Recall that awareness of an action or a player is understood as the ability to reason about them – being part of the language expressing the reasoning of the aware player. This explains why if Alice can reason that Bob can reason about some statement then Alice already has that statement in the language they use and hence Alice can reason about that statement as well. In order to understand the other direction implied by this assumption we interpret higher order awareness as existence in the language. Hence, if Alice is aware that Bob is aware of an action then Bob being aware of the action is part of the language. In the general framework in Feinberg (2004c) this need not imply that Bob must actually be aware of the action since Bob may have several identities some of which are not aware of the action. However, since Bob's awareness of the action is part of the language, we must have that there exists a manifestation of Bob that *is* aware of this action. This is the hypothetical manifestation that Alice considers possible. Furthermore, Alice will be unaware of any other manifestation if she is unaware that Bob might be unaware of this action. In the restricted framework of games considered here we do not allow for several manifestations of a player since uncertainties about awareness or dynamics are excluded. Hence, the unique "awareness type" of each player implies that Bob must be aware of the action if Alice is aware that he is.[6]

2. For $\theta = (i_1, ..., i_k, i_{k+1}, ..., i_n)$ such that $i_k = i_{k+1}$ for some $k$ we have $I_\theta = I_{\bar{\theta}}$ and $\alpha_\theta = \alpha_{\bar{\theta}}$ where $\bar{\theta} = (i_1, ..., i_{k-1}, i_{k+1}, ..., i_n)$.

   Here we require that Alice is aware of everything that she is aware that she is aware of. Note, that the first condition implies that what she is aware that she is aware of, she must be aware of. In addition, there is higher order awareness of that.

3. For all $\theta = (i_1, ..., i_n) \in \bar{\Theta}$ we have $i_n \in I_\theta$.

   The third condition states that Alice is aware of her own existence and that if there is higher order awareness of Alice being aware of something, i.e. of Alice's reasoning, then there is the same high order awareness of Alice's awareness of herself and in particular awareness of Alice.

   The following conditions relate to the unawareness construction in the context of the game $\Gamma = (I, \prod_{i \in I} A_i, u_i)$.

   Denote by $A_i^\theta = A_i \bigcap \alpha_\theta$ the set of actions of player $i$ that the state of awareness $\theta$ allows.

4. $A_i^\theta \subset \alpha_{\theta\hat{}i}$.

   This condition states that Alice is aware of her own actions and that there is high order awareness that she is aware of the actions available to her as perceived from that high order awareness.

5. $i \in I_\theta$ implies $A_i^\theta \neq \emptyset$.

   The final condition states that if there is awareness of a player then there must be awareness of one of his actions. Hence, the awareness of the existence of a player implies awareness that the player participates in the game.

[6] See Feinberg (2004c) for further discussion of this assumption.

**Definition 1** *A collection* $\mathcal{U} = \{I_\theta, \alpha_\theta\}_{\theta\in\Theta}$ *which satisfies the consistency conditions* $1-3$ *above is called an* unawareness construction *or an* awareness construction*.*

We note that the inverse of condition 5 follows from conditions 4 and 3. In addition, conditions 3 and 5 imply that $\alpha_\theta \neq \emptyset \Leftrightarrow I_\theta \neq \emptyset$ which otherwise is an additional assumption made for general unawareness constructions.

**Definition 2** *A normal form game* $\Gamma = (I, \prod_{i\in I} A_i, u_i)$ *and an unawareness construction* $\mathcal{U}$ *with players* $I$ *and* $\alpha = \bigcup_{i\in I} A_i$ *which satisfies the consistency conditions* $4-5$ *above is called* a game with incomplete awareness *and will be denote by* $\Gamma^{\mathcal{U}}$*.*

This construction for high order awareness is based on the representation of reasoning with interactive awareness provided in Feinberg (2004c). In that paper we used conditions $1-3$ to define the properties of high order unawareness. We then considered the language for reasoning based on the subjective views of reasoning agents in a game as proposed in Feinberg (2004a). Compounding the awareness construction on the subjective framework for reasoning yields a language that only allows for statements where agents reason about things they are aware of. Hence the interpretation we use here for awareness as the ability to reason is derived from the syntactic representation in Feinberg (2004c).[7]

For any non-empty high order view of awareness $\theta \in \bar{\Theta}$ we can consider all the possible continuations of higher order awareness. Considering an unawareness construction restricted to these continuations is itself an unawareness construction. We denote this construction by $\mathcal{U}^\theta = \{I_{\theta\hat{}\bar{\theta}}, \alpha_{\theta\hat{}\bar{\theta}}\}_{\bar{\theta}\in\Theta}$ where $\theta\hat{}\bar{\theta}$ is the concatenation of the two words.

The normal form game corresponding to the view of awareness $\theta$ is defined as $\Gamma_\theta = (I_\theta, \prod_{i\in I_\theta} A_i^\theta, u_i^\theta)$ where $I_\theta$ and $A_i^\theta$ are already defined. For the definition of $u_i^\theta$ we first distinguish a specific action $d_i \in A_i$ for every $i \notin I_\theta$. We define $u_k^\theta$ as the projection of $u_k$ to $\prod_{i\in I_\theta} A_i^\theta$ fixing the action of players $j \notin I_\theta$ as $d_j$.

$$u_k^\theta(\{a_i\}_{i\in I_\theta}) = u_k(\{a_i\}_{i\in I_\theta}, \{d_j\}_{j\notin I_\theta}) \tag{3}$$

Distinguishing an action $d_i$ for a player of which there is unawareness is based on the context of the game. This action corresponds to a default, or non-participation, action. For example, in an auction if Alice is unaware that Bob participates we can set her payoff – as far as she is aware of the game – conditional on Bob opting out of the auction.

[7] Feinberg (2004c) also contains a definition of dynamic games with unawareness using the syntactic approach. In fact, actions in the game are represented as atomic statements. However, the epistemic form of a dynamic game is much more complicated than the structures provided here. Furthermore, there is no direct definition of a solution for the epistemic form of dynamic games with unawareness. In order to discuss players behavior we resorted to epistemic characterization of solutions as provided by Feinberg (2004b).

**Proposition 3** *For every high order view of awareness $\theta \in \bar{\Theta}$ the normal form game $\Gamma_\theta$ with the unawareness construction $\mathcal{U}^\theta$ constitute the game with incomplete awareness $\Gamma_\theta^{\mathcal{U}^\theta}$.*

See the Appendix for the proof of this proposition.

# 3 Extended Nash Equilibria of Games with Incomplete Awareness

A game with incomplete awareness can be seen as a collection of normal form games – the games as viewed from every high order awareness. When considering a solution we associate a mixed strategy with each non-empty high order awareness. For example, if we consider Alice being aware of Bob being aware of the game $\Gamma_{(Alice,Bob)}$ we will associate a mixed strategy for Bob in this game. This will be the strategy that Alice will assume Bob is playing in what she views as the game that he is aware of.

The extended Nash equilibrium is defined as a collection of mixed strategies for every order of awareness, such that at every view of the game a best response is played to how other players are playing the game they aware of, where this awareness is captured at the high order considered. Hence, we ask that Alice play a best response to the strategy assigned to Bob in the game that Alice is aware that Bob is aware of. We also require that if at two states of awareness the two games with incomplete awareness as viewed from the two states coincide, then the strategies assigned with the states must be identical. In other words, the solution may only depend on the game with unawareness being considered an not at which high order awareness the game is considered.

**Definition 4** *An extended equilibrium of a game with incomplete awareness $\Gamma^{\mathcal{U}}$ assigns to each state of awareness $\theta = (i_1, ..., i_n)$ a mixed strategy $\sigma^\theta \in \Delta(A_{i_n}^\theta)$ such that:*

- *$\sigma^\theta$ is a best response to $\{\sigma^{\theta\hat{}j}\}_{j \in I_\theta \setminus i_n}$*
- *For all $\theta, \bar{\theta}$ with the same last member $i \in I$, if $\Gamma_\theta^{\mathcal{U}^\theta} = \Gamma_{\bar{\theta}}^{\mathcal{U}^{\bar{\theta}}}$ then $\sigma^\theta = \sigma^{\bar{\theta}}$*

Note that the solution assigns a strategy to only a single player at every level of awareness $\theta = (i_1, ..., i_n)$, since the game $\Gamma_\theta$ is the game that $i_1$ views that $i_2$ views that ... that $i_n$ views as the game being played. Hence, the high order view is of $i_n$'s action in $\Gamma_\theta$. In particular, the strategy profile comprised of the each single player's strategy in the games $\{\Gamma_i\}_{i \in I}$ can be viewed as the actual strategy profile associated with the game of incomplete awareness $\Gamma$.

The justification for the term extended Nash equilibrium comes from the following result which states that when there is no unawareness, the solutions coincide.

**Proposition 5** *If at some state of awareness the game is viewed as a standard normal form game (all higher orders of awareness have the same view of the game) then the extended equilibrium dictates that from that point of awareness onward the players are playing a Nash equilibrium of that normal form game. Formally, if for some* $\theta$ *we have that* $\alpha_\theta = \alpha_{\theta\hat{}\bar{\theta}}$ *for all* $\bar{\theta}$ *with* $I_{\theta\hat{}\bar{\theta}} \neq \emptyset$ *then* $\{\sigma_j^{\theta\hat{}j}\}_{j\in I_\theta}$ *forms a Nash equilibrium of* $\Gamma_\theta$*.*

When viewed as the countable collection of normal form games associated with each order of awareness it is not clear why an extended Nash equilibrium should exists. Indeed, as Example 8 below demonstrates, there need not be a bound on the level of iteration of awareness where the view of the game is modified due to introduction of yet higher orders of awareness. Hence, since at every order of awareness $\theta = (i_1, ..., i_n)$ a player $i_n$ is playing a best response to strategies associated with higher order of awareness, we need to show that such a simultaneous best response at all $\theta \in \bar{\Theta}$ exists. Fortunately, the structure of awareness yields an existence theorem.

**Theorem 6** *Every game with incomplete awareness has an extended equilibrium.*

The proof of Theorem 6 is based on the following proposition which states that each game with incomplete awareness contains only a finite number of views of the game, in the sense that there is one out of a finite number of games with unawareness associated with each high order of awareness. The proof of this proposition and of Theorem 6 appear in the Appendix.

**Proposition 7** *For every unawareness construction* $\mathcal{U}$ *with a finite set of actions* $\alpha$ *and players* $I$*, there is a finite number of views of the unawareness construction, i.e., there exist a finite* $m$ *and* $\theta_1, ..., \theta_m$ *such that for all* $\theta \in \bar{\Theta}$ *there is an* $i \in \{1, ..., m\}$ *with* $\mathcal{U}^\theta = \mathcal{U}^{\theta_i}$*.*

The following example demonstrates how a game with incomplete awareness may have unboundedly high orders of awareness that have a strictly restricted view of the game.

**Example 8** *Consider a three players game* $\Gamma$ *with players* $(\{1,2,3\}$ *and respective sets of actions* $(\{a_1, b_1\}, \{a_2, b_2\}, \{a_3, b_3\})$*. Assume there is common awareness of the existence of all the players, i.e. for all* $\theta$ *we have* $I_\theta = \{1,2,3\}$*. We define* $\mathcal{U}$ *by setting*

$$\alpha_\theta = \left\{ \begin{array}{cc} (\{a_1, b_1\}, \{a_2, b_2\}, \{a_3, b_3\}) & \text{3 does not appear in } \theta \\ (\{a_1\}, \{a_2, b_2\}, \{a_3, b_3\}) & \text{3 appears in } \theta \end{array} \right). \tag{4}$$

*The first condition defining a game with incomplete awareness is satisfied since there is common awareness of the set of players and since $\alpha_\theta$ is monotone – containing all actions if player 3's awareness is not considered and containing all but the action $b_1$ of player 1 once the awareness of player 3 is considered. Condition 2 holds since the example defines $\alpha_\theta$ only based on which players appear in $\theta$ and not the order or frequency of appearances. Condition 3 is satisfied trivially. It suffices to check condition 4 in the case where $i = 1$ since otherwise $\alpha_{\theta\hat{}i} \cap A_i = A_i$. Since 3 is a member of $\theta$ if and only if it is a member of $\theta\hat{}1$ we also have that $\alpha_{\theta\hat{}1} = \alpha_\theta$ hence the condition must hold. The last condition also holds trivially. Hence the game $\Gamma$ (with arbitrary payoffs) with the awareness construction described above constitute a game with incomplete awareness.*

Note that for every $\theta$ that contains only the awareness of players 1 and 2 we have that $\Gamma_\theta \neq \Gamma_{\theta\hat{}3}$ hence the awareness view is not set after any given finite iteration of high order awareness. However, as Proposition 7 requires, in this example we have only two possible views of the game with unawareness. For every $\theta$ we have either $\Gamma_\theta^{\mathcal{U}^\theta} = \Gamma^{\mathcal{U}}$ if 3 does not appear in $\theta$, or $\Gamma_\theta^{\mathcal{U}^\theta} = \Gamma_{(3)}^{\mathcal{U}^{(3)}}$ if 3 appears in $\theta$.

We conclude this section by pointing out some properties of extended Nash equilibria. We are mainly interested in how the strategies assigned by an extended equilibrium to each player at the first order awareness of the game relate to strategies in the normal form game at hand.

We first note that pure strategy Nash equilibria are maintained as long as all players are aware of all the actions constituting the equilibrium.

**Proposition 9** *Let $\Gamma = (I, \prod_{i\in I} A_i, u_i)$ be a normal form game and let $\mathcal{U}$ be such that $\Gamma^{\mathcal{U}}$ is a game with incomplete awareness. If $\tilde{\sigma} = \{\tilde{\sigma}_i \in \Delta(A_i)\}_{i\in I}$ is a Nash equilibrium of $\Gamma$ and $\tilde{\sigma}_i(a_i) \geq 0$ implies $a_i \in \alpha_\theta$ for all $\theta$ then there is an extended Nash equilibrium of $\Gamma^{\mathcal{U}}$ such that $\sigma^{(i)} = \tilde{\sigma}_i$ for all $i \in I$.*

The proof of this proposition follows immediately by associating the mixed strategy $\tilde{\sigma}_i$ with every high order awareness that terminates with $i$. By the assumption that there is common awareness of the support of $\tilde{\sigma}$ this mapping is feasible and results with an extended Nash equilibrium since every player plays against the same strategy with possibly eliminated pure strategies outside of the support of the equilibrium strategy, hence players play best responses in all the games $\Gamma_\theta$. Note that common awareness of the support implies that $I_\theta = I$ for all $\theta$.

It is worthwhile noting that the common awareness of the support cannot be replaced by awareness, even if one considers pure strategy Nash equilibria and under the assumption of common awareness of the set of players $I$. For example, consider $\Gamma$ to be the game depicted in (1) with common awareness of the set of players $I = \{Alice, Bob\}$, Alice and Bob both being aware of all actions, Bob being aware that Alice is aware of all actions but Alice being aware that Bob is aware only of her action $a_1$. For all other order of awareness consider the maximal sets allowable under the conditions of a game with incomplete awareness. In any extended Nash equilibrium we have that Alice finds that Bob must play $b_2$ which implies that her best response is $a_3$ which is not part of the support of any Nash equilibrium of the original game.

# 4 Discussion and Extensions

The main contribution of the paper is the extension of Nash equilibrium to a new class of games with incomplete awareness. Both game forms and solution concepts are natural avenues for extensions of this work. Extending the framework to games with incomplete awareness and incomplete information is one such possibility. While Feinberg (2004c) presents a framework for the epistemic form of a dynamic game with unawareness, a more direct semantic framework that combines Harsanyi's incomplete information constructions with unawareness might yield itself more easily to analysis[8]. Alternatively, one can consider refinements to the extended Nash equilibrium, e.g. via perturbations whose impact is correlated with the unawareness construction.

The question also arises in what way do game with incomplete awareness differ from games with incomplete information. We point out that unawareness is distinctively different from assigning zero probability as is clearly revealed in an interactive setting:

> If Alice is aware that Bob is aware of an event, then Alice must be aware of the event as well, on the other hand, if Alice assigns positive probability that Bob assigns positive probability to an event we would generally not impose that Alice assign positive probability to the event.

However, both notions are similar in the the natural hierarchy created when we wish to incorporate one player's awareness (belief) about other players awareness (belief). In fact, unawareness can be seen as an extreme version of incomplete information in the sense that the support of beliefs usually will not coincide at various high order levels of awareness and for various players. In this sense we expect almost always the common prior assumption to be violated, hence the starting point is orthogonal to Harsanyi's consistency condition.

[8] In Feinberg (2004c) a solution is accessed indirectly via its epistemic characterization.

Resolving the relationship between uncertainty and unawareness would be central for an extension to games that express both notions and in particular uncertainty about others unawareness.

Finally, the notion of unawareness assumed in this work is just one interpretation of the phenomena of unawareness in economic interaction. A variety of other conditions could be assumed for the implications of high order awareness. For example, we could relax condition 1 and require that if Alice is aware that Bob is aware of $a$ then Alice is aware of $a$ without also requiring that Bob must indeed be aware of $a$. This will avoid the need to use two different manifestations of Bob to capture the possibility that Bob is unaware of $a$ as suggested in Feinberg (2004c). This relaxed conditions would lead to a new notion of unawareness, albeit one that does not currently have an epistemic formulation. On the other hand, one could also try and formulate conditions that correspond to existing alternative interactive awareness frameworks, such as Heifetz, Meier and Schipper (2003) and Li (2003). The question is whether alternative formulations can be supported by an epistemic framework, or what notions of unawareness in games can arise from alternative epistemic formulations of interactive awareness.

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# Appendix

**Proof of Proposition 3.** We need to show that conditions $1-5$ hold for $\Gamma_\theta$ with the unawareness construction $\mathcal{U}^\theta$. Since $\theta \in \bar{\Theta}$ we have that conditions $1-3$ hold from $\mathcal{U}$ being an unawareness construction. Note that $\theta\hat{}\tilde{\theta} \succeq \theta\hat{}\hat{\theta}$ if and only if $\tilde{\theta} \succeq \hat{\theta}$. Also, for every $\tilde{\theta}$ we have that

$$(A_i^\theta)^{\tilde{\theta}} = A_i^\theta \cap \alpha_{\theta\hat{}\tilde{\theta}} = A_i \cap \alpha_\theta \cap \alpha_{\theta\hat{}\tilde{\theta}} = A_i \cap \alpha_{\theta\hat{}\tilde{\theta}} = A_i^{\theta\hat{}\tilde{\theta}} \tag{5}$$

where the third equality follows from condition 1. Since $\Gamma$ with $\mathcal{U}$ satisfy conditions 4 and 5 we have that (5) implies that these conditions also hold for $\Gamma_\theta$ with $\mathcal{U}^\theta$ as required. ■

**Proof of Proposition 5.** Let $\theta$ be such that every continuation retains the same game, i.e. there is no higher order unawareness. Formally, we have that for all $\tilde{\theta}$ with members from $I_\theta$ the equality $\Gamma_\theta = \Gamma_{\theta\hat{}\tilde{\theta}}$ holds. In particular, we have $I_\theta = I_{\theta\hat{}\tilde{\theta}}$ and $\alpha_\theta = \alpha_{\theta\hat{}\tilde{\theta}}$. From the definition of an extended Nash equilibrium for all $j \in I_\theta$ and $\tilde{\theta}$ with members from $I_\theta$ we have

$$\sigma_j^{\theta\hat{}j} = \sigma^{\theta\hat{}\tilde{\theta}\hat{}j}. \tag{6}$$

From the definition of the extended equilibrium we have that for all $j \in I_\theta$, $\sigma_j^{\theta\hat{}j}$ is a best response to $\{\sigma^{\theta\hat{}j\hat{}k}\}_{k \in I_{\theta\hat{}j} \setminus j}$. But (6) implies that $\sigma_j^{\theta\hat{}j}$ is a best response to $\{\sigma^{\theta\hat{}k}\}_{k \in I_\theta \setminus j}$ and the proof is complete. ■

Before we prove the main theorem we first prove that there exists only a finite number of views of an unawareness construction. Throughout this Appendix we fix an unawareness construction $\mathcal{U}$ with finite sets $I$ and $\alpha$. For clarity and to minimize notation we will assume throughout that $I_\theta = I$ for all $\theta$. The proofs carry as stated for the general case.

**Proof of Proposition 7.** Assume by way of contradiction that there is a countable sequence of state of unawareness $\{\theta_n\}_{n=1}^\infty$ that offer distinct views of the unawareness construction. Formally, for every $n < k$ we must have a $\tilde{\theta}$ such that

$$\alpha_{\theta_n \hat{} \tilde{\theta}} \neq \alpha_{\theta_k \hat{} \tilde{\theta}}. \tag{7}$$

From Lemma 10 there is a countable subsequence $\{\theta_{n_k}\}_{k=1}^{\infty}$ such that $\theta_{n_k} \preceq \theta_{n_{k+1}}$. In particular, for all $\tilde{\theta}$ we have $\theta_{n_k} \hat{} \tilde{\theta} \preceq \theta_{n_{k+1}} \hat{} \tilde{\theta}$. From condition 1 of the definition of an unawareness construction we have for every $k$ and $\tilde{\theta}$ that

$$\alpha_{\theta_{n_k} \hat{} \tilde{\theta}} \supset \alpha_{\theta_{n_{k+1}} \hat{} \tilde{\theta}}. \tag{8}$$

From (7) we have for every $k > 1$ there exists some $\tilde{\theta}_k$ such that

$$\alpha_{\theta_{n_{k-1}} \hat{} \tilde{\theta}_k} \neq \alpha_{\theta_{n_k} \hat{} \tilde{\theta}_k} \tag{9}$$

and for this particular $\tilde{\theta}_k$ we have that

$$\alpha_{\theta_{n_1} \hat{} \tilde{\theta}_k} \supset ... \supset \alpha_{\theta_{n_{k-1}} \hat{} \tilde{\theta}_k} \supsetneq \alpha_{\theta_{n_k} \hat{} \tilde{\theta}_k}. \tag{10}$$

Since (10) holds for every $k > 2$ we have a subsequence $\{\theta_{n_k}\}_{k=1}^{\infty}$ and a sequence $\{\tilde{\theta}_k\}_{k=2}^{\infty}$ such that for all $k = 2, 3, ...$ we have

$$\alpha_{\theta_{n_k} \hat{} \tilde{\theta}_k} \neq \alpha_{\theta_{n_j} \hat{} \tilde{\theta}_k} \qquad \forall j < k. \tag{11}$$

Using Lemma 10 once more, we can find a subsequence $\{\tilde{\theta}_{k_l}\}_{l=1}^{\infty}$ such that $\tilde{\theta}_{k_l} \preceq \tilde{\theta}_{k_{l+1}}$ and considering the same subset of indices for $\theta_{n_k}$ we have for all $l$

$$\begin{aligned} \alpha_{\theta_{n_{k_l}} \hat{} \tilde{\theta}_{k_l}} &\neq \alpha_{\theta_{n_{k_j}} \hat{} \tilde{\theta}_{k_l}} \qquad \forall j < l, && (12) \\ \theta_{n_{k_l}} &\preceq \theta_{n_{k_{l+1}}}, && (13) \\ \tilde{\theta}_{k_l} &\preceq \tilde{\theta}_{k_{l+1}}. && (14) \end{aligned}$$

From (13) and (14) and since concatenation with the same word preserves the order $\preceq$ we have that for all $l$

$$\theta_{n_{k_l}} \hat{} \tilde{\theta}_{k_{l+1}} \preceq \theta_{n_{k_{l+1}}} \hat{} \tilde{\theta}_{k_{l+1}} \preceq \theta_{n_{k_{l+1}}} \hat{} \tilde{\theta}_{k_{l+2}}. \tag{15}$$

From condition 1 in the definition of an awareness construction and from (15) we have for every $l$

$$\alpha_{\theta_{n_{k_l}} \hat{} \tilde{\theta}_{k_{l+1}}} \supset \alpha_{\theta_{n_{k_{l+1}}} \hat{} \tilde{\theta}_{k_{l+1}}} \supset \alpha_{\theta_{n_{k_{l+1}}} \hat{} \tilde{\theta}_{k_{l+2}}}. \tag{16}$$

Since every $\alpha_\theta$ is a subset of the finite set $\alpha$, we conclude that there exists a $t$ such that

$$\alpha_{\theta_{n_{k_t}}}\hat{\ }\tilde{\theta}_{k_{t+1}} = \alpha_{\theta_{n_{k_{t+1}}}}\hat{\ }\tilde{\theta}_{k_{t+1}}. \tag{17}$$

Since (17) contradicts (12) we have reached the desired contradiction and the proof is complete. ■

We make use of the following Lemma due to Higman (1952).

**Lemma 10** *For every sequence of states of awareness $\{\theta_k\}_{k=1}^{\infty}$ we can find a countable subsequence $\{\theta_{n_k}\}_{k=1}^{\infty}$ such that $\theta_{n_k} \prec \theta_{n_{k+1}}$, i.e., each word $\theta_{n_k}$ can be obtained by deleting some members of the word $\theta_{n_{k+1}}$.*

**Proof.** This Lemma follows immediately from Theorem 4.4 in Higman (1952)[9]. Higman shows (as a special case of his finite basis property theorems) that given a finite alphabet $I$, every set of words $X$ from this alphabet has a finite subset $X_0$ such that for every word $w \in X$ one can find a word $w_0 \in X_0$ such that the letters of $w_0$ occur in $w$ in their right order, though not necessarily consecutively. In particular, let $X = \{\theta_k\}_{k=1}^{\infty}$, from Higman's theorem there exists a finite subset $X_0 \subset X$ such that from each word in $\theta \in X$ one can obtain at least one of the words in $X_0$ by eliminating some members in $\theta$. Since $X_0$ is finite and $X$ is countable there exists a word in $X_0$ denoted $\theta_{n_1}$ that can be imbedded in an infinite subsequence of words from $X \setminus X_0$. Hence from every countable sequence of words we can find a subsequence such that the first word in the subsequence can be obtained from *every* word that follows by eliminating some members. We can now consider the subsequence from the second word onwards and find a subsequence such that the second word can be imbedded in all the words that follow. Maintaining the same first element $\theta_{n_1}$ we now have that the first two words can be imbedded in every word that follows. By induction the required subsequence is derived. ■

We are grateful to Michael Ostrovsky for suggesting the following direct proof of Higman's theorem:

**Direct proof of Higman's theorem.**

By induction on $k$ - the number of letters in the alphabet (main induction). For $k = 1$, the claim is obvious. Suppose it is true for $k$ up to $n$. Let us show that it is also true for $k = n + 1$.

[9]Higman's results also solve the problem posed in Erdös (1949). See also Rado (1954) and Nash-Williams (1965) for related work and further generalizations.

**Lemma 11** *Any infinite sequence $w_i$ of words (made up of $k = n + 1$ different letters) contains two words, $w_{i_1}$ and $w_{i_2}$, such that $i_1 < i_2$ and $w_{i_1} \preceq w_{i_2}$.*

**Proof.** By induction on $l$ - the length of the shortest word in the sequence. $l = 1$. Take the one-letter word. Without loss of generality, the letter is $A$. Eliminate all the words that go before that word from the sequence; we now have $w_1 = A$. If any other word in the remaining sequence contains the letter $A$, we are done. If not, then the sequence $(w_2, w_3, w_4, \dots)$ is made up of only $n = k - 1$ different letters, and by the assumption of the main induction, this sequence contains an increasing subsequence $(w_{j_1}, w_{j_2}, \dots)$ with any two words, e.g., $w_{j_1}$ and $w_{j_2}$, satisfying the requirements.

Suppose the Lemma holds for all $l$ up to $m$. Let us show that it is also true for $l = m+1$. Take the shortest word in the sequence. Without the loss of generality, it is the first word in the sequence, and the first letter in this word is $A$. If there is only a finite number of other words that contain the letter $A$, then the remaining infinite subsequence is made up of only $n$ different letters and we are done. Otherwise, drop all the words that do not contain the letter $A$ from the sequence. For each remaining word $w_i$, let $L_i$ be the part of the word that precedes the first occurrence of $A$ in the word, and $R_i$ be the part that follows the first occurrence of $A$ (e.g., if $w_i = BCADCAB$, then $L_i = BC$ and $R_i = DCAB$; if $w_i = ABC$, then $L_i$ is the empty word, and $R_i = BC$). Note that all words in the sequence $(L_1, L_2, \dots, L_i, \dots)$ are made up of only $n$ different letters, and so there exists an increasing subsequence $(L_{i_1}, L_{i_2}, \dots)$ such that for any $t$, $i_t < i_{t+1}$ and $L_{i_t} \preceq L_{i_{t+1}}$. Note also that since we assumed that $w_1$ is the shortest word and starts with an $A$, we can let $i_1 = 1$ - the empty word is smaller than any other word.

Now, consider the corresponding sequence $(R_{i_1}, R_{i_2}, \dots)$. The shortest word in this sequence has length of at most $m$ (because $R_1$, by construction, has length $m$), and therefore, by the minor induction assumption there exist $u$ and $v$ such that $u < v$ and $R_{i_u} \preceq R_{i_v}$. But we also know that, by construction, $L_{i_u} \preceq L_{i_v}$, and so $w_{i_u} \preceq w_{i_v}$, and we are done with the proof of the Lemma.

■

We can now finish the proof of the step of the induction of Higman's theorem. Take any sequence of words made up of $k = n + 1$ different letters. Consider all words $w_i$ in this sequence such that there does not exist $j > i$ such that $w_i \preceq w_j$. There can be at most a finite number of such words - otherwise, the subsequence formed from these words would be a counter-example to the Lemma. Let $w_h$ be the last one of these words in the sequence, so that for any $j > h$ there exists $k > j$ such that $w_j \preceq w_k$. It is now possible to construct an infinite increasing subsequence, e.g., take the subsequence $(w_{i_t})$ such that $i_1 = h + 1$ and for all $t > 1$, $i_t = min_{i > i_{t-1}}\{i | w_{i_{t-1}} \preceq w_i\}$.

■

Now that the finiteness of the possible states of awareness is established the proof of the existence of an extended Nash equilibrium is easily obtained.

**Proof of Theorem 6.** Consider a game with incomplete awareness $\Gamma^{\mathcal{U}}$. Where $\Gamma = (I, \prod_{i \in I} A_i, u_i : \prod_{k \in I} A_k \to R)$. From Proposition 7 we have that for every $i \in I$ there is a finite set $\{\theta_{i,1}, ..., \theta_{i,l(i)}\}$ such that the last member of each $\theta_{i,l}$ is $i$ and such that for every $\tilde{\theta}$ whose last member is $i$ we have some $1 \leq l \leq l(i)$ such that $\mathcal{U}^{\tilde{\theta}} = \mathcal{U}^{\theta_{i,l}}$. We assume that $l(i)$ is minimal for every $i \in I$, i.e., $\{\theta_{i,1}, ..., \theta_{i,l(i)}\}$ are distinct states of awareness.

Consider the following normal form game. The set of players in this game is given by

$$\mathcal{N} = \bigcup_{i \in I} \bigcup_{l=1}^{\infty} \{\theta_{i,l}\} \tag{18}$$

the action set for each player $\theta_{i,l}$ is given by

$$\mathcal{A}_{\theta_{i,l}} = \alpha_{\theta_{i,l}} \cap A_i \tag{19}$$

For every $\theta_{i,l}$ consider the states of awareness $\{\theta_{i,l}\hat{\ }j\}_{j \in I_{\theta_{i,l}} \setminus \{i\}}$. From the minimality of $l(j)$ for every $j \in I_{\theta_{i,l}}$, each member $\theta_{i,l}\hat{\ }j$ of this set has a unique $\theta_{j,m_j}$, $1 \leq m_j \leq l(j)$ such that $\mathcal{U}^{\theta_{i,l}\hat{\ }j} = \mathcal{U}^{\theta_{j,m_j}}$.

We define the payoff function for each player in this game by

$$\mathcal{P}_{\theta_{i,l}}(\{a_{j,m_j}\}_{j \in I_{\theta_{i,l}}}) = u_i^{\theta_{i,l}}(\{a_{j,m}\}_{j \in I_{\theta_{i,l}}}) \tag{20}$$

here $u_i^{\theta_{i,l}}$ is the payoff to $i$ in the game $\Gamma^{\theta_{i,l}}$.

The game $\mathcal{G} = (\mathcal{N}, \mathcal{A}, \mathcal{P})$ constitutes a finite normal form game, hence it has a Nash equilibrium, we complete the proof by showing that such a Nash equilibrium defines an extended Nash equilibrium of the game $\Gamma^{\mathcal{U}}$.

Consider a Nash equilibrium of the game $\mathcal{G}$ given by the mixed strategies $\sigma_{\theta_{i,l}}$ for each player $\theta_{i,l}$ in $\mathcal{G}$. We define for every $\theta$ the mixed strategy $\sigma^{\theta} = \sigma_{\theta_{i,l}}$ in the game with incomplete awareness where $\theta_{i,l}$ is the unique player in $\mathcal{G}$ such that $\mathcal{U}^{\theta_{i,l}} = \mathcal{U}^{\theta}$. By the minimality of $l(i)$ we have that whenever $\mathcal{U}^{\theta} = \mathcal{U}^{\bar{\theta}}$ and both $\theta, \bar{\theta}$ have the same last member, then $\sigma^{\theta} = \sigma^{\bar{\theta}}$ hence the second condition for an extended equilibrium is satisfied. Furthermore, $\sigma^{\theta}$ is a best response to $\{\sigma^{\theta\hat{\ }j}\}_{j \in I_{\theta} \setminus i}$ since the continuations $\theta\hat{\ }j$ are uniquely determined by $\theta_{i,l}$ which corresponds to $\theta$ and the strategies $\sigma^{\theta\hat{\ }j}$ were defined as $\sigma^{\theta\hat{\ }j} = \sigma_{\theta_{j,m}}$ for each uniquely determined $\theta_{j,m}$ corresponding to $\theta\hat{\ }j$. From the definition of the game $\mathcal{G}$ and since $\sigma_{\theta_{i,l}}$ is a best response to corresponding $\sigma_{\theta_{j,m}}$ we have that $\sigma^{\theta}$ is a best response to $\{\sigma^{\theta\hat{\ }j}\}_{j \in I_{\theta} \setminus i}$ and the first condition for an extended Nash equilibrium is satisfied and the existence proof is complete. ■

We note that the Nash equilibria of auxiliary normal form game $\mathcal{G}$ defined in the proof of Theorem 6 completely characterizes the set of extended Nash equilibria for the game $\Gamma^{\mathcal{U}}$. This follows from observing that all minimal sets states of awareness for states $\theta$ terminating with $i$ are isomorphic, in the sense that there is an isomorphism between each pair of sets for every $i \in I$ such that the concatenations $\theta\hat{\ }j$ are mapped to a unique state of awareness in a manner that respect the isomorphism.