On Pricing Kernels and Dynamic Portfolios

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Abstract

We investigate the structure of the pricing kernels in a general dynamic investment setting by making use of their duality with the self financing portfolios. We generalize the variance bound on the intertemporal marginal rate of substitution introduced in Hansen and Jagannathan (1991) along two dimensions, first by looking at the variance of the pricing kernels over several trading periods, and second by studying the restrictions imposed by the market prices of a set of securities.

The variance bound is the square of the optimal Sharpe ratio which can be achieved through a dynamic self financing strategy. This Sharpe ratio may be further enhanced by investing dynamically in some additional securities. We exhibit the kernel which yields the smallest possible increase in optimal dynamic Sharpe ratio while agreeing with the current market quotes of the additional instruments.

Keywords: Pricing Kernel, Sharpe Ratio, Self Financing Portfolio, Variance-Optimal Hedging.

1 Introduction

The duality between pricing kernels and portfolio payoffs is the key to many fundamental results in asset pricing theory. In a one-period setting, a pricing kernel is a random variable m_{t+1} which satisfies the equality

$$E_t\left[m_{t+1}w_{t+1}\right] = R^f_{t,t+1}E_t[m_{t+1}]w_t \tag{1}$$

for every portfolio with payoff w_{t+1} at time $(t+1)$ and value w_t at time t, where $R^f_{t,t+1}$ and E_t denote respectively the (gross) risk free rate from t to $(t+1)$ and the conditional expectation operator corresponding to the information available at time t. Harrison and Kreps (1979) show that the existence of a pricing kernel is equivalent to the law of one price while the absence of arbitrage corresponds to the existence of a positive pricing kernel.

If we know the prices today and the payoffs tomorrow of a set of securities, then a positive pricing kernel m_{t+1} consistent with these securities provides an efficient method to produce contingent claim prices in an arbitrage free framework. The kernel m_{t+1} yields an arbitrage free price F_t today for a payoff F_{t+1} tomorrow through the equation

$$R^f_{t,t+1}E_t[m_{t+1}]F_t = E_t\left[m_{t+1}F_{t+1}\right].$$

This technique is especially useful when the market is incomplete and the claim F_{t+1} cannot be obtained as the payoff of a portfolio based on the primitive securities.

Every positive pricing kernel yields however a different arbitrage free price system, and in many situations the resulting range of contingent claim prices is so wide as to be of little practical use. It is then natural to seek a rationale to reduce the set of admissible pricing kernels, and in turn the range of corresponding prices. The quest for such a rationale is a central theme in asset pricing theory. Bernardo and Ledoit (2000) show for instance that setting upper and lower bounds to a pricing kernel in every state of the world controls the maximum gain-loss ratio of every investment strategy. Balduzzi and Kallal (1997) consider the restrictions imposed by the risk premia assigned by the pricing kernels on some arbitrary sources of risk.

The variance bound on the pricing kernels introduced in Hansen and Jagannathan (1991) is another important consequence of the duality between kernels and portfolios. The square of the Sharpe ratio of every portfolio is smaller than the variance of every pricing kernel, once properly normalized, and equality obtains for a unique portfolio whose payoff is also itself a pricing kernel. This result is useful in two ways. On the one hand, the variance of every pricing kernel yields an upper bound to the Sharpe ratios which portfolio managers may expect to obtain in the market. On the other hand the Sharpe ratio of any portfolio is a lower bound to the variance of the pricing kernels, and this allows to reject the asset pricing theories for which the discount factor does not display enough variation across the states of nature. Bekaert and Liu (2001) give an extensive account of the growing use of these bounds in financial economics.

In view of this result, Cochrane and Saá-Requejo (2000) reduce the set of admissible pricing kernels by rejecting candidates with large variance on the ground that they may give rise to abnormal good deals in the form of investment opportunities with large Sharpe ratios. They reason that although positive pricing kernels with large variance do not create arbitrage opportunities, they are nevertheless suspicious and should be discarded. Cochrane and Saá-Requejo (2000) compute the upper and lower bounds for the price of a contingent claim when a variance bound is imposed on the kernels.

Our contribution is to extend the investigation of the duality between investment strategies and pricing kernels from a single period to several consecutive trading periods. A pricing kernel from time t to horizon T is a random variable m_T which satisfy the equality

$$E_s\left[m_T w_T\right] = R_s^f E_s[m_T] w_s, \tag{2}$$

for every intermediate period s between t and T and for every self financing portfolio whose value varies from w_s to w_T between time s and horizon T. We denote here R_s^f the risk free rate from s to T.

The time dimension of this duality has so far been limited to the description of the information set implicit in the conditional expectation of Equation 1. We generalize the variance bound of Hansen and Jagannathan (1991) to a multiperiod setting by showing that the standard deviation of the intertemporal marginal rate of substitution over a span of trading periods is larger than the optimal Sharpe ratio available over the corresponding investment horizon through dynamic self financing strategies. Every investment span gives rise to a different variance bound, and it is legitimate to expect a sharper restriction on the pricing kernels than the one which results from a single trading period.

The asset pricing results in the literature which follow from restrictions on the pricing kernels are derived through a repeated use of a single period analysis. This is for instance the case in both Bernardo and Ledoit (2000) and Cochrane and Saá-Requejo (2000) who compute contingent claim price bounds recursively. They cannot deal with a constraint on the kernels which is defined over several periods and which cannot be written as a succession of constraints on the one period intertemporal marginal rates of substitution.

An important example of such a constraint is the observation of the current market prices of a set of new securities on top of the original ones. These quotes may be the only information available about the price process of the new securities, and it is logical to restrict the kernels to the ones which agree with them. If the payoffs of the new securities span several trading periods, this constraint cannot be written in a convenient time separable way. Our multiperiod analysis handles these constraints and allows us to exhibit the sharper variance bounds which they generate.

We propose a theory of pricing kernels in a general dynamic investment environment. We describe the structure of the pricing kernels which are consistent with the stochastic evolution of a finite number of securities. Equation 2 shows that the set of pricing kernels is the dual of the set of the self financing portfolios which invest in these securities. We show that the pricing kernel with minimum conditional variance over a span of trading periods is the unique kernel which is also the final value of a self financing portfolio. This investment strategy happens to be dynamically mean-variance efficient. The analysis of this duality yields a number of results, both on the pricing kernels and on the dynamic investment strategies.

As explained above, positive pricing kernels allow to derive the price dynamics of new instruments in an arbitrage free framework. This technique is also often described as the choice of a risk neutral probability distribution in which discounted security prices are martingales. The new instruments can for instance be derivatives written on the original securities. We take a partial equilibrium point of view and we assume that the new securities have no effect on the dynamics of the original ones. The introduction of additional instruments may therefore only enhance the efficient frontier available through dynamic trading.

This increase in efficiency depends on the price dynamics of the new instruments. We show that if the price process followed by the new instruments is derived from a pricing kernel consistent with the original securities, then the increase in the optimal dynamic Sharpe ratio is a function of the extent to which the new instruments help dynamically replicate the kernel. The maximum gain in efficiency is obtained once the kernel is perfectly replicated with both the original and the additional securities so that it becomes the final value of a self financing strategy. The maximum dynamic Sharpe ratio is then the standard deviation of the pricing kernel. This also proves that the standard deviation of a given pricing kernel is an upper bound to the dynamic Sharpe ratio which can be reached through dynamic self financing strategies which invest in a arbitrarily large number of instruments, provided that the price process of these instruments is derived from the given kernel.

Once the pricing kernel is perfectly replicated, no more mean-variance efficiency gain may be expected from the introduction of new securities and the strategy which replicates the kernel belongs to the enhanced efficient frontier. If we use a pricing kernel which is already the final value of a self financing strategy based on the original securities in the first place, then no efficiency gain is possible right from the start. This means that every new instrument is priced by this kernel in such a way as to be useless for the construction of a dynamically mean-variance efficient strategy. The pricing kernel with minimum variance is the only kernel enjoying this property. This special kernel corresponds therefore to a min-max in terms of dynamic Sharpe ratio. Cochrane and Saá-Requejo (2000) have proposed to eliminate dynamics which create "good deals", where they define a good deal as an investment strategy with a large instantaneous Sharpe ratio. The minimum-variance kernel extends this methodology to an intertemporal Sharpe ratio. It generates conservative dynamics which do not allow any increase in Sharpe ratio, thereby eliminating "good deals" in a dynamic sense.

Besides its interpretation in terms of portfolio management, the minimum-variance pricing kernel has received attention in the finance literature for another related issue: the variance-optimal hedging of a contingent claim. Schweizer (1995) derives the price of a contingent claim from the cost of its optimal replication by means of self financing strategies. Optimality is measured by a quadratic loss function. This price happens to be identical to the one derived from the minimum-variance pricing kernel. The importance of the variance-optimal hedging strategy is highlighted by the fact that every pricing kernel can be written as the variance optimal hedge residual of a contingent claim.

We prove that the cost of the variance-optimal hedge of a security does not change as new hedging instruments are introduced, as long as these instruments are themselves priced according to the cost of their variance-optimal hedge, that is if their price dynamics is derived from the minimum-variance pricing kernel.

We next investigate the situation where, on top of the original securities, the current market prices of a set of additional securities are available. These new instruments could typically be a set of actively traded calls and puts written on the original securities. In line with the option pricing literature, we shall sometimes refer to the collection of these prices as a smile. We illustrate the significance of this situation by considering two dynamic investment problems, the dynamic management of a portfolio on the one hand, and the pricing and hedging of a contingent claim on the other hand.

We consider first a fund manager who trades in a finite number of securities and who considers investing in derivative instruments written on them. Markets are frictionless and perfectly competitive and we assume that the manager knows the price dynamics of the underlying securities. Although she observes the prices of all traded securities every period, she does not know the future price dynamics of the derivative instruments. The manager could for instance be an equity portfolio manager who is considering investing in convertible bonds written on the shares in which she is trading. The manager faces several interconnected questions. Which derivatives should she select? Which price dynamics will they follow? How should she optimally manage her portfolio with the new instruments? Which performance gain can she expect from expanding her investment scope?

Consider now an investment banker who is seeking to price and hedge an exotic derivative instrument written on some underlying securities. The banker knows the price process followed by the underlying securities, and he observes the market quotes of a set of actively traded derivatives written on them, for instance vanilla calls and puts, but he does not know their price dynamics. The exotic derivative is not actively traded and no market price is readily available. The banker seeks to use the traded derivatives, together with the underlying securities, in order to hedge the exotic instrument. He is confronted with several questions, echoing the questions raised by the fund manager. Which price dynamics will follow the traded derivatives? At which price should he deal in the exotic instrument? Which its the best replication strategy, using both the underlying securities and the traded derivatives?

In a complete market setting, the questions raised by both the fund manager and the investment banker find immediate answers. For every derivative instrument, only one price dynamics is consistent with absence of arbitrage, and it is given by the value process of its exact replication strategy. No performance gain can be expected in the management of a portfolio by the introduction of new securities since the opportunity set is not changed by the addition of redundant securities. There is no need either for the banker to hedge the exotic instrument with the traded derivatives since it is already perfectly replicated with the underlying securities. In an incomplete market setting however, exact replication is typically not possible and many price dynamics for the new instruments may be consistent with the observed market quotes and the principle of absence of arbitrage. An important question arises as to which rationale allows to reduce the choice among admissible price dynamics. We offer a rationale which answers the concerns of both the fund manager and the investment banker.

Following again the logic of limiting good deals in a dynamic sense, we characterize the kernel which yields a minimum increase in optimum Sharpe ratio while agreeing with the prices of the instruments for which market quotes are available. Drawing on the duality with the dynamic portfolios, we describe the efficient investment strategies which corresponds to this kernel. They solve a max-min problem in terms of dynamic Sharpe ratio. These strategies keep a fixed quantity of every quoted instrument, on top of an investment in the L^2 minimum portfolio for the original securities.

The constraint of matching the smile reduces the set of admissible pricing kernels and leads to a higher variance bound on the kernels. We describe this set and we show that the increase in the variance bound is given by the distance, in the metric of the variance-optimal hedge residuals, between the observed market quotes of the instruments and the cost of their variance-optimal hedge.

We show that the pricing kernel which limits dynamic good deals while agreeing with the smile is also optimal in terms of variance-optimal hedge for two reasons. First it prices a contingent claim as close as possible to the cost of its variance-optimal hedge. Second this price is the initial value of a constrained optimal replication strategy. In both cases, the constrained optimality corresponds to a min-max where we consider the worst possible contingent claim. We show finally that the contingent claim price derived from this kernel is equal to the value of the variance-optimal hedge of the claim, when the dynamic hedging strategy uses both the original securities and the instruments of the smile.

The paper is organized as follows. Sections 2 to 4 describe the self financing portfolios and their mean-variance properties. They draw heavily on Henrotte (2001) which provides an extensive account of the structure of these dynamic investment strategies. Section 5 studies the structure of the pricing kernels and generalizes the Hansen and Jagannathan (1991) variance bound to a multiperiod setting. Section 6 explains how to price additional securities in an incomplete market setting while avoiding mean-variance good deals in a dynamic sense. It relates the increase in the slope of the efficient frontier with the extent to which the additional securities help replicate the kernel. Section 7 studies the pricing kernels and the price dynamics which are consistent with the constraint of matching the market quotes of a given set of securities. We derive a lower bound to the variance of these kernels and we describe the minimum increase in the optimum dynamic Sharpe ratio implied by this constraint. This lower bound and this minimum are reached for a pricing kernel and an efficient dynamic strategy which we describe in Section 8. We propose this dynamics as a solution to our two investment problems in incomplete markets, the mean-variance management of a portfolio and the optimal replication of a contingent claim.

2 Dynamic Portfolios

2.1 Initial Market Structure

We consider a finite number n of underlying securities traded in a frictionless and competitive market over a set of discrete times with finite horizon. We index the trading dates by the integers between 0 and a final horizon T. Information is described by a filtration $\mathcal{F} \stackrel{\text{def.}}{=} \{\mathcal{F}_t\}_{0\leq t\leq T}$ over a probability space $(\Omega, \mathcal{F}_T, P)$.

Throughout the article, equalities and inequalities between random variables are understood to hold P almost surely. We denote respectively $E[F]$ and $E_t[F]$ the expected value and the conditional expectation with respect to $\mathcal{F}_t$ of a random variable F in $L^1(P)$. We let $L^2_t(P)$ and $L^2_t(P; I\!R^n)$ be respectively the space of random variables and random vectors in $I\!R^n$ which are both measurable with respect to $\mathcal{F}_t$ and in $L^2(P)$. If f_t is positive and measurable with respect to $\mathcal{F}_t$, we define $L^2_t(P, f_t)$ as the set of random variables F such that f_tF belongs to $L^2_t(P)$. We define in the same way $L^2_t(P, f_t; I\!R^n)$ for random vectors in $I\!R^n$. We close this list of technical notations by letting $x'y$ denote the usual scalar product of two vectors x and y in $I\!R^n$.

An unspecified numeraire is fixed every period and we let p_t be the vector of prices of the n securities in this numeraire at time t. We let d_t be the numeraire dividend distributed by the securities at time t. The owner of one unit of security i at time t is entitled to receive the dividend d^i_{t+1} in numeraire the next period. We let $\phi_t \stackrel{\text{def.}}{=} (p_t + d_t)$ be the cum-dividend price vector of the securities at time t. The vector processes $\{p_t\}_{0\leq t\leq T}$, $\{d_t\}_{0\leq t\leq T}$, and $\{\phi_t\}_{0\leq t\leq T}$ are adapted to the filtration $\mathcal{F}$.

We do not rule out that some security might be redundant at some trading period and in some state of the world but we do impose that the law of one price holds. For the remainder of the article, we shall assume that the following two assumptions are satisfied.

Assumption 1 *Prices and returns of the securities do not vanish. For every period t between 0 and T and for every period s between 1 and T the price vectors p_t and ϕ_s are P almost surely different from the null vector.*

Assumption 2 *Law of one price. For every period t between 0 and $(T-1)$, and for every random vectors X_t and Y_t in $I\!R^n$ measurable with respect to $\mathcal{F}_t$, the equality $\phi'_{t+1}X_t = \phi'_{t+1}Y_t$ implies $p'_tX_t = p'_tY_t$.*

2.2 Self Financing Portfolios

A dynamic portfolio X starting at time t is a process in $I\!R^n$ adapted to $\mathcal{F}$ and indexed by time s with $t \leq s \leq (T-1)$, where X_s^i represents the number of units of security i held in portfolio X at time s. We let $w(X)$ be the value process of portfolio X, naturally defined by $w_s(X) \stackrel{\text{def.}}{=} p'_sX_s$ for $s \leq (T-1)$ and we let $w_T(X) = \phi'_TX_{T-1}$.

We say that a dynamic portfolio X starting at time t is self financing at time s whenever $w_s(X) = \phi'_sX_{s-1}$ and that it is self financing whenever it is self financing from $(t+1)$ to T. We remark that the definition of the final value of the strategy implies that a dynamic portfolio is always self financing at time T.

It is easily checked that the law of one price implies that two self financing portfolios with identical final values at time T share the same value process. This property will allow us later to identify two such dynamic portfolios.

Henrotte (2001) characterizes the set of self financing dynamic portfolios starting at time t with the property that their final wealth at time T is in $L^2(P)$. Saving on notation, we denote $\mathcal{X}_t$ this set with no explicit reference to T since which we shall keep this final horizon constant throughout our analysis. We also let $w_T(\mathcal{X}_t) \stackrel{\text{def.}}{=} \{w_T(X)\ ;\ X \in \mathcal{X}_t\}$ be the set in $L^2(P)$ of terminal wealths of portfolios in $\mathcal{X}_t$. Besides the self financing condition, no restriction is imposed on the value process of the portfolios at periods prior to the final horizon.

Henrotte (2001) builds a positive process h by backward induction from the final value $h_T = 1$ at time T. This process plays a central role in the description of the structure of $\mathcal{X}_t$, and more generally in the mean-variance analysis. It is closely linked to the notion of dynamic Sharpe ratio and it can be interpreted as a correction lens for myopic investors.

We denote N^+ the Moore-Penrose generalized inverse of a symmetric matrix N in $I\!R^n \times I\!R^n$. The matrix N^+ is itself symmetric, commutes with N, and satisfies[1]

$$NN^+N = N, \tag{3}$$

$$N^+NN^+ = N^+. \tag{4}$$

If N is a random matrix measurable with respect to $\mathcal{F}_t$, then $N^+(\omega)$ is defined for every ω in Ω and N^+ is also measurable with respect to $\mathcal{F}_t$.

Proposition 1 *The adapted process h defined by $h_T = 1$ at time T and the backward equation*

$$h_t \stackrel{def.}{=} \left(p_t' N_t^+ p_t\right)^{-1} \tag{5}$$

with $N_t \stackrel{def.}{=} E_t\left[h_{t+1}\phi_{t+1}\phi_{t+1}'\right]$ for $0 \leq t \leq (T-1)$, is well defined, P almost surely positive, and satisfies $\phi_t \in L_t^2(P, \sqrt{h_t}; I\!R^n)$ for every period t between 0 and T as soon as the following two conditions are met:

(a). $\phi_T \in L^2(P; I\!R^n)$;

(b). $d_t \in L_t^2(P, \sqrt{h_t}; I\!R^n)$ *for every period t with $0 \leq t \leq (T-1)$.*

The following properties then hold.

(i). For every dynamic portfolio $X \in \mathcal{X}_t$ the process $\{h_s w_s(X)^2\}_{t \leq s \leq T}$ is a submartingale, that is, for every period s with $t \leq s \leq (T-1)$ we have

$$h_s w_s(X)^2 \leq E_s\left[h_{s+1} w_{s+1}(X)^2\right] \leq E_s\left[h_T w_T(X)^2\right].$$

(ii). The set $\mathcal{X}_t$ is the set of self financing dynamic portfolios starting at time t such that $w_s(X) \in L_s^2(P, \sqrt{h_s})$ for every period $t \leq s \leq T$.

(iii). The set $w_T(\mathcal{X}_t)$ is closed in $L^2(P)$.

Condition (b) of Proposition 1 involves the variable h_t which is derived recursively through Equation 5. The following lemma provides a sufficient condition independent of h.

[1]see Theil (1983) for a general description of the Moore-Penrose inverse.

Lemma 1 *If ϕ_T is an element of $L^2(P; \mathbb{R}^n)$ (Condition (a) of Proposition 1), then d_t belongs to $L^2_t(P, \sqrt{h_t}; \mathbb{R}^n)$ for every period t from 0 to $(T-1)$ (Condition (b) of Proposition 1) if one security, say Security k, pays no dividend and is such that $(p^k_T/p^k_t)d_t$ is an element of $L^2(P; \mathbb{R}^n)$ for every period t from 0 to $(T-1)$.*

For the remainder of the article, we assume that Conditions (a) and (b) of Proposition 1 are satisfied so that the results of this proposition apply.

Assumption 3 *Conditions (a) and (b) of Proposition 1 are satisfied.*

Two equations will prove useful. For every period t between 0 and $(T-1)$,

$$\phi_{t+1} = N_t N_t^+ \phi_{t+1}, \tag{6}$$

and the law of one price implies then that

$$p_t = N_t N_t^+ p_t. \tag{7}$$

The process h acts as a weight which regularizes the prices and the values of the self financing portfolios in $\mathcal{X}_t$ every period. Once we multiply those processes by the square root of h, they all have finite second moments every period. Henrotte (2001) shows that the process h is the largest process with value $h_T = 1$ at horizon T having this regularization property.

3 Optimal Replication

This section investigates the replication properties of the self financing dynamic portfolios. We first show how to construct a dynamic strategy which best replicates a payoff F_T at horizon T, starting from a wealth w_t at time t. The loss function which we choose at horizon T is the norm of $L^2(P)$, which is well defined for the portfolios in $\mathcal{X}_t$. We then study the cost and quality of the optimal replication and we show that the value process of the optimal solution is unique. When the final payoff F_T is zero, we obtain as a special case the minimum L^2 portfolio which is the hedging numeraire used by Gouriéroux et al. (1998). We show that are analysis can be extended to deal with the optimal replication of securities described by a sequence of contingent cash flows instead of a single final payoff. We introduce interest rates by mean of default free zero coupon bonds and we relate our work with the concept of variance-optimal signed martingale measure introduced in Schweizer (1995).

3.1 Construction of an Optimal Replication

The optimal L^2 replication of a contingent claim involves a mixture of forward and backward equations. We derive first the cost of the optimal replication every period in a backward way, and we then use this process in order to construct the optimal replicating strategy through a forward equation.

Proposition 2 *For every period t such that $0 \leq t \leq (T-1)$, for every function w_t in $L^2_t(P, \sqrt{h_t})$, and for every function F_T in $L^2(P)$:*

$$(8)\qquad \begin{aligned} &\textit{essinf}\, E_t\left[(F_T - w_T(X))^2\right] = E_t\left[\left(F_T - w_T(X^{t,w_t,F_T})\right)^2\right] = h_t(F_t - w_t)^2 + g_t. \\ &s.\ t. \begin{cases} X \in \mathcal{X}_t \\ w_t(X) = w_t \end{cases} \end{aligned}$$

F_t and g_t are defined by backward induction by $g_T \overset{def.}{=} 0$ and for $t \leq s \leq (T-1)$,

$$\begin{aligned} F_s &\overset{def.}{=} p_s' N_s^+ E_s\left[h_{s+1}F_{s+1}\phi_{s+1}\right], \\ g_s &\overset{def.}{=} E_s\left[g_{s+1}\right] + E_s\left[h_{s+1}F_{s+1}^2\right] - E_s\left[h_{s+1}F_{s+1}\phi_{s+1}'\right] N_s^+ E_s\left[h_{s+1}F_{s+1}\phi_{s+1}\right]. \end{aligned}$$

For every period s between time t and $(T-1)$, $F_s \in L^2_s(P, \sqrt{h_s})$, g_s is measurable with respect to $\mathcal{F}_s$, $g_s \in L^1(P)$, and $g_s \geq 0$. The dynamic portfolio X^{t,w_t,F_T} is defined recursively by

$$(9)\qquad X_t^{t,w_t,F_T} \overset{def.}{=} h_t(w_t - F_t)N_t^+ p_t + N_t^+ E_t\left[h_{t+1}F_{t+1}\phi_{t+1}\right],$$

$$(10)\qquad X_s^{t,w_t,F_T} \overset{def.}{=} h_s\left(\phi_s' X_{s-1}^{t,w_t,F_T} - F_s\right) N_s^+ p_s + N_s^+ E_s\left[h_{s+1}F_{s+1}\phi_{s+1}\right],$$

for $(t+1) \leq s \leq (T-1)$. The dynamic portfolio X^{t,w_t,F_T} belongs to $\mathcal{X}_t$, starts at time t with initial wealth w_t, and satisfies

$$(11)\qquad E_s\left[\left(F_T - w_T(X^{t,w_t,F_T})\right)^2\right] = h_s\left(F_s - w_s(X^{t,w_t,F_T})\right)^2 + g_s$$

for every period s between t and T.

It is easily checked that if F_T, F_T^a, and F_T^b are in $L^2(P)$, if w_t, w_t^a, and w_t^b are in $L^2_t(P, \sqrt{h_t})$, and if γ_t is measurable with respect to $\mathcal{F}_t$ with $\gamma_t F_T$ in $L^2(P)$ and $\gamma_t w_t$ in $L^2_t(P, \sqrt{h_t})$, then

$$(12)\qquad X^{t,w_t^a,F_T^a} + X^{t,w_t^b,F_T^b} = X^{t,w_t^a+w_t^b,F_T^a+F_t^b},$$

$$(13)\qquad \gamma_t X^{t,w_t,F_T} = X^{t,\gamma_t w_t,\gamma_t F_T}.$$

It is clear from Proposition 2 that the optimization program

$$\begin{aligned}&\text{essinf } E_t\left[(F_T - w_T(X))^2\right]\\&\text{s. t. } X \in \mathcal{X}_t\end{aligned}$$

is solved in X^{t,F_t,F_T} with g_t as optimal value. The initial value F_t can be seen as the initial cost of the best replication strategy, while g_t describes the quality of this replication.

We remark that the construction of both F_s and g_s from F_T in Proposition 2 is respectively linear and quadratic and does not depend on the starting time t as long as $t \leq s$. This allows us to construct both a linear operator Q_t and a quadratic operator G_t for every period t between 0 and T from the space of random variables in $L^2(P)$ to the space of random variables measurable with respect to $\mathcal{F}_t$ such that $Q_t(F_T) \stackrel{\text{def.}}{=} F_t$ and $G_t(F_T) \stackrel{\text{def.}}{=} g_t$ as defined recursively in Proposition 2. This Proposition shows that $Q_t(F_T)$ belongs to $L_t^2(P, \sqrt{h_t})$ while $G_T(F_T)$ is an element of $L^1(P)$. At time T, the operators Q_T and G_T are trivially respectively the identity and the null operator. We still denote G_t the bilinear operator obtained through polarization of G_t and defined by

$$G_t(F_T^a, F_T^b) = \frac{1}{2}\left(G_t(F_T^a + F_T^b) - G_t(F_T^a) - G_t(F_T^b)\right)$$

for F_T^a and F_T^b in $L^2(P)$. Equation 11 of Proposition 2 writes

$$G_s(F_T) = E_s\left[\left(F_T - w_T(X^{t,w_t,F_T})\right)^2\right] - h_s\left(Q_s(F_T) - w_s(X^{t,w_t,F_T})\right)^2$$

for any initial wealth level w_t in $L_t^2(P, \sqrt{h_t})$. In particular for $s = t$ and $w_t = Q_t(F_T)$ we obtain

$$G_t(F_T) = E_t\left[\left(F_T - w_T(X^{t,Q_t(F_T),F_T})\right)^2\right]. \tag{14}$$

so that, by polarization,

$$G_t(F_T^a, F_T^b) = E_t\left[\left(F_T^a - w_T(X^{t,Q_t(F_T^a),F_T^a})\right)\left(F_T^b - w_T(X^{t,Q_t(F_T^b),F_T^b})\right)\right]. \tag{15}$$

The following lemma lists some properties of these operators which will be used throughout our analysis.

Lemma 2 *Let s and t be two periods such that $t \leq s \leq T$, let F_T be a random variable in $L^2(P)$, and let w_t be a random variable in $L_t^2(P, \sqrt{h_t})$.*

(i). For every dynamic portfolio X in $\mathcal{X}_t$, $Q_s(w_T(X)) = w_s(X)$.

(ii). $G_t(F_T) = 0$ *if and only if* $F_T \in w_T(\mathcal{X}_t)$. *For every dynamic portfolio* X *in* $\mathcal{X}_t$, $G_t(w_T(X), F_T) = 0$.

(iii). $h_s w_s(X^{t,w_t,0}) Q_s(F_T) = E_s\left[w_T(X^{t,w_t,0}) F_T\right]$.

(iv). *For every dynamic portfolio* X *in* $\mathcal{X}_s$,

$$h_s\left(Q_s(F_T) - w_s(X^{t,w_t,F_T})\right) w_s(X) = E_s\left[\left(F_T - w_T(X^{t,w_t,F_T})\right) w_T(X)\right].$$

3.2 Uniqueness of the Optimal Replication

The next result shows that the optimization problem 8 of Proposition 2 has a unique solution, at least in terms of value at time T, and therefore also in terms of value process. We recall that we cannot expect to obtain a unique portfolio because we do not rule out redundancy between the securities.

Lemma 3 *We consider a period* t *between* 0 *and* $(T-1)$, *an initial wealth* w_t *in* $L_t^2(P, \sqrt{h_t})$, *and a payoff* F_T *in* $L^2(P)$. *For every dynamic portfolio* Y *in* $\mathcal{X}_t$, *the equality* $w_T(Y) = w_T(X^{t,w_t,F_T})$ *holds* P *almost surely on the set* $A_t(Y)$ *in* $\mathcal{F}_t$ *defined by*

$$A_t(Y) \overset{def.}{=} \quad \{\omega \in \Omega \text{ such that: } E_t\left[(F_T - w_T(Y))^2\right] = h_t(Q_t(F_T) - w_t)^2 + G_t(F_T) \text{ and } w_t(Y) = w_t\}.$$

3.3 L^2 Minimum Portfolio

One special choice of final payoff F_T and initial wealth w_t at time t will prove important, it is obtained for $F_T = 0$ and $w_t = 1/\sqrt{h_t}$. Notice that for this choice of initial wealth $\sqrt{h_t} w_t = 1$, which belongs to $L^2(P)$. We introduce the simplified notation $X^t \overset{\text{def.}}{=} X^{t,1/\sqrt{h_t},0}$ and $w_s^t \overset{\text{def.}}{=} w_s(X^t)$ for $t \leq s \leq T$. Equations 9 and 10 of Proposition 2 show that the self financing strategy X^t is obtained by investing every period s between t and $(T-1)$ the value w_s^t in the portfolio $h_s N_s^+ p_s$ whose value at time s is $h_s p_s' N_s^+ p_s = 1$ and $X_s^t = h_s w_s^t N_s^+ p_s$. The self financing condition implies that $w_{s+1}^t = \phi_{s+1}' X_s^t$ so that

$$w_{s+1}^t = w_s^t h_s \phi_{s+1}' N_s^+ p_s. \tag{16}$$

Proposition 2 proves that this strategy yields a dynamic portfolio in $\mathcal{X}_t$ which satisfies

$$\text{(17)} \qquad \begin{array}{l} \operatorname{essinf} E_t\left[w_T(X)^2\right] = E_t\left[(w_T^t)^2\right] = h_t(w_t^t)^2 = 1. \\ \text{s. t.} \begin{cases} X \in \mathcal{X}_t \\ w_t(X) = 1/\sqrt{h_t} \end{cases} \end{array}$$

Statement (iii) of Lemma 2 with $s = t$ and $w_t = 1/\sqrt{h_t}$ implies that

$$\text{(18)} \qquad Q_t(F_T) = \frac{1}{\sqrt{h_t}} E_t\left[w_T^t F_T\right],$$

which shows that Q_t is a positive operator whenever w_T^t is itself positive. If w_s^t does not vanish at time s between t and T, we also have

$$\text{(19)} \qquad Q_s(F_T) = \frac{1}{h_s w_s^t} E_s\left[w_T^t F_T\right].$$

3.4 Replication of a Sequence of Cash Flows

We generalize our analysis from a single payoff at horizon T to a sequence of contingent cash flows every period up to T. We consider a period t between 0 and $(T-1)$ and we let $f = \{f_s\}_{t+1 \le s \le T}$ be a sequence of cash flows from $(t+1)$ up to T adapted to $\mathcal{F}$.

We say that a dynamic portfolio X starting at time t finances the cash flow f_s at time s with $s \le (T-1)$ whenever $w_s(X) = \phi_s' X_{s-1} - f_s$ and that it finances the sequence of cash flow f whenever it finances the cash flows f_s from $(t+1)$ to $(T-1)$. At the last period, we recall that we have defined the final value of a dynamic portfolio X by the equation $w_T(X) = \phi_T' X_{T-1}$.

We create a one to one operator θ_f on the set of dynamic portfolios starting at time t as follows. For every dynamic portfolio X starting at time t, we let $Y = \theta_f(X)$ be the dynamic portfolio starting at time t defined by $Y_t = X_t$ at time t and

$$\text{(20)} \qquad Y_s = X_s - \left(\sum_{u=t+1}^{s} f_u \sqrt{h_u} w_s^u\right) h_s N_s^+ p_s$$

for $(t+1) \le s \le (T-1)$. The following lemma yields some first properties of this operator.

Lemma 4 *Let X and Y be two dynamic portfolios starting at time t such that $Y = \theta_f(X)$.*

(i). The portfolio X is self financing if and only if the portfolio Y finances the sequence of cash flows f.

(ii). $w_t(Y) = w_t(X)$ *and* $(f_T - w_T(Y)) = (F_T - w_T(X))$ *with*

$$F_T = \sum_{s=t+1}^{T} f_s \sqrt{h_s} w_T^s,$$

where we let $w_T^T = 1$.

We remark that the payoff F_T is obtained at time T by investing every cash flow of the sequence f in the L^2 minimum strategy up to time T.

We let $\mathcal{X}_t(f)$ be the set of dynamic portfolios starting at time t which finance f and which end up at horizon T with a value in $L^2(P)$. The following proposition proves the equivalence between the variance-optimal replication of F_T through self financing portfolios in $\mathcal{X}_t$ and the L^2 optimal replication of the sequence f by means of dynamic strategies in $\mathcal{X}_t(f)$. Some integrability condition on the sequence f are needed for this result.

Proposition 3 *Let* $f = \{f_s\}_{t+1 \leq s \leq T}$ *be a sequence of cash flows such that* f_s *belongs to* $L_s^2(P, \sqrt{h_s})$ *for every period* s *from* $(t+1)$ *to* T. *The payoff* $F_T = \sum_{s=t+1}^{T} f_s \sqrt{h_s} w_T^s$ *is in* $L^2(P)$ *and the mapping* θ_f *is one to one from* $\mathcal{X}_t$ *to* $\mathcal{X}_t(f)$. *For every initial wealth* w_t *in* $L_t^2(P, \sqrt{h_t})$, *we have*

$$\begin{array}{lcl} \textit{essinf } E_t\left[(f_T - w_T(Y))^2\right] & = & \textit{essinf } E_t\left[(F_T - w_T(X))^2\right] \\ s.\ t. \begin{cases} Y \in \mathcal{X}_t(f) \\ w_t(Y) = w_t \end{cases} & & s.\ t. \begin{cases} X \in \mathcal{X}_t \\ w_t(X) = w_t \end{cases} \end{array}$$

and the first program is solved in $Y = \theta_f\left(X^{t,w_t,F_T}\right)$.

The optimal replication strategies for the two equivalent optimization programs of Proposition 3 start with an identical initial wealth at time t equal to $Q_t(F_T)$ and lead to the same replication error described by $G_t(F_T)$. The next lemma explains how both the optimal hedging cost $Q_t(F_T)$ and the optimal hedging quality $G_t(F_T)$ can be directly computed from the sequence f.

Lemma 5 *Let* $f = \{f_s\}_{t+1 \leq s \leq T}$ *be a sequence of cash flows such that* f_s *is in* $L_s^2(P, \sqrt{h_s})$ *for every period* s *from* $(t+1)$ *to* T *and let* $F_T = \sum_{s=t+1}^{T} f_s \sqrt{h_s} w_T^s$. *We define the processes* $\bar{f} = \{\bar{f}_s\}_{t \leq s \leq T}$ *and* $\bar{g} = \{\bar{g}_s\}_{t \leq s \leq T}$ *from the sequence* f *by backward induction as follows.*

We let $\bar{f}_T = \bar{g}_T \stackrel{def.}{=} 0$*, and*

$$\begin{aligned}
\bar{f}_s &\stackrel{def.}{=} p_s' N_s^+ E_s \left[h_{s+1}(\bar{f}_{s+1} + f_{s+1})\phi_{s+1}\right], \\
\bar{g}_s &\stackrel{def.}{=} E_s \left[\bar{g}_{s+1}\right] + E_s \left[h_{s+1}(\bar{f}_{s+1} + f_{s+1})^2\right] \\
&\quad - E_s \left[h_{s+1}(\bar{f}_{s+1} + f_{s+1})\phi_{s+1}'\right] N_s^+ E_s \left[h_{s+1}(\bar{f}_{s+1} + f_{s+1})\phi_{s+1}\right],
\end{aligned}$$

for $t \leq s \leq (T-1)$*. For every period* s *between* t *and* T *we have*

$$\begin{aligned}
Q_s(F_T) &= \sum_{u=t+1}^{s} f_u \sqrt{h_u} w_s^u + \bar{f}_s, \\
G_s(F_T) &= \bar{g}_s,
\end{aligned}$$

with the convention that $\sum_{u=t+1}^{s} f_u \sqrt{h_u} w_s^u = 0$ *when* $s = t$ *so that* $Q_t(F_T) = \bar{f}_t$*.*

3.5 Interest Rates

We introduce from now on a money market. For the rest of the article we assume that Security 1 is a risk free zero coupon bond paying a unique dividend of one unit of numeraire at maturity T.

Assumption 4 *For every period* t *between* 0 *and* $(T-1)$ *the price* p_t^1 *of the zero coupon bond is positive.*

Since the price of the bond is assumed positive, we define $R_t^f \stackrel{\text{def.}}{=} 1/p_t^1$ for $0 \leq t \leq (T-1)$. R_t^f is the nominal risk free return from investing in the zero coupon bond from time t up to horizon T, for our choice of numeraire every period. This buy and hold strategy belongs to $\mathcal{X}_t$, we denote it $\mathbf{1}_t$. We remark that $w_s(\mathbf{1}_t) = Q_s(1) = p_s^1 = 1/R_s^f$ for $s \geq t$ and we learn from Statement (i) of Proposition 1 that $h_t(p_t^1)^2 \leq E_t\left[h_{t+1}(p_{t+1}^1)^2\right] \leq 1$. We define $H_t \stackrel{\text{def.}}{=} h_t/(R_t^f)^2$ so that, with this normalization, this last inequality writes

$$H_t \leq E_t\left[H_{t+1}\right] \leq 1$$

and the normalized process H is a positive submartingale.

Notice that we have

$$\begin{aligned}
&\operatorname{essinf} E_t\left[w_T(X)^2\right] = H_t, \\
&\text{s. t.} \begin{cases} X \in \mathcal{X}_t \\ w_t(X) = p_t^1 = 1/R_t^f \end{cases}
\end{aligned}$$

which is reached in $\sqrt{H_t}X^t$.

We derive from Statement (iii) of Lemma 2 with $F_T = 1$ and $w_t = 1/\sqrt{h_t}$,

$$E_s\left[w_T^t\right] = \frac{h_s w_s^t}{R_s^f} \quad \text{for } t \leq s \leq T, \tag{21}$$

$$E_t\left[w_T^t\right] = \frac{\sqrt{h_t}}{R_t^f} = \sqrt{H_t}. \tag{22}$$

We remark that the sufficient condition of Lemma 1 which requires that $(p_T^k/p_t^k)d_t$ be in $L^2(P; I\!R^n)$ holds with $k = 1$ as soon as $R_t^f d_t$ is in $L^2(P; I\!R^n)$ for every period t between 0 and $(T-1)$. This is the case for instance if R_t^f is bounded and d_t belongs to $L^2(P; I\!R^n)$.

3.6 Variance-Optimal Martingale Measure

We have seen that the operator Q_t is positive as soon as the final wealth $w_T^t = w_T(X^t)$ of the L^2 minimum portfolio X^t is itself positive. We show that when this happens, the cost $Q_s(F_T)$ at time s between t and $(T-1)$ of the optimal replication of a payoff F_T in $L^2(P)$ can be expressed as the discounted conditional expectation of F_T in a probability distribution different from the original probability P. This new probability distribution is called the minimum variance probability distribution or the variance-optimal martingale probability.

We first notice from Equation 21 that if w_T^t is positive, then the value w_s^t of the strategy X^t at time s is also positive. Statement (iii) of Lemma 2, together with Equation 21, yields the following result

$$Q_s(F_T) = \frac{1}{R_s^f}\frac{E_s\left[w_T^t F_T\right]}{E_s\left[w_T^t\right]}. \tag{23}$$

If f is a positive random variable in $L^1(P)$, we denote P^f and E^f the probability distribution and its corresponding expectation operator obtained from the original probability P by means of the positive Radon-Nikodym derivative $f/E[f]$. For every random variable F such that fF is in $L^1(P)$ we have $E^f[F] = E[fF]/E[f]$ and $E_t^f[F] = E_t[fF]/E_t[f]$.

We use this construct here with $f = w_T^t$, which is in $L^1(P)$. We obtain

$$Q_s(F_T) = \frac{1}{R_s^f}E_s^{w_T^t}\left[F_T\right],$$

which shows that $Q_s(F_T)$ can indeed be written as a discounted expectation in the modified probability distribution $P^{w_T^t}$.

One can usually not expect w_T^t to be positive when the cum-dividend prices assume unbounded values. This fact has been noted in Schweizer (1995). When this happens, the minimum variance probability becomes the variance-optimal signed martingale measure and the operator Q_t, although still well defined, is not positive.

In a continuous time setting, Gouriéroux et al. (1998) shows that w_T^t is always positive as long as prices follow continuous semimartingales with no dividend distribution. They assume a no arbitrage condition which is more strict than the law of one price.

4 Mean-Variance Portfolio Selection

We summarize in this section the mean-variance properties of self financing dynamic portfolios. We consider in this section a time period t between 0 and $(T-1)$ and we study the notions of dynamic Sharpe ratio and efficient frontier conditioned on the information at date t.

For every dynamic portfolio X in $\mathcal{X}_t$, we denote $\mathrm{SR}_t(X)$ the Sharpe ratio conditioned on the information at time t, which results from following the self financing investment strategy X from time t to horizon T. We let

$$\mathrm{SR}_t(X) \overset{\text{def.}}{=} \frac{E_t\left[w_T(X)\right] - R_t^f w_t(X)}{\sqrt{\mathrm{Var}_t\left[w_T(X)\right]}}$$

when $\mathrm{Var}_t\left[w_T(X)\right]$ is non zero and we set $\mathrm{SR}_t(X) \overset{\text{def.}}{=} 0$ whenever $\mathrm{Var}_t\left[w_T(X)\right] = 0$.

We denote $R_t(X) \overset{\text{def.}}{=} w_T(X)/w_t(X)$ the total return from period t to horizon T of a dynamic portfolio X in $\mathcal{X}_t$ with non vanishing wealth $w_t(X)$ at date t. In particular we have $R_t^f = R_t(\mathbf{1}_t)$ when $X = \mathbf{1}_t$ is the strategy which invests without rebalancing in the default free zero coupon bond with maturity T from time t on. If $w_t(X)$ and $\mathrm{Var}_t\left[w_T(X)\right]$ are P almost surely different from zero, we also have

$$\mathrm{SR}_t(X) = \frac{E_t\left[R_t(X)\right] - R_t^f}{\sqrt{\mathrm{Var}_t\left[R_t(X)\right]}},$$

the usual definition of a Sharpe ratio.

Our definition of return is not innocuous. The choice of non annualized simple total return allows us to bring together in an common framework the theories of dynamic replication and of dynamic mean-variance analysis. This nice convergence may not hold for other specifications of the returns.

We let the dynamic mean-variance efficient frontier at time t with horizon T, which we denote $\mathcal{EF}_t$, be the set of portfolios in $\mathcal{X}_t$ which are solution to the optimization program

$$\begin{aligned} &\operatorname{essinf} \operatorname{Var}_t \left[R_t(X)\right] \\ &\text{s. t.} \begin{cases} X \in \mathcal{X}_t \\ w_t(X) = w_t \\ E_t \left[R_t(X)\right] = \mathcal{R}_t \end{cases} \end{aligned}$$

for some expected return target $\mathcal{R}_t$ measurable with respect to $\mathcal{F}_t$ and some positive initial wealth w_t in $L^2_t(P, \sqrt{h_t})$.

Henrotte (2001) shows that the optimum dynamic Sharpe ratio from time t to horizon T, conditioned on the information available at time t, writes

$$\mathrm{SR}_t \stackrel{\text{def.}}{=} \sqrt{\left(\frac{1}{H_t} - 1\right)}$$

and

$$\begin{aligned} &\operatorname{esssup} \mathrm{SR}_t(X)^2 = \mathrm{SR}_t(X^t)^2 = \left(\mathrm{SR}_t\right)^2. \\ &\text{s. t. } X \in \mathcal{X}_t \end{aligned}$$

The optimum dynamic Sharpe ratio obtains for the portfolios on the efficient frontier $\mathcal{EF}_t$. Under some regularity condition, every efficient portfolio on $\mathcal{EF}_t$ can be identified with a combination of the strategy X^t and the zero-coupon bond with maturity T. In particular the strategy X^t belongs to the efficient frontier $\mathcal{EF}_t$.

5 Pricing Kernels

We let PK_t be the set of pricing kernels corresponding to the dynamics of the underlying securities from period t until the horizon T. It is defined as the set of random variables m_T in $L^2(P)$ such that

$$E_s \left[m_T w_T(X)\right] = R^f_s E_s \left[m_T\right] w_s(X), \tag{24}$$

for every period s between t and T and for every dynamic portfolio X in $\mathcal{X}_s$. We do not require any a priori positivity condition on the pricing kernels and PK_t is a vector subspace of $L^2(P)$. For every period s between t and T we denote $m_s \stackrel{\text{def.}}{=} E_s[m_T]$ the conditional expectation at time s of a pricing kernel m_T in PK_t and we define

$$\mathrm{PK}^0_t \stackrel{\text{def.}}{=} \left\{m_T \in \mathrm{PK}_t \;\text{ such that }\; m_t = 0\right\}.$$

The following lemma proves that a variable m_T in $L^2(P)$ is a pricing kernel if and only if it "prices" correctly the n securities from one trading period to the next.

Lemma 6 *A random variable m_T in $L^2(P)$ is a pricing kernel in PK_t if and only if*

$$E_s\left[R^f_{s+1} m_{s+1} \phi_{s+1}\right] = R^f_s m_s p_s$$

for every period s between t and $(T-1)$.

5.1 Structure of Pricing Kernels

The next proposition describes the structure of the set PK_t of pricing kernels. The notion of conditional orthogonality will be useful. We say that two random variables f and g respectively in $L^2(P)$ and $L^2(P)$ are conditionally orthogonal at time t if and only if $E_t[fg] = 0$. If A is a subset of $L^2(P)$, we let $A^{\perp_t}$ be the set of random variables in $L^2(P)$ conditionally orthogonal at time t with every random variable in A.

Proposition 4 *For every period t and s such that $0 \le t \le s \le (T-1)$,*

(i). $w_T^t \in PK_t$ *and PK_t is therefore not reduced to zero.*

(ii). $PK_t^0 = w_T(\mathcal{X}_t)^{\perp_t}$.

(iii). Every pricing kernel m_T in PK_t satisfies

$$\frac{m_s}{\sqrt{H_s}} = \sqrt{h_s} Q_s(m_T)$$

so that m_s belongs to $L^2_s(P, 1/\sqrt{H_s})$.

(iv). PK_t is the set of random variables m_T in $L^2(P)$ which can be written

$$m_T = m_T^0 + \xi_t w_T^t$$

for some random variables m_T^0 in PK_t^0 and ξ_t in $L^2_t(P)$. In other words

$$PK_t = w_T(\mathcal{X}_t)^{\perp_t} + \left(L^2_t(P) \times w_T^t\right)$$

where $L^2_t(P) \times w_T^t \overset{def.}{=} \{\xi_t w_T^t \ \text{ with } \ \xi_t \in L^2_t(P)\}$.

(v). $PK_t \cap w_T(\mathcal{X}_t) = L^2_t(P) \times w_T^t$.

(vi). $PK_t = \left\{ \left(F_T - w_T(X^{t,w_t,F_T}) \right) \text{ with } F_T \in L^2(P) \text{ and } w_t \in L_t^2(P, \sqrt{h_t}) \right\}$ *and*
$PK_t^0 = \left\{ \left(F_T - w_T(X^{t,Q_t(F_T),F_T}) \right) \text{ with } F_T \in L^2(P) \right\}$.

(vii). The two sets PK_t^0 *and* PK_t *are closed in* $L^2(P)$ *and* $w_T(\mathcal{X}_t) = \left(PK_t^0 \right)^{\perp_t}$.

Statement (i) proves that w_T^t is a pricing kernel and Statement (iv) shows that every other pricing kernel in PK_t can be decomposed as the sum of this pricing kernel and a pricing kernel conditionally orthogonal to the final values of the dynamic portfolios. Statement (v) proves that the final value of the L^2 minimum portfolio X^t, possibly normalized by an initial value ξ_t in $L_t^2(P)$, is the only pricing kernel which is also the final value of a dynamic portfolio in $\mathcal{X}_t$. Statement (vi) proves that a pricing kernels is the minimum-variance hedge residual of some contingent claim. The L^2 minimum portfolio obtains for instance for $F_T = 0$ and $w_t = -1/h_t$.

The next lemma describes how a pricing kernel in PK_t evaluates a payoff in $L^2(P)$ possibly outside $w_T(\mathcal{X}_t)$.

Lemma 7 *We consider two periods t and s such that $0 \le t \le s \le T$, a pricing kernel m_T in PK_t and a payoff F_T in $L^2(P)$, we have*

$$E_s\left[m_T F_T\right] = R_s^f m_s Q_s(F_T) + G_s\left(m_T, F_T\right). \tag{25}$$

We remark that when F_T is in $w_T(\mathcal{X}_t)$ and writes $w_T(X)$, Equation 25 is Equation 24, since $Q_s(w_T(X)) = w_s(X)$ and $G_s(m_T, w_T(X)) = 0$, according to Statements (i) and (ii) of Lemma 2.

5.2 Variance Bounds on Pricing Kernels

We have seen in Section 4 that the dynamic portfolio X^t is mean-variance optimal within $\mathcal{X}_t$. The next proposition shows that w_T^t is also L^2 optimal within PK_t. This results yields intertemporal bounds on the variance of the pricing kernels.

Proposition 5 *We consider two periods t and s such that $0 \le t \le s \le T$. Every pricing kernel m_T in PK_t satisfies the equalities*

$$E_s\left[m_T^2\right] = \frac{m_s^2}{H_s} + G_s\left(m_T\right), \tag{26}$$

$$Var_s\left[m_T\right] = m_s^2 (SR_s)^2 + G_s(m_T). \tag{27}$$

In particular every pricing kernel m_T in PK_t satisfies the inequalities

$$\frac{m_s^2}{H_s} \leq E_s\left[m_T^2\right], \tag{28}$$

$$m_s^2 (SR_s)^2 \leq Var_s[m_T]. \tag{29}$$

Inequalities 28 and 29 become equalities if $m_T = \xi_t w_T^t$ with ξ_t in $L_t^2(P)$, and, for $s = t$, if and only if $m_T = \xi_t w_T^t$ with ξ_t in $L_t^2(P)$.

It results from Proposition 5 that if ξ_t is a random variable in $L_t^2(P)$, then the pricing kernel $\xi_t w_T^t$ solves

$$\begin{aligned} &\operatorname{essinf} E_t\left[m_T^2\right] = \xi_t^2. \\ &\text{s. t.} \begin{cases} m_T \in \mathrm{PK}_t \\ m_t = \sqrt{H_t}\xi_t \end{cases} \end{aligned} \tag{30}$$

We also derive that if $\bar{m}_t$ is a random variable in $L_t^2(P, 1/\sqrt{H_t})$, then the pricing kernel $(\bar{m}_t/\sqrt{H_t})w_T^t$ solves

$$\begin{aligned} &\operatorname{essinf} \mathrm{Var}_t[m_T] = \bar{m}_t^2 \mathrm{SR}_t^2. \\ &\text{s. t.} \begin{cases} m_T \in \mathrm{PK}_t \\ m_t = \bar{m}_t \end{cases} \end{aligned} \tag{31}$$

Inequality 29 provides a series of variance bounds on pricing kernels within PK_t for every intermediate period s between t and T as a function of the optimal dynamic Sharpe ratio between s and horizon T. Equation 27 identifies the distance to the bound to the quality of the replication of the kernel, as measured by $G_s(m_T)$. Excess variance of a pricing kernel is due to its component which is conditionally orthogonal to the space $w_T(\mathcal{X}_t)$ of payoffs which can be reached through self financing strategies. When the kernel is in $w_T(\mathcal{X}_t)$, as it is the case for w_T^t, the replication is perfect and the inequality becomes an equality.

6 Extension of the Investment Scope

The analysis of the self financing portfolios and their pricing kernels which we developed so far will help us now tackle a central issue in incomplete markets. We study the implications of selecting a price process for some derivative instruments in a way which is consistent with the dynamic behavior of their underlying securities. We focus on two related investment problems, the dynamic management of a portfolio on the one hand, and the optimal hedging of a contingent claim on the other hand. We deal in this section with basic issues and we postpone until the next one the analysis of the additional constraint imposed by a smile.

In addition to the original n securities, we consider n_x new securities which distribute some numeraire dividends every period described by the vector process $\{d_t^x\}_{1\leq t\leq(T-1)}$. For every period t between 0 and $(T-2)$, the owner of one unit of security j at time t, receives the next period the quantity $d_{t+1}^{x,j}$ of numeraires as dividend. A time $(T-1)$, one unit of security j gives right to the final payoff ϕ_T^x at time T. One can think of ϕ_T^x as the sum of a dividend and a residual value. We assume that the dividend process $\{d_t^x\}_{1\leq t\leq(T-1)}$ and the final payoff ϕ_T^x are given and known. We further assume that the dividend process is adapted to $\mathcal{F}$, that the final payoff ϕ_T^x is a random vector in $L^2(P; \mathbb{R}^{n_x})$, and that for every period t between 1 and $(T-1)$ and for every index j the dividend $d_t^{x,j}$ belongs to $L_t^2(P, \sqrt{h_t})$. These new instruments may be for instance derivatives written on the original securities, in which case the dividends and the final payoff are functions of the prices of the original securities. We do not however limit ourselves to this special situation.

We consider a period t between 0 and $(T-1)$ and we let the vector processes in $\mathbb{R}^{n_x}$ $\{p_s^x\}_{t\leq s\leq(T-1)}$ and $\{\phi_s^x\}_{t\leq s\leq(T-1)}$ be respectively the ex and cum dividend price dynamics of the new securities between t and $(T-1)$. We say that this price dynamics starting at time t is admissible if it is adapted to $\mathcal{F}$, if $\phi_s^x = (p_s^x + d_s^x)$ every period, and if it satisfies the law of one price together with the prices of the original n securities. In line with Assumption 2, this last requirement means that for every period s between t and $(T-1)$ and for every vector (u, v) in $\mathbb{R}^n \times \mathbb{R}^{n_x}$ measurable with respect to $\mathcal{F}_s$, the equality $\phi_{s+1}' u + (\phi_{s+1}^x)' v = 0$ implies $p_s' u + (p_s^x)' v = 0$. It is a weak notion of absence of arbitrage, the minimum structure which we need in order to apply our dynamic mean-variance analysis.

We shall limit our investigations to admissible price dynamics for the new securities. Let us remark that our approach is purely partial equilibrium. We do not study for instance how the introduction of the new securities changes the price dynamics of the original ones, a complex and fascinating question. The denomination "original" and "new" securities is therefore somewhat misleading, it is only a convenient way to describe the extension of the investment scope.

6.1 Admissible Price Dynamics

A first question is the existence and the construction of an admissible price dynamics for the new securities. The following lemma shows that an admissible price dynamics may be derived from a positive pricing kernel for the original securities. It is well known that such a positive kernel prevents the existence of arbitrage opportunities, as would result from any violation to the law of one price. The proof of this lemma, left in the Appendix, is a straightforward application of Lemma 6.

Lemma 8 *Let m_T be a positive pricing kernel in PK_t. The processes $\{p_s^x, \phi_s^x\}_{t\leq s\leq(T-1)}$ defined by the backward equations*

$$
(32) \qquad \begin{aligned} p_s^x &= E_s\left[R_{s+1}^f m_{s+1}\phi_{s+1}^x\right] / \left(R_s^f m_s\right), \\ \phi_s^x &= p_s^x + d_s^x, \end{aligned}
$$

form an admissible price dynamics for the new securities.

Even when no positive kernel is available, and in particular even if we do not know if w_T^t is positive, it is possible to create an admissible price dynamics for the new securities.

Lemma 9 *The processes $\{\bar{p}_s^x, \bar{\phi}_s^x\}_{t\leq s\leq(T-1)}$ in $I\!R^{n_x}$ defined for every index j from one to n_x by the backward equations*

$$
\begin{aligned} \bar{p}_s^{x,j} &= p_s' N_s^+ E_s\left[h_{s+1}\bar{\phi}_{s+1}^{x,j}\phi_{s+1}\right], \\ \bar{\phi}_s^{x,j} &= \bar{p}_s^{x,j} + d_s^{x,j}, \end{aligned}
$$

form an admissible price dynamics for the new securities.

We remark that if we describe the dividends starting from $(t+1)$ and the final payoff of security j as a sequence of cash flows with $f = \{f_s\}_{t+1\leq s\leq T}$ defined by $f_s = d_s^{x,j}$ for s between $(t+1)$ and $(T-1)$ and $f_T = \phi_T^{x,j}$, then the process $\{\bar{p}_s^{x,j}\}_{t\leq s\leq(T-1)}$ coincides with the process $\{\bar{f}_s\}_{t\leq s\leq(T-1)}$ defined in Section 3.4. In particular if we let

$$
(33) \qquad F_T^x \stackrel{\text{def.}}{=} \sum_{s=t+1}^{T-1} \sqrt{h_s} w_T^s d_s^x + \phi_T^x,
$$

then we learn from Lemma 5 that $\bar{p}_t^{x,j} = Q_t(F_T^{x,j})$. This means that $\bar{p}_s^{x,j}$ represents the cost at time s of the optimal replication of the sequence of cash flows generated by new security j from $(s+1)$ up to horizon T.

We also remark that for every pricing kernel m_T in PK_t we have

$$E_t\left[m_T F_T^x\right] = E_t\left[\sum_{s=t+1}^{T-1} R_s^f m_s d_s^x + m_T \phi_T^x\right]. \tag{34}$$

6.2 Extended Asset Structure

For an admissible price dynamics $\{p_s^x, \phi_s^x\}_{t \leq s \leq (T-1)}$, we consider the extended asset structure between time t and horizon T which consists in the n original securities together with the n_x payoffs priced according to the dynamics $\{p_s^x, \phi_s^x\}_{t \leq s \leq (T-1)}$. We denote p_s^e and ϕ_s^e the corresponding ex and cum dividend prices at time s. The first n components of the vectors p_s^e and ϕ_s^e are respectively p_s and ϕ_s while their last n_x components are respectively p_s^x and ϕ_s^x.

The extended asset structure satisfies both Assumptions 1 and 2 and Conditions (a) and (b) of Proposition 1. The zero coupon bond is a security of the original asset structure and it remains traded in the extended one. The results of Sections 2 to 5 can therefore be brought to bear, with period t corresponding to the initial trading period 0 in these sections.

We set $h_T^e = h_T = 1$ and for $t \leq s \leq (T-1)$, we let h_s^e, H_s^e, $\mathcal{X}_s^e$, Q_s^e, G_s^e, PK_s^e, $X^{s,e}$, $w_T^{s,e}$, SR_s^e be the counterparts to h_s, H_s, $\mathcal{X}_s$, Q_s, G_s, PK_s, X^s, w_T^s, SR_s for the extended asset structure. Notice that for every period s between t and $(T-1)$ we have $H_s^e \stackrel{\text{def.}}{=} h_s^e/(R_s^f)^2$ and

$$\mathrm{SR}_s^e \stackrel{\text{def.}}{=} \sqrt{\left(\frac{1}{H_s^e} - 1\right)}.$$

It is clear that PK_s^e is a subset of PK_s. The next lemma, another direct application of Lemma 6, shows that a necessary and sufficient condition for a pricing kernel in PK_t to belong to PK_t^e is to "price" correctly the new securities.

Lemma 10 *Let m_T be a pricing kernel m_T in PK_t. The following three statements are equivalent.*

(i). m_T belongs to PK_t^e.

(ii). For every period s between t and $(T-1)$,

$$E_s\left[R_{s+1}^f m_{s+1} \phi_{s+1}^x\right] = R_s^f m_s p_s^x.$$

(iii). For every period s between t and $(T-1)$,

$$R_s^f m_s p_s^x = E_s \left[\sum_{u=s+1}^{T-1} R_u^f m_u d_u^x + m_T \phi_T^x \right].$$

If m_T is a pricing kernel in PK_t^e for the extended asset structure then we derive from Statement (iii) of Lemma 10 and Equation 34 that

$$E_t [m_T F_T^x] = R_t^f m_t p_t^x. \tag{35}$$

6.3 Sharpe Ratio Improvement

The optimum dynamic Sharpe ratio may only increase as a result of the extension of the investment set, which means that for every period s between t and $(T-1)$ we have $\mathrm{SR}_s \leq \mathrm{SR}_s^e$ and $H_s^e \leq H_s$. The following result quantifies this increase in terms of pricing kernels, it is a direct application of Equation 27 to the extended asset structure. We recall that $w_T^{t,e}$ is the value at time T of the L^2 minimum portfolio $X^{t,e}$ in the set of self financing strategies $\mathcal{X}_t^e$ for the extended asset structure.

Result 1 *For every pricing kernel m_T in PK_t^e and for every period s between t and $(T-1)$,*

$$m_s^2 \left[(SR_s^e)^2 - (SR_s)^2 \right] = G_s(m_T) - G_s^e(m_T)$$

and in particular

$$(SR_t^e)^2 - (SR_t)^2 = \frac{G_t(w_T^{t,e})}{H_t^e}.$$

Result 1 tells us that the optimum dynamic Sharpe ratio increases inasmuch as the pricing kernels for the extended asset structure are better replicated with the help of the new securities. The increase in the square of the Sharpe ratio is also directly related to the distance between the L^2 minimum portfolio for the extended asset structure and the final values of the self financing strategies based on the initial securities, as measured by $G_t(w_T^{t,e})$.

If, as in Lemma 8, a positive pricing kernel m_T is used in order to generate the price dynamics of an increasing number of new instruments, then the Sharpe ratio increases as long as $G_s^e(m_T)$ decreases and the new instruments help replicate the kernel. Once enough instruments have been introduced so that m_T is perfectly replicated, the optimum dynamic Sharpe ratio ceases to increase as new instruments are added. The optimum dynamic Sharpe ratio from s to T reaches then its maximum possible value given by

$$(\mathrm{SR}_s^e)^2 = \mathrm{Var}_s[m_T/m_s].$$

With no clear indication on which pricing kernel to choose, a fund manager runs the risk of picking a kernel with too large a variance, leading to large potential increases in performance for some carefully selected new instruments. We next investigate the conservative situation which corresponds to a min-max in terms of dynamic Sharpe ratio. We study the admissible price dynamics which yields the lowest possible increase in Sharpe ratio for the corresponding optimal dynamic strategy. Without any smile constraint, it is possible to avoid any mean-variance good deal altogether.

6.4 Absence of Good Deal

We consider an admissible price dynamics $\{p_s^x, \phi_s^x\}_{t\leq s\leq(T-1)}$ and the extended asset structure which it generates from time t up to horizon T. The following propositions characterize the situation where no gain in dynamic Sharpe ratio may be expected from trading in the new securities.

Proposition 6 *$SR_t^e = SR_t$ if and only if the following equivalent conditions hold.*

(i). $w_T^{t,e} = w_T^t$.

(ii). w_T^t *belongs to* PK_t^e.

When this happens, $p_t^x = \bar{p}_t^x$.

If no good deal is available from t to T, it seems intuitive that no good deal should exist between a later trading date s and T. We only prove this fact for the periods s such that the value w_s^t of the L^2 minimum strategy does not vanish.

Proposition 7 *If $SR_t^e = SR_t$, then at every period s between t and $(T-1)$ such that w_s^t does not vanish we have $SR_s^e = SR_s$.*

In particular, when w_T^t is positive, we know that w_s^t never vanishes and we may further characterize the absence of good deal at time t in terms of the entire price process of the new instruments.

Proposition 8 *If w_T^t is positive, then $SR_t^e = SR_t$ if and only if $p_s^x = \bar{p}_s^x$ for every period s between t and $(T-1)$. When this happens, we also have $SR_s^e = SR_s$ for $t \leq s < T$.*

The following proposition deals with the general case where we do not know whether w_T^t is positive. It shows that no good deal is available at time t and at later trading periods if the price dynamics is derived from the optimal hedging cost of the new instruments, which is admissible according to Lemma 9.

Proposition 9 *The following two statements are equivalent.*

(i). $SR_s^e = SR_s$ for every period s between t and $(T-1)$.

(ii). $p_s^x = \bar{p}_s^x$ for every period s between t and $(T-1)$.

In view of Result 1, no gain in Sharpe ratio can be expected for this dynamics from the introduction of new securities because the pricing kernel w_T^t in PK_t^e is already the final value of a dynamic portfolio constructed with the original securities.

The use of the pricing kernel w_T^t, or of the dynamics $\{\bar{p}_s^x\}_{t \leq s \leq (T-1)}$, can therefore be justified on the ground of avoiding "good-deal" in the form of an increase in the slope of the dynamic efficient frontier due to the introduction of new securities. This provides an additional appeal to this special pricing kernel, in addition to its interpretation as pricing securities according to the cost of their optimal replications.

The next proposition proves that the cost of the variance optimal hedge of a security is not affected by the introduction of new hedging instruments, as long as these instruments are themselves priced according to the cost of their variance optimal hedge.

Proposition 10 *If $p_s^x = \bar{p}_s^x$ for every period s between t and $(T-1)$ then the operators Q_s^e and Q_s are identical for every period s between t and $(T-1)$.*

7 Smile Consistent Kernels and Dynamics

We consider again a period t between 0 and $(T-1)$ and the n_x new securities described by their dividends and final payoffs. A smile at time t is a random vector S_t^x in $\mathbb{R}^{n_x}$, measurable with respect to $\mathcal{F}_t$, which describes the prices of the n_x new securities at period t. We start by studying the set of pricing kernels which are consistent with the smile and we provide a lower bound on the variance of these kernels. We then study the admissible price dynamics for the new securities between period t and horizon T which agree with the smile S_t^x at time t, and we derive a lower bound on the optimum dynamic Sharpe ratio for the corresponding extended market structure.

7.1 Smile Consistent Pricing Kernels

A pricing kernel consistent with the smile S_t^x at time t is a pricing kernel m_T in PK_t which satisfies

$$R_t^f m_t S_t^x = E_t\left[\sum_{s=t+1}^{T-1} R_s^f m_s d_s^x + m_T \phi_T^x\right].$$

According to Equation 34, this is equivalent to the requirement that $E_t\left[m_T F_T^x\right] = R_t^f m_t S_t^x$. We let $\mathrm{PK}_t(S_t^x)$ be the set of pricing kernels consistent with the smile S_t^x at time t. We give conditions for the set $\mathrm{PK}_t(S_t^x)$ to be non empty and we study the properties of the pricing kernels in $\mathrm{PK}_t(S_t^x)$.

It will be useful to extend the domain of the operators Q_t and G_t from random variables to random vectors. If F_T^a and F_T^b are two random vectors respectively in $L^2(P; I\!R^{n_a})$ and $L^2(P; I\!R^{n_b})$, we let $Q_t(F_T^a)$ be the random vector in $I\!R^{n_a}$ such that $(Q_t(F_T^a))_i \stackrel{\text{def.}}{=} Q_t((F_T^a)_i)$ and we let $G_t(F_T^a, F_T^b)$ be the random matrix of size $n_a \times n_b$ such that $(G_t(F_T^a))_{i,j} \stackrel{\text{def.}}{=} Q_t((F_T^a)_i, (F_T^b)_j)$. We further denote $G_t(F_T^a)$ the symmetric matrix $G_t(F_T^a, F_T^a)$. The following inequality will prove useful.

Lemma 11 *Let F_T^a and F_T^b be respectively a random variable in $L^2(P)$ and a random vector in $L^2(P; I\!R^{n_b})$, then*

$$G_t(F_T^a, F_T^b) G_t(F_T^b)^+ G_t(F_T^b, F_T^a) \leq G_t(F_T^a).$$

In the same spirit as above, if w_t^a is a random vector in $L_t^2(P, \sqrt{h_t}; I\!R^{n_a})$ and if F_T^a is a vector payoff in $L^2(P; I\!R^{n_a})$, we let X^{t, w_t^a, F_T^a} be the random matrix of size $n \times n_a$ whose ith column is the random vector $X^{t,(w_t^a)_i,(F_T^a)_i}$ in $I\!R^n$ which describes the optimal dynamic replication of the payoff $(F_T^a)_i$ starting at time t with wealth $(w_t^a)_i$. It is then natural to let

$w_s(X^{t,w_t^a,F_T^a})$ represent the value process of these n_a dynamic portfolios, a random vector in $I\!R^{n_a}$ such that $w_s(X^{t,w_t^a,F_T^a})_i \stackrel{\text{def.}}{=} w_s(X^{t,(w_t^a)_i,(F_T^a)_i})$.

We recall that the vector payoff F_T^x represents the final value of the strategies which reinvest every period the dividends of the new securities in the L^2 minimum portfolio up to horizon T. We let $F_t^x \stackrel{\text{def.}}{=} Q_t(F_T^x)$ be the cost at time t of the optimal replication of F_T^x, we have earlier obtained that $F_t^x = \bar{p}_t^x$. We let $M_T^{t,x} \stackrel{\text{def.}}{=} F_T^x - w_T(X^{t,F_t^x,F_T^x})$ represent the gap at maturity T between the vector payoff F_T^x and its optimal mean-variance replication strategy which starts at time t. Notice that $Q_t(M_T^{t,x}) = 0$ and that Equation 15 implies that $G_t(F_T^x) = E_t\left[M_T^{t,x}(M_T^{t,x})'\right]$.

The next lemma yields some first results on $\text{PK}_t(S_t^x)$. We define the random vector Λ_t^x and the random variables K_t^x and H_t^x by

$$\begin{aligned}
\Lambda_t^x &\stackrel{\text{def.}}{=} \sqrt{h_t}G_t(F_T^x)^+\left(S_t^x - F_t^x\right), \\
K_t^x &\stackrel{\text{def.}}{=} h_t\left(S_t^x - F_t^x\right)' G_t(F_T^x)^+\left(S_t^x - F_t^x\right) = \Lambda_t^{x\prime} G_t(F_T^x)\Lambda_t^x, \\
H_t^x &\stackrel{\text{def.}}{=} \frac{H_t}{(1+K_t^x)}.
\end{aligned}$$

Lemma 12 *Let m_T be a pricing kernel in $PK_t(S_t^x)$. We have*

$$R_t^f m_t\left(S_t^x - F_t^x\right) = G_t\left(F_T^x, m_T\right), \tag{36}$$

$$\frac{m_t^2}{H_t}K_t^x \leq G_t\left(m_T\right), \tag{37}$$

and m_t belongs to $L_t^2(P, 1/\sqrt{H_t^x})$.

For a random variable $\bar{m}_t$ in $\mathcal{F}_t$, we let

$$m_T^{t,x}(\bar{m}_t) \stackrel{\text{def.}}{=} \frac{\bar{m}_t}{\sqrt{H_t}}\left((\Lambda_t^x)' M_T^{t,x} + w_T^t\right).$$

This kernel will play a central role in our analysis, we list here some basic properties.

Lemma 13 *For every random variable $\bar{m}_t$ in $L_t^2(P, 1/\sqrt{H_t^x})$ the random variable $m_T^{t,x}(\bar{m}_t)$ is a pricing kernel in PK_t which satisfies $E_t[m_T^{t,x}(\bar{m}_t)] = \bar{m}_t$ and*

$$E_t\left[\left(m_T^{t,x}(\bar{m}_t)\right)^2\right] = \frac{\bar{m}_t^2}{H_t^x}.$$

For $\bar{m}_t$ in $L^2_t(P, 1/\sqrt{H^x_t})$ and $t \leq s \leq T$ we let $m^{t,x}_s(\bar{m}_t) \stackrel{\text{def.}}{=} E_s\left[m^{t,x}_T(\bar{m}_t)\right]$. The next condition will be shown to be necessary and sufficient for the existence of smile consistent pricing kernels.

Condition 1 *$(S^x_t - F^x_t) = G_t(F^x_T)G_t(F^x_T)^+ (S^x_t - F^x_t)$.*

Proposition 11 *(i). If the smile S^x_t satisfies Condition 1, then for every random variable $\bar{m}_t$ in $L^2_t(P, 1/\sqrt{H^x_t})$, the pricing kernel $m^{t,x}_T(\bar{m}_t)$ is in $PK_t(S^x_t)$. In particular $PK_t(S^x_t)$ contains pricing kernels m_T such that m_t does not vanish.*

(ii). Reciprocally, if there exists a pricing kernel m_T in $PK_t(S^x_t)$ such that m_t does not vanish, then the smile S^x_t satisfies Condition 1.

We next investigate the L^2 properties of pricing kernels in $\mathrm{PK}_t(S^x_t)$. This will provide a lower bound on the variance of the kernels which are consistent with the smile.

7.2 Variance Bound with a Smile

The following proposition proves that the kernels $m^{t,x}_T$ have minimum L^2 norm within $\mathrm{PK}_t(S^x_t)$.

Proposition 12 *Let us assume that Condition 1 is satisfied. Every pricing kernel m_T in $PK_t(S^x_t)$ satisfies the following three inequalities:*

$$\frac{m_t^2}{H_t} K^x_t \leq G_t\left(m_T\right), \tag{38}$$

$$\frac{m_t^2}{H^x_t} \leq E_t\left[m_T^2\right], \tag{39}$$

$$m_t^2 (SR_t)^2 + m_t^2 (R^f_t)^2 \left(S^x_t - F^x_t\right)' G_t(F^x_T)^+ \left(S^x_t - F^x_t\right) \leq Var_t\left[m_T\right]. \tag{40}$$

These three inequalities become equalities if and only if $m_T = m^{t,x}_T(m_t)$.

When m_t does not vanish, Inequality 40 writes also

$$(\mathrm{SR}_t)^2 + (R^f_t)^2 \left(S^x_t - F^x_t\right)' G_t(F^x_T)^+ \left(S^x_t - F^x_t\right) \leq \mathrm{Var}_t\left[\frac{m_T}{m_t}\right].$$

This inequality describes how the smile constraint sharpens the variance bound on the marginal rate of substitution which we derived in Proposition 5. The increase in the bound is a function of the distance, in the metric described by the matrix $G_t(F_T^x)^+$, between the observed prices S_t^x of the instruments in the smile and the cost F_t^x of their optimal mean-variance replication.

In the simple case where $n_x = 1$ and the smile data is limited to one instrument, the increase in the square of the Sharpe ratio writes $(S_t^x - F_t^x)^2/G_t(F_T^x)$. It is large when the quality of the replication is high and the difference between F_t^x and S_t^x is large. Intuitively, this says that it is "costly" for a pricing kernel to produce prices which deviate much from the cost of the replication when this replication is very good, as this would imply a very large variance for the kernel.

7.3 Optimum Dynamic Sharpe Ratio with a Smile

We recall that an admissible price dynamics starting at time t for the new securities is a couple of vector processes $\{p_s^x, \phi_s^x\}_{t \le s \le (T-1)}$ in $I\!R^{n_x}$ adapted to $\mathcal{F}$ which satisfies $\phi_s^x = p_s^x + d_s^x$ and such that, together with the price processes of the original securities, they satisfy the law of one price. We say that a price dynamics for the new securities is consistent with the smile S_t^x at time t if it is admissible and if it satisfies $p_t^x = S_t^x$. The next proposition studies the existence of smile consistent price dynamics. It shows in particular the necessity of Condition 1.

Proposition 13 *(i). Let m_T be a positive pricing kernel in $PK_t(S_t^x)$. The price dynamics $\{p_s^x, \phi_s^x\}_{t \le s \le (T-1)}$ defined by the backward equations*

$$\begin{aligned} p_s^x &= E_s\left[R_{s+1}^f m_{s+1} \phi_{s+1}^x\right] / \left(R_s^f m_s\right), \\ \phi_s^x &= p_s^x + d_s^x, \end{aligned}$$

is consistent with the smile S_t^x at time t.

(ii). If there exists a price dynamics consistent with the smile S_t^x at time t, then the smile S_t^x satisfies Condition 1.

We derive from Propositions 11 and 13 that a sufficient condition for the existence of a price dynamics consistent with the smile is Condition 1, together with the requirement that the kernel $m_T^{t,x}\left(\sqrt{H_t^x}\right)$ be positive.

We now assume that there exists a price dynamics consistent with the smile S_t^x at time t. We let $\{p_s^x, \phi_s^x\}_{t \le s \le (T-1)}$ be such a consistent price dynamics and we consider the extended asset structure which it generates between time t and horizon T.

We learn from Proposition 13 that Condition 1 is satisfied and we know from Proposition 11 that the pricing kernel $m_T^{t,x}(\bar{m}_t)$ is an element of $\mathrm{PK}_t(S_t^x)$, for every variable $\bar{m}_t$ in $L_t^2(P, 1/\sqrt{H_t^x})$. The set $\mathrm{PK}_t(S_t^x)$ is therefore not empty.

Statement (iii) of Lemma 10 proves that if m_T is a pricing kernel for the extended asset structure, then

$$R_t^f m_t p_t^x = E_t \left[\sum_{s=t+1}^{T-1} R_s^f m_s d_s^x + m_T \phi_T^x \right].$$

Since $p_t^x = S_t^x$, we obtain that m_T belongs to $\mathrm{PK}_t(S_t^x)$. This proves that the set PK_t^e of pricing kernels for the extended asset structure is a subset of $\mathrm{PK}_t(S_t^x)$.

The next proposition provides a lower bound to the optimum dynamic Sharpe ratio of the extended asset structure. Notice that for every period s between t and $(T-1)$ we have $\mathrm{SR}_s^e = \sqrt{1/H_s^e - 1}$. We define similarly $\mathrm{SR}_s^x \stackrel{\text{def.}}{=} \sqrt{1/H_s^x - 1}$ and we remark that

$$(\mathrm{SR}_t^x)^2 = (\mathrm{SR}_t)^2 + (R_t^f)^2 \, (S_t^x - F_t^x)' \, G_t(F_T^x)^+ \, (S_t^x - F_t^x) \, .$$

Proposition 14 *Let $\{p_s^x, \phi_s^x\}_{t \le s \le (T-1)}$ be a price dynamics for the new securities which is consistent with the smile S_t^x at time t. The corresponding extended asset structure satisfies $0 < H_t^e \le H_t^x \le H_t \le 1$ and its optimum dynamic Sharpe ratio from t to T satisfies $SR_t \le SR_t^x \le SR_t^e$. The minimum increase in the square of the optimum dynamic Sharpe ratio from the original asset structure to the extended one is given by*

$$(SR_t^x)^2 - (SR_t)^2 = (R_t^f)^2 \, (S_t^x - F_t^x)' \, G_t(F_T^x)^+ \, (S_t^x - F_t^x) \, .$$

Every pricing kernel m_T in PK_t^e satisfies the inequality

$$m_t^2 (SR_t^x)^2 \le Var_t \left[m_T \right].$$

The next section investigates situations where the Sharpe ratio reaches its smile constrained lower bound SR_t^x.

8 Two Dynamic Investment Problems

Our analysis will help us answer the questions raised by the fund manager and the investment banker who are seeking a rationale for selecting a price dynamics for some new instruments in an incomplete market setting. A basic requirement is to avoid working with price dynamics who create arbitrage opportunities. A second objective is to be consistent with the market quotes of some liquid derivative instruments. The positive pricing kernels in $\mathrm{PK}_t(S_t^x)$ fulfill these requirements. When markets are incomplete however, these kernels are usually not unique, and an additional rationale is needed in order to pick a "good" candidate. The fund manager is afraid of generating spurious dynamic good deals, while the banker would like to keep a close link between the price of a security and the cost of its dynamic hedge.

We show in this section that when the kernel $m_T^{t,x}$ is positive, it meets these two concerns. For the fund manager, it generates a smile consistent price dynamics which yields the smallest possible increase in Sharpe ratio. For the investment banker, it produces derivative prices which are as close as possible to the hedging cost under the constraint of the smile.

8.1 Portfolio Management and the Smile

We first consider a price dynamics $\{p_s^x, \phi_s^x\}_{t \le s \le (T-1)}$ consistent with the smile S_t^x at time t and the corresponding extended asset structure which it generates. We study the situation where the optimum dynamic Sharpe ratio of the extended asset structure reaches its theoretical lower bound, as described in Proposition 14.

The kernel $m_T^{t,x}$ plays here again a crucial role. We know from Proposition 13 that Condition 1 is satisfied and from Proposition 11 that $m_T^{t,x}(\sqrt{H_t^x})$ is a smile consistent pricing kernel in $\mathrm{PK}_t(S_t^x)$.

We recall that $F_T^{x,j}$ corresponds to the final payoff of the self financing strategy which holds one unit of the new security j from t to T and reinvests every dividend distributed by this security in the L^2 minimum portfolio up to horizon T. As a result, the random variable

$$m_T^{t,x}\left(\sqrt{H_t^x}\right) = \frac{\sqrt{H_t^x}}{\sqrt{H_t}}\left((\Lambda_t^x)' M_T^{t,x} + w_T^t\right)$$

is the final value of a self financing dynamic portfolio starting at t which combines on the one hand some constant quantities of the n_x new securities given by the vector $(\sqrt{H_t^x}/\sqrt{H_t})\Lambda_t^x$, and on the other hand a portfolio based on the n original securities. Since, according to Lemma 13, the payoff $m_T^{t,x}\left(\sqrt{H_t^x}\right)$ is in $L^2(P)$, it is also in $w_T(\mathcal{X}_t^e)$ and there exists a dynamic portfolio Y^e in $\mathcal{X}_t^e$ such that $m_T^{t,x}\left(\sqrt{H_t^x}\right) = w_T(Y^e)$. Its value at time t is

$$w_t(Y^e) = \frac{\sqrt{H_t^x}}{\sqrt{H_t}}\left((\Lambda_t^x)'(S_t^x - F_t^x) + \frac{1}{\sqrt{h_t}}\right) = \frac{1}{R_t^f\sqrt{H_t^x}}.$$

Proposition 15 *Let $\{p_s^x, \phi_s^x\}_{t\leq s\leq (T-1)}$ be a price dynamics consistent with the smile S_t^x at time t. The optimum dynamic Sharpe ratio SR_t^e of the extended asset structure reaches its minimum value SR_t^x if and only if one of the following equivalent conditions holds.*

(i). $m_T^{t,x}\left(\sqrt{H_t^x}\right) = w_T^{t,e}$.

(ii). There exists a vector Λ_t in $I\!R^{n_x}$ measurable with respect to $\mathcal{F}_t$ such that the strategy $X^{t,e}$ has a value process which is identical to the one of a self financing strategy in $\mathcal{X}_t^e$ which holds constant quantities of the new securities given by the vector Λ_t and $\Lambda_t' M_T^{t,x}$ is in $L^2(P)$.

(iii). The pricing kernel $m_T^{t,x}\left(\sqrt{H_t^x}\right)$ belongs to PK_t^e.

(iv). For every period s between time t and $(T-1)$,

$$E_s\left[R_{s+1}^f m_{s+1}^{t,x}\left(\sqrt{H_t^x}\right)\phi_{s+1}^x\right] = R_s^f m_s^{t,x}(\sqrt{H_t^x}) p_s^x.$$

Because mean-variance efficient portfolios for the extended asset structure are fixed combinations from t to T of the strategy $X^{t,e}$ and the risk free bond, Statement (ii) implies that every efficient strategy from t to T for the extended asset structure keeps a fixed quantity of each new security in portfolio.

We next investigate if reaching the lower bound of the Sharpe ratio at time t implies that an equivalent lower bound is reached at a later trading date s, for the smile given by p_s^x. For every period s between t and $(T-1)$ we let

$$\begin{aligned}
K_s^x &\overset{\text{def.}}{=} h_s\,(p_s^x - \bar{p}_s^x)'\, G_s (F_T^x)^+\,(p_s^x - \bar{p}_s^x)\,,\\
H_s^x &\overset{\text{def.}}{=} \frac{H_s}{(1+K_s^x)},\\
\mathrm{SR}_s^x &\overset{\text{def.}}{=} \sqrt{\left(\frac{1}{H_s^x} - 1\right)}.
\end{aligned}$$

Proposition 16 *Let $\{p_s^x, \phi_s^x\}_{t\leq s\leq (T-1)}$ be a price dynamics consistent with the smile S_t^x at time t. If $SR_t^e = SR_t^x$ then at every period s between t and $(T-1)$ such that $m_s^{t,x}(\sqrt{H_t^x})$ is P almost surely different from zero, we have $SR_s^e = SR_s^x$.*

Combining these results with Propositions 11 and 13, we obtain that when the kernel $m_T^{t,x}\left(\sqrt{H_t^x}\right)$ is positive, it generates a smile consistent price dynamics which avoids good deals every period.

Result 2 *If the pricing kernel* $m_T^{t,x}\left(\sqrt{H_t^x}\right)$ *is positive and if the smile* S_t^x *satisfies Condition 1, then the price dynamics* $\{p_s^x, \phi_s^x\}_{t\leq s\leq (T-1)}$ *defined recursively by*

$$\begin{aligned} p_s^x &= E_s\left[R_{s+1}^f m_{s+1}^{t,x}\left(\sqrt{H_t^x}\right)\phi_{s+1}^x\right] / \left(R_s^f m_s^{t,x}(\sqrt{H_t^x})\right), \\ \phi_s^x &= p_s^x + d_s^x, \end{aligned}$$

for every period s *between time* t *and* $(T-1)$ *is consistent with the smile* S_t^x *at time* t*. It generates an extended asset structure such that* $SR_s^e = SR_s^x$ *for every period* s *between time* t *and horizon* $(T-1)$*.*

8.2 Hedging and Pricing with the Smile

We now turn our attention to the problem of hedging and pricing derivatives with the constraint of a smile. We assume here that both w_T^t and $m_T^{t,x}\left(\sqrt{H_t^x}\right)$ are positive.

For a pricing kernel m_T in PK_t such that $m_t \overset{\text{def.}}{=} E_t[m_T]$ is positive, we let $J_t(F_T; m_T)$ represent the quality of the optimal L^2 replication of the payoff F_T in $L^2(P)$ under the constraint that the replication starts at time t with a value derived from the kernel m_T. Formally we let

$$\begin{aligned} &\text{essinf } E_t\left[(F_T - w_T(X))^2\right] \overset{\text{def.}}{=} J_t(F_T; m_T). \\ &\text{s. t. } \begin{cases} X \in \mathcal{X}_t \\ w_t(X) = \frac{E_t[m_T F_T]}{R_t^f m_t} \end{cases} \end{aligned}$$

We know from Proposition 2 that $J_t(F_T; m_T) = h_t(w_t(X) - Q_t(F_T))^2 + G_t(F_T)$. When $m_T = w_T^t$, Equation 23 proves that $w_t(X) = Q_t(F_T)$ and $J_t(F_T; m_T)$ reaches its minimum over the set of pricing kernels with positive conditional expectation at time t with $J_t(F_T; w_T^t) = G_t(F_T)$. The following proposition shows that the kernel $m_T^{t,x}\left(\sqrt{H_t^x}\right)$ solves a min-max problem in terms of replication quality over all possible normalized payoffs in $L^2(P)$ at horizon T.

Proposition 17 *If the smile* S_t^x *satisfies Condition 1, then the optimization program*

$$\underset{\begin{cases} m_T \in PK_t(S_t^x) \\ m_t > 0 \end{cases}}{\textit{essinf}} \quad \underset{\begin{cases} F_T \in L^2(P) \\ E_t\left[F_T^2\right] = 1 \end{cases}}{\textit{esssup}} \quad J_t(F_T; m_T) - J_t(F_T; w_T^t)$$

is solved for the pricing kernel $m_T^{t,x}\left(\sqrt{H_t^x}\right)$ *with minimum value* K_t^x *with*

$$K_t^x = h_t\left(S_t^x - F_t^x\right)' G_t(F_T^x)^+ \left(S_t^x - F_t^x\right).$$

Since $J_t(F_T; m_T) - J_t(F_T; w_T^t) = h_t(w_t(X) - Q_t(F_T))^2$ with $w_t(X) = E_t[m_T F_T]/(R_t^f m_t)$, the min-max problem of Proposition 17 can also be interpreted as selecting the smile consistent kernel which produces contingent claim prices as close as possible to the cost of the optimal unconstrained hedge. This result proves the constrained optimality of the kernel $m_T^{t,x}\left(\sqrt{H_t^x}\right)$, both in terms of hedging and in terms of pricing.

We now look at the optimal hedge for the extended asset structure generated by the kernel $m_T^{t,x}\left(\sqrt{H_t^x}\right)$. We show that this kernel generates prices which correspond to the cost of the optimal replication constructed with both the original and the new securities.

Proposition 18 *Let us assume that* $m_T^{t,x}\left(\sqrt{H_t^x}\right)$ *is positive and that the smile* S_t^x *satisfies Condition 1. Let us consider the extended asset structure which this pricing kernel generates through the smile consistent price dynamics* $\{p_s^x, \phi_s^x\}_{t \leq s \leq (T-1)}$ *defined by*

$$p_s^x = \frac{E_s\left[R_{s+1}^f m_{s+1}^{t,x}\left(\sqrt{H_t^x}\right)\phi_{s+1}^x\right]}{R_s^f m_s^{t,x}(\sqrt{H_t^x})}$$

and $\phi_s^x = p_s^x + d_s^x$ *for every period* s *between time* t *and* $(T-1)$*. For every payoff* F_T *in* $L^2(P)$*, the price generated by the pricing kernel* $m_T^{t,x}\left(\sqrt{H_t^x}\right)$ *coincides with the cost of the variance-optimal replication of* F_T *which uses both the original and the new securities, that is*

$$Q_s^e(F_T) = \frac{E_s\left[m_T^{t,x}\left(\sqrt{H_t^x}\right)F_T\right]}{R_s^f m_s^{t,x}(\sqrt{H_t^x})} \tag{41}$$

for every period s *between time* t *and horizon* T*. Furthermore at time* t *we have*

$$Q_t^e(F_T) = Q_t(F_T) + (S_t^x - F_t^x)' \, G_t\left(F_T^x\right)^+ G_t\left(F_T^x, F_T\right). \tag{42}$$

We check that for $F_T = F_T^x$ in Equation 42, Condition 1 implies that $Q_t^e(F_T^x) = S_t^x$, and Equation 41 with $s = t$ confirms the consistency of the kernel $m_T^{t,x}\left(\sqrt{H_t^x}\right)$ with the smile S_t^x at time t.

Appendix to Section 3

Proof of Lemma 2. We only prove the second part of Statement (ii) which is not in Henrotte (2001). We use Equation 15 with $F_T^a = w_T(X)$ and $F_T^b = F_T$. The equality $G_t(w_T(X), F_T) = 0$ results from the fact that $Q_t(w_T(X)) = w_t(X)$ and $w_T(X) = w_T(X^{t,w_t(X),w_T(X)})$. □

Proof of Lemma 4. Statement (i). Let X be a self financing dynamic strategy starting at time t and let $Y = \theta_f(X)$. For every period s from $(t+1)$ up to $(T-1)$ we have

$$w_s(Y) = p_s' Y_s = p_s' X_s - \left(\sum_{u=t+1}^{s} f_u \sqrt{h_u} w_s^u \right) h_s p_s' N_s^+ p_s.$$

We know from Equation 5 that $h_s p_s' N_s^+ p_s = 1$. The fact that X is self financed at time s implies that $p_s' X_s = \phi_s' X_{s-1}$. Since furthermore $\sqrt{h_s} w_s^s = 1$, we derive

$$w_s(Y) = \phi_s' X_{s-1} - \left(\sum_{u=t+1}^{s-1} f_u \sqrt{h_u} w_s^u \right) - f_s.$$

For $u \leq (s-1)$ we know from Equation 16 that $w_s^u = w_{s-1}^u h_{s-1} \phi_s' N_{s-1}^+ p_{s-1}$ and therefore

$$\begin{aligned} w_s(Y) &= \phi_s' X_{s-1} - \left(\sum_{u=t+1}^{s-1} f_u \sqrt{h_u} w_{s-1}^u h_{s-1} \phi_s' N_{s-1}^+ p_{s-1} \right) - f_s \\ &= \phi_s' \left[X_{s-1} - \left(\sum_{u=t+1}^{s-1} f_u \sqrt{h_u} w_{s-1}^u \right) h_{s-1} N_{s-1}^+ p_{s-1} \right] - f_s \\ &= \phi_s' Y_{s-1} - f_s, \end{aligned}$$

which proves that Y finances the sequence of cash flows f. Reciprocally, the same equations proves that X is self financing as soon as Y finances f.

Statement (ii). The equality $w_t(Y) = w_t(X)$ results from the fact that $Y_t = X_t$. At time T, and since Equation 16 implies that $w_T^u = w_{T-1}^u h_{T-1} \phi_T' N_{T-1}^+ p_{T-1}$ for $u \leq (T-1)$, an analysis similar to the one developed above yields

$$\begin{aligned} w_T(Y) - f_T &= \phi_T' Y_{T-1} - f_T \\ &= \phi_T' X_{T-1} - \left(\sum_{u=t+1}^{T-1} f_u \sqrt{h_u} w_{T-1}^u h_{T-1} \phi_T' N_{T-1}^+ p_{T-1} \right) - f_T \end{aligned}$$

$$= \quad w_T(X) - \sum_{u=t+1}^{T-1} f_u \sqrt{h_u} w_T^u - f_T$$

$$= \quad w_T(X) - \sum_{u=t+1}^{T} f_u \sqrt{h_u} w_T^u,$$

and we obtain that $(w_T(Y) - f_T) = (w_T(X) - F_T)$. □

Proof of Proposition 3. The fact that the payoff F_T belongs to $L^2(P)$ results from the equalities

$$E\left[(f_s\sqrt{h_s}w_T^s)^2\right] = E\left[h_s f_s^2 E_s\left[(w_T^s)^2\right]\right] = E\left[h_s f_s^2\right],$$

and the fact that every cash flow f_s belongs to $L_s^2(P, \sqrt{h_s})$. Since both f_T and F_T are in $L^2(P)$ and since, according to Statement (ii) of Lemma 4, $(f_T - w_T(Y)) = (F_T - w_T(X))$, $w_T(Y)$ is in $L^2(P)$ if and only if $w_T(X)$ is itself in $L^2(P)$. We conclude with Statement (i) of Lemma 4 that the mapping θ_f is one to one from $\mathcal{X}_t$ to $\mathcal{X}_t(f)$. The equivalence between the two optimization programs is then a direct consequence of the properties of the mapping θ_f. □

Proof of Lemma 5. We prove these results by backward induction. We first deal with the hedging cost. At time T, we check that $Q_T(F_T) = F_T = \sum_{s=t+1}^{T} f_s\sqrt{h_s}w_T^s$. Let us assume that

$$Q_{s+1}(F_T) = \sum_{u=t+1}^{s+1} f_u\sqrt{h_u}w_{s+1}^u + \bar{f}_{s+1}$$

for $t \leq s \leq (T-1)$. Since $\sqrt{h_{s+1}}w_{s+1}^{s+1} = 1$, we also have

$$Q_{s+1}(F_T) = \sum_{u=t+1}^{s} f_u\sqrt{h_u}w_{s+1}^u + \bar{f}_{s+1} + f_{s+1}.$$

Using the equations $w_{s+1}^u = w_s^u h_s \phi_{s+1}' N_s^+ p_s$, $N_s^+ N_s N_s^+ = N_s^+$, and the definition of $\bar{f}_s$, we derive

$$\begin{aligned} Q_s(F_T) &= p_s' N_s^+ E_s\left[h_{s+1} Q_{s+1}(F_T)\phi_{s+1}\right] \\ &= \sum_{u=t+1}^{s} f_u\sqrt{h_u} p_s' N_s^+ E_s\left[h_{s+1} w_{s+1}^u \phi_{s+1}\right] + p_s' N_s^+ E_s\left[h_{s+1}(\bar{f}_{s+1} + f_{s+1})\phi_{s+1}\right] \\ &= \sum_{u=t+1}^{s} f_u\sqrt{h_u} p_s' N_s^+ E_s\left[h_{s+1}\phi_{s+1}\phi_{s+1}'\right] N_s^+ p_s h_s w_s^u + \bar{f}_s \end{aligned}$$

$$
\begin{aligned}
&= \sum_{u=t+1}^{s} f_u \sqrt{h_u} p_s' N_s^+ p_s h_s w_s^u + \bar{f}_s \\
&= \sum_{u=t+1}^{s} f_u \sqrt{h_u} w_s^u + \bar{f}_s,
\end{aligned}
$$

and this proves the desired backward induction.

For the hedging quality, $\bar{g}_T = G_T(F_T) = 0$ at time T and we assume that $\bar{g}_{s+1} = G_{s+1}(F_T)$ for $t \leq s \leq (T-1)$. We know from Proposition 2 that

$$
\begin{aligned}
G_s(F_T) &= E_s\left[G_{s+1}(F_T)\right] + E_s\left[h_{s+1}F_{s+1}^2\right] - E_s\left[h_{s+1}F_{s+1}\phi_{s+1}'\right] N_s^+ E_s\left[h_{s+1}F_{s+1}\phi_{s+1}\right] \\
&= E_s\left[G_{s+1}(F_T)\right] + E_s\left[h_{s+1}\left(F_{s+1} - \phi_{s+1}' N_s^+ E_s\left[h_{s+1}F_{s+1}\phi_{s+1}\right]\right)^2\right],
\end{aligned}
$$

where $F_{s+1} = Q_{s+1}(F_T)$. According to our previous result,

$$
\begin{aligned}
Q_{s+1}(F_T) &= \sum_{u=t+1}^{s} f_u \sqrt{h_u} w_{s+1}^u + \bar{f}_{s+1} + f_{s+1} \\
&= \sum_{u=t+1}^{s} f_u \sqrt{h_u} h_s w_s^u \phi_{s+1}' N_s^+ p_s + \bar{f}_{s+1} + f_{s+1} \\
&= \phi_{s+1}' Z_s + \bar{f}_{s+1} + f_{s+1},
\end{aligned}
$$

where $Z_s = \sum_{u=t+1}^{s} f_u \sqrt{h_u} h_s w_s^u N_s^+ p_s$ is a vector measurable with respect to $\mathcal{F}_s$. We compute

$$
\begin{aligned}
F_{s+1} - \phi_{s+1}' N_s^+ E_s\left[h_{s+1}F_{s+1}\phi_{s+1}\right] &= \phi_{s+1}' Z_s - \phi_{s+1}' N_s^+ N_s Z_s \\
&+ \bar{f}_{s+1} + f_{s+1} - \phi_{s+1}' N_s^+ E_s\left[h_{s+1}\left(\bar{f}_{s+1} + f_{s+1}\right)\phi_{s+1}\right]
\end{aligned}
$$

and since, according to Equation 6, $N_s N_s^+ \phi_{s+1} = \phi_{s+1}$, we obtain that

$$
\begin{aligned}
G_s(F_T) &= E_s\left[G_{s+1}(F_T)\right] \\
&\quad + E_s\left[h_{s+1}\left(\bar{f}_{s+1} + f_{s+1} - \phi_{s+1}' N_s^+ E_s\left[h_{s+1}\left(\bar{f}_{s+1} + f_{s+1}\right)\phi_{s+1}\right]\right)^2\right] \\
&= E_s\left[\bar{g}_{s+1}\right] + E_s\left[h_{s+1}\left(\bar{f}_{s+1} + f_{s+1}\right)^2\right] \\
&\quad - E_s\left[h_{s+1}\left(\bar{f}_{s+1} + f_{s+1}\right)\phi_{s+1}'\right] N_s^+ E_s\left[h_{s+1}\left(\bar{f}_{s+1} + f_{s+1}\right)\phi_{s+1}\right].
\end{aligned}
$$

This proves that $G_s(F_T) = \bar{g}_s$ and concludes the backward induction proof. □

Appendix to Section 5

Proof of Lemma 6. Let m_T be a pricing kernel in PK_t and let s be a trading period between t and $(T-1)$. For every index i from 1 to n, there exists a self financing portfolio Y^i in $\mathcal{X}_s$ such that $w_s(Y^i) = p_s^i$ and $w_{s+1}(Y^i) = \phi_{s+1}^i$. We create indeed this portfolio by holding one unit of security i at time s, and by investing the value ϕ_{s+1}^i of the portfolio at time $(s+1)$ in the L^2 minimum portfolio X^{s+1} up to horizon T. This strategy is obviously self financing. Since the portfolio X^{s+1} is worth $1/\sqrt{h_{s+1}}$ at time $(s+1)$, the value at time T of the strategy writes $w_T(Y^i) = \phi_{s+1}^i \sqrt{h_{s+1}} w_T^{s+1}$ and satisfies

$$E\left[w_T(Y^i)^2\right] = E\left[h_{s+1}(\phi_{s+1}^i)^2 E_{s+1}\left[(w_T^{s+1})^2\right]\right] = E\left[h_{s+1}(\phi_{s+1}^i)^2\right],$$

which is finite according to Proposition 1. We conclude that Y^i is indeed in $\mathcal{X}_s$. Since m_T is in PK_t, we have

$$\begin{aligned} E_{s+1}[m_T w_T(Y^i)] &= R_{s+1}^f m_{s+1} \phi_{s+1}^i, \\ E_s[m_T w_T(Y^i)] &= R_s^f m_s p_s^i, \end{aligned}$$

and we conclude that

$$E_s\left[R_{s+1}^f m_{s+1} \phi_{s+1}\right] = R_s^f m_s p_s.$$

The reciprocal is easily obtained by backward induction on s, making use of the self financing condition at every trading period. □

Proof of Proposition 4. Statement (i). The payoff w_T^t is an element of $w_T(\mathcal{X}_t)$ and is therefore in $L^2(P)$ so that w_T^t is in $L^2(P)$. Consider a period s between t and T and a dynamic portfolio X in $\mathcal{X}_s$. We know from Statement (iii) of Lemma 2 with $w_t = 1/\sqrt{h_t}$ and $F_T = w_T(X)$ that $E_s\left[w_T^t w_T(X)\right] = h_s w_s^t Q_s(w_T(X))$ and we conclude with Statement (i) of Lemma 2 and Equation 21 that $E_s\left[w_T^t w_T(X)\right] = R_s^f E_s\left[w_T^t\right] w_s(X)$ which proves that w_T^t is a pricing kernel in PK_t.

Statement (ii). If m_T is an element of PK_t^0 then for every portfolio X in $\mathcal{X}_t$

$$E_t\left[m_T w_T(X)\right] = R_t^f m_t w_t(X) = 0$$

since $m_t = 0$ and m_T belongs to $w_T(\mathcal{X}_t)^{\perp_t}$.

Reciprocally, consider a payoff m_T in $w_T(\mathcal{X}_t)^{\perp_t}$, a period s between t and T, and a portfolio X in $\mathcal{X}_s$. For every event A_s in $\mathcal{F}_s$, we create a dynamic portfolio Y starting at time t with zero wealth in the following way. We do not invest until time s. At time s we do nothing until the horizon T in case the event A_s does not occur. If the event A_s occurs at time s, we purchase the portfolio $\sqrt{H_s}X_s$ and we borrow its cost $\sqrt{H_s}w_s(X)$ by selling $R_s^f\sqrt{H_s}w_s(X)$ units of zero coupon bonds. We then follow the self financing strategy of $\sqrt{H_s}X$ until time T when we redeem the bond.

This dynamic portfolio Y is clearly self financing and starts indeed in t with zero wealth. Its final value $w_T(Y)$ is given by $w_T(Y) = 1_{A_s}\sqrt{H_s}\left(w_T(X) - R_s^f w_s(X)\right)$. It is an element of $L^2(P)$ since $\sqrt{H_s}$ is bounded, $w_T(X)$ is in $L^2(P)$, and $\sqrt{H_s}R_s^f w_s(X) = \sqrt{h_s}w_s(X)$ is in $L^2(P)$.

according to Statement (ii) of Proposition 1. We conclude that Y is an element of $\mathcal{X}_t$ and since m_T is in $w_T(\mathcal{X}_t)^{\perp_t}$, we obtain

$$E_t\left[1_{A_s}\sqrt{H_s}m_T\left(w_T(X) - R_s^f w_s(X)\right)\right] = 0.$$

This equality holds for every event A_s in $\mathcal{F}_s$ and therefore

$$E_s\left[\sqrt{H_s}m_T\left(w_T(X) - R_s^f w_s(X)\right)\right] = 0$$

and since $\sqrt{H_s} > 0$ we conclude that $E_s\left[m_T w_T(X)\right] = R_s^f E_s\left[m_T\right] w_s(X)$ and m_T is in PK_t.

The strategy $\mathbf{1}_t$ which consists in holding the zero coupon bond from time t on is an element of $\mathcal{X}_t$ with final payoff 1. Since m_T is conditionally orthogonal to this strategy, we derive that $E_t\left[m_T\right] = 0$ and m_T is indeed an element of PK_t^0.

Statement (iii). First notice that if m_T is in PK_t, m_T belongs to $L^2(P)$ so that m_T is in $L^2(P)$. Equation 18 shows that $Q_s\left(m_T\right) = (1/\sqrt{h_s})E_s\left[w_T^s m_T\right]$. Since m_T is in PK_t and w_T^s is in $w_T(\mathcal{X}_s)$ with $w_s^s = 1/\sqrt{h_s}$, we compute

$$Q_s\left(m_T\right) = \frac{1}{\sqrt{h_s}}R_s^f m_s w_s^s = \frac{1}{h_s}R_s^f m_s$$

and eventually $m_s/\sqrt{H_s} = \sqrt{h_s}Q_s(m_T)$ which is in $L^2(P)$ according to Proposition 2.

Statement (iv). We check first that if ξ_t is an element of $L_t^2(P)$, the product $\xi_t w_T^t$ is in $L^2(P)$ as required. Indeed

$$E\left[\left(\xi_t w_T^t\right)^2\right] = E\left[\left(\xi_t w_T^t\right)^2\right] = E\left[{\xi_t}^2 E_t\left[(w_T^t)^2\right]\right] = E\left[{\xi_t}^2\right] < \infty$$

since $E_t\left[(w_T^t)^2\right] = 1$. It is now clear that $\xi_t w_T^t$ and the sum $m_T^0 + \xi_t w_T^t$ is in PK_t for every variable m_T^0 in PK_t^0.

Reciprocally, if m_T is a pricing kernel in PK_t, we let $\xi_t = m_t/\sqrt{H_t}$. We know from Statement (iii) that ξ_t is in $L^2_t(P)$. As seen above, the product $\xi_t w^t_T$ is therefore in PK_t and so is the difference $m^0_T = m_T - \xi_t w^t_T$. We check that

$$\begin{aligned} E_t\left[m^0_T\right] &= E_t\left[m_T - \xi_t w^t_T\right] = m_t - \xi_t E_t\left[w^t_T\right] \\ &= m_t - \frac{m_t}{\sqrt{H_t}}\sqrt{H_t} = 0, \end{aligned}$$

since $E_t\left[w^t_T\right] = \sqrt{H_t}$ from Equation 22. This proves that m^0_T is a element of PK^0_t and that m_T is in $\mathrm{PK}^0_t + \left(L^2_t(P) \times w^t_T\right)$. Eventually we conclude with Statement (ii) that $\mathrm{PK}_t = w_T(\mathcal{X}_t)^{\perp_t} + \left(L^2_t(P) \times w^t_T\right)$.

Statement (v). We have already seen that $L^2_t(P) \times w^t_T$ is a subset of PK_t. It is also a subset of $w_T(\mathcal{X}_t)$ since for every element ξ_t of $L^2_t(P)$, the product $\xi_t w^t_T$ is in $L^2(P)$ and corresponds to the value at time T of the self financing portfolio $\xi_t X^t$.

Reciprocally, if m_T is an element of $\mathrm{PK}_t \bigcap w_T(\mathcal{X}_t)$, we know that, as an element of PK_t, it writes $m_T = m^0_T + \xi_t w^t_T$, with m^0_T in $w_T(\mathcal{X}_t)^{\perp_t}$ and ξ_t in $L^2_t(P)$. Since both $\xi_t w^t_T$ and m_T are in $w_T(\mathcal{X}_t)$, so is m^0_T and we obtain that $E_t\left[(m^0_T)^2\right] = 0$. We conclude that $m^0_T = 0$ and $m_T = \xi_t w^t_T$ is an element of $L^2_t(P) \times w^t_T$.

Statement (vi). We consider a payoff F_T in $L^2(P)$, an initial wealth w_t at time t in $L^2_t(P, \sqrt{h_t})$ and a period s between t and T. We let $m_T = \left(F_T - w_T(X^{t,w_t,F_T})\right)$, obviously an element of $L^2(P)$. We apply Statement (iv) of Lemma 2 successively with $X = \mathbf{1}_s$, the strategy in $\mathcal{X}_s$ which buys and holds one unit of the zero coupon from time s until maturity T, and with X any dynamic portfolio in $\mathcal{X}_s$. We obtain

$$m_s = E_s\left[m_T\right] = \frac{h_s}{R^f_s}\left(Q_s(F_T) - w_s(X^{t,w_t,F_T}))\right)$$

and $E_s\left[m_T w_T(X)\right] = R^f_s m_s w_s(X)$. This proves that m_T is a pricing kernel in PK_t. If w_t is chosen equal to $Q_t(F_T)$, then

$$m_t = \frac{h_t}{R^f_t}\left(Q_t(F_T) - w_t(X^{t,Q_t(F_T),F_T})\right) = 0$$

and m_T belongs to PK^0_t.

Reciprocally, if m^0_T is a pricing kernel in PK^0_t, then m^0_T is in $L^2(P)$ and the final wealth $w_T(X^{t,Q_t(m^0_T),m^0_T})$ is an element of $w_T(\mathcal{X}_t)$. We have obtained above that the variable $\left(m^0_T - w_T(X^{t,Q_t(m^0_T),m^0_T})\right)$ is an element in PK^0_t and this proves that $w_T(X^{t,Q_t(m^0_T),m^0_T})$ is also in PK_t^0. Since $\mathrm{PK}_t^0 = w_T(\mathcal{X}_t)^{\perp_t}$ we conclude that $w_T(X^{t,Q_t(m_T^0),m_T^0}) = 0$ and if we choose $F_T = m_T^0$, then $m_T^0 = \left(F_T - w_T(X^{t,Q_t(F_T),F_T})\right)$.

We consider now a pricing kernel m_T in PK_t. From Statement (iv) we write $m_T = m_T^0 + \xi_t w_T^t$ with m_T^0 in PK_t^0 and ξ_t in $L_t^2(P)$. We let $F_T = m_T^0$ and we have seen that $m_T^0 = \left(F_T - w_T(X^{t,Q_t(F_T),F_T})\right)$. We compute $-\xi_t w_T^t = -\xi_t w_T(X^{t,1/\sqrt{h_t},0}) = w_T(X^{t,-\xi_t/\sqrt{h_t},0})$ so that, following Equation 12,

$$\begin{aligned} m_T &= F_T - w_T\left(X^{t,Q_t(F_T),F_T} + X^{t,-\xi_t/\sqrt{h_t},0}\right) \\ &= F_T - w_T\left(X^{t,Q_t(F_T)-\xi_t/\sqrt{h_t},F_T}\right). \end{aligned}$$

We conclude that $m_T = \left(F_T - w_T(X^{t,w_t,F_T})\right)$ with $F_T = m_T^0$ and $w_t = Q_t(F_T) - \xi_t/\sqrt{h_t}$.

Statement (vii). We first prove that $\mathrm{PK}_t^0 = w_T(\mathcal{X}_t)^{\perp_t}$ is closed in $L^2(P)$. Consider a sequence $\{m_T^{0,n}\}_{n\geq 0}$ in $w_T(\mathcal{X}_t)^{\perp_t}$ which converges in $L^2(P)$ to m_T^0. We prove that m_T^0 belongs to $w_T(\mathcal{X}_t)^{\perp_t}$. Let X be any dynamic portfolio in $\mathcal{X}_t$. The sequence $\{E_t[m_T^{0,n} w_T(X)]\}_{n\geq 0}$ is null for every n and converges in $L^1(P)$ to $E_t\left[m_T^0 w_T(X)\right]$ since

$$E\left[\left|E_t\left[m_T^{0,n} w_T(X)\right] - E_t\left[m_T^0 w_T(X)\right]\right|\right] \leq \|w_T(X)\|_{L^2(P)} \|\left(m_T^{0,n} - m_T^0\right)\|_{L^2(P)}.$$

Therefore $E_t\left[m_T^0 w_T(X)\right] = 0$ and m_T^0 belongs to $w_T(\mathcal{X}_t)^{\perp_t}$ which is closed in $L^2(P)$.

We consider next a sequence $\{m_T^n\}_{n\geq 0}$ in PK_t which converges in $L^2(P)$ to m_T. From Statement (iv) we find two sequences $\{m_T^{0,n}\}_{n\geq 0}$ and $\{\xi_t^n\}_{n\geq 0}$ respectively in $w_T(\mathcal{X}_t)^{\perp_t}$ and in $L_t^2(P)$ such that $m_T^n = m_T^{0,n} + \xi_t^n w_T^t$ for every $n \geq 0$. The sequence $\{m_T^n\}_{n\geq 0}$ is a Cauchy sequence in $L^2(P)$ and we compute

$$\begin{aligned} E_t\left[(m_T^n - m_T^m)^2\right] &= E_t\left[\left(m_T^{0,n} - m_T^{0,m} + (\xi_t^n - \xi_t^m) w_T^t\right)^2\right] \\ &= E_t\left[\left(m_T^{0,n} - m_T^{0,m}\right)^2\right] + (\xi_t^n - \xi_t^m)^2\, E_t\left[\left(w_T^t\right)^2\right] \\ &\quad +2\,(\xi_t^n - \xi_t^m)\, E_t\left[\left(m_T^{0,n} - m_T^{0,m}\right) w_T^t\right] \\ &= E_t\left[\left(m_T^{0,n} - m_T^{0,m}\right)^2\right] + (\xi_t^n - \xi_t^m)^2, \end{aligned}$$

since $E_t\left[\left(w_T^t\right)^2\right] = 1$ and $E_t\left[\left(m_T^{0,n} - m_T^{0,m}\right) w_T^t\right] = 0$. We obtain

$$E\left[(m_T^n - m_T^m)^2\right] = E\left[\left(m_T^{0,n} - m_T^{0,m}\right)^2\right] + E\left[(\xi_t^n - \xi_t^m)^2\right],$$

which shows that both $\{m_T^{0,n}\}_{n\geq 0}$ and $\{\xi_t^n\}_{n\geq 0}$ are Cauchy sequences which converge respectively to m_T^0 and ξ_t in $L^2(P)$.

On the one hand we know that m_T^0 belongs to $w_T(\mathcal{X}_t)^{\perp_t}$ because this set is closed in $L^2(P)$. On the other hand ξ_t is measurable with respect to $\mathcal{F}_t$ and belongs to $L_t^2(P)$. It is easily checked that the sequence $\{\xi_t^n w_T^t\}_{n\geq 0}$ converges in $L^2(P)$ to $\xi_t w_T^t$ so that $m_T = m_T^0 + \xi_t w_T^t$ and, according to Statement (iv), m_T belongs to PK_t. We conclude that PK_t is closed in $L^2(P)$.

Since $\mathrm{PK}_t^0 = w_T(\mathcal{X}_t)^{\perp_t}$, it is clear that $w_T(\mathcal{X}_t)$ is a subset of $\left(\mathrm{PK}_t^0\right)^{\perp_t}$. Let us prove the reverse inclusion by considering a random variable F_T in $\left(\mathrm{PK}_t^0\right)^{\perp_t}$. We let $F_t = Q_t(F_T)$. The final value $w_T(X^{t,F_t,F_T})$ is in $w_T(\mathcal{X}_t)$ and therefore $\left(F_T - w_T(X^{t,F_t,F_T})\right)$ is in $\left(\mathrm{PK}_t^0\right)^{\perp_t}$. We know from Statement (vi) that $\left(F_T - w_T(X^{t,F_t,F_T})\right)$ is in PK_t^0 and we conclude that $E_t\left[\left(F_T - w_T(X^{t,F_t,F_T})\right)^2\right] = 0$, which proves that $F_T = w_T(X^{t,F_t,F_T})$ and that F_T belongs to $w_T(\mathcal{X}_t)$. □

Proof of Lemma 7. Equation 15 yields

$$\begin{aligned}
G_s\left(m_T, F_T\right) &= E_s\left[\left(m_T - w_T(X^{s,Q_s(m_T),m_T})\right)\left(F_T - w_T(X^{s,Q_s(F_T),F_T})\right)\right] \\
&= E_s\left[m_T F_T\right] - E_s\left[m_T w_T(X^{s,Q_s(F_T),F_T})\right] \\
&\quad - E_s\left[\left(F_T - w_T(X^{s,Q_s(F_T),F_T})\right) w_T(X^{s,Q_s(m_T),m_T})\right].
\end{aligned}$$

The last term vanishes since $\left(F_T - w_T(X^{s,Q_s(F_T),F_T})\right)$ belongs to PK_s^0 as seen in Statement (vi) of Proposition 4 and $w_T(X^{s,Q_s(m_T),m_T})$ is a payoff in $w_T(\mathcal{X}_s)$. Eventually we obtain

$$G_s(m_T, F_T) = E_s\left[m_T F_T\right] - R_s^f m_s w_s(X^{s,Q_s(F_T),F_T}) = E_s\left[m_T F_T\right] - R_s^f m_s Q_s(F_T)$$

which is Equation 25. □

Proof of Proposition 5. If we set $F_T = m_T$ in Equation 25, we obtain Equation 26 since $\sqrt{h_s} Q_s(m_T) = m_s/\sqrt{H_s}$, according to Statement (iii) of Proposition 4. Equation 27 results then from Equation 26 and the definition of SR_s, we have

$$\begin{aligned}
\mathrm{Var}_s\left[m_T\right] &= E_s\left[m_T^2\right] - \left(E_s[m_T]\right)^2 \\
&= G_s\left[m_T\right] + m_s^2\left(\frac{1}{H_s} - 1\right) \\
&= G_s\left(m_T\right) + m_s^2(\mathrm{SR}_s)^2.
\end{aligned}$$

The two inequalities are a direct consequence of Equations 26 and 27 and the fact that $G_s(m_T)$ is nonnegative. If $m_T = \xi_t w_T^t$ with ξ_t in $L_t^2(P)$, then $m_T = w_T(\xi_t X^t)$ and m_T belongs to $w_T(\mathcal{X}_t)$, and therefore also to $w_T(\mathcal{X}_s)$. According to Statement (ii) of Lemma 2, $G_s(m_T) = 0$ and both inequalities are equalities.

For $s = t$, equality obtains in both cases if and only if $G_t(m_T) = 0$. Statement (ii) of Lemma 2 proves that this happens if and only if m_T belongs to $w_T(\mathcal{X}_t)$, or, according to Statement (v) of Proposition 4, if and only if m_T belongs to the set $L_t^2(P) \times w_T^t$. □

Appendix to Section 6

Proof of Lemma 8. We show that the law of one price holds from t to T. We consider a period s between t and $(T-1)$ and a vector (u, v) in $I\!R^n \times I\!R^{n_x}$ measurable with respect to $\mathcal{F}_s$ such that $\phi_{s+1}' u + (\phi_{s+1}^x)' v = 0$. It results from Lemma 6 and from Equation 32 that

$$\begin{aligned} E_s\left[R_{s+1}^f m_{s+1} \phi_{s+1}\right] &= R_s^f m_s p_s, \\ E_s\left[R_{s+1}^f m_{s+1} \phi_{s+1}^x\right] &= R_s^f m_s p_s^x. \end{aligned}$$

The equality $\phi_{s+1}' u + (\phi_{s+1}^x)' v = 0$ implies that

$$E_s\left[R_{s+1}^f m_{s+1}\left(\phi_{s+1}' u + (\phi_{s+1}^x)' v\right)\right] = R_s^f m_s\left(p_s' u + (p_s^x)' v\right) = 0$$

and we conclude that $p_s' u + (p_s^x)' v = 0$ since m_s is positive. □

Proof of Lemma 9. We consider a period s between t and $(T-1)$ and a vector (u, v) in $I\!R^n \times I\!R^{n_x}$ measurable with respect to $\mathcal{F}_s$ such that $\phi_{s+1}' u + (\bar{\phi}_{s+1}^x)' v = 0$. We know from Proposition 2 and from Equation 7 that

$$\begin{aligned} E_s\left[h_{s+1} \bar{\phi}_{s+1}^x \phi_{s+1}'\right] N_s^+ p_s &= F_s^x, \\ E_s\left[h_{s+1} \phi_{s+1} \phi_{s+1}'\right] N_s^+ p_s &= p_s. \end{aligned}$$

The equality $\phi_{s+1}' u + (\bar{\phi}_{s+1}^x)' v = 0$ implies that

$$0 = E_s\left[h_{s+1}\left(u' \phi_{s+1} + v' \bar{\phi}_{s+1}^x\right) \phi_{s+1}'\right] = u' p_s + v' \bar{p}_s^x$$

and we conclude that $p_s' u + (\bar{p}_s^x)' v = 0$. □

Proof of Proposition 6. We assume that $\mathrm{SR}_t^e = \mathrm{SR}_t$ and we show that $w_T^{t,e} = w_T^t$ by proving that $E_t\left[(w_T^{t,e} - w_T^t)^2\right] = 0$. We know from Equation 17 that $E_t\left[(w_T^t)^2\right] =$

$E_t\left[(w_T^{t,e})^2\right] = 1$. According to Statement (i) of Proposition 4, $w_T^{t,e}$ is in PK_t^e, and therefore also in PK_t and Equation 24 yields $E_t\left[w_T^{t,e}w_T^t\right] = R_t^f E_t\left[w_T^{t,e}\right] w_t^t$. It results from Equation 22 that $E_t\left[w_T^{t,e}\right] = \sqrt{H_t^e}$ and since $R_t^f w_t^t = R_t^f/\sqrt{h_t} = 1/\sqrt{H_t}$, we derive that $E_t\left[w_T^{t,e}w_T^t\right] = \sqrt{H_t^e}/\sqrt{H_t}$. Now $\mathrm{SR}_t^e = \mathrm{SR}_t$ implies that $H_t^e = H_t$ and $E_t\left[w_T^{t,e}w_T^t\right] = 1$ and we conclude that

$$E_t\left[(w_T^{t,e} - w_T^t)^2\right] = E_t\left[(w_T^{t,e})^2\right] + E_t\left[(w_T^t)^2\right] - 2E_t\left[w_T^{t,e}w_T^t\right] = 0.$$

If $w_T^{t,e} = w_T^t$, then Statement (ii) holds since, according to Statement (i) of Proposition 4, $w_T^{t,e}$ is an element of PK_t^e.

If $w_T^t \in \mathrm{PK}_t^e$, then since w_T^t is a self financing strategy in $w_T(\mathcal{X}_t)$, it is also an element of $w_T(\mathcal{X}_t^e)$. According to Statement (v) of Proposition 4, there exists ξ_t in $L_t^2(P)$ such that $w_T^t = \xi_t w_T^{t,e}$. We know however that $E_t\left[(w_T^t)^2\right] = E_t\left[(w_T^{t,e})^2\right] = 1$, and we conclude that $\xi_t = 1$ and that $w_T^{t,e} = w_T^t$. Since, according to Equation 22, $E_t\left[w_T^t\right] = \sqrt{H_t}$ and $E_t\left[w_T^{t,e}\right] = \sqrt{H_t^e}$, we conclude that $H_t^e = H_t$ and $\mathrm{SR}_t^e = \mathrm{SR}_t$.

If w_T^t is a pricing kernel in PK_t^e, then we learn from Equation 35 that $E_t\left[w_T^t F_T^x\right] = R_t^f E_t\left[w_T^t\right] p_t^x = \sqrt{h_t} p_t^x$. According to Equation 18, this implies that for every security j we have $p_t^{x,j} = Q_t(F_T^{x,j})$ and since $Q_t(F_T^{x,j}) = \bar{p}_t^{x,j}$, we conclude that $p_t^x = \bar{p}_t^x$. □

Proof of Proposition 7. We assume that $\mathrm{SR}_t^e = \mathrm{SR}_t$ and we consider a period s such that w_s^t does not vanish. The payoff w_T^t is clearly both in PK_s^e and in $w_T(\mathcal{X}_s^e)$. According to Statement (v) of Proposition 4, there exists ξ_s in $L_s^2(P)$ such that $w_T^t = \xi_s w_T^{s,e}$. From the construction of X^t and X^s and the fact that w_s^t does not vanish, we derive that

$$w_T^s = \frac{w_s^s}{w_s^t} w_T^t = \frac{1}{\sqrt{h_s} w_s^t} w_T^t = \frac{\xi_s}{\sqrt{h_s} w_s^t} w_T^{s,e}.$$

Since, according to Equation 17, $E_s\left[(w_T^s)^2\right] = E_s\left[(w_T^{s,e})^2\right] = 1$, we obtain that $w_T^s = w_T^{s,e}$ and we conclude with Proposition 6 that $\mathrm{SR}_s^e = \mathrm{SR}_s$. □

Proof of Proposition 8. If w_T^t is positive, then w_s^t is positive for every period s between t and $(T-1)$ since $E_s\left[w_T^{s,e}\right] = h_s w_s^t / R_s^f$. If $\mathrm{SR}_t^e = \mathrm{SR}_t$, then we conclude from Proposition 7 that $\mathrm{SR}_s^e = \mathrm{SR}_s$, and from Proposition 6 that $p_s^x = \bar{p}_s^x$.

Reciprocally, if $p_s^x = \bar{p}_s^x$ at every period s between t and $(T-1)$, then $\phi_{s+1}^x = p_{s+1}^x + d_{s+1}^x = \bar{\phi}_{s+1}^x$ and $p_s^{x,j} = p_s' N_s^+ E_s\left[h_{s+1}\phi_{s+1}^{x,j}\phi_{s+1}\right]$ for every security j. Since $R_s^f E_s[w_T^t] = h_s w_s^t$ and

$w_{s+1}^t = w_s^t h_s \phi_{s+1}' N_s^+ p_s$ we have

$$R_s^f E_s[w_T^t] p_s^{x,j} = h_s w_s^t p_s^{x,j} = h_s w_s^t p_s' N_s^+ E_s\left[h_{s+1}\phi_{s+1}^{x,j}\phi_{s+1}\right] = E_s\left[h_{s+1} w_{s+1}^t \phi_{s+1}^{x,j}\right].$$

We obtain that for every period s between t and $(T-1)$

$$E_s\left[R_{s+1}^f E_{s+1}[w_T^t]\phi_{s+1}^x\right] = R_s^f E_s[w_T^t] p_s^x.$$

According to Lemma 10, this proves that w_T^t belongs to PK_t^e and we conclude with Proposition 6 that $\mathrm{SR}_t^e = \mathrm{SR}_t$. □

Proof of Proposition 9. The fact that Statement (i) implies Statement (ii) results directly from Proposition 6. Let us assume that $p_s^x = \bar{p}_s^x$ and that $\phi_{s+1}^x = \bar{\phi}_{s+1}^x$ for every period s between t and $(T-1)$. We check by backward induction that $h_s^e = h_s$ for every period s between t and $(T-1)$. Let us assume therefore that $h_{s+1}^e = h_{s+1}$. We seek to prove that $h_s^e = h_s$.

We know from Equation 5 of Proposition 1 that $1/h_s^e = (p_s^e)' E_s\left[h_{s+1}\phi_{s+1}^e(\phi_{s+1}^e)'\right]^+ p_s^e$. We consider the following matrix A_s, measurable with respect to $\mathcal{F}_s$,

$$A_s = \begin{pmatrix} Id & 0 \\ B_s & Id \end{pmatrix},$$

with $B_s = -E_s\left[h_{s+1}\bar{\phi}_{s+1}^x \phi_{s+1}'\right] N_s^+$. Since A_s is invertible, we remark that

$$(A_s')^{-1} E_s\left[h_{s+1}\phi_{s+1}^e(\phi_{s+1}^e)'\right]^+ A_s^{-1} = \left(A_s E_s\left[h_{s+1}\phi_{s+1}^e(\phi_{s+1}^e)'\right] A_s'\right)^+$$

and we have

$$\begin{aligned}
\frac{1}{h_s^e} &= (p_s^e)' E_s\left[h_{s+1}\phi_{s+1}^e(\phi_{s+1}^e)'\right]^+ p_s^e \\
&= (p_s^e)' A_s' (A_s')^{-1} E_s\left[h_{s+1}\phi_{s+1}^e(\phi_{s+1}^e)'\right]^+ A_s^{-1} A_s p_s^e \\
&= (A_s p_s^e)' \left(A_s E_s\left[h_{s+1}\phi_{s+1}^e(\phi_{s+1}^e)'\right] A_s'\right)^+ A_s p_s^e \\
&= (A_s p_s^e)' \left(E_s\left[h_{s+1}(A_s\phi_{s+1}^e)(A_s\phi_{s+1}^e)'\right]\right)^+ A_s p_s^e.
\end{aligned}$$

We know from Lemma 9 that $\bar{p}_s^x = E_s\left[h_{s+1}\bar{\phi}_{s+1}^x \phi_{s+1}'\right] N_s^+ p_s = -B_s p_s$ and therefore

$$A_s p_s^e = \begin{pmatrix} p_s \\ \bar{p}_s^x + B_s p_s \end{pmatrix} = \begin{pmatrix} p_s \\ 0 \end{pmatrix}$$

and

$$A_s \phi^e_{s+1} = \left(\begin{array}{c} \phi_{s+1} \\ \bar{\phi}^x_{s+1} + B_s \phi_{s+1} \end{array} \right).$$

The block diagonal terms of the matrix $E_s\left[h_{s+1}(A_s\phi^e_{s+1})(A_s\phi^e_{s+1})'\right]$ vanish since

$$E_s\left[h_{s+1}\phi_{s+1}(\bar{\phi}^x_{s+1} + B_s\phi_{s+1})'\right] = E_s\left[h_{s+1}\phi_{s+1}(\bar{\phi}^x_{s+1})'\right] - E_s\left[h_{s+1}N_s N_s^+ \phi_{s+1}(\bar{\phi}^x_{s+1})'\right]$$

and $N_s N_s^+ \phi_{s+1} = \phi_{s+1}$, as seen in Equation 6. As a result, we have

$$\left(E_s\left[h_{s+1}(A_s\phi^e_{s+1})(A_s\phi^e_{s+1})'\right]\right)^+ = \left(\begin{array}{cc} N_s^+ & 0 \\ 0 & C_s \end{array} \right),$$

for some matrix C_s, and

$$\frac{1}{h^e_s} = (A_s p^e_s)' \left(E_s\left[h_{s+1}(A_s\phi^e_{s+1})(A_s\phi^e_{s+1})'\right]\right)^+ A_s p^e_s = p_s N_s^+ p_s.$$

Since $1/h_s = p_s N_s^+ p_s$, we obtain that $h^e_s = h_s$, which concludes the backward induction proof. □

Proof of Proposition 10. We us assume that $p^x_s = \bar{p}^x_s$ for every period s between t and $(T-1)$ and we check by backward induction that for every random variable F_T in $L^2(P)$ we have $Q^e_s(F_T) = Q_s(F_T)$ for every period s between t and T. At time T we easily have $Q^e_T(F_T) = Q_T(F_T) = F_T$. Let us assume that $Q^e_{s+1}(F_T) = Q_{s+1}(F_T)$ and let F_{s+1} be this common value. We know from Proposition 2 that

$$\begin{aligned} Q_s(F_T) &= p'_s N_s^+ E_s\left[h_{s+1}F_{s+1}\phi_{s+1}\right], \\ Q^e_s(F_T) &= (p^e_s)' E_s\left[h_{s+1}\phi^e_{s+1}(\phi^e_{s+1})'\right]^+ E_s\left[h_{s+1}F_{s+1}\phi^e_{s+1}\right], \end{aligned}$$

since, according to Proposition 9, $h^e_{s+1} = h_{s+1}$. Drawing from the same analysis and from the same notations as in the proof of Proposition 9, we write

$$\begin{aligned} Q^e_s(F_T) &= (A_s p^e_s)' E_s\left[h_{s+1}(A_s\phi^e_{s+1})(A_s\phi^e_{s+1})'\right]^+ E_s\left[h_{s+1}F_{s+1}(A_s\phi^e_{s+1})\right] \\ &= \left(\begin{array}{cc} p'_s & 0 \end{array} \right) \left(\begin{array}{cc} N_s^+ & 0 \\ 0 & C_s \end{array} \right) \left(\begin{array}{c} E_s\left[h_{s+1}F_{s+1}\phi_{s+1}\right] \\ D_s \end{array} \right), \end{aligned}$$

for some random vector D_s in $\mathbb{R}^{n_x}$. We conclude that $Q^e_s(F_T) = p'_s N_s^+ E_s\left[h_{s+1}F_{s+1}\phi_{s+1}\right] = Q_s(F_T)$ which concludes the proof by backward induction. □

Appendix to Section 7

Proof of Lemma 11. We shall use the fact that if Σ is a σ-algebra on the probability space (Ω, P) and if X is a random variable in $L^2(P)$ and Y is a random vector in $L^2(P; \mathbb{R}^n)$, then if we let $N = E[YY'|\Sigma]$ and $Z = E[XY'|\Sigma]\, N^+ Y$, then Z is a random variable in $L^2(P)$ and $E[Z^2|\Sigma] \leq E[X^2|\Sigma]$. This fact results from the following two equations:

$$E\left[X^2|\Sigma\right] - E[XY'|\Sigma]\, N^+ E[YX|\Sigma] = E\left[(X-Z)^2|\Sigma\right] \geq 0,$$

and

$$E\left[Z^2|\Sigma\right] = E[XY'|\Sigma]\, N^+ E[YX|\Sigma] \leq E\left[X^2|\Sigma\right].$$

We apply this result with $\Sigma = \mathcal{F}_t$ and

$$\begin{aligned} X &= M_T^{t,a} = F_T^a - w_T\left(X^{t,Q_t(F_T^a),F_T^a}\right), \\ Y &= M_T^{t,b} = F_T^b - w_T\left(X^{t,Q_t(F_T^b),F_T^b}\right). \end{aligned}$$

Equations 14 and 15 show that

$$\begin{aligned} G_t(F_T^a) &= E_t\left[(M_T^{t,a})^2\right], \\ G_t(F_T^a, F_T^b) &= E_t\left[M_T^{t,a}(M_T^{t,b})'\right], \end{aligned}$$

and we obtain that $Z = G_t(F_T^a, F_T^b) G_t(F_T^b)^+ M_T^b$ is an element of $L^2(P)$ and that

$$G_t(F_T^a, F_T^b) G_t(F_T^b)^+ G_t(F_T^b, F_T^a) \leq G_t(F_T^a),$$

which proves Lemma 11. □

Proof of Lemma 12. From the definition of $\mathrm{PK}_t(S_t^x)$ and Equation 25 we derive respectively

$$\begin{aligned} E_t[m_T F_T^x] &= R_t^f m_t S_t^x, \\ E_t[m_T F_T^x] &= R_t^f m_t F_t^x + G_t(F_T^x, m_T), \end{aligned}$$

hence

$$R_t^f m_t (S_t^x - F_t^x) = G_t(F_T^x, m_T),$$

which is Equation 36.

We now apply Lemma 11 with $F_T^a = (m_T)$ and $F_T^b = F_T^x$ and we obtain

$$(R_t^f m_t)^2 \left(S_t^x - F_t^x\right)' G_t(F_T^x)^+ \left(S_t^x - F_t^x\right) \le G_t\left(m_T\right)$$

which writes $(m_t^2/H_t)K_t^x \le G_t(m_T)$ and yields Inequality 37. Since $G_t(m_T)$ is in $L^1(P)$, this inequality shows that $(m_t^2/H_t)K_t^x$ is also in $L^1(P)$. Statement (iii) of Proposition 4 proves that m_t^2/H_t belongs to $L^1(P)$ and therefore so does $(m_t^2/H_t)(1+K_t^x) = m_t^2/H_t^x$. □

Proof of Lemma 13. We start by showing that $m_T^{t,x}(\bar{m}_t)$ is a random variable in $L^2(P)$. According to Statement (vi) of Proposition 4, every component of $M_T^{t,x}$ is in $\mathrm{PK}_t^0 = w_T(\mathcal{X}_t)^{\perp_t}$. In particular, $E_t\left[w_T^t M_T^{t,x}\right] = 0$ and

$$\begin{aligned} E_t\left[\left(m_T^{t,x}(\bar{m}_t)\right)^2\right] &= E_t\left[\left(\frac{\bar{m}_t}{\sqrt{H_t}}(\Lambda_t^x)' M_T^{t,x}\right)^2\right] + E_t\left[\left(\frac{\bar{m}_t}{\sqrt{H_t}} w_T^t\right)^2\right] \\ &= \frac{\bar{m}_t^2}{H_t}(\Lambda_t^x)' G_t(F_T^x)\Lambda_t^x + \frac{\bar{m}_t^2}{H_t} E_t\left[\left(w_T^t\right)^2\right] \\ &= \frac{\bar{m}_t^2}{H_t}(1+K_t^x) = \frac{\bar{m}_t^2}{H_t^x}. \end{aligned}$$

Since $\bar{m}_t$ is in $L_t^2(P, 1/\sqrt{H_t^x})$, we conclude that $m_T^{t,x}(\bar{m}_t)$ is indeed in $L^2(P)$. Notice also that $E_t\left[m_T^{t,x}(\bar{m}_t)\right] = \bar{m}_t$ since $E_t\left[M_T^{t,x}\right] = 0$ and $E_t\left[w_T^t\right] = \sqrt{H_t}$.

Next we show that $m_T^{t,x}(\bar{m}_t)$ is a pricing kernel in PK_t. We derive from above that the random variable $(\bar{m}_t/\sqrt{H_t})(\Lambda_t^x)' M_T^{t,x}$ is in $L^2(P)$ and in $\mathrm{PK}_t^0 = w_T(\mathcal{X}_t)^{\perp_t}$. Since H_t^x is smaller than H_t, the ratio $(\bar{m}_t/\sqrt{H_t})$ is in $L_t^2(P)$ and we conclude with Statement (iv) of Proposition 4 that $m_T^{t,x}(\bar{m}_t)$ is a pricing kernel in PK_t. □

Proof of Proposition 11. Statement (i). We know from Lemma 13 that $m_T^{t,x}(\bar{m}_t)$ is a pricing kernel in PK_t. We check that Condition 1 implies that $E_t\left[m_T^{t,x}(\bar{m}_t)F_T^x\right] = R_t^f \bar{m}_t S_t^x$. We compute

$$\begin{aligned} E_t\left[m_T^{t,x}(\bar{m}_t) M_T^{t,x}\right] &= \frac{\bar{m}_t}{\sqrt{H_t}} E_t\left[M_T^{t,x}\left(M_T^{t,x}\right)'\right]\Lambda_t^x + \frac{\bar{m}_t}{\sqrt{H_t}} E_t\left[w_T^t M_T^{t,x}\right] \\ &= \frac{\bar{m}_t}{\sqrt{H_t}}\sqrt{h_t} G_t(F_T^x) G_t(F_T^x)^+ \left(S_t^x - F_t^x\right) + \frac{\bar{m}_t}{\sqrt{H_t}}\sqrt{h_t} Q_t(M_T^{t,x}) \\ &= R_t^f \bar{m}_t \left(S_t^x - F_t^x\right). \end{aligned}$$

The last equation results from Condition 1, and the fact that $Q_t(M_T^{t,x}) = 0$. Since $m_T^{t,x}(\bar{m}_t)$ is a pricing kernel in PK_t, we know that

$$E_t\left[m_T^{t,x}(\bar{m}_t) w_T\left(X^{t,F_t^x,F_T^x}\right)\right] = R_t^f \bar{m}_t F_t^x$$

and we obtain that $E_t\left[m_T^{t,x}(\bar{m}_t)F_T^x\right] = R_t^f \bar{m}_t S_t^x$, which proves that $m_T^{t,x}(\bar{m}_t)$ is an element of $\mathrm{PK}_t(S_t^x)$.

Statement (ii). We consider a pricing kernel m_T in $\mathrm{PK}_t(S_t^x)$ such that $m_t = E_t[m_T]$ does not vanish. We know from Equation 36 of Lemma 12 that

$$G_t\left(F_T^x, m_T\right) = R_t^f m_t\left(S_t^x - F_t^x\right).$$

Since the variables m_t and R_t^f do not vanish, Condition 1 holds if we show that

$$G_t(F_T^x)G_t(F_T^x)^+ G_t\left(F_T^x, m_T\right) = G_t\left(F_T^x, m_T\right),$$

or equivalently, if we prove that $G_t\left(F_T^x, m_T\right) = G_t\left(\bar{F}_T^x, m_T\right)$, where we let

$$\bar{F}_T^x = G_t(F_T^x)G_t(F_T^x)^+ F_T^x.$$

We check that $G_t(F_T^x) = G_t(\bar{F}_T^x) = G_t(F_T^x, \bar{F}_T^x)$, and therefore that $G_t(F_T^x - \bar{F}_T^x) = 0$. We derive from Statement (ii) of Lemma 2 that every component of the random vector $(F_T^x - \bar{F}_T^x)$ is a payoff in $w_T(\mathcal{X}_t)$. We conclude with the same statement that $G_t\left(\left(F_T^x - \bar{F}_T^x\right), m_T\right) = 0$, and this proves that Condition 1 is satisfied. □

Proof of Proposition 12. Inequality 38 is Inequality 37 of Lemma 12. Inequalities 39 and 40 result then respectively from Equations 26 and 27 of Proposition 5, together with the fact that $(1 + K_t^x)/H_t = 1/H_t^x$ and the equality

$$\frac{K_t^x}{H_t} = (R_t^f)^2\left(S_t^x - F_t^x\right)' G_t(F_T^x)^+\left(S_t^x - F_t^x\right).$$

It is clear that these inequalities become jointly equalities when $(m_t^2/H_t)K_t^x = G_t(m_T)$. Let us prove that this happens if and only if $m_T = m_T^{t,x}(m_t)$.

We know from Lemma 12 that m_t belongs to $L_t^2(P, 1/\sqrt{H_t^x})$ and from Statement (i) of Proposition 11 that $m_T^{t,x}(m_t)$ is a pricing kernel in $\mathrm{PK}_t(S_t^x)$. We have obtained in Lemma 12 that a pricing kernel m_T in $\mathrm{PK}_t(S_t^x)$ satisfies the equality $G_t(F_T^x, m_T) = R_t^f m_t\left(S_t^x - F_t^x\right)$. This fact, together with some elementary algebra, prove that

$$G_t\left(m_T - \frac{m_t}{\sqrt{H_t}}(\Lambda_t^x)' M_T^{t,x}\right) = G_t\left(m_T\right) - \frac{m_t^2}{H_t}K_t^x.$$

According to Statement (ii) of Lemma 2, the equality $(m_t^2/H_t)K_t^x = G_t(m_T)$ is therefore equivalent to the fact that $(m_T) - (m_t/\sqrt{H_t})(\Lambda_t^x)' M_T^{t,x}$ is a dynamic portfolio in $w_T(\mathcal{X}_t)$.

Since both m_T and $(m_t/\sqrt{H_t})\,(\Lambda_t^x)' M_T^{t,x}$ are in PK_t, the equivalent condition says that the variable $m_T - (m_t/\sqrt{H_t})(\Lambda_t^x)' M_T^{t,x}$ is an element of the set $\mathrm{PK}_t \bigcap w_T(\mathcal{X}_t)$ which, according to Statement (v) of Proposition 4, is also $L_t^2(P) \times w_T^t$. Therefore the equivalent condition writes $m_T = (m_t/\sqrt{H_t})(\Lambda_t^x)' M_T^{t,x} + \xi_t w_T^t$ with ξ_t in $L_t^2(P)$. Since then $E_t[m_T] = \xi_t \sqrt{H_t}$ we conclude that $\xi_t = (m_t/\sqrt{H_t})$ and the equivalent condition writes $m_T = (m_t/\sqrt{H_t})\left((\Lambda_t^x)' M_T^{t,x} + w_T^t\right) = m_T^{t,x}(m_t)$ as desired. □

Proof of Proposition 13. Statement (i). Lemma 8 has established the admissibility of the proposed price dynamics. Lemma 10 shows that $R_t^f m_t p_t^x = E_t[m_T F_T^x]$ and since m_T belongs to $\mathrm{PK}_t(S_t^x)$, we obtain that $R_t^f m_t p_t^x = R_t^f m_t S_t^x$. Since m_T is positive, we conclude that $p_t^x = S_t^x$.

Statement (ii). We first show that for every random vectors X_t in $I\!R^{n_x}$ measurable with respect to $\mathcal{F}_t$, the equality $X_t' M_T^{t,x} = 0$ implies $X_t'(S_t^x - F_t^x) = 0$. We do this by proving by backward induction that the equality

$$X_t' \left(\phi_s^x + \sum_{u=t+1}^{s-1} d_u^x \sqrt{h_u} w_s^u - w_s\left(X^{t,F_t^x,F_T^x}\right) \right) = 0$$

holds for every period s between $(t+1)$ and T, where we set the sum $\sum_{u=t+1}^{s-1} d_u^x \sqrt{h_u} w_s^u$ to zero for $s = (t+1)$. The equality holds for $s = T$ since $M_T^{t,x} = F_T^x - w_T\left(X^{t,F_t^x,F_T^x}\right)$. Let us assume that

$$X_t' \left(\phi_{s+1}^x + \sum_{u=t+1}^{s} d_u^x \sqrt{h_u} w_{s+1}^u - w_{s+1}\left(X^{t,F_t^x,F_T^x}\right) \right) = 0.$$

The self financing condition at time $(s+1)$ of the n_x portfolios described by the matrix X^{t,F_t^x,F_T^x} implies that $w_{s+1}\left(X^{t,F_t^x,F_T^x}\right) = \phi_{s+1}' X_s^{t,F_t^x,F_T^x}$. We also know that $w_{s+1}^u = h_s w_s^u p_s' N_s^+ \phi_{s+1}$ and we obtain

$$X_t' \left(\phi_{s+1}^x + \sum_{u=t+1}^{s} d_u^x \sqrt{h_u} h_s w_s^u p_s' N_s^+ \phi_{s+1} - \phi_{s+1}' X_s^{t,F_t^x,F_T^x} \right) = 0.$$

The law of one price for the extended asset structure implies that

$$X_t' \left(p_s^x + \sum_{u=t+1}^{s} d_u^x \sqrt{h_u} h_s w_s^u p_s' N_s^+ p_s - p_s' X_s^{t,F_t^x,F_T^x} \right) = 0$$

which also writes

$$X_t' \left(p_s^x + \sum_{u=t+1}^{s-1} d_u^x \sqrt{h_u} w_s^u + d_s^x - w_s\left(X^{t,F_t^x,F_T^x}\right) \right) = 0.$$

Finally, since $\phi_s^x = p_s^x + d_s^x$, we obtain

$$X_t' \left(\phi_s^x + \sum_{u=t+1}^{s-1} d_u^x \sqrt{h_u} w_s^u - w_s \left(X^{t,F_t^x,F_T^x} \right) \right) = 0$$

as desired. For $s = (t+1)$ this equation writes $X_t' \left(\phi_{t+1}^x - w_{t+1} \left(X^{t,F_t^x,F_T^x} \right) \right) = 0$. The law of one price from t to $(t+1)$ implies that $X_t' \left(p_t^x - w_t \left(X^{t,F_t^x,F_T^x} \right) \right) = 0$ and since the price dynamics is consistent with smile, $p_t^x = S_t^x$ and we conclude that $X_t'(S_t^x - F_t^x) = 0$ as claimed.

We consider now a random vectors X_t in $I\!R^{n_x}$ measurable with respect to $\mathcal{F}_t$. Since $G_t(F_T^x) = E_t \left[M_T^{t,x} (M_T^{t,x})' \right]$ and $G_t(F_T^x) G_t(F_T^x)^+ G_t(F_T^x) = G_t(F_T^x)$, some simple algebra shows that

$$E_t \left[\left(X_t' M_T^{t,x} - X_t' G_t(F_T^x) G_t(F_T^x)^+ M_T^{t,x} \right)^2 \right] = 0,$$

which proves that $(X_t - G_t(F_T^x)^+ G_t(F_T^x) X_t)' \, M_T^{t,x} = 0$. According to our first result, we obtain that $(X_t - G_t(F_T^x)^+ G_t(F_T^x) X_t)' \, (S_t^x - F_t^x) = 0$, which also writes

$$X_t' \left((S_t^x - F_t^x) - G_t(F_T^x) G_t(F_T^x)^+ \, (S_t^x - F_t^x) \right) = 0.$$

Since this last equation is true for every random vectors X_t in $I\!R^{n_x}$ measurable with respect to $\mathcal{F}_t$, we conclude that $(S_t^x - F_t^x) = G_t(F_T^x) G_t(F_T^x)^+ \, (S_t^x - F_t^x)$ and the smile satisfies Condition 1. □

Proof of Proposition 14. We let $\bar{m}_t = \inf(\sqrt{H_t^x}, \sqrt{H_t^e})$ so that both $(\bar{m}_t / \sqrt{H_t^x})$ and $(\bar{m}_t / \sqrt{H_t^e})$ are in $L_t^2(P)$, and we let $\xi_t = \bar{m}_t / \sqrt{H_t^e}$. Proposition 5 applied to the extended asset structure proves that the pricing kernel $m_T = \xi_t w_T^{t,e}$ is in PK_t^e and satisfies $m_t = \sqrt{H_t^e} \xi_t = \bar{m}_t$ and $E_t[m_T^2] = \xi_t^2$. Since PK_t^e is a subset of $\mathrm{PK}_t(S_t^x)$, the pricing kernel m_T is also in $\mathrm{PK}_t(S_t^x)$. We learn from Equation 39 of Proposition 12 that $m_t^2 / H_t^x \leq E_t[m_T^2]$, and therefore $(\bar{m}_t^2 / H_t^x) \leq \xi_t^2 = (\bar{m}_t^2 / H_t^e)$. Since $\bar{m}_t$ is positive, we conclude that $H_t^e \leq H_t^x \leq H_t$ and

$$\mathrm{SR}_t = \sqrt{\left(\frac{1}{H_t} - 1 \right)} \leq \mathrm{SR}_t^x = \sqrt{\left(\frac{1}{H_t^x} - 1 \right)} \leq \mathrm{SR}_t^e = \sqrt{\left(\frac{1}{H_t^e} - 1 \right)}.$$

We now compute

$$(\mathrm{SR}_t^x)^2 - (\mathrm{SR}_t)^2 \quad = \quad \frac{1}{H_t^x} - \frac{1}{H_t} = \frac{1}{H_t} \left(\frac{H_t}{H_t^x} - 1 \right) = \frac{K_t^x}{H_t}$$

$$= \frac{h_t}{H_t} (S_t^x - F_t^x)' G_t(F_T^x)^+ (S_t^x - F_t^x)$$
$$= (R_t^f)^2 (S_t^x - F_t^x)' G_t(F_T^x)^+ (S_t^x - F_t^x) .$$

which yields the desired result. The last statement results directly from Equation 40 of Proposition 12 and the fact that PK_t^e is a subset of $\mathrm{PK}_t(S_t^x)$. □

Appendix to Section 8

Proof of Proposition 15. We first remark that the equality between SR_t^x and SR_t^e is equivalent to the fact that H_t^e and H_t^x are themselves identical.

$H_t^e = H_t^x$ **implies (i).** We apply Lemma 3 to the extended asset structure, with $F_T = 0$, $w_t = w_t(X^{t,e}) = (1/\sqrt{h_t^e})$, and $Y = Y^e$ in $\mathcal{X}_t^e$. Since

$$E_t\left[(F_T - w_T(Y))^2\right] = E_t\left[(w_T(Y^e))^2\right] = 1 = h_t^e w_t^2$$

and

$$w_t(Y) = w_t(Y^e) = \frac{1}{R_t^f\sqrt{H_t^x}} = \frac{1}{R_t^f\sqrt{H_t^e}} = \frac{1}{\sqrt{h_t^e}} = w_t,$$

the set $A_t(Y^e)$ has probability one and $m_T^{t,x}\left(\sqrt{H_t^x}\right) = w_T(Y^e) = w_T^{t,e}$.

(i) implies (ii). If $w_T(X^{t,e}) = m_T^{t,x}\left(\sqrt{H_t^x}\right) = w_T(Y^e)$ then iterated use of the law of one price and the self financing condition proves by backward induction that $w_s(X^{t,e}) = w_s(Y^e)$ for every period s between t and T and one can choose $\Lambda_t = (\sqrt{H_t^x}/\sqrt{H_t})\Lambda_t^x$. The equality

$$E_t\left[\left(\Lambda_t' M_T^{t,x}\right)^2\right] = \frac{(\Lambda_t^x)' G_t(F_T^x)\Lambda_t^x}{1 + (\Lambda_t^x)' G_t(F_T^x)\Lambda_t^x}$$

proves that $\Lambda_t' M_T^{t,x}$ is in $L^2(P)$.

(i) implies (iii). Statement (i) of Proposition 4 applied to the extended asset structure proves that $w_T^{t,e}$ is an element in PK_t^e. If $m_T^{t,x}\left(\sqrt{H_t^x}\right) = w_T^{t,e}$, then $m_T^{t,x}\left(\sqrt{H_t^x}\right)$ is in PK_t^e.

(iii) implies $H_t^e = H_t^x$**.** On the one hand we have proved in Proposition 14 that $H_t^e \leq H_t^x$. On the other hand, since $m_T^{t,x}\left(\sqrt{H_t^x}\right)$ belongs to PK_t^e, Optimization Program 30 applied to the extended asset structure yields

$$\frac{H_t^x}{H_t^e} \leq E_t\left[\left(m_T^{t,x}\left(\sqrt{H_t^x}\right)\right)^2\right].$$

Since $E_t\left[(m_T^{t,x}(\sqrt{H_t^x}))^2\right] = 1$, we conclude that $H_t^x \leq H_t^e$ so that $H_t^e = H_t^x$ as desired.

(ii) implies $H_t^e = H_t^x$. We assume that the strategy $X^{t,e}$ has a value process which is identical to the one of a self financing strategy in $\mathcal{X}_t^e$ which holds the constant quantities Λ_t of the new securities from time t up to horizon T. Every period s between $(t+1)$ and $(T-1)$, this self financing strategy reinvests the dividend $\Lambda_t' d_s^x$ distributed by the new securities in the original securities. If we define the sequence of cash flows $f = \{f_s\}_{t+1 \leq s \leq T}$ by $f_s = -\Lambda_t' d_s^x$ for s between $(t+1)$ and $(T-1)$ and $f_T = 0$, then there must exist a dynamic portfolios Y starting at time t which only invests in the original securities, which finances f, and such that $w_T(X^{t,e}) = \Lambda_t' \phi_T^x + w_T(Y)$.

Let X be the dynamic portfolio starting at time t such that $\theta_f(X) = Y$. We learn from Lemma 4 that X is a self financing portfolio which only invests in the original securities and that $w_T(Y) = (w_T(X) - F_T)$ with

$$\begin{aligned}
F_T &= \sum_{s=t+1}^{T} f_s \sqrt{h_s} w_T^s \\
&= -\Lambda_t' \left(\sum_{s=t+1}^{T-1} d_s^x \sqrt{h_s} w_T^s \right) \\
&= \Lambda_t' \phi_T^x - \Lambda_t' F_T^x.
\end{aligned}$$

We obtain that $w_T(X^{t,e}) = \Lambda_t' F_T^x + w_T(X)$ and therefore $w_T(X^{t,e}) = \Lambda_t' M_T^{t,x} + w_T(Z)$ where $Z = X + X^{t,F_t^x,F_T^x} \Lambda_t$ is also a self financing portfolio which only invests in the original securities. Since both $w_T(X^{t,e})$ and $\Lambda_t' M_T^{t,x}$ are in $L^2(P)$, so is $w_T(Z)$, and we conclude that Z is a self financing portfolio in $\mathcal{X}_t$.

According to Statement (i) of Proposition 4, $w_T(X^{t,e})$ is a pricing kernel in PK_t^e and therefore also in PK_t. Statement (iv) of the same proposition proves that there exists ξ_t in $L_t^2(P)$ such that $w_T(X^{t,e}) = \Lambda_t' M_T^{t,x} + \xi_t w_T^t$. We know from Equation 22 that $E_t[w_T(X^{t,e})] = \sqrt{H_t^e}$ and therefore we obtain that $\xi_t = (\sqrt{H_t^e}/\sqrt{H_t})$.

Since $w_T^{t,e} = w_T(X^{t,e})$ is in PK_t^e, it is consistent with the smile and we have

$$\begin{aligned}
S_t^x &= \frac{1}{\sqrt{h_t^e}} E_t\left[w_T^{t,e} F_T^x \right] \\
&= \frac{1}{\sqrt{h_t^e}} G_t(F_T^x)\Lambda_t + \frac{1}{\sqrt{h_t^e}} \frac{\sqrt{H_t^e}}{\sqrt{H_t}} E_t\left[w_T^t F_T^x \right] \\
&= \frac{1}{\sqrt{h_t^e}} G_t(F_T^x)\Lambda_t + \frac{\sqrt{h_t}}{\sqrt{h_t^e}} \frac{\sqrt{H_t^e}}{\sqrt{H_t}} F_t^x \\
&= \frac{1}{\sqrt{h_t^e}} G_t(F_T^x)\Lambda_t + F_t^x
\end{aligned}$$

and $(S_t^x - F_t^x) = (1/\sqrt{h_t^e})G_t(F_T^x)\Lambda_t$. Condition 1 implies that we also have $(S_t^x - F_t^x) = (1/\sqrt{h_t})G_t(F_T^x)\Lambda_t^x$ and we obtain that $G_t(F_T^x)\sqrt{H_t^e}\Lambda_t^x = G_t(F_T^x)\sqrt{H_t}\Lambda_t$.

Let $\epsilon_T = (\sqrt{H_t^x}/\sqrt{H_t^e})w_T^{t,e} - m_T^{t,x}\left(\sqrt{H_t^x}\right)$. We have

$$\epsilon_T = \frac{\sqrt{H_t^x}}{\sqrt{H_t^e}\sqrt{H_t}}\left(\sqrt{H_t}\Lambda_t - \sqrt{H_t^e}\Lambda_t^x\right)' M_T^{t,x}$$

and

$$E_t\left[\epsilon_T^2\right] = \frac{H_t^x}{H_t^e H_t}\left(\sqrt{H_t}\Lambda_t - \sqrt{H_t^e}\Lambda_t^x\right)' G_t(F_T^x)\left(\sqrt{H_t}\Lambda_t - \sqrt{H_t^e}\Lambda_t^x\right) = 0.$$

This proves that $\epsilon_T = 0$ and $m_T^{t,x}\left(\sqrt{H_t^x}\right) = (\sqrt{H_t^x}/\sqrt{H_t^e})w_T^{t,e}$. We know from Lemma 13 that $E_t\left[\left(m_T^{t,x}\left(\sqrt{H_t^x}\right)\right)^2\right] = 1$, and since $E_t\left[(w_T^{t,e})^2\right] = 1$, we conclude that $H_t^e = H_t^x$.

(iii) is equivalent to (iv). This equivalence results directly from Lemma 10. □

Proof of Proposition 16. If $H_t^e = H_t^x$, then we know from Statement (iii) of Proposition 15 that $m_T^{t,x}\left(\sqrt{H_t^x}\right)$ is in PK_t^e, and therefore also in PK_s^e. We have already seen that the payoff $m_T^{t,x}\left(\sqrt{H_t^x}\right)$ is in $w_T(\mathcal{X}_t^e)$, it is therefore also in $w_T(\mathcal{X}_s^e)$. We obtain that $m_T^{t,x}\left(\sqrt{H_t^x}\right)$ is in $\mathrm{PK}_t^e \bigcap w_T(\mathcal{X}_t^e)$ and, according to Statement (v) of Proposition 4,

$$m_T^{t,x}\left(\sqrt{H_t^x}\right) = \frac{m_s^{t,x}\left(\sqrt{H_t^x}\right)}{\sqrt{H_s^e}} w_T^{s,e}, \tag{43}$$

where we recall that $m_s^{t,x}\left(\sqrt{H_t^x}\right) \stackrel{\text{def.}}{=} E_s\left[m_T^{t,x}\left(\sqrt{H_t^x}\right)\right]$.

We let $F_T^{s,x} = \sum_{u=s+1}^{T-1}\sqrt{h_u}w_T^u d_u^x + \phi_T^x$. Since $m_T^{t,x}\left(\sqrt{H_t^x}\right)$ is in PK_s^e, we have

$$E_s\left[m_T^{t,x}\left(\sqrt{H_t^x}\right)F_T^{s,x}\right] = R_s^f m_s^{t,x}\left(\sqrt{H_t^x}\right)p_s^x,$$

and since $m_T^{t,x}\left(\sqrt{H_t^x}\right)$ is in PK_t and $Q_s(F_T^{s,x}) = \bar{p}_s^x$, we derive from Equation 25 of Lemma 7 that

$$E_s\left[m_T^{t,x}\left(\sqrt{H_t^x}\right)F_T^x\right] = R_s^f m_s^{t,x}\left(\sqrt{H_t^x}\right)\bar{p}_s^x + G_s\left(F_T^{s,x}, m_T^{t,x}\left(\sqrt{H_t^x}\right)\right).$$

We also compute

$$G_s\left(F_T^{s,x}, m_T^{t,x}\left(\sqrt{H_t^x}\right)\right) = \frac{\sqrt{H_t^x}}{\sqrt{H_t}}G_s\left(F_T^{s,x}, (\Lambda_t^x)'M_T^{t,x} + w_T^t\right) = \frac{\sqrt{H_t^x}}{\sqrt{H_t}}G_s(F_T^x)\Lambda_t^x,$$

since

$$G_s\left(F_T^{s,x}, (\Lambda_t^x)'M_T^{t,x}\right) = G_s\left(F_T^x - \sum_{u=t+1}^{s} d_u^x\sqrt{h_u}w_T^u, (\Lambda_t^x)'M_T^{t,x}\right) = G_s(F_T^x)\Lambda_t^x.$$

Combining these results we obtain

$$R_s^f m_s^{t,x}\left(\sqrt{H_t^x}\right)(p_s^x - \bar{p}_s^x) = \frac{\sqrt{H_t^x}}{\sqrt{H_t}} G_s(F_T^x)\Lambda_t^x,$$

from which we derive

$$\left(R_s^f m_s^{t,x}\left(\sqrt{H_t^x}\right)\right)^2 K_s^x = \frac{h_s H_t^x}{H_t}(\Lambda_t^x)' G_s(F_T^x)\Lambda_t^x.$$

We compute

$$G_s\left(m_T^{t,x}\left(\sqrt{H_t^x}\right)\right) = \frac{H_t^x}{H_t} G_s\left((\Lambda_t^x)' M_T^{t,x} + w_T^t\right) = \frac{H_t^x}{H_t}(\Lambda_t^x)' G_s(F_T^x)\Lambda_t^x,$$

and we obtain that

$$G_s\left(m_T^{t,x}\left(\sqrt{H_t^x}\right)\right) = \frac{(R_s^f)^2\left(m_s^{t,x}\left(\sqrt{H_t^x}\right)\right)^2}{h_s} K_s^x = \frac{\left(m_s^{t,x}\left(\sqrt{H_t^x}\right)\right)^2}{H_s} K_s^x.$$

According to Proposition 12, and since the kernel $m_T^{t,x}\left(\sqrt{H_t^x}\right)$ is consistent with the smile p_s^x at time s, this last equation proves that $m_T^{t,x}\left(\sqrt{H_t^x}\right) = m_T^{s,x}\left(m_s^{t,x}\left(\sqrt{H_t^x}\right)\right)$. Since $m_s^{t,x}\left(\sqrt{H_t^x}\right)$ is P almost surely different from zero, we also have

$$m_T^{s,x}\left(\sqrt{H_s^x}\right) = \frac{\sqrt{H_s^x}}{m_s^{t,x}\left(\sqrt{H_t^x}\right)} m_T^{s,x}\left(m_s^{t,x}\left(\sqrt{H_t^x}\right)\right)$$

and therefore

$$m_T^{s,x}\left(\sqrt{H_s^x}\right) = \frac{\sqrt{H_s^x}}{m_s^{t,x}\left(\sqrt{H_t^x}\right)} m_T^{t,x}\left(\sqrt{H_t^x}\right).$$

We derive from Equation 43 that

$$m_T^{s,x}\left(\sqrt{H_s^x}\right) = \frac{\sqrt{H_s^x}}{\sqrt{H_s^e}} w_T^{s,e}.$$

Since $E_s\left[(w_T^{s,e})^2\right] = 1$ and, according to Lemma 13, $E_s\left[(m_T^{s,x}\left(\sqrt{H_s^x}\right))^2\right] = 1$, we conclude that $H_s^e = H_s^x$. □

Proof of Proposition 17. Let us consider a pricing kernel m_T in $\mathrm{PK}_t(S_t^x)$ such that $m_t > 0$ and a payoff F_T in $L^2(P)$ such that $E_t\left[F_T^2\right] = 1$. Let $w_t = E_t\left[m_T F_T\right]/(R_t^f m_t)$ be the value of the payoff F_T derived from the kernel m_T. According to Equation 25 of Lemma 7, $(w_t - Q_t(F_T)) = G_t(m_T, F_T)/(R_t^f m_t)$, and since $J_t(F_T; m_T) - J_t(F_T; w_T^t) = h_t\left(w_t - Q_t(F_T)\right)^2$, we obtain that

$$J_t(F_T; m_T) - J_t(F_T; w_T^t) = \frac{h_t}{\left(R_t^f m_t\right)^2} G_t(m_T, F_T)^2 = \frac{H_t}{m_t^2} G_t(m_T, F_T)^2.$$

Following Statement (iv) of Proposition 4, we decompose the kernel m_T as $m_T = m_T^0 + (m_t/\sqrt{H_t})w_T^t$, where m_T^0 is a kernel in PK_t^0. According to Statement (ii) of Lemma 2 and Equation 15, we have

$$G_t(m_T, F_T) = G_t(m_T^0, F_T) = E_t\left[m_T^0\left(F_T - w_T(X^{t,Q_t(F_T),F_T})\right)\right] = E_t\left[m_T^0 F_T\right].$$

For m_T fixed, the maximum value of $G_t(m_T, F_T)^2$ over random variables F_T such that $E_t\left[F_T^2\right] = 1$ is $E_t[(m_T^0)^2]$ and we obtain that

$$\operatorname*{esssup}_{\begin{cases} F_T \in L^2(P) \\ E_t\left[F_T^2\right] = 1 \end{cases}} J_t(F_T; m_T) - J_t(F_T; w_T^t) = (H_t/m_t^2)E_t\left[\left(m_T^0\right)^2\right].$$

Since $E_t[m_T^2] = E_t[(m_T^0)^2] + (m_t^2/H_t)$, we have

$$\frac{H_t}{m_t^2}E_t\left[\left(m_T^0\right)^2\right] = \frac{H_t}{m_t^2}E_t[m_T^2] - 1$$

and we conclude with Inequality 39 of Proposition 12 that the program

$$\operatorname*{essinf}_{\begin{cases} m_T \in \mathrm{PK}_t(S_t^x) \\ 0 < E_t\left[m_T\right] \end{cases}} (H_t/m_t^2)E_t\left[\left(m_T\right)^2\right] - 1$$

is solved for the pricing kernel $m_T^{t,x}\left(\sqrt{H_t^x}\right)$ with minimum value $(H_t/H_t^x) - 1 = K_t^x$. □

Proof of Proposition 18. Since the smile satisfies Condition 1, the kernel $m_T^{t,x}\left(\sqrt{H_t^x}\right)$ is in $\mathrm{PK}_t(S_t^x)$ and since it is positive, we learn from Proposition 13 that the proposed price dynamics is consistent with the smile. We obtain From Proposition 15 that $m_T^{t,x}\left(\sqrt{H_t^x}\right)$ is equal to $w_T^{t,e}$, the value at horizon T of the L^2 minimum strategy for the extended asset structure. Equation 41 results then directly from Equation 23.

According to Equation 25 and since $m_t^{t,x}(\sqrt{H_t^x}) = \sqrt{H_t^x}$, we compute

$$Q_t^e(F_T) = \frac{E_t\left[m_T^{t,x}\left(\sqrt{H_t^x}\right)F_T\right]}{R_t^f m_t^{t,x}(\sqrt{H_t^x})} = Q_t(F_T) + \frac{1}{R_t^f\sqrt{H_t^x}}G_t\left(m_T^{t,x}(\sqrt{H_t^x}), F_T\right).$$

We also derive from Statement (ii) of Lemma 2 that

$$G_t\left(m_T^{t,x}(\sqrt{H_t^x}), F_T\right) = \frac{\sqrt{H_t^x}}{\sqrt{H_t}}G_t\left((\Lambda_t^x)' M_T^{t,x} + w_T^t, F_T\right) = \frac{\sqrt{H_t^x}}{\sqrt{H_t}}(\Lambda_t^x)' G_t\left(F_T^x, F_T\right)$$

and we conclude that

$$\begin{aligned} Q_t^e(F_T) &= Q_t(F_T) + \frac{1}{R_t^f \sqrt{H_t^x}} \frac{\sqrt{H_t^x}}{\sqrt{H_t}} (\Lambda_t^x)' G_t\left(F_T^x, F_T\right) \\ &= Q_t(F_T) + \frac{\sqrt{h_t}}{R_t^f \sqrt{H_t}} \left(S_t^x - F_t^x\right)' G_t\left(F_T^x\right)^+ G_t\left(F_T^x, F_T\right) \\ &= Q_t(F_T) + \left(S_t^x - F_t^x\right)' G_t\left(F_T^x\right)^+ G_t\left(F_T^x, F_T\right), \end{aligned}$$

which is Equation 42. □

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