An Ex-Post Efficient Auction*

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(First version: October 1998)

July 1999

Abstract

An analogue of Vickrey's (1961) multi-unit auction is provided when bidders have interdependent values. The analogue is strategically equivalent to a collection of *two-bidder single-unit* second-price auctions and it possesses an ex-post efficient equilibrium. As an application of this result, it is shown that the FCC auction possesses an efficient equilibrium in the case of homogeneous goods. Conditions are provided under which the new auction (and also the FCC auction) revenue-dominates all ex-post equilibria of ex-post efficient individually rational mechanisms.

* We wish to thank Larry Ausubel for very helpful discussions. Both authors gratefully acknowledge financial support from the National Science Foundation (SBR-9709392) and the US-Israel Binational Science Foundation (grant# 95-00023/1).

1. Introduction

Our interest lies in extending Vickrey's (1961) ex-post efficient multi-unit auction to environments with interdependent values.[1] In addition, we investigate the revenue properties of this new auction and provide conidtions under which it is revenue-dominant (in a sense to be explained below). As a by-product of our analysis we establish the ex-post efficiency of the recently designed FCC auction as well as its revenue-dominance under the same conditions and in the same sense.

When bidders have private values and downward-sloping demand, Vickrey's (1961) celebrated auction renders it a dominant strategy for each bidder to report his true value for each unit. Moreover, the outcome of Vickrey's auction is then guaranteed to be ex-post efficient. When bidders' values are interdependent, bidders may have private information that affects others' values. As a result, bidders are not only ignorant of others' values, they are also ignorant of *their own* values. Consequently, the strategy "bid your value" is no longer feasible, let alone dominant. In addition, straightforward examples demonstrate that even in equilibrium Vickrey's auction can fail to yield an ex-post efficient outcome. Thus, the introduction of interdependencies in values calls for a modification of Vickrey's auction if ex-post efficiency is to be achieved.

The interdependent-value auction we present does produce ex-post efficient outcomes and is based upon an important result due to Maskin (1992). He shows that even with interdependent values, a *single-unit* Vickrey auction between two bidders yields an ex-post efficient allocation. To exploit this result in the *multi-unit* setting, our auction, in effect, consists of finitely many single-unit auctions between pairs of bidders contingent upon the incremental units that are at stake for each of them. The main difference in our auction from Vickrey's is that bidders engage in two rounds of bidding and submit *multiple bids for each unit* in the second round. Nonetheless, in the case of private values our auction effectively reduces to Vickrey's (1961) and the equilibrium we display reduces to the dominant bid-your-value strategies.

As in standard auctions, the rules of our auction are independent of the details of the environment. In particular, the rules do not depend in any way on the functional forms of the bidders' utilities or upon the joint distribution of their private information. This is in contrast, for example, to the direct revelation mechanism approach. Moreover, the equilibrium we display requires no more

[1] By interdependent values, we refer to situations in which there may be a common value component to the bidders' valuations of the auctioned goods.

sophistication from the bidders than they would need to play well in a *single-unit* Vickrey auction against a *single* other bidder.

We also establish a sense in which our auction is revenue-dominant. Call a pure strategy Bayesian Nash equilibrium an *ex-post equilibrium* if for every vector of player types the joint action specified by the Bayesian equilibrium strategy is a Nash equilibrium even after the types are revealed.[2] Under certain conditions, any ex-post equilibrium of an ex-post efficient and ex-post individually rational selling mechanism yields weakly less revenue *ex-post* than the ex-post equilibrium of the ex-post efficient and ex-post individually rational auction we present. This revenue result is closely related to a similar result due to Krishna and Perry (1998).

Finally, we consider the FCC's spectral auction. While the auction employed by the FCC is considered to have performed well, both in terms of revenue and efficiency, this view has largely been based upon intuition gleaned from single-unit auctions. The theoretical properties of the significantly more complex FCC multi-unit auction are simply not known. However, we show that the ex-post efficient equilibrium of our generalized Vickrey auction forms the basis of an ex-post efficient equilibrium of the FCC auction and that, as a by-product, *the FCC auction is ex-post efficient and revenue-dominant* under the same conditions and in the same sense as above. We stress that one of these conidtions is that the units for sale are homogeneous and noncomplementary.

1.1. Other Related Work

Dasgupta and Maskin (1999) provide a selling mechanism that achieves ex-post efficiency when there are a finite number of possibly *heterogeneous* goods. But even when the goods are homogeneous and demand is downward-sloping Dasgupta and Maskin's mechanism remains significantly more complex than the auction we offer here. Their selling mechansim requires the agents to submit valuation correspondences (in effect, a continuum of potential preference profiles). The designer is then required to calculate the entire set of fixed points of their product. When there are multiple fixed points, a possibility even in their equilibrium, the agents must simultaneously announce the "correct" one to the auctioneer.[3] To

[2] An ex-post equilibrium has the desirable practical feature that no bidder, subsequent to the auction, regrets his equilibrium bids. This remains so even after finding out the winning bids of the other bidders. (Such information is not always public during an auction, while it is typically made so afterward.)

[3] Even though, as in any other coordination game, simultaneously announcing any other fixed point is also an equilibrium in the subgame.

enusre that the agents can indeed identify the "correct" fixed point, Dasgupta and Maskin must impose additional restrictions on the agents' valuation functions. No such restrictions are required when our auction is employed.

Perry and Reny's (1997, 1999) work on the linkage principle in multi-unit auctions provides an equilibrium analysis of the two-bidder two-unit Vickrey auction with interdependent values. Their anlysis is drawn upon here to demonstrate that even though Vickrey's (1961) auction was designed with private values in mind, it continues to yield an ex-post efficient outcome when there are two bidders and any number of units even when the bidders' values are interdependent.[4]

Ausubel (1997) constructs an ascending auction for the many-bidder multi-unit setting. While Ausubel's ascending auction possesses an ex-post efficient equilibrium in the case of *private values*, when values are interdependent ex-post efficiency cannnot be guaranteed unless the the bidders are ex-ante symmetric, their signals are affiliated and they have flat demand schedules up to a fixed capacity. Our auction requires none of these restrictions. Ausubel (1997) also provides a direct revelation mechanism implementing an ex-post efficient allocation in the many-bidder multi-unit setting with downward-sloping demand.

Jehiel and Moldovanu (1998) study ex-post efficient implementation in economic environments in which the agents' signals can be multi-dimensional. They show that when the signals are of dimension two or above, unless the agents' signal marginal rates of substitutions coincide, ex-post efficient implementation is typically not possible. An early insight into the difficulties posed by multi-dimensional signals can be found in Maskin (1992). Jehiel and Moldovanu also provide some positive results when signals are of dimension one.[5]

While our assumption that the bidders' signals are one-dimensional may be reasonable in the present case of homogeneous goods, it is less so otherwise. Consequently, an important open problem is the design of an auction that yields incentive efficient outcomes (i.e. efficient subject to incentive constraints) in the presence of multi-dimensional signals. On the other hand, Dasgupta and Maskin (1999) show that under certain conditions, obtaining *incentive efficiency* with *multi-dimensional* signals can be reduced to the problem of obtaining *ex-*

[4] While the bidders' strategies are no longer dominant, they do constitute an ex-post equilibrium and reduce to the dominant bid-your-value strategies when values happen to be private.

[5] Their positive results are derived under the assumption that utilities are linear in signals and that either there are effectively two alternatives or a continuum. In the latter case additional assumptions are imposed. For example, it is assumed that the alternatives are linearly ordered so that every agent's marginal utility with respect to his own signal is a strictly increasing function of the alternatives.

post efficiency with *one-dimensional* signals. Under such conditions, our auction will yield incentive-efficient outcomes even though the bidders' signals are multi-dimensional.

2. Preliminaries

There are K units of an identical good to be distributed among N bidders. Each bidder i privately receives a one-dimensional signal, $x_i \in [0,1]$. The entire vector of signals determines the bidders' marginal values. So, if the vector of signals is $x = (x_1, ..., x_N)$, then bidder i's marginal value for a k^{th} unit is denoted by $m_{ik}(x)$. Consequently, bidders do not know their own marginal values. We shall maintain the following assumptions throughout. For all $i, j = 1, 2, ..., N,\ k, l = 1, 2, ..., K$ and all $x \in [0,1]^N$,

A.1 $m_{ik}(x) \geq m_{ik+1}(x)$

A.2 $m_{ik}(x)$ is weakly increasing in x.

A.3 For $i \neq j$, if $m_{ik}(x) > m_{jl}(x)$ then the inequality remains strict when x_i rises or x_j falls and all other components of x remain unchanged.

The last of these is a weak single-crossing condition. For example, it is satisfied whenever one's own signal affect one's marginal values at least as much as it affects others' marginal values.[6]

An *ex-post equilibrium* of a Bayesian game is a joint strategy (i.e. for each player, a mapping from his types to actions) with the property that for each vector of types, the joint action specified by the strategies constitutes a Nash equilibrium of the game in which that vector of types is common knowledge. Note that an ex-post equilibrium, while not necessarily dominant, remains a Bayesian-Nash equilibrium for any prior distribution over types. Because of this, and because we shall focus on ex-post equilibria, there will be no need to explicitly specify the prior distribution over the bidders' signals in the sequel.

[6] As Maskin (1992) notes, the single-crossing property is necessary. For example, if there are two bidders and one unit, and 1's valuation is $x_1 + 2x_2$ while 2's is $2x_1 + x_2$, and the signals are i.i.d. then it is impossible to implement an ex-post efficient outcome. The intuition is straightforward. Efficiency requires giving the good to the bidder with the lowest signal. Consequently, when one's signal increases the good becomes more valuable to him yet he is less likely to receive it. This is not incentive compatible.

3. Three Illustrative Cases

Important features of the general case can be obtained by a careful examination of the cases involving two bidders and one unit (2x1), two bidders and two units (2x2), and three bidders and one unit (3x1). We shall maintain assumptions A.1-A.3 and in addition strengthen A.3 for the purposes of illustrating these cases by assuming that for every $x \in [0,1]^N$, every distinct pair of bidders i and j, and every $k, l = 1, ..., K$, there is a unique solution, α, to:

$$m_{ik}(\alpha, x_{-i}) = m_{jl}(\alpha, x_{-i}) \tag{3.1}$$

Call this strengthened version of A.3, A.3$^+$

We begin with the case in which there are $N = 2$ bidders and $K = 1$ unit of the good.

3.1. The 2x1 Case (Maskin 1992)

With just one unit and two bidders, we can simplify the notation somewhat. Let x denote bidder 1's signal and y denote 2's. Both are elements of $[0,1]$. Given the vector of signals (x, y), denote bidder i's (marginal) value for the good by $m_i(x, y)$. With this notation, (3.1) says that for every pair of signals $(x, y) \in [0,1]^2$, there are unique numbers, α and β such that

$$\begin{aligned} m_1(x,\alpha) &= m_2(x,\alpha), \text{ and} \\ m_1(\beta,y) &= m_2(\beta,y). \end{aligned} \tag{3.2}$$

Given x, and y, the solutions α and β have the following significance. Ex-post efficiency requires bidder 1 to obtain the good when $m_1(x, y) > m_2(x, y)$. Assumption A.3$^+$ together with (3.2) imply that this will be the case if and only if $x > \beta$ and $y < \alpha$. Consequently, (see Figure 3.1 for α) we have

$$m_1(x,y) > m_2(x,y) \Leftrightarrow m_1(x,\alpha) \geq m_1(x,y) > m_2(x,y) \geq m_2(\beta,y). \tag{3.3}$$

[Figure 3.1 here]

So, if we were to conduct a Vickrey auction and bidder 1 were to bid $b_1(x) = m_1(x, \alpha)$ and bidder 2 were to bid $b_2(y) = m_2(\beta, y)$, the outcome would be efficient. Moreover, these bids would be in equilibrium even if the bidders knew the vector of signals. Indeed, when $m_1(x,y) > m_2(x,y)$, bidder 1 receives the object, worth $m_1(x,y)$ to him ex-post and he must pay $m_2(\beta,y)$, leaving him with positive surplus. Agent 2 receives a payoff of zero ex-post and would have to raise her bid above $m_1(x,\alpha)$ to obtain the object. Because this would require her to pay $m_1(x,\alpha)$, an amount strictly above her ex-post value of the object, $m_2(x,y)$, she can do no better than obtain a zero payoff. Based upon this it is straightforward to demonstrate the following.

Proposition 3.1. *Suppose that for every x and $y \in [0,1]$, bidder one submits the bid $b_1(x) = m_1(x,\alpha)$ and bidder two submits the bid $b_2(y) = m_2(\beta,y)$, where α and β satisfy (3.2).*[7] *Then the outcome of the Vickrey auction is ex-post efficient and the bid functions $(b_1(\cdot), b_2(\cdot))$ constitute an ex-post equilibrium.*

In equilibrium, each bidder bids his value for the good conditional on the other bidder's signal being that which would render him indifferent between winning and losing. When the two bidders are symmetric, this reduces to the equilibrium derived in Milgrom and Weber (1981).

In conclusion then, a standard Vickrey auction serves to yield an ex-post efficient outcome in this case. We now discuss the case of $N = 2$ bidders and $K = 2$ units to show that the same conclusion applies.[8] As we shall see, this is so because with two-bidders Vickrey's multi-unit auction decomposes into a collection of single-unit second-price auctions.

3.2. The 2x2 Case (Perry and Reny 1997)

Returning to our original notation for the marginal values, (3.1) says that for every pair of signals $(x,y) \in [0,1]^2$ there are unique numbers $\alpha, \beta, \gamma, \delta \in [0,1]$ such that

$$\begin{aligned} m_{11}(x,\alpha) &= m_{22}(x,\alpha) \\ m_{22}(\beta,y) &= m_{11}(\beta,y) \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} m_{12}(x,\gamma) &= m_{21}(x,\gamma) \\ m_{21}(\delta,y) &= m_{12}(\delta,y) \end{aligned} \tag{3.5}$$

[7] Hence, α and β are functions of x and y, respectively.

[8] A similar analysis of this 2x2 case is contained in Perry and Reny (1997), while a worked example can be found in Perry and Reny (1999).

Before proceeding, recall that in the present quasi-linear utility context, efficiency amounts to total surplus maximization. In the one-unit setting, efficiency clearly requires giving the unit to the bidder who values it most. Put differently, it is efficient to assign the unit to bidder 1 when the opportunity cost of doing so, namely bidder 2's value, is no more than bidder 1's value. This *opportunity cost* perspective (in terms of foregone total surplus) is most helpful in understanding the 2x2 case.

When two units are available, the opportunity cost of assigning bidder 1 at least *one unit* is bidder 2's marginal value for a *second* unit. Conversely, the opportunity cost of assigning bidder 2 a *second unit* is bidder 1's marginal value for a first unit. Hence, efficiency requires bidder 1 to receive at least one unit when $m_{11}(x, y) > m_{22}(x, y)$. So, one way to determine whether bidder 1 ought to receive at least one unit is to consider the fictitious setting in which there is but a single unit available and bidder 1's value for it is $m_{11}(x, y)$ and bidder 2's value for it is $m_{22}(x, y)$.

Similarly, the opportunity cost of assigning bidder 1 a second unit is bidder 2's marginal value for a first unit, and vice versa. Therefore we can determine whether bidder 1 ought to receive both units by considering the fictitious setting in which a single unit is available and bidder 1's value for it is $m_{12}(x, y)$ and bidder 2's is $m_{21}(x, y)$.

It is a remarkable fact that Vickrey's (1961) multi-unit auction essentially reduces, in this 2x2 case, to two single-unit Vickrey auctions, each corresponding to one of the two fictitious settings above. The reason for this is that in a 2x2 Vickrey auction, a bidder must pay the other bidder's lower bid for a first unit and must pay the other bidder's higher bid for a second unit.[9]

Indeed, consider conducting a Vickrey auction. The logic of the two fictitious single-unit settings suggests that bidder 1's bid on a first unit, $b_{11}(x)$, and bidder 2's bid on a second unit, $b_{22}(y)$, are given by the corresponding single-unit bids when the single-unit value functions are $m_{11}(\cdot)$ and $m_{22}(\cdot)$. That is,

$$b_{11}(x) = m_{11}(x, \alpha) \text{ and } b_{22}(y) = m_{22}(\beta, y),$$

where α and β satisfy (3.4). Similarly, bidder 1's bid on a second unit, $b_{12}(x)$, and bidder 2's bid on a first unit, $b_{21}(y)$, are given by

$$b_{12}(x) = m_{12}(x, \gamma) \text{ and } b_{21}(y) = m_{21}(\delta, y)$$

[9] In this 2x2 setting Vickrey's (1961) auction is as follows. Each bidder submits bids on each unit. The highest two of the four bids are deemed winning. A winning bidder pays the sum total of the other bidder's losing bids.

where γ and δ satisfy (3.5). Figure 3.2 shows bidder 1's bids.

[Figure 3.2 here]

To see that these bid functions constitute an ex-post equilibrium and lead to an ex-post efficient allocation, recall from the 2x1 analysis that

$$\begin{aligned} m_{11}(x,y) &> m_{22}(x,y) \Leftrightarrow b_{11}(x) \geq m_{11}(x,y) > m_{22}(x,y) \geq b_{22}(y), \\ &\text{and} \\ m_{12}(x,y) &> m_{21}(x,y) \Leftrightarrow b_{12}(x) \geq m_{12}(x,y) > m_{21}(x,y) \geq b_{21}(y). \end{aligned}$$

Moreover, note that $b_{11}(x) \geq b_{12}(x)$ and $b_{21}(y) \geq b_{22}(y)$.[10] Consequently, if say, $m_{11}(x,y) > m_{22}(x,y)$ and $m_{12}(x,y) > m_{21}(x,y)$, then $b_{11}(x) > b_{22}(y)$ and $b_{12}(x) > b_{21}(y)$, so that bidder 1 will obtain both units in the auction, which is ex-post efficient. Moreover, because bidder 1 must pay $b_{22}(y)$ for a first unit and $b_{21}(y)$ for a second, he earns positive surplus on both units ex-post and so cannot improve his ex-post payoff. Similarly, bidder 2 cannot increase her ex-post payoff since acquiring a first unit would cost $b_{12}(x)$, which is above her ex-post marginal value for a first unit, and acquiring a second unit would cost an additional $b_{11}(x)$, which is above her ex-post marginal value for a second unit.

Thus, once again, a Vickrey auction suffices to yield an ex-post efficient outcome. Indeed, it is straightforward to extend the above analysis to show that Vickrey's (1961) multi-unit auction is ex-post efficient when there are two bidders and any number of units. Moreover, in the two-bidder setting Vickrey's auction naturally decomposes (as above) into a collection of two-bidder single-unit second-price auctions. This observation will prove useful in the sequel. But important new considerations arise when there are three or more bidders. Indeed, with just three bidders and a single unit Vickrey's auction is no longer guaranteed to yield ex-post efficiency, as the following example due to Dasgupta and Maskin (1999) demonstrates.

3.3. The 3x1 Case

Introduce a third bidder with signal $z \in [0,1]$, and denote bidder i's marginal value for the single unit by $m_i(x,y,z)$. Suppose that

$$m_1(x,y,z) = 3x + y + 2z$$

[10] This is evident from Figure 3.2 for bidder 1's bids and follows in general from A.1 and the fact that $\alpha \geq \gamma$ and $\beta \geq \delta$.

$$\begin{aligned} m_2(x,y,z) &= 2x+3y+z \\ m_3(x,y,z) &= 3z. \end{aligned}$$

Note that when $x = y = 1/2$, it is efficient to give the unit to bidder 1 when $z > 1/2$ and to bidder 2 when $z < 1/2$. But this cannot be achieved by a Vickrey auction since the bids of bidders 1 and 2 are independent of 3's signal.

One way to overcome this difficulty is to give the bidders an opportunity to condition their bids on one another's signals. The auction we shall present does so by introducing two rounds of bidding where second round bids can be conditioned on those made in the first round.[11]

4. The Main Result: An Ex-Post Efficient Auction

As we have seen, Maskin (1992) has shown that Vickrey's second-price auction is efficient in the two-bidder single-unit case, and our result from Section 3.2 shows that Vickrey's multi-unit auction is efficient when there are two bidders and any number of units. While the auction below can handle any number of bidders, in light of the above results it is most useful when there are three or more.

We now describe our auction and display an ex-post equilibrium that is ex-post efficient. An important feature of the auction is that it incorporates two rounds of bidding. However, the only role of the first-round bids is to provide the bidders with information about one anothers' signals. Thus, *for simplicity only*, we shall ask the bidders to report their signals in the first round. Our analysis extends immediately to the case in which the bidders submit bids, not signals, in the first round. (See Remark 1 below.)

After each bidder receives his private signal, the auction proceeds in two rounds. In the first round each bidder i submits a report, r_i, of his signal to the auctioneer. The vector of reported signals is then revealed. In the second round, each bidder i submits a collection of bids $\{b_{ik}^{jl}\}$ where j runs through all other bidders, and l and k run through all units $1, 2, ..., K$, where $l + k \leq K + 1$. The second-round bid b_{ik}^{jl} can be interpreted as the bid that bidder i would submit in a two-bidder, single-unit second-price auction when a k^{th} unit is at stake for him, and an l^{th} unit is at stake for his competitor on the unit, bidder j.

While the first-round reports will affect the second-round bids, only the second-round bids are employed by the auctioneer to determine the allocation and payments. Suppose then that the collection of second-round bids is $\{b_{ik}^{jl}\}_{i,j,k,l}$. The

[11] Section 4.3 below demonstrates how our auction handles the present three-bidder example.

auctioneer allocates the K units one at a time as follows. A bidder i qualifies to receive a k^{th} unit if for all bidders $j \neq i$ who so far have been allocated fewer than $l \in \{1, ..., K\}$ units,

$$b_{ik}^{jl} \geq b_{jl}^{ik}. \tag{4.1}$$

If more than one bidder qualifies to receive a particular unit, it is allocated to one of them at random.[12] If no bidder qualifies, then the unit is allocated to one of the N bidders at random.[13]

After the allocation process is completed, payments are made as follows. If bidder i is allocated zero units he pays nothing, while if he is allocated $k > 0$ units he pays $p_{i1} + ... + p_{ik}$, where p_{ik} denotes the $K - k + 1$st-largest number among the set of second-round bids $\{b_{jl}^{ik}\}_{j \neq i,\, l=1,...,K}$.[14]

Note that this auction can be implemented without knowledge of the players' strategies, signals or marginal value functions.

4.1. The Equilibrium Bids

We now introduce the bids that will be submitted by the bidders in equilibrium.[15] As we have mentioned, bidder i's equilibrium bid, b_{ik}^{jl}, is that which he would submit in a two-bidder, single-unit second-price auction when a k^{th} unit is at stake for him, and an l^{th} unit is at stake for his competitor on the unit, bidder j. These are defined below.

For every $i, j = 1, 2, ..., N$, every $k, l = 1, 2, ..., K$ and every vector of signals $x \in [0, 1]^N$, precisely one of the following must hold, where (x_i, x_{-j}) denotes the vector of signals that results when x_j in x is replaced by x_i, so that x_i appears twice[16]:

(i) $m_{ik}(x_i, x_{-j}) < m_{jl}(x_i, x_{-j})$
(ii) $m_{ik}(x_i, x_{-j}) > m_{jl}(x_i, x_{-j})$
(iii) $m_{ik}(x_i, x_{-j}) = m_{jl}(x_i, x_{-j})$.

[12] Any tie-breaking rule will do.

[13] For example, suppose there are three bidders and one unit. The notation b_{ik}^{jl} can then be simplified to b_i^j. If $b_1^2 > b_2^1$ and $b_2^3 > b_3^2$ and $b_3^1 > b_1^3$, then no bidder would qualify to receive the unit. As we show, this cannot occur in the equilibrium we display.

[14] Repeated bids count. For example, if the set of bids is $\{2, 2, 2, 1, 1, 0\}$, then the second and third highest bids are both 2, while the fifth highest bid is 1.

[15] The strategies below appear more complex than those in the examples of Section 3 above. This is due to our use now of the more permissive A.3 versus our earlier use of A.3$^+$. Nonetheless, the essence of the strategies remains the same and they reduce to those in Section 3 under A.3$^+$.

[16] For example, when there are three bidders, $(x_1, x_{-2}) = (x_1, x_1, x_3)$.

Accordingly, define[17]

$$\hat{b}_{ik}^{jl}(x_{-j}) = \begin{cases} \inf_\alpha m_{ik}(\alpha, x_{-j}), \text{ s.t. } m_{ik}(\alpha, x_{-j}) < m_{jl}(\alpha, x_{-j}), & \text{if (i) holds} \\ \sup_\alpha m_{ik}(\alpha, x_{-j}), \text{ s.t. } m_{ik}(\alpha, x_{-j}) > m_{jl}(\alpha, x_{-j}), & \text{if (ii) holds} \\ m_{ik}(x_i, x_{-j}), & \text{if (iii) holds} \end{cases} \tag{4.2}$$

As in Section 3, $\hat{b}_{ik}^{jl}(x_{-j}) = m_{ik}(\alpha^*, x_{-j})$ is the bid that bidder i would submit in a second-price auction against bidder j for a single unit when bidder i values the unit according to $m_{ik}(\cdot)$, bidder j values the unit according to $m_{jl}(\cdot)$, and x_{-j} is the vector of signals of all bidders but j.

As a matter of notation, let x_{-ij} denote the vector resulting from the removal of components i and j from the vector x.

4.2. Efficiency

Theorem 4.1. *Consider the auction given above. Under A.1-A.3 the following is an ex-post equilibrium. Given the vector of signals, x, bidder i reports x_i in round one. In round two, if the vector of first-round reports is r, then bidder i submits the collection of bids $\{\hat{b}_{ik}^{jl}(x_i, r_{-ij})\}_{j,k,l}$. Moreover, this yields an ex-post efficient allocation.*

Remark 1. The incentive to reveal one's signal in the first round is weak. Indeed, in equilibrium a bidder's payoff is independent of his first-round report *regardless of the others' reports*. In addition, in practice it would be more natural for the bidders to submit bids, not signals, in the first round. We now take care of both concerns. In addition to A.1-A.3, assume that for all i and k, $m_{ik}(0) = 0$, and that $m_{ik}(x_i, 0, ..., 0)$ is strictly increasing in x_i. Let m_{ik}^{-1} denote its inverse. Also assume for each i that $x_i = 0$ occurs with probability zero. Change the first round of the auction so that each bidder i submits K bids, $\beta_{i1}, ..., \beta_{iK}$. If some bidder submits K bids of zero, the auction ends with the units assigned according to Vickrey's (1961) auction. Otherwise, the auctioneer reveals every bidder i's first-unit bid β_{i1}.[18] Second-round bids are submitted as in the original auction with the

[17] Note that it would not be equivalent when say (i) holds, to define $\hat{b}_{ik}^{jl}(x_{-j})$ as $\min_\alpha m_{ik}(\alpha, x_{-j})$, s.t. $m_{ik}(\alpha, x_{-j}) \leq m_{jl}(\alpha, x_{-j})$ since equality can occur at multiple values of α. To render the condition equivalent, one would then have to consider the largest value of α yielding equality.

[18] Revealing the bids on any other unit would work just as well.

proviso that $b_{ik}^{jl} \geq \beta_{ik}$ for all i, k, j, l. An equilibrium of this auction is for bidder i with signal x_i to set each first-round bid $\beta_{ik} = m_{ik}(x_i, 0, ...0)$. In the second round, each bidder i infers bidder j's signal according to $r_j = m_{j1}^{-1}(\beta_{j1})$ given the revealed bids β_{j1} (which needn't be in equilibrium), and bidder i then submits the collection of bids as given in Theorem 4.1. This is clearly an ex-post efficient ex-post equilibrium since we have only replaced the direct revelation of the signals by an equilibrium revelation of them through the first-round bids. However, this equilibrium provides somewhat stonger incentives in the first round. Indeed, were the auction to end in the first round, a possibility in equilibrium (albeit one with probability zero), one might regret bidding below $m_{ik}(x_i, 0, ..., 0)$ on some unit k since one might end up losing the unit when winning it would have provided positive surplus. On the other hand, bidding above $m_{ik}(x_i, 0, ..., 0)$ in the first round runs the risk that one will be constrained to bid higher than is optimal in the second round.

4.3. The Auction in Action: A worked example

When there are three bidders and one unit, Dasgupta and Maskin's (1998) example of Section 3.3 demonstrates that Vickrey's auction can fail to yield an ex-post efficient allocation. It is therefore instructive to see how the auction we have defined above works in the three-bidder one-unit case in general, and in Dasgupta and Maskin's example in particular. Suppose then that the bidders' vector of signals is (x, y, z). The equilibrium is as follows:
Each bidder reports his true signal in the first round. If the reported (perhaps untruthful) signals are (x', y', z'), then in the second round[19]

- 1 submits the pair of bids $\hat{b}_1^2(x, z')$ and $\hat{b}_1^3(x, y')$
- 2 submits the pair of bids $\hat{b}_2^1(y, z')$ and $\hat{b}_2^3(y, x')$
- 3 submits the pair of bids $\hat{b}_3^1(z, y')$ and $\hat{b}_3^2(z, x')$

The auctioneer then allocates the unit to a "winner." For example, bidder 1 is deemed the winner if

$$\hat{b}_1^2(x, z') > \hat{b}_2^1(y, z') \text{ and } \hat{b}_1^3(x, y') > \hat{b}_3^1(z, y').$$

[19]Because there is only one unit, the notation b_{i1}^{j1} is simplified to b_i^j. Thus, b_i^j is interpreted as bidder i's bid against bidder j.

Bidder 1 would then receive the unit and make the payment

$$\max\{\hat{b}_2^1(y,z'),\hat{b}_3^1(z,y')\}.$$

In particular, when the marginal values for the three bidders are those from Dasgupta and Maskin's example in Section 3.3, namely

$$\begin{aligned} m_1(x,y,z) &= 3x+y+2z \\ m_2(x,y,z) &= 2x+3y+z \\ m_3(x,y,z) &= 3z, \end{aligned}$$

the equilibrium second-round bids are:

$$\begin{aligned} \text{Bidder 1 —}\quad \hat{b}_1^2(x;z') &= \frac{7}{2}x+\frac{5}{2}z' \\ \hat{b}_1^3(x;y') &= 9x+3y' \\ \\ \text{Bidder 2 —}\quad \hat{b}_2^1(y;z') &= \max(7y-z',3y+z') \\ \hat{b}_2^3(y;x') &= \frac{9}{2}y+3x' \\ \\ \text{Bidder 3 —}\quad \hat{b}_3^1(z;y') &= b_3^2(z;x')=3z. \end{aligned}$$

Consequently, when $x=y=1/2$ and the equilibrium is followed, bidder 1 wins the auction when z is slightly above $1/2$ and pays slightly more than 3, while bidder 2 wins the auction when z is slightly below $1/2$ and pays slightly less than 3.[20]

We now apply Theorem 4.1 to the FCC auction.

[20] The reader might have noticed that bidder 2's bid against bidder 1, b_2^1, is not monotone in bidder 3's signal. The intuition for this is related to the winner's curse. As 3's signal increases, bidder 1's value increases faster than 2's. Consequently, winning against bidder 1, when 3's signal is high, implies that 1's signal is *lower* than would be implied when winning against bidder 1 when 3's signal is low. Thus, while 2's value rises with 3's signal cetris paribus, 2's value falls as 3's signal rises *conditional on outbidding bidder 1.* So, as 3's signal rises 2's bid against 1 falls. But bidder 2 can be only so pessimisitc about 1's signal. When 3's signal is high enough so that 2 can outbid 1 only when 1's signal is zero, 2's bid will thereafter increase with with 3's signal as the negative inference effect from 1's signal is no longer present. Of course, when 3's signal reaches this point and higher, bidder 1 is sure to outbid bidder 2 in equilibrium.

5. The FCC Auction

In the present section we shall use the results above to establish that in our environment, the recently designed FCC spectral auction posesses an ex-post efficient equilibrium. Formally, the auction we consider is a simplification of the actual FCC auction. In particular, we do not insist on minimum bid increments, nor do we incorpoarate activity rules. While these features of the FCC auction are helpful in practice, they are unnecessary from a theoretical perspective. Nonetheless, the equilibrium we display for the simplified FCC auction below is, for any $\varepsilon > 0$, an ε -equilibrium when either one or both of the above rules are added, so long as the minimum bid increment is small enough relative to ε. While the goods are perfect substitutes, for bidding purposes they are distinguished by name. The names are simply good 1, good 2, ..., good K.

The (simplified) FCC auction is as follows. After each bidder receives his private signal, the auction proceeds in rounds. In every round each bidder may submit a bid on any number of goods. Bidders can submit whatever bids they want, but their bid on a good is ignored if their previous round bid on it was higher. Thus, in effect, bids cannot decrease. However, bidders can withdraw their bids. But if they do so they must pay the difference between their bid and the resulting sale price of the good. All bids are made public after each round. Ties in high bids submitted in different rounds are broken in favor of the most recent high bidder, while ties in high bids made in the same round are broken at random at the end of that round. Consequently after each round there is at most one high bidder on each good. The auction then proceeds to the next round. The auction ends after the first round in which the high bidder and high bid on each good does not change. The high bidders then pay their bids on the units they have won.

Throughout this section we assume that $m_{ik}(x) \geq \underline{m} > 0$ for every i, k and x.

5.1. The Equilibrium

The equilibrium we shall construct for the FCC auction is based upon the equilibrium provided in Section 4.1. In particular, along the equilibrium path, each bidder bids his generalized Vickrey prices. Accordingly, let $p_{ik}(x_{-i})$ denote the $K - k + 1$st-highest number among $\{\hat{b}_{jl}^{ik}(x_{-i})\}$, where each $\hat{b}_{jl}^{ik}(x_{-i})$ is defined by (4.2). So defined, $p_{ik}(x_{-i})$ is the price that bidder i must pay for a kth unit of the good in the equilibrium of Section 4.1, when the others' signals are x_{-i}. We are now ready to describe the FCC auction equilibrium.

In the first round the signals are revealed by submitting bids below $\underline{m}$ in a monotonic way. Specifically, if bidder i's signal is x_i he bids $\underline{m} \cdot x_i$ on some single good.[21] These bids can then be inverted to yield the *round 1 vector of revealed signals* x^1. In any round $r+1 \geq 2$, define the *current assignment* (of goods to bidders) as one that is efficient given the round r vector of revealed signals, x^r (already defined for $r=1$ and defined below for $r>1$).[22] Let $g_{i1} < g_{i2} < ... < g_{in_i}$ denote the goods currently assigned to bidder i. Thus, n_i denotes the number of goods currently assigned to bidder i.

We wish to emphasizie that according to our definition, the goods currently assigned to bidder i *need bear no relation to those goods on which bidder i is the high bidder.*[23] We urge the reader to keep this in mind when digesting the equilibrium strategies below.

The *round r vector of revealed signals,* x^r, is determined inductively as follows: For every bidder i who in the previous round bid *more* on some good than dictated by the equilibrium,[24] replace the ith component of x^{r-1} by i's highest possible signal, 1. This then yields the vector x^r.[25]

The equilibrium is defined inductively. Consider round $r+1 \geq 2$. For every i and every $k = 1, ..., n_i$:

I. If in all previous rounds after round 1 and for each good, no bidder bid more on any good than dictated by the equilibrium (so $x^r = x^1$), then
(i) bidder i bids $p_{ik}(x^r_{-i})$ on good g_{ik}, and
(ii) if $p_{ik}(x^r_{-i}) \leq b^{ik}_{jl}(x^r_{-i})$ for some bidder $j \neq i$ who is currently assigned $l-1$ goods, then j bids $p_{ik}(x^r_{-i})$ on good g_{ik}, unless this is no higher than the previous-round high bid on good g_{ik}.

II. Otherwise, let p^* denote the lowest competitive price (i.e. the $K+1$st-highest marginal value among the bidders) given x^r, and let p_i denote the $K-n_i+1$st-highest marginal value among all bidders but i, given x^r.
(iii) If b was the previous-round high bid on good g_{ik} (set $b = p_i$ if there were no previous-round bids on good g_{ik}), then bidder i bids $\min(b, p^*)$ on good g_{ik}.

[21] Different bidders can choose different goods, but need not.

[22] When there is more than one good and because the goods are perfect substitutes, there are many efficient assignments of goods (now distinguished by *name*) to bidders. The current assignment selects one of these and, being part of the equilibrium, this selection is common knowledge among the bidders.

[23] Although in equilibrium, bidders will ultimately win the goods assigned to them.

[24] This includes a bidder merely *submitting* a bid on a good when according to the equilibrium the bidder is to refrain from bidding on that good.

[25] Of course, if x_i is already equal to 1 no change results.

(iv) If $m_{jl}(x^r) = p_i$ for some bidder $j \neq i$ currently assigned $l - 1$ goods,[26] then bidder j bids p_i on each good assigned to bidder i having a strictly lower previous-round high bid.
(v) If bidder i is a high bidder on a good, say k, not currently assigned to him, and the second-highest bid on it is at least p^*, then bidder i withdraws his bid on good k.

This completes the description of the equilibrium strategies. The idea behind them is as follows. Along the equilibrium path, the signals are revealed in the first round of bidding. This renders the efficient allocations common knowledge among the bidders and in equilibrium they coordinate on a single efficient allocation. In the second round the bidders bid their generalized Vickrey prices on the goods (efficiently) assigned to them, while other bidders match these bids to ensure that these prices are minimal. The ties are broken at random. In the third round only the efficient bidders again submit their previous round bids. So if all ties were previously broken in their favor, no new high bidders result and the auction ends. Otherwise in the next round the efficient bidders once again place the same bids. Because now no new high bidders will result, the auction ends. This results in an ex-post efficient allocation with the winners paying the same generalized Vickrey prices as defined in Section 4. Thus the outcome and revenue in this equilibrium of the FCC auction is the same as that in the two-round auction presented in Section 4.

Off of the equilibrum path the competition becomes more severe (and severe enough to render the equilibrium path optimal for each bidder). The severity of the competition is expressed in two ways. First, a bidder who bids above his equilibrium bid is assumed to have the highest possible signal (and so the highest possible values given the signals of the others).[27] Consequently, not only is this bidder now presumed to have a higher willingness to pay, all other bidders' willingness to pay increases as well. Second, in order to win a good, all bidders must now pay at least its opportunity cost.[28] Because both of these effects increase the price a bidder must pay for a good above its equilibrium price $p_{ik}(x_{-i})$, no bidder can gain by deviating from the equilibrium path. Finally, because the assignment of goods to bidders is always efficient (with respect to the current

[26] By the definition of p_i, some such j and l must exist.

[27] Less severe adjustments in the bidders' beliefs are also consistent with this equilibrium. The present specification merely keeps things simple.

[28] i.e. The highest marginal value for that unit among the other bidders given the current vector of revealed signals.

round's vector of revealed signals) and in each round bidders are expected to bid up to the competitive price on each good assigned to them, it is optimal for those not assigned goods who are high bidders on them above the competitive price to withdraw their bids on those units when the competitive price is reached, but no sooner. We summarize this discussion in the following result. The straightforward proof is left to the reader.

Theorem 5.1. *The FCC auction strategies described above constitute a sequentially rational Nash equilibrium and result in an ex-post efficient allocation.*[29]

Remark 2. Despite the fact that the goods are identical, in equilibrium their sale prices typically will not be. Once bidders have bid their Vickrey prices, potential competitors have no incentive to bid on units whose prices are low. They recognize that the current winner is willing to bid up to the competitive price for each unit. Moreover, because this is common knowledge, *it would be suboptimal for the current winner to fail to do so.* Consequently, the FCC auction need not produce uniform-price auction outcomes in the homogeneous-goods case as is sometimes argued.

Remark 3. The strategies we have provided are more natural when it is efficient for each bidder to receive a significant number of goods. They can be less natural when it is efficient for some bidders to receive no goods. Such bidders might have "nothing to lose" by bidding on low-priced goods up to their value for a first unit. Of course, bidders do not employ such strategies in our equilibrium, and so when these strategies are natural our equilibrium might not be.[30]

Remark 4. One might wonder about the purpose of introducing the auction of Section 4 when the FCC auction is capable of producing ex-post efficient outcomes. First, it should be noted that it was the analysis of the Section 4 auction that led to the discovery of the FCC's efficient equilibrium. Second, and perhaps more to the point, as noted in Remark 3, the efficient equilibrium of the FCC auction needn't always be reasonable. On the other hand, the efficient equilibrium displayed in Section 4.1 is always reasonable. Moreover, even when the efficient equilibrium of Section 4.1 is unique in undominated strategies (e.g. in the private values case), there are inefficient undominated equilibria of the FCC auction. In this sense, the

[29] By sequentially rational, we mean that there exist beliefs that render the strategy-belief pair sequentially rational. Moreover, because the strategies are in Nash equilibrium, the beliefs can be chosen to conform to Bayes' rule along the equilibrium path.

[30] We thank Larry Ausubel for prompting us to include this discussion.

FCC auction might well be less likely to lead to efficiency in practice than the auction provided in Section 4.

6. Revenue

Although our primary concern is efficiency, we now consider the revenue generated by the auction presented in Section 4 above.[31] At various points below, we will need to strengthen assumptions A.2 and A.3 as follows: For every i, k

A.2′ $m_{ik}(x)$ is continuously differentiable and weakly increasing in x.

A.3′ For $i \neq j$, if $m_{ik}(x) \geq m_{jl}(x)$ then the inequality is strict when x_i rises or x_j falls and all other components of x remain unchanged.

A (direct) *selling mechanism* is a pair (q, c), where $q(x) = (q_1(x), ..., q_N(x))$ and $c(x) = (c_1(x), ..., c_N(x))$ for every vector of bidder signals $x \in [0, 1]^N$. Each $q_i(x)$ is a vector of probablities, with $q_{ik}(x)$ denoting the probability that bidder i receives at least k units of the good if the vector of reported signals is x. Each $c_i(x)$ is the payment that bidder i must make given the report x. We shall refer to q as the probability assignment function. A selling mechanism is *ex-post incentive compatible* if truth-telling is an ex-post equilibrium,[32] and it is *ex-post individually rational* if truth-telling implies that every bidder's ex-post utility is non negative.

The result to follow is an ex-post revenue-equivalence theorem. The essentials of the proof follow Myerson (1981) quite directly.[33]

Theorem 6.1. *Suppose that A.1, A.2′ and A.3 hold. Consider any two ex-post incentive-compatible and ex-post individually rational selling mechanisms with the same probability assignment functions. If whenever a bidder receives the lowest possible signal he is, ex-post, indifferent between the two mechanisms,*

[31] Krishna and Perry (1998) provides a general analysis of revenue-maximization subject to efficiency when the bidders' signals are independent. The present analysis follows similar lines. However, we are able to drop the independence assumption because of our focus on ex-post revenue and ex-post equilibria.

[32] As with Bayesian equilibrium, there is a revelation principle for ex–post equilibrium. Consequently, it is without loss that we restrict attention to direct selling mechanisms.

[33] A small technical caveat however is that unlike Myerson's (1981) treatment, the bidders' utilities here are not linear in their signals. Consequently, a bidder's indirect utility need not be a convex function of his signal. It is, however, Lipschitz in his signal and this is enough to push the proof through.

then the two mechanisms yield the seller the same ex-post revenue; while if he strictly prefers one mechanism over the other, the one yields the seller strictly less ex-post revenue than does the other.

Because under A.3′ the criterion of ex-post efficiency almost everywhere determines the probability assignment function, the above ex-post revenue-equivalence theorem immediately yields the following corollary.

Corollary 6.2. *Under A.1, A.2′ and A.3′, the ex-post efficient auction of Section 4 yields maximal ex-post revenue among ex-post incentive-compatible ex-post individually rational and ex-post efficient selling mechanisms so long as each bidder receives zero ex-post utility when his signal is zero.*

This leads to the following constrained optimal revenue result.

Theorem 6.3. *Suppose that A.1, A.2′ and A.3′ hold. Also, suppose that for all distinct pairs of bidders i, j and all numbers of units k, l, $m_{ik}(0, 0, x_{-ij}) = m_{jl}(0, 0, x_{-ij})$ for all x_{-ij}. Then the Section 4.1 equilibrium of the auction of Section 4 raises at least as much ex-post revenue as is raised in any ex-post efficient ex-post individually rational ex-post equilibrium of any other selling mechanism.*

Remark 5. The additional condition on marginal values is satisfied in both of the following particular instances with the second being more general than the first:
(i) There are two bidders and $m_{ik}(0, 0) = 0$ for $i = 1, 2$ and all k.
(ii) There are N bidders and $m_{ik}(0, 0, x_{-ij}) = 0$ for all $i \neq j$, all k and all x_{-ij}.

Remark 6. Because the equilibrium of the FCC auction provided in the previous section yields the same revenue as the auction of Section 4, the FCC auction is revenue dominant in the same sense and under the same conditions as the auction of Section 4.

The following example shows that when the conditions of Theorem 6.3 fail, some bidder might receive positive utiltiy when his signal is zero and so the Section 4.1 equilibrium need not generate maximal ex-post revenue.

Example 6.4. *There are two bidders, 1 and 2, with signals x and y respectively, both in $[0, 1]$, and a single unit. Bidder 1's value is $m_1(x, y) = 2 + 3x + y$ and 2's is $m_2(x, y) = 2x + 2y$. Because there are just two bidders, according to the* *equilibrium of Section 4.1 the bids submitted in the second round are independent of the first-round announcements. Bidder 1 bids* $\hat{b}_1(x) = 3 + 3x$, *and bidder 2 bids* $\hat{b}_2(y) = 2y$. *Consequently, the outcome is ex-post efficient since bidder 1 always wins the auction and the seller's ex-post revenue is* $2y$. *But note that even when* $x = 0$ *bidder 1 wins the unit and pays strictly less than his value for it, thereby earning strictly positive utility. To see that the seller can then earn even more ex-post revenue, note that if bidder 2 instead bids* $\tilde{b}_2(y) = 1+y$, *the strategies remain in ex-post equilibrium and the outcome remains ex-post efficient. Consequently, the seller earns strictly more ex-post revenue whenever* $y < 1$.

The example evidently indicates that the equilibrium strategies provided in Section 4.1 above are not unique. This is related to the well known issue of non uniqueness of equilibrium in Vickrey's auction in the private values case. Of course, in that setting each bidder has a unique undominated strategy and so the multiplicity problem is rendered moot through a simple dominance argument. But when values are interdependent bidders do not know their own values and so dominance arguments lose much of their power. So, in contrast to the private values setting, multiple equilibria in undominated strategies cannot in general be ruled out here. On the other hand, the particular strategies displayed in Section 4.1 do possess some noteworthy features. They are always undominated while strategies that yield the seller more revenue need not be.[34] Consequently, the strategies provided in Section 4.1 reduce to Vickrey's bid-your-value strategies in the private values case while those yielding more revenue need not. But perhaps their most important feature is that they remain ex-post best replies *regardless of which ex-post efficient ex-post equilibrium strategies the other bidders use.* For instance, in the above example bidder 1's strategy, $\hat{b}_1(x)$, remains an ex-post best-reply and the outcome remains ex-post efficient no matter which of the two equilibrium strategies bidder 2 chooses to employ. Consequently, the strategies provided in Section 4.1 are focal in a natural sense.

[34] To see the latter, one need only consider a private value setting with two bidders and one unit. The unit is worth $1 + x$ to bidder 1 when his signal is x, and it is always worth zero to bidder 2. The unique undominated equilibrium has both bidders bidding their value so that the seller earns no revenue. A more favorable equilibrium for the seller has bidder 2 bidding just below unity (a dominated strategy) and bidder 1 again bidding his value.

7. Proofs

The following lemma is central to proving Theorem 4.1. It shows that the bid functions proposed above in (4.2) can be employed to allocate the units efficiently, in effect, by conducting a collection of single-unit second-price auctions between pairs of bidders. The proof is contained in the appendix.

Lemma 7.1. *The functions, $\hat{b}_{ik}^{jl}(\cdot)$, defined in (4.2) satisfy the following condition:*

$$\text{If } \hat{b}_{ik}^{jl}(x_{-j}) \geq \hat{b}_{jl}^{ik}(x_{-i}), \text{ then } \hat{b}_{ik}^{jl}(x_{-j}) \geq m_{ik}(x) \geq m_{jl}(x) \geq \hat{b}_{jl}^{ik}(x_{-i}).$$

Proof of Theorem 4.1. Suppose that the true vector of signals is x, and that each bidder behaves as in the statement of the theorem. We first show that the allocation is ex-post efficient.

Because the auction rules always result in the allocation of all K units, efficiency will obtain so long as when bidder i is allocated a kth unit $m_{ik}(x)$ is greater than or equal to all but perhaps $K-k$ of the others' marginal values. Now, in equilibrium each bidder reports his true signal in the first round. Hence, the equilibrium second-round bids are $\{\hat{b}_{ik}^{jl}(x_{-j})\}_{i,j,k,l}$. Therefore, if bidder i is allocated a k^{th} unit, then according to the auction rules

$$\hat{b}_{ik}^{jl}(x_{-j}) \geq \hat{b}_{jl}^{ik}(x_{-i}) \tag{7.1}$$

for all bidders $j \neq i$ who are allocated fewer than l units. Consequently, (7.1) holds for all but possibly $K-k$ elements of $\{(j,l)\}_{j\neq i, l=1,\ldots,K}$. Lemma 7.1 then implies that

$$\hat{b}_{ik}^{jl}(x_{-j}) \geq m_{ik}(x) \geq m_{jl}(x) \geq \hat{b}_{jl}^{ik}(x_{-i}) \tag{7.2}$$

for all but possibly $K-k$ elements of $\{(j,l)\}_{j\neq i, l=1,\ldots,K}$. But this means that $m_{ik}(x)$ is greater than or equal to all but perhaps $K-k$ marginal values of the other bidders, as desired. So, if the proposed strategies are followed, the allocation will be ex-post efficient. We now demonstrate that the proposed strategies constitute an ex-post equilibrium.

Let $\hat{p}_{ik}$ denote the $K-k+1^{st}$ largest bid in the set $\{\hat{b}_{jl}^{ik}(x_{-i})\}_{j\neq i,\, l=1,\ldots,K}$. If the others employ the proposed strategies, then winning a k^{th} unit requires bidder i to pay an additional $\hat{p}_{ik}$, an amount that is independent of his strategy. We'd like to show that in the proposed equilibrium, $m_{ik}(x) \geq \hat{p}_{ik}$ for every k^{th} unit won by bidder i, and $m_{ik}(x) \leq \hat{p}_{ik}$ for every k^{th} unit not won by bidder i. This being so, bidder i can do no better, ex-post, than to follow his proposed equilibrium strategy. The proposed strategies would then constitute an ex-post equilibrium.

Because the former inequality follows immediately from (7.2), it remains only to show that $m_{ik}(x) \leq \hat{p}_{ik}$ if bidder i does not receive a k^{th} unit. Now, if i does not receive a kth unit, at least $K-k+1$ units are allocated to other bidders, and so according to the auction rules,

$$\hat{b}^{ik}_{jl}(x_{-i}) \geq \hat{b}^{jl}_{ik}(x_{-j}) \tag{7.3}$$

for every bidder $j \neq i$ who receives at least l units. Consequently, (7.3) holds for at least $K-k+1$ elements of $\{(j,l)\}_{j\neq i, l=1,\ldots,K}$. Lemma 7.1 then implies that

$$\hat{b}^{ik}_{jl}(x_{-i}) \geq m_{jl}(x) \geq m_{ik}(x) \geq \hat{b}^{jl}_{ik}(x_{-j})$$

for at least $K-k+1$ elements of $\{(j,l)\}_{j\neq i, l=1,\ldots,K}$. But this means that $m_{ik}(x)$ is less than or equal to at least $K-k+1$ members of $\{\hat{b}^{ik}_{jl}(x_{-i})\}_{j\neq i,\, l=1,\ldots,K}$, i.e. that $m_{ik}(x) \leq \hat{p}_{ik}(x)$. ■

Proof of Theorem 6.1. Fix an ex-post incentive-compatible selling mechanism (q,c). Given a vector of reported signals, x, recall that $q_{ij}(x)$ denotes the probability that bidder i receives at least j units and $c_i(x)$ denotes i's payment.

Define

$$U_i(x) = q_i(x) \cdot m_i(x) - c_i(x), \tag{7.4}$$

where $q_i(x) = (q_{i1}(x), ..., q_{iK}(x))$ and $m_i(x) = (m_{i1}(x), ..., m_{iK}(x))$. So defined, $U_i(x)$ is bidder i's ex-post utility in equilibrium when the vector of signals is x.

Because truth-telling is an ex-post equilibrium, the following ex-post incentive-compatibility constraint must be satisfied for every $x_{-i} \in [0,1]^{N-1}$:

$$U_i(x) \geq q_i(r_i, x_{-i}) \cdot m_i(x) - c_i(r_i, x_{-i}) \tag{7.5}$$

for every $r_i \in [0,1]$. A straightforward adaptation of standard arguments (see e.g. Myerson (1981)), (7.5) and the continuity (by A.2′) of the derivative of each $m_{ik}(x)$ imply that (i) $U_i(x_i, x_{-i})$ is Lipschitz in x_i on $[0,1]$ and (ii) whenever $\partial U_i(x_i, x_{-i})/\partial x_i$ exists,

$$\frac{\partial U_i(x_i, x_{-i})}{\partial x_i} = q_i(x_i, x_{-i}) \cdot D_i m_i(x_i, x_{-i}), \tag{7.6}$$

where $D_i m_i(x_i, x_{-i})$ denotes the K-vector whose jth component is $\partial m_{ij}(x_i, x_{-i})/\partial x_i$.

Being Lipschitz in x_i, $U_i(\cdot, x_{-i})$ can be recovered from its derivative for each fixed x_{-i}. And because Lipschitz functions are differentiable almost everywhere we may use (7.6) to conclude that

$$U_i(x_i, x_{-i}) = U_i(0, x_{-i}) + \int_0^{x_i} q_i(s, x_{-i}) \cdot D_i m_i(s, x_{-i}) ds \tag{7.7}$$

for every $x = (x_i, x_{-i})$. Combining (7.4) and (7.7) the seller's ex-post revenue can be written as

$$\sum_i c_i(x) = \sum_{i,j} q_{ij}(x) m_{ij}(x) - \sum_i \int_0^{x_i} q_i(s, x_{-i}) \cdot D_i m_i(s, x_{-i}) ds - \sum_i U_i(0, x_{-i}). \tag{7.8}$$

Thus the seller's ex-post revenue depends only upon the probability assignment function q and the utility of each bidder when his signal is as low as possible, and it is strictly decreasing in the latter. ■

Proof of Theorem 6.3. If a selling mechanism's truth-telling (without loss by the revelation principle) ex-post equilibrium is ex-post efficient, then for every vector of signals x, its probability assignment function $\{q_{ij}(x)\}_{ij}$ must solve

$$\max \sum_{i,j} q_{ij}(x) m_{ij}(x) \tag{7.9}$$

subject to $\sum_{i,j} q_{ij}(x) \leq K$ and $0 \leq q_{ij}(x) \leq 1$ for all i, j. Moreover for every x_{-i}, A.1 and A.3$'$ imply that $q_{ij}(\cdot, x_{-i})$ is, for all but perhaps one $x_i \in [0, 1]$, uniquely determined, taking on the value 0 or 1. Consequently, the values of the first two terms in (7.8) are unique among ex-post efficient ex-post incentive-compatible selling mechanisms. Because ex-post individual rationality requires $U_i(0, x_{-i}) \geq 0$ for every i and every x_{-i}, it suffices to show that in the Section 4.1 ex-post efficient ex-post equilibrium of the auction of Section 4 each bidder obtains zero ex-post utility whenever his signal is zero (i.e. $U_i(0, x_{-i}) = 0$ for every x_{-i}).

So, suppose that $x_i = 0$ and that the others' signals are x_{-i}. If bidder i wins a kth unit in equilibrium, then according to the proof of Theorem 4.1 he pays no more than his value, $m_{ik}(0, x_{-i})$, for that unit. Moreover, according to the auction rules he must pay $\hat{b}^{ik}_{jl}(x_{-i}) = m_{jl}(\alpha, x_{-i})$ for it for some j and l, and some $\alpha \in [0, 1]$. But because $m_{jl}(0, 0, x_{-ij}) = m_{ik}(0, 0, x_{-ij})$, the single crossing property A.3$'$ implies that $m_{jl}(0, x_{-i}) \geq m_{ik}(0, x_{-i})$ and so $\hat{b}^{ik}_{jl}(x_{-i}) = m_{jl}(\alpha, x_{-i}) \geq m_{jl}(0, x_{-i}) \geq m_{ik}(0, x_{-i})$. But this means that bidder i pays exactly $m_{ik}(0, x_{-i})$ for the unit and hence obtains exactly zero (ex-post) surplus on every unit won. ■

Appendix

PROOF OF LEMMA 7.1. It suffices to show that,
(i) $m_{ik}(x) > m_{jl}(x)$ implies $\hat{b}^{jl}_{ik}(x_{-j}) \geq m_{ik}(x) > m_{jl}(x) \geq \hat{b}^{ik}_{jl}(x_{-i})$, and
(ii) $m_{ik}(x) = m_{jl}(x)$ implies that both $m_{ik}(x)$ and $m_{jl}(x)$ lie between $\hat{b}^{jl}_{ik}(x_{-j})$ and $\hat{b}^{ik}_{jl}(x_{-i})$.

The proofs of (i) and (ii) are considered separately, each consisting of a number of cases.

PROOF OF (i). Suppose that

$$m_{ik}(x) > m_{jl}(x). \qquad (a.1)$$

There are 3 cases to consider.

Case I. $m_{ik}(x_i, x_{-j}) < m_{jl}(x_i, x_{-j})$. In this case,

$$\begin{aligned} \hat{b}^{jl}_{ik}(x_{-j}) &= \inf_{\alpha} m_{ik}(\alpha, x_{-j}) \text{ s.t. } m_{ik}(\alpha, x_{-j}) < m_{jl}(\alpha, x_{-j}) \\ &\geq m_{ik}(x) \end{aligned}$$

because $m_{ik}(\alpha, x_{-j}) < m_{jl}(\alpha, x_{-j})$, (a.1) and A.3 imply that $\alpha > x_j$. The monotonicity of $m_{ik}(\cdot, x_{-j})$ then yields $m_{ik}(\alpha, x_{-j}) \geq m_{ik}(x)$.

Case II. $m_{ik}(x_i, x_{-j}) > m_{jl}(x_i, x_{-j})$. In this case,

$$\begin{aligned} \hat{b}^{jl}_{ik}(x_{-j}) &= \sup_{\alpha} m_{ik}(\alpha, x_{-j}) \text{ s.t. } m_{ik}(\alpha, x_{-j}) > m_{jl}(\alpha, x_{-j}) \\ &\geq m_{ik}(x), \text{ since by (a.1) } \alpha = x_j \text{ is feasible.} \end{aligned}$$

Case III. $m_{ik}(x_i, x_{-j}) = m_{jl}(x_i, x_{-j})$. In this case,

$$\hat{b}^{jl}_{ik}(x_{-j}) = m_{ik}(x_i, x_{-j}) = m_{jl}(x_i, x_{-j}).$$

Consequently, A.3 and (a.1) imply that $x_i > x_j$, so that the monotonicity of $m_{ik}(\cdot)$ yields $\hat{b}^{jl}_{ik}(x_{-j}) \geq m_{ik}(x)$.

Thus we have shown that (a.1) implies $\hat{b}^{jl}_{ik}(x_{-j}) \geq m_{ik}(x)$. A similar argument establishes that $m_{jl}(x) \geq \hat{b}^{ik}_{jl}(x_{-i})$.

PROOF OF (ii). Suppose that

$$m_{ik}(x) = m_{jl}(x) \text{ and } x_i \geq x_j \qquad (a.2)$$

Then by A.3, $m_{ik}(x_i, x_{-j}) \leq m_{jl}(x_i, x_{-j})$. Hence there are just two cases to consider.

Case I. $m_{ik}(x_i, x_{-j}) < m_{jl}(x_i, x_{-j})$. In this case,

$$\begin{aligned}\hat{b}_{ik}^{jl}(x_{-j}) &= \inf_{\alpha} m_{ik}(\alpha, x_{-j}) \text{ s.t. } m_{ik}(\alpha, x_{-j}) < m_{jl}(\alpha, x_{-j}) \\ &\geq m_{ik}(x)\end{aligned}$$

because $m_{ik}(\alpha, x_{-j}) < m_{jl}(\alpha, x_{-j})$, (a.2) and A.3 imply that $\alpha > x_j$. Consequently, by monotonicity, $m_{ik}(\alpha, x_{-j}) \geq m_{ik}(x)$.
Case II. $m_{ik}(x_i, x_{-j}) = m_{jl}(x_i, x_{-j})$. In this case, monotonicity and $(a.2)$ yield

$$\hat{b}_{ik}^{jl}(x_{-j}) = m_{ik}(x_i, x_{-j}) \geq m_{ik}(x).$$

Thus we have shown that (a.2) implies $\hat{b}_{ik}^{jl}(x_{-j}) \geq m_{ik}(x)$. A similar argument establishes that $m_{jl}(x) \geq \hat{b}_{jl}^{ik}(x_{-i})$, completing the proof. ■

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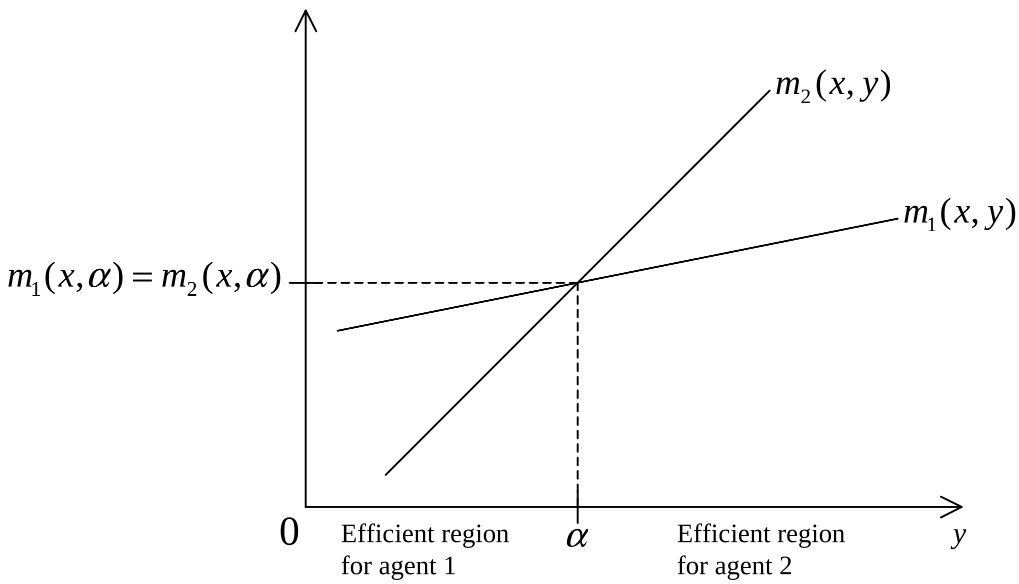
$m_2(x,y)$
$m_1(x,y)$
$m_1(x,\alpha) = m_2(x,\alpha)$
0
Efficient region
for agent 1
α
Efficient region
for agent 2
y

Figure 3.1

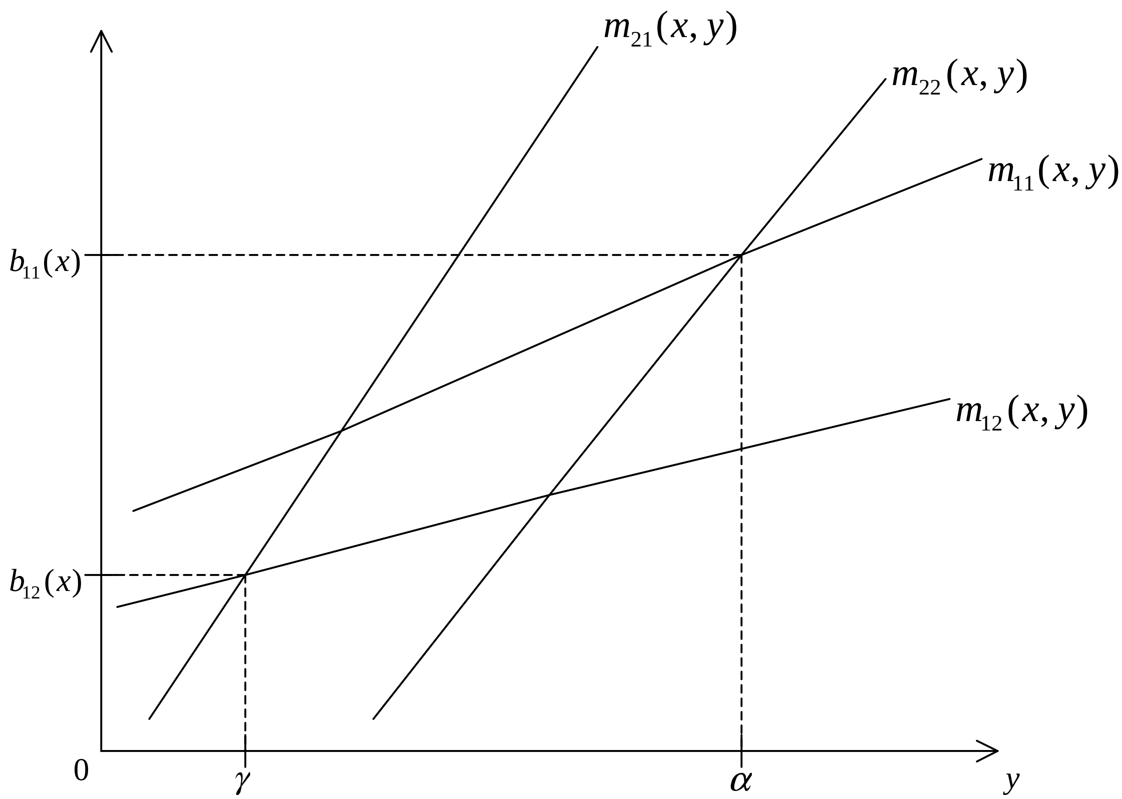

Figure 3.2