

# Beauty Contests, Bubbles and Iterated Expectations in Asset Markets\*

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## Abstract

In a financial market where traders are risk averse and short lived, and prices are noisy, asset prices today depend on the average expectation today of tomorrow's price. Thus (iterating this relationship) the date 1 price equals the date 1 average expectation of the date 2 average expectation of the date 3 price. This will *not* in general equal the date 1 average expectation of the date 3 price. We show how this failure of the law of iterated expectations for average belief can help understand the role of higher order beliefs in a fully rational asset pricing model and explain over-reaction to (noisy) public information.

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“...professional investment may be likened to those newspaper competitions in which the competitors have to pick out the six prettiest faces from a hundred photographs, the prize being awarded to the competitor whose choice most nearly corresponds to the average preferences of the competitors as a whole; so that each competitor whose choice most nearly corresponds to the average preferences of the competitors as a whole: so that each competitor has to pick, not those faces which he himself finds prettiest, but those which he thinks likeliest to catch the fancy of the other competitors, all of whom are looking at the problem from the same point of view. It is not a case of choosing those which, to the best of one’s judgement, are really the prettiest, nor even those which average opinion genuinely thinks the prettiest. We have reached the third degree where we devote our intelligences to anticipating what average opinion expects the average opinion to be. And there are some, I believe, who practise the fourth, fifth and higher degrees.” Keynes (1936), page 156.

“The history of speculative bubbles begins roughly with the advent of newspapers. One can assume that, although the record of these early newspapers is mostly lost, they regularly reported on the first bubble of any consequence, the Dutch tulipmania of the 1630s. Although the news media - newspapers, magazines, and broadcast media, along with their new outlets on the Internet - present themselves as detached observers of market events, they are themselves an integral part of these event. Significant market events generally occur only if there is similar thinking among large groups of people, and the news media are essential vehicles for the spread of ideas.” Shiller (2000), page 71.

## 1. Introduction

Keynes (1936) introduced the influential metaphor of financial markets as a beauty contest. An implication of the metaphor is that an understanding of financial markets requires an understanding not just of market participants' beliefs about assets' future returns, but also an understanding of market participants' beliefs about other market participants' beliefs, and higher order beliefs. Judging by how often the above passage from Keynes is quoted in academic and non-academic circles, many people find the metaphor highly suggestive. Yet the theoretical literature on asset pricing has, on the whole, failed to develop models that validate the role of higher order beliefs in asset pricing.<sup>1</sup> One purpose of our paper is to illuminate the role of higher order expectations in an asset pricing context, and thereby to explore the extent to which Keynes's beauty contest metaphor is valid as a guide for thinking about asset prices.

The second purpose of this paper is to explore the idea of asset market bubbles as an excessive reaction to (noisy) public information, in a rational model. Chapter 4 of Shiller (2000) is devoted to the idea that the news media, by propagating information in a public way, may create or exacerbate asset market bubbles by coordinating market participants' expectations. News stories without much information content may play a role akin to "sunspots" - i.e., payoff irrelevant signals that coordinate players' expectations. If public information suggests that payoffs will be high then this can lead to high asset prices even if many traders have private information that the true value is low.

Neither phenomena make an appearance in standard competitive asset pricing models with a representative investor. Asset prices in such settings reflect the discounted expected value of returns from the asset, suitably adjusted for risk. Since we believe that both phenomena alluded to above are consistent with competitive asset pricing models, our explanation must include an account of why asset prices in a competitive market may fail to reflect solely the discounted expected returns.

A key feature of the representative investor model of asset prices that makes higher order expectations redundant is the martingale property of asset prices. The price of an asset today is the discounted expected value of the asset's terminal payoff with respect to an equivalent martingale measure conditional on the information available to the representative individual today. This allows the folding back of future outcomes to the present in coming up with today's price. Accompanying the martingale property is the law of iterated expectations in which the

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<sup>1</sup>Some exceptions are discussed in section 5.

representative investor's expectation today of his expectation tomorrow of future returns is equal to his expectation today of future returns.

But if there is differential information between investors so that there is some role for the *average* expectations about returns, the folding back of future outcomes to the present cannot easily be achieved. In general, average expectations fail to satisfy the law of iterated expectations. It is *not* the case that the average expectation today of the average expectation tomorrow of future returns is equal to the average expectation of future returns. The key observation in this paper is not only that the law of iterated expectations fails to hold for average opinion when there is differential information, but that its failure follows a systematic pattern that ties in with the disproportionate impact of the media and other sources of public information.

Suppose that an individual has access to both private and public information about an asset's returns, and they are of equal value in predicting the asset's returns. Thus in predicting the asset's returns, the individual would put equal weight on private and public signals. Now suppose that the individual is asked to guess what the average expectation of the asset's returns is. Since he knows that others have also observed the same public signal, the public signal is a better predictor of average opinion, he will put more weight on the public signal than on the private signal. Thus if individuals' willingness to pay for an asset is related to their expectations of average opinion, then we will automatically get asset prices overweighting public information relative to the private information. Thus *any* model where higher order beliefs play a role in pricing assets will automatically deliver the conclusion that there is an excess reliance on public information.

Even in a single-period rational expectations asset pricing model, the price is a biased signal of the true liquidation value of the asset when the model is modified by the inclusion of a public signal. The price puts excessive weight on the public signal relative to the true liquidation value. This bias towards the public signal is reminiscent of the result in Morris and Shin (2002) where the coordination motive of the agents induces a disproportionate role for the public signal. Although there is no explicit coordination motive in the rational expectations equilibrium, the fact that the public signal enters into everyone's demand function means that it still retains some value for forecasting the aggregate demand above and beyond its role in estimating the liquidation value. Another way of expressing this is to say that, whereas the noise in the individual traders' private signals get "washed out" when demand is aggregated across traders, the noise term in the public signal is not similarly washed out. Thus, the noise in the public signal is still useful in

forecasting aggregate demand, and hence the price.

Our main focus, however, is on the multi-period asset pricing context. Do asset prices reflect average opinion, and average opinion about average opinion, in the manner that Keynes suggests? We will describe one simple standard asset pricing model where this is the case. We look at a dynamic, noisy rational expectations asset pricing model, of the type developed by Singleton (1987), Brown and Jennings (1989), Grundy and McNichols (1989) and He and Wang (1995). An asset will pay a one off dividend in period 3. In periods 1 and 2, the asset is traded by short lived traders, who live for only one period, and observe both public and private signals. A noisy supply function ensures that asset prices are not fully revealing. We show that as the noise in asset prices becomes large, the average asset price in period 2 converges to the average expectation of the dividend; and the average asset price in period 1 converges to the average expectation of the average expectation of the dividend. This result readily extends to an arbitrary number  $k$  of trading periods / generations of short-lived traders. The average asset price will equal the  $k$ th order average expectation of the dividend. For large  $k$ , private information will not be reflected in asset prices.

Noisy rational expectations equilibria in the standard constant absolute risk aversion/normally distributed returns (CARA-normal) model have a number of well-known conceptual problems; and the limit we focus on - where the noise in the supply function becomes large - is an extreme case. We believe this case is nonetheless interesting to study because there is a very simple and transparent account about how higher order beliefs come to be reflected in asset prices. We want a minimal model that is fully rational and highlights the role of the key assumption, short-lived traders. Similar conclusions do and would result in models with more detailed analysis of market microstructure. For example, one reason for short horizons is that individuals' funds are managed by professionals and inefficiencies resulting from the agency problem give rise to short horizons (Allen and Gorton (1993)). In the concluding section 5, we discuss some of the existing literature with similar conclusions. While the model in this paper is too stylized to directly apply to time series data on asset prices, we believe that the insights may help interpret what is going on in empirical work using computational dynamic noisy rational expectations models, such as the pioneering work of Singleton (1987) and the recent work (in a currency market context) of Bacchetta and van Wincoop (2002).

The insights of this paper are relevant beyond the asset pricing application. An old literature, dating back to Phelps (1983) and Townsend (1983), looked at

dynamic models where agents follow linear decision rules but their choices depend on others' choices and their heterogeneous expectations about future realizations of economic variables. As a consequence, forward looking iterated average expectations matter. The CARA-normal noisy rational expectations asset prices of this paper inherit both the linear decision rules and the forward looking iterated average expectations. In section 4, we explore the connection in more detail. One insight highlighted in this paper is that forward looking iterated average expectations have a rich structure *even when there is no learning*. Thus while learning is an interesting (and unavoidable) phenomenon in its own right, it may be interesting to understanding the role of dynamic higher order beliefs independently of learning. There has been a recent resurgence of work looking at heterogeneous expectations in dynamic macroeconomics where this insight might be relevant<sup>2</sup>.

## 2. Asymmetric Information and Iterated Expectations

For any random variable  $\theta$ , let  $E_{it}(\theta)$  be player  $i$ 's expectation of  $\theta$  at date  $t$ ; write  $\bar{E}_t(\theta)$  for the average expectation of  $\theta$  at time  $t$ ; and write  $E_t^*(\theta)$  for the public expectation of  $\theta$  at time  $t$  (i.e., the expectation of  $\theta$  conditional on public information only; in a partition model, this would be conditional on the meet of players' information).

We know that individual and public expectations satisfy the law of iterated expectations:

$$E_{it}(E_{i,t+1}(\theta)) = E_{it}(\theta)$$

and  $E_t^*(E_{t+1}^*(\theta)) = E_t^*(\theta)$ .

But the analogous property for average expectations will typically fail under asymmetric information. In other words, we will typically have

$$\bar{E}_t(\bar{E}_{t+1}(\theta)) \neq \bar{E}_t(\theta).$$

This is most easily seen by considering the case where there is no learning. Suppose  $\theta$  is distributed normally with mean  $y$  and variance  $\frac{1}{\alpha}$ . Each player  $i$  in a continuum observes a signal  $x_i = \theta + \varepsilon_i$ , where  $\varepsilon_i$  is distributed in the population with mean

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<sup>2</sup>For example, Amato and Shin (2002), Hellwig (2002), Pearlman and Sargent (2002), Stasavage (2002) and Woodford (2001).

0 and variance  $\frac{1}{\beta}$ . Suppose that this is all the information available *at all dates*. Then we may drop the date subscripts. Now observe that

$$\begin{aligned}
E_i(\theta) &= \frac{\alpha y + \beta x_i}{\alpha + \beta} \\
\bar{E}(\theta) &= \frac{\alpha y + \beta \theta}{\alpha + \beta} \\
E_i(\bar{E}(\theta)) &= \frac{\alpha y + \beta E_i(\theta)}{\alpha + \beta} \\
&= \frac{\alpha y + \beta \left(\frac{\alpha y + \beta x_i}{\alpha + \beta}\right)}{\alpha + \beta} \\
&= \left(1 - \left(\frac{\beta}{\alpha + \beta}\right)^2\right) y + \left(\frac{\beta}{\alpha + \beta}\right)^2 x_i \\
\bar{E}(\bar{E}(\theta)) &= \left(1 - \left(\frac{\beta}{\alpha + \beta}\right)^2\right) y + \left(\frac{\beta}{\alpha + \beta}\right)^2 \theta
\end{aligned}$$

Iterating this operation, one can show that

$$\bar{E}^k(\theta) = \left(1 - \left(\frac{\beta}{\alpha + \beta}\right)^k\right) y + \left(\frac{\beta}{\alpha + \beta}\right)^k \theta.$$

Note that (1) the expectation of the expectation is biased towards the public signal  $y$ : that is,

$$\text{sign}(\bar{E}(\bar{E}(\theta)) - \bar{E}(\theta)) = \text{sign}(y - \bar{E}(\theta));$$

and (2) as  $k \rightarrow \infty$ ,  $\bar{E}^k(\theta) \rightarrow y$ .<sup>3</sup> Putting back the time subscripts, we have

$$\bar{E}_t(\bar{E}_{t+1}(\theta)) = \left(1 - \left(\frac{\beta}{\alpha + \beta}\right)^2\right) y + \left(\frac{\beta}{\alpha + \beta}\right)^2 \theta \neq \frac{\alpha y + \beta \theta}{\alpha + \beta} = \bar{E}_t(\theta).$$

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<sup>3</sup>Property (1) does not hold for all distributions: one can construct examples where it fails to hold. However, property (2) holds independently of the normality assumption: this is, for any random variable and information system with a common prior, the average expectation of the average expectations... of the random variable converges to the expectation of the random variable conditional on public information (see Samet (1998)).

and

$$\bar{E}_t \left( \bar{E}_{t+1} \left( \dots \bar{E}_{T-2} \left( \bar{E}_{T-1} (\theta) \right) \right) \right) = \left( 1 - \left( \frac{\beta}{\alpha + \beta} \right)^{T-t} \right) y + \left( \frac{\beta}{\alpha + \beta} \right)^{T-t} \theta.$$

Now suppose that there is an asset that has liquidation value  $\theta$  at date  $T$ . Suppose - in the spirit of the Keynes beauty contest - that the asset is priced according to the asset pricing formula

$$p_t = \bar{E}_t (p_{t+1}). \quad (2.1)$$

Then we would have

$$p_t = \left( 1 - \left( \frac{\beta}{\alpha + \beta} \right)^{T-t} \right) y + \left( \frac{\beta}{\alpha + \beta} \right)^{T-t} \theta.$$

This implies that, given the realization of the public signal, the period  $t$  price is biased toward the public signal relative to fundamentals. It also implies that, unconditional on the realized public signal, the period  $t$  price is normally distributed with mean  $\theta$  and variance  $\frac{1}{\alpha} \left( 1 - \left( \frac{\beta}{\alpha + \beta} \right)^{T-t} \right)$ . Thus the more trading periods there are, the higher the variance of the price. This is despite the fact that (by assumption) no new information is being revealed.

So far, we have given no justification for asset pricing formula (2.1), other than an appeal to the authority of Keynes. Furthermore, our assumption in this section that there is no learning will not be consistent in a rational model with traders observing prices. We would like to describe an asset pricing model that generates asset pricing formula (2.1), or something like it, and deals with the issue of learning from prices. We do this in the next section.

### 3. Rational Expectations with Short Lived Traders

In this section, we work with a noisy rational expectations model of the type developed for static models by Grossman (1976), Hellwig (1980) and Diamond and Verrecchia (1981) and extended to dynamic model settings by Singleton (1987), Brown and Jennings (1989), Grundy and McNichols (1989) and He and Wang (1995). In particular, the main model of this section is a special case of the “myopic trader” model of Brown and Jennings (1989).

There is a unit mass of traders, indexed by the unit interval  $[0, 1]$ . There are three periods, 1, 2 and 3. In period 3, an asset will be liquidated, where



the liquidation value is  $\theta$ . The initial information of traders is exactly as in the previous section:  $\theta$  is distributed normally with mean  $y$  and variance  $\frac{1}{\alpha}$ ; each trader  $i$  in the continuum observes a signal  $x_i = \theta + \varepsilon_i$ , where  $\varepsilon_i$  is distributed in the population with mean 0 and variance  $\frac{1}{\beta}$ .

The asset is traded twice, in periods 1 and 2. We denote by  $p_1$  and  $p_2$  the price of the asset in periods 1 and 2 respectively. In each trading period, we assume that there is an exogenous noisy supply of the asset,  $s_t$ , distributed normally with mean 0 and precision  $\gamma_t$ . The traders have identical preferences, with constant absolute risk aversion utility function  $u(w) = -e^{-\frac{w}{\tau}}$ . Parameter  $\tau$  is the reciprocal of the absolute risk aversion, and we shall refer to it as the traders' risk tolerance. It is initially assumed each trader lives for only one period. New traders born in period 2 inherit the private signals of the traders that they are replacing.

This economy will have at least one linear rational expectations equilibrium, as shown by Brown and Jennings (1989, theorem 1). In the short-lived trader case, we can characterize the linear equilibrium in terms of the prices  $p_1, p_2$  in the two trading periods. We will denote by  $E_s(\cdot)$  the expectation with respect to the supply shocks  $(s_1, s_2)$ , so that  $E_s(s_1) = E_s(s_2) = 0$ . We then have our first result.

**Proposition 3.1.** *In the short-lived trader model, there exist constants  $w$  and  $z$  with  $0 < w < 1, 0 < z < 1$  such that*

$$\begin{aligned} E_s(p_1) &= (1 - wz)y + wz\theta \\ E_s(p_2) &= (1 - z)y + z\theta \end{aligned}$$

*In particular, as  $(\gamma_1, \gamma_2) \rightarrow (0, 0)$ , we have*

$$\begin{aligned} E_s(p_1) &\rightarrow \left(\frac{\beta}{\alpha + \beta}\right)^2 \theta + \left(1 - \left(\frac{\beta}{\alpha + \beta}\right)^2\right) y \\ E_s(p_2) &\rightarrow \left(\frac{\beta}{\alpha + \beta}\right) \theta + \left(\frac{\alpha}{\alpha + \beta}\right) y \end{aligned}$$

There are several noteworthy features of this result. Mean prices taken over realizations of the supply shocks  $(s_1, s_2)$  are given by convex combinations of the true liquidation value  $\theta$  and the ex ante mean  $y$ . In this sense, both prices are biased signals of the true liquidation value  $\theta$ . The distribution of prices is biased towards the public signal  $y$  relative to the true liquidation value  $\theta$ . Moreover,

the extent of the bias is worse for the first period price. The first period price puts more weight on the public signal  $y$  than does the second period price. In particular, when the noise in the supply of the asset becomes large (so that the informational role of price is diminished), the expressions for mean price converge to the expressions  $\bar{E}(\theta)$  and  $\bar{E}\bar{E}(\theta)$ , explored in the previous section. Thus, we have a specific example where iterations of the average expectations operator  $\bar{E}(\cdot)$  determines the price, and hence where Keynes's beauty contest metaphor applies. The argument for proposition 3.1 is given in appendix A, but we will present an informal argument later in this section for the limiting case where the supply noise becomes large.

The fact that price is biased towards the public signal also appears in a single period version of our model. In the single period model, the mean of the linear rational expectations price  $p$  over the realizations of the supply shock  $s$  is given by

$$E_s(p) = \frac{\alpha y + (\beta + \tau^2 \beta^2 \gamma) \theta}{\alpha + \beta + \tau^2 \beta^2 \gamma} \quad (3.1)$$

so that  $E_s(p)$  is a convex combination of  $y$  and  $\theta$ . Thus,  $E_s(p) \neq \theta$ , so that price is a biased signal of true liquidation value. However, this bias disappears when either  $\beta \rightarrow \infty$ , so that the private information of traders swamps the public signal  $y$ , or when  $\tau \rightarrow \infty$ , when traders become risk neutral in the limit, or when  $\gamma \rightarrow \infty$  when the supply noise disappears. However, as long as traders are risk averse and the public signal has some information value relative to the private signals, price is a biased signal of  $\theta$ .

This bias towards the public signal has some similarities with the result in Morris and Shin (2002) where the coordination motive of the agents induces a disproportionate role for the public signal. Although there is no explicit coordination motive in the rational expectations equilibrium, the fact that the public signal enters into everyone's demand function means that  $y$  still retains some value for forecasting the aggregate demand, above and beyond its role in estimating the liquidation value  $\theta$ . Another way of expressing this is to say that, whereas the noise in the individual traders' private signals  $x_i$  gets "washed out" when demand is aggregated across traders, the noise term in the public signal (the difference between  $y$  and  $\theta$ ) is not similarly washed out. Thus, the noise in the public signal is still useful in forecasting aggregate demand, and hence the price.

The detailed argument for proposition 3.1 follows the familiar linear solution method that relies on (i) the linearity of conditional expectations for jointly normal random variables, and (ii) the linearity of the demand functions arising from

CARA utility functions. We thus relegate the argument to the appendix. However, it is worthwhile sketching an informal argument for the limiting case in proposition 3.1.

As the noise in the supply becomes larger, the informational content of prices become more and more diluted, so that in period 2, each trader  $i$  will not have learned much from either first or second period prices. Thus in period 2, trader  $i$ 's belief concerning  $\theta$  is close to someone who has not observed either price. So, as an approximation, trader  $i$  believes that  $\theta$  is distributed normally with mean

$$\left(\frac{\beta}{\alpha + \beta}\right) x_i + \left(\frac{\alpha}{\alpha + \beta}\right) y$$

and precision  $\alpha + \beta$ . By the well-known formula for the asset demand of a CARA consumer facing normal returns, trader  $i$ 's demand will be

$$\tau(\alpha + \beta) \left( \left(\frac{\beta}{\alpha + \beta}\right) x_i + \left(\frac{\alpha}{\alpha + \beta}\right) y - p_2 \right).$$

Integrating over all traders  $i \in [0, 1]$ , the total demand will be

$$\tau(\alpha + \beta) \left( \left(\frac{\beta}{\alpha + \beta}\right) \theta + \left(\frac{\alpha}{\alpha + \beta}\right) y - p_2 \right).$$

Market clearing requires

$$\tau(\alpha + \beta) \left( \left(\frac{\beta}{\alpha + \beta}\right) \theta + \left(\frac{\alpha}{\alpha + \beta}\right) y - p_2 \right) = s_2.$$

This implies the pricing equation

$$p_2 = \left(\frac{\beta}{\alpha + \beta}\right) \theta + \left(\frac{\alpha}{\alpha + \beta}\right) y - \frac{s_2}{\tau(\alpha + \beta)}. \quad (3.2)$$

Taking expectations with respect to  $s_2$  gives us our result.

In period 1, each trader  $i$  will know that  $p_2$  is determined approximately by equation (3.2). His expectation of  $p_2$  based on the public signal  $y$  and his own private signal  $x_i$  will then be

$$\begin{aligned} & \left(\frac{\beta}{\alpha + \beta}\right) \left( \left(\frac{\beta}{\alpha + \beta}\right) x_i + \left(\frac{\alpha}{\alpha + \beta}\right) y \right) + \left(\frac{\alpha}{\alpha + \beta}\right) y \\ &= \left(\frac{\beta}{\alpha + \beta}\right)^2 x_i + \left(1 - \left(\frac{\beta}{\alpha + \beta}\right)^2\right) y. \end{aligned}$$

If  $\text{Var}_{1i}(p_2)$  is the variance of  $p_2$  conditional on the information available to trader  $i$  at date 1, his demand will be

$$\frac{\tau}{\text{Var}_{1i}(p_2)} \left( \left( \frac{\beta}{\alpha + \beta} \right)^2 x_i + \left( 1 - \left( \frac{\beta}{\alpha + \beta} \right)^2 \right) y - p_1 \right).$$

The conditional variance of  $p_2$  at date 1 will be identical across traders, and so we may write  $\text{Var}_1(p_2)$  for the common conditional variance of  $p_2$  across traders. Then, aggregate demand will be

$$\frac{\tau}{\text{Var}_1(p_2)} \left( \left( \frac{\beta}{\alpha + \beta} \right)^2 \theta + \left( 1 - \left( \frac{\beta}{\alpha + \beta} \right)^2 \right) y - p_1 \right).$$

and market clearing requires

$$\frac{\tau}{\text{Var}_1(p_2)} \left( \left( \frac{\beta}{\alpha + \beta} \right)^2 \theta + \left( 1 - \left( \frac{\beta}{\alpha + \beta} \right)^2 \right) y - p_1 \right) = s_1.$$

This implies the pricing equation

$$p_1 = \left( \frac{\beta}{\alpha + \beta} \right)^2 \theta + \left( 1 - \left( \frac{\beta}{\alpha + \beta} \right)^2 \right) y - s_1 \frac{\text{Var}_1(p_2)}{\tau}.$$

Integrating out  $s_1$  gives us our limiting result.

To be sure, the limiting results in proposition 3.1 should be interpreted with caution. As the supply noise in the two periods become very large, the distributions of  $p_1$  and  $p_2$  themselves will become very dispersed. In the limit, both prices have degenerate distributions in which variances become infinite. However, proposition 3.1 also shows that even away from the limit, when prices *are* informative about  $\theta$ , the first period price shows a greater bias towards the ex ante mean than does the second period price. The constants  $w$  and  $z$  are unwieldy expressions in general, but they are analogous to the ratio  $\beta/(\alpha + \beta)$  that figure in the iterated average expectations operator.

For simplicity, we proved a result for a model with two periods of asset trading. However, the result will extend straightforwardly to a world with  $T$  rounds of asset trading (just open the market for  $T$  periods, with a new supply shock in each period). Again, we obtain pricing formula (2.1).<sup>4</sup>

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<sup>4</sup>He and Wang (1995) have studied a quite general dynamic noisy rational expectations model with many periods of trading. There does not exist an existence result for this setting (even with short-lived traders), but it would be straightforward to establish existence for sufficiently small  $\gamma$ .

It is important to compare this model with what would happen if traders were long-lived, i.e., the same traders were in the market at dates 1 and 2 and maximize the utility of final period consumption. This more complex case is again analyzed formally in the appendix. We present here the limiting results as the supply noise becomes large.

**Proposition 3.2.** *In the long-lived trader model, as  $(\gamma_1, \gamma_2) \rightarrow (0, 0)$ ,*

$$E_s(p_1) = E_s(p_2) = \left( \frac{\beta}{\alpha + \beta} \right) \theta + \left( \frac{\alpha}{\alpha + \beta} \right) y$$

Again, we provide a formal limiting argument in appendix A and here present the simple heuristic argument “in the limit”. Period 2 will look identical to the short-lived trader model. So consider the problem of a trader in the first period. He can anticipate what his second period demand for the asset will be. His first period demand is not the same as in the short-lived trader model, because of his hedging demand. Even if he thought that holding the asset from period 1 to period 2 had a zero or negative expected return, he might want to buy a positive amount of the asset because it hedges the risk that he will be taking on in period 2. It turns out that in this case, this hedging demand implies that the period 1 trader will purchase his expected demand for the asset in the next period, as a hedge. The net effect of this is that he will behave as if the period 2 trading opportunity does not exist. Of course, this makes sense, since  $p_2$  is very noisy in the limit, and much noisier than  $\theta$ .

The comparison of the short-run and long-run trader models is instructive. The expected price is biased towards the public signal in the first trading round of the short-run trader model relative to the long-run trader model. Short horizons are generating an over-reliance on public information. To be sure, our comparison between short-run and long-run models is somewhat stark. However, it is worth noting that even in a long lived asset market, all that is required to get the qualitative features of the short-run model is that *some* traders consume before all the expected future dividends reflected in the assets they are trading are realized. This is surely a realistic assumption. We could easily introduce hybrid traders who, with probability  $\lambda$ , will face liquidity needs and be forced to consume at date 2, and, with probability  $1 - \lambda$ , will not consume until period 3. The model would move smoothly between the two limits discussed here as we let  $\lambda$  vary between 0 and 1.

An unsatisfactory (but standard) feature of the noisy REE model considered here is the assumption that there is an exogenous noisy supply. This feature is

particularly unsatisfactory because we are interested in cases where prices are not very revealing, and we achieved this here by letting the noisy supply become large. There are various (more complicated) devices in the literature for allowing prices to not be fully revealing, and this is the only function the noisy supply is playing in this model.<sup>5</sup> We could have fixed the variance of the noisy supply and let the traders become very risk averse (i.e., let  $\tau \rightarrow 0$ ). In this case, traders' demand would be very price insensitive and again prices would be non-revealing. Identical results would follow.

#### 4. Forecasting the Forecasts of Others

The multi-period rational expectations equilibrium is a sophisticated example of a setting in which agents attempt to forecast others' forecasts. In period 1, traders are attempting to forecast price at period 2, which in turn depends on the forecasts of traders in period 2 concerning the liquidation value  $\theta$ . What makes the problem rather complicated is the fact that the information sets of the traders consist not only of the exogenous signals  $y$  and  $\{x_i\}$ , but also the endogenous signals - i.e. the prices - generated by the actions of the traders themselves. However, we have seen that essential features of the Keynes beauty contest survive in modified form.

This suggests that it would be instructive to study the effect of iterative forecasts in isolation from other features of the problem, so as to gauge the importance of this element by itself, and to see how much of the total effect can be attributed to the problem of forecasting the forecasts of others. This is the task that we will take on in this section. By restricting the information of the agents to exogenous signals only, we will analyze the specific features of iterative forecasts.

It should be noted at the outset that the problem of forecasting the forecasts of others has a long history in economics. It has been an important theme in macroeconomic models with asymmetric information, for instance. The Lucas-Phelps island economy model (Phelps (1970), Lucas (1972, 1973)) is perhaps the first formalization of such a problem, and Townsend (1978, 1983), Phelps (1983) and others have commented on the importance of this issue in solving for the aggregate laws of motion for the economy as a whole. Our analysis below has some bearing on this, and related areas of the literature.

Let us consider a model where time is indexed by  $\{0, 1, 2, \dots, T + 1\}$ . Agents live for two periods and agents of the same generation are indexed by the unit

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<sup>5</sup>See, for example, Ausubel (1990) and Wang (1994).

interval  $[0, 1]$ , so that at any date other than the first or last, there is a unit mass of young agents and a unit mass of old agents.

At date zero, the random variable  $\theta$  is chosen by nature, where  $\theta$  is drawn from a normal distribution with mean  $y_0$  and precision  $\alpha_0$ . No one observes this realization, but all agents receive a private signal  $x_i = \theta + \varepsilon_i$  where  $\varepsilon_i$  is normal with mean zero and precision  $\beta$ . In addition, there is a public signal  $y_t$  at date  $t$  which has mean  $\theta$  and precision  $\alpha_t$  which is observable to agents at date  $t$  and to all agents that come later. Thus, for agent  $i$  alive at date  $t$ , his information set consists of

$$\{x_i, y_0, y_1, y_2, \dots, y_t\}$$

The agents have no other information.

At the final date  $T + 1$ , the realization of  $\theta$  is revealed. At date  $T$ , the young agents try to forecast this realization. When agent  $i$  announces the forecast  $p_{iT}$ , his payoff at date  $T + 1$  is then

$$-(p_{iT} - \theta)^2$$

At earlier dates  $t < T$ , the young agents try to forecast the average forecast of the next generation of agents. Thus, at date  $t$ , agent  $j$  announces the forecast  $p_{jt}$  in order to maximize his payoff at  $t + 1$  which is given by

$$-(p_{jt} - \bar{p}_{t+1})^2$$

where  $\bar{p}_{t+1} = \int_{k \in [0,1]} p_{k,t+1} dk$ , the average forecast of the young generation in the next period.

The assumption that the information available to agents consist only of the exogenous signals  $\{x_i, y_0, y_1, \dots, y_{t-1}\}$  leads directly to the result that the average  $\bar{p}_t$  of decisions at date  $t$  is given by

$$\bar{p}_t = \bar{E}_t \bar{E}_{t+1} \dots \bar{E}_{T-1} \bar{E}_T (\theta)$$

where  $\bar{E}_t(\cdot)$  is the average expectation based on the signals  $\{x_i, y_0, y_1, \dots, y_t\}$ . The argument is by induction. At the penultimate date  $T$ , the optimal choice is the conditional expectation of  $\theta$  based on signals  $\{x_i, y_0, y_1, \dots, y_T\}$ , so that

$$p_{iT} = E_{iT}(\theta)$$

Hence

$$\bar{p}_T = \bar{E}_T(\theta)$$

Thus, suppose at date  $t + 1$  that

$$\bar{p}_{t+1} = \bar{E}_{t+1} \cdots \bar{E}_{T-1} \bar{E}_T (\theta)$$

then the optimal decision of  $i$  is the conditional expectation of  $\bar{p}_{t+1}$  based on the signals  $\{x_i, y_0, y_1, \dots, y_t\}$ . So,

$$p_{i,t} = E_{it} \bar{E}_{t+1} \cdots \bar{E}_{T-1} \bar{E}_T (\theta)$$

Thus, averaging over all  $i$ ,

$$\bar{p}_t = \bar{E}_t \bar{E}_{t+1} \cdots \bar{E}_{T-1} \bar{E}_T (\theta)$$

as claimed.

We can solve explicitly for the iterated average expectations of  $\theta$ , and investigate how the relative weights on the private and public signals vary over time. Let us first state the general result, and then examine a special case that is somewhat easier to interpret. The proof of the following proposition is presented separately in appendix B.

**Proposition 4.1.**

$$\bar{E}_t \bar{E}_{t+1} \cdots \bar{E}_{T-1} \bar{E}_T (\theta) = \lambda_{t,T} \cdot \theta + (1 - \lambda_{t,T}) \frac{\sum_{\tau=0}^t \alpha_\tau y_\tau}{\sum_{\tau=0}^t \alpha_\tau} \quad (4.1)$$

where

$$\lambda_{t,T} = \sum_{i=t}^{T-1} \left[ \frac{\sum_{\tau=0}^t \alpha_\tau}{\sum_{\tau=0}^i \alpha_\tau} \cdot \frac{\alpha_{i+1}}{\sum_{\tau=0}^{i+1} \alpha_\tau} \cdot \prod_{j=t}^i \frac{\beta}{\sum_{\tau=0}^j \alpha_\tau + \beta} \right] + \frac{\sum_{\tau=0}^t \alpha_\tau}{\sum_{\tau=0}^T \alpha_\tau} \prod_{j=t}^T \frac{\beta}{\sum_{\tau=0}^j \alpha_\tau + \beta}$$

This is a somewhat unwieldy expression, but we can get a better feel for the magnitudes by considering a special case. Thus, consider the case where

$$\beta = \alpha_0 = \alpha_1 = \cdots = \alpha_T$$

so that all the public signals are of equal precision, and the private signal has the same precision as a typical public signal. Then, we have

$$\lambda_{t,T} = (t+1) \sum_{i=t}^{T-1} \frac{(t+i)! (t+i-2)!}{((t+i+2)!)^2} + (T-t+1) \frac{t+1}{T+1} \frac{(t+1)!}{(T+2)!} \quad (4.2)$$



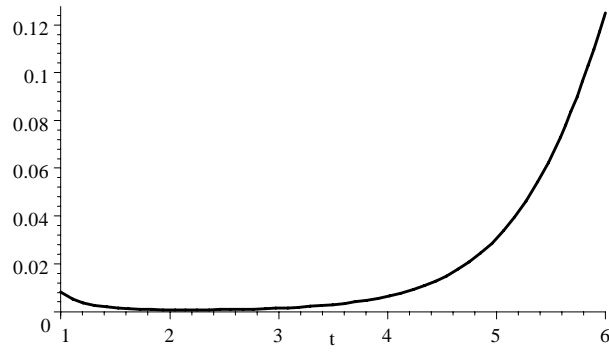


Figure 4.1: Weight on  $x_i$  for  $T = 6$

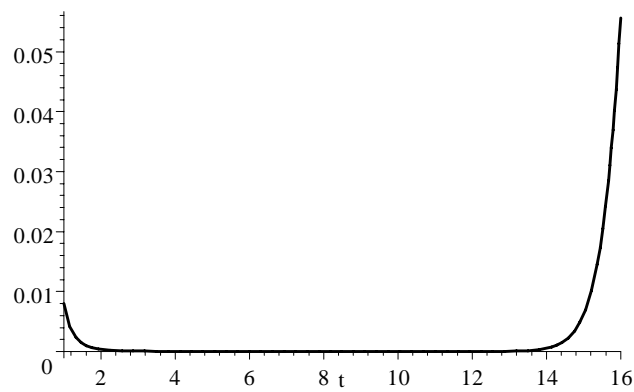


Figure 4.2: Weight on  $x_i$  for  $T = 16$

We plot two instances of this weight, the first for  $T = 6$ , and the second for  $T = 16$ .

The weight on the private signal is non-monotonic. At first, the weight on the private signal is decreasing, as the newly arriving public signals swamp the informational value of the private signal. For dates in the middle of the span of time, the weight on the private signal is virtually zero. However, as the terminal date looms closer, the weight on the private signal increases. The intuition for the increasing weight on the private signal lies in the fact that, as  $t$  becomes larger, the number of layers of the average expectations operator  $\overline{E}$  starts to diminish, so that an agent's action bears a closer resemblance to his best forecast of  $\theta$  itself, rather than the iterated average expectations of  $\theta$ . This "tug of war" between the agent's best estimate of  $\theta$  versus his motive to second guess the forecasts of others gives rise to the U-shaped weight on the private signal  $x_i$ .

## 5. Discussion

Our paper has a number of antecedents that touch upon the main themes that we have introduced here. A number of papers have examined the role of higher order beliefs in asset pricing. However, fully rational models are typically somewhat special and hard to link to standard asset pricing models (see, e.g., Allen, Morris and Postlewaite (1993), Morris, Postlewaite and Shin (1995) and Biais and Boessarts (1998)).

A number of authors have noted that agents will not act on private information if they do not expect that private information to be reflected in asset prices at the time that they sell the asset (e.g., Froot, Scharfstein and Stein (1992)). This phenomenon is clearly related to the horizons of traders in the market (see, e.g., Dow and Gorton (1994)). Tirole (1982) emphasized the importance of myopic traders in breaking down the backward induction argument against asset market bubbles. The behavioral approach exploits this feature to the full by assuming that rational (but impatient) traders forecast the beliefs of irrational traders (e.g., De Long, Shleifer, Summers and Waldmann (1990)). But since irrationality is by no means a necessary ingredient for higher order beliefs to matter it seems useful to have a model where rational agents are worried about the forecasts of other *rational* agents.

Asset market bubbles are often explained in models where there is some indeterminacy, and then public but payoff irrelevant events ("sunspots") determine

the outcome. But these models are often used to proxy situations when there is apparent over-reaction to public and slight payoff relevant events. There is some evidence of over-reaction to public announcements in the finance literature (see, e.g., Kim and Verrecchia (1991)).

The existence of an equivalent martingale measure in the standard representative investor asset pricing model has become a cornerstone of modern finance since the early contributions, such as Harrison and Kreps (1979). In some case, it is possible to extend the martingale property to differential information economies. For instance, Duffie and Huang (1986) showed that as long as there is one agent who is more informed than any other, we can find an equivalent martingale measure. In general, however, the existence of an equivalent martingale measure cannot be guaranteed. Duffie and Kan (1991) give an example of an economy with true asymmetric information - i.e., no agent who is more informed than any other - where there is *no* equivalent martingale measure.<sup>6</sup> In our case, the average expectations operator fails to satisfy the law of iterated expectations. So, if the average expectations operator is also the pricing operator, then we know that there cannot be an equivalent martingale measure. This failure of the martingale property is, of course, hardly surprising. What we want to emphasize is that the martingale property fails for average expectations in a systematic way (e.g., there is a bias towards public information).<sup>7</sup>

The arguments that we have presented in this paper combine all the above ingredients. The noisy rational expectations model with short-lived traders exhibits the following features: prices reflected average expectations of average expectations of asset returns; prices are overly sensitive to public information; and traders underweighted their private information. The admittedly extreme but rather standard model we used highlights the fact that these three phenomena are closely linked. We believe they should be linked in a wide array of asset pricing models, including rational competitive models.

## APPENDIX A

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<sup>6</sup>More precisely, there is no “universal equivalent martingale measure” - i.e., no one probability distribution that could be used to price assets conditional on each trader’s information.

<sup>7</sup>In Harrison and Kreps (1978), the martingale asset pricing formula fails because there are short sales constraints and the asset price depends in each period on the most optimistic expectation in the economy. Most optimistic expectations also fail to satisfy a martingale property in a systematic way (they are a submartingale).

In this appendix, we will provide a detailed argument for propositions 3.1 and 3.2. In order to help the reader to get a firmer interpretation, we will first provide a solution to the single period model, in which there is only one trading stage.

**Single period trading model.**

The notation is identical to the model of the text, except that we may remove the time subscripts due to the one-shot nature of the trading. Thus, suppose that price is a linear function of  $y$ ,  $\theta$  and  $s$  given by

$$p = \kappa (\lambda y + \mu \theta - s) \tag{5.1}$$

Then

$$\frac{1}{\kappa \mu} (p - \kappa \lambda y) = \theta - \frac{s}{\mu}$$

is normal with mean  $\theta$  and precision  $\mu^2 \gamma$ . We may regard this as the public signal given by the price  $p$ . A trader  $i$  has access to two additional signals - the public signal  $y$  and his private signal  $x_i$ . The joint normality of the random variables implies that his posterior expectation of  $\theta$  is the convex combination of the three signals weighted by the respective precisions. Thus, denoting by  $E_i(\theta)$  trader  $i$ 's posterior expectation of  $\theta$ , we have

$$\begin{aligned} E_i(\theta) &= \frac{\alpha y + \beta x_i + \mu^2 \gamma \cdot \frac{1}{\kappa \mu} (p - \kappa \lambda y)}{\alpha + \beta + \mu^2 \gamma} \\ &= \frac{(\alpha - \mu \lambda \gamma) y + \beta x_i + \frac{\mu \gamma}{\kappa} p}{\alpha + \beta + \mu^2 \gamma} \end{aligned}$$

The conditional variance of  $\theta$  is given by

$$\frac{1}{\alpha + \beta + \mu^2 \gamma}$$

Thus, trader  $i$ 's demand for the asset is

$$\begin{aligned} &\tau \left( \alpha + \beta + \mu^2 \gamma \right) \left( \frac{(\alpha - \mu \lambda \gamma) y + \beta x_i + \frac{\mu \gamma}{\kappa} p}{\alpha + \beta + \mu^2 \gamma} - p \right) \\ &= \tau \left( (\alpha - \mu \lambda \gamma) y + \beta x_i - \left( \alpha + \beta + \mu \gamma \left( \mu - \frac{1}{\kappa} \right) \right) p \right) \end{aligned}$$

Aggregate demand is then

$$\tau \left( (\alpha - \mu \lambda \gamma) y + \beta \theta - \left( \alpha + \beta + \mu \gamma \left( \mu - \frac{1}{\kappa} \right) \right) p \right)$$

Market clearing implies that this is equal to supply  $s$ . Solving for  $p$ , we have

$$p = \frac{(\alpha - \mu\lambda\gamma)y + \beta\theta - \frac{1}{\tau}s}{\alpha + \beta + \mu\gamma\left(\mu - \frac{1}{\kappa}\right)}$$

Thus, comparing coefficients with (5.1), we can solve for the parameters  $\mu$ ,  $\lambda$  and  $\kappa$ . They are

$$\begin{aligned}\mu &= \tau\beta \\ \lambda &= \frac{\tau\alpha}{1 + \beta\gamma\tau^2} \\ \kappa &= \frac{1 + \tau^2\beta\gamma}{\tau(\alpha + \beta + \tau^2\beta^2\gamma)}\end{aligned}$$

Substituting into (5.1) gives us the equation cited in the text.

We now provide an argument for propositions 3.1 and 3.2 by solving the two period trading case. The model solved here is a small variation on the models of Brown and Jennings (1989) and Grundy and McNichols (1989). For completeness, we report a self-contained argument, but those papers and Brunnermeier (2001) can be consulted for more detail.

### The short lived trader model

We first solve the short lived trader model in four steps. We assume that prices follows linear rules and deduce the resulting public and private information in periods one and two (in steps 1 and 2). Second, by backward induction for what the linear rules must be in the two periods (in steps 3 and 4).

#### STEP 1: LEARNING FROM FIRST PERIOD PRICES

Assume that period 1 prices follow a linear rule

$$p_1 = \kappa_1(\lambda_1 y + \mu_1 \theta - s_1) \tag{5.2}$$

Observe that

$$\frac{1}{\kappa_1\mu_1}(p_1 - \kappa_1\lambda_1 y) = \theta - \frac{1}{\mu_1}s_1, \tag{5.3}$$

so

$$\frac{1}{\kappa_1\mu_1}(p_1 - \kappa_1\lambda_1 y)$$

is distributed normally with mean  $\theta$  and precision  $\mu_1^2\gamma_1$ . Thus at period 1, based on prior information alone, we will have that  $\theta$  is distributed normally with mean

$$\begin{aligned}\bar{y}_2 &= \frac{\alpha y + \frac{\mu_1\gamma_1}{\kappa_1}(p_1 - \kappa_1\lambda_1 y)}{\alpha + \mu_1^2\gamma_1} \\ &= \frac{(\alpha - \mu_1\gamma_1\lambda_1)y + \frac{\mu_1\gamma_1}{\kappa_1}p_1}{\alpha + \mu_1^2\gamma_1}\end{aligned}$$

and precision

$$\bar{\alpha}_2 = \alpha + \mu_1^2\gamma_1.$$

An individual  $i$  who in addition observes private signal  $x_i$  will believe that  $\theta$  is normally distributed with mean

$$E_i^1(\theta) = \frac{(\alpha - \mu_1\gamma_1\lambda_1)y + \beta x_i + \frac{\mu_1\gamma_1}{\kappa_1}p_1}{\alpha + \beta + \mu_1^2\gamma_1}$$

and precision

$$\alpha + \beta + \mu_1^2\gamma_1$$

## STEP 2: LEARNING FROM SECOND PERIOD PRICES

Now assume that second period prices follow a linear rule:

$$p_2 = \kappa_2(\lambda_2\bar{y}_2 + \mu_2\theta - s_2)$$

Again, we have that

$$\frac{1}{\kappa_2\mu_2}(p_2 - \kappa_2\lambda_2\bar{y}_2)$$

is distributed normally with mean  $\theta$  and precision  $\mu_2^2\gamma_2$ .

An individual  $i$  who in addition observes private signal  $x_i$  will believe that  $\theta$  is normally distributed with mean

$$\begin{aligned}E_i^2(\theta) &= \tag{5.4} \\ & \left( \frac{\bar{\alpha}_2 - \lambda_2\mu_2\gamma_2}{\bar{\alpha}_2 + \beta + \mu_2^2\gamma_2} \right) \bar{y}_2 + \left( \frac{\beta}{\bar{\alpha}_2 + \beta + \mu_2^2\gamma_2} \right) x_i + \left( \frac{\frac{\mu_2\gamma_2}{\kappa_2}}{\bar{\alpha}_2 + \beta + \mu_2^2\gamma_2} \right) p_2 \\ &= \left\{ \begin{aligned} & \left( \frac{\alpha + \mu_1^2\gamma_1 - \lambda_2\mu_2\gamma_2}{\alpha + \mu_1^2\gamma_1 + \beta + \mu_2^2\gamma_2} \right) \left( \frac{(\alpha - \mu_1\gamma_1\lambda_1)y + \frac{\mu_1\gamma_1}{\kappa_1}p_1}{\alpha + \mu_1^2\gamma_1} \right) \\ & + \left( \frac{\beta}{\alpha + \mu_1^2\gamma_1 + \beta + \mu_2^2\gamma_2} \right) x_i \\ & + \left( \frac{\frac{\mu_2\gamma_2}{\kappa_2}}{\alpha + \mu_1^2\gamma_1 + \beta + \mu_2^2\gamma_2} \right) p_2 \end{aligned} \right\}\end{aligned}$$

$$= \left\{ \begin{array}{l} \left( \frac{\alpha + \mu_1^2 \gamma_1 - \lambda_2 \mu_2 \gamma_2}{\alpha + \mu_1^2 \gamma_1 + \beta + \mu_2^2 \gamma_2} \right) \left( \frac{\alpha - \mu_1 \gamma_1 \lambda_1}{\alpha + \mu_1^2 \gamma_1} \right) y \\ \left( \frac{\alpha + \mu_1^2 \gamma_1 - \lambda_2 \mu_2 \gamma_2}{\alpha + \mu_1^2 \gamma_1 + \beta + \mu_2^2 \gamma_2} \right) \frac{\mu_1 \gamma_1}{\kappa_1 (\alpha + \mu_1^2 \gamma_1)} p_1 \\ + \left( \frac{\beta}{\alpha + \mu_1^2 \gamma_1 + \beta + \mu_2^2 \gamma_2} \right) x_i \\ + \left( \frac{\frac{\mu_2 \gamma_2}{\kappa_2}}{\alpha + \mu_1^2 \gamma_1 + \beta + \mu_2^2 \gamma_2} \right) p_2 \end{array} \right\}$$

and precision

$$\begin{aligned} & \bar{\alpha}_2 + \beta + \mu_2^2 \gamma_2 \\ = & \alpha + \mu_1^2 \gamma_1 + \beta + \mu_2^2 \gamma_2. \end{aligned}$$

STEP 3: SOLVING FOR SECOND PERIOD PRICES

Individual  $i$ 's demand for the asset will be

$$\tau \left( \bar{\alpha}_2 + \beta + \mu_2^2 \gamma_2 \right) \left[ \left( \frac{\bar{\alpha}_2 - \lambda_2 \mu_2 \gamma_2}{\bar{\alpha}_2 + \beta + \mu_2^2 \gamma_2} \right) \bar{y}_2 + \left( \frac{\beta}{\bar{\alpha}_2 + \beta + \mu_2^2 \gamma_2} \right) x_i + \left( \frac{\frac{\mu_2 \gamma_2}{\kappa_2}}{\bar{\alpha}_2 + \beta + \mu_2^2 \gamma_2} \right) p_2 - p_2 \right]$$

Collecting terms and simplifying, we have

$$\tau \left[ (\bar{\alpha}_2 - \lambda_2 \mu_2 \gamma_2) \bar{y}_2 + \beta x_i - \left( \bar{\alpha}_2 + \beta + \mu_2 \gamma_2 \left( \mu_2 - \frac{1}{\kappa_2} \right) \right) p_2 \right]$$

Total demand for the asset will be

$$\tau \left[ (\bar{\alpha}_2 - \lambda_2 \mu_2 \gamma_2) \bar{y}_2 + \beta \theta - \left( \bar{\alpha}_2 + \beta + \mu_2 \gamma_2 \left( \mu_2 - \frac{1}{\kappa_2} \right) \right) p_2 \right]$$

Market clearing implies that this equals  $s_2$ , i.e., rearranging,

$$p_2 = \frac{(\bar{\alpha}_2 - \lambda_2 \mu_2 \gamma_2) \bar{y}_2 + \beta \theta - \frac{1}{\tau} s_2}{\bar{\alpha}_2 + \beta + \mu_2 \gamma_2 \left( \mu_2 - \frac{1}{\kappa_2} \right)}$$

So

$$\mu_2 = \tau \beta \tag{5.5}$$

$$\lambda_2 = \frac{\bar{\alpha}_2 \tau}{1 + \beta \gamma_2 \tau^2} = \frac{(\alpha + \mu_1^2 \gamma_1) \tau}{1 + \beta \gamma_2 \tau^2} \tag{5.6}$$

$$\kappa_2 = \frac{\frac{1}{\tau} + \tau \beta \gamma_2}{\bar{\alpha}_2 + \beta + \tau^2 \beta^2 \gamma_2} = \frac{\frac{1}{\tau} + \tau \beta \gamma_2}{\alpha + \mu_1^2 \gamma_1 + \beta + \tau^2 \beta^2 \gamma_2} \tag{5.7}$$

This implies that the second period price is normally distributed with mean

$$\frac{\alpha + \mu_1^2 \gamma_1}{\alpha + \mu_1^2 \gamma_1 + \beta + \tau^2 \beta^2 \gamma_2} y + \frac{\beta (1 + \tau^2 \beta \gamma_2)}{\alpha + \mu_1^2 \gamma_1 + \beta + \tau^2 \beta^2 \gamma_2} \theta$$

and variance

$$\left( \frac{\frac{1}{\tau} + \tau \beta \gamma_2}{\alpha + \mu_1^2 \gamma_1 + \beta + \tau^2 \beta^2 \gamma_2} \right)^2 \frac{1}{\gamma_2}.$$

Define  $z$  as

$$z \equiv \frac{\beta (1 + \tau^2 \beta \gamma_2)}{\alpha + \mu_1^2 \gamma_1 + \beta + \tau^2 \beta^2 \gamma_2}$$

Then,  $p_2$  can be written as a linear combination of  $y$ ,  $\theta$  and  $s_2$  where

$$p_2 = (1 - z) y + z \theta - \frac{s_2}{\tau \left( \bar{\alpha}_2 + \beta + \mu_2 \gamma_2 \left( \mu_2 - \frac{1}{\kappa_2} \right) \right)} \quad (5.8)$$

Integrating out the supply shock  $s_2$ , we have that the mean of  $p_2$  is

$$(1 - z) y + z \theta$$

as claimed in proposition 3.1.

#### STEP 4: SOLVING FOR FIRST PERIOD PRICES

For the short-lived trader, the demand for the asset in period 1 is

$$\frac{\tau (E_{1i}(p_2) - p_1)}{\text{Var}_{1i}(p_2)}$$

where  $E_{1i}(p_2)$  is the  $i$ 's conditional expectation of  $p_2$  at date 1 and  $\text{Var}_{1i}(p_2)$  is  $i$ 's conditional variance of  $p_2$  at date 1. From (5.8), trader  $i$ 's demand is given by

$$\begin{aligned} & \frac{\tau (z E_{1i}(\theta) + (1 - z) y - p_1)}{\text{Var}_{1i}(p_2)} \\ &= \frac{\tau}{\text{Var}_{1i}(p_2)} \left( z \left( \frac{(\alpha - \mu_1 \lambda_1 \gamma_1) y + \beta x_i + \frac{\mu_1 \gamma_1}{\kappa_1} p_1}{1/\text{Var}_{i1}(\theta)} \right) + (1 - z) y - p_1 \right) \end{aligned}$$

where

$$\text{Var}_{1i}(\theta) = \frac{1}{\alpha + \beta + \mu_1^2 \gamma_1}$$



is  $i$ 's conditional variance of  $\theta$  based on information at date 1. The conditional variances  $\text{Var}_{1i}(p_2)$  and  $\text{Var}_{1i}(\theta)$  are identical across traders, and so we can write them simply as  $\text{Var}_1(p_2)$  and  $\text{Var}_1(\theta)$ . Integrating over all traders, the aggregate demand is given by

$$\frac{\tau \text{Var}_1(\theta)}{\text{Var}_1(p_2)} \left\{ \begin{array}{l} y [z(\alpha - \mu_1 \lambda_1 \gamma_1) + (1-z)(\alpha + \beta + \mu_1^2 \gamma_1)] \\ + z \beta \theta \\ - p_1 [(\alpha + \beta + \mu_1^2 \gamma_1) - z \frac{\mu_1 \gamma_1}{\kappa_1}] \end{array} \right\}$$

Market clearing implies that this is equal to  $s_1$ . Rearranging in terms of  $p_1$  and comparing coefficients with (5.2) we have

$$\begin{aligned} \kappa_1 \mu_1 &= z \left( \frac{\beta + \mu_1^2 \gamma_1}{\alpha + \beta + \mu_1^2 \gamma_1} \right) \\ \kappa_1 \lambda_1 &= 1 - z \left( \frac{\beta + \mu_1^2 \gamma_1}{\alpha + \beta + \mu_1^2 \gamma_1} \right) \\ \kappa_1 &= \frac{\text{Var}_1(p_2)}{\tau \text{Var}_1(\theta) [(\alpha + \beta + \mu_1^2 \gamma_1) - z \frac{\mu_1 \gamma_1}{\kappa_1}]} \end{aligned}$$

Thus, defining

$$w \equiv \frac{\beta + \mu_1^2 \gamma_1}{\alpha + \beta + \mu_1^2 \gamma_1}$$

we can express first period price as a linear combination of  $y$ ,  $\theta$  and  $s_1$  where

$$p_1 = (1 - wz)y + wz\theta - s_1 \frac{\text{Var}_1(p_2)}{\tau \text{Var}_1(\theta) [(\alpha + \beta + \mu_1^2 \gamma_1) - z \frac{\mu_1 \gamma_1}{\kappa_1}]}$$

Integrating out the supply noise  $s_1$ , we have

$$E_s(p_1) = (1 - wz)y + wz\theta$$

as claimed in proposition 3.1.

To complete the proof of proposition 3.1, note that as the supply noise becomes large we have  $\gamma_1 \rightarrow 0$  and  $\gamma_2 \rightarrow 0$ , while the informativeness of first period prices given by  $\mu_1$  also falls. Hence

$$\begin{aligned} w &\rightarrow \frac{\beta}{\alpha + \beta} \\ z &\rightarrow \frac{\beta}{\alpha + \beta} \end{aligned}$$

so that we obtain the limiting results in proposition 3.1.

### The long lived trader model

Now consider the long-lived trader model. The analysis is unchanged until we derive the first period prices (step 4). Now trader's anticipations of their asset purchases in period two create "hedging demand". To deduce first period demand, we need to know trader  $i$ 's beliefs about the joint distribution of  $p_2$  and  $E_{2i}(\theta)$  at date 1. Letting

$$\eta_i = \theta - E_{1i}(\theta)$$

we have

$$\begin{aligned} p_2 &= \kappa_2 (\lambda_2 \bar{y}_2 + \mu_2 \theta - s_2) \\ &= \kappa_2 \left( \begin{array}{l} \lambda_2 \left( \frac{(\alpha - \mu_1 \gamma_1 \lambda_1) y + \frac{\mu_1 \gamma_1}{\kappa_1} p_1}{\alpha + \mu_1^2 \gamma_1} \right) \\ + \mu_2 \left( \frac{\alpha - \mu_1 \gamma_1 \lambda_1}{\alpha + \beta + \mu_1^2 \gamma_1} y + \frac{\beta}{\alpha + \beta + \mu_1^2 \gamma_1} x_i + \frac{\frac{\mu_1 \gamma_1}{\kappa_1} p_1 + \eta_i}{\alpha + \beta + \mu_1^2 \gamma_1} \right) \\ - s_2 \end{array} \right) \\ &= \left\{ \begin{array}{l} \kappa_2 \left( \lambda_2 \left( \frac{\alpha - \mu_1 \gamma_1 \lambda_1}{\alpha + \mu_1^2 \gamma_1} \right) + \mu_2 \left( \frac{\alpha - \mu_1 \gamma_1 \lambda_1}{\alpha + \beta + \mu_1^2 \gamma_1} \right) \right) y \\ + \kappa_2 \mu_2 \left( \frac{\beta}{\alpha + \beta + \mu_1^2 \gamma_1} \right) x_i + \\ \kappa_2 \left( \lambda_2 \left( \frac{\frac{\mu_1 \gamma_1}{\kappa_1}}{\alpha + \mu_1^2 \gamma_1} \right) + \mu_2 \left( \frac{\frac{\mu_1 \gamma_1}{\kappa_1}}{\alpha + \beta + \mu_1^2 \gamma_1} \right) \right) p_1 \\ + \kappa_2 \mu_2 \eta_i - \kappa_2 s_2 \end{array} \right\} \end{aligned} \quad (5.9)$$

So trader  $i$ 's expected value of  $p_2$  at date 1 is

$$E_{1i}(p_2) = \left\{ \begin{array}{l} \kappa_2 \left( \lambda_2 \left( \frac{\alpha - \mu_1 \gamma_1 \lambda_1}{\alpha + \mu_1^2 \gamma_1} \right) + \mu_2 \left( \frac{\alpha - \mu_1 \gamma_1 \lambda_1}{\alpha + \beta + \mu_1^2 \gamma_1} \right) \right) y \\ + \kappa_2 \mu_2 \left( \frac{\beta}{\alpha + \beta + \mu_1^2 \gamma_1} \right) x_i + \\ \kappa_2 \left( \lambda_2 \left( \frac{\frac{\mu_1 \gamma_1}{\kappa_1}}{\alpha + \mu_1^2 \gamma_1} \right) + \mu_2 \left( \frac{\frac{\mu_1 \gamma_1}{\kappa_1}}{\alpha + \beta + \mu_1^2 \gamma_1} \right) \right) p_1 \end{array} \right\}.$$

Recall from (5.4) that trader  $i$ 's expected value of  $\theta$  at period two will be

$$E_{2i}(\theta) = \left\{ \begin{array}{l} \left( \frac{\alpha + \mu_1^2 \gamma_1 - \lambda_2 \mu_2 \gamma_2}{\alpha + \mu_1^2 \gamma_1 + \beta + \mu_2^2 \gamma_2} \right) \left( \frac{\alpha - \mu_1 \gamma_1 \lambda_1}{\alpha + \mu_1^2 \gamma_1} \right) y \\ \left( \frac{\alpha + \mu_1^2 \gamma_1 - \lambda_2 \mu_2 \gamma_2}{\alpha + \mu_1^2 \gamma_1 + \beta + \mu_2^2 \gamma_2} \right) \frac{\mu_1 \gamma_1}{\kappa_1 (\alpha + \mu_1^2 \gamma_1)} p_1 \\ + \left( \frac{\beta}{\alpha + \mu_1^2 \gamma_1 + \beta + \mu_2^2 \gamma_2} \right) x_i \\ + \left( \frac{\frac{\mu_2 \gamma_2}{\kappa_2}}{\alpha + \mu_1^2 \gamma_1 + \beta + \mu_2^2 \gamma_2} \right) p_2 \end{array} \right\}. \quad (5.10)$$

The expected value of the expected value of  $\theta$  at period 2 is

$$E_{1i}(E_{2i}(\theta)) = \left\{ \begin{array}{l} \left( \frac{(\alpha + \mu_1^2 \gamma_1 - \lambda_2 \mu_2 \gamma_2)}{(\alpha + \mu_1^2 \gamma_1 + \beta + \mu_2^2 \gamma_2)} \left( \frac{\alpha - \mu_1 \gamma_1 \lambda_1}{\alpha + \mu_1^2 \gamma_1} \right) y \right. \\ \left. \left( \frac{(\alpha + \mu_1^2 \gamma_1 - \lambda_2 \mu_2 \gamma_2)}{(\alpha + \mu_1^2 \gamma_1 + \beta + \mu_2^2 \gamma_2)} \frac{\mu_1 \gamma_1}{\kappa_1 (\alpha + \mu_1^2 \gamma_1)} p_1 \right) \right. \\ \left. + \left( \frac{\beta}{\alpha + \mu_1^2 \gamma_1 + \beta + \mu_2^2 \gamma_2} \right) x_i \right. \\ \left. + \left( \frac{\frac{\mu_2 \gamma_2}{\kappa_2}}{\alpha + \mu_1^2 \gamma_1 + \beta + \mu_2^2 \gamma_2} \right) \left\{ \begin{array}{l} \kappa_2 \left( \lambda_2 \left( \frac{\alpha - \mu_1 \gamma_1 \lambda_1}{\alpha + \mu_1^2 \gamma_1} \right) + \mu_2 \left( \frac{\alpha - \mu_1 \gamma_1 \lambda_1}{\alpha + \beta + \mu_1^2 \gamma_1} \right) \right) y \right. \\ \left. + \kappa_2 \mu_2 \left( \frac{\beta}{\alpha + \beta + \mu_1^2 \gamma_1} \right) x_i + \right. \\ \left. \kappa_2 \left( \lambda_2 \left( \frac{\frac{\mu_1 \gamma_1}{\kappa_1}}{\alpha + \mu_1^2 \gamma_1} \right) + \mu_2 \left( \frac{\frac{\mu_1 \gamma_1}{\kappa_1}}{\alpha + \beta + \mu_1^2 \gamma_1} \right) \right) p_1 \right\} \right. \end{array} \right\}. \quad (5.11)$$

This equals:

$$\left\{ \begin{array}{l} \left[ \frac{\alpha + \mu_1^2 \gamma_1 - \lambda_2 \mu_2 \gamma_2}{\alpha + \mu_1^2 \gamma_1 + \beta + \mu_2^2 \gamma_2} \frac{\alpha - \mu_1 \gamma_1 \lambda_1}{\alpha + \mu_1^2 \gamma_1} + \left( \frac{\frac{\mu_2 \gamma_2}{\kappa_2}}{\alpha + \mu_1^2 \gamma_1 + \beta + \mu_2^2 \gamma_2} \right) \kappa_2 \left( \lambda_2 \frac{\alpha - \mu_1 \gamma_1 \lambda_1}{\alpha + \mu_1^2 \gamma_1} + \mu_2 \frac{\alpha - \mu_1 \gamma_1 \lambda_1}{\alpha + \beta + \mu_1^2 \gamma_1} \right) \right] y \\ + \left[ \frac{\alpha + \mu_1^2 \gamma_1 - \lambda_2 \mu_2 \gamma_2}{\alpha + \mu_1^2 \gamma_1 + \beta + \mu_2^2 \gamma_2} \frac{\mu_1 \gamma_1}{\kappa_1 (\alpha + \mu_1^2 \gamma_1)} + \left( \frac{\frac{\mu_2 \gamma_2}{\kappa_2}}{\alpha + \mu_1^2 \gamma_1 + \beta + \mu_2^2 \gamma_2} \right) \kappa_2 \left( \lambda_2 \frac{\frac{\mu_1 \gamma_1}{\kappa_1}}{\alpha + \mu_1^2 \gamma_1} + \mu_2 \frac{\frac{\mu_1 \gamma_1}{\kappa_1}}{\alpha + \beta + \mu_1^2 \gamma_1} \right) \right] p_1 \\ + \left[ \frac{\beta}{\alpha + \mu_1^2 \gamma_1 + \beta + \mu_2^2 \gamma_2} + \left( \frac{\frac{\mu_2 \gamma_2}{\kappa_2}}{\alpha + \mu_1^2 \gamma_1 + \beta + \mu_2^2 \gamma_2} \right) \kappa_2 \mu_2 \left( \frac{\beta}{\alpha + \beta + \mu_1^2 \gamma_1} \right) \right] x_i \end{array} \right\}$$

Now  $E_{1i}(p_2) - p_1$  equals

$$\begin{aligned} & \left\{ \begin{array}{l} \kappa_2 \left( \lambda_2 \left( \frac{\alpha - \mu_1 \gamma_1 \lambda_1}{\alpha + \mu_1^2 \gamma_1} \right) + \mu_2 \left( \frac{\alpha - \mu_1 \gamma_1 \lambda_1}{\alpha + \beta + \mu_1^2 \gamma_1} \right) \right) y \\ + \kappa_2 \mu_2 \left( \frac{\beta}{\alpha + \beta + \mu_1^2 \gamma_1} \right) x_i + \\ \kappa_2 \left( \lambda_2 \left( \frac{\frac{\mu_1 \gamma_1}{\kappa_1}}{\alpha + \mu_1^2 \gamma_1} \right) + \mu_2 \left( \frac{\frac{\mu_1 \gamma_1}{\kappa_1}}{\alpha + \beta + \mu_1^2 \gamma_1} \right) \right) p_1 \end{array} \right\} - p_1 \quad (5.12) \\ & = \left\{ \begin{array}{l} \kappa_2 \left( \lambda_2 \left( \frac{\alpha - \mu_1 \gamma_1 \lambda_1}{\alpha + \mu_1^2 \gamma_1} \right) + \mu_2 \left( \frac{\alpha - \mu_1 \gamma_1 \lambda_1}{\alpha + \beta + \mu_1^2 \gamma_1} \right) \right) y \\ + \kappa_2 \mu_2 \left( \frac{\beta}{\alpha + \beta + \mu_1^2 \gamma_1} \right) x_i + \\ - \left[ 1 - \kappa_2 \left( \lambda_2 \left( \frac{\frac{\mu_1 \gamma_1}{\kappa_1}}{\alpha + \mu_1^2 \gamma_1} \right) + \mu_2 \left( \frac{\frac{\mu_1 \gamma_1}{\kappa_1}}{\alpha + \beta + \mu_1^2 \gamma_1} \right) \right) \right] p_1 \end{array} \right\} \end{aligned}$$

and  $E_{1i}(E_{2i}(\theta)) - E_{1i}(p_2)$  equals

$$\begin{aligned}
& \left\{ \begin{aligned} & \left[ \left( \frac{\alpha + \mu_1^2 \gamma_1 - \lambda_2 \mu_2 \gamma_2}{\alpha + \mu_1^2 \gamma_1 + \beta + \mu_2^2 \gamma_2} \frac{\alpha - \mu_1 \gamma_1 \lambda_1}{\alpha + \mu_1^2 \gamma_1} + \left( \frac{\mu_2 \gamma_2}{\kappa_2} \right) \kappa_2 \left( \lambda_2 \frac{\alpha - \mu_1 \gamma_1 \lambda_1}{\alpha + \mu_1^2 \gamma_1} + \mu_2 \frac{\alpha - \mu_1 \gamma_1 \lambda_1}{\alpha + \beta + \mu_1^2 \gamma_1} \right) \right] y \\ & + \left[ \left( \frac{\alpha + \mu_1^2 \gamma_1 - \lambda_2 \mu_2 \gamma_2}{\alpha + \mu_1^2 \gamma_1 + \beta + \mu_2^2 \gamma_2} \right) \frac{\mu_1 \gamma_1}{\kappa_1 (\alpha + \mu_1^2 \gamma_1)} + \left( \frac{\mu_2 \gamma_2}{\kappa_2} \right) \kappa_2 \left( \lambda_2 \frac{\mu_1 \gamma_1}{\alpha + \mu_1^2 \gamma_1} + \mu_2 \frac{\mu_1 \gamma_1}{\alpha + \beta + \mu_1^2 \gamma_1} \right) \right] p_1 \\ & + \left[ \left( \frac{\beta}{\alpha + \mu_1^2 \gamma_1 + \beta + \mu_2^2 \gamma_2} \right) + \left( \frac{\mu_2 \gamma_2}{\kappa_2} \right) \kappa_2 \mu_2 \left( \frac{\beta}{\alpha + \beta + \mu_1^2 \gamma_1} \right) \right] x_i \end{aligned} \right\} \\
& - \left\{ \begin{aligned} & \left[ \kappa_2 \left( \lambda_2 \left( \frac{\alpha - \mu_1 \gamma_1 \lambda_1}{\alpha + \mu_1^2 \gamma_1} \right) + \mu_2 \left( \frac{\alpha - \mu_1 \gamma_1 \lambda_1}{\alpha + \beta + \mu_1^2 \gamma_1} \right) \right) y \right] \\ & + \left[ \kappa_2 \mu_2 \left( \frac{\beta}{\alpha + \beta + \mu_1^2 \gamma_1} \right) x_i + \right. \\ & \left. \kappa_2 \left( \lambda_2 \left( \frac{\mu_1 \gamma_1}{\alpha + \mu_1^2 \gamma_1} \right) + \mu_2 \left( \frac{\mu_1 \gamma_1}{\alpha + \beta + \mu_1^2 \gamma_1} \right) \right) p_1 \right] \end{aligned} \right\} \\
= & \left\{ \begin{aligned} & \left[ \left( \frac{\alpha + \mu_1^2 \gamma_1 - \lambda_2 \mu_2 \gamma_2}{\alpha + \mu_1^2 \gamma_1 + \beta + \mu_2^2 \gamma_2} \right) \left( \frac{\alpha - \mu_1 \gamma_1 \lambda_1}{\alpha + \mu_1^2 \gamma_1} \right) \right. \\ & \left. + \left( \left( \frac{\mu_2 \gamma_2}{\kappa_2} \right) - 1 \right) \kappa_2 \left( \lambda_2 \left( \frac{\alpha - \mu_1 \gamma_1 \lambda_1}{\alpha + \mu_1^2 \gamma_1} \right) + \mu_2 \left( \frac{\alpha - \mu_1 \gamma_1 \lambda_1}{\alpha + \beta + \mu_1^2 \gamma_1} \right) \right) \right] y \\ & + \left[ \left( \frac{\alpha + \mu_1^2 \gamma_1 - \lambda_2 \mu_2 \gamma_2}{\alpha + \mu_1^2 \gamma_1 + \beta + \mu_2^2 \gamma_2} \right) \frac{\mu_1 \gamma_1}{\kappa_1 (\alpha + \mu_1^2 \gamma_1)} \right. \\ & \left. + \left( \left( \frac{\mu_2 \gamma_2}{\kappa_2} \right) - 1 \right) \kappa_2 \left( \lambda_2 \left( \frac{\mu_1 \gamma_1}{\alpha + \mu_1^2 \gamma_1} \right) + \mu_2 \left( \frac{\mu_1 \gamma_1}{\alpha + \beta + \mu_1^2 \gamma_1} \right) \right) \right] p_1 \\ & + \left[ \left( \frac{\beta}{\alpha + \mu_1^2 \gamma_1 + \beta + \mu_2^2 \gamma_2} \right) \right. \\ & \left. + \left( \left( \frac{\mu_2 \gamma_2}{\kappa_2} \right) - 1 \right) \kappa_2 \mu_2 \left( \frac{\beta}{\alpha + \beta + \mu_1^2 \gamma_1} \right) \right] x_i \end{aligned} \right\}
\end{aligned} \tag{5}$$

Now the variance of  $\theta$  at period 2 will be

$$\xi = \frac{1}{\alpha + \mu_1^2 \gamma_1 + \beta + \mu_2^2 \gamma_2}$$

The variance of  $p_2$  for a trader in period 1 is

$$\zeta = \frac{\kappa_2^2 \mu_2^2}{\alpha + \beta + \mu_1^2 \gamma_1} + \frac{\kappa_2^2}{\gamma_2}$$

At period 1,  $E_{2i}(\theta)$  is perfectly correlated with  $p_2$ . The variance of  $E_{2i}(\theta)$  is equal to  $\psi^2$  times the variance of  $p_2$ , where

$$\psi = \frac{\frac{\mu_2 \gamma_2}{\kappa_2}}{\alpha + \mu_1^2 \gamma_1 + \beta + \mu_2^2 \gamma_2}$$

Using the formula in Brown and Jennings (1989) (see also Brunnermeier (2001), page 110), trader  $i$ 's demand for the asset will be

$$\tau \left[ \left( \frac{1}{\zeta} + \frac{(1 - \psi)^2}{\xi} \right) (E_{1i}(p_2) - p_1) + \left( \frac{1 - \psi}{\xi} \right) (E_{1i}(E_{2i}(\theta)) - E_{1i}(p_2)) \right]$$

Now observe that as  $\gamma_1 \rightarrow 0$  and  $\gamma_2 \rightarrow 0$ , we have  $\xi \rightarrow \frac{1}{\alpha+\beta}$ ,  $\zeta \rightarrow \frac{\kappa_2^2}{\gamma_2}$  and  $\psi \rightarrow \frac{\mu_2 \gamma_2}{\alpha+\beta}$ . So  $\frac{1}{\xi} + \frac{(1-\psi)^2}{\xi} \rightarrow \alpha + \beta$  and  $\frac{1-\psi}{\xi} \rightarrow \alpha + \beta$ . Also observe from (5.12) and (5.13), that as  $\gamma_1 \rightarrow 0$  and  $\gamma_2 \rightarrow 0$ ,

$$\begin{aligned} E_{1i}(p_2) - p_1 &\rightarrow \left( \left( 1 - \left( \frac{\beta}{\alpha + \beta} \right)^2 \right) y + \left( \frac{\beta}{\alpha + \beta} \right)^2 x_i \right) - p_1 \\ E_{1i}(E_{2i}(\theta)) - E_{1i}(p_2) &\rightarrow \left( \left( \frac{\alpha}{\alpha + \beta} \right) y + \left( \frac{\beta}{\alpha + \beta} \right) x_i \right) - \left( \left( 1 - \left( \frac{\beta}{\alpha + \beta} \right)^2 \right) y + \left( \frac{\beta}{\alpha + \beta} \right)^2 x_i \right) \end{aligned}$$

Thus total demand for the asset is

$$\tau(\alpha + \beta) \left( \left( \frac{\alpha}{\alpha + \beta} \right) y + \left( \frac{\beta}{\alpha + \beta} \right) \theta - p_1 \right).$$

This is the same as in period two, so we get the same distribution of  $p_1$ .

## APPENDIX B

In this appendix, we provide a proof of proposition 4.1. The proof is by induction. First observe that

$$E_{iT}(\theta) = \frac{\beta x_i + \sum_{\tau=0}^T \alpha_\tau y_\tau}{\beta + \sum_{\tau=0}^T \alpha_\tau}$$

so

$$I_{T,T}(\theta) = \bar{E}_T(\theta) = \frac{\beta \theta + \sum_{\tau=0}^T \alpha_\tau y_\tau}{\beta + \sum_{\tau=0}^T \alpha_\tau}$$

This shows that (4.1) holds for  $t = T$ . Now we argue by backward induction that (4.1) holds for  $t = 0, \dots, T - 1$ . Suppose that (4.1) holds for  $t \geq 1$ .

$$E_{i,t-1}(I_{t,T}(\theta)) = E_{i,t-1} \left( (\lambda_{t,T}) \theta + (1 - \lambda_{t,T}) \left( \frac{\alpha_t y_t}{\sum_{\tau=0}^t \alpha_\tau} \right) + (1 - \lambda_{t,T}) \left( \frac{\sum_{\tau=0}^{t-1} \alpha_\tau y_\tau}{\sum_{\tau=0}^t \alpha_\tau} \right) \right)$$

$$\begin{aligned}
&= \left( (\lambda_{t,T}) + (1 - \lambda_{t,T}) \left( \frac{\alpha_t}{\sum_{\tau=0}^t \alpha_\tau} \right) \right) E_{i,t-1}(\theta) + (1 - \lambda_{t,T}) \left( \frac{\sum_{\tau=0}^{t-1} \alpha_\tau y_\tau}{\sum_{\tau=0}^t \alpha_\tau} \right) \\
&= \left\{ \left( (\lambda_{t,T}) + (1 - \lambda_{t,T}) \left( \frac{\alpha_t}{\sum_{\tau=0}^t \alpha_\tau} \right) \right) \left( \frac{\beta x_i + \sum_{\tau=0}^{t-1} \alpha_\tau y_\tau}{\beta + \sum_{\tau=0}^{t-1} \alpha_\tau} \right) \right. \\
&\quad \left. + (1 - \lambda_{t,T}) \left( \frac{\sum_{\tau=0}^{t-1} \alpha_\tau y_\tau}{\sum_{\tau=0}^t \alpha_\tau} \right) \right\} \\
&= \left\{ \left( (\lambda_{t,T}) + (1 - \lambda_{t,T}) \left( \frac{\alpha_t}{\sum_{\tau=0}^t \alpha_\tau} \right) \right) \left( \frac{\beta}{\beta + \sum_{\tau=0}^{t-1} \alpha_\tau} \right) x_i \right. \\
&\quad \left. + \left( 1 - \left( (\lambda_{t,T}) + (1 - \lambda_{t,T}) \left( \frac{\alpha_t}{\sum_{\tau=0}^t \alpha_\tau} \right) \right) \right) \left( \frac{\beta}{\beta + \sum_{\tau=0}^{t-1} \alpha_\tau} \right) \frac{\sum_{\tau=0}^{t-1} \alpha_\tau y_\tau}{\sum_{\tau=0}^{t-1} \alpha_\tau} \right\}
\end{aligned}$$

Thus

$$\begin{aligned}
I_{t-1,T}(\theta) &= \bar{E}_{t-1}(I_{t,T}(\theta)) \\
&= \left\{ \left( (\lambda_{t,T}) + (1 - \lambda_{t,T}) \left( \frac{\alpha_t}{\sum_{\tau=0}^t \alpha_\tau} \right) \right) \left( \frac{\beta}{\beta + \sum_{\tau=0}^{t-1} \alpha_\tau} \right) \theta \right. \\
&\quad \left. + \left( 1 - \left( (\lambda_{t,T}) + (1 - \lambda_{t,T}) \left( \frac{\alpha_t}{\sum_{\tau=0}^t \alpha_\tau} \right) \right) \right) \left( \frac{\beta}{\beta + \sum_{\tau=0}^{t-1} \alpha_\tau} \right) \frac{\sum_{\tau=0}^{t-1} \alpha_\tau y_\tau}{\sum_{\tau=0}^{t-1} \alpha_\tau} \right\}
\end{aligned}$$

So

$$\lambda_{t-1,T} = \left( \frac{\alpha_t}{\sum_{\tau=0}^t \alpha_\tau} \right) \left( \frac{\beta}{\beta + \sum_{\tau=0}^{t-1} \alpha_\tau} \right) + (\lambda_{t,T}) \left( \frac{\sum_{\tau=0}^{t-1} \alpha_\tau}{\sum_{\tau=0}^t \alpha_\tau} \right) \left( \frac{\beta}{\beta + \sum_{\tau=0}^{t-1} \alpha_\tau} \right)$$

$$\begin{aligned}
&= \left\{ \begin{aligned} &\frac{\alpha_t}{\sum_{\tau=0}^t \alpha_\tau} \frac{\beta}{\beta + \sum_{\tau=0}^{t-1} \alpha_\tau} \\ &+ \left[ \frac{\sum_{\tau=0}^{t-1} \alpha_\tau}{\sum_{\tau=0}^t \alpha_\tau} \frac{\beta}{\beta + \sum_{\tau=0}^{t-1} \alpha_\tau} \right] \left( \sum_{i=t}^{T-1} \left[ \frac{\sum_{\tau=0}^t \alpha_\tau}{\sum_{\tau=0}^i \alpha_\tau} \frac{\alpha_{i+1}}{\sum_{\tau=0}^{i+1} \alpha_\tau} \prod_{j=t}^i \frac{\beta}{\sum_{\tau=0}^j \alpha_\tau + \beta} \right] + \frac{\sum_{\tau=0}^t \alpha_\tau}{\sum_{\tau=0}^T \alpha_\tau} \prod_{j=t}^T \frac{\beta}{\sum_{\tau=0}^j \alpha_\tau + \beta} \right) \end{aligned} \right\} \\
&= \left( \sum_{i=t-1}^{T-1} \left[ \frac{\sum_{\tau=0}^{t-1} \alpha_\tau}{\sum_{\tau=0}^i \alpha_\tau} \frac{\alpha_{i+1}}{\sum_{\tau=0}^{i+1} \alpha_\tau} \prod_{j=t-1}^i \frac{\beta}{\sum_{\tau=0}^j \alpha_\tau + \beta} \right] + \frac{\sum_{\tau=0}^{t-1} \alpha_\tau}{\sum_{\tau=0}^T \alpha_\tau} \prod_{j=t-1}^T \frac{\beta}{\sum_{\tau=0}^j \alpha_\tau + \beta} \right)
\end{aligned}$$

This establishes that (4.1) holds for  $t - 1$ .

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