Bidding with Securities: Auctions and Security Design†

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ABSTRACT. We study *security-bid auctions* in which bidders compete for an asset by bidding with securities. That is, they offer payments that are contingent on the realized value of the asset being sold. The value depends on investment by the winner, thus creating the possibility of moral hazard. Such auctions are commonly used, both formally and informally. In formal auctions, the seller generally restricts bids to an ordered set, such as an equity share or royalty rate. By restricting the bids, standard auction formats such as first and second-price auctions can be used. In informal settings with competing buyers, the seller does not commit to a mechanism upfront. Rather, bidders offer securities and the seller chooses the most attractive bid, based on his beliefs, ex-post.

We characterize equilibrium payoffs and bidding strategies in this setting, and show that an informal mechanism yields the lowest possible revenues across all mechanisms. We also determine the optimal formal mechanism, and show that the security design is more important than the auction format. We show that the revenue equivalence principle (that expected revenues are independent of the auction format) holds if the set of permissible securities is ordered and convex (such as equity). Otherwise, it need not hold. For example, when bidders offer standard debt securities, a second-price auction is superior. On the other hand, if bidders compete on the conversion ratio of convertible debt, a first-price auction yields higher revenues. Finally, we examine how different forms of moral hazard impact our results.

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1. Introduction

Auction theory and its applications have become increasingly important as an area of economic research over the last twenty years. As a result, we now have a better understanding of how the structure of an auction affects its outcome. Almost all the existing literature studies the case when bidders use cash payments, so that the value of a bid is not contingent on future events.

In a few cases, such as art auctions, the realized value is subjective and cannot be used as a basis for payment; however, this is the exception. In many important applications, the realization of the future cash flow generated by the asset can be used in determining the actual payment. That is, the bids can be securities whose values are derived from the future cash flow. We call this setting a *security-bid auction*, and show that for these auctions, the design of the securities can be more important than design of the auction itself.

Formal auctions of this type are commonly used in government sales of oil leases, wireless spectrum, highway building contracts, and lead-plaintiff auctions. Informal auctions of this type (in the sense that formal auction rules are not set forth in advance) are common in the private sector. Examples include authors selling publishing rights, entrepreneurs selling their firm to an acquirer or soliciting venture capital, and sports associations selling broadcasting rights.[1]

The major difference between a formal and an informal mechanism is the level of commitment by the seller. In an informal mechanism, bidders choose which securities to offer, and the seller selects the most attractive offer ex-post. In this case, the auction contains the elements of a signaling game because the seller may infer bidders' private information from their security choices when evaluating their offers. In a formal mechanism the seller restricts bidders to use securities from a pre-specified ordered set, such as an equity share or royalty rate. The seller is committed not to accept an offer outside this set. The seller also commits to an auction format, such as a first or second-price auction. One of our main results is that the revenues from an informal mechanism are the lowest across a large set of possible mechanisms. In other words, the seller benefits from *any* form of commitment. Moreover, we show how to rank security designs and auction formats in terms of their impact on the seller's revenues.

In our model, several agents compete for the right to undertake a project. Bidders are endowed with private signals regarding the value they can expect from the project. The structure is similar to an independent private values model, so that different bidders expect different payoffs upon winning. The model differs from standard auction models,

[1] See Hendricks and Porter (1988) for a discussion of oil lease auctions, in which royalty rates are commonly used. In wireless spectrum auctions, the bids are effectively debt securities (leading in some cases to default). Highway building contracts are often awarded through "build, operate, and transfer" agreements to the bidder that offers to charge the lowest toll for a pre-specified period. See Fisch (2001) for the use of contingency-fee auctions in the selection of the lead plaintiff in class action suits. In mergers, acquisitions and venture capital agreements, equity and other securities are commonly used (see Martin (1996)). McMillan (1991) describes the auction of the broadcast rights to the Olympic games, where bids contained revenue-sharing clauses. Similarly, publishing contracts include advance and royalty payments.

in that bids are securities. Bidders offer derivatives in which the underlying value is the future payoff of the project. Because the winner may make investments or take other actions that affect this future payoff, there is also the possibility of moral hazard.

One may conjecture that since there exists a cash value of each security, there may be equivalence between security-bid and cash auctions. That is, perhaps the results from standard auction theory carry over, replacing each security by its cash value. However, unlike cash bids, the value of a security bid depends upon the bidder's private information. This difference can have important consequences as the following simple example demonstrates:[2]

> Consider an auction in which two bidders, Alice and Bob, compete for a project. The project requires an initial fixed and non-random investment that is equivalent to \$1M. Alice expects that if she undertakes the project then on average it would yield revenues of \$3M; Bob expects that future revenues will equal only \$2M. Hence, Alice sees a profit of \$2M while Bob sees a profit of \$1M. Assuming these estimates are private values, in a standard second-price auction it is a dominant strategy for bidders to bid their reservation values. As a result, Alice would win the auction and pay Bob's bid, \$1M.
>
> Now suppose that rather than bidding with cash, the bidders compete by offering a fraction of the future revenues. As we later discuss, it is again a dominant strategy for bidders to bid their reservation values. Alice offers 2/3 of future revenues while Bob offers 1/2. As a result, Alice wins the auction and pays according to Bob's bid; that is, she gives up one-half of the future revenues. This yields a higher payoff for the auctioneer; $(1/2)\times\$3M = \$1.5M > \$1M$.

This example highlights the fact that the use of security bids has non-trivial implications even with a standard second-price auction. There are two key considerations in the design of a security-bid auction: the choice of the security and the format itself. An important conclusion of this paper is that while revenues may differ by the choice of format (e.g., first-price versus second-price) the choice of security design can have a much larger impact.

Another significant consideration in many applications is moral hazard, as the action of the winning bidder (and, in some applications, the seller) may impact the realized cash flow. This may restrict the choice of security design, for if the winner has a small stake in future payoff he may choose to under-invest in the project. For example, we show that if the bidder must make a non-contractible investment in the project, it will not be optimal for the seller to reimburse the bidder for the investment.

The structure of the paper is as follows. The basic model is described in Section 2. We begin our analysis in Section 3 by examining formal mechanisms which consist of both an auction format and a security design. There we establish the following results:

- We characterize super-modularity conditions under which a monotone – and hence efficient – equilibrium is the unique outcome for the first and second-price auctions.

[2] This example is based on Hansen (1985).

- First we compare security designs holding fixed the auction format (first or second-price). We show that for either format, the seller's expected revenues are positively related to the "steepness" of the securities. As a result, debt contracts minimize the seller's expected payoffs while call options maximize it.

- Fixing the security design, we then consider the role of the auction format. We show that the relationship between the security set and its convex hull determines the ranking between first and second-price formats. For *sub-convex sets* – which include, for example, the set of debt securities – we show that a second-price auction yields higher expected revenues than a first-price auction. Alternatively, if the set is *super-convex* (e.g., call options), the reverse conclusion holds and first-price auctions are superior. However, we find the effect of the auction format to be small relative to the security design.

- We then ask whether the Revenue Equivalence principle for cash auctions, which states that expected revenues are independent of the auction format, can be extended to security bid auctions. We show it holds if the ordered set of securities is convex. This is true for important classes of securities, such as equity.

- Finally we combine these results to show that the first-price auction with call options maximizes the seller's revenue, while the first-price format with debt minimizes it, *over a general set of auction mechanisms.*

In the second part of the paper (Section 4), we consider the case in which the seller is unable to commit ex-ante to formal auction mechanism. Instead he accepts all bids and chooses the security that is optimal ex-post. As mentioned above, in this case the task of selecting the winning bid is not trivial; it involves a signaling game in which the seller uses his beliefs to rank the different securities and choose the most attractive one. Our main result in this section is as follows:

- In the unique equilibrium satisfying standard refinements of off-equilibrium beliefs, bidders use only debt securities. Moreover, the outcome is equivalent to a first-price auction. As a result we conclude that this ex-post maximization yields the worst possible outcome for the seller!

The intuition is that debt provides the cheapest way for a high type to signal his quality. Thus, bidders find it optimal to compete using debt.

In the final sections of the paper we discuss bidding with combinations of cash and securities, as well as the introduction of moral hazard. We demonstrate that the main insights of our analysis carry over to these settings. In particular, we show that:

- If the bidder's investment in the project is unverifiable and subject to moral hazard, then it is not optimal for the seller to offer the winner cash compensation to the bidder for this investment.

- If bidders can combine cash payments with their bids, this effectively "flattens" their bids and reduces the expected revenues of the seller.

We complete the paper with a discussion of other forms of moral hazard, an extension of the model and results to incorporate reservation prices, and an extension of the model to allow for bidders to have correlated assessments of the asset's value.

Related Literature

Hansen (1985) was the first to examine the use of securities in an auction setting. He shows that a second-price mechanism that is based on equity payments yields higher expected revenues than a cash-based auction.

Riley (1988) and Rhodes-Kropf and Viswanathan (2000) focus on first-price auctions in a setting that is similar to the model we study in the first part of the paper. They show that securities yield higher revenues than a cash-based auction. However, this is conditional upon the existence of a separating equilibrium in which a higher type bids a higher security. For example in Rhodes-Kropf and Viswanathan (2000), there always exists a pooling equilibrium and in some cases it is the unique outcome. This is because they assume that the project does not require any costly inputs – thus the lowest type can offer 100% of the proceeds to the seller and breakeven. Thus, a low type is always willing to imitate the bid of a high type. We use a framework that is closer to Hansen (1985), in which the project requires costly inputs. In this case, we show that under certain conditions the first-price auction has a unique equilibrium, and it is separating.

One reason security-bid auctions may not have received greater attention in the literature is perhaps due to Cremer (1987), who argues that the seller can extract the entire surplus if he can "buy" the winning bidder. Specifically, the seller can offer cash to the bidder to cover the costs of any required investment, and ask all bidders to reveal their type. The seller then offers the project to highest type in exchange for its full value. Since bidders earn zero profits regardless, truthful reporting is incentive compatible.

Reimbursing the winning bidder is extremely fragile to the introduction of moral hazard. We show that if the bidder's investment is not verifiable, then the seller will not offer such a contract. If it were offered, bidders would all claim the highest type, collect the compensation, but then choose not to invest in the project. For bidders to invest, it must be that their compensation is contingent on the outcome of the project. Thus Cremer's approach is infeasible and the issue of security and auction design remains relevant.[3]

Board (2002), Che and Gale (2000), Rhodes-Kropf and Viswanathan (2002), and Zheng (2001) consider auctions with financially constrained bidders who use debt, or external financing, in their bids. Hence, while bids maybe expressed in terms of cash, they are in fact contingent claims and are thus examples of the security-bids that we examine here.

Garmaise (2001) studies a security-bid auction in the context of a financing problem for an entrepreneur. The entrepreneur commits to rank securities according to some announced beliefs regarding the distribution of the cash flows. He examines a common value environment and obtains a partial characterization of the equilibrium in a binary model (two bidders, two types, two values).

Other related literature includes McAfee and McMillan (1987), who solve for the optimal mechanism in a model with a moral hazard problem. The optimal mechanism is a combination of debt and equity, with the mixture depending on the distribution of types. Laffont and Tirole (1987) examine a similar model.

[3] Samuelson (1987) points to some additional problems in the implementation of the Cremer mechanism as it may yield an inefficient outcome if there is any "noise" regarding bidders' preferences.

Some of our results are also related to the security design literature. DeMarzo and Duffie (1999) consider the ex-ante security design problem faced by an issuer who will face a future liquidity need. They show that debt securities are optimal because they have the greatest liquidity. DeMarzo (2002) extends this result to the case in which the issuer learns his private information prior to the design of the security, as is the case here. The security design results of this paper are also related to the results of Nachman and Noe (1994). They consider a situation in which the seller is obligated to raise a fixed amount of capital, which leads to a pooling equilibrium using debt securities. None of these models consider security design in a competitive setting like the auction environment considered here.

2. The Model

Signals and Values

There are n risk neutral bidders who compete for the rights to a project. The project requires an investment by the winner of $X > 0$. For tractability, we assume that this cost is non-random and equal across bidders. Conditional on being undertaken by bidder i, the project yields a stochastic future payoff Z_i. Bidders have private signals regarding Z_i, which we denote by V_i. The seller is also risk neutral, and cannot undertake the project independently. The interest rate is normalized to zero.

We make the following standard economic assumptions on the signals and payoffs:

> **ASSUMPTION A.** The private signals $V = (V_1, \ldots, V_n)$ and payoffs $Z = (Z_1, \ldots, Z_n)$ satisfy the following properties:
>
> 1. The private signals V_i are i.i.d. with density $f(v)$ with support $[v_L, v_H]$.
>
> 2. Conditional on $V = v$, the payoff Z_i has density $h(z|v_i)$ with full support $[0, \infty)$.
>
> 3. (Z_i, V_i) satisfy the strict Monotone Likelihood Ratio Property (SMLRP); that is, the likelihood ratio $h(z|v)/h(z|v')$ is strictly increasing in z if $v > v'$.[4]

The important economic assumptions contained above are, first, that the private signals of other bidders are not informative regarding the signal or payoff of bidder i. This does not imply a pure private value setting – there may be an additional common value component that is common knowledge across all bidders. Second, because Z_i is not bounded away from zero, the project payoff cannot be used to provide a completely riskless payment to the seller. Finally, the private signal V_i is "good news" about the project payoff Z_i. This is the standard strict version of the affiliation assumption (see Milgrom and Weber (1982)).

Given the above assumptions, we normalize (without loss of generality) the private signals so that

$$E[\, Z_i \mid V_i \,] - X = V_i \,.$$

[4] This is equivalent to the log-supermodularity of h, which can be written as $\frac{\partial^2}{\partial z \partial v} \log h(z \mid v) > 0$ assuming differentiability.

Thus, we can interpret the signal as the NPV of the project.

To simplify our analysis, we make several additional technical assumptions regarding differentiability and integrability:

> **ASSUMPTION B.** The conditional density function $h(z|v)$ is twice differentiable in z and v. In addition, the functions $z\, h(z|v)$, $|\, z\, h_v(z|v)\,|$ and $|\, z\, h_{vv}(z|v)\,|$ are integrable on $z \in (0, \infty)$.

These assumptions are weak, and allow us to take derivatives "through" expectation operators. As a concrete example, we can consider the following payoff structure:

$$Z_i = \theta\,(X + V_i) \tag{1}$$

where θ is independent of V and log-normal with a mean of 1.[5] Here we can interpret θ as the project risk.

Feasible Bids

The focus of this paper is on the case in which bids are securities. Bidders compete for the project by offering the seller a share of the final payoff. That is, the bids are in terms of derivative securities, in which the underlying asset is the future payoff of the project Z_i. Bids can be described as function $S(z)$, indicating the payment to the seller when the project has final payoff z.

We make the following assumptions regarding the set of feasible bids:

> **DEFINITION.** A *feasible* security bid is described by a function $S(z)$, such that S is non-decreasing, $z - S(z)$ is non-decreasing, and $0 \le S(z) \le z$.

The assumption that $S(z) \le z$ implies limited liability for the bidder; only the underlying asset can be used to pay the seller. We assume, for now, that bidders do not have access to cash or other assets that they can pledge as payment; they can only transfer property rights in the project. We will generalize the setting to allow for cash payments in Section 5.

On the other hand, requiring $S(z) \ge 0$ implies limited liability for the seller; the seller cannot commit to pay the bidder except through a share of the project payoff.[6] This rules out a solution a-la Cremer (1987). However, as we show in Section 5, this is optimal for the seller if the bidder's investment X is not verifiable.

Finally, we require both the seller's and the bidder's payment to be non-decreasing in the payoff of the project. Absent this, parties will have an incentive to "sabotage" the project and destroy output, or alternatively to artificially inflate the output.[7] Together, these requirements are equivalent to $S(0) = 0$, S is continuous, and $S'(z) \in [0, 1]$ almost

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[5] More generally, what is required for the SMLRP is that $\log(\theta)$ have a log-concave density function.

[6] In some instances, the seller may not have the resources to do so, which may in fact be the motivation for selling off the project.

[7] For example, if $S(z)$ is somewhere decreasing so that $S(z_0) > S(z_1)$ for $z_0 < z_1$, the bidder can artificially inflate the cash flows of the project from z_0 to z_1 via a short-term loan from a third party and get the payoff $z_0 - S(z_1)$.

everywhere. Thus, we admit standard sets of securities such as equity and debt contracts.[8] For example, some standard contracts that we consider in our analysis include:

1. Equity: The seller receives some fraction $\alpha \in [0,1]$ of the payoff: $S(z) = \alpha z$.
2. Debt: The seller is promised a face value $d \geq 0$, secured by the project: $S(z) = \min(z, d)$.
3. Convertible Debt: The seller is promised a face value $d \geq 0$, secured by the project, or a fraction $\alpha \in [0,1]$ of the payoff: $S(z) = \max(\alpha z, \min(z, d))$. (This is equivalent to a debt plus royalty rate contract.)
4. Levered Equity: The seller receives a fraction $\alpha \in [0,1]$ of the payoff, after debt with face value $d \geq 0$ is paid: $S(z) = \alpha \max(z - d, 0)$. (This is equivalent to a royalty agreement in which the bidder recoups some costs upfront.)
5. Call Option: The seller receives a call option on the firm with strike price k: $S(z) = \max(z - k, 0)$. Higher bids correspond to lower strike prices. (This equivalent to the bidder retaining a debt claim.)

Given any security S, we define $ES(v) \equiv E[S(Z_i) \mid V_i = v]$ to denote the excepted payoff of security S conditional on the bidder having value $V_i = v$. Thus, the expected payoff to seller if the bid S is accepted from bidder i is $ES(V_i)$. On the other hand, the bidder's expected payoff is given by $V_i - ES(V_i)$. Thus, we can interpret V_i as the independent, private value for bidder i, and $ES(V_i)$ as the payment offered. The key difference from a standard auction, of course, is that the seller does not know the value of the bids, but only the security bid, S. The seller must infer the value of this security. Since the security S is monotone, the value of the security is increasing with the signal V_i of the bidder, as we show below:

LEMMA 1. The value of the security $ES(v)$ is twice differentiable. For $S \neq 0$, $ES'(v) > 0$, and for $S \neq Z$, $ES'(v) < 1$.

Mergers and Acquisitions

Thus far we have interpreted the setting as one in which bidders compete for the right to undertake a project. We remark, however, that the model can also be applied to mergers and acquisitions. In this case, the bidders are rival firms, each competing to take over the target company (the seller). We interpret X as the stand-alone value of the acquiring firm plus any acquisition related costs, and V_i as the bidder's estimate of the synergy value of the acquisition (i.e. the value of the target once acquired). The bids in this case represent the securities offered to the target shareholders.

[8] Aside from these motivations, the assumptions above on the set of bids are typical of those made in the security design literature (e.g. DeMarzo and Duffie (1999), Hart and Moore (1995), and Nachman and Noe (1994)). These assumptions therefore make it easier to compare our results to the prior literature.

3. Formal Auctions with Ordered Securities

In many auctions, bidders compete by offering "more" of a certain security. For example, they compete by offering more debt or more equity. We begin our analysis by examining formal auctions in which the seller restricts the bids to elements of a well-ordered set of securities. Bidders compete by offering a higher security.

There are two main reasons why sellers restrict the set of securities that are admissible as bids in the auction. First, it allows them to use standard auction formats – such as first or second-price – to allocate the object and to determine the payments. Without an imposed structure, ranking different securities is very difficult and depends on the beliefs of the seller. There is no objective notion of the "highest" bid.

The second reason a seller may want to restrict the set of securities is that it can enhance revenues. We will demonstrate this result by first (in this section) studying the revenues from auctions with ordered sets of securities and then (in section 4) comparing this to the revenues from auctions in which the seller cannot commit to a restricted set and bidders can bid using any feasible security.

Before presenting the technical details of the analysis, we consider an example that illustrates our main results.

Example: Comparison of Revenues Across Securities and Auction Formats

Two bidders compete for a project that requires an upfront investment of $X = 100$. The NPV of the project if run by bidder i is V_i, where V_i is uniform on the interval [20, 110]. The project is risky, however, with final value Z_i which is lognormal with mean $X + V_i$ and volatility of 50%.

Total surplus is maximized by allocating the project to the highest type, in this case leading to an expected value of $E[\max(V_1, V_2)] = 80$. This is the maximum expected revenue achievable by any auction. On the other hand, using a cash auction, the expected revenue is given by $E[\min(V_1, V_2)] = 50$ (which is the same for first and second-price auctions by revenue equivalence). Next, we calculate the revenues for different security designs and auction formats numerically. See Figure 1.

	Expected Seller Revenues	
Security Type	First-price Auction	Second-price Auction
Debt	50.05	50.14
Equity	58.65	58.65
Call Option	74.53	74.49

Figure 1: Expected Revenues for Different Security Designs and Auction Formats

Several observations can be made, which coincide with our main results of this section:

1. Fixing the auction format (first or second-price), revenues increase moving from debt to equity to call options. In section 3.2 we will define a notion of "steepness" for securities and show that steeper securities lead to higher revenues.

2. The auction format is irrelevant for an equity auction. While the format does make a difference for debt and call options, the rankings are reversed. In section 3.3 we will generalize these observations and show precisely when revenue equivalence will hold or fail. Overall, though, the impact of the auction format on revenues is minor compared to the security design.

3. Among the mechanisms examined the first-price auction with debt yields the lowest expected revenues while the first-price auction with call options yields the highest expected revenues. In section 3.4 we shall see that these are the worst and best possible mechanisms in a broad class of security-bid auctions.

3.1. Securities, Auctions and Mechanisms

The first step in our analysis is to formalize the notion of an ordered set of securities. An ordered collection of securities can be defined by a function $S(s,z)$, where $s \in [s_0, s_1]$ is the "index" of the security, and $S(s,\cdot)$ is a feasible security. That is, $S(s,z)$ is the payment of security s when the output of the project has value z.

For the collection of securities to be ordered, we require that its value, for any type, is increasing in s. Then, a bid of s dominates a bid of s' if $s > s'$. We would also like to allow for a sufficient range of bids so that for the lowest bid, every bidder earns a non-negative profit, while for the highest bid, no bidder earns a positive profit. This leads to the following formal requirements for an ordered set of securities:

> **DEFINITION.** The function $S(s,z)$ for $s \in [s_0,s_1]$ defines an *ordered set of securities* if:
>
> 1. $S(s,\cdot)$ is a feasible security.
> 2. For all v, $ES_1(s, v) > 0$.
> 3. $ES(s_0,v_L) \le v_L$ and $ES(s_1,v_H) \ge v_H$.

Examples of ordered sets include the sets of (levered) equity and (convertible) debt, indexed by the equity share or debt amount, and call options, indexed by the strike price.

Given an ordered set of securities, it is straightforward to generalize the standard definitions of a first and second-price auction to our setting:

> **FIRST-PRICE AUCTION:** Each agent submits a security. The bidder who submitted the highest security (highest s) wins and pays according to his security.
>
> **SECOND-PRICE AUCTION:** Each agent submits a security. The bidder who submitted the highest security (highest s) wins and pays according to the second-highest security (second-highest s). [9]

Next, we characterize the equilibria for both types of auction formats. We are interested in the case for which these equilibria are efficient; that is, the case for which the highest

[9] Note that with private values, the second-price auction is equivalent to an English auction.

value bidder wins the auction. For second-price auctions this is straightforward; the standard characterization of the second-price auction with private values generalizes to:

> **LEMMA 2.** The unique equilibrium in weakly undominated strategies in the second-price auction is for a bidder i who has value $V_i = v$ to submit security $s(v)$ such that $ES(s(v),v) = v$. The equilibrium strategy $s(v)$ is strictly increasing.

The above lemma implies that similar to a standard second-price auction, each bidder submits bids according to his true value. We now turn our attention to the first-price auction. Incentive compatibility in the first-price auction implies that no bidder gains by mimicking another type, so that $s(v)$ satisfies

$$U(v) \equiv \max_{\hat{v}} F^{n-1}(\hat{v})(v - ES(s(\hat{v}), v)) = F^{n-1}(v)(v - ES(s(v), v)) \tag{2}$$

where $U(v)$ is the expected payoff of type v. The first-order condition of (2) then leads to a differential equation for s. However, and additional assumption is required to guarantee the second-order conditions hold:

> **ASSUMPTION C.** For all (s, v) such that the bidder earns a positive expected profit, i.e. $v - ES(s,v) > 0$, the profit function is log-supermodular:
>
> $$\tfrac{\partial^2}{\partial v \partial s} \log\left[v - ES(s, v)\right] > 0\,.$$

With this assumption we have the following generalization of the standard characterization of the first-price auction to our setting:

> **LEMMA 3.** There exists a unique symmetric equilibrium for the first-price auction. It is strictly monotone, differentiable, and it is the unique solution to the following differential equation:
>
> $$s'(v) = \frac{(n-1)f(v)}{F(v)} \times \frac{\left[v - ES(s(v), v)\right]}{ES_1(s(v), v)}$$
>
> together with the boundary condition $ES(s(v_L), v_L) = v_L$.

Thus, given Assumption C, Lemma 3 characterizes the first-price auction and shows that it is efficient. Of course, the question remains regarding how restrictive is Assumption C.[10] It is a joint restriction on the set of securities and the conditional distribution of Z. It can be shown to hold generally in the lognormal setting (1) in the case of debt, equity, and levered equity securities with $d \leq X$. It can be established numerically for other types of securities, such as call options, under suitable parameter restrictions – for example, it holds in the numerical example computed earlier.

The first and second-price auctions are two standard auction mechanisms. They share the features that the highest bid wins, and only the winner pays. The first property is necessary for efficiency, and the second is natural in our setting, since only the winner

[10] Assumption C is the same as a standard condition on the utility function used in other papers on auctions: for example Maskin and Riley (1984) use it to show existence and uniqueness of equilibria with risk averse bidders. Still, the addition of this assumption in order to prove efficiency in first-price auctions suggests that the efficiency of allocations may be more fragile in first-price auctions than in second-price auctions.

can use the assets of the project to collateralize the payment. One can construct many other auction mechanisms, however, that share these properties. For example, one can consider third-price auctions, or auctions where the winner pays an average of the bids, etc. Below we define a broad class of mechanisms that will encompass these examples:

> **DEFINITION.** A *General Symmetric Mechanism* (GSM) is a symmetric incentive compatible mechanism in which the highest type wins, and pays a security chosen at random from a given set $\boldsymbol{S}$. The randomization can depend on the realization of types, but not on the identity of the bidders (so as to be symmetric).

The first-price auction fits this description, with no randomization (the security is a function of your type). In the second-price auction, the security you pay depends upon the realization of the second-highest type. GSMs also allow for more complicated payment schemes that depend on all of the bids.

It will be useful in what follows to derive a basic characterization of the incentive compatibility condition for a GSM. We show that any GSM can be converted into an equivalent mechanism in which the winner pays a security that depends only on his reported type without further randomization. This observation will be crucial in comparing revenues across mechanisms.

> **LEMMA 4.** Incentive compatibility in a GSM implies the existence of securities $\hat{S}_v$ in the convex hull of $\boldsymbol{S}$ such that
>
> $$v \in \arg\max_{v'} F^{n-1}(v')\left(v - E\hat{S}_{v'}(v)\right).$$
>
> Thus, it is equivalent to a GSM in which the winner pays the *non-random* security $\hat{S}_v$.

This result will allow us to compare revenues across different mechanisms by studying the relationship between the set of securities $\boldsymbol{S}$ and its convex hull.

3.2. Ranking Security Designs

Recall from Figure 1 that the seller's revenues varied greatly with the security design. As we will show, the revenues of different designs depend upon the "steepness" of the securities. To do so, we need to formalize the notion of the steepness of a set of securities. Intuitively, one security is steeper than another if it crosses that security from below. Thus, we introduce the following definition:

> **DEFINITION.** Security S_1 strictly crosses security S_2 from below if $ES_1(v^*) = ES_2(v^*)$ implies $ES_1'(v^*) > ES_2'(v^*)$. An ordered set of securities $\boldsymbol{S_1}$ is *steeper* than an ordered set $\boldsymbol{S_2}$ if for all $S_1 \in \boldsymbol{S_1}$ and $S_2 \in \boldsymbol{S_2}$ with $S_1 \neq S_2$, S_1 strictly crosses S_2 from below.

The following technical lemma is useful in identifying a strict crossing by relating it to the shape of the underlying securities:

LEMMA 5. (Single Crossing) A sufficient condition for S_1 to strictly cross S_2 from below is that $S_1 \neq S_2$, and there exists z^* such that $S_1(z) \leq S_2(z)$ for $z < z^*$ and $S_1(z) \geq S_2(z)$ for $z > z^*$.

Using this lemma, we can compare security payoffs directly to see that a call options is steeper than equity, which in turn is steeper than debt. See Figure 2.

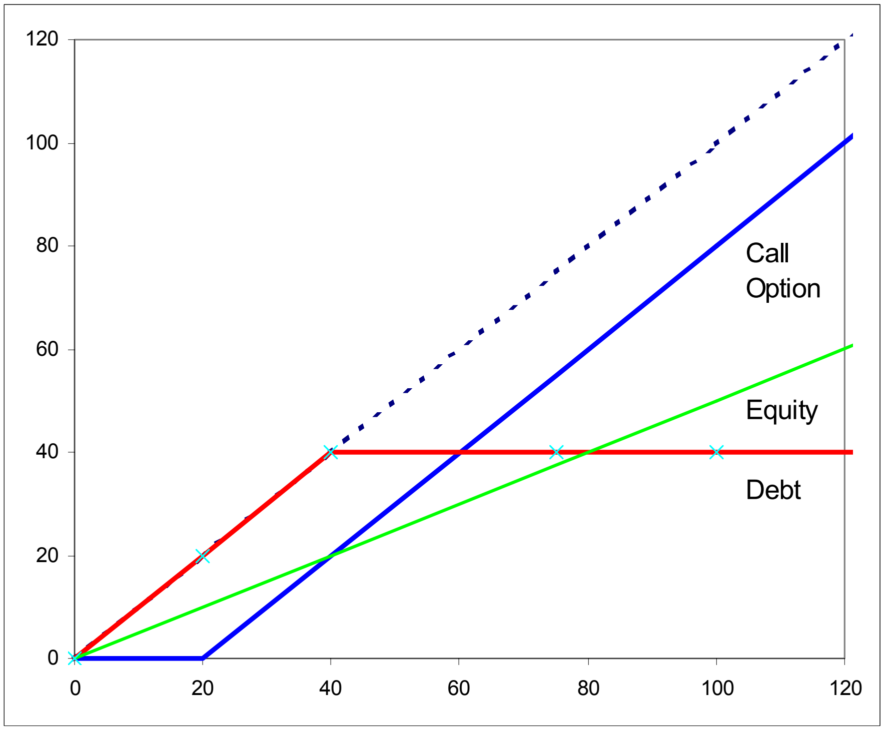

Figure 2: Payoff Diagrams for Call Options, Equity and Debt

Why is steepness related to auction revenues? Consider a second-price auction, where the winning bidder with type V^1 pays the security bid by the second highest type V^2. That is, the winner pays $ES(s(V^2), V^1)$. Since bidders bid their reservation value in a second-price auction, $ES(s(V^2), V^2) = V^2$. Hence, the security design impacts revenues only through the difference,

$$ES(s(V^2), V^1) - ES(s(V^2), V^2)$$

which is just the sensitivity of the security to the true type; i.e., its steepness.

More generally, steepness enhances competition between bidders since even with the same bid, a higher type will pay more. This is the essence of the "Linkage Principle," first used by Milgrom and Weber (1982) to rank auction formats for cash auctions when types are affiliated.[11] It follows from applying the envelope theorem to the incentive condition (2) for a first-price auction, to get

[11] See also Krishna (2002) for a nice summary and discussion. The typical use of the linkage principle is in settings for which bidders' signals are affiliated. Interestingly, the same argument can be applied to rank security auctions when types are independent. The reason is that with security-bid auctions, unlike cash auctions, even when types are independent the expected payment of the winner depends on his true type, not just his bid. This was pointed out by Riley (1988) in the context of royalty rates.

$$U'(v) = F^{n-1}(v)(1 - ES_2(s(v), v)) \;. \tag{3}$$

Therefore, bidders' payoffs are lower the higher is $ES_2(s(v), v)$; i.e., the steeper the security. This leads to the following main result:

> **PROPOSITION I.** Suppose the ordered set of securities $\boldsymbol{S_1}$ is steeper than $\boldsymbol{S_2}$. Then for either a first or second-price auction, for *any* realization of types, the seller's revenues are higher using $\boldsymbol{S_1}$ than using $\boldsymbol{S_2}$.

As a result, flat securities, like debt, lead to low expected revenues, and steep securities, like call options, lead to high expected revenues. In fact, the since debt and call options are the flattest and steepest possible securities, we have the following:

> **COROLLARY.** For a first or second-price auction, standard debt yields the lowest possible expected revenues, and call options yield the highest possible expected revenues, for any realization of types.

PROOF: Since debt has slope 1 and then 0, it crosses any other feasible security from above. Call options have slope 0 and then 1, and so cross any other security from below. The result then follows directly from Proposition I. ◆

Note that these rankings are for any realization of types, and hence are stronger than the usual comparison based on an expectation over types.

3.3. Ranking Auction Formats

We now turn to examining the revenue consequences of using a first or second-price auction format, holding fixed the security design. The key distinction between these formats is that from the point of view of the winner, in a second-price auction the payment is according to a random security determined by the second highest bid. That is, the winner in a second-price auction pays a convex combination of securities.

As a consequence, the revenue differences across formats will stem from the differences between the set of securities and its convex hull. Using the idea of strict crossing, we can identify two important types of sets of securities, sub-convex and super-convex, for which we can rank the revenues from first-price and second-price auctions:

> **DEFINITION.** An ordered set of securities $\boldsymbol{S} = \{S(s,\cdot) : s \in [s_0, s_1]\}$ is *super-convex* if it is steeper than its convex hull. It is *sub-convex* if it is less steep than its convex hull.

Not every set falls into one of the above categories. Still, there are some important examples of sub- and super-convex sets:

> **LEMMA 6.** The set of standard debt contracts is sub-convex. The set of convertible debt contracts indexed by the equity share α, the set of levered equity contracts indexed by leverage, and call options are super-convex sets.

Based on the above characterization, we can again use the Linkage Principle to rank the expected revenues of first and second-price auctions. Here the proof relies on **LEMMA 4**, which allows us to interpret the second-price format as a first-price mechanism in the convex hull of the set of securities:

> **PROPOSITION II.** If the ordered set of securities is sub-convex, then the first-price auction yields strictly lower expected revenues than the second-price auction. If the ordered set of securities is super-convex, the first-price auction yields strictly higher expected revenues than the second-price auction.

Revenue Equivalence for Convex Sets of Securities

In our setting of independent private values and risk neutrality, a well-known and important result for cash auctions is the Revenue Equivalence Principle. It implies that the choice of the auction format is irrelevant when the ultimate allocation is efficient.[12] Unlike cash auctions, however, revenue equivalence does not hold for general security auctions, as we have already seen from **PROPOSITION II**. Here we ask whether it can be recovered for some classes of securities – that is, what is special about cash?

From **PROPOSITION II**, revenue equivalence fails in one direction for a super-convex set, and in the opposite direction for a sub-convex set. Hence, a natural candidate is a set in the middle; i.e., a convex set:

> **DEFINITION.** An ordered set of securities $\boldsymbol{S}$ is *convex* if it is equal to its convex hull.

In fact, convex sets of securities have a simple characterization – each security is a convex combination of the lowest security s_0 and the highest security s_1.[13] Thus, each security can be thought of as s_0 plus some "equity shares" of the security $(s_1 - s_0)$, and so it can be thought of as a generalization of a standard equity auction. Our main result in this section is that under convexity, the Revenue Equivalence Principle for auctions continues to hold.

> **PROPOSITION III (REVENUE EQUIVALENCE).** Every efficient equilibrium of a general symmetric mechanism (GSM) with securities from an ordered convex set yields the same expected revenues.

Note that this is a stronger statement than equivalence between a first and second-price auction, as it holds for any symmetric mechanism. Also note that the standard envelope argument behind Revenue Equivalence does not extend directly to security auctions. For cash, there is no linkage between the true type and the bidder's expected payment when types are independent, so revenues only depend upon the allocation.[14] That is not the case with security-bids, as we have seen. However, when the security set is convex, the "replication" argument used in the proof below implies that the *expected* linkage across all mechanisms is identical.

---

[12] Vickery (1961), Myerson (1981), Riley and Samuelson (1981).

[13] To see why, note that since the set is convex, for each λ there exists a mapping s:[0, 1] $\to$ $[s_0, s_1]$ such that $S(s(\lambda), z) = (1-\lambda)\, S(s_0, z) + \lambda\, S(s_1, z)$. Then $s(0) = s_0$, $s(1) = s_1$ and since the set is ordered and $s_0 \neq s_1$, $s(\lambda)$ is strictly increasing. Thus, the result follows if $s(\lambda)$ is continuous. But since $ES(s(\lambda),v) = (1-\lambda)\, ES(s_0, v) + \lambda\, ES(s_1, v)$ is continuous, so is $s(\lambda)$ since $ES(s,v)$ is strictly increasing in s.

[14] That is, in the case of cash auctions, (3) reduces to $U'(v) = F^{n-1}(v)$.

PROOF: In a GSM, the winner pays according to a random security. From Lemma 4, the expected payment by type v reporting v' can be written as $E\hat{S}_{v'}(v)$, where $\hat{S}_{v'}$ is in the convex hull of the ordered set of securities S. Since S is convex, we can define $s^*(v')$ such that

$$S(s^*(v'),\cdot) = \hat{S}_{v'}(\cdot) .$$

Because S is ordered, incentive compatibility implies $s^*(v)$ must be strictly increasing; otherwise a bidder could raise the probability of winning without increasing the expected payment. Thus, $s^*(v)$ defines an efficient equilibrium for the first-price auction. The result then follows from the uniqueness of equilibrium in the first-price auction. ◆

Thus, we have shown that the important property needed for the revenue equivalence principle is that the securities be ordered and convex. This is true for cash, but also true more generally for equity-type auctions. The result also allows us to weaken our condition for the existence of an efficient equilibrium in the first-price auction:

> **COROLLARY.** Even absent Assumption C, given a convex ordered set of securities, there exists an efficient symmetric equilibrium in a first-price auction with the same expected revenues as in a second-price auction.

PROOF: Assumption C is not required for existence of the second-price auction. Then, we can use the construction in the proof to generate an equivalent equilibrium for the first-price auction. ◆

3.4. Best and Worst Mechanisms

We can combine the results of the previous two sections to determine the best and worst security design and format combinations. Note that, since debt is a sub-convex set, from Proposition II the first-price auction is inferior to the second-price auction, and conversely for call options, which are super-convex. The following proposition establishes that a first-price auction with debt and with call options bound the range of outcomes for the seller for a broad class of mechanisms.

> **PROPOSITION IV.** A first-price auction with call options yields the highest expected revenues amongst all general symmetric mechanisms. A first-price auction with standard debt yields the lowest expected revenues amongst all general symmetric mechanisms.

PROOF: The proof is identical to that of Proposition II, except that instead of the second-price auction we consider a general symmetric mechanism over some subset of the feasible securities. The result follows from the fact that a call option contract is steeper and a standard debt contract is flatter than any convex combination of feasible securities. ◆

Of course, this optimality is with respect to the feasible set of securities we have allowed thus far. This feasible set may differ with different assumptions regarding moral hazard and limited liability.

For example, if bidders can pay cash, the same methodology establishes that a cash auction is the worst possible auction for the seller. This is because cash, which is insensitive to type, is even flatter than standard debt securities (see Section 6.1 for a treatment of this case.).

Alternatively, the seller may be able to increase revenues by using securities that are even more leveraged than call options. For example, the seller might pay the bidder upfront for additional equity. This is related to the result of Cremer (1987). We analyze this possibility in Section 5, and show that it is not robust to moral hazard considerations. Indeed, even call options may be too levered in some settings of moral hazard, as we discuss in section 6.3.

4. Informal Auctions: The Signaling Game

In the previous section we considered formal auctions in which bidders are restricted to choose securities from a specific well-ordered set. In reality, there is often no such restriction. That is, the seller is unable to commit to ignore offers that are outside the set. As a result, the seller will consider all bids, choosing the most attractive bid ex-post. In this case, the "security design" is in the hands of the bidders, who can choose to bid using any feasible security.

Without the structure of a well-ordered set, once the bids are submitted there is no obvious notion of a "highest" bid. In this case, the seller faces the task of choosing one of the submitted bids. Since there is no ex-ante commitment by the seller to a decision rule, the seller will choose the winning bid that offers the highest expected payoff. Since the payoff of the security depends on the bidder's type, the seller's choice will depend upon his beliefs regarding the bid each type submits in equilibrium. Thus, this setting has the features of a classic signaling game.

Formally, in this section we examine an auction setting in which:

1. Bidders submit simultaneous bids that are feasible securities.
2. The seller chooses a winning bid from the set submitted.
3. The winner pays his bid and runs the project.

In a sequential equilibrium of this game, the seller will choose the bid that he believes has the highest expected payoff. Thus, this is a generalization of the standard first-price auction, where the ranking now depends upon the seller's beliefs.

Refining Beliefs – The D1 Criterion

As with general signaling games, there are many equilibria of this game if we do not impose any restrictions on the beliefs of the seller when an "unexpected" bid is observed. The standard refinement of beliefs in the signaling literature is the notion of strategic stability, introduced by Kohlberg and Mertens (1986). For our purposes, a weaker refinement, known as D1, is sufficient to identify a unique equilibrium.[15] The D1 refinement (see Cho and Kreps (1987), Cho and Sobel (1990)) is a refinement commonly

[15] Strictly speaking, D1 is defined for discrete type spaces. However, it can be naturally extended to continuous types (see, e.g., Ramey 1996).

used in the security design literature.[16] Intuitively, the D1 refinement criterion requires that if the seller observes an out-of-equilibrium bid, the seller should believe the bid came from the type "most eager" to make the deviation.

In order to define the D1 criterion in our context, we introduce the following notation. First, let S^i be the random variable representing the security bid by bidder i, which will depend on V_i. For any feasible security S, let $R^i(S)$ be the *scoring rule* assigned by the seller, representing the expected revenues the seller anticipates from that security, given his beliefs. Along the equilibrium path, the seller's beliefs are correct, so that the scoring rule satisfies

$$R^i(S) = E\left[ES(V_i) \,\middle|\, S^i = S \right]. \tag{4}$$

Given the seller's scoring rule, R^i, it must also be the case in equilibrium that bidders are bidding optimally. That is, conditional on $V_i = v$, S^i solves

$$U^i(v) = \max_S P^i(R^i(S))(v - ES(v)), \tag{5}$$

where $P^i(r)$ is the probability that r is the highest score.[17] Thus, $U^i(v)$ is the equilibrium expected payoff for bidder i with type v.

Suppose the seller observes an out-of-equilibrium bid, so that the score is not determined by (4). Which types would be most likely to gain from such a bid? For each type v, we can determine the minimum probability of acceptance, $B^i(S,v)$, that would make bidding S attractive:

$$B^i(S,v) = \min\left\{ p : p\,(v - ES(v)) \geq U^i(v) \right\}.$$

Then the D1 criterion requires that the seller believe that a deviation to security S came from the types which would find S attractive for the lowest probability:[18]

$$R^i(S) \in ES(\arg\min_v B^i(S,v)). \tag{6}$$

Thus, a sequential equilibrium satisfying the D1 criterion for the auction game can be described by scoring rules R^i and bidding strategy S^i for all i satisfying (4)-(6).

Equilibrium Characterization

We now turn to characterizing the equilibrium of the auction game. As is standard in the auction setting, we will focus on symmetric equilibria.[19] Our main result is that if the bidders are unrestricted, the resulting symmetric equilibrium of the informal auction is equivalent to a first-price auction with standard debt securities. Note that, from Proposition IV, this implies that the seller's expected revenues are the lowest possible from any general auction mechanism.

[16] See, e.g. Nachman and Noe (1994) and DeMarzo and Duffie (1999).
[17] If there are ties, we require that P^i be consistent with *some* tie-breaking rule.
[18] We have economized on notation here. If the set of minimizers is not unique, the score is in the convex hull of $ES(v)$ for v in the set of minimizers.
[19] That is, we restrict attention to equilibria in which the bidders use symmetric strategies and the seller uses the same scoring rule for all players.

The intuition for standard debt is as follows. Consider any equilibrium security S_v bid by type v. If type v were instead to offer a standard debt contract with the same expected payoff, how would the seller respond? Because the debt contract is flatter than any non-debt contract, types $v' > v$ would find the debt contract cheaper, whereas types $v' < v$ would find the debt contract more expensive than S_v. That is, only those types $v' > v$ would benefit from such a deviation. By D1, this implies that the score can only increase if the bidder switches to debt, so that debt is an optimal bid.

We now proceed with a formal statement of our results. We continue to maintain Assumption C for standard debt, so that existence of an efficient equilibrium of the first-price auction is assured. Then we have

> **PROPOSITION V.** Given symmetric strategies, there is a unique equilibrium of the informal auction satisfying D1. This equilibrium is equivalent, in both payoffs and strategies, to the equilibrium of a first-price auction in standard debt contracts.

Again, we can now combine this result with the result of the previous section to formalize the value of the seller's ability to commit to a restricted set of securities.

> **COROLLARY.** If the seller can commit to a formal auction with a an ordered set of securities other than debt contracts, then expected revenues are strictly greater than without such commitment.

PROOF: Follows immediately from Proposition IV and Proposition V. ◆

5. Moral Hazard: Non-Contractible Investment

Thus far, we have restricted bids to satisfy $S(0) \geq 0$. When the seller has cash or other assets, this rules out the following simple strategy: auction off the rights to a fraction ε of the cash flows, and reimburse the winner directly for the investment X. By making ε small, the seller can extract almost all of the available surplus. This is essentially the mechanism proposed by Cremer (1987).

This solution, however, is not observed in practice. A likely reason is moral hazard: if the winner's investment is not fully contractible, and if the winner receives only a small fraction of future revenues, then he may under-invest. A real-world illustration of the importance of moral hazard is provided by several oil lease auctions run by the U.S. Department of the Interior (see Binmore and Klemperer (2002)). In these auctions, bidders bid high royalty rates. As a result, many of the oil fields were left undeveloped despite the fact that it was economically efficient to do so. Bidders did not capture enough of the revenues to warrant their private investment. In the end, the government was left with almost no revenues.

In this section we extend our model to allow for moral hazard with regard to the investment by the winning bidder. We show that in this case, the restriction $S(0) \geq 0$ is without loss of generality – it is never optimal for the seller to offer any reimbursement to the bidder for non-contractible investment.

Formally, suppose that after the winning bid is determined, the winning bidder can choose whether to invest X. If X is invested, the payoff of the project is Z as before, and his payment to the seller is $S(Z)$. If X is not invested, the payoff is 0, and his payment to the seller is $S(0)$. Note that if $S(0) \geq 0$, none of our earlier analysis changes. The bidder's payoff is non-positive with no investment, whereas at the equilibrium bid the payoff is non-negative with investment. Thus, the option not to invest is never exercised.

But suppose instead a bid with $S(0) < 0$ is allowed. In fact, suppose such a bid wins with positive probability. Then every bidder, including the lowest type, can earn positive profits by simply making such a bid and then not investing. But if bidders do not invest and $S(0) < 0$, the seller loses money.[20] As a result, one would expect the seller to choose not to use such securities. This is confirmed with the following result:

> **PROPOSITION VI.** Suppose that the investment X is not contractible and the seller allows for some securities that specify a transfer to the buyer for some realizations of Z, (i.e. $S(0) < 0$). Then,

1. In a first and second-price formal auction (with an ordered set of securities):
    a. If $S(s_1,0) \geq 0$ then with probability 1 the winning bid satisfies $S(s,0) \geq 0$. That is, competition between bidders rules out reimbursement.
    b. If $S(s_1,0) < 0$ then all bidders bid s_1 and do not invest, leading to negative revenues for the seller.
2. Any mechanism in which bids with $S(0) < 0$ win with positive probability cannot be efficient.
3. In an informal auction, securities with $S(0) < 0$ are used with probability 0.

Thus, when X is not contractible, we can rule out reimbursement from the seller – it would either not occur in equilibrium or not be in the seller's best interest. Thus, we can restrict $S(0) \geq 0$ even if the seller has cash.

What if X is partially contractible? That is, suppose $X = X_1 + X_2$, where the investment X_1 can be verified. If the seller has cash, then it will typically be optimal for the seller to offer reimbursement for X_1. For example, the sets of debt, (levered) equity, and call option securities are all steeper if we replace each security S by $S - X_1$. This increases leverage and the seller's expected revenues. Indeed, the steepest possible security in this case is a levered call option:

$$S(k, z) = -X_1 + \max(z - k, 0).$$

In other words, the bidder *receives* a cash payment of X_1 together with a debt claim, and gives up an equity claim on the project.

[20] It might be tempting to think this problem can be solved by making $S(0) = 0$ but then $S(z) < 0$ for z close to but not equal to 0. But such a security violates our restriction that S is monotone, which followed from the bidder's ability to artificially inflate profits. In this case, the bidder would not invest, and then artificially inflate the profits to z and receive compensation $-S(z) > 0$ from the seller.

6. Other Extensions

We finish this section with a discussion of several extensions of the basic model: partial cash bids, reservation prices, other forms of moral hazard, and correlation between bidders' types.

6.1. Partial Cash Bids

Thus far, we have assumed that bidders can only bid claims over the future cash flows of the project. Suppose in addition that the bidders have cash or other assets that can also be traded with market value B. For simplicity, we consider the case where B is known and common to all bidders; Che and Gale (2000), Rhodes-Kropf and Viswanathan (2002), and Zheng (2001) consider models where the cash amount is heterogeneous and privately known.

In terms of the assumptions of our model, the effect of adding cash is to relax the limited liability restriction for bidders. Rather than requiring $0 \le S(z) \le z$, we require

$$0 \le S(z) \le z + B.$$

How does this relaxation of the constraint affect our results? Essentially, cash allows us to make the flattest securities even flatter. Specifically, the flattest securities are now debt claims on the total assets of the bidder (cash + project), defined by

$$S^D(d, z) = \min(d, B + z) = \min(d, B) + \min((d - B)^+, z).$$

As the decomposition above reveals, we can think of this security as an immediate cash payment (up to B), plus a standard debt claim on the project (for amounts above B). Since these new securities are flatter than in the absence of cash, the results in the paper imply that the expected revenues will be lower than before.

Note that if B is large enough, a pure cash auction is possible. This has the lowest possible expected revenues for the seller. For this to be the case, it is necessary that

$$B \ge E\left[V_{n-1}^*\right] \qquad \text{for a first-price auction,}$$

and

$$B \ge v_H > E\left[V_{n-1}^*\right] \qquad \text{for a second-price auction,}$$

where V_{n-1}^* is the maximum type for $n-1$ bidders. This requirement is stricter for a second-price auction, so that a first-price auction yields strictly lower revenues for the seller as long as $B < v_H$. These results are consistent with Board (2002), who considers debt auctions and shows that they yield higher revenues than cash auctions, with the smallest effect for first price auctions.

6.2. Reservation Prices

In this section we discuss briefly how reservation prices can be incorporated into our analysis. Commitment to a reservation price can improve the seller's revenues, and even absent commitment, a reservation price may be relevant if selling the project entails an opportunity cost.

In the case of formal auctions, we assumed earlier that the lowest security, s_0, was such that all types earn non-negative profits; that is, $ES(s_0,v_L) \leq v_L$. A reservation price is equivalent to assuming that s_0 restricts that set of types that can profitably participate. In particular, if we choose s_0 so that

$$ES(s_0, v_r) = v_r$$

for some type $v_r \in [v_L, v_H]$, then v_r is the reservation price[21], and types below v_r will not be allocated the project.

All of our results regarding formal auctions generalize to this case. In particular, holding fixed the reservation price, the seller's revenues are higher using steeper securities. In addition, first-price auctions are superior for super-convex securities, second-price auctions are superior for sub-convex securities, and revenue equivalence holds for convex ordered securities.

The proof of these results follows exactly as the proofs in Section 3: we apply the linkage principle to compare $U'(v)$ across settings, but now use the boundary condition $U(v_r) = 0$ rather than $U(v_L) = 0$.

A similar generalization applies to informal auctions. There, since there is no commitment, it is natural to interpret v_r as the outside opportunity for the seller. Thus, the seller will not sell unless he believes the best security is worth more than v_r. Consider the standard debt security S_r^d defined by $ES_r^d(v_r) = v_r$. Types below v_r would lose money submitting this security. Thus, in a D1 equilibrium, the seller must believe the bidder submitting S_r^d is at least type v_r, and this bid would be accepted. The remainder of the proof can be extended to show that the unique D1 equilibrium is a first-price standard debt auction with reservation price v_r, which has the lowest possible revenues for any auction mechanism with this reservation price.

6.3. Moral Hazard

Thus far, we have included minimal moral hazard restrictions in the model: payoffs to both parties must be weakly increasing to avoid incentives to sabotage or artificially inflate the project cashflows, and $S(0)$ must be non-negative to maintain incentives for the buyer to invest X. However, depending on the environment, more stringent moral hazard constraints might be appropriate. In this section we discuss the implications of several standard forms of moral hazard.

We note at the outset that depending on the setting, both the buyer and the seller may be subject to moral hazard. For example, it is natural to think of moral hazard on the part of the buyer in the case of oil lease and wireless spectrum auctions, where the seller (the government) is generally not involved in the project operations ex post. On the other hand, for an author auctioning publishing rights, or an entrepreneur seeking funding from

[21] We refer to this as a "price" since it is the minimum amount the seller will accept in order to sell the object.

competing venture capitalists, moral hazard on the part of the seller is likely to be relevant.

Cash Flow Diversion. Consider the case in which one of the parties has the opportunity to divert cash flows from the project ex-post. Suppose that each dollar the agent diverts leads to a private payoff for the agent of $\delta < 1$. This is a standard agency setting in which the agent can siphon off resources inefficiently, but enjoy a private gain.[22] By the usual revelation principle type of argument, we can restrict attention to securities that do not induce the agent to divert cash flows. This is equivalent to adding an additional constraint that the agent has an equity stake of at least δ on the payoff of the project.

Thus, if the bidder can divert the cash flows, we must add the restriction

$$S'(z) \leq 1 - \delta$$

to the requirements for a feasible security.[23] Note that this implies that the payoff of the winning bidder is bounded below by

$$E[\, Z_i - (1-\delta)\, Z_i\,] - X = \delta\, (X + V_i) - X.$$

If δ is too large, this would imply positive profits even for the lowest type, and we would not have a separating (and thus efficient) equilibrium. To rule this out, we assume that $\delta < X/(X + v_H)$.

What is the effect of this additional moral hazard consideration? First, it reduces the maximal steepness of a feasible security. In this case, the security type that maximizes the seller's expected revenue is levered equity, with an equity share $\alpha = 1 - \delta$. Because this is less steep than call options, competition between bidders is reduced and the maximum possible revenues the seller can achieve using the optimal mechanism will be strictly lower as a result of this moral hazard.

On the other hand, if the seller cannot commit to a formal auction, this constraint prevents bidders from using standard debt. The flattest securities are now securities of the form

$$S(d, z) = \min(d, (1-\delta)\, z).$$

Since these securities are not as flat as standard debt, the revenues of the seller are in fact enhanced by this restriction. In other words, moral hazard benefits the seller by allowing him to commit not to accept standard debt contracts.

On the other hand, if the seller can divert cash flows, the constraint becomes $S'(z) \geq \delta$. Now, the flattest securities are a combination of debt plus equity, $S(d, z) = \min(d, z) + \delta \max(z - d, 0)$. These lead to higher revenues than standard debt. The steepest securities are equity plus options, $S(d, z) = \delta\, z + (1-\delta) \max(z - d, 0)$. These lead to lower revenues than call options.

Similar arguments can be used to analyze the impact of other forms of moral hazard. In general, moral hazard concerns will limit the steepness/flatness of the securities that can

[22] This is a special case of Lacker and Weinberg's (1989) model of "costly state falsification."

[23] The security S need not be everywhere differentiable – in this case we mean that $(1-\delta)$ is a supergradient of S.

be used by bidders, reducing the revenues of an optimal auction, and possibly increasing revenues in an informal auction. We discuss several additional examples below.

Asset Substitution. A common moral hazard concern when debt securities are used is that the equity holder may increase the riskiness of the project (asset substitution). In the extreme case in which it is possible for an agent to add arbitrary mean-preserving spreads to the project payoff, we can restrict the set of feasible securities so that the agent's payoff is a concave function of the underlying cash flows.[24]

Thus, if the seller can add risk, levered equity and call options are no longer feasible. The optimal auction involves the steepest possible concave security, equity, which gives lower expected revenues for the seller. On the other hand, the unrestricted informal auction is unaffected (debt is concave).

Alternatively, if the bidder can add risk to the project, in order that his residual payoff is concave we must restrict the securities to be convex. In this case, in the unrestricted informal auction the bidder will use equity (the flattest convex security), enhancing the seller's expected revenues.

Costly State Verification and Costly Effort. In the costly state verification framework of Townsend (1979) or the costly effort model of Innes (1990), having the agent issue debt and retain the equity is an optimal security design in mitigating the moral hazard problem.[25] In an auction framework, our results show that this is consistent with maximizing the auction revenues if the agent with moral hazard is the seller. Moreover, in the informal auction, these considerations will move bidders away from debt, enhancing competition in the auction.

Again, when the agent with the moral hazard is the bidder, the results are reversed. Now, the optimal auction must tradeoff auction revenues (which favor issuing equity) with moral hazard concerns (which favor issuing debt). On the other hand, in the unrestricted auction, debt will be used. This is efficient from a moral hazard standpoint, but results in decreased auction competition.

6.4. Affiliated Private Values

Our model thus far is based on the classic independent private value framework. We discuss here how our results would change if we consider a somewhat more general framework in which bidders' private values are affiliated.[26] For example, there may exist a common factor that affects profitability independent of the identity of the bidder. One possible specification of this is

$$V_i = V + \varepsilon_i \tag{7}$$

[24] See Ravid and Spiegel (1997) for a model of this sort.
[25] McAfee and McMillan (1987) also consider an optimal mechanism with full commitment from the seller; without the constraint $0 < S(z) < z$ and a costly effort model of moral hazard. The optimal mechanism is under some conditions a linear function that consists of a fixed fee and an equity share. However, the menu of securities they propose depends critically on the underlying distribution of types and the effort cost function (in particular, it is not detail-free), and hence is difficult to implement.
[26] That is, V_i is good news regarding V_j in the sense of MLRP. See Milgrom and Weber (1982).

where ε_i are i.i.d. and have a log-concave density function. This is consistent with private values as long as V_i is a sufficient statistic for the distribution of Z_i; that is, the information of other bidders is not useful to bidder i.[27]

With affiliated private values, under the same assumptions as before, we have the following extension of our results for auctions with an ordered set of securities:

- There exists a unique symmetric equilibrium for both the first and the second-price auction. The equilibrium in both cases is efficient and is similar to the one we derive for the case of independent types.
- By the linkage principle, affiliation increases revenues for the second price auction relative to the first price auction. As a result, for both convex and sub-convex sets of securities the second-price auction yields higher expected revenues.[28]
- Fixing the mechanism (first or second-price), steeper sets of securities yield higher revenues for the seller.
- First-price auctions with debt securities yields the lowest revenues among a general class of mechanisms.[29]

If the seller is unable to commit ex-ante to a formal mechanism, one can show that again bidders use debt:

- In the unique equilibrium satisfying standard refinements of off-equilibrium beliefs, bidders use only debt securities. Moreover, the outcome is equivalent to a first-price auction. As a result we conclude that ex-post maximization yields the worst possible outcome for the seller.

Thus, the main aspects of our analysis generalize to the affiliated private values case.

It would also be interesting to explore the properties of security-bid auctions when there is a common value component to the project. (For example, in the context of (7) above, suppose the payoff Z_i also depends on V.) This setting introduces new complexities. For example, consider the case in which the seller is unable to commit to a formal mechanism. In the case, the seller's valuation of the security bid by bidder i depends on the security bid by bidder j, taking it outside the setting of a standard signaling game. We plan to explore this setting in future work.

7. Conclusion

We have examined an aspect of bidding relatively ignored in auction theory – the fact that bidders' payments often depend on the realization of future cash flows. This embeds a security design problem within the auction setting. First we analyzed formal auctions in which the seller chooses the security design and restricts bidders to bid only using

[27] For example, V might consist of market data that is available to all bidders, and ε_i is private information about bidder i's own profitability.

[28] In particular, "Revenue Equivalence" does not hold, which is true in this setting even for cash auctions.

[29] The class of mechanisms is general symmetric mechanisms in which the payment is weakly increasing in the bids of losing bidders.

securities in an ordered set. This enables a simple ranking of the securities and the use of standard auction formats. We showed conditions for which Revenue Equivalence holds, and determine the optimal and worst format and security design combinations. In particular, we showed that revenues are increasing in the steepness of the securities, and demonstrated that the first-price debt auction yields the lowest revenues, whereas a first-price auction with call options yields the highest revenues, across a broad class of possible mechanisms.

Next we considered informal auctions in which the seller does not restrict the set of securities or the mechanism ex ante, but chooses the most attractive bid ex post. In this case, security design is in the hands of the bidders. We show that this yields the lowest possible expected revenues for the seller, and is equivalent to a first-price debt auction. Thus there are strong incentives for the seller to be actively involved in the auction design and select the securities that can be used.

Finally, we generalized our results to incorporate aspects of moral hazard, partial cash bids, reservation prices, and affiliated values. All of our main insights and results are robust to these features.

There are several important potential extensions of our results that are likely to be important in applications. First, it would be useful to allow for asymmetries, both in bidders' valuations and costs. We believe that our result showing that debt is the outcome for the informal auction generalizes to this setting. However, for formal mechanisms, we are confronted with the relative lack of theoretical results in the presence of asymmetries, even in the case of cash auctions.

Also, for some applications such as mergers and acquisitions, other considerations such as taxes and accounting treatment are likely to be important. For example, the deferral of taxes possible with an equity-based transaction may give rise to the use of equity bids even in an unrestricted setting. Our results are useful, in that they imply that this tax preference can also lead to enhanced revenues for the seller.

8. Appendix

PROOF OF LEMMA 1: Differentiability follows from dominated convergence under our assumptions. Then, for any z^*,

$$ES'(v) = \int S(z)\, h_v(z \mid v)\, dv = \int \left[S(z) - S(z^*)\right] \left[\frac{h_v(z \mid v)}{h(z \mid v)}\right] h(z \mid v)\, dv\,,$$

since $\int S(z^*)\, h_v(z \mid v)\, dv = S(z^*) \frac{\partial}{\partial v} \int h(z \mid v)\, dv = 0$.

From SMLRP, [h_v / h] is strictly increasing in z. Therefore, we can choose z^* so that

$$\left[\frac{h_v(z \mid v)}{h(z \mid v)}\right] \gtrless 0 \quad \text{if and only if} \quad z \gtrless z^*.$$

Then, since S is weakly increasing,

$$\left[S(z)-S(z^*)\right]\left[\frac{h_v(z\,|\,v)}{h(z\,|\,v)}\right]\geq 0,$$

and the inequality is strict for z such that $S(z) \neq S(z^*)$. This set has positive measure since $S \neq 0$ and Z has full support conditional on v. Hence, $ES'(v) > 0$. The proof of $ES'(v) < 1$ is identical, substituting $Z - S(Z)$ for S. ◆

PROOF OF LEMMA 2: The proof that $s(v)$, which solves $ES(s,v) = v$, is the unique weakly undominated strategy is standard. Differentiating $ES(s(v),v) = v$ yields,

$$s'(v)=\frac{\frac{\partial}{\partial v}\left[v-ES(s,v)\right]}{\frac{\partial}{\partial s}ES(s,v)}.$$

Thus s strictly increasing in v follows, since S is increasing in s, and from Lemma 1, $\frac{\partial}{\partial v}\left[v-ES(s,v)\right] > 0$ as long as $S(s,Z) \neq Z$ (which is not possible in equilibrium since $X > 0$). ◆

PROOF OF LEMMA 3: Let $P(s)$ be the probability of winning with a bid of s, and $\pi(s, v) = \log(P(s)) + \log(v - ES(s, v))$. Then

$$s(v)\in \arg\max_s P(s)(v-ES(s,v)) = \arg\max_s \pi(s,v).$$

By Assumption C the objective in the second expression is strictly supermodular, and so by Topkis (1978), any selection $s(v)$ is increasing in v. If $s(v)$ were constant on an interval, then the highest type in that interval can increase his bid marginally and increase his probability of winning, and thus his payoff, by a discrete amount. Thus, $s(v)$ is strictly increasing. This implies $P(s(v)) = F_n(v) \equiv F(v)^{n-1}$.

Continuity of s follows since otherwise a type just above a discontinuity could gain by lowering his bid. For differentiability, note that we can rewrite the bidder's optimality condition as

$$v\in \arg\max_{v'} F_n(v')(v-ES(s(v'),v)).$$

Letting $u(s, v) = v - ES(s,v)$, this implies that for any $v' > v$,

$$F_n(v)u(s(v),v)\geq F_n(v')u(s(v'),v)=F_n(v')\left[u(s(v),v)+u_1(s^*,v)(s(v')-s(v))\right],$$

for some s^* between $s(v)$ and $s(v')$. Since $u_1 < 0$, this can be rewritten as

$$\frac{F_n(v')-F_n(v)}{v'-v}\times\frac{u(s(v),v)}{-F_n(v')u_1(s^*,v)}\leq\frac{s(v')-s(v)}{v'-v}.$$

Changing the roles of v and v' yields, for some s^{**} between $s(v)$ and $s(v')$,

$$\frac{F_n(v')-F_n(v)}{v'-v}\times\frac{u(s(v'),v')}{-F_n(v)u_1(s^{**},v')}\geq\frac{s(v')-s(v)}{v'-v}.$$

Taking limits establishes the differential equation for s.

For the boundary condition, note that $P(s(v_L)) = 0$, and since all types earn non-negative profits $ES(s(v_L), v_L) \leq v_L$. But if the inequality were strict, the lowest type could raise his bid and earn positive profits with positive probability.

Having established uniqueness, it remains to verify existence by establishing the sufficiency of the bidder's first order condition. Consider any s' such that $s(v_L) < s' < s(v)$. There exists $v_L < v' < v$ such that $s(v') = s'$. Thus, by Assumption C,

$$\pi_s(s', v) > \pi_s(s', v') = 0.$$

A similar argument shows that for $s(v) < s' < s(v_H)$, $\pi_s(s', v) < 0$. Hence, π is quasiconcave in s and the first order condition is sufficient. ◆

PROOF OF LEMMA 4: Using the revelation principle, note that if type v reports v' he will win with probability $F^{n-1}(v')$. His expected payoff conditional on winning is equal to $(v - T(v, v'))$, where $T(v, v')$ is the expected payment by type v when he reports v'. Thus, type v will choose v' to maximize $F^{n-1}(v')\left(v - T(v,v')\right)$. Thus, we need to establish the correct form for T.

Letting V_{-i}^* be the highest type excluding i, bidder i wins with report v' if $V_{-i}^* < v'$. Let $\tilde{S}_{v'} \in \boldsymbol{S}$ be the random security that he will pay if he wins. Then define

$$\hat{S}_{v'}(z) = E\left[\tilde{S}_{v'}(z) \,\middle|\, V_{-i}^* \leq v'\right], \tag{8}$$

a security in the convex hull of $\boldsymbol{S}$ (which does not depend on i by symmetry). This is the "expected security" paid with a report of v'. Using the fact that types are independent and that Z_i and V_{-i} are independent given V_i (private values), we then have that:

$$\begin{aligned}
T(v,v') &= E\left[\tilde{S}_{v'}(Z_i) \,\middle|\, V_i = v, V_{-i}^* \leq v'\right] \\
&= E\left[E\left[\tilde{S}_{v'}(Z_i) \,\middle|\, Z_i, V_i = v, V_{-i}^* \leq v'\right] \,\middle|\, V_i = v, V_{-i}^* \leq v'\right] \\
&= E\left[E\left[\tilde{S}_{v'}(Z_i) \,\middle|\, Z_i, V_{-i}^* \leq v'\right] \,\middle|\, V_i = v, V_{-i}^* \leq v'\right] \\
&= E\left[\hat{S}_{v'}(Z_i) \,\middle|\, V_i = v, V_{-i}^* \leq v'\right] \\
&= E\hat{S}_{v'}(v)
\end{aligned}$$

This completes the proof. ◆

PROOF OF LEMMA 5: Let $H(z) = S_1(z) - S_2(z)$. Then if $EH(v^*) = 0$,

$$\begin{aligned}
EH'(v^*) &= \int H(z)\, h_v(z \mid v^*)\, dv = \int H(z) \left[\frac{h_v(z \mid v^*)}{h(z \mid v^*)}\right] h(z \mid v^*)\, dv \\
&= \int H(z) \left[\frac{h_v(z \mid v^*)}{h(z \mid v^*)} - \frac{h_v(z^* \mid v^*)}{h(z^* \mid v^*)}\right] h(z \mid v^*)\, dv
\end{aligned}$$

From SMLRP, $[\, h_v / h \,]$ is strictly increasing in z. Therefore,

$$H(z)\left[\frac{h_v(z\mid v^*)}{h(z\mid v^*)}-\frac{h_v(z^*\mid v^*)}{h(z^*\mid v^*)}\right]\geq 0\,,$$

and the inequality is strict on the set $\{z : S_1(z) \neq S_2(z)\}$. Thus, $EH'(v^*) > 0$. ◆

PROOF OF PROPOSITION I: Consider a second price auction. From Lemma 2, the winner is the highest type, V^1, and pays the second highest security bid by the second highest type, V^2. Let S_1 be the security bid by type V^2 under $\boldsymbol{S_1}$ and S_2 be the security bid by type V^2 under $\boldsymbol{S_2}$. Since $ES(s(v), v) = v$ in a second-price auction, $V^2 = ES_1(V^2) = ES_2(V^2)$. But then, since $\boldsymbol{S_1}$ is steeper, S_1 strictly crosses S_2 from below and so $ES_1(V^1) > ES_2(V^1)$. Thus, the seller's expected revenues are higher under $\boldsymbol{S_1}$.

For the first price auction, let $U^j(v)$ be the equilibrium payoff for type v, and S_v^j the security bid, with the set $\boldsymbol{S_j}$. First, note that efficiency implies $U^1(v_L) = U^2(v_L) = 0$. Since

$$U^j(v) \equiv \max_{\hat{v}} F^{n-1}(\hat{v})(v - ES_{\hat{v}}^j(v)) = F^{n-1}(v)(v - ES_v^j(v))\,,$$

if $U^1(v) = U^2(v)$, we have $ES_v^1(v) = ES_v^2(v)$. But if $\boldsymbol{S_1}$ is steeper than $\boldsymbol{S_2}$, $ES_v^1{}'(v) > ES_v^2{}'(v)$. Thus, from the envelope condition,

$$U^1{}'(v) = F^{n-1}(v)(1 - ES_v^1{}'(v)) \;<\; F^{n-1}(v)(1 - ES_v^2{}'(v)) = U^2{}'(v)\,.$$

Hence, $U^1(v) < U^2(v)$ for $v > v_L$. Since bidders' payoffs are lower, the seller's expected revenue is higher for each realization of the winning type under $\boldsymbol{S_1}$. ◆

PROOF OF LEMMA 6: For debt securities, consider any feasible security S_2. If $S_2(z) > \min(d, z)$, then $z > d$ and so $S_2(z') > \min(d, z')$ for all $z' > z$. Hence $\min(d, z)$ crosses S_2 from above.

For levered equity, note that a convex combination of these securities for different levels of leverage is a security $S_2(z)$ that is convex in z with maximum slope α. Thus, any levered equity security crosses S_2 from below. A similar argument applies to call options, and to convertible debt when indexed by the equity share α. ◆

PROOF OF PROPOSITION II: Consider the direct revelation game corresponding to the two auctions. Let S_v^1 be the security bid in the first-price auction, and let S_v^2 be the *expected* security payment in the second-price auction for a winner with type v, defined according to (8) in the proof of Lemma 4. Then, if the set of securities is super-convex, S_v^1 crosses S_v^2 from below, and the identical argument used in the proof of Proposition I for first-price auctions can be applied to prove that the seller's expected revenues are higher in the first-price auction. The proof for sub-convex sets is identical, with the inequalities reversed. ◆

PROOF OF PROPOSITION V: We prove our result in several steps.

STEP 1: The equilibrium from a first-price debt auction is a D1 equilibrium in the unrestricted auction.

To show this, we need to demonstrate that given the strategies from the debt auction, there is a set of beliefs satisfying D1 that support this equilibrium in the unrestricted auction. We construct the beliefs R using (4) and (6). If (6) does not produce a unique score, we can choose the lowest one. We now show that this supports the equilibrium.

STEP 1A: For debt contracts, the score is increasing in the face value of the debt. That is, $R(S^d)$ is strictly increasing in d, where $S^d(z) = \min(d,z)$.

Note that

$$\arg\min_v B(S^d, v) = \arg\max_v \frac{v - ES^d(v)}{U(v)}.$$

Because the objective function is strictly log-supermodular, from Topkis (1978) we know that v is weakly increasing with d. Thus, (6) implies $R(S^d)$ is strictly increasing in d.

STEP 1B: R supports an equilibrium in the unrestricted auction.

Consider any deviation to a debt contract. From Step 1a, the probability of winning the auction is the same as in a first-price auction. Since we have a first-price equilibrium, there is no gain to the deviation.

Consider a deviation to a non-debt contract S. Let v be the highest type in the set that minimizes $B(S,v)$. Then $ES(v) \geq R(S)$. It is sufficient to show that type v does not find the deviation to S profitable; i.e. to show that $P(R(S)) \leq B(S,v)$.

Find d such that $ES^d(v) = ES(v)$. Then $B(S^d,v) = B(S,v)$. From Lemma 5, types $v' < v$ find S^d more expensive than S, so that $B(S^d,v') > B(S,v') \geq B(S,v)$. Therefore, $\arg\min_{v'} B(S^d, v') \geq v$, and so from (6),

$$R(S^d) \geq ES^d(v) \geq R(S).$$

Thus, if a deviation to S is profitable, so is a deviation to S^d. But this contradicts the fact that no deviation to a debt contract is profitable.

STEP 2: A symmetric D1 equilibrium in the unrestricted auction has the same payoffs as the equilibrium of the first-price debt auction.

Our method of proof is to show that any non-debt bids can be replaced with an equivalent debt bid without changing the equilibrium.

STEP 2A: If S is not a debt contract, then at most one type uses this security.

Suppose not, so that $v_1 < v_2$ are the lowest and highest types that use S. Then, by (4), $R(S) = ES(v^*)$ for some $v_1 < v^* < v_2$. Consider the debt contract S^d with the same cost for type v^*; i.e. such that $ES^d(v^*) = ES(v^*)$. From Lemma 5, types $v < v^*$ find the S^d more expensive than S, so that $B(S^d, v) > P(R(S))$. Therefore, $\arg\min_v B(S^d, v) \geq v^*$. Thus, from (6), $R(S^d) \geq ES^d(v^*) = R(S)$. This implies that v_2 bidding S contradicts (5), since S^d is strictly cheaper and yields a weakly higher score.

STEP 2B: If type v uses contract S, and if $ES^d(v) = ES(v)$, then $P(R(S^d)) = P(ES^d(v)) = P(R(S))$ and S^d is also optimal for type v.

Note first that $P(R(S^d)) \leq P(R(S))$ by (5). However, by the same argument used in the previous step, $B(S^d, v') > P(R(S))$ for $v' < v$, and hence $R(S^d) \geq ES^d(v) = R(S)$. Thus, $P(R(S^d)) = P(R(S))$. Since the cost and probability of acceptance are the same, bid S^d is also optimal for type v.

STEP 2C: $R(S^d)$ is strictly increasing in d. (Same proof as Step 1a.)

STEP 2D: Let $d(v)$ be an optimal face value of debt for each type v in Step 2b. Then $d(v)$ is strictly increasing in v.

From Step 2b, there exists $d(v)$ that solves

$$U(v) = P(R(S^{d(v)}))(v - ES^{d(v)}(v)) = \max_d P(R(S^d))(v - ES^d(v)).$$

From Assumption C, the objective function is strictly log-supermodular, so that $d(v)$ is weakly increasing in v.

Suppose $d(v)$ were constant on an interval. Then, from step 2b, all types on that interval have the same probability of winning. Since strategies are symmetric, this implies they must tie. But then the highest type can increase his debt marginally, increase his score (by Step 2c), and raise his probability of winning by a discrete amount. This contradicts the optimality of $d(v)$.

STEP 2E: If type v uses contract S, then $R(S^{d(v)}) = R(S)$.

From Step 2b, $R(S^d) \geq R(S)$. But if $R(S^d) > R(S)$, this implies d is optimal for some type $v' > v$, contradicting Step 2d.

STEP 2F: Bidding $d(v)$ is an equilibrium in the unrestricted auction, and in a first-price debt auction.

From Step 2d and 2e, payoffs and scores are unchanged. Thus, it is an equilibrium in the unrestricted auction.

Since $R(S^d)$ is strictly increasing, $P(R(S^d)) = \text{Prob}(d \text{ is highest debt bid})$. Thus, this is also an equilibrium of a first-price auction.

STEP 3: In a symmetric D1 equilibrium in the unrestricted auction, almost every bid is a debt contract.

From Step 2 and Lemma 3, the equilibrium payoff from type v is given by

$$U(v) = F^{n-1}(v)\,(v - ES^{d(v)}(v)),$$

and so U is differentiable. Suppose type $v \in (v_L, v_H)$ bids S in equilibrium. Then by a standard envelope argument,

$$U'(v) = F^{n-1}(v)(1 - ES'(v)).$$

Thus, $ES(v) = ES^{d(v)}(v)$ and $ES'(v) = ES^{d(v)\prime}(v)$. Thus, from Lemma 5, $S = S^{d(v)}$. ◆

PROOF OF PROPOSITION VI:

Case 1: First and Second Price Auctions

Let s' be a bid that wins with positive probability such that $S(s',0) < 0$. Then, due to the moral hazard problem, submitting this bid earns strictly positive profits for any type, since any type can simply not invest and collect $-S(s',0)$ in a first-price auction, or even more in a second-price auction (since the second highest bid is below s'). Thus, by incentive compatibility, all equilibrium bids earn positive profits.

Let $\underline{s}$ be the lowest bid submitted. Then the above implies this bid must win with positive probability. Since it is the lowest bid, this implies a tie -- that is, $\underline{s}$ is submitted with positive probability. But then raising the bid slightly would lead to a discrete jump in the probability of winning and hence in profits.

Incentive compatibility therefore implies $\underline{s} = s_1$, the highest possible bid. If $S(s_1,0) \geq 0$, this contradicts the existence of s'. If $S(s_1,0) < 0$, then all types bid s_1. But at s_1, all types lose money if they run the project. Therefore, all types bid s_1, do not invest, and collect $-S(s_1, 0) > 0$ from the seller.

Case 2: General Mechanisms

In an efficient mechanism, the lowest type wins with zero probability and so earns zero expected profits. Since lowest type can claim to be any type and not invest, it must be the case that no type with positive probability of winning pays a security with $S(0) < 0$ with positive probability.

Case 3: Informal Auctions

Here the result follows immediately from our result that an equilibrium will always use the flattest possible securities. If $S(0) < 0$, we can "flatten" the security by raising $S(0)$ and flattening it elsewhere. ◆

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