The Value of Information in Matching Tournaments*

Anne-Katrin Roesler†

September 30, 2013

Abstract

We study a two-sided, one-to-one matching market with a finite number of agents, *candidates* and *firms*. There is incomplete information on both sides of the market and, a priori, candidates are uncertain about their own characteristics. We present a model and methods that allow to analyze the value of information in matching tournaments in which agents engage in wasteful signaling to compete for match partners. A higher information level of market participants has two, possibly opposing, effects. It increases expected total match output but may also increase investments in wasteful signaling. Overall welfare may be decreasing but we identify conditions under which it is increasing. We characterize and compare the socially optimal and candidate optimal levels of information. We find that, if the number of candidates is large enough, the candidate optimal level is below the socially optimal information level.

JEL classification: C78, D82, D83

Keywords: Matching, Information, Signaling, Stochastic orderings

*I would like to thank Benny Moldovanu for advice throughout the project. I also thank Andrei Barbos, Dirk Bergemann, Maximilian Conze, Ian Jewitt, Daniel Krähmer, Larry Samuelson, Nora Szech, Alexander Westkamp and Philipp Wichardt for valuable comments and suggestions. The paper has benefited from the comments of participants at seminars in Bonn and Saarbrücken, and at the Conference of Economic Design and the Spring Meeting of Young Economists 2013. Financial support from the DFG through Bonn Graduate School of Economics is gratefully acknowledged.

†Department of Economics, University of Bonn, 53113 Bonn, Germany; and Department of Economics, Yale University, 30 Hillhouse Avenue, New Haven, CT 06511, USA; anne-katrin.roesler@yale.edu

1 Introduction

Starting with the seminal papers by Gale and Shapley (1962), Shapley and Shubik (1971) and Becker (1973), there is a vast literature on matching markets. Insights from this research have not only helped to better understand school choice, college admission and labor markets but even lead to a redesign of some of these markets.[1]

Most of the theoretical analysis of matching markets focuses on settings in which agents know their preferences over potential match alternatives. However, it is often the case that agents have preferences over characteristics of match partners, but the actual realization of these characteristics is private information. For instance, in job markets candidates compete for the most attractive jobs and firms try to recruit the most qualified workers.[2] Both, the individual features of jobs, as well as the abilities of candidates, are usually private information to the agents. They are revealed through – typically wasteful – signaling, for example, job advertisements, applications and interviews. Moreover, candidates are often uncertain about their own market value a priori. They may not know their ranking compared to their competitors, or how their abilities are evaluated by the other side of the market.

In this paper, we provide a theoretical analysis of this situation. In particular, we investigate how the information level of market participants affects the equilibrium outcome. We illustrate that a higher information level has two, possibly opposing, effects: On the one hand, it allows for a better allocation, on the other hand, investments in wasteful signaling may increase. As a consequence, better information of market participants does not always improve welfare.

We consider a two-sided, one-to-one matching market with a finite number of agents on both sides of the market. Our model is a modification of the marriage market formulated in Becker (1973) with two-sided incomplete

[1] Prominent examples include the kidney paired donation program in the US, motivated by Roth et al. (2004, 2005), and school choice in New York and Boston (cf. Abdulkadiroglu et al. (2005, 2009))

[2] Which jobs are considered to be the most desirable may differ across professions and even across individuals. In this paper, we make the simplifying assumption that there exist common criteria to measure the desirability of a job, for example a low workload-salary ratio.

information and one side of the market whose agents are a priori uncertain about their own characteristics. Job markets are a typical application, and we refer to the two sides of the market as *candidates* and *firms*.

Our model has a two-period structure. In the first period agents obtain an informative, private signal about their individual characteristics and update their beliefs accordingly. Agents then enter a matching tournament as considered in Hoppe et al. (2009). Such a tournament is a generalization of a contest, in which prizes are replaced by matching opportunities and agents invest in non-productive, costly signaling à la Spence (1973) to compete for match partners. There exists a separating equilibrium in which agents are matched positively assortatively according to their signals. In our analysis we focus on this equilibrium.

We investigate the impact of the information level of market participants on equilibrium outcome, expected match output, signaling investments and welfare. Moreover, we characterize the socially optimal information level and compare it to the candidate optimal level which maximizes candidates' aggregate welfare. This social welfare criterion only takes the utilities of agents on one side of the market into account. Our comparison therefore illustrate differences between one- and two-sided matching markets. In addition, the results provide insights into the role of information management in these markets. A designer who is constrained to a given matching mechanism but can invoke a certain information level of market participants would implement the socially optimal level.[3]

A higher information level of candidates leads to more differentiation among candidates in the matching tournament. This allows for a better allocation which results in larger expected match surplus. More differentiation among candidates also raises the stakes for the firms of being matched to a better or worse partner which increases firms' expected signaling investments. The effect on candidates' expected signaling investments is less clear cut and depends on certain features of the market. For a higher information level of candidates, the marginal benefit from obtaining a better match is increasing for high-ranked candidates, whereas it is decreasing for low-ranked

[3] An example are standardized tests that provide students with information about their ranking in the market. The actual level of information of market participants depends on the precision of these tests.

candidates.[4] This results in high-ranked candidates increasing their investments in signaling whereas lower ranked candidates may invest less. If candidates constitute the long side of the market, only high-ranked candidates are matched in equilibrium and aggregate expected signaling is increasing. If there are more firms than candidates and the distribution of firms' characteristics has a increasing hazard rate, the effect on low-ranked candidates is dominant and candidates' aggregate expected signaling investments are decreasing.

Welfare of agents on the side of the market receiving more information is always increasing in expectation. However, we show that, due to the trade-off between a better allocation and a potential increase in wasteful signaling, more information of candidates may result in a decrease of total expected welfare.[5] We identify conditions which guarantee that total expected welfare is increasing.

Finally, we characterize the candidate optimal and the socially optimal level of information in matching tournaments, if information is costly. The relation between these two information levels is driven by the following feature which is only prevalent in two-sided markets: Agents do not only impose externalities on agents on the same side of the market but also on agents on the other side of the market. In particular, a candidate puts a positive externality on firms by providing a match opportunity. However, he also imposes a negative externality since better match opportunities lead to more competition among firms which results in higher signaling investments. For a higher information level of candidates, expected match surplus increases and also firms' share thereof. But firms face a more competitive environment in the sense that there is a stronger diversification between "winners" and "loosers" of the matching tournament. Consequently, firms' aggregate expected investments in signaling are increasing. If there is a large number of candidates, the first effect on firms' expected match surplus is dominant and the socially optimal information level is higher than the candidate optimal level. We obtain the reversed relation if candidates constitute the short side of the market and the distribution of firms' characteristics has an increasing

[4] To be precise we mean non-increasing here. If there are more candidates than firms, for the lowest-ranked candidates the marginal benefit does not change because they are never matched in equilibrium

[5] The sum of utilities of agents on both sides of the market

hazard rate.

Related literature The papers most closely related to our work are Hoppe et al. (2009) and Ganuza and Penalva (2010). Hoppe et al. (2009) study a matching tournament with two-sided incomplete information in which privately informed agents invest in costly signaling to compete for match partners. The authors discuss comparative statics and compare random and assortative matching. In our model we introduce an information stage and identify a higher information level of agents as a source of more heterogeneity in agents' types. This link is established by the observation that more informative signals yield a more dispersed distribution of conditional expectations. We use this insight to define a precision criterion similar to the one in Ganuza and Penalva (2010) to measure the informativeness of signals in the information stage. We then follow the elegant approach of Hoppe et al. (2009) and adopt methods from reliability theory to analyze the effects of different levels of information in the matching tournament.

Our model incorporates not only the matching tournament of Hoppe et al. (2009) but also the private values auction setting of Ganuza and Penalva (2010) as a special case. If there is only one firm and a finite number of candidates, our matching model corresponds to the private value auction setting. The methods from reliability used to obtain our results directly imply the results in Ganuza and Penalva (2010) about the effect of more precise information on the expected value and information rents of the winning bidder. The methods even yield slightly stronger results as was also pointed out in Shaked et al. (2012)

Our paper is also related to the literature on matching tournaments which were introduced by Cole et al. (1992, 1995). This research usually considers markets with a continuum of agents on both sides, and studies complete information models in which agents preferences, types and match values are common knowledge. Our analysis differs in that we investigate a small market with a finite number of agents, and incomplete information.

Some recent papers, like Mailath et al. (2013) or Hopkins (2012), consider incomplete information models and issues of screening and signaling which arise therein. In a setting with costly signaling, Hopkins (2012) studies comparative static effects if agents face a more competitive environment. Stronger

competition is modeled as a change in the type distribution in terms of first-order stochastic dominance. Similarly, pre-match investments, as studied in Cole et al. (2001), Peters and Siow (2002) or Mailath et al. (2013), generate first-order effects on agents' types. By contrast, in our analysis, investments in information yield second-order effects in the sense that a higher level of information leads to a more dispersed distribution of agents' types in the second-stage matching tournament.

This paper is also connected to the literature on information disclosure in auctions. Ganuza and Penalva (2010) discuss the effects of different information levels of buyers in a second-price auction, whereas Bergemann and Pesendorfer (2007) and Eso and Szentes (2007) adopt a mechanism design perspective. In a private values environment, these papers discuss the revenue maximizing information structure and selling mechanism for the seller. Similar to Ganuza and Penalva (2010) we focus on a given mechanism and study how different information levels affect the equilibrium. That is, in our setting the designer is constrained to a given matching mechanism, which he cannot adjust to the information provided to market participants. However, the most notable distinction between information disclosure in our matching setting and auctions is the following: a seller faces a trade-off between efficiency and having to leave information rents to the buyers, whereas in our matching market the trade-off is between allocative efficiency and wasteful signaling investments.

Outline The rest of the paper is organized as follows. In Section 2 we present the model and, for exogenously given levels of information, properties of the symmetric separating equilibrium in the matching tournament. We introduce a precision criterion to measure the informational content of information technologies in Section 3 and discuss the comparative statics effects of a higher information level of market participants. Section 4 contains our results on the candidate optimal and socially optimal levels of information. All proofs are relegated to the appendix. In the appendix, we also provide a discussion of different precision criteria (cf. Appendix C).

2 The Model

We consider a two-sided market with a finite set of candidates, $\mathcal{I} = \{1, \ldots, n\}$, and a finite set of firms, $\mathcal{J} = \{1, \ldots, k\}$. Characteristics of candidates, x_i, and firms, y_i, are determined by iid draws from interval $[0, \overline{x}]$, respectively $[0, \overline{y}]$. If $\overline{x}$ or $\overline{y}$ are infinity, characteristics are drawn from $[0, \infty)$. Agents' characteristics are independently distributed with prior distribution F_X for candidates and F_Y for firms. We assume throughout the paper that, $F_X(0) = F_Y(0) = 0$, F_X and F_Y are continuously differentiable with positive densities, $f_X > 0$ and $f_Y > 0$, on the support.

If candidate i is matched with firm j, the match output is $2x_i y_j$ which is equally split among match partners. This multiplicative model implies that, under complete information, all candidates agree on the ranking of firms and all firms agree on a ranking of the candidates. Thus, *positive assortative matching* is the allocation which maximizes expected match output.

We assume that there is incomplete information on both sides of the market. Firms' characteristics are private information to firms whereas candidates do not know their characteristics ex-ante. That is, candidates have only distributional knowledge about the value of their own characteristics as well as of the characteristics of all other agents.[6]

We consider a two-period model which consists of an *information stage* followed by a *matching stage*.

2.1 Information Stage

In the first period, the *information stage*, every candidate observes a private signal realization which is informative, but typically noisy, about his own characteristic, from an information technology Θ.

Formally, an information technology is a signal Θ with typical realizations $\theta \in \left[0, \overline{\theta}\right]$ which is characterized by a family of distributions $\{G(\theta|x)\}_{x \in X}$ where

$$G(\theta|x) := Pr(\Theta \leq \theta | X = x),$$

is the probability that a candidate with characteristic x receives a signal

[6] We consider this model to simplify notation. It is straightforward to extend the current model such that agents on both sides of the market are uncertain about their characteristics. A discussion is available from the author upon request.

realization $\theta' \leq \theta$. We assume that for every $x \in X$, $G(\cdot|x)$ admits a density function $g(\cdot|x)$. Together with the prior distribution F_X, an information technology induces a joint distribution (an *information structure*) on (X, Θ). Agents update their beliefs according to Bayes' rule and the posterior distribution of X conditional on $\Theta = \theta$ is $G(x|\theta)$.

We assume that signals satisfy the strict monotone likelihood ratio property introduced by Milgrom (1981) which implies that candidates with high characteristics are more likely to observe a high signal realization than candidates with lower characteristics.

Assumption 1. (MLRP) $\{G(\theta|x)\}_{x\in\mathcal{X}}$ *have the strict* monotone likelihood ratio property *(MLRP), that is, for every* $x > x'$, $\frac{g(\theta|x)}{g(\theta|x')}$ *is strictly increasing in* θ.

This implies that for every $x > x'$, $G(\cdot|x)$ first-order stochastically dominates $G(\cdot|x')$. Consequently, the conditional expectations $E[X|\theta]$ are strictly increasing in θ.

Every information technology, Θ, results in a distribution of conditional expected values for candidates. This is represented by the random variable $\widehat{X} := E[X|\Theta]$ with distribution function $H(\hat{x}) := Pr(\{\theta|E[X|\theta] \leq \hat{x}\})$. A simple information technology which is commonly used in the literature is the following *truth-or-noise* technology.

Example 1 (Truth-or-noise technology)**.** Let X with distribution F_X represent the state of the world. A truth-or-noise technology provides with some probability $\alpha \in [0, 1]$ a perfectly informative signal $\theta = x$ and with probability $(1 - \alpha)$ pure noise, independently drawn from prior distribution F_X. The receiver cannot distinguish which kind of signal he observes. For signal realization θ, the conditional expected value is $E[X|\theta] = \alpha\theta + (1-\alpha)E[X]$. $\triangle$

2.2 Matching Tournament

In the second period, the *matching stage*, the agents on both sides of the market enter a matching tournament, a contest in which prizes are replace by match opportunities. In the matching tournament, all agents simultaneously choose an individual investment which serves as a costly signal of their

characteristic. As in Spence (1973) we assume that investments are non-productive, that is, they solely serve as an observable signal of the agent's unobservable characteristic but do not have an effect on the match output. This corresponds to the case of *valueless signaling* in Hopkins (2012). It should be noted that we use the term 'signaling investments' here solely to avoid confusion with signals in the information stage. Agents are ranked based on their signaling investments to then be matched positively assortatively. In case of equal investments, we assume random tie-breaking. We follow the common practice in the matching tournament literature and do not specify the allocation mechanism which is used.[7]

If candidate i sends signal b and is matched with firm j his payoff is

$$x_i y_j - b.$$

In our setting, candidates do not know their characteristics a priori. Thus, in the second stage matching tournament, candidates can only condition their investment decisions on the information obtained in the information stage. For a given prior distribution, F_X, the information technology Θ determines the distributions of candidates' conditional expected characteristics in the matching tournament, $H^{\Theta}(\widehat{X})$ which is common knowledge. The individual expected values conditional on the private signal realizations, $\hat{x}_i = E[X|\theta_i]$, are private information of the candidates. Similarly, for firms, characteristic y_j is private information to firm j.

If a candidate with conditional expected value $\hat{x}_i$ is matched to a firm with characteristic y_j, given the linearity of our model, the expected match value is $\hat{x}_i y_j$ for each of the match partners.

2.3 Separating Equilibrium

In this section we review some results of Hoppe et al. (2009) who establish the existence of a symmetric separating equilibrium in matching tournaments and provide formulas for expected total output, signaling investments and welfare in equilibrium. By adapting their results to our setting, we obtain

[7] A matching mechanism which would achieve this outcome is, for example, the firm- (respectively candidate-) proposing deferred acceptance algorithm, assuming that agents rank their potential match-partners according to the observed investments.

the existence of a symmetric separating equilibrium in the matching stage in which firms with higher characteristics and candidates with higher conditional expected characteristics invest more. By the MLRP such strategies are increasing in signal realizations received by candidates in the information stage.

Theorem 1 (Hoppe et al. (2009))**.** *Given the assumptions in Section 2, in the matching tournament, there exists a symmetric separating equilibrium in monotone strategies in which agents with higher (conditional expected) characteristics choose higher signaling investments.*

In our analysis we focus on this separating equilibrium which implements the positive assortative matching with respect to (conditional expected) characteristics of agents. This matching is stable, given the information available in the market after the information stage. Before we can provide the formulas for expected total output, investments and welfare in equilibrium we need to introduce some more notation.

For a sample $X_1, \ldots, X_n$ let

$$X_{1:n} \geq_{st} \cdots \geq_{st} X_{n:n}$$

be the corresponding order statistics where $\geq_{st}$ indicates first-order stochastic dominance. That is, $X_{1:n}$ denotes the highest order statistics, which represents the distribution of the maximal value among n iid draws.[8]

For given market sizes n, k, priors F_X, F_Y and information technology Θ set

$$\mu_{i:n}^{\Theta} := E\left[\widehat{X}_{i:n}\right] \quad \text{and} \quad \eta_{i:k} := E\left[Y_{i:k}\right].$$

That is, $\mu_{i:n}^{\Theta}$ denotes the expected value of the ith order statistics of the conditional expected characteristics given information technology Θ. Table 1 displays the formulas for expected total output, investments and welfare in equilibrium.

Hoppe et al. (2009) relate agents' investments to Vickrey-payments. We recapitulate this interpretation here to provide the reader with some intuition

[8]Hereby, we adopt the notation which is used in most economics literature. It should be noted that, by contrast, Hoppe et al. (2009) employ the standard notation in statistics in which $X_{n:n}$ denotes the highest order statistics.

	candidates	**firms**
exp. total output	$O = 2\sum_{i=1}^{\min} \mu_{i:n}^{\Theta} \cdot \eta_{i:k}$	
exp. total investments	$S_c = \sum_{i=1}^{\min} i\big(\eta_{i:k} - \eta_{i+1:k}\big)\mu_{i+1:n}^{\Theta}$	$S_f = \sum_{i=1}^{\min} i\big(\mu_{i:n}^{\Theta} - \mu_{i+1:n}^{\Theta}\big)\eta_{i+1:k}$
exp. total welfare	$W_c = \sum_{i=1}^{\min} i \cdot \big(\mu_{i:n}^{\Theta} - \mu_{i+1:n}^{\Theta}\big)\eta_{i:k}$	$W_f = \sum_{i=1}^{\min} i \cdot \big(\eta_{i:k} - \eta_{i+1:k}\big)\mu_{i:n}^{\Theta}$
exp. aggregate welfare	$W = \sum_{i=1}^{\min} i \cdot \big(2\mu_{i:n}^{\Theta}\eta_{i:k} - \mu_{i+1:n}^{\Theta}\eta_{i:k} - \mu_{i:n}^{\Theta}\eta_{i+1:k}\big)$	

Table 1: Formulas for expected total output, aggregate welfare, and total investments and welfare for candidates and firms, respectively, for given prior distributions, information technology Θ, and market sizes, n, k.
Here, $\min := \min\{k, n\}$

for the formulas in Table 1 which we will use repeatedly in our analysis. After the information stage, the situation for candidates in the matching tournament is as if they are in a contest competing for k heterogeneous prizes. For a candidate receiving signal realization θ, the values of the prizes are $E[X|\theta] \cdot \eta_{1:k}, \ldots, E[X|\theta] \cdot \eta_{k:k}$. The allocation of the matching tournament is positive assortative with respect to signaling investments. Since our model is linear, expected payoffs are already determined (up to a constant) by the allocation rule. A candidate who receives signal realization 0 does not invest in signaling in the matching tournament, is matched to the lowest firm with certainty and his expected payoff is $E[X|0] \cdot \eta_{n:k}$ (which is 0 if $n > k$). Consequently, by the revenue equivalence theorem, expected total payments (here: expected total signaling investments of candidates) must be the same as in a VCG-mechanism.[9]

In the VCG-mechanism, each candidate has to pay an amount equal to the negative externalities he imposes on the other candidates. For a profile of signal realizations $\theta_1 \geq \cdots \geq \theta_n$, refer to the candidate receiving signal θ_i as *type* i. The presence of type i does not affect candidates who receive a higher signal than himself but he imposes a negative externality on all can-

[9]The well-known Vickrey-Clarke-Groves (VCG) mechanisms due to Vickrey (1961), Clarke (1971) and Groves (1973)

didates receiving a lower signal realization. Each of those candidates would be assigned to a higher match-partner if type i were not present. It follows that the expected signaling investment, s_i, of the ith type is:

$$s_i = \sum_{j=i}^{\min\{k,n\}} \mu_{j+1:n} \cdot (\eta_{j:k} - \eta_{j+1:k}). \tag{1}$$

Summing up over all i we obtain the formula for total expected investments of candidates.

3 Precision of information technologies

We now introduce a criterion, *precision*, to compare information technologies in terms of their informational content and to discuss comparative static effects of a higher level of information of candidates on total match output, total signaling investments and welfare (cf. Subsection 3.1).

Given our assumption that signals have the monotone likelihood ratio property, the natural informativeness criterion to use is the concept of *effectiveness* introduced by Lehmann (1988).[10] Mizuno (2006) shows that for a more effective signal about X the resulting distribution of conditional expectations is more dispersed. We use this observation to define a precision criterion in terms of properties of the resulting distribution of conditional expectations. Our criterion is similar to the notions of *integral* and *supermodular precision* in Ganuza and Penalva (2010) and is based on the following insight: for a completely random signal, i.e., an information technology which is uninformative about X, the resulting conditional expectation is always the ex-ante mean. For a more informative technology, the resulting distribution of conditional expectations will be more responsive to the signal realization and thus result in a more variable distribution of conditional expectations. As a formal variability criterion we use the star order which requires single-crossing of the quantiles.

[10] Persico (2000) refers to this concept as *accuracy*. Effectiveness applies to monotone decision problems and requires less restrictive conditions than *sufficiency* (Blackwell (1951)) to compare signals in terms of their informativeness. A more formal presentation and a discussion of the relation to other informativeness criteria is provided in Appendix C.

Definition 1. (i) For random variables Z_1 and Z_2 with distributions H_1 and H_2 and interval support we say that Z_2 is greater than Z_1 in the *star order*, $Z_2 \geq_* Z_1$, if $H_2^{-1}(H_1(z_1))$ is *starshaped* in z_1 on the support of Z_1. This is equivalent to

$$\frac{H_2^{-1}(u)}{H_1^{-1}(u)} \text{ is increasing in } u \in (0,1).$$

(ii) We say that, for a given prior F_X, information technology Θ_2 is more *precise* than Θ_1, denoted $\Theta_2 \succsim_* \Theta_1$, if $E[X|\Theta_2]$ is greater in the star order than $E[X|\Theta_1]$.

$$\Theta_2 \succsim_* \Theta_1 \text{ if } [X|\Theta_2] \geq_* [X|\Theta_1]$$

We say that agents have a *higher information level* if the private signal realizations which agents receive in the information stage originate from a more precise information technology.

For random variables with the same finite mean, if Z_2 is greater than Z_1 in the star order then Z_2 is a mean-preserving spread of Z_1, and, moreover, Z_2 is at least as negatively skewed as Z_1 on average. For our concept of precision we thus obtain that, roughly put, for a more precise information technology the resulting distribution of conditional expectations is more dispersed and skewed. For the truth-or-noise technologies of Example 1, it holds that Θ_α is more precise than Θ_β if and only if $\alpha \geq \beta$.[11]

Lemma 1 presents two important properties of the conditional expectations from information technologies which are ordered in terms of precision. A more precise information technology results in larger expected spacings of the mean order statistics of the conditional expectations. Moreover, the vectors of mean order statistics of the conditional expectations satisfy a "single-crossing condition". In particular, switching to a more precise information technology results in an increase of the highest order statistics of the conditional expectations whereas the expected values of lower order statistics will decrease. The following lemma states these results as they apply to our setting.

[11] This follows from theorem 4.B.5 in Shaked and Shanthikumar (2007) and the fact that $\frac{\beta\theta+(1-\beta)E[X]}{\alpha\theta+(1-\alpha)E[X]}$ is increasing in θ if and only if $\beta > \alpha$.

Lemma 1. *(i) If information technology Θ_2 is more precise than Θ_1 then for $i = 1, \ldots, n-1$*

$$\mu_{i:n}^{\Theta_2} - \mu_{i+1:n}^{\Theta_2} \geq \mu_{i:n}^{\Theta_1} - \mu_{i+1:n}^{\Theta_1}$$

(ii) If information technology Θ_2 is more precise than Θ_1 then for all $n \in \mathbb{N}$

$$\frac{\mu_{i:n}^{\Theta_1}}{\mu_{i:n}^{\Theta_2}} \text{ is increasing in } i.$$

Both properties can be thought of as implications of the more dispersed distribution of conditional expectations which results from a more precise information technology.

3.1 The comparative statics effects of higher precision

We now analyze the effects of a change in the information level of candidates on total match output, investments and welfare.

In our model agents are a priori uncertain about their own characteristics and the outcome depends on the signals received by candidates in the information stage. Thus, from an ex-ante perspective, the matching mechanism yields a lottery over all possible matchings of candidates to firms. In the two extreme cases, candidates either receive no information about their own characteristics, which results in random matching, or they observe a perfectly informative signal and are matched positive assortatively in equilibrium. Since signals have the MLRP, candidates with high characteristics are more likely to receive a high signal in the information stage. Therefore, for a higher information level of candidates the probability that candidates with high characteristics are matched with firms of a similar ranking increases. This effect – which results in an increase of total match output – is also observed in empirical studies.[12]

[12]For example, in their study Hoxby and Turner (2013) provide a subgroup of high-school seniors with additional information about their college opportunities and find that for students who received information the probability to enroll in a college that matches their abilities increases significantly.

Proposition 1. *Total match output is increasing in signal precision:*

$$\Theta_2 \succsim_* \Theta_1 \quad \Rightarrow \quad O(\Theta_2) \geq O(\Theta_1) \quad \forall k,\, n \in \mathbb{N}$$

It should be noted that even though the effect of a higher information level of candidates on total match output is always positive, the match output for some agents may decrease. In general, more precise information results in a more dispersed distribution of conditional expectations of candidates. Therefore, the expected match output for agents with high characteristics increases whereas it decreases for those with low characteristics.

Another implication of a higher information level of candidates is that low-type candidates reduce their signaling investments[13] whereas high-type agents may invest more (cf. Equation 1). On an aggregate level it is not clear a priori which of these effects is dominant and some of the following results will depend on distributional properties of firms' characteristics.

We show that if candidates are on the long side of the market and the number of candidates is sufficiently large, then, candidates' total expected signaling investments are increasing in the level of information, irrespective of the distribution of firms' characteristics. However, total signaling is decreasing if there are more firms than candidates and the distribution of prizes F_Y has an *increasing hazard rate* (IHR), that is, if $\frac{f_Y(y)}{1-F_Y(y)}$ is monotone increasing in y.[14] Candidates' total welfare is always increasing in the level of information.

Theorem 2 (Candidates' expected total signaling investments and welfare)**.** *For information technologies Θ_1, Θ_2 such that Θ_2 is more precise than Θ_1*

(i) There exists some $\hat{n} \in \mathbb{N}$, $\hat{n} \geq k$ such that for all $n \geq \hat{n}$, total signaling of candidates is increasing in information precision, i.e.,

$$S_c(\Theta_2) \geq S_c(\Theta_1) \quad \forall\, n \geq \hat{n}.$$

[13]Candidates with zero investments do not adjust their investments

[14]This distributional property is a common assumption in the mechanism design literature. In statistics and reliability theory it is known as an *increasing failure rate*.

If the distribution of firms' characteristics, F_Y, has an increasing hazard rate, then total signaling of candidates is decreasing in information precision for $n \leq k$.

$$S_c(\Theta_2) \leq S_c(\Theta_1) \quad \textit{if } n \leq k \quad \textit{and } F_Y \textit{ is IHR.}$$

(ii) Candidates' total welfare is increasing in information precision,

$$W_c(\Theta_2) \geq W_c(\Theta_1) \quad \forall\, n,\, k \in \mathbb{N}.$$

The results incorporate as special cases both, the comparative statics results on heterogeneity of Hoppe et al. (2009) and the results of Ganuza and Penalva (2010) on expected valuation and informational rents of the winning bidder, and seller's expected revenue.

The private values auction setting of Ganuza and Penalva (2010) corresponds to the case of $k = 1$ and $Y \equiv 1$ and bidders being the candidates in our model. By Proposition 1 the expected valuation of the winning bidder, $E\left[\widehat{X}_{1:n}\right] = \frac{1}{2} \cdot O$, is increasing in the level of information, and so is his expected information rent, $E\left[\widehat{X}_{1:n} - \widehat{X}_{2:n}\right] = W_c$, and seller's expected revenue, $E\left[\widehat{X}_{2:n}\right] = S_c$,for n sufficiently large (cf. Theorem 2).

The intuition of the private values auction setting extends to matching tournaments. For a higher information level of candidates, the total informational rents of candidates, captured by W_c, are always increasing. Total expected signaling investments S_c represent the aggregate externalities candidates' impose on agents on the same side of the market. If there is a small number of candidates, $n < k$, then also the low-type candidates (i.e. those receiving one of the lowest signal realizations among a profile $\theta = (\theta_1, \ldots, \theta_n)$) are matched in the separating equilibrium of Theorem 1. For a higher information level in the market, the expected characteristics of the low-type candidates decrease. Consequently, for these candidates the (marginal) benefit in match output from being matched with a firm of higher ranking reduces and thus also the expected externalities imposed on them by other candidates. For high-type candidates the effect is reversed and externalities imposed on them are non-decreasing. If the distribution of firms' characteristics has an increasing hazard rate, the externalities imposed on low-type candidates have

a higher impact on aggregate signaling than those imposed on high-type candidates (cf. Theorem 7 and Table 1). As a result, for a higher information level of candidates the expected total signaling investments of candidates are decreasing.

However, if there are more candidates than firms, competition among candidates is so strong that only the higher-type candidates will be matched in a separating equilibrium. Therefore, the externalities imposed on high-type candidates will have more impact on total signaling investments and, for a sufficiently large number of candidates, this will result in total signaling investments of candidates being increasing in the level of information of market participants. Competition may even be so strong that the investment of each individual candidate type is nondecreasing in information. This can be established by combining Lemma 3 with Equation 1.

We can also characterize the effect of a higher information level of candidates on firms' expected total welfare and investments:

Theorem 3 (Firms' expected total welfare and investments)**.** *For information technologies Θ_1, Θ_2 such that Θ_2 is more precise than Θ_1:*

(i) Total expected signaling of firms is increasing in information precision,

$$S_f(\Theta_2) \geq S_f(\Theta_1) \quad \forall\, n,\, k \in \mathbb{N}.$$

(ii) Firms' expected total welfare is increasing in information precision if F_Y has a decreasing hazard rate (DHR).
If F_Y has an increasing hazard rate, then there exists some $\hat{n} > k$ such that firms' expected total welfare is increasing in information precision for all $n \geq \hat{n}$ whereas it is decreasing for $n \leq k$:

$$W_f(\Theta_2) \leq W_f(\Theta_1) \quad \text{if } k \geq n \text{ and } F_Y \text{ is IHR}$$
$$W_f(\Theta_2) \geq W_f(\Theta_1) \quad \text{if } F_Y \text{ is IHR and } n \geq \hat{n} > k; \quad \text{or, if } F_Y \text{ is DHR.}$$

If candidates have a higher information level, firms face a sample of potential match partners with a more heterogeneous distribution of conditional expected characteristics. Consequently, the expected difference between the match value of being paired with one of two candidates whose ranking differs

only by one increases which results in firms increasing their investments in signaling. This is also true on an individual level, that is, every firm will increase their investments. Among the firms which are matched in equilibrium, the match output of high ranked firms is increasing in the information level of candidates. For lower ranked firms it will typically be decreasing, unless there is a much larger number of candidates than firms. Thus, lower ranked firms will be worse off if candidates' hold more precise information whereas high ranked firms may profit.[15] The effect on total welfare of firms depends on which of these effects is dominant, and hinges on the distribution of firms' characteristics and the sizes of the two sides of the market.

Hoppe et al. (2009) show that random matching may be welfare superior to assortative matching. However in many settings agents are already partially informed and matching will be neither purely random nor perfectly assortative. We investigate this situation and discuss if it is welfare improving to provide candidates with additional and hence more precise information. Our results thus complement the analysis in Hoppe et al. (2009). From the previous analysis we know that, if the distribution of firms' attributes has a decreasing hazard rate, candidates' and firms' total welfare is increasing in their level of information technology and so is aggregate welfare. However, if the distribution of firms' characteristics has an increasing hazard rate and the number of candidates is small, then firms' total welfare is decreasing in candidates' information level and the effect on aggregate welfare is not clear.

To better understand these informational effects, it helps to decompose aggregate welfare as $W = o + (W_f - S_c)$; the sum of total match output of candidates, $o = \frac{1}{2}O$, and aggregate externalities imposed by candidates on other agents, $W_f - S_c$. Here, S_c captures the aggregate externalities candidates impose on each other, whereas W_f captures the aggregate externalities imposed on firms, i.e., agents on the other side of the market. Thus, the effect of a higher level of information of candidates on total welfare consists of the effect on candidates' match output and the change in the aggregate externalities candidates impose on all agents. By Proposition 1 we know that total

[15] This may serve as a formal rationale for the claim often raised in parent empowerment, that providing parents with more information results in them making more informed choices which – in the long run – will increase school quality.

match output is increasing in candidates' information level whereas $W_f - S_c$ may be decreasing. In this case the effect on aggregate welfare depends on which of the two effects is dominant.

The following example shows that increasing the information level of candidates may result in a decrease of aggregate welfare.

Example 2. Consider a matching market with three candidates and three firms, $n = k = 3$. Candidates' characteristics are standard uniformly distributed, $X_i \overset{iid}{\sim} U[0,1]$, and the information technology is a truth-or-noise technology Θ_α with precision level α. In this setting, the conditional expected characteristics of candidates are uniformly distributed on $\left[\frac{1}{2}(1-\alpha),\ \frac{1}{2}(1+\alpha)\right]$, and the corresponding vector of posterior mean-order statistics is $(\mu^\alpha_{1:3}, \mu^\alpha_{2:3}, \mu^\alpha_{3:3}) = \left(\frac{1}{2} + \frac{1}{4}\alpha, \frac{1}{2}, \frac{1}{2} - \frac{1}{4}\alpha\right)$. That is, from an ex-ante perspective, the expected posterior characteristic of the highest candidate is $\frac{1}{2} + \frac{1}{4}\alpha$. Suppose firms' characteristics are represented by the vector $(\eta_{1:3}, \eta_{2:3}, \eta_{3:3}) = \left(\frac{2}{3}, \frac{1}{2}, \frac{1}{3}\right)$. Table 2 illustrates expected output, aggregate signaling and welfare of candidates and firms for the given specifications.

$O = \frac{3}{2} + \frac{1}{12}\alpha$	
$S_c = \frac{3}{12} - \frac{1}{12}\alpha$	$S_f = \frac{7}{24}\alpha$
$W_c = \frac{1}{2} + \frac{1}{6}\alpha$	$W_f = \frac{3}{4} - \frac{5}{24}\alpha$
$W = \frac{5}{4} - \frac{1}{24}\alpha$	

Table 2: Expected total output, signaling investments and welfare for candidates and firms, for $n = k = 3$, $X_i \overset{iid}{\sim} U[0,1]$, a truth-or-noise technology of precision level α, and firms' types $(\eta_{1:3}, \eta_{2:3}, \eta_{3:3}) = \left(\frac{2}{3}, \frac{1}{2}, \frac{1}{3}\right)$.

In this setting, as the information level α increases, the externalities imposed on the lowest-ranked candidate, $2 \cdot \left(\frac{1}{2} - \frac{1}{4}\alpha\right) \cdot \frac{1}{6}$, decrease whereas those imposed on the middle-ranked candidate are constant. Consequently, candidates' aggregate signaling, $S_c = \frac{3}{12} - \frac{1}{12}\alpha$, is decreasing in the information level α. Moreover, as the information level of candidates increases, aggregate welfare of firms, $W_f = \frac{3}{4} - \frac{5}{24}\alpha$, decreases. In total, we obtain that the

negative effect on firms dominates the positive effect on candidates:

$$\begin{aligned}\frac{\partial W}{\partial \alpha} &= \frac{\partial o}{\partial \alpha} + \frac{\partial W_f}{\partial \alpha} - \frac{\partial S_c}{\partial \alpha} \\ &= \frac{1}{12} - \frac{5}{24} + \frac{1}{12} = -\frac{1}{24} < 0.\end{aligned}$$

A notable feature of this example is, that even though candidates' aggregate signaling investments are decreasing in the level of information, the negative effect on firms' welfare is so strong that aggregate welfare is decreasing. △

We now identify three conditions which each individually guarantee that aggregate welfare is increasing in candidates' information level. The most prominent sufficient conditions is that candidates constitute the long side of the market.

Theorem 4. *Aggregate welfare is increasing in information precision if one of the following conditions is satisfied:*

(i) F_Y *has a decreasing hazard rate,*

(ii) there are fewer firms than candidates, $k < n$*, or*

(iii) $n < k$ *and* f_Y *is monotone decreasing.*[16]

Some of the results in this section can already be established for signals being ordered in terms of *integral precision*, a less stringent precision criterion. More details can be found in Appendix C.

4 Optimal Level of Precision

We now take the analysis one step further and assume that, in the information stage, the precision of the information technologies is not exogenously given, but can be chosen before the information stage. We call this game the *precision and matching tournament*.

In a precision and matching tournament, first, candidates' information technology Θ_α is chosen from a set $\mathcal{S}_\Theta$ of feasible information technologies,

[16]Every absolute continuous DHR random variable has a decreasing density function. But there also exist IHR distributions with monotone decreasing density functions (cf. Bagnoli and Bergstrom (2005))

either collectively by the candidates or by a social planner. The information technology Θ_α is then implemented and, in the information stage, every candidate obtains a private informative signal from Θ_α. Agents update their beliefs according to Bayes' rule before they enter the matching stage in which the assignment is determined by a matching tournament. The timing in the precision and matching tournament is illustrated in Figure 1.

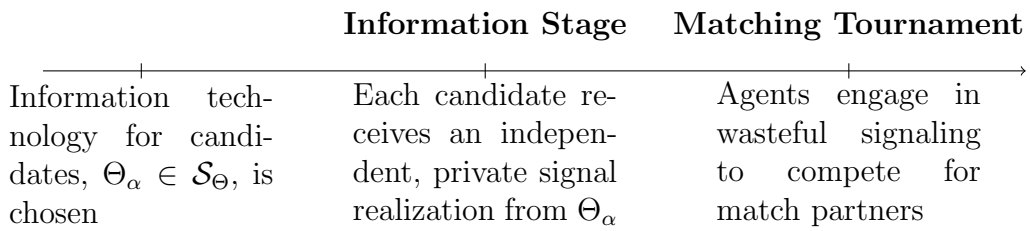

Figure 1: Timing in the *precision and matching tournament*

We characterize the socially optimal information level, α^{so}, which maximizes aggregate welfare of all agents in the market, and compare it to the *candidate optimal* information level, α^{co}, which maximizes candidates total welfare. This is the optimal information level in a one-sided market in which only candidates are active agents. It is also the information level a designer who only cares about the well-being of candidates would want to implement in a two-sided market.[17] Focusing on these two information levels allows to isolate the effects which originate from the two-sidedness of the matching market and are not prevalent in one-sided markets.

If information is costless it is easy to see from our previous discussion that the candidate optimal information level is to be perfectly informed (cf. Theorem 2). However, this is not necessarily the socially optimal information level since total welfare may be decreasing in information precision (cf. Example 2). The same is true for the *firm-optimal* information level, i.e., the precision of candidates' information which maximizes firms total welfare. In any case, if information is costless, the firm-optimal and the socially optimal

[17]This applies to setting in which there is a lobby group representing agents on one side of the market. Examples include the parent empowerment movement or labor unions. It should be noted that candidates would also choose α^{co} if they could coordinate on a common information level. e.g. by collectively choosing an information technology.

level of information will always be extreme, that is, either full information or no information.

4.1 Costly Precision

We now consider the case when information is costly. To formally analyze this case, let $\mathcal{S}_\Theta$ be a set of feasible information technologies which is totally ordered in terms of precision. That is, there exists some $\mathcal{A} \subseteq [0, \infty)$ such that $\mathcal{S}_\Theta = \{\Theta_\alpha\}_{\alpha \in \mathcal{A}}$ and Θ_α is more precise than $\Theta_{\alpha'}$ if and only if $\alpha > \alpha'$. Information technology Θ_α is characterized by $\{G^\alpha(\theta|x)\}_{x \in \mathcal{X}}$. For ease of presentation, we restrict attention to linear information models in which conditional expectations are convex combinations of the signal, Θ, and the ex-ante mean. That is, $E[X|\Theta] = \alpha\Xi + (1-\alpha)E[X]$, $\alpha \in [0, 1]$. The natural indexation in this case is to denote by Θ_α the information technology that results in $E[X|\Theta] = \alpha\Xi + (1-\alpha)E[X]$.

The following condition on the distribution of signals guarantees that for all precision levels $\alpha \in (0, 1)$, the distribution and density functions of the posterior estimates, H^α and h^α, are continuously differentiable in the precision level α:

Assumption 2. *The marginal distribution of signal realizations $G(\theta)$ is twice continuously differentiable in θ.*

We assume that information costs have a 'pay per signal' structure. That is, for information technology $\Theta_\alpha \in \mathcal{S}_\Theta$ of precision $\alpha \in [0, 1]$, in the information stage, every candidate who receives a signal from Θ_α has to pay $c(\alpha) \in \mathbb{R}^+$. Precision costs are increasing in α and capture e.g. investments in time or resources to generate or collect information. Let the precision-cost-function $c : [0, 1] \to [0, \infty)$ be increasing and continuously differentiable with $c(0) = 0$ and $c'(0) = 0$. Since every candidate obtains exactly one signal, total costs for information technology Θ_α are $C(\alpha) := nc(\alpha)$.

Given Assumption 2 expected total welfare of candidates and expected aggregate welfare are continuously differentiable in the level of information of candidates (cf. Lemma 5). Under the following single-crossing conditions the marginal cost function crosses the marginal gain functions ($\frac{\partial W_c}{\partial \alpha}$ for can-

didates and $\frac{\partial W}{\partial \alpha}$ for the social planner) at most once and from below.

(SC$_\mathbf{C}$): $\frac{\partial c}{\partial \alpha} / \frac{\partial W_c}{\partial \alpha}$ is strictly increasing in $\alpha \in (0,1)$

(SC): $\frac{\partial c}{\partial \alpha} / \frac{\partial W}{\partial \alpha}$ is strictly increasing in $\alpha \in (0,1)$ whenever $\frac{\partial W}{\partial \alpha} > 0$.

For $\mathcal{S}_\Theta$ being the set of truth-or-noise technologies and $X_i \overset{iid}{\sim} U[0,1]$, **(SC$_\mathbf{C}$)** and **(SC)** are satisfied for convex precision costs.

We now characterize the relation between the candidate optimal and the socially optimal information level if information is costly.

Theorem 5. *In a precision and matching tournament, suppose Assumption 2, **(SC$_C$)** and **(SC)** are satisfied.*

Then, the socially optimal information level of candidates is higher than the candidate optimal level, $\alpha^{co} \leq \alpha^{so}$, if $n > k$ and n is sufficiently large, or if the distribution of firms' characteristics F_Y has a decreasing hazard rate.

If $n \leq k$ and the distribution of firms' characteristics has an increasing hazard rate, then the socially optimal information level is lower than the candidate optimal level, $\alpha^{so} \leq \alpha^{co}$.

It is not surprising that in a precision and matching tournament the candidate optimal and socially optimal levels of precision do not coincide. Information of candidates imposes an externality on firms. For a higher information level a better allocation can be achieved which increases total match output. However, more information also leads to more differentiation among candidates which increases competition among firms. This results in higher signaling investments of firms. The relation between the candidate optimal and the socially optimal level of information depends on whether the overall effect of a higher information level of candidates on firms is positive or negative.

As is shown in Theorem 5, if the uninformed agents constitute the short side of the market or in markets with approximately the same number of agents on both sides, it depends on the distribution of characteristics of agents on the informed side whether the candidate optimal information level is above or below the socially optimal level. If the uninformed side of the market is the long side of the market and this side is sufficiently large, then a higher information level increases expected total welfare of both sides of

the market and consequently the socially optimal information level is higher than the candidate optimal level.

5 Conclusion

In this paper we studied the impact of the level of information of market participants in a matching tournament. We illustrated that for a higher information level of candidates there is a trade-off between the benefits from allowing for a better allocation and potential losses due to a possible increase in wasteful signaling. In particular, we provided an example which showed that the negative effect on firms may be so strong that overall welfare decreases, even in cases in which a higher information level of candidates leads to a reduction of candidates' investments in wasteful signaling. However, we showed that in markets in which the a priori uninformed agents constitute the long side of the market an increase in the information level will always improve welfare. These results are complemented by a discussion and comparison of the socially optimal and the candidate optimal information level, if information is costly.

Our results are obtained by identifying a relation between the concepts used in Hoppe et al. (2009) and Ganuza and Penalva (2010) which allows to combine the elegant methods of both papers. We think that linking these methods will be useful beyond establishing the results in this paper. In particular, the link provides a new approach to study settings in which the information held by market participants is endogenous, including the analysis of information acquisition and disclosure in auctions and matching markets.

In our analysis we study the impact of the information level of market participants for a given matching mechanism. It would also be interesting to compare different matching mechanisms with respect to their responsiveness to different information levels of candidates. For example, one could analyze different sharing rules of the match values in the matching tournament, or let match partners bargain about how to share the match output. More generally, it would be interesting to adopt a mechanism design perspective with the intention to characterize the mechanism which provides the strongest incentives to establish a high information level in the market, or alternatively, performs best in terms of overall welfare.

Appendix

A Technical Prerequisites

In this section we present the main techniques used to prove our results. The methods stem from statistics and reliability theory. Shaked and Shanthikumar (2007) provide a comprehensive treatment of order statistics whereas Marshall et al. (2011) is a good reference for the theory of majorization. If not indicated otherwise, all definitions and theorems stated in this sections can be found in these two books.

Definition 2. Consider two ordered n-dimensional vectors $\mathbf{a} = (a_1, \ldots, a_n)$, and $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n$ such that $a_1 \geq \cdots \geq a_n$ and $b_1 \geq \ldots b_n$. We say that $\mathbf{a}$ *submajorizes* $\mathbf{b}$, ($\mathbf{a} \succ_{sub} \mathbf{b}$), if

$$\sum_{i=1}^{m} a_i \geq \sum_{i=1}^{m} b_i \quad \text{for all } m = 1, \ldots, n. \tag{2}$$

If in addition Equation 2 holds with equality for $m = n$ we say that $\mathbf{a}$ *majorizes* $\mathbf{b}$, ($\mathbf{a} \succ \mathbf{b}$).

A function $\phi : \mathbb{R}^n \supseteq A \longrightarrow \mathbb{R}$ is *Schur-convex* (resp. *Schur-concave*) if, whenever $\mathbf{a} \succ \mathbf{b}$ then $\phi(\mathbf{a}) \geq \phi(\mathbf{b})$ (resp. $\phi(\mathbf{a}) \leq \phi(\mathbf{b})$).
If $\mathbf{a} \succ_{sub} \mathbf{b}$ then $\phi(\mathbf{a}) \geq \phi(\mathbf{b})$ for every Schur-convex and increasing function ϕ.

The following two theorems from Barlow and Proschan (1966) are the counterparts of theorem 1 and 2 in Hoppe et al. (2009).

Theorem 6. *Let X and Y be integrable random variables with equal means and $F(0) = G(0) = 0$. Then if $X \leq_* Y$*

(i) The vector of mean order statistics of Y, $(E[Y_{1:n}], \ldots, E[Y_{n:n}])$ majorizes *the vector of mean order statistics of X for all $n \geq 1$. That is,*

$$(E[Y_{1:n}], \ldots, E[Y_{n:n}]) \succ (E[X_{1:n}], \ldots, E[X_{n:n}])$$

(ii) For $1 \leq r \leq n$:

$$\sum_{i=r}^{n} i \cdot (E[X_{i:n}] - E[X_{i+1:n}]) \geq \sum_{i=r}^{n} i \cdot (E[Y_{i:n}] - E[Y_{i+1:n}])$$

(iii) For $a_1 \leq \cdots \leq a_n$:

$$\sum_{i=1}^{n} a_i \cdot i \cdot (E[X_{i:n}] - E[X_{i+1:n}]) \geq \sum_{i=1}^{n} a_i \cdot i \cdot (E[Y_{i:n}] - E[Y_{i+1:n}])$$

By Cal and Carcamo (2006) Theorem 6 (i) is also true for random variables ordered in terms of the convex order $X \leq_{cx} Y$.

Theorem 7. *Let F be IHR with $F(0) = 0$. Then, for fixed n, the normalized spacings of order statistics $i \cdot (X_{i:n} - X_{i+1:n})$ are stochastically increasing in $i = 1, \ldots, n$. That is:*

$$(X_{1:n} - X_{2:n}) \leq_{st} 2 \cdot (X_{2:n} - X_{3:n}) \leq_{st} \cdots \leq_{st} n \cdot (X_{n:n} - X_{n+1:n})$$

If F is DHR, then the normalized spacings are stochastically decreasing in i.

We will also need the following result

Theorem 8. *Let X, Y be nonnegative random variables with distribution functions F and G, respectively, such that $F(0) = G(0) = 0$. If $X \leq_* Y$ then*

- $\frac{E[Y_{i:n}]}{E[X_{i:n}]}$ *is decreasing in i, and*
- $\frac{E[Y_{i:n}]}{E[X_{i:n}]}$ *is increasing in n.*

B Proofs

Proof of Lemma 1. (i) is a direct corollary of the following lemma from Shaked and Shanthikumar (2007)

Lemma 2. *Let X and Y be nonnegative random variables. If $X \leq_* Y$, then*

$$E[X_{i:n} - X_{i+1:n}] \leq E[Y_{i:n} - Y_{i+1:n}]$$

for $i = 1, \ldots, n-1$.

(ii) is a direct corollary of theorem 3.6 in Barlow and Proschan (1966).

□

Proof of Proposition 1. If $\Theta_2 \succsim_* \Theta_1$ then $(\mu_{1:n}^{\Theta_2}, \ldots, \mu_{n:n}^{\Theta_2}) \succ (\mu_{1:n}^{\Theta_1}, \ldots, \mu_{n:n}^{\Theta_1})$, by Theorem 6 (i).
For $k \geq n$, $O = \sum_{i=1}^{n} \eta_{i:k}\mu_{i:n}$ is Schur-convex in the vector of mean order statistics of candidates' characteristics and consequently if $\Theta_2 \succsim_* \Theta_1$ then $O(\Theta_2) \geq O(\Theta_1)$.

For $k < n$, $O = \sum_{i=1}^{k} \eta_{i:k}\mu_{i:n}$ is Schur-convex in the *truncated* vector of mean-order statistics of candidates' characteristics. A higher information level of candidates only results in (weak) submajorization of the truncated vectors of mean-order statistics, i.e.

$$\Theta_2 \succsim_* \Theta_1 \quad \Rightarrow \quad \mu^{\Theta_2}\big|_{\leq k} \succ_{sub} \mu^{\Theta_1}\big|_{\leq k}$$

where $\mu|_{\leq k} = (\mu_{1:n}, \ldots, \mu_{k:n})$. Since O is increasing and Schur-convex it follows that $O(\Theta_2) \geq O(\Theta_1)$. □

In order to prove Theorem 2 we first establish a technical Lemma. To state and prove it we need the following concept and fact.

Definition 3. Let X and Y be two random variables with distribution functions F and G, respectively. We say that X is smaller than Y in the *convex order*, $X \leq_{cx} Y$, if

$$E\left[\varphi(X)\right] \leq E\left[\varphi(Y)\right]$$

for all convex functions $\varphi : \mathbb{R} \to \mathbb{R}$ for which the expectations exist.

Fact 1 (Theorem 3.A.5 in Shaked and Shanthikumar (2007))**.** *The following conditions are each sufficient and necessary for $X \leq_{cx} Y$*

$$\int_0^p \left(G^{-1}(u) - F^{-1}(u)\right) du \leq 0 \quad \forall\, p \in [0,\, 1] \qquad \textit{and} \tag{3}$$

$$\int_p^1 \left(G^{-1}(u) - F^{-1}(u)\right) du \geq 0 \quad \forall\, p \in [0,\, 1] \tag{4}$$

Lemma 3. *Let X, Y be random variables with continuous differentiable distributions F and G and equal means, such that $X \leq_* Y$.*[18] *Then, for every $k \in \mathbb{N}$ there exists some $\hat{n}_k$ such that*

$$E\left[X_{k:n}\right] \leq E\left[Y_{k:n}\right] \quad \forall\, n \geq \hat{n}_k.$$

Proof. The methods in this proof are similar to the ideas used to prove theorem 1 in Ganuza and Penalva (2010).
We can apply the probability integral transformation[19] to obtain the following simple formula for the kth order statistics of X:

$$E\left[X_{k:n}\right] = \frac{n!}{(k-1)!(n-k)!} \int_0^1 F^{-1}(u) u^{n-k} (1-u)^{k-1} \mathrm{d}\, u$$

Set $\phi(u) := G^{-1}(u) - F^{-1}(u)$. Since G and F are continuously differentiable, by the inverse function theorem F^{-1} and G^{-1} are continuous and so is ϕ.

Suppose $\phi(u) \neq 0$ on subset of $[0,1]$ with nonempty interior.[20] Let $L := \{u \in X\,[0,1] :\ \phi(u) < 0\}$ and $\overline{u} := \sup\{L\}$. Equation 3 and Equation 4, continuity of ϕ and the assumption that $\phi(u) \neq 0$ on a subset of $[0,1]$ of positive measure imply that $\overline{u} \in (0,\, 1)$. We obtain that there exist $p_1,\, p_2 \in (\overline{u},\, 1]$ such that $\phi(u) > 0$ for all $u \in [p_1,\, p_2]$. Set $c_1 := \min_{u \in [0, p_1]}\{\phi(u)(1-u)^{k-1}\}$ and $c_2 := \min_{u \in [p_2, 1]}\{\phi(u)(1-u)^{k-1}\}$. By construction $c_1 < 0$ and $c_2 > 0$. This yields:

$$\begin{aligned} E\left[Y_{k:n}\right] - E\left[X_{k:n}\right] =& k\binom{n}{k} \int_0^1 \left(G^{-1}(u) - F^{-1}(u)\right) u^{n-k}(1-u)^{k-1} \mathrm{d}\, u \\ \geq& \frac{n!}{(k-1)!(n-k+1)!} p_2^{n-k+1} \left[\left(\frac{p_1}{p_2}\right)^{n-k+1} (c_1 - c_2) + c_2\right] \end{aligned}$$

Set $\hat{n} := \lceil k - 1 + \frac{\ln\left(\frac{c_2}{c_2 - c_1}\right)}{\ln\left(\frac{p_1}{p_2}\right)} \rceil$ where $\lceil x \rceil$ denotes the smallest natural number

[18] For the proof it suffices for X and Y to be ordered in terms of the convex order, $X \leq_{cx} Y$.

[19] For every random variable X with continuous c.d.f. F and density f, the transformed random variable $F(X)$ has a standard uniform distribution, $F(X) \sim U\,[0,1]$

[20] The case $\phi(u) = 0$ a.e. is trivial.

greater or equal than x. It follows that:

$$\frac{n!}{(k-1)!(n-k+1)!}p_2^{n-k+1}\left[\left(\frac{p_1}{p_2}\right)^{n-k+1}(c_1-c_2)+c_2\right]\geq 0 \quad \forall\, n\geq \hat{n}$$

□

Proof of Theorem 2.

(i) <u>*Part 1:*</u> By Lemma 3 there exists some $\hat{n} > k+1$ such that for every $n \geq \hat{n}$, $\mu_{k+1:n}^{\Theta_2} - \mu_{k+1:n}^{\Theta_1} \geq 0$. Since $\mu_{k+1:n}^{\Theta_1} \geq 0$, it follows that

$$\frac{\mu_{k+1:n}^{\Theta_2}}{\mu_{k+1:n}^{\Theta_1}} \geq 1 \quad \forall n \geq \hat{n}.$$

By Theorem 8, $\frac{\mu_{i:n}^{\Theta_2}}{\mu_{i:n}^{\Theta_1}}$ is decreasing in i for every n and it follows that $\mu_{i:n}^{\Theta_2} - \mu_{i:n}^{\Theta_1} \geq 0$ for all $i \leq k+1$. We obtain that for all $n \geq \hat{n} > k$:

$$\begin{aligned} S_c(\Theta_2) - S_c(\Theta_1) &= \sum_{i=1}^{\min\{n,k\}} (\eta_{i:k} - \eta_{i+1:k}) \cdot \left(\mu_{i+1:n}^{\Theta_2} - \mu_{i+1:n}^{\Theta_1}\right) \\ &= \sum_{i=1}^{k} \underbrace{(\eta_{i:k} - \eta_{i+1:k})}_{>0} \cdot \underbrace{\left(\mu_{i+1:n}^{\Theta_2} - \mu_{i+1:n}^{\Theta_1}\right)}_{\geq 0} \geq 0 \end{aligned}$$

<u>*Part 2:*</u> If F_Y has an IHR, then by Theorem 7 the normalized spacings $i(\eta_{i:k} - \eta_{i+1:k})$ are stochastically increasing in i.

Set $\widetilde{S}_c := \sum_{i=0}^{n} i(\eta_{i:k} - \eta_{i+1:k})\mu_{i+1:n}$. Then, for $n \leq k$, $S_c = \widetilde{S}_c$ and $\widetilde{S}_c$ is Schur-concave in the vector of mean order statistics of candidates' characteristics. It follows that S_c is decreasing (non-increasing) in the level of information of candidates.

(ii) For $k < n$, by Lemma 1(i) and since $\eta_{i+1:k} \geq 0$ for all $i = 1, \ldots, n-1$ we obtain:

$$W_c(\Theta_2) = \sum_{i=1}^{k} i\left(\mu_{i:n}^{\Theta_2} - \mu_{i+1:n}^{\Theta_2}\right)\eta_{i:k} \geq \sum_{i=1}^{k} i\left(\mu_{i:n}^{\Theta_1} - \mu_{i+1:n}^{\Theta_1}\right)\eta_{i:k} = W_c(\Theta_1).$$

Set $a_i := -\eta_{i:k}$. Then, for $n \leq k$, applying Theorem 6 (iii) yields

$$\Theta_2 \succsim_* \Theta_1 \quad \Rightarrow \quad W_c(\Theta_2) \geq W_c(\Theta_1).$$

□

Proof of Theorem 3. (i) Analogous to the proof of Theorem 2 (ii) whereas the case-by-case analysis is now for $k \leq n$ and $k > n$.

(ii) If F_Y has a DHR, by Theorem 7 the normalized spacings $i(\eta_{i:k} - \eta_{i+1:k})$ are stochastically decreasing in i. Consequently, W_f is Schur-convex in the vector of conditional mean order statistics of candidates. By Theorem 6(i) it follows that $W_f(\Theta_2) \geq W_f(\Theta_1)$, if Θ_2 is more precise than Θ_1. The results for F_Y being IHR follow from arguments analogous to those used to prove Theorem 2 (i).

□

Proof of Theorem 4. $W = W_c + W_f$. Rearranging terms yields:

$$W(\Theta) = \underbrace{\left[\sum_{i=1}^{\min\{n,k\}} i \cdot \left(\mu^{\Theta}_{i:n} - \mu^{\Theta}_{i+1:n}\right)(\eta_{i:k} - \eta_{i+1:k})\right]}_{W_f - S_c} + \underbrace{\sum_{i=1}^{\min\{n,k\}} \mu^{\Theta}_{i:n}\eta_{i:k}}_{0.5O}$$

By Proposition 1 we know that total match output is increasing in precision. Whether overall welfare is increasing or decreasing in the level of candidates' information depends on the effect on $W_f - S_c$, and, for the case that $W_f - S_c$ is decreasing, on which of these effects dominates.

(i) If F_Y is DHR, both W_c and W_f are increasing in precision (cf. Theorem 2 and Theorem 3) and so is $W = W_c + W_f$.

(ii) Suppose $k < n$. Let Θ_2, Θ_1 be two information technologies such that Θ_2 is more precise than Θ_1. Set

$$\Delta\left(W_c - S_f\right) := \left[W_c(\Theta_2) - S_f(\Theta_2)\right] - \left[W_c(\Theta_1) - S_f(\Theta_1)\right].$$

We obtain

$$\Delta\left(W_c - S_f\right) = \sum_{i=1}^{k} \underbrace{i \cdot \left(\eta_{i:k} - \eta_{i+1:k}\right)}_{>0} \underbrace{\left[\left(\mu_{i:n}^{\Theta_2} - \mu_{i+1:n}^{\Theta_2}\right) - \left(\mu_{i:n}^{\Theta_1} - \mu_{i+1:n}^{\Theta_1}\right)\right]}_{>0 \text{ by Lemma 1}}$$

This shows that for $k < n$, $W_c - S_f$ is non-decreasing in signal precision which implies that total welfare is increasing.

(iii) Suppose $n < k$ and f_Y is monotone decreasing. In this case

$$W = \sum_{i=1}^{n} \mu_{i:n}^{\Theta} \eta_{i:k} + \left[\sum_{i=1}^{n} i \cdot \left(\mu_{i:n}^{\Theta} - \mu_{i+1:n}^{\Theta}\right)\left(\eta_{i:k} - \eta_{i+1:k}\right)\right].$$

We use the following result which establishes that the spacings of order statistics from random variables with monotone density functions can be ordered in terms of stochastic dominance.[21]

Lemma 4. *Let $Y_1, \ldots, Y_n$ be independently, identically distributed random variables with finite support and density function f_Y. Then,*

- *if f_Y is monotone increasing (non-decreasing)*

$$Y_{i:n} - Y_{i+1:n} \leq_{st} Y_{i+1:n} - Y_{i+2:n} \quad \forall\, i = 1, \ldots, n-2$$

- *if f_Y is monotone decreasing (non-increasing)*

$$Y_{i:n} - Y_{i+1:n} \geq_{st} Y_{i+1:n} - Y_{i+2:n} \quad \forall\, i = 1, \ldots, n-2$$

It follows directly that if f_Y is monotone decreasing, the expected spacings of mean order statistics $\left(\eta_{i:k} - \eta_{i+1:k}\right)$, $i = 1, \ldots, k-1$ are decreasing in i. Setting $a_i := -\left(\eta_{i:k} - \eta_{i+1:k}\right)$, by Theorem 6 (iii) we obtain $(W_f - S_c)(\Theta_2) \geq (W_f - S_c)(\Theta_1)$, for $\Theta_2 \succsim_* \Theta_1$. It follows that overall welfare is increasing in candidates' information level.

□

Lemma 5. *For linear information technologies, under Assumption 2, for all $\alpha \in (0,1)$, H^α and h^α are continuously differentiable in the precision level α. Moreover, O, S_c, S_f, W_c, W_f are continuously differentiable in $\alpha \in (0,1)$.*

[21] This result can be found in Shaked and Shanthikumar (2007).

Proof. For $\alpha \neq 0$, set $\phi(\alpha, w) := \frac{w-(1-\alpha)E(X)}{\alpha}$. Then, for linear information technologies and $\alpha \neq 0$, $H^\alpha(w) = G(\phi(\alpha, w))$, and $h^\alpha(w) = \frac{1}{\alpha} g(\phi(\alpha, w))$, for $\alpha = 0$, $H = G$. By Assumption 2 and since $\phi(w, \alpha)$ is continuously differentiable in $\alpha \in (0, 1)$, $H^\alpha(w)$ and $h^\alpha(w)$ are continuously differentiable in α. Moreover, if H^α and h^α are continuously differentiable in α then so are the distributions of order statistics $H^\alpha_{i:n}$. The densities $h^\alpha_{i:n}$ are continuous in α for all $i = 1, \ldots, n$. This implies that the conditional mean order statistics $E\left[\widehat{X}^\alpha_{i:n}\right]$ are continuously differentiable in α. It follows that W, W_c, S_f, S_c, O are continuously differentiable in α. □

Proof of Theorem 5. The marginal value of information for candidates is $\frac{\partial W_c}{\partial \alpha}$ and the socially marginal value is $\frac{\partial W}{\partial \alpha} = \frac{\partial W_c}{\partial \alpha} + \frac{\partial W_f}{\partial \alpha}$. By Theorem 3, if F_Y is DHR or if $n \geq \hat{n} > k$ then $\frac{\partial W_f}{\partial \alpha} > 0$ and it follows that, at any information level $\alpha \in (0, 1)$, $\frac{\partial W}{\partial \alpha} > \frac{\partial W_c}{\partial \alpha}$. However, if F_Y is IHR and $n \leq k$, then social marginal gains from higher precision are lower than the marginal gains for candidates, $\frac{\partial W}{\partial \alpha} < \frac{\partial W_c}{\partial \alpha}$.

In the *precision and matching tournament* with costly precision, the optimization problem for candidates is:

$$\max_{\alpha \in [0,1]} \{U_c(\alpha) = W_c(\Theta_\alpha) - n \cdot c(\alpha)\}$$

and for the social planner:

$$\max_{\alpha \in [0,1]} \{U_{SP}(\alpha) = W(\Theta_\alpha) - n \cdot c(\alpha)\}$$

Given our assumptions on the cost function and Assumption 2, U_c and U_{SP} are continuously differentiable in α. By the extreme value theorem this guarantees the existence of a solution to the optimization problem of candidates, respectively the social planner. The single-crossing conditions, **(SC)** and **($\mathbf{SC_C}$)**, establish uniqueness.

If U_c is increasing on $[0, 1]$, then the optimal level of precision for candidates is $\alpha^{co} = 1$, otherwise it is characterized by:

$$\left.\frac{\partial W_c}{\partial \alpha}\right|_{\alpha=\alpha^{co}} = n \left.\frac{\partial c}{\partial \alpha}\right|_{\alpha=\alpha^{co}} \tag{I}$$

Analogous reasoning shows that the unique socially optimal level of preci-

sion is either $\alpha^{so} \in \{0, 1\}$ or an interior solution exists which is characterized by:

$$\left.\frac{\partial W}{\partial \alpha}\right|_{\alpha=\alpha^{so}} = n \cdot \left.\frac{\partial c}{\partial \alpha}\right|_{\alpha=\alpha^{so}} \tag{II}$$

Suppose F_Y, is DHR or $n \geq \hat{n} > k$. In this case, at any information level $\widetilde{\alpha}$, the marginal gains for candidates from higher precision are lower than the social marginal gains, $\left.\frac{\partial W_c}{\partial \alpha}\right|_{\alpha=\widetilde{\alpha}} < \left.\frac{\partial W}{\partial \alpha}\right|_{\alpha=\widetilde{\alpha}}$. Given uniqueness of the candidate optimal and the socially optimal level of precision we obtain $\alpha^{co} \leq \alpha^{so}$.
The result for F_Y being IHR and $n < k$ follows by analogous reasoning. □

C Discussion and relation to other informativeness criteria

Given our assumption that signals have the monotone likelihood ratio property the natural informativeness criterion to use is the concept of *effectiveness* introduced by Lehmann (1988).[22] The basic idea behind this concept is that for a given state space X, information technology Θ_2 is more informative about X than Θ_1, if the conditional distribution of Θ_2 is more dependent on X than that of Θ_1. Formally,

Definition 4 (Effectiveness, Lehmann (1988))**.** Given X, let Θ_1 and Θ_2 be two signals which satisfy the MLRP. Then Θ_2 is said to be **more effective** than Θ if for all θ

$$F^{-1}_{\Theta_2}(F_{\Theta_1}(\theta|x)|x) \quad \text{is nondecreasing in } x.$$

Mizuno (2006) shows that for a more effective signal about X the resulting distribution of conditional expectations is more dispersed.

Theorem 9 (Mizuno (2006))**.** *If signals satisfy the MLRP, then if Θ_2 is more effective than Θ_1, it follows that Θ_2 is more integral precise than Θ_1 for all priors.*

[22] Persico (2000) refers to this concept as *accuracy*. Effectiveness applies to monotone decision problems and requires less restrictive conditions than *sufficiency* (Blackwell (1951)) to compare signals in terms of their informativeness.

Our definition of precision is similar to the notion of *integral* and *supermodular precision* in Ganuza and Penalva (2010) but the stochastic orders used to define these concepts differ. Our *precision* criterion in Definition 1 is based on the star order whereas *integral precision* is based on the *convex order* and *supermodular precision* is based on the *dispersive order*.[23] We briefly discuss the relation of these criteria which amounts to analyzing the relation of the stochastic orders.[24]

Let X and Y be two random variables with interval support and distribution functions F and G, respectively. We write $X \leq_* Y$ for X being smaller than Y in the star order, and $X \leq_{cx} Y$ and $X \leq_{disp} Y$ for X and Y ordered in terms of the convex, respectively dispersive order.

For a random variable X and signals Θ_1 and Θ_2, by the law of iterated expectations $E[E[X|\Theta_1]] = E[E[X|\Theta_1]] = \mu$. Consequently, in our informational setting we always compare random variables with finite and equal means. In this case the *convex order* is equivalent to the concept of second order stochastic dominance. Moreover, the dispersive order and the star order are both stronger than the convex order, that is

$$\begin{aligned} X \leq_{disp} Y \quad &\Rightarrow \quad X \leq_{cx} Y \text{ and} \\ X \leq_* Y \quad &\Rightarrow \quad X \leq_{cx} Y. \end{aligned}$$

For the star order and the dispersive order the following relation holds:

$$X \leq_* Y \quad \Leftrightarrow \quad \log X \leq_{disp} \log Y \tag{5}$$

Thus, the star order and the dispersive order are in general not nested. However, under some conditions they are. Let $X \leq_{st} Y$ indicate that Y first-order stochastically dominates X.

[23] For a formal definition of these concepts, see Shaked and Shanthikumar (2007) or Ganuza and Penalva (2010)

[24] For further insights on the relation to other informativeness criteria, like *sufficiency* (Blackwell (1951)) or *accuracy*, respectively *effectiveness* (Lehmann (1988), Persico (2000)) we refer the reader to the discussion in Ganuza and Penalva (2010).

Lemma 6. *For nonnegative random variables, X and Y with distribution functions F and G, respectively,*

(i) if $X \leq_{st} Y$, then $X \leq_* Y$ implies $X \leq_{disp} Y$.

(ii) if F and G are absolutely continuous with $F(0) = G(0) = 0$ and $f(0) \geq g(0) > 0$, then $X \leq_* Y$ implies $X \leq_{disp} Y$.

Some of the results of Subsection 3.1 can already be establish for signals ordered in terms of integral precision, a less stringent requirement than precision in terms of Definition 1. This is true for all results which only use Theorem 6 (i) and Lemma 3. In particular this includes the results that expected total match output is increasing in the information level of candidates (Proposition 1) and that expected total signaling of candidates and expected total welfare of firms is increasing for n sufficiently large (Theorem 2 (i) and Theorem 3 (ii)).

References

Abdulkadiroglu, Atila, Parag A. Pathak, and Alvin E. Roth (2009), "Strategy-Proofness versus Efficiency in Matching with Indifferences: Redesigning the NYC High School Match." *American Economic Review*, 99, 1954–1978.

Abdulkadiroglu, Atila, Parag A. Pathak, Alvin E. Roth, and Tayfun Sonmez (2005), "The Boston Public School Match." *American Economic Review*, 95, 368–371.

Bagnoli, Mark and Ted Bergstrom (2005), "Log-Concave Probability and its Applications." *Economic Theory*, 26, 445–469.

Barlow, Richard E. and Frank Proschan (1966), "Inequalities for Linear Combinations of Order Statics from Restricted Families." *The Annals of Mathematical Statistics*, 37, 1574–1592.

Becker, Gary S. (1973), "A Theory of Marriage: Part I." *Journal of Political Economy*, 91, 813–846.

Bergemann, Dirk and Martin Pesendorfer (2007), "Information Structures in Optimal Auctions." *Journal of Economic Theory*, 137, 580–609.

Blackwell, D. (1951), "Comparison of Experiments." In *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability* (J. Neyman, ed.), 93–102, University of California Press, Berkeley, CA.

Cal, Jesus de la and Jarvier Carcamo (2006), "Stochastic Orders and Majorization of Mean Order Statistics." *Jounal of Applied Probability*, 43, 704–712.

Clarke, Edward H. (1971), "Multipart Pricing of Public Goods." *Public Choice*, 11, 17–33.

Cole, Harold L., George J. Mailath, and Andrew Postlewaite (1992), "Social Norms, Savings Behavior, and Growth." *Journal of Political Economy*, 100, 1092–1125.

Cole, Harold L., George J. Mailath, and Andrew Postlewaite (1995), "Incorporating Concern for Relative Wealth into Economic Models." *Federal Reserve Bank of Minneapolis Quarterly Review*, 19, 12–21.

Cole, Harold L., George J. Mailath, and Andrew Postlewaite (2001), "Efficient Non-Contractibel Investments in Large Economies." *Journal of Economic Theory*, 101, 333–373.

Eso, Peter and Balazs Szentes (2007), "Optimal Information Disclosure in Auctions and the Handicap Auction." *Review of Economic Studies*, 74, 705–731.

Gale, David and Lloyd S. Shapley (1962), "College Admissions and the Stability of Marriage." *The American Mathematical Monthly*, 69, 9–15.

Ganuza, Juan-Jose and Jose S. Penalva (2010), "Signal Orderings Based on Dispersion and the Supply of Private Information in Auctions." *Econometrica*, 78, 1007–1030.

Groves, Theodore (1973), "Incentives in Teams." *Econometrica*, 41, 617–631.

Hopkins, Ed (2012), "Job Market Signalling of Relative Position, or Becker Married to Spence." *Jounal of the European Economic Association*, 10, 290–322.

Hoppe, Heidrun, Benny Moldovanu, and Aner Sela (2009), "The Theory of Assortative Matching Based on Costly Signals." *The Review of Economic Studies*, 76, 253–281.

Hoxby, Caroline and Sarah Turner (2013), "Expanding College Opportunities for High-Achieving, Low Income Students." Technical report, SIEPR Discussion Paper 12-014.

Lehmann, E. (1988), "Comparing Location Experiments." In *The Annals of Statistics*, volume 16, 521–533.

Mailath, George J, Andrew Postlewaite, and Larry Samuelson (2013), "Pricing and Investments in Matching Markets." *Theoretical Economic*, 8, 535–590.

Marshall, Albert W., Ingram Olkin, and Barry C. Arnold (2011), *Inequalities: Theory of Majorization and its Applications*, 2nd edition. Springer Series in Statistics.

Milgrom, Paul R. (1981), "Good News and Bad News: Representation Theorems and Applications." *The Bell Journal of Economics*, 12, 380–391.

Mizuno, Toshihide (2006), "A Relation between Positive Dependence of Signal and the Variability of Conditional Expectation Given Signal." *Journal of Applied Probability*, 43, 1181–1185.

Persico, Nicola (2000), "Information Acquisition in Auctions." *Econometrica*, 68, 135–148.

Peters, Michael and Aloysius Siow (2002), "Competing Premarital Investments." *Journal of Political Economy*, 110, 592–608.

Roth, Alvin E., Tayfun Sonmez, and Utku Unver (2004), "Kidney Exchange." *Quarterly Journal of Economics*, 119, 457–488.

Roth, Alvin E., Tayfun Sonmez, and Utku Unver (2005), "Pairwise Kidney Exchange." *Journal of Economic Theory*, 125, 151–188.

Shaked, Moshe and J. George Shanthikumar (2007), *Stochastic Orders*. Springer Series in Statistics, Springer, New York, NY.

Shaked, Moshe, Miguel A. Sordo, and Alfonso Suarez-Llorens (2012), "Global Dependence Stochastic Orders." *Methodology and Computing in Applied Probability*, 14, 617–648.

Shapley, Lloyd S. and Martin Shubik (1971), "The Assignment Game I: the Core." *International Journal of Game Theory*, 1, 111–130.

Spence, Michael (1973), "Job Market Signaling." *Quarterly Journal of Economics*, 87, 296–332.

Vickrey, William (1961), "Counterspeculation, Auctions, and Competitive Sealed Tenders." *The Journal of Finance*, 16, 8–37.