Uncertainty Traps

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[PRELIMINARY AND INCOMPLETE]

Abstract

We develop a quantitative theory of endogenous uncertainty and business cycles. In the model, higher uncertainty about fundamentals discourages investment but agents can learn from the actions of others. Therefore, in times of low activity information flows slowly and uncertainty stays high, further discouraging investment. This creates room for uncertainty traps — self-reinforcing episodes of high uncertainty and low activity. We characterize conditions that give rise to these events. Negative shocks to average productivity or beliefs may have permanent effects on the level of activity through the persistence of uncertainty. We also characterize optimal policy interventions. The socially efficient allocation can be implemented with aggregate-beliefs dependent subsidies, but under certain conditions it necessarily features uncertainty traps. We embed these forces into a standard quantitative model of the business cycle to evaluate the impact of uncertainty traps.

JEL Classification: E32, D83

1 Introduction

One of the central features of macroeconomic activity is its high persistence. The NBER definition of business cycles implies that it takes for the U.S. economy close to 40 months on average to recover from a through until the next peak. Business cycles are also asymmetric: it takes about 17 months for the economy to move from peak to through, so that recoveries last on average more than twice as long as downfalls. These features have been visible during in the 2007-2009 crisis. The unemployment rate increased from 4.4% in May 2007 to 10% in October 2009 and has barely decreased to a level 7.9% in early 2013.

What explains these prolonged declines in economic activity? In this paper, we develop a quantitative theory of endogenous uncertainty and business cycles to explain these phenomena. The theory captures two key forces. First, higher uncertainty about fundamentals discourages investment. Second, economic agents can learn from the actions of others. The interaction between these two forces creates room for uncertainty traps — self reinforcing episodes of high uncertainty and low activity. In times of low activity information flows slowly and uncertainty stays high, further discouraging investment. This explains why low activity may persist under good fundamentals.

We first develop a baseline theory that includes only the essential features of the mechanism, and then we extend the model in various dimensions for a quantitative evaluation. In the model, firms choose to undertake an irreversible investment whose return depends on an imperfectly observed fundamental. Beliefs about that fundamental are common to all firms, but can be regularly updated using various signals. Formally, we define uncertainty as the variance of the prior about the fundamental. Information, in turn, diffuses through a simple social learning channel: the higher the number of firms that invests, the larger the number of signals received by firms and the stronger the reduction in their uncertainty.

This environment naturally produces an interaction between beliefs and economic activity. Firms are more likely to invest if they hold more optimistic or less uncertain beliefs about the fundamental. Therefore, low uncertainty is associated with a high investment rate. At the same time, the law of motion for beliefs depends on the investment rate through social learning. When few firms invest, uncertainty rises and the firms' optimism, captured by the mean of the beliefs distribution, is less likely to fluctuate.

Using this setup we demonstrate the existence of uncertainty traps. Formally, we define an uncertainty trap as the coexistence of multiple stationary points in the joint dynamics of uncertainty and economic activity for a given mean of the distribution of beliefs. We refer to these fixed points as regimes. Due to the complementarity between investment and information diffusion, in high-activity regimes there is low uncertainty and in low-activity regimes there is high uncertainty. Despite this multiplicity, the recursive equilibrium is uniquely pinned down by the stochastic evolution of the mean level of beliefs. But, because of it, the unique equilibrium is prone to nonlinear dynamics and asymmetries. For example, the long-run response to a temporary negative shock becomes considerably more protracted when its magnitude is above some threshold. The economy quickly recovers after a small temporary shock, but it may permanently shift into a low activity regime after a large shock of the same duration. In turn, a positive temporary shock of sufficient magnitude can put the economy back on track.

As in other theories of social learning, there are inefficiently low levels of investment because agents do not internalize the effect of their actions on common information. This inefficiency naturally creates room for welfare-enhancing policy interventions. To find these policies, we study the problem of a constrained planner that is subject to the same informational constraints as private agents. We find that the socially constrained-efficient allocation can be implemented with aggregate-beliefs dependent subsidies. For example, it could be desirable to subsidize investment in times of high uncertainty and low activity. However, under certain conditions, the optimal policy does not eliminate the uncertainty traps. Therefore, while policy interventions may be desirable, they do not necessarily eradicate the nonlinearities generated by the complementarity between uncertainty and economic activity.

After characterizing the model, we evaluate the quantitative importance of the uncertainty traps. For that, we extend the baseline model to bring it closer to general real business cycle models. Among other features, we generalize the capital accumulation process by adding an intensive margin. We also introduce a risk-averse representative household with endogenous labor supply. To estimate the importance of uncertainty traps, we compare the outcomes from our extended model with a restricted setup in which uncertainty is not allowed to adjust endogenously. In preliminary numerical exercises, we find that uncertainty traps make economic downturns more persistent and pronounced relative to a framework with fixed uncertainty.

The emphasis on the wait-and-see effect of uncertainty on investment is shared with a recent literature that studies how changes in the volatility of productivity shocks affects the economy, such as Arellano et al. (2012), Bachmann and Bayer (2009), Bloom (2009), Bloom et al. (2012), and Schaal (2012). Two features set us apart from that literature. First, these papers focus on uncertainty induced by time-varying volatility in productivity. In contrast, our learning approach enables us to dissociate subjective uncertainty from volatility in fundamentals. While in our setup volatility generates uncertainty, there can also be periods of high uncertainty with constant volatility. Second, in our analysis the movements in uncertainty are endogenous. That literature focuses, in contrast, on exogenous volatility shocks to productivity. These two distinguishing features create the additional propagation of shocks that we explore in the paper.

The notion of uncertainty in this paper seems justified in the face of systematic references by businessmen and commentators to high levels of uncertainty in the aftermath the 2007-2009 recession despite the decline in several measures of volatility.[1] Indeed, our theory allows for uncertainty to persist in a context with low volatility. The advantage of allowing for endogenous uncertainty movements (as opposed to exogenous volatility shocks) is that endogenous uncertainty is better able to deliver persistent macroeconomic series. Because high volatility events are short-lived, models that focus on that type of shock are hard to reconcile with the persistence of recessions. In contrast,

[1]Measures of aggregate and idiosyncratic volatility such as the VIX volatility index have substantially declined since 2009, as shown in Schaal (2012). Another interesting source of uncertainty suggested in the literature, from which we abstract in this paper, is policy uncertainty. See Baker et al. (2012); Fernández-Villaverde et al. (2011).

subjective uncertainty traps can deliver persistence in a low-volatility context.

Endogenous movements in uncertainty can be modeled in different ways. We make this notion operative using a simple concept of social learning. Intuitively, we envision firms holding bits and pieces of information about a shared fundamental; when a firm invests or hires, its actions reveal information about the state of the economy to other agents. Hence, our analysis relates to papers on fads and herding in the tradition of Banerjee (1992), Bikhchandani et al. (1992), and Chamley and Gale (1994). A number of studies, such as Cunningham (2004), Kaustia and Knüpfer (2009), Khang (2012), and Patnam (2011), empirically document the relevance of social learning in various contexts such as investment in the semiconductor industry, stock market entry decisions, housing purchases, and R&D expenditures. Social learning about technology has also been demonstrated to be important in other contexts such as economic development, as shown by Foster and Rosenzweig (1995) and Hausmann and Rodrik (2003).

Our analysis also relates to a theoretical macroeconomic literature that studies environments with learning from market outcomes such as Rob (1991), Caplin and Leahy (1993), Zeira (1994), Veldkamp (2005), Ordonez (2009), and Amador and Weill (2010), as well as to papers on endogenous volatility over the business cycle, such as Bachmann and Moscarini (2011) and DErasmo and Boedo (2011). Specially related is the analysis in Van Nieuwerburgh and Veldkamp (2006). In their model, agents hold beliefs about a fundamental and the signal-to-noise ratio varies procyclically; this delays recoveries because agents discount new information more heavily in recessions. However, in that paper, uncertainty about the fundamental provides a weak feedback and the economy quickly learns its way out of a recession. The key feature that distinguishes our analysis is the presence of irreversible investments. The option value created by irreversibilities offers a strong additional motive for agents to defer investment in uncertain times. The interaction between social learning and irreversible investment leads, in our setup, to nearly permanent effects of uncertainty on economic activity.

The paper is structured as follows. Section 2 presents the baseline model and the definition of the recursive equilibrium. Section 3 characterizes the partial-equilibrium investment decision of an individual firm, the uniqueness of the equilibrium, the existence of uncertainty traps, and the welfare implications. Section 4 features the preliminary quantitative analysis using an extended model. Section 5 concludes. Proofs are relegated to the appendix.

2 Suggestive Evidence

The central channel in the theory is the feedback between uncertainty and investment. We argue that the inactivity of firms during recessions slows down the diffusion of information, creating uncertainty and discouraging further investment. Therefore, the model predicts that recessions are times where both uncertainty and firm inactivity are high, and that these features may persist even if productivity has recovered. In this section we provide first-pass evidence consistent with these features of the model.

2.1 Uncertainty over the Business Cycle

The literature that studies the impact of uncertainty shocks establishes that the variance in idiosyncratic shocks to productivity increases during bad times. Bloom et al. (2012) demonstrate that the dispersion of plant- and industry- level shocks to productivity is counter-cyclical and peaks in recessions. Other commonly used measures of firm-level volatility, such as the VIX index of volatility in stock market returns or the dispersion in firm level sales, reproduce the same pattern.

Our theory is more specifically concerned with subjective uncertainty. Direct measures of subjective uncertainty are also available and exhibit similar counter-cyclical patterns. Different surveys ask respondents to assess the main reasons why they prefer to postpone economic decisions. According to the National Federation of Independent Business (2012), 40% of answers rank "economic uncertainty" as the most critical problem that they faced in 2012. A more systematic evidence comes from the Michigan Survey of Consumers, which shows a peak during recessions in the percent of consumers who state that "uncertain future" is the main reason to postpone purchases of durable goods in the United States (see figure 1). A similar pattern is observed in the UK, where firm's perceived uncertainty increases during recessions according to the CBI Industrial Trends Survey. Leduc and Liu (2012) argue that these subjective measures of uncertainty are countercyclical.

Other measures of subjective uncertainty with strong counter-cyclical patterns are the variance of ex-post forecast errors about economic conditions and the dispersion of beliefs featured in Bachmann et al. (2013). In bad times, agents hold more heterogeneous beliefs about future economic conditions. Our model allows for short-lived dispersion of beliefs among economic agents within each period, and predicts that the within-period variance of beliefs is larger when uncertainty is high and economic activity is low.

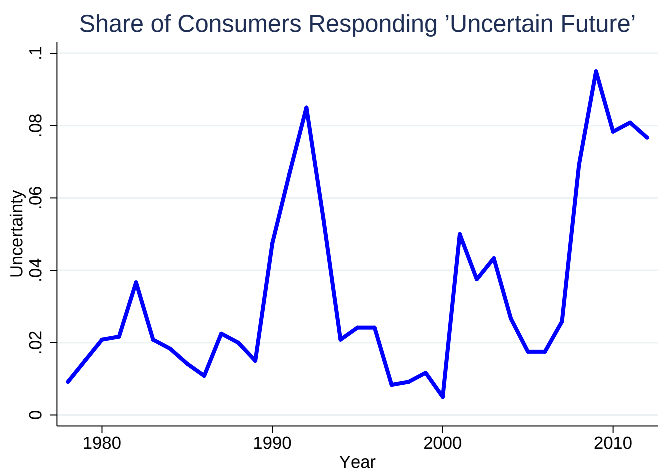


Figure 1: Subjective Uncertainty over Time (Source: Michigan Survey of Consumers)

2.2 Share of Zeros in Investment over the Business Cycle

A second piece of evidence consistent with the basic mechanism that we present concerns the incidence of firm inactivity during recessions. While aggregate investment naturally is counter-cyclical, we emphasize a microeconomic channel based on the inactivity of firms. Because firms face indivisible investment choices, their incentives to invest are low when uncertainty is high. In turn, we posit that lack of activity slows down diffusion of information. Therefore, we expect to see a higher fraction of firms that do not invest when the level of activity is low.

The literature on lumpy adjustment studies the distribution of investment rates over the business cycle, but it focuses more on investment spikes rather than zeros. In this literature, Cooper and Haltiwanger (2006) report that 8% of plant-year observations in the United States between 1972 and 1998 have investment rates below 1% in absolute value. For us, the important question is how this fraction varies over the business cycle. Gourio and Kashyap (2007) report the share of investment zeros for the US and Chile between 1975 and 2000, arguing that in both countries the share of exact or near-zeros is "strongly countercyclical". They report a correlation between aggregate investment and the share of investment rates close to zero of -0.94 in the US and -0.56 in Chile. Since aggregate investment is strongly procyclical, this implies that the share of zeros or near-zeros in firm-level investment is countercyclical.

We complement this evidence with data on the prevalence of exact and near-zeros in investment for a longer time series that includes the current recession. For that, we use quarterly data from Compustat between 1975 and 2012. We follow Eisfeldt and Rampini (2006) in using the variable Property, Plant and Equipment as proxy for physical capital at the firm level. Figure 2 shows the share of firms with zero or near-zero investment rates. To compute these figures, we first restrict the Compustat dataset to firms with non-missing investment rates at the quarterly level and to quarters with at least 500 such firms. Then, we calculate the share of zeros or near-zeros in each quarter. The figure shows the average shares across all quarters in each year, distinguishing between all firms and firms in manufacturing only.

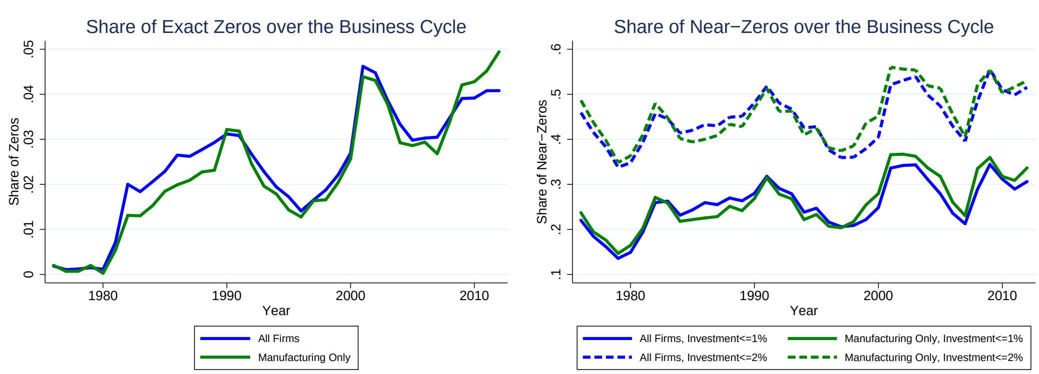


Figure 2: Share of firms with Zero or Near-Zero Investment (Source: Compustat)

On average, 2% of firms display zero investment rates at the quarterly level. In turn, the share of inactive firms is countercyclical. For the years in which the series overlaps with the Gourio and Kashyap (2007) data, both series display similar properties.[2] Interestingly, for the recent recession investment inactivity spikes and remains relatively high after economic activity has recovered. In 2012, 4.9% of firms display zero change in capital. Similar patterns are observed for the share of firms near zero investment in absolute value. For example, the average share of firms with investment rate below 1% is 33%, and with investment rate below 2% is 53%, and both measures peak in bad times. Figure 3 shows the positive correlation between uncertainty from the Michigan Consumer Survey and the share of firms with zero investment from Compustat. The years corresponding to the 2007-2009 crisis and its aftermath appear on the upper right area of the graph.

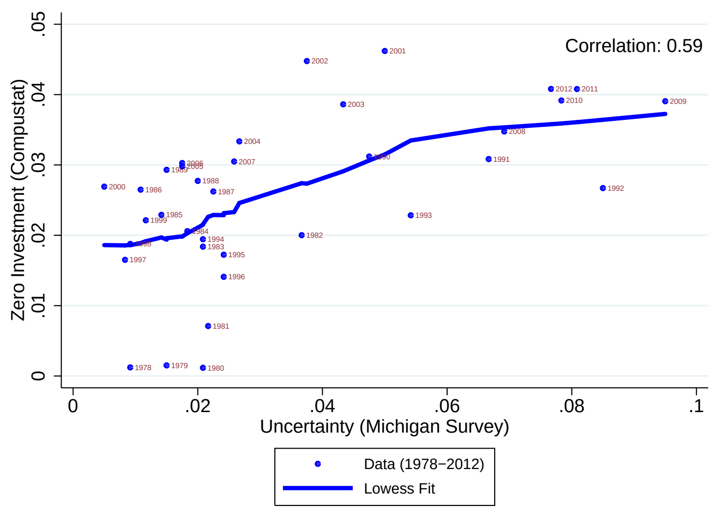


Figure 3: Uncertainty and Share of Firms with Zero Investment

3 Baseline Model

We present a stylized model that features only the necessary ingredients to generate uncertainty traps. The intuitions from this simple model as well as the laws of motion governing the dynamics of uncertainty will carry through to the full model that we use for quantitative analysis.

3.1 Population and Technology

There is a large, fixed number of firms $\bar{N}$ indexed by $n \in \{1, \ldots, \bar{N}\}$. Each firm has an investment opportunity that produces output x_n. Output x_n is the sum of two components: a persistent aggregate component θ which denotes the economy's fundamental as well as an idiosyncratic

[2] Gourio and Kashyap (2007) use establishment level data from the Census Bureau's Annual Survey of Manufacturers. The correlation between the share of firms with investment rate below 2% in their data and in Compustat is 0.65.

transitory component ε_n^x,

$$x_n = \theta + \varepsilon_n^x.$$

Time is discrete. The aggregate term follows a random walk, so that the next-period's fundamental is

$$\theta' = \theta + \varepsilon^\theta. \tag{1}$$

The innovations $\left(\varepsilon^\theta,\ \varepsilon_n^x\right)$ are independent and normally distributed,

$$\varepsilon^\theta \sim \mathcal{N}\left(0, \gamma_\theta^{-1}\right) \text{ and } \varepsilon_n^x \sim \mathcal{N}\left(0, \gamma_x^{-1}\right).$$

We let $u(x)$be the flow payoff to the firm when it invests. Firms have constant absolute risk-aversion[3],

$$u(x) = \frac{1}{a}\left(1 - e^{-ax}\right),$$

where a is the coefficient of absolute risk aversion.

To produce, firms need to incur a fixed cost f. This cost is an i.i.d. draw in every period from the continuous cumulative distribution F with mean $\bar{f}$ and variance σ^f. Therefore, firms solve an optimal stopping time problem. In each period, given the fixed cost f and the beliefs about the returns to the investment that we specify below, firms can either wait or invest. When a firm invests, it pays the fixed cost f, produces x_n, and exits the economy. We assume that investing firms are immediately replaced with new firms that hold an investment opportunity. In this way, the mass of firms remains constant over time.[4]

3.2 Timing and Information

At the beginning of a period firms do not observe the fundamental θ but hold beliefs about its distribution. We call "uncertainty" the variance of the prior beliefs about θ. Because the fundamental and the signals are normally distributed and information is public, all firms start the period with a common, normally distributed prior about θ,

$$\theta \mid \mathcal{I} \sim \mathcal{N}\left(\mu, \gamma^{-1}\right),$$

where $\mathcal{I}$ is the information set at the beginning of the period. The mean of the beliefs distribution μ captures the optimism of agents about the state of the economy, while γ is the precision of information. A lower γ means that firms have higher uncertainty. In each period, the aggregate state space of the economy reduces to the common beliefs (μ, γ).

[3]The assumption of risk aversion is not necessary for the results. We include it for technical reasons in the general-equilibrium uniqueness proofs. In the simulation of the baseline model, we show that the key properties of the model carry through with risk neutrality. In the full quantitative model, risk aversion arises from a standard stochastic discount factor derived from risk averse households.

[4]The assumption that firms exit when they invest is for tractability of the baseline model and it can be relaxed.

Firms may learn about the fundamental θ in various ways as time unfolds. First, firms learn by producing. We let $N \in \{1, \ldots, \bar{N}\}$ be the endogenous number of firms that invests in a given period and I be the set of such firms. When firm n invests, output x_n is observed by every firm but the fundamental θ cannot be distinguished from the idiosyncratic term ε_n^x. Therefore, production is a noisy signal of the fundamental. Because of the Gaussianity assumption, the information about the fundamental conveyed by each firm's output is summarized by the public signal

$$X \equiv \frac{1}{N} \sum_{n \in I} x_n = \theta + \varepsilon_N^X, \tag{2}$$

where

$$\varepsilon_N^X \equiv \frac{1}{N} \sum_{n \in I} \varepsilon_n^x \sim \mathcal{N}\left(0, (N\gamma_x)^{-1}\right).$$

Average output, X, summarizes all the information provided by the distribution of firm-level output. It is also important that the precision of this signal, $N\gamma_x$, increases with the number of investing firms, N. The higher is N, the more precise is the information collected by each individual firm.

In addition, firms learn about the fundamental from a public signal Y observed at the end of each period,

$$Y = \theta + \varepsilon^y,\ \varepsilon^y \sim \mathcal{N}\left(0, \gamma_y^{-1}\right). \tag{3}$$

This captures the information released by statistical agencies or the media.

The timing of events is summarized in Figure 4.

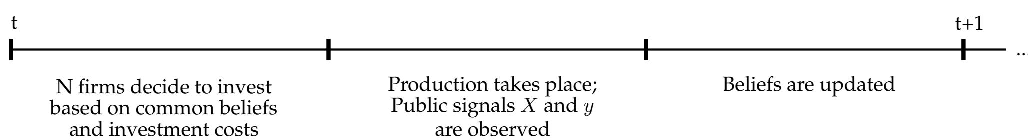


Figure 4: Timing of events

3.3 Firm Problem

The value of a firm with an investment opportunity that starts the period with beliefs (μ, γ) and a draw f of the fixed investment cost is

$$V(\mu, \gamma, f) = \max\left\{V^W(\mu, \gamma), V^I(\mu, \gamma) - f\right\}, \tag{4}$$

where $V^W(\mu, \gamma)$ is the value of the firm if it waits until the next period and $V^I(\mu, \gamma)$ is the value of the firm after incurring the investment cost f.

We assume that the number of firms $\bar{N}$ is large enough that firms behave competitively. Specifically, they do not internalize the impact of their decisions on aggregate information. The firm's problem yields an investment rule $\chi(\mu,\gamma,f) \in \{0,1\}$, such that[5]

$$\chi(\mu,\gamma,f) = \begin{cases} 1 & \text{if invests } \Leftrightarrow V^{I}(\mu,\gamma) - f \geq V^{W}(\mu,\gamma) \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

When a firm waits, it starts the next period with new beliefs (μ',γ') about the fundamental and a new fixed cost draw f'. Therefore, the value of waiting is

$$V^{W}(\mu,\gamma) = \beta\mathbb{E}\left[V\left(\mu',\gamma',f'\right)\right]. \tag{6}$$

In turn, when a firm invests it receives output x and exits. Therefore, the value of investing, net of the fixed cost, is

$$\begin{aligned} V^{I}(\mu,\gamma) &= \mathbb{E}\left[u(x)\,|\mu,\gamma\right] \\ &= \mathbb{E}\left[\frac{1}{a}\left(1-e^{-a\cdot x}\right)|\mu,\gamma\right] = \frac{1}{a}\left(1-e^{-a\mu+\frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}\right). \end{aligned} \tag{7}$$

3.4 Law of Motion for Common Beliefs (μ,γ)

Firms start the period with beliefs (μ,γ). By period's end, they have observed the public signals X and Y defined in (2) and (3). After that, standard rules for Bayesian updating imply that the common posterior belief about θ is normally distributed with mean and precision of information equal to

$$\begin{aligned} \mu_{post} &= \frac{\gamma\cdot\mu + \gamma_y\cdot Y + N\gamma_x\cdot X}{\gamma+\gamma_y+N\gamma_x}, \\ \gamma_{post} &= \gamma+\gamma_y+N\gamma_x. \end{aligned}$$

These standard updating rules have a straightforward interpretation: the mean of the posterior belief is a precision-weighted average of the past belief μ and the new signals, Y and X, whereas its precision is the sum of the precision of the prior belief, γ, and the precision of the the new signals.

$\{\mu_{post},\gamma_{post}\}$ define the beliefs by period's end. Before the next period starts, the fundamental θ receives an innovation as defined in (1). Therefore, at the beginning of the new period the prior about the fundamental θ' is normally distributed with mean and inverse of uncertainty equal to

$$\mu' = \mu_{post} = \frac{\gamma\cdot\mu + \gamma_y\cdot Y + N\gamma_x\cdot X}{\gamma+\gamma_y+N\gamma_x}, \tag{8}$$

$$\gamma' \equiv \Gamma(N,\gamma) = \left(\frac{1}{\gamma_{post}}+\frac{1}{\gamma_\theta}\right)^{-1} = \left(\frac{1}{\gamma+\gamma_y+N\gamma_x}+\frac{1}{\gamma_\theta}\right)^{-1}. \tag{9}$$

Conditions (8) and (9) are the laws of motions for beliefs. The first moment, μ', depends on

[5] We assume that firms choose to invest in the case of indifference. This assumption is innocuous as these events happen with probability 0.

the aggregate public signals X and Y. The number of investing firms, N, determines the quality of the public signal X. In turn, the precision of information γ' solely evolves based on N. The higher is N, the more precise the public signal X is and the higher the precision of the prior beliefs on θ'. For future reference, and because it is a key object in our analysis, we let $\Gamma(N,\gamma)$ in 9 be the law of motion of the precision of information.

3.5 Law of Motion for the Number of Investing Firms N

We have so far introduced the firm's problem and the law of motion for the aggregate state given a number of investing firms N. Of course, the process for N must be consistent with the individual choices of firms. The number of firms that invests satisfies

$$N(\mu,\gamma) = \sum_{n=1}^{\bar{N}} \chi(\mu,\gamma,f_n) = \sum_{n=1}^{\bar{N}} \mathbb{I}\left(V^I(\mu,\gamma) - f_n \geq V^W(\mu,\gamma)\right). \tag{10}$$

Because the investment rule $\chi(\mu,\gamma,f_n)$ is a random function of the fixed cost, the number of investing firms is stochastic and depends on the realization of the shocks $\{f_n\}_{1\leq n\leq \bar{N}}$. Therefore, at the beginning of a period and before these shocks are realized, N is perceived as following a binomial distribution,

$$N(\mu,\gamma) \sim Bin\left(\overline{N}, p(\mu,\gamma)\right).$$

The probability of investment for each firm, $p(\mu,\gamma)$, must equal the probability of receiving a fixed cost such that investment is profitable. From (5), this implies a fixed cost below the threshold $V^I(\mu,\gamma) - V^W(\mu,\gamma)$, so that

$$p(\mu,\gamma) = F\left(V^I(\mu,\gamma) - V^W(\mu,\gamma)\right).$$

As the total number of firms grows large, the fraction of firms that invests in every period becomes a deterministic function of aggregate beliefs,

$$\frac{N(\mu,\gamma)}{\bar{N}} \xrightarrow{a.s} p(\mu,\gamma). \tag{11}$$

This dependence of the level of activity on aggregate beliefs justifies our initial statement that $\{\mu,\gamma\}$ are the sole aggregate states of the economy. From the perspective of an individual firm, when the total number of firms is large, N is a deterministic function of aggregate beliefs. In turn, the mean of aggregate beliefs μ evolves stochastically according to (8), but the law of motion for the precision of information $\gamma' = \Gamma(N(\mu,\gamma),\gamma)$, defined in (9), is a deterministic function of $\{\mu,\gamma\}$.

3.6 Recursive Equilibrium

We are ready to define a recursive equilibrium

Definition 1. An equilibrium consists of the policy function $\chi(\mu,\gamma,f)$, value functions $V(\mu,\gamma,f)$,

$V^W(\mu,\gamma)$, $V^I(\mu,\gamma)$, laws of motions for aggregate beliefs $\{\mu',\gamma'\}$, and a number of investing firms $N(\mu,\gamma)$, such that

1. the value function $V(\mu,\gamma,f)$ solves (4), with $V^W(\mu,\gamma)$ and $V^I(\mu,\gamma)$ defined according to (6) and (7), generating a policy function $\chi(\mu,\gamma,f)$ in (5);

2. the aggregate beliefs $\{\mu,\gamma\}$ evolve according to (8) and (9); and

3. the number $N(\mu,\gamma)$ of firms that invest is given by (10).

4 Equilibrium Characterization and Uncertainty Traps

We start by characterizing the partial-equilibrium investment decision of a firm given the laws of motions for beliefs. Because of the irreversibility of the investment, firms are less likely to invest when uncertainty is high. After establishing the existence and uniqueness of the recursive equilibrium, we characterize its key properties. Specifically, we examine the interaction between the option value of investment and social learning. This interaction creates episodes of self-sustaining uncertainty and low activity, which we call uncertainty traps. We provide sufficient conditions on the parameters that guarantee the existence of uncertainty traps, we discuss the type of aggregate dynamics that they imply, and we characterize their policy implications.

4.1 Investment Rule Given the Evolution of Beliefs

The investment rule $\chi(\mu,\gamma,f)$ crucially depends on how beliefs evolve. We establish two simple lemmas about the aggregate beliefs process.

Evolution of Mean Beliefs Using (8) we can characterize the stochastic process for the mean of the prior distribution of beliefs about the fundamental.

Lemma 1. *Mean beliefs μ follow a random walk with time-varying volatility s,*

$$\mu' = \mu + s(N,\gamma)\,\varepsilon,$$

where $s(N,\gamma) = \left(\frac{1}{\gamma} - \frac{1}{\gamma+\gamma_y+N\gamma_x}\right)^{\frac{1}{2}}$ and $\varepsilon \sim \mathcal{N}(0,1)$.

Mean beliefs capture the optimism of agents about the fundamental, and they evolve stochastically due to the the arrival of new information. The volatility of mean beliefs is time-varying because the amount of information that firms collect over time is endogenous. The volatility $s(N,\gamma)$ depends negatively on the current precision of beliefs γ. In times of low uncertainty, when the precision of beliefs is high, Bayesian updaters place less weight on new information, making mean beliefs less sensitive to the cycle. The volatility of optimism also depends positively on the number of active firms, N. When N is large, information flows faster, making beliefs more likely to jump. Through this effect, the volatility of mean beliefs is lower in recessions.

Evolution of Uncertainty The precision of beliefs γ is specially important as it embodies the dynamics of uncertainty. The precision of beliefs is random only as a result of the finiteness of the number of firms, which vanishes as their total number grows large. Conditioning on the realization of N, the dynamics of precision γ is deterministic and allows for a simple analytical characterization.

Lemma 2. *The precision of next-period beliefs γ' increases with N and γ. For a given N, the law of motion for the precision of beliefs $\gamma' = \Gamma(N, \gamma)$ admits a unique stable fixed point in γ.*

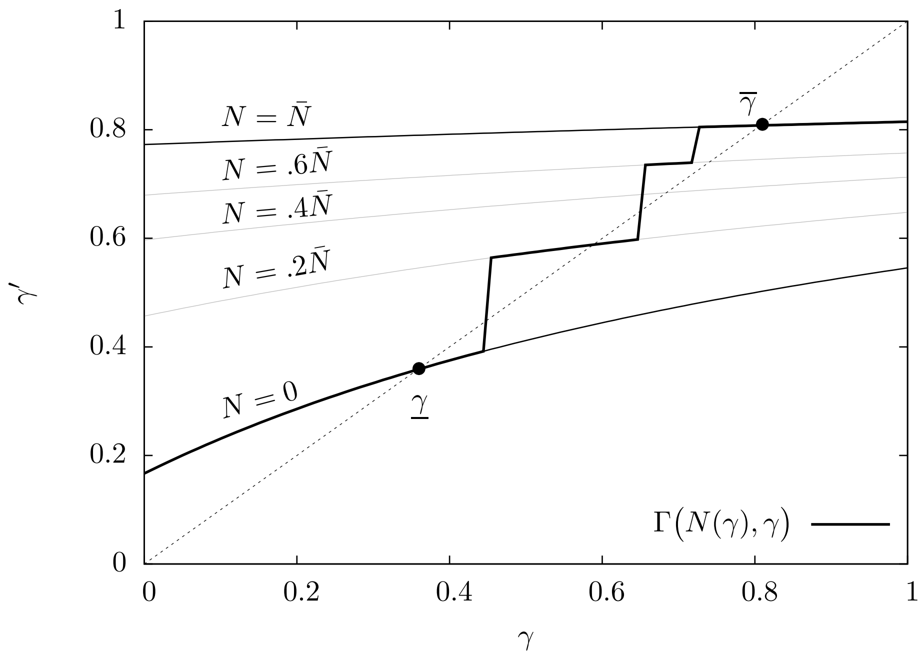


Figure 5: Example of dynamics for belief precision γ

Figure 5 depicts $\Gamma(N, \gamma)$ for different values of N that range from $N = 0$ to $N = \bar{N}$. As stated in Lemma 2, an increase in the level of activity raises the next-period precision of information γ' for any precision of information in the current period. In the example of the figure, it is evident that the support of the ergodic distribution of γ must be bounded between $\underline{\gamma}$ and $\bar{\gamma}$, i.e., the levels for the inverse of uncertainty corresponding to $N = 0$ and $N = \bar{N}$.

In equilibrium, N is endogenous and varies with γ. Assuming momentarily that $N(\gamma)$ is an increasing step function, the figure illustrates how the feedback from uncertainty to investment opens up the possibility of multiple stationary points in the dynamics of beliefs precision (or its inverse, uncertainty). For the chosen path of $N(\gamma)$, the function $\gamma' = \Gamma(N(\gamma), \gamma)$ depicted by the solid line in the figure has three stable fixed points. Below, we formally establish that this type of multiplicity is a generic feature of the recursive equilibrium.

Optimal Timing of Investment How does the individual investment decision depend on beliefs? Intuitively, more optimistic priors, in the form of higher mean beliefs μ, should raise aggregate investment due to the higher opportunity cost of delays. In turn, uncertainty may reduce investment for two reasons. First, higher uncertainty reduces the expected payoff of the investment due to risk aversion. Second, because the investment is costly and irreversible there is an option value of waiting. This creates an extra reason to wait when uncertainty is high in order to gather additional information and avoid downside risks. More formally, this delay occurs because, as highlighted in Lemma 1, mean beliefs are more volatile when uncertainty is high. Since the difference between the value of waiting defined in (6) and the value of investing defined in (7) is a convex function of mean beliefs, the higher volatility in mean beliefs caused by higher uncertainty makes waiting more attractive than investing.

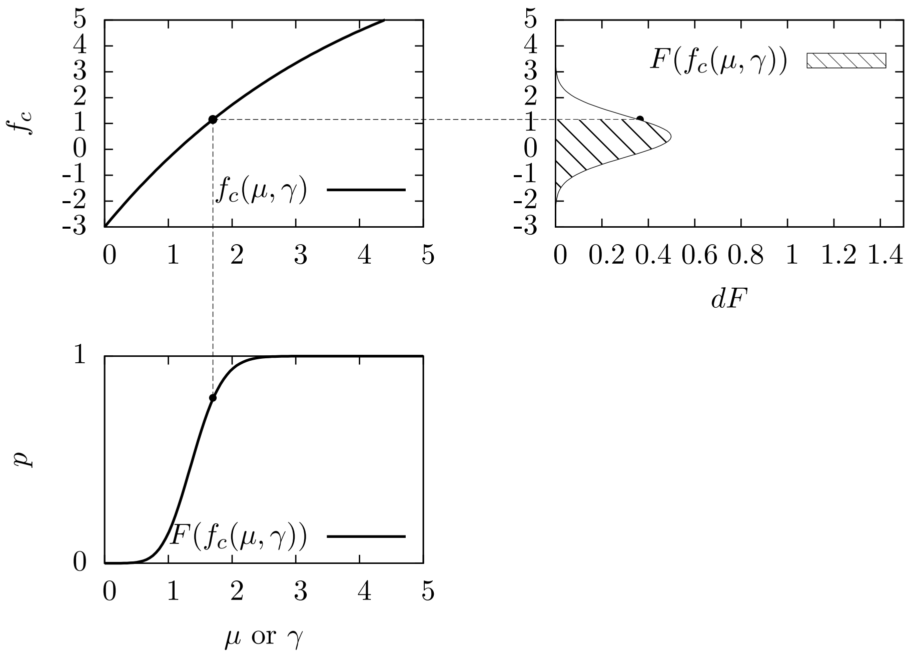


Figure 6: Investment probability as a function of beliefs

The following proposition formally establishes that this intuition is valid and provides a characterization of the optimal investment behavior.

Proposition 1. *Under the regularity condition in assumption 1 stated in the appendix, given a random number of investing firms* $N(\mu,\gamma) \sim Bin\left(\overline{N}, p(\mu,\gamma)\right)$ *for some* $p(\mu,\gamma)$ *and for* γ_x *sufficiently low, there exists a unique cutoff for the fixed costs* $f_c(\mu,\gamma) \in \mathbb{R} \cup \{-\infty,\infty\}$ *such that firms invest if and only if* $f \leq f_c(\mu,\gamma)$*. The cutoff* $f_c(\mu,\gamma)$ *is strictly increasing with* μ *and* γ*.*

This partial-equilibrium result characterizes the investment rule given the random number of investing firms $N(\mu,\gamma)$ and the laws of motion for μ and γ. Firms invest if and only if the idiosyncratic fixed cost falls below the threshold $f_c(\mu,\gamma)$. Figure 6 depicts how the probability of investment is affected by the beliefs μ and γ. The upper-left panel shows the threshold as function of the mean level of beliefs or the precision of information. For each level of beliefs, the bottom panel shows the probability of investing, which corresponds to the shaded area below the fixed-cost distribution in the right panel. Crucially for what follows, the probability of investment decreases with uncertainty.

4.2 Equilibrium Uniqueness

We have established in Lemmas 1 and 2 how beliefs depend on the number of investing firms, and in Proposition 1 how firms' investment decision is affected by beliefs. From the latter, a randomly chosen firm invests with probability $F(f_c(\mu,\gamma))$ given an arbitrary function $p(\mu,\gamma)$. To fulfill the recursive equilibrium definition, we need that $p(\mu,\gamma) = F(f_c(\mu,\gamma))$. The next proposition states that a unique equilibrium satisfies this condition.

Proposition 2. *Under the regularity conditions in assumptions 1 and 2 stated in the appendix, and for γ_x small enough, a recursive equilibrium exists and is unique. The expected number of investing firms is increasing in mean beliefs μ and decreasing in uncertainty (i.e., increasing in the precision of beliefs γ).*

Figure 7 depicts the expected number of investing firms as a function of beliefs (μ,γ). The partial-equilibrium results from Proposition 1 carry through to the general equilibrium: investment is more likely as firms are more optimistic about the fundamental (μ high) or less uncertain (γ high). In turn, as we illustrated in the example of figure 5, when the number of firms that invests increases with γ there may be multiple fixed points in the joint dynamics of activity and uncertainty. In what follows, we demonstrate that this multiplicity is a generic feature of the equilibrium and that it leads to persistent dynamics.

4.3 Uncertainty Traps

We describe here the core mechanism of the paper. We assume at this point that the total number of firms $\bar{N}$ is large enough, so that

$$\frac{N(\mu,\gamma)}{\bar{N}} \simeq p(\mu,\gamma) = F(f_c(\mu,\gamma)).$$

With this assumption we can treat N as a deterministic function of beliefs, ignoring fluctuations due to the finiteness in the number of firms.[6]

We formally define an uncertainty trap as the existence of multiple stationary points in the dynamics of belief precision γ given a level of mean beliefs μ.

Definition 2. Given mean beliefs μ_0, there is an uncertainty trap if there are at least two locally stable fixed points in the dynamics of beliefs precision $\gamma' = \Gamma(N(\mu_0,\gamma),\gamma)$.

[6] Of course, we must be careful in taking this limit to ensure that agents remain uncertain. See the appendix for a formal statement.

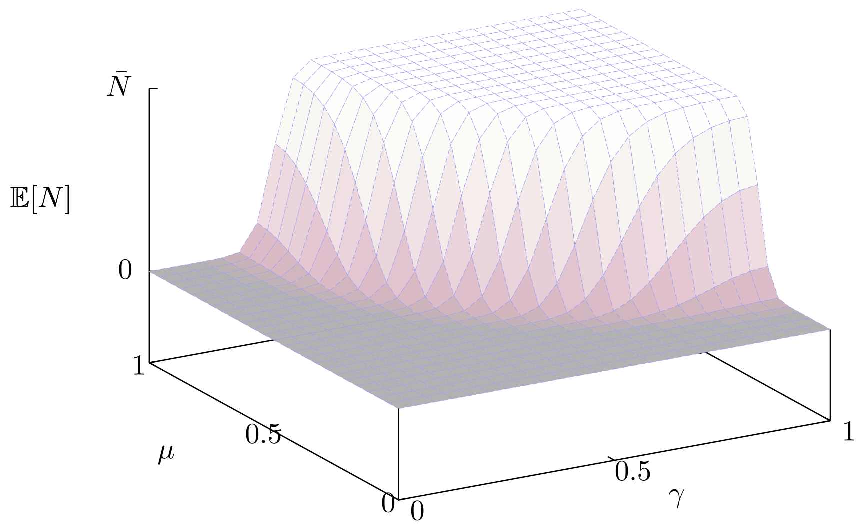


Figure 7: Example of aggregate investment pattern

Uncertainty traps, defined as multiple fixed points in the dynamics of uncertainty given a level of mean beliefs, do not imply that there are multiple equilibria in the model. In fact, in Proposition 2 we have already established that the recursive equilibrium is unique. While multiple values of γ may satisfy the requirement that $\gamma = \Gamma(N(\mu, \gamma), \gamma)$ given a value of μ, the recursive equilibrium is unique because mean beliefs μ evolve stochastically, as characterized in Lemma 1. We defined to each fixed point in the dynamics of beliefs as a regime.

Do uncertainty traps necessarily arise? We formally establish the existence of a range of values for μ for which the economy necessarily features uncertainty traps. An important condition on the parameters which guarantees the existence of uncertainty traps is that the dispersion in the distribution of fixed costs, σ^f, is not too large. This ensures a strong enough feedback from information to investment.[7]

Proposition 3. *Under the conditions of Proposition 2 and for σ^f small enough, there exists a non-empty interval $[\mu_l, \mu_h]$ such that, for all $\mu_0 \in (\mu_l, \mu_h)$, the economy features an uncertainty trap with at least two regimes $\gamma_l(\mu_0) < \gamma_h(\mu_0)$. Regime γ_l (γ_h) is characterized by high (low) uncertainty and low (high) investment.*

Figure 8 offers an example for the law of motion of γ given different values of μ. For the range of values of μ in $[\mu_l, \mu_h]$, the dynamics of belief precision admits two locally stable regimes. This is the range highlighted in Proposition (3). For values of μ above μ_h, the dynamics of beliefs only

[7] Intuitively, as the distribution of fixed costs becomes less dispersed, the number of investing firms $N(\mu, \gamma)$ becomes steeper with changes in beliefs. See the appendix for details.

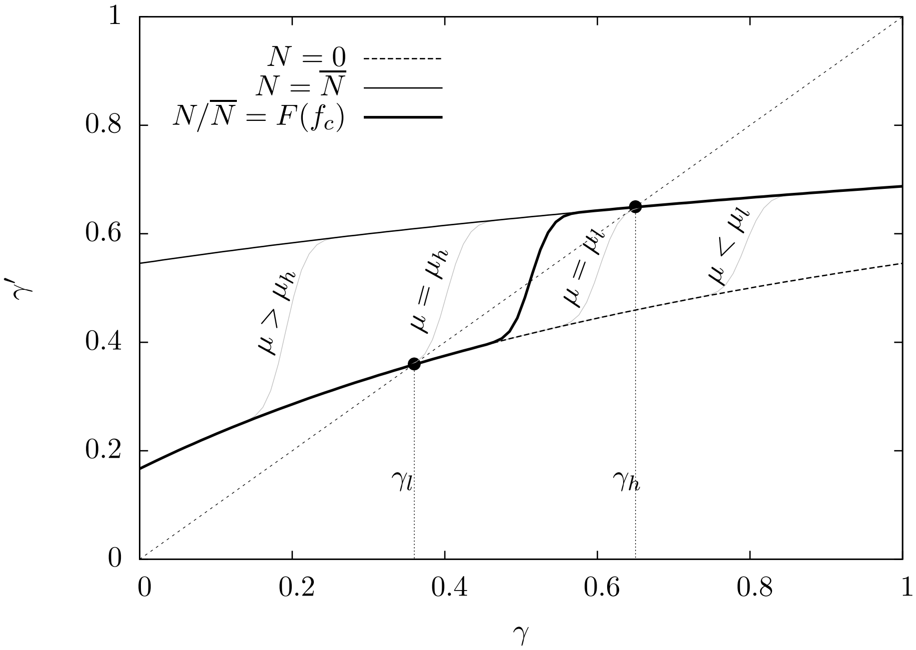


Figure 8: Example of dynamics for precision γ

admits the high-activity regime, while for values below μ_l, it only admits the low-activity regime. Hence, for a non-negligible range of mean beliefs, the economy may fluctuate between extremes of no activity or full activity. Of course, the mean of the beliefs distribution evolve in the background following a random walk. Therefore, the economy may remain in one of the two regimes for a while, but eventually escaping if μ drifts sufficiently far away. Proposition 3 establishes that the situation depicted in this figure is a generic feature of the equilibrium.

Uncertainty traps give rise to non-linear aggregate dynamics, business cycle asymmetries and shocks that may have near permanent effect on the economy. Figure 9 illustrates these effects though various simulations based on the example from Figure 8. The top panel presents three different series of shocks to the mean beliefs μ. The three series start from the high activity/low uncertainty regime. The economy is hit at t=5 by a negative shock to mean beliefs, either due to a particularly bad realization of the public signals or the fundamental. The economy returns to normal from t=10 onwards. Across the three series, what varies is the magnitude of the initial drop.

The middle and bottom panels show the response of belief precision γ and the number of investing firms N. The total number of firms that invest prior to the shock equals $\bar{N}$. The solid gray line represents a small temporary shock. After the shock hits, firms still find it profitable to invest, the number of investing firms remains equal to $\bar{N}$, and the precision of beliefs is unaffected. When the economy is hit by a temporary shock of slightly larger magnitude, some firms stop investing, leading to a gradual increase in uncertainty. As uncertainty rises, investment goes down even further and the economy starts to drift towards the low regime. However, when mean beliefs

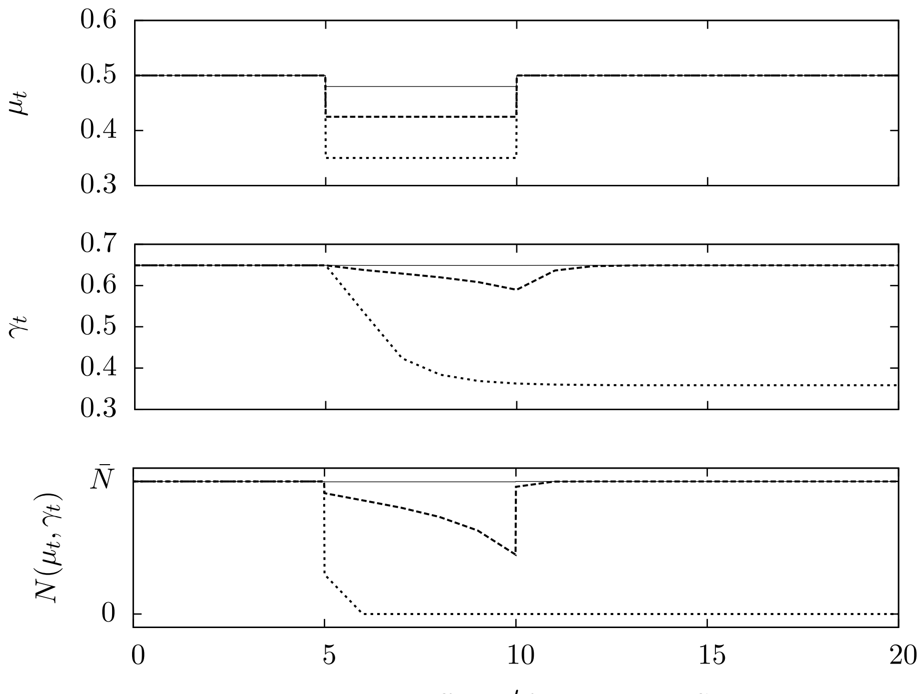


Figure 9: Persistent Effects of Temporary Shocks

recover, the precision of information and the number of active firms quickly return to the high-activity regime. In contrast, when the economy is hit by an even larger temporary shock, such as the dotted line, the number of firms delaying investment is large enough to produce a self-sustaining increase in uncertainty. The economy quickly shifts to the low-activity regime and remains trapped there even after mean beliefs recover to the starting position.

How does the economy escape an uncertainty trap? Figure (10) depicts the evolution of the economy after it is hit by the large shock from Figure (9). As before, the large negative shock hitting the economy from periods 5 to 10 pushes the economy into the low activity-high uncertainty regime. Eventually, the economy receives positive signals that lead to a temporary increase in mean beliefs between periods 20 and 25, maybe because of positive realizations of the fundamental. When the temporary increase in average beliefs is not sufficiently strong, the recovery is short-lived. However, when the temporary increase is sufficiently large, the economy reverts back to the high-activity regime. Once again, temporary shocks of sufficient magnitude to the fundamental may lead to permanent effects on the economy.

A number of additional lessons can be drawn from these simulations. First, uncertainty only matters in cases where firms do not have an overwhelming preference for either investing or waiting. Second, in this framework uncertainty is a by-product of recessions. This result echoes some of the empirical findings from Bachmann et al. (2013). Third, as in models with learning in the spirit of Van Nieuwerburgh and Veldkamp (2006), this theory provides an explanation for asymmetries in business cycles. Finally, the simulations also highlight that agents can be uncertain about the fundamentals without necessarily being uncertain about endogenous variables. For example, in

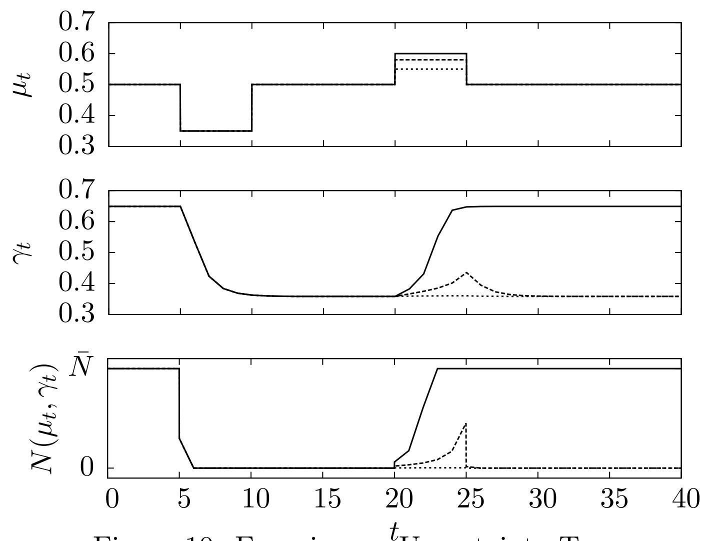


Figure 10: Escaping an Uncertainty Trap

the low activity regime, firms can predict the level of investment because they know that they are uncertain and trapped in the bad equilibrium. This highlights a potential difficulty with identifying uncertainty in the data. As implied by the model, accuracy of forecasts may possibly be a bad proxy for uncertainty about fundamentals.

4.4 Policy Implications

The existence information frictions raises the question of efficiency. In the decentralized equilibrium firms invest less often than they should from a social perspective because they do not internalize that their actions release information to the rest of the economy. Proposition 4 shows that the decentralized economy is constrained inefficient. But a simple policy instrument such as an investment subsidy that only depends on current beliefs (μ, γ) is sufficient to make firms internalize their impact on the rest of the economy and implements the efficient allocation.

Proposition 4. *The decentralized competitive equilibrium is constrained inefficient and the symmetric, socially efficient allocation can be implemented with positive investment subsidies $\tau(\mu, \gamma)$. In turn, when γ_x and σ^f are small, the efficient allocation is still subject to uncertainty traps.*

Proposition 4 implies that firms are more likely to invest in the efficient allocation. It does not mean, though, that uncertainty traps cannot arise in the constrained-efficient allocation. Even though such situations seem to occur due to a lack of coordination between firms, there are cases when the planner cannot do any better than the decentralized economy and ends up trapped in a similar fashion. If the planner does not have any additional information than agents in the economy, it is still optimal to wait when uncertainty is too high. Hence, there still exists a strong complementarity between information and the level of activity in the constrained efficient-allocation, although uncertainty traps might be less likely to arise in that case.

5 Quantitative Analysis [Preliminary]

To quantify the impact of uncertainty traps we enrich the model in several dimensions. We introduce a Cobb-Douglas production function with labor and capital as inputs. We also introduce an intensive margin for investment, so that firms can now precisely choose their capital stock, and a representative household that supplies labor and owns the firms. These additions allow us to evaluate the effect of uncertainty traps on employment and wages.

5.1 Technology

Now, each firm operates a Cobb-Douglas production function to produce the unique consumption good. A firm n employing l units of labor and using k units of capital produces output

$$q_n = e^y k_n^\alpha l_n^{1-\alpha},$$

where

$$\begin{aligned} y &= \theta + \varepsilon^y \\ \theta' &= \rho_\theta \theta + \varepsilon^\theta \end{aligned}$$

with distributions

$$\begin{aligned} &\varepsilon^\theta \sim \mathcal{N}\left(0, (1-\rho_\theta^2)\gamma_\theta^{-1}\right) \\ &\varepsilon^y \sim \mathcal{N}\left(0, \gamma_y^{-1}\right). \end{aligned}$$

As before, the stochastic process θ is the fundamental. Now, it follows an AR(1) process instead of a random walk as in our baseline model. Next-period capital for surviving firms evolves according to

$$k_n' = (1 - \delta + i_n)k_n,$$

where δ denotes the depreciation rate. When a firm invests, it must pay a fixed cost $f_n k_n$ in units of the final good and a variable cost $c(i_n)k_n$. As in the baseline model, $f \geq 0$ is an i.i.d. random variable with density $g_f(f)$. The function c is strictly convex and continuously differentiable. By making the fixed and variable investment costs proportional to the current capital stock, we can solve the investment problem independently from the capital stock and preserve tractability.

A firm exits with probability $1-\omega$. In that case, a fraction $1-\xi$ of its capital is destroyed while the remainder is assigned to a new entering firm. A firm can only invest if it has an investment opportunity. Firms without an opportunity randomly receive one with probability q. A firm that does not exert its investment opportunity carries it into the following period, but each firm can only hold one investment opportunity at a time.

5.2 Timing and Information

The timing and the information diffusion process closely follow from the baseline model. At the beginning of a period, firms hold beliefs about the state of the fundamental,

$$\theta \mid \mathcal{I} \sim \mathcal{N}\left(\mu, \gamma^{-1}\right).$$

As before, μ and γ denote the mean and the precision of these beliefs. A firm that invests obtains a signal $x_n = \theta + \epsilon_n^x$ about the fundamental. As in the baseline model, we denote by N the endogenous number of firms that decide to invest in a given period. We assume that, at the end of the period, a firm observes the decision of all the other firms in the economy. This is equivalent to observing the public signal

$$X = \frac{1}{N}\sum_{n \in I} x_n = \theta + \varepsilon_N^X$$

where I is the set of firms that invest, and $\varepsilon_N^X \equiv \sim \mathcal{N}\left(0, (N\gamma_x)^{-1}\right)$. Importantly, the precision of this aggregate public signal is increasing in the number of firms that decide to invest in the current period. Firms also learn about the fundamental by observing the aggregate productivity y.

The timing of events is as follow:

1. All firms share the same prior distribution over the fundamental θ.

2. Firms that do not hold an investment opportunity receive one with probability q.

3. Firms that hold an investment opportunity observe their fixed cost f_n and decide whether or not to invest. If they pay the cost, they invest i_n.

4. The firms that invested receive a signal:

   - Each firm n that invests observes a private signal $x_n = \theta + \varepsilon_n^x$.
   - After observing x_n, each firm chooses labor l_n. In equilibrium, this

5. The common shock y is revealed and observed by everyone. Actions are observed. All firms produce and markets clear.

6. Each firm survives with probability ω. If a firm dies, it is replaced by a new firm with capital equal to a fraction ξ of the exiting firm.

7. Agents update their beliefs for next period.

The key difference with the baseline model is that, now, the labor decision of firms is revealing of their private information. As before, all firms decide whether or not to invest based on their common information and fixed cost. Those who invest, also decide how much capital to add. Once the investment has taken place, firms observe a signal and, based on that signal, hire a number of workers. Because the number of workers hired by a firm is monotonic in the signals, and because labor is observed, the information diffused through the economy again increases linearly with the number of firms that invest.

5.3 Representative Household

A representative household consumes C units of the final good and supplies L units of labor to firms in a competitive environment. The household holds wealth in firm shares and maximizes life-time expected utility given its budget constraint. Optimality leads to standard labor-consumption choices satisfying

$$w(\Omega, X, y) = -\frac{U_L(C(\Omega, X, y), L(\Omega, X, y))}{U_C(C(\Omega, X, y), L(\Omega, X, y))}$$

and as well as,

$$p(\Omega, X, y) = U_C(C(\Omega, X, y), L(\Omega, X, y)),$$

where we denote the state of the economy at the beginning of each period by Ω. With preferences given by $U = \frac{C^{1-\sigma}}{1-\sigma} - hL$, the wage is

$$w(\Omega, X, y) = \frac{h}{C^{-\sigma}}$$

and the price satisfies

$$p(\Omega, X, y) = C^{-\sigma}.$$

5.4 Firm Problem

We let$K$ be the average stock of capital in the economy and Q be the fraction of firms with an investment opportunity. Then, the aggregate state at the beginning of the period as $\Omega = \{\mu, \gamma, K, Q\}$. The value function of a firm with an opportunity to invest is

$$V^1(\Omega, k) = \mathbb{E}_f \left[\max \left\{ V^w(\Omega, k), \max_i \mathbb{E}_x \left[V^i(\Omega, x, i, f, k) \right] \right\} \right],$$

where the value of waiting is

$$V^w(\Omega, k) = \max_l \mathbb{E}_{y,X} \left[p(\Omega, X, y) \left(e^y k^\alpha l^{1-\alpha} - l w(\Omega, X, y) \right) + \omega \beta V^1 \left(\Omega', (1-\delta) k \right) \mid \Omega \right]$$

and the value of investing is

$$V^i(\Omega, x, i, f, k) = \max_l \mathbb{E}_{y,X}\left[p(\Omega, X, y)\left(e^y k^\alpha l^{1-\alpha} - lw(\Omega, X, y) - c(i)k - fk\right)\right.$$
$$\left. + \omega\beta\left[qV^1\left(\Omega', (1-\delta+i)k\right) + (1-q)V^0\left(\Omega', (1-\delta+i)k\right)\right] \mid \Omega, x\right].$$

Similarly, the value of a firm without an investment opportunity is

$$V^0(\Omega, k) = \max_l \mathbb{E}_{y,X}\left[p(\Omega, X, y)\left(e^y k^\alpha l^{1-\alpha} - lw(\Omega, X, y)\right)\right.$$
$$\left. + \omega\beta\left[qV^1\left(\Omega', (1-\delta)k\right) + (1-q)V^0\left(\Omega', (1-\delta)k\right)\right] \mid \Omega\right].$$

To simplify these expressions, it is useful to introduce the following notation

$$\begin{aligned}
\bar{w} &= \mathbb{E}_{y,X}\left[w(\Omega, X, y) \mid \Omega\right], \\
\bar{ep} &= \mathbb{E}_{y,X}\left[e^y p(\Omega, X, y) \mid \Omega\right], \\
\bar{pw} &= \mathbb{E}_{y,X}\left[p(\Omega, X, y) w(\Omega, X, y) \mid \Omega\right]
\end{aligned}$$

and

$$\begin{aligned}
\bar{w}_x &= \mathbb{E}_{y,X}\left[w(\Omega, X, y) \mid \Omega, x\right], \\
\bar{ep}_x &= \mathbb{E}_{y,X}\left[e^y p(\Omega, X, y) \mid \Omega, x\right], \\
\bar{pw}_x &= \mathbb{E}_{y,X}\left[p(\Omega, X, y) w(\Omega, X, y) \mid \Omega, x\right].
\end{aligned}$$

The labor demand decision of firms implies that all value functions are linear in k. We can therefore write the values per unit of capital as

$$\begin{aligned}
v^1(\Omega) &= \mathbb{E}_f\left[\max\left\{v^w(\Omega), \max_i \mathbb{E}_x\left[v^i(\Omega, x, i)\right] - \bar{p}f\right\}\right], \\
v^0(\Omega) &= \alpha\left(\bar{ep}\right)^{\frac{1}{\alpha}}\left(\frac{1-\alpha}{\bar{pw}}\right)^{\frac{1-\alpha}{\alpha}} + (1-\delta)\omega\beta\mathbb{E}_{y,X}\left[qv^1\left(\Omega'\right) + (1-q)v^0\left(\Omega'\right) \mid \Omega\right], \\
v^w(\Omega) &= \alpha\left(\bar{ep}\right)^{\frac{1}{\alpha}}\left(\frac{1-\alpha}{\bar{pw}}\right)^{\frac{1-\alpha}{\alpha}} + (1-\delta)\omega\beta\mathbb{E}_{y,X}\left[v^1\left(\Omega'\right) \mid \Omega\right], \\
v^i(\Omega, x, i) &= \alpha\left(\bar{ep}_x\right)^{\frac{1}{\alpha}}\left(\frac{1-\alpha}{\bar{pw}_x}\right)^{\frac{1-\alpha}{\alpha}} - \bar{p}_x c(i) \\
&+ (1-\delta+i)\omega\beta\mathbb{E}_{y,X}\left[qv^1\left(\Omega'\right) + (1-q)v^0\left(\Omega'\right) \mid \Omega, x\right].
\end{aligned}$$

Finally, the policy function for investment is

$$i(\Omega, x) = c'^{-1}\left(\frac{\omega\beta}{\bar{p}_x}\mathbb{E}_{y,X}\left[qv^1\left(\Omega'\right) + (1-q)v^0\left(\Omega'\right) \mid \Omega\right]\right),$$

and a firm decides to invest if and only if

$$\mathbb{E}_x\left[v^i(\Omega, x, i)\right] - v^w(\Omega) > \bar{p}f.$$

5.5 Aggregation

To aggregate the model analytically we take the limit as the total number of firms goes to infinity and the precision of each individual signal goes to 0 in a way that keeps the precision of information unaltered.[8] In the limit, each firm disregards its private information at the hiring stage. Therefore we ignore the value function's dependence on x and proceed with the aggregation. Under these assumptions, the laws of motion of information perceived by agents replicate those in the baseline model. The only differences is that, now, the next-period prior is adjusted by the fact that the fundamental follows an AR(1) process instead of a random walk,

$$\begin{aligned}
\mu' &= \rho\frac{\gamma\mu + \gamma_y y + N\gamma_x X}{\gamma + \gamma_y + N\gamma_x} \\
\gamma' &= \left(\frac{\rho^2}{\gamma + \gamma_y + N\gamma_x} + (1-\rho^2)\sigma_\theta^2\right)^{-1}.
\end{aligned}$$

The fraction of firms that invest, $\tilde{N}(\Omega)$, satisfies

$$\tilde{N}(\Omega) = QG_f\left(\left(\max_i \mathbb{E}_x\left[v^i(\Omega, i)\right] - v^w(\Omega)\right)/\bar{p}\right).$$

The average capital by firm K evolves according to

$$K'(\Omega) = K\left(\omega\left(1 - \delta + N(\Omega)i(\Omega)\right) + (1-\omega)\xi\right).$$

The law of motion of the fraction of firms with an investment opportunity is:

$$Q'(\Omega) = \omega\left(Nq + (Q-N) + (1-Q)q\right) + (1-\omega)q$$

The average output is given by

$$Y(\Omega, X, y) = Ke^y\left((1-\alpha)\frac{\bar{e}p_n}{\bar{p}w_n}\right)^{\frac{1-\alpha}{\alpha}}.$$

Finally, the consumption of the representative household is

$$C(\Omega, X, y) = Y(\Omega, X, y) - K\int_f g_f(f)(c(i(\Omega)) + f)1\left(v^w(\Omega) < \max_i \mathbb{E}_x\left[v^i(\Omega, i)\right] - \bar{p}f\right)df.$$

[8] More precisely, we assume that the precision of each private signal is $\gamma_x = \bar{N}\tilde{\gamma}_x$, and then take the limit as $\bar{N} \to \infty$ but. See the appendix for details

5.6 Simulation (Preliminary)

The model can be solved numerically by iterating on the aggregate variables w, p and N, and the laws of motion K' and Q'. All these objects are functions of the state space Ω. The algorithm consists of an outer loop that iterates on the vector $(w, p, \tilde{N}, K', Q')$ and an inner loop that iterates on the value function and the policy decision. The linear structure of the model allows us to solve the full general equilibrium exactly without using approximations in the spirit of Krussell-Smith.

To evaluate the quantitative influence of uncertainty traps we perform a preliminary simulation. The goal of the simulation is to illustrate that uncertainty traps can create additional persistence in the macroeconomic aggregates.[9] The parameters values for the simulation are in Table 1.

Parameter	Value
Preference for leisure	$h = 2$
Discount factor	$\beta = 0.95$
Capital intensity	$\alpha = 1/3$
Depreciation rate	$\delta = 1\%$
Persistence of fundamental	$\rho_\theta = 0.98$
Precision of ergodic distribution of fundamental	$\gamma_\theta = 10$
Precision of productivity shocks	$\gamma_y = 20$
Precision of public signal (if all firms enter)	$\gamma_x = 200$
Distribution of fixed cost	$\text{pdf}(f) \propto f^\zeta$ for $f \in [0, f_{max}]$
Parameter of distribution of fixed cost	$\zeta = 100$
Upper bound of the support of fixed cost	$f_{max} = 0.01$
Probability of receiving an investment opportunity	$q = 10\%$
Fraction of capital that new firms keep	$\xi = 10\%$

Table 1: Parameter values for the numerical simulations

Once the value functions and the prices have been computed, we can examine the entry behavior of firms for different beliefs (μ, γ) about the fundamental. Figure 11 shows the number of active firms as function of beliefs. As in the baseline model, the number of active firms increases both with mean beliefs and with the precision of information. As we demonstrated in the simpler setup, this opens up the possibility of uncertainty traps.

The effect of the uncertainty trap more clear when the shocks to the economy are large. To illustrate this, consider the evolution of the economy when it is hit by a small shock to the fundamental, equal to 1/10 of its standard deviation. The impulse response functions to this shock are shown in the solid lines of Figure 12. The dashed line represents the evolution of this economy when the precision of beliefs is kept fixed at its initial level. Therefore, the difference between the two lines corresponds to the the additional impact of endogenous uncertainty. Because a small fraction of firms stops investing, the impact on uncertainty is not sufficiently strong and the economy behaves similarly with fixed and with endogenous uncertainty.

[9] In current work, we are developing a proper calibration exercise. So far, this is a simulation for illustrative purposes.

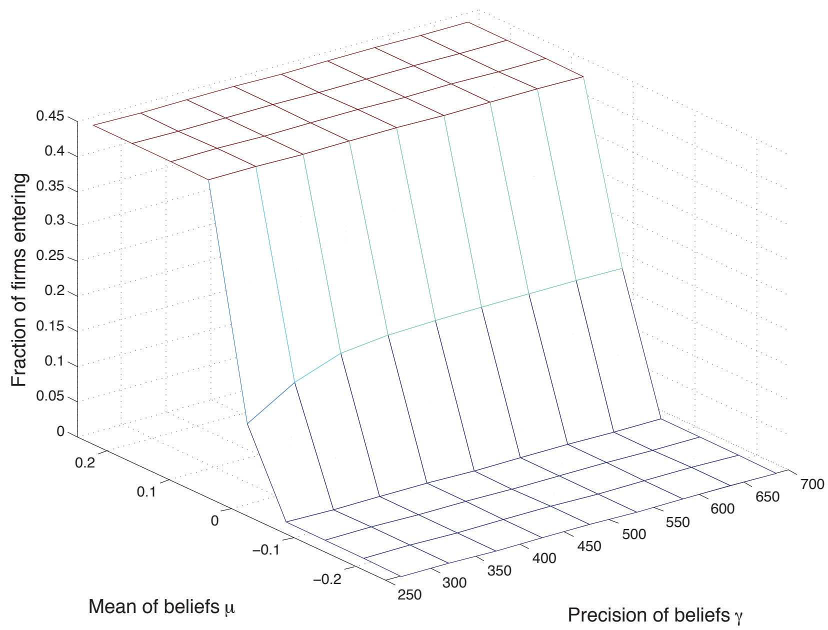
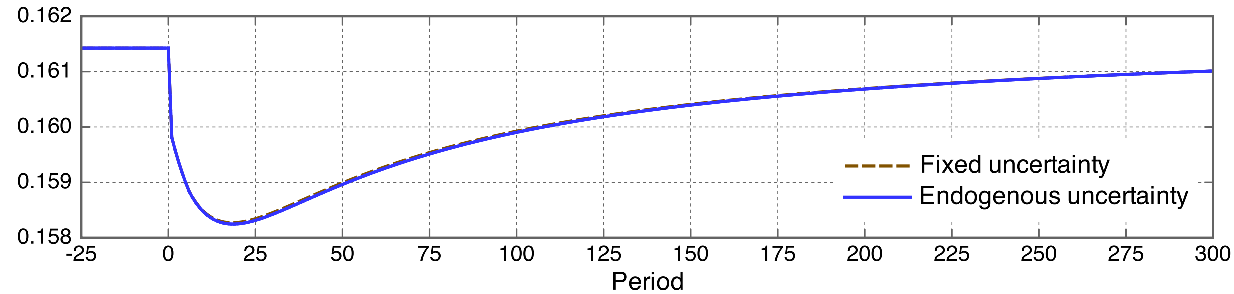


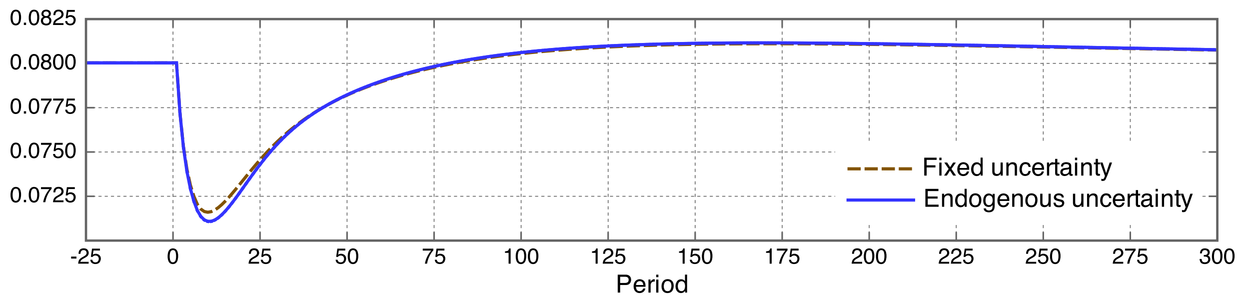
Figure 11: Equilibrium entry behavior of firms $N(\mu, \gamma, K, Q)$. For this figure, K and Q are fixed at their steady state values.

In contrast, Figure 13 shows the outcome after a larger shock to the fundamental, equal to one standard deviation of its distribution. Now, uncertainty has a visible larger effect on the economy. The downturn is deeper and lasts longer when uncertainty can adjust. Now, the fraction of firms investing declines substantially and so does the release of information. When the mean of the prior recovers, firms remain uncertain and choose to wait.

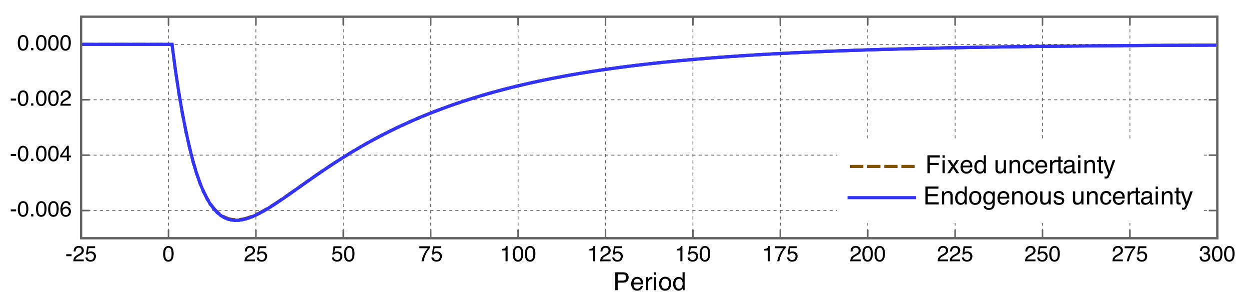
To sum up, these preliminary simulations with the extended model show that the uncertainty traps which we have identified in our theory are also visible in a fully fledged general-equilibrium model of the business cycle. In current work, we plan to explore this in a calibration of the extended model.



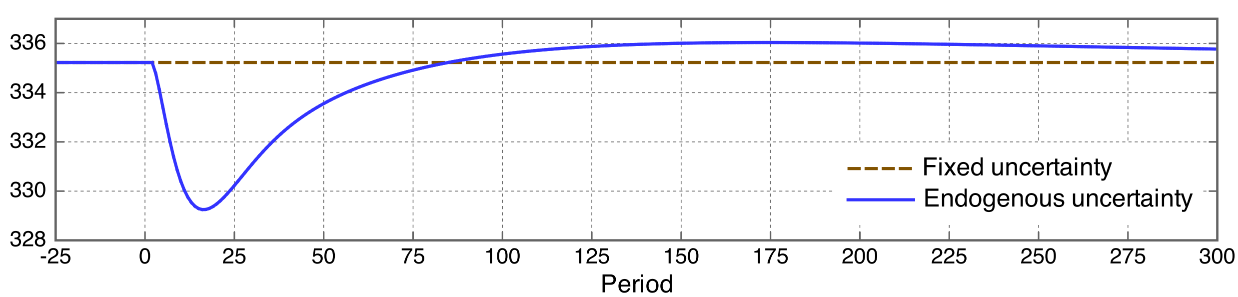
(a) Output



(b) Fraction of firms entering

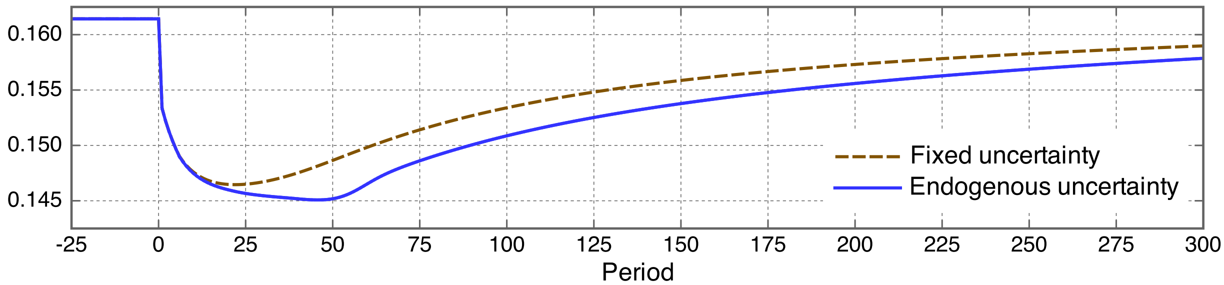


(c) Mean of beliefs μ

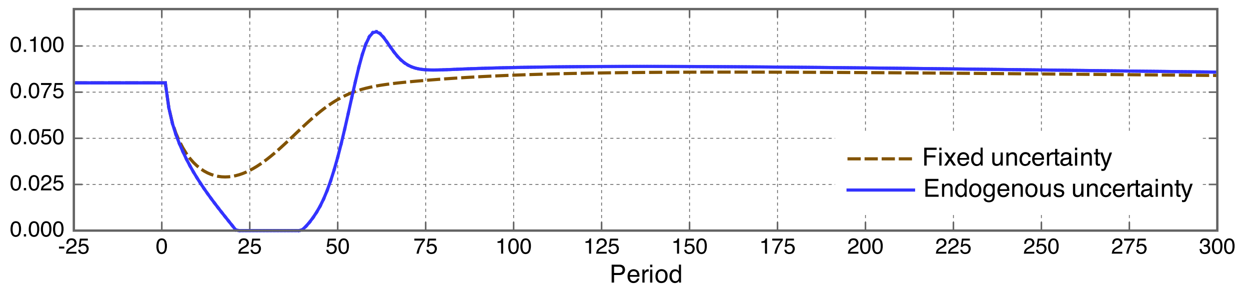


(d) Precision of beliefs γ

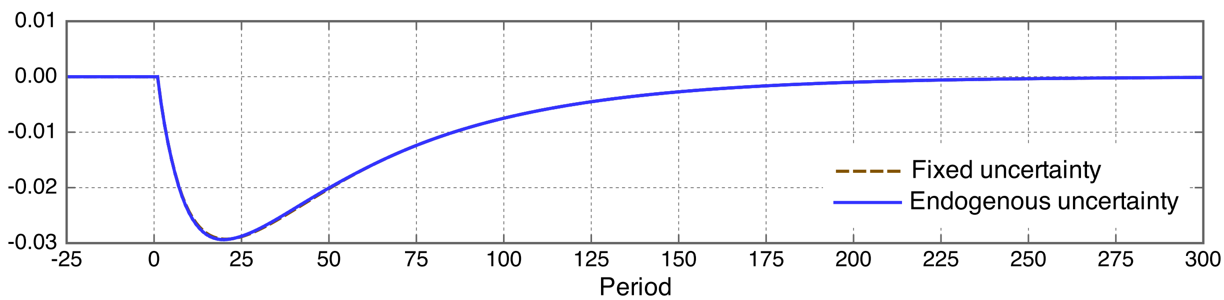
Figure 12: Evolution of the economy after a *small* shock to θ



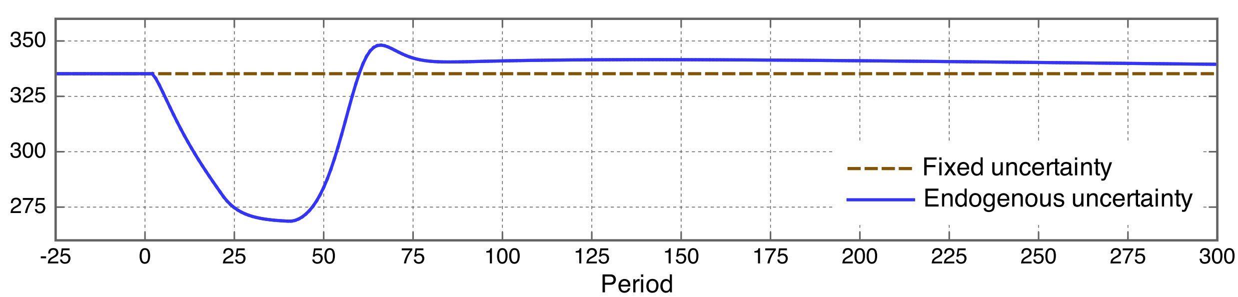
(a) Output



(b) Fraction of firms entering



(c) Mean of beliefs μ



(d) Precision of beliefs γ

Figure 13: Evolution of the economy after a *large* shock to θ

6 Conclusion

We developed a quantitative theory of endogenous uncertainty that embeds social learning in a standard model of irreversible investment. In the model, agents receive private information about some fundamental and learn from the actions of others. During periods of high economic activity, a large amount of information is revealed. In periods of low economic activity, information diffuses slowly and uncertainty rises. High uncertainty encourages firms to defer investment in the future, slowing the diffusion of information further.

We show that this interaction between social learning and the option value of investment creates a powerful complementarity between information and economic activity. We derive conditions under which this complementarity is strong enough to sustain two distinct locally stable steady states in the dynamics of activity and uncertainty for a given level of optimism: a high activity/low uncertainty regime and a low activity/high uncertainty one. We demonstrate that the equilibrium of the model is still unique, but that this multiplicity creates nonlinear dynamics in which recessions can have a near permanent impact on the economy.

We explore the robustness of this mechanism in a quantitative version of the model that nests the key components of our setup in a more traditional model of business cycles. Despite being at a very preliminary stage, our simulations show some encouraging evidence that the effects of uncertainty are robust and able to provide a sizable propagation mechanism with substantial persistence of recessions. We are currently working on a full-fledged calibration and an empirical study to complement these results.

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A Appendix

Proofs of the lemmas and propositions from the theoretical part are provided here. We start with a number of assumptions and definitions.

Assumption 1. *Parameters are such that* $\beta e^{\frac{a^2}{2\gamma_\theta}} < 1$.

This assumption guarantees that a number of effects highlighted in the baseline model are unambiguous and in particular that the option value of waiting is strong enough to dominate other forces. It states in particular the following: risk aversion (a) and aggregate risk ($\frac{1}{\gamma_\theta}$) cannot be too large, otherwise the incentives for waiting are greatly reduced (the condition states that the risk induced by waiting never outweighs the option value of waiting).

Assumption 2. *F is continuous, twice differentiable with bounded first and second derivatives.*

We impose a number of regularity conditions on the cumulative distribution of investment costs to guarantee that the equilibrium number of investing firms $N(\mu,\gamma) \sim Bin\left(\bar{N}, F\left(f_c(\mu,\gamma)\right)\right)$ is well-behaved.

Definition 3. Define the following bounds and set:

1. Let $\bar{\gamma}$ be the unique strictly positive solution of

$$\bar{\gamma} = \left(\frac{1}{\bar{\gamma}+\gamma_y+\bar{N}\gamma_x}+\frac{1}{\gamma_\theta}\right)^{-1} = \Gamma\left(\bar{N},\bar{\gamma}\right),$$

and $\underline{\gamma}$ the unique strictly positive solution of

$$\underline{\gamma} = \left(\frac{1}{\underline{\gamma}+\gamma_y}+\frac{1}{\gamma_\theta}\right)^{-1} = \Gamma\left(0,\underline{\gamma}\right),$$

2. Let $\mathcal{S} = [\underline{\mu},\bar{\mu}] \times [\underline{\gamma},\bar{\gamma}]$, where $\underline{\mu}$ and $\bar{\mu}$ are some arbitrary but large bounds on μ.

We restrict (μ,γ) to be in the domain $\mathcal{S}$. Unfortunately, since shocks are normally distributed, imposing bounds on μ restricts the equilibrium and makes the Bayesian learning formulas not completely consistent with rationality. We ignore that matter for practical purposes as these bounds are necessary to the fixed point theorems. With bounds large enough, this should be a good enough approximation.

We define the set $\mathcal{P}$ in which the probability $p(\mu,\gamma) = F\left(f_c(\mu,\gamma)\right)$ that a firm invests will lie:

Definition 4. Let $\mathcal{P}$ be the set of twice-continuously differentiable functions $p : (\mu,\gamma) \in \mathcal{S} \longrightarrow \mathbb{R}$ such that p has bounded first and second derivatives: $\forall (\mu,\gamma) \in \mathcal{S}, |p_\mu(\mu,\gamma)| \leq \bar{p}_\mu$, $|p_\gamma(\mu,\gamma)| \leq \bar{p}_\gamma$, and $|p_{xy}(\mu,\gamma)| \leq \bar{p}_{xy}$ for $(x,y) \in \{\mu,\gamma\}^2$.

We also define the set $\mathcal{G}$ in which the firm's surplus of waiting compared to investing will lie:

Definition 5. Let $\mathcal{G}$ be the set of twice-continuously differentiable functions G of $(\mu, \gamma, f) \in \mathcal{S} \times \mathbb{R} \longrightarrow \mathbb{R}$ such that

1. G is weakly decreasing and convex in μ,
2. G is weakly decreasing in γ,
3. G is Lipschitz continuous of constant 1 in f,
4. G has bounded first and second derivatives: $\forall (\mu, \gamma, f), |G_x(\mu, \gamma)| \leq \bar{G}_x$ and $|G_{xy}(\mu, \gamma)| \leq \bar{G}_{xy}$ for $(x, y) \in \{\mu, \gamma, f\}^2$.

The bounds on the derivatives of N depend on those of G and vice-versa in equilibrium. In the final version, we will need clean expressions (TO DO). In fact, differentiability may not obtain, so we will have to rewrite the proofs with Lipschitz-continuous functions or perhaps semi-differentiability.

Let us now turn to the proofs.

Lemma 1. *Mean beliefs μ follow a random walk with time-varying volatility,*

$$\mu^{'} = \mu + s(N, \gamma)\varepsilon^{\mu},$$

where $s(N, \gamma) = \left(\frac{1}{\gamma} - \frac{1}{\gamma + \gamma_y + N\gamma_x}\right)^{\frac{1}{2}}$ *and* $\varepsilon \sim \mathcal{N}(0, 1)$.

Proof. Conditioning on the realization of N and on current information, expression (8) tells us that μ' is normally distributed (it is the sum of a constant, μ, and two normally distributed signals, Y and X. Let us characterize its mean and variance:

$$\mathbb{E}[\mu'|\mu, \gamma, N] = \mathbb{E}\left[\frac{\gamma \cdot \mu + \gamma_y \cdot Y + N\gamma_x \cdot X}{\gamma + \gamma_y + N\gamma_x}|\mu, \gamma, N\right] = \mu,$$

$$\begin{aligned} Var(\mu'|\mu, \gamma, N) &= Var\left(\frac{(\gamma_y + N\gamma_x)\theta + \gamma_y \varepsilon^y + N\gamma_x \varepsilon_N^X}{\gamma + \gamma_y + N\gamma_x}|\mu, \gamma, N\right) \\ &= \left(\frac{\gamma_y + N\gamma_x}{\gamma + \gamma_y + N\gamma_x}\right)^2 \frac{1}{\gamma} + \left(\frac{\gamma_y}{\gamma + \gamma_y + N\gamma_x}\right)^2 \frac{1}{\gamma_y} + \left(\frac{N\gamma_x}{\gamma + \gamma_y + N\gamma_x}\right)^2 \frac{1}{N\gamma_x} \\ &= \frac{1}{\gamma} - \frac{1}{\gamma + \gamma_y + N\gamma_x}. \end{aligned}$$

We can therefore write μ' as the sum of μ plus some normal innovation ε^{μ} with a variance that depends on N and γ, N being random, distributed according to a binomial law. □

Proposition 1 (full). *For $p(\mu, \gamma) \in \mathcal{P}$ such that $N(\mu, \gamma) \sim Bin\left(\overline{N}, p(\mu, \gamma)\right)$, under assumption 1 and for γ_x sufficiently low, for all $(\mu, \gamma) \in \mathcal{S}$, there exists a unique cutoff value $f_c(\mu, \gamma) \in \mathbb{R} \cup \{-\infty, \infty\}$, such that firms invest if and only if $f \leq f_c(\mu, \gamma)$. In addition, $f_c(\mu, \gamma)$ is a strictly increasing function of μ and γ.*

Proof. Using the definition 4, write the mapping $\mathcal{T}^p : \mathcal{G} \longrightarrow \mathcal{G}$ for $G(\mu,\gamma,f) = V(\mu,\gamma,f) - \left(V^I(\mu,\gamma) - f\right)$ as follows:

$$\mathcal{T}^p G(\mu,\gamma) = \max\ \left\{0, \beta E\left[G(\mu',\gamma',f') + V^I(\mu',\gamma') - f\right] - V^I(\mu',\gamma') + f\right\}.$$

Substitute with the stopping value $V^I(\mu,\gamma) = \frac{1}{a}\left(1 - e^{-a\mu + \frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}\right)$:

$$\begin{aligned}
\mathcal{T}^p G(\mu,\gamma,f) &= \max\ \left\{0, \beta E\left[G(\mu',\gamma',f') + \frac{1}{a}\left(1 - e^{-a\mu' + \frac{a^2}{2}\left(\frac{1}{\gamma'}+\frac{1}{\gamma_x}\right)}\right) - f\right] - V^I(\mu,\gamma) + f\right\} \\
&= \max\ \left\{0, \beta E\left[G(\mu',\gamma',f') + \frac{1}{a}\left(1 - e^{-a(\mu+s\cdot\varepsilon) + \frac{a^2}{2}\left(\frac{1}{\gamma+\gamma_y+N\gamma_x}+\frac{1}{\gamma_\theta}+\frac{1}{\gamma_x}\right)}\right) - f\right]\right. \\
&\qquad \left. -\frac{1}{a}\left(1 - e^{-a\mu + \frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}\right) + f\right\} \\
&= \max\ \left\{0, \beta\left[E\left(G(\mu',\gamma',f')\right) + \frac{1}{a}\left(1 - e^{-a\mu + \frac{a^2}{2}\left(\frac{1}{\gamma+\gamma_y+N\gamma_x}+\frac{1}{\gamma_\theta}+\frac{1}{\gamma_x}+s^2\right)}\right) - \bar{f}\right]\right. \\
&\qquad \left. -\frac{1}{a}\left(1 - e^{-a\mu + \frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}\right) + f\right\} \\
&= \max\ \left\{0, \underbrace{\frac{1}{a}e^{-a\mu + \frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}\left(1 - \beta e^{\frac{a^2}{2\gamma_\theta}}\right) + f - (1-\beta)\frac{1}{a} - \beta\bar{f} + \beta E\left[G(\mu',\gamma',f')\right]}_{\equiv C^p(G)\ \text{(continuation value)}}\right\}
\end{aligned}$$

where the notation $s(N(\mu,\gamma),\gamma) = \left(\frac{1}{\gamma} - \frac{1}{\gamma+\gamma_y+N(\mu,\gamma)\gamma_x}\right)^{\frac{1}{2}}$ denotes the standard deviation of μ' given (μ,γ) and N. Under assumption 1, the first term in the braces is positive: it is a strictly decreasing, convex function of μ, and strictly decreasing function of γ.

We are going to show that $\mathcal{T}^p$ is a well defined mapping from $\mathcal{G}$ to $\mathcal{G}$. We show in addition that it satisfies the Blackwell conditions, so that it is a contraction with a unique fixed point. Indeed, $\mathcal{T}^p$ satisfies the following properties:

1. Monotonicity: if $G_1(\mu,\gamma,f) \leq G_2(\mu,\gamma,f)$ for all $(\mu,\gamma) \in \mathcal{S}$, $f \in \mathbb{R}$ then

$$\begin{aligned}
&\frac{1}{a}e^{-a\mu + \frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}\left(1 - \beta e^{\frac{a^2}{2\gamma_\theta}}\right) + f - (1-\beta)\frac{1}{a} - \beta\bar{f} + \beta E\left[G_1(\mu',\gamma',f')\right] \quad \leq \\
&\frac{1}{a}e^{-a\mu + \frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}\left(1 - \beta e^{\frac{a^2}{2\gamma_\theta}}\right) + f - (1-\beta)\frac{1}{a} - \beta\bar{f} + \beta E\left[G_2(\mu',\gamma',f')\right]
\end{aligned}$$

so that $(\mathcal{T}^p G_1)(\mu,\gamma,f) \leq (\mathcal{T}^p G_2)(\mu,\gamma,f)$ for all $(\mu,\gamma) \in \mathcal{S}$, $f \in \mathbb{R}$;

2. Discounting: for $K \geq 0$,

$$\begin{aligned}
&\tfrac{1}{a}e^{-a\mu + \frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}\left(1 - \beta e^{\frac{a^2}{2\gamma_\theta}}\right) + f - (1-\beta)\tfrac{1}{a} - \beta\bar{f} + \beta E\left[G(\mu',\gamma',f') + K\right] \\
= \ &\tfrac{1}{a}e^{-a\mu + \frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}\left(1 - \beta e^{\frac{a^2}{2\gamma_\theta}}\right) + f - (1-\beta)\tfrac{1}{a} - \beta\bar{f} + \beta E\left[G(\mu',\gamma',f')\right] + \beta K \\
\leq \ &(\mathcal{T}^p G)(\mu,\gamma,f) + \beta K
\end{aligned}$$

so that $\left[\mathcal{T}^p (G+K)\right](\mu,\gamma,f) \le \left(\mathcal{T}^p G\right)(\mu,\gamma,f) + \beta K$ for all $(\mu,\gamma)\in\mathcal{S},\ f\in\mathbb{R}$.

Therefore, $\mathcal{T}^p$ is a contraction mapping that admits a unique fixed point. We only need to check that it is well defined, i.e. that it remains in set $\mathcal{G}$.

- **Decreasing in μ**

We want to show that $\mathcal{T}^p G$ decreases with μ. The term $\frac{1}{a}e^{-a\mu+\frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}\left(1-\beta e^{\frac{a^2}{2\gamma_\theta}}\right)$ decreases with μ, so we only need to check that $\beta E\left[G\left(\mu',\gamma',f'\right)\right]$ does as well. Denote $\pi_N^{\bar{N}}(p)=\binom{\bar{N}}{N}p^N(1-p)^{\bar{N}-N}$. Pick $\mu_1<\mu_2$. We use the following notation:

$$E_i\left[G\left(\mu_j+s\varepsilon,\Gamma,f'\right)\right] \;=\; \sum_{N=1}^{\bar{N}}\pi_N^{\bar{N}}\left(p\left(\mu_i,\gamma\right)\right)\int G\left(\mu_j+s\left(N,\gamma\right)\varepsilon,\Gamma\left(N,\gamma\right),f'\right)d\Phi\left(\varepsilon\right)dF\left(f'\right)$$

with $s(N,\gamma)=\left(\frac{1}{\gamma}-\frac{1}{\gamma+\gamma_y+N\gamma_x}\right)^{\frac{1}{2}}$ and $\Gamma(N,\gamma)=\left(\frac{1}{\gamma+\gamma_y+N\gamma_x}+\frac{1}{\gamma_\theta}\right)^{-1}$ and $\varepsilon\sim\mathcal{N}(0,1)$ for $i,j=1,2$. A change in μ has a direct effect on G as a function of μ and another effect through the expectation $p(\mu,\gamma)$. We are interested in the sign of

$$\begin{aligned}
& E_2\left[G\left(\mu_2+s\varepsilon,\Gamma,f'\right)\right]-E_1\left[G\left(\mu_1+s\varepsilon,\Gamma,f'\right)\right]\\
= \;& \underbrace{E_2\left[G\left(\mu_2+s\varepsilon,\Gamma,f'\right)\right]-E_2\left[G\left(\mu_1+s\varepsilon,\Gamma,f'\right)\right]}_{\le 0 \text{ because } G \text{decreasing}}+E_2\left[G\left(\mu_1+s\varepsilon,\Gamma,f'\right)\right]-E_1\left[G\left(\mu_1+s\varepsilon,\Gamma,f'\right)\right].
\end{aligned}$$

The first term is negative, which is good for us. The only nuisance term is the second one coming from the change in beliefs. We are going to show that it is a $O(\gamma_x)$, i.e. negligible when γ_x is small. Using the notation $g_N(\mu,\gamma)\equiv\int G\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f'\right)d\Phi(\varepsilon)\,dF(f')$ and $\Pi_N(p)=\sum_{n=1}^{N}\pi_n^{\bar{N}}(p)$, rewrite the following:

$$E_p\left[G\left(\mu+s\varepsilon,\Gamma,f'\right)\right] \;=\; \sum_{N=1}^{\bar{N}}\pi_N^{\bar{N}}(p)\underbrace{\int G\left(\mu+s\left(N,\gamma\right)\varepsilon,\Gamma\left(N,\gamma\right),f'\right)d\Phi\left(\varepsilon\right)dF\left(f'\right)}_{\equiv g_N(\mu,\gamma)}$$

Summing by parts:

$$E_p\left[G\left(\mu+s\varepsilon,\Gamma,f'\right)\right]=g_{\bar{N}}\left(\mu,\gamma\right)-\sum_{N=1}^{\bar{N}-1}\Pi_N\left(p\right)\cdot\left(g_{N+1}-g_N\right)\left(\mu,\gamma\right).$$

Taking the derivative with respect to p, we obtain

$$\frac{\partial}{\partial p}E_p\left[G\left(\mu+s\varepsilon,\Gamma,f'\right)\right]=-\sum_{N=1}^{\bar{N}-1}\Pi_N'\left(p\right)\cdot\left(g_{N+1}-g_N\right)\left(\mu,\gamma\right)$$

Let us now control the terms $(g_{N+1} - g_N)(\mu, \gamma)$. To that end, let us look at the following term:

$$\begin{aligned}
& \left| G\left(\mu + s\left(N+1,\gamma\right)\varepsilon, \Gamma\left(N+1,\gamma\right), f'\right) - G\left(\mu + s\left(N,\gamma\right)\varepsilon, \Gamma\left(N,\gamma\right), f'\right) \right| \\
\leq & \left| G\left(\mu + s\left(N+1,\gamma\right)\varepsilon, \Gamma\left(N+1,\gamma\right), f'\right) - G\left(\mu + s\left(N,\gamma\right)\varepsilon, \Gamma\left(N+1,\gamma\right), f'\right) \right| \\
& + \left| G\left(\mu + s\left(N,\gamma\right)\varepsilon, \Gamma\left(N+1,\gamma\right), f'\right) - G\left(\mu + s\left(N,\gamma\right)\varepsilon, \Gamma\left(N,\gamma\right), f'\right) \right| \\
\leq & \bar{G}_\mu \left| \varepsilon \left(s\left(N+1,\gamma\right) - s\left(N,\gamma\right)\right)\right| + \bar{G}_\gamma \left| \Gamma\left(N+1,\gamma\right) - \Gamma\left(N,\gamma\right)\right|
\end{aligned}$$

where we have used the fact that G is bi-Lipschitz continuous. Therefore,

$$\begin{aligned}
\left| \left(g_{N+1} - g_N\right)(\mu,\gamma)\right| & \leq \bar{G}_\mu \left| \left(s\left(N+1,\gamma\right) - s\left(N,\gamma\right)\right)\right| \int |\varepsilon|\, d\Phi\left(\varepsilon\right) + \bar{G}_\gamma \left| \Gamma\left(N+1,\gamma\right) - \Gamma\left(N,\gamma\right)\right| \\
& \leq \bar{G}_\mu \left| \left(s\left(N+1,\gamma\right) - s\left(N,\gamma\right)\right)\right| + \bar{G}_\gamma \left| \Gamma\left(N+1,\gamma\right) - \Gamma\left(N,\gamma\right)\right|.
\end{aligned}$$

We just need to check that these terms are small:

$$\left| s\left(N+1,\gamma\right) - s\left(N,\gamma\right)\right| \leq \frac{\gamma_x}{\left(\underline{\gamma} + \gamma_y\right)^2} = O\left(\gamma_x\right)$$

$$\begin{aligned}
\left| \Gamma\left(N+1,\gamma\right) - \Gamma\left(N,\gamma\right)\right| & = \frac{\gamma_\theta^2 \gamma_x}{\left(\gamma_\theta + \gamma + \gamma_y + N\gamma_x\right)\left(\gamma_\theta + \gamma + \gamma_y + \left(N+1\right)\gamma_x\right)} \\
& \leq \frac{\gamma_\theta^2 \gamma_x}{\left(\gamma_\theta + \underline{\gamma} + \gamma_y\right)^2} = O\left(\gamma_x\right).
\end{aligned}$$

We can now go back to the nuisance term

$$\left| \frac{\partial}{\partial p} E_p \left[G\left(\mu + s\varepsilon, \Gamma, f'\right)\right]\right| = O\left(\gamma_x\right),$$

where we have used the fact that $\Pi_N'(p)$ is a polynomial of degree $\bar{N} - 1$ and is thus bounded on the interval $[0,1]$. We may now conclude. For γ_x small enough, the continuation value $C^p(G)$ will be strictly decreasing. Its derivative,

$$\frac{\partial}{\partial \mu} C^p\left(G\right) = \underbrace{-e^{-a\mu + \frac{a^2}{2}\left(\frac{1}{\gamma} + \frac{1}{\gamma_x}\right)}\left(1 - \beta e^{\frac{a^2}{2\gamma_\theta}}\right)}_{< -e^{-a\bar{\mu} + \frac{a^2}{2}\frac{1}{\gamma}}\left(1 - \beta e^{\frac{a^2}{2\gamma_\theta}}\right) < 0} + \beta \underbrace{E_{p(\mu,\gamma)}\left[\frac{\partial G}{\partial \mu}\right]}_{\leq 0} + \beta \underbrace{\frac{\partial p}{\partial \mu}}_{\leq \bar{p}_\mu} \cdot \underbrace{\frac{\partial}{\partial p} E_p\left[G\left(\mu + s\varepsilon, \Gamma, f'\right)\right]}_{O(\gamma_x)}$$

is strictly negative for γ_x small enough. Therefore, for a small enough γ_x, $\mathcal{T}^p G$ is weakly decreasing in μ.

- **Decreasing in γ**

The proof is based on the same argument as the one we just developed. The derivative of the continuation value with respect to γ is:

$$\frac{\partial}{\partial \gamma} C^p(G) = \underbrace{-\frac{a}{2\gamma^2} e^{-a\mu+\frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}\left(1-\beta e^{\frac{a^2}{2\gamma_\theta}}\right)}_{<-\frac{a}{2\bar{\gamma}} e^{-a\bar{\mu}+\frac{a^2}{2}\frac{1}{\bar{\gamma}}}\left(1-\beta e^{\frac{a^2}{2\gamma_\theta}}\right)<0} + \beta \underbrace{E_{p(\mu,\gamma)}\left[\frac{\partial G}{\partial \gamma}\right]}_{\leq 0} + \beta \underbrace{\frac{\partial p}{\partial \gamma}}_{\leq \bar{p}_\gamma} \cdot \underbrace{\frac{\partial}{\partial p} E_p\left[G\left(\mu+s\varepsilon, \Gamma, f'\right)\right]}_{O(\gamma_x)}.$$

Therefore, for γ_x small enough, the derivative is strictly negative, bounded away from 0. The whole mapping $\mathcal{T}^p G$ is thus weakly decreasing in γ.

- **Lipschitz in f of constant 1**

Pick $f_1 < f_2$. This is trivially true:

$$\left|\mathcal{T}^p G(\mu,\gamma,f_2) - \mathcal{T}^p G(\mu,\gamma,f_1)\right| \leq \left|\left[C^p(G)\right](\mu,\gamma,f_2) - \left[C^p(G)\right](\mu,\gamma,f_1)\right| = \left|f_2 - f_1\right|.$$

- **Convex in μ**

Take the second derivative of the continuation value:

$$\begin{aligned}\frac{\partial^2}{\partial \mu^2} C^p(G) &= \underbrace{a e^{-a\mu+\frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}\left(1-\beta e^{\frac{a^2}{2\gamma_\theta}}\right)}_{\underbrace{\geq a e^{-a\bar{\mu}+\frac{a^2}{2}\frac{1}{\bar{\gamma}}}\left(1-\beta e^{\frac{a^2}{2\gamma_\theta}}\right)>0}_{\text{independent of}_{\gamma_x}}} + \underbrace{\beta E_{p(\mu,\gamma)}\left[\frac{\partial^2 G}{\partial \mu^2}\right]}_{\geq 0} \\ &+ \beta \underbrace{\frac{\partial p}{\partial \mu}}_{\leq \bar{p}_\mu} \cdot \frac{\partial}{\partial p} E_p\left[\frac{\partial}{\partial \mu} G\left(\mu+s\varepsilon,\Gamma,f'\right)\right] + \beta \underbrace{\frac{\partial^2 p}{\partial \mu^2}}_{\leq \bar{p}_{\mu\mu}} \cdot \frac{\partial^2}{\partial p^2} E\left[G\left(\mu+s\varepsilon,\Gamma,f'\right)\right].\end{aligned}$$

The claim is proved by showing that the terms $\frac{\partial}{\partial p} E_p\left[\frac{\partial}{\partial \mu} G\left(\mu+s\varepsilon,\Gamma,f'\right)\right]$ and $\frac{\partial^2}{\partial p^2} E\left[G\left(\mu+s\varepsilon,\Gamma,f'\right)\right]$ are $O(\gamma_x)$. The argument extends the one developed in the proof for monotonicity.

$$\frac{\partial^2}{\partial p^2} E_p\left[G\left(\mu+s\varepsilon,\Gamma,f'\right)\right] = -\sum_{N=1}^{\bar{N}-1} \Pi''_N(p) \cdot \underbrace{\left(g_{N+1}-g_N\right)(\mu,\gamma)}_{=O(\gamma_x)} = O(\gamma_x)$$

since $\Pi''_N(p)$ is a polynomial of degree $\bar{N}-2$ in p, therefore bounded on $[0,1]$. Defining $g_N^{(\mu)}(\mu,\gamma) \equiv \int G_\mu\left(\mu+s(N,\gamma)\varepsilon, \Gamma(N,\gamma), f'\right) d\Phi(\varepsilon)\, dF(f')$, we have

$$E_p\left[\frac{\partial}{\partial \mu} G\left(\mu+s\varepsilon,\Gamma,f'\right)\right] = \sum_{N=1}^{\bar{N}} \pi_N^{\bar{N}}(p)\, g_N^{(\mu)}(\mu,\gamma) = g_{\bar{N}}^{(\mu)}(\mu,\gamma) - \sum_{N=1}^{\bar{N}-1} \Pi_N(p) \cdot \left(g_{N+1}^{(\mu)} - g_N^{(\mu)}\right)(\mu,\gamma).$$

Taking the derivative:

$$\frac{\partial}{\partial p} E_p\left[\frac{\partial}{\partial \mu} G\left(\mu+s\varepsilon,\Gamma,f'\right)\right] = -\sum_{N=1}^{\bar{N}-1} \Pi'_N(p) \cdot \left(g_{N+1}^{(\mu)} - g_N^{(\mu)}\right)(\mu,\gamma).$$

By a similar argument as before

$$\begin{aligned} & \left|G_\mu\left(\mu+s\left(N+1,\gamma\right)\varepsilon,\Gamma\left(N+1,\gamma\right),f'\right)-G_\mu\left(\mu+s\left(N,\gamma\right)\varepsilon,\Gamma\left(N,\gamma\right),f'\right)\right| \\ \leq\ & \bar{G}_{\mu\mu}\left|\varepsilon\left(s\left(N+1,\gamma\right)-s\left(N,\gamma\right)\right)\right|+\bar{G}_{\mu\gamma}\left|\Gamma\left(N+1,\gamma\right)-\Gamma\left(N,\gamma\right)\right|, \end{aligned}$$

so that

$$\left|\left(g_{N+1}^{(\mu)}-g_N^{(\mu)}\right)(\mu,\gamma)\right| \leq \bar{G}_{\mu\mu}\left|\left(s\left(N+1,\gamma\right)-s\left(N,\gamma\right)\right)\right|+\bar{G}_{\mu\gamma}\left|\Gamma\left(N+1,\gamma\right)-\Gamma\left(N,\gamma\right)\right|=O\left(\gamma_x\right).$$

The whole term $E_p\left[\frac{\partial}{\partial\mu}G\left(\mu+s\varepsilon,\Gamma,f'\right)\right]$ is thus a $O\left(\gamma_x\right)$, so for γ_x small enough, the second derivative of the continuation value $C^p\left(G\right)$ is strictly positive, uniformly bounded away from 0. Convexity (weak) is thus preserved by the mapping $\mathcal{T}^pG$.

- **Regularity conditions: continuity, differentiability and boundedness**

Skip for now.

- **Existence and monotonicity of $f_c\left(\mu,\gamma\right)$**

The existence of the cutoff is trivially guaranteed by the fact that the continuation utility is linear in f. Firms invest iff

$$\begin{aligned} C^p\left(G\right)=\ & \frac{1}{a}e^{-a\mu+\frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}\left(1-\beta e^{\frac{a^2}{2\gamma_\theta}}\right)+f-\left(1-\beta\right)\frac{1}{a}-\beta\bar{f}+\beta E\left[G\left(\mu',\gamma',f'\right)\right]\leq 0 \\ \Leftrightarrow f\leq\ & -\frac{1}{a}e^{-a\mu+\frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}\left(1-\beta e^{\frac{a^2}{2\gamma_\theta}}\right)+\left(1-\beta\right)\frac{1}{a}+\beta\bar{f}-\beta E\left[G\left(\mu',\gamma',f'\right)\right]\equiv f_c\left(\mu,\gamma\right). \end{aligned}$$

Notice that $f_c\left(\mu,\gamma\right)=f-\left[C^p\left(G\right)\right]\left(\mu,\gamma,f\right)$, so that the threshold inherits a number of properties from the continuation value. In particular, $f_c\left(\mu,\gamma\right)$ is strictly increasing in μ and γ, strictly concave in μ for γ_x small enough. $\square$

We now turn to the general equilibrium results. Define the mapping:

Definition 6. Let $\mathcal{M}$ be the mapping from $p:\mathcal{P}\longrightarrow\mathcal{P}$ such that

$$\forall\left(\mu,\gamma\right)\in\mathcal{S},\quad\left(\mathcal{M}p\right)\left(\mu,\gamma\right)=F\left(f_c^p\left(\mu,\gamma\right)\right)$$

where f_c is defined as

$$f_c^p\left(\mu,\gamma\right)=-\frac{1}{a}e^{-a\mu+\frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}\left(1-\beta e^{\frac{a^2}{2\gamma_\theta}}\right)+\left(1-\beta\right)\frac{1}{a}+\beta\bar{f}-\beta E\left[G^p\left(\mu',\gamma',f'\right)\right]$$

and G^p is the unique fixed point of the mapping

$$G^p=\mathcal{T}^pG^p.$$

Proposition 2 (full). *Under assumption 1, 2 and for γ_x small enough, an equilibrium exists and is unique, i.e. there exists a unique fixed point $p^* \in \mathcal{P}$ such that $\mathcal{M}p^* = p^*$. In addition, the investment decision of firms is characterized by the threshold $f_c^*(\mu,\gamma)$, strictly increasing in μ and γ.*

Proof. Unfortunately for our purpose, the mapping $\mathcal{M}$ does not satisfy the Blackwell conditions. In particular, monotonicity does not obtain under the current assumptions. If $p_1(\mu,\gamma) \leq p_2(\mu,\gamma)$ for all $(\mu,\gamma) \in \mathcal{S}$, even though the effect on G^{p_i} is small, it is in general ambiguous and I cannot conclude anything with regard to f_c^p. We are still going to show that $\mathcal{M}$ defines a contraction from $\mathcal{P}$ to $\mathcal{P}$.

First, let us check that it is a well-defined mapping. Recall the definition of f_c^p:

$$f_c^p(\mu,\gamma) = -\frac{1}{a}e^{-a\mu+\frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}\left(1-\beta e^{\frac{a^2}{2\gamma_\theta}}\right)+(1-\beta)\frac{1}{a}+\beta\bar{f}-\beta E\left[G^p\left(\mu',\gamma',f'\right)\right]$$

Given that $G^p \in \mathcal{G}$, f_c^p inherits a number of nice properties: it is twice-continuously differentiable, with bounded first and second derivatives. Under assumption 2,$\mathcal{M}p = F(f_c^p)$ preserves these properties. So, $\mathcal{M}$ is a well-defined mapping from $\mathcal{P}$ to $\mathcal{P}$. TO DO: check that the actual bounds $\bar{p}_\mu$, $\bar{p}_\gamma$, etc., are consistent with our assumptions on $\mathcal{G}$.

We will now show that $\mathcal{M}$ defines a contraction. Take $p_1, p_2 \in \mathcal{P}$. We must first control the term$\| G^{p_1} - G^{p_2} \|$. Start with some $G \in \mathcal{G}$, we have:

$$\begin{aligned}\mathcal{T}^{p_i}G(\mu,\gamma,f) &= \max\left\{0, \frac{1}{a}e^{-a\mu+\frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}\left(1-\beta e^{\frac{a^2}{2\gamma_\theta}}\right)+f-(1-\beta)\frac{1}{a}-\beta\bar{f}\right.\\ &\quad \left.+\beta E_{p^i}\left[G\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f'\right)\right]\right\}\end{aligned}$$

How does $\mathcal{T}^{p_2}G$ compare to $\mathcal{T}^{p_1}G$? First, remember that using the notation $\Pi_N(p) = \sum_{n=1}^{N}\pi_n^{\bar{N}}(p)$ we can write the following:

$$\begin{aligned}E_p\left[G\left(\mu+s\varepsilon,\Gamma,f'\right)\right] &= \sum_{N=1}^{\bar{N}}\pi_N^{\bar{N}}(p)\underbrace{\int G\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f'\right)d\Phi(\varepsilon)\,dF\left(f'\right)}_{\equiv g_N(\mu,\gamma)}\\ &= g_{\bar{N}}(\mu,\gamma) - \sum_{N=1}^{\bar{N}-1}\Pi_N(p)\cdot\left(g_{N+1}-g_N\right)(\mu,\gamma)\end{aligned}$$

so that we can control the term:

$$\begin{aligned}
&\left|E_{p_2}\left[G\left(\mu+s\left(N,\gamma\right)\varepsilon,\Gamma\left(N,\gamma\right),f^{'}\right)\right]-E_{p_1}\left[G\left(\mu+s\left(N,\gamma\right)\varepsilon,\Gamma\left(N,\gamma\right),f^{'}\right)\right]\right| \\
= &\left|\sum_{N=1}^{\bar{N}-1}\left[\Pi_N\left(p_2\right)-\Pi_N\left(p_1\right)\right]\cdot\left(g_{N+1}-g_N\right)\left(\mu,\gamma\right)\right| \\
\leq &\ \gamma_x\left(\frac{\bar{G}_\mu}{\left(\underline{\gamma}+\gamma_y\right)^2}+\frac{\bar{G}_\gamma\gamma_\theta^2}{\left(\gamma_\theta+\underline{\gamma}+\gamma_y\right)^2}\right)\sum_{N=1}^{\bar{N}-1}\left|\Pi_N\left(p_2\right)-\Pi_N\left(p_1\right)\right|
\end{aligned}$$

where I have used some results established in proposition 1. The probability $\Pi_N\left(p\right)$ is a polynomial in p of degree $\bar{N}$, it is continuous on the compact $[0,1]$ and therefore uniformly continuous: $\parallel \Pi_N\left(p_2\right)-\Pi_N\left(p_1\right)\parallel\leq B_N\parallel p_2-p_1\parallel$. Therefore, there exists a constant $B\geq 0$ such that

$$\left|E_{p_2}\left[G\left(\mu+s\left(N,\gamma\right)\varepsilon,\Gamma\left(N,\gamma\right),f^{'}\right)\right]-E_{p_1}\left[G\left(\mu+s\left(N,\gamma\right)\varepsilon,\Gamma\left(N,\gamma\right),f^{'}\right)\right]\right|\leq B\gamma_x\parallel p_2-p_1\parallel.$$

What does it imply for $\left(\mathcal{T}^{p_2}G-\mathcal{T}^{p_1}G\right)\left(\mu,\gamma,f\right)$? Assume WLOG that

$$E_{p_2}\left[G\left(\mu+s\left(N,\gamma\right)\varepsilon,\Gamma\left(N,\gamma\right),f^{'}\right)\right]\geq E_{p_1}\left[G\left(\mu+s\left(N,\gamma\right)\varepsilon,\Gamma\left(N,\gamma\right),f^{'}\right)\right].$$

Then three cases arise:

1. Either the max is reached at 0 for both and $\left(\mathcal{T}^{p_2}G-\mathcal{T}^{p_1}G\right)\left(\mu,\gamma,f\right)=0$;

2. Or:

$$\begin{aligned}
\frac{1}{a}e^{-a\mu+\frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}\left(1-\beta e^{\frac{a^2}{2\gamma_\theta}}\right)+f-\left(1-\beta\right)\frac{1}{a}-\beta\bar{f}+\beta E_{p_2}\left[G\left(\mu+s\left(N,\gamma\right)\varepsilon,\Gamma\left(N,\gamma\right),f^{'}\right)\right] &> 0 \\
\frac{1}{a}e^{-a\mu+\frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)}\left(1-\beta e^{\frac{a^2}{2\gamma_\theta}}\right)+f-\left(1-\beta\right)\frac{1}{a}-\beta\bar{f}+\beta E_{p_1}\left[G\left(\mu+s\left(N,\gamma\right)\varepsilon,\Gamma\left(N,\gamma\right),f^{'}\right)\right] &\leq 0
\end{aligned}$$

   so that $\left(\mathcal{T}^{p_1}G\right)\left(\mu,\gamma,f\right)=0$. In that case,

$$\begin{aligned}
\left|\left(\mathcal{T}^{p_2}G-\mathcal{T}^{p_1}G\right)\left(\mu,\gamma,f\right)\right| &\leq \beta\left\{E_{p_2}\left[G\left(\mu+s\left(N,\gamma\right)\varepsilon,\Gamma\left(N,\gamma\right),f^{'}\right)\right]-E_{p_1}\left[G\left(\mu+s\left(N,\gamma\right)\varepsilon,\Gamma\left(N,\gamma\right),f^{'}\right)\right]\right\} \\
&\leq \beta B\gamma_x\parallel p_2-p_1\parallel
\end{aligned}$$

3. Or, last case, the max is reached on the right-hand side and the inequality directly applies

$$\left|\left(\mathcal{T}^{p_2}G-\mathcal{T}^{p_1}G\right)\left(\mu,\gamma,f\right)\right|\leq\beta B\gamma_x\parallel N_2-N_1\parallel.$$

Now, we must find out what happens after a number of iterations. To lighten notation, let us denote $G_n^i\equiv\left(\mathcal{T}^{p_i}\right)^nG$.

$$
\begin{aligned}
& \left| E_{p_2}\left[G_1^2\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f^{'}\right)\right]-E_{p_1}\left[G_1^1\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f^{'}\right)\right]\right| \\
= & \left| E_{p_2}\left[G_1^2\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f^{'}\right)\right]-E_{p_2}\left[G_1^1\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f^{'}\right)\right]\right. \\
& \left.+E_{p_2}\left[G_1^1\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f^{'}\right)\right]-E_{p_1}\left[G_1^1\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f^{'}\right)\right]\right| \\
\leq & \ \beta B\gamma_x \parallel p_2-p_1 \parallel + \left| E_{p_2}\left[G_1^1\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f^{'}\right)\right]-E_{p_1}\left[G_1^1\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f^{'}\right)\right]\right| \\
& \text{(apply same argument as above)} \\
\leq & \ (1+\beta)B\gamma_x \parallel p_2-p_1 \parallel
\end{aligned}
$$

Recursively, one can show that for $n \geq 1$:

$$
\parallel (\mathcal{T}^{p_2})^n G - (\mathcal{T}^{p_1})^n G \parallel \leq \beta \frac{1-\beta^n}{1-\beta} B\gamma_x \parallel p_2-p_1 \parallel
$$

Since operators $\mathcal{T}^{p_i}$ are contractions, we can take the limit:

$$
\parallel G^{p_2} - G^{p_1} \parallel \leq \frac{\beta}{1-\beta} B\gamma_x \parallel p_2-p_1 \parallel . \tag{12}
$$

Now, what does this imply for $\parallel f_c^{p_2} - f_c^{p_1} \parallel$? Recall the definition of the threshold:

$$
f_c^{p_i}(\mu,\gamma) = -\frac{1}{a} e^{-a\mu+\frac{a^2}{2}\left(\frac{1}{\gamma}+\frac{1}{\gamma_x}\right)} \left(1-\beta e^{\frac{a^2}{2\gamma_\theta}}\right) + (1-\beta)\frac{1}{a} + \beta\bar{f} - \beta E_{p_i}\left[G^{p_i}\left(\mu',\gamma',f'\right)\right]
$$

Therefore, we have:

$$
\begin{aligned}
\left|f_c^{p_2}(\mu,\gamma)-f_c^{p_1}(\mu,\gamma)\right| = & \ \beta \left| E_{p_2}\left[G^{p_2}\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f^{'}\right)\right]-E_{p_1}\left[G^{p_1}\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f^{'}\right)\right]\right| \\
= & \ \beta \left| E_{p_2}\left[G^{p_2}\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f^{'}\right)\right]-E_{p_2}\left[G^{p_1}\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f^{'}\right)\right]\right. \\
& \left.+E_{p_2}\left[G^{p_1}\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f^{'}\right)\right]-E_{p_1}\left[G^{p_1}\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f^{'}\right)\right]\right| \\
\leq & \ \beta\left(\frac{\beta}{1-\beta}B\gamma_x \parallel p_2-p_1 \parallel + B\gamma_x \parallel p_2-p_1 \parallel\right) \\
\leq & \ \frac{\beta}{1-\beta}B\gamma_x \parallel p_2-p_1 \parallel
\end{aligned}
$$

The mapping is thus also continuous in f_c, since we have:

$$
\parallel f_c^{p_2} - f_c^{p_1} \parallel \leq \frac{\beta}{1-\beta} B\gamma_x \parallel p_2-p_1 \parallel .
$$

We just have to conclude now with the mapping $\mathcal{M}$:

$$\begin{aligned}
\left|\left(\mathcal{M}p_2-\mathcal{M}p_1\right)(\mu,\gamma)\right| &= \left|F\left(f_c^{p_2}(\mu,\gamma)\right)-F\left(f_c^{p_1}(\mu,\gamma)\right)\right| \\
&= \left|F'\left(\tilde{f}\right)\left(f_c^{p_2}(\mu,\gamma)-f_c^{p_1}(\mu,\gamma)\right)\right| \\
&\quad \left(\text{for some } \tilde{f}\in\left[f_c^{p_1}(\mu,\gamma),f_c^{p_2}(\mu,\gamma)\right]\right) \\
&\leq \frac{\beta}{1-\beta}B\gamma_x \parallel F' \parallel \cdot \parallel p_2-p_1 \parallel .
\end{aligned}$$

This tells us that the mapping $\mathcal{M}$ is continuous as long as F' is bounded, which is guaranteed by assumption 2. But the best is yet to come: we can choose γ_x such that

$$\frac{\beta}{1-\beta}B\gamma_x \parallel F' \parallel < 1.$$

In that case, the mapping $\mathcal{M}$ is actually a contraction! By the contraction mapping theorem, this guarantees the existence and uniqueness of the equilibrium $\mathcal{N}^*$ for small values of γ_x. □

Proposition 3 (full). *Under assumption 1, 2 and for γ_x small enough, there exists a non-empty interval $[\mu_l,\mu_h]$, such that for all $\mu\in(\mu_l,\mu_h)$ and for σ^f low enough, there exists at least two locally stable stationary points $\gamma_l(\mu)<\gamma_h(\mu)$ in the dynamics of γ. Equilibrium γ_l (resp. γ_h) is characterized by high uncertainty (resp. low) and low investment (resp. high).*

Proof. Define the function

$$\varphi_\mu^N(\gamma)=\left(\frac{1}{\gamma+\gamma_y+N(\mu,\gamma)\gamma_x}+\frac{1}{\gamma_\theta}\right)^{-1}-\gamma.$$

By continuity of $N=\bar{N}\cdot p(\mu,\gamma)$, $\varphi_\mu^N(\gamma)$ is a continuous function. Notice first that $\varphi_\mu^N(\underline{\gamma})\geq 0$ and $\varphi_\mu^N(\bar{\gamma})\leq 0$:

$$\begin{aligned}
\varphi_\mu^N(\underline{\gamma}) &= \left(\frac{1}{\underline{\gamma}+\gamma_y+N(\mu,\underline{\gamma})\gamma_x}+\frac{1}{\gamma_\theta}\right)^{-1}-\underline{\gamma} \\
&\geq \left(\frac{1}{\underline{\gamma}+\gamma_y}+\frac{1}{\gamma_\theta}\right)^{-1}-\underline{\gamma}=0
\end{aligned}$$

$$\begin{aligned}
\varphi_\mu^N(\bar{\gamma}) &= \left(\frac{1}{\bar{\gamma}+\gamma_y+N(\mu,\bar{\gamma})\gamma_x}+\frac{1}{\gamma_\theta}\right)^{-1}-\bar{\gamma} \\
&\leq \left(\frac{1}{\bar{\gamma}+\gamma_y+\gamma_x}+\frac{1}{\gamma_\theta}\right)^{-1}-\bar{\gamma}=0
\end{aligned}$$

We are going to show that when σ_f is low, there exists a range $[\mu_l,\mu_h]$ such that for any $\mu^*\in(\mu_l,\mu_h)$, we can always find two points $\gamma_1<\gamma_2$ with $\gamma_1,\gamma_2\in(\underline{\gamma},\bar{\gamma})$ such that $\varphi_{\mu^*}^N(\gamma_1)<0$ and $\varphi_{\mu^*}^N(\gamma_2)>0$. This will imply, by the Intermediate Value Theorem, that there exists two values $\gamma_l^*<\gamma_h^*$ with

$\underline{\gamma} \leq \gamma_l^* < \gamma_1$ and $\gamma_2 < \gamma_h^* \leq \bar{\gamma}$ such that $\varphi_\mu^N(\gamma_l^*) = \varphi_\mu^N(\gamma_h^*) = 0$, i.e. two distinct stationary points in the dynamics of precision γ.

An important step in this proof is established in lemma 3 below, where we prove that as σ^f goes to 0 the cutoff $f_c^{\sigma^f}$ converges uniformly towards some limit f_c^0 and that the number of investing firms converges pointwise to the limit $N^0(\mu, \gamma) = \bar{N}\mathbb{1}\left(\bar{f} \leq f_c^0(\mu, \gamma)\right)$.

We must first find a range of values for μ in which we are guaranteed to have multiple stationary points for γ. We are going to use the fact that $f_c^{\sigma^f}$ is strictly increasing in μ and γ at a bounded rate. Recall the definition:

$$f_c^{\sigma^f}(\mu, \gamma) = -\frac{1}{a} e^{-a\mu + \frac{a^2}{2}\left(\frac{1}{\gamma} + \frac{1}{\gamma_x}\right)} \left(1 - \beta e^{\frac{a^2}{2\gamma_\theta}}\right) + (1 - \beta)\frac{1}{a} + \beta\bar{f} - \beta E_{\sigma^f}\left[G^{\sigma^f}(\mu', \gamma', f')\right].$$

Since G^{σ^f} has bounded derivatives, we can find upper and lower bounds for the derivatives of $f_c^{\sigma^f}$ in μ and γ that are strictly positive, as long as γ_x is low enough. Denote these bounds $\overline{f}_\mu$,$\underline{f}_\mu$ and $\overline{f}_\gamma$,$\underline{f}_\gamma$. The derivatives are:

$$\begin{aligned} 0 < \underline{f}_\mu \leq \frac{\partial}{\partial\mu} f_c^{\sigma^f}(\mu, \gamma) &= e^{-a\mu + \frac{a^2}{2}\left(\frac{1}{\gamma} + \frac{1}{\gamma_x}\right)} \left(1 - \beta e^{\frac{a^2}{2\gamma_\theta}}\right) + \beta E\left[\frac{\partial}{\partial\mu} G^{\sigma^f}\right] + O(\gamma_x) \leq \overline{f}_\mu \\ 0 < \underline{f}_\gamma \leq \frac{\partial}{\partial\gamma} f_c^{\sigma^f}(\mu, \gamma) &= \frac{a}{2\gamma^2} e^{-a\mu + \frac{a^2}{2}\left(\frac{1}{\gamma} + \frac{1}{\gamma_x}\right)} \left(1 - \beta e^{\frac{a^2}{2\gamma_\theta}}\right) + \beta E\left[\frac{\partial}{\partial\gamma} G^{\sigma^f}\right] + O(\gamma_x) \leq \overline{f}_\gamma \end{aligned}$$

Since f_c^0 is the uniform limit of continuous functions, it is continuous. The limit f_c^0 may not be differentiable, but it is bi-Lipschitz continuous with Lipschitz constants $\left(\underline{f}_\mu, \overline{f}_\mu\right)$ and $\left(\underline{f}_\gamma, \overline{f}_\gamma\right)$. We know therefore that for μ low, $f_c^0(\mu, \bar{\gamma}) < \bar{f}$ (remember that $\bar{f}$ is the mean of the fixed cost distribution), and that for μ high, $f_c^0(\mu, \bar{\gamma}) > \bar{f}$. By the Intermediate Value theorem, we know that there exists a point μ_l at which $f_c^0(\mu_l, \bar{\gamma}) = \bar{f}$. Since f_c^0 is strictly increasing in γ, we have that $f_c^0(\mu_l, \underline{\gamma}) < \bar{f}$. Using the fact that f_c^0 is bi-Lipschitz continuous, we have the following inequality:

$$f_c^0(\mu, \underline{\gamma}) \quad \leq \quad f_c^0(\mu_l, \underline{\gamma}) + \bar{f}_\mu \cdot (\mu - \mu_l).$$

Define $\mu_h = \mu_l + \frac{\bar{f} - f_c(\mu_l, \underline{\gamma})}{\bar{f}_\mu} > \mu_l$. Then, for any $\mu \in (\mu_l, \mu_h)$:

$$f_c^0(\mu, \underline{\gamma}) \leq f_c^0(\mu_l, \underline{\gamma}) + \bar{f}_\mu \cdot (\mu - \mu_l) < \bar{f} < f_c^0(\mu, \bar{\gamma}).$$

We will now show that the interval (μ_l, μ_h) is a range of values for μ in which we are guaranteed to have two steady-states. Pick any $\mu^* \in (\mu_l, \mu_h)$. Then $f_c^0(\mu^*, \underline{\gamma}) < \bar{f}$ (meaning that $N^0(\mu^*, \underline{\gamma}) = 0$) and $f_c^0(\mu^*, \bar{\gamma}) > \bar{f}$ (meaning $N^0(\mu^*, \bar{\gamma}) = \bar{N}$). By continuity of f_c^0, we can pick (γ_1, γ_2) with $\underline{\gamma} < \gamma_1 < \gamma_2 < \bar{\gamma}$, such that $f_c^0(\mu^*, \gamma_1) < \bar{f}$ and $f_c^0(\mu^*, \gamma_2) > \bar{f}$. Therefore, $N^0(\mu^*, \gamma_1) = 0$ and

$N^0(\mu^*, \gamma_2) = \bar{N}$. We have:

$$\begin{aligned}\varphi_{\mu^*}^{N^0}(\gamma_1) &= \left(\frac{1}{\gamma_1 + \gamma_y + N^0(\mu^*, \gamma_1)\gamma_x} + \sigma_\theta^2\right)^{-1} - \gamma_1 \\ &= \left(\frac{1}{\gamma_1 + \gamma_y} + \sigma_\theta^2\right)^{-1} - \gamma_1 < \left(\frac{1}{\underline{\gamma} + \gamma_y} + \sigma_\theta^2\right)^{-1} - \underline{\gamma} = 0\end{aligned}$$

$$\begin{aligned}\varphi_{\mu^*}^{N^0}(\gamma_2) &= \left(\frac{1}{\gamma_2 + \gamma_y + N^0(\mu^*, \gamma_2)\gamma_x} + \sigma_\theta^2\right)^{-1} - \gamma_2 \\ &= \left(\frac{1}{\gamma_2 + \gamma_y + \gamma_x} + \sigma_\theta^2\right)^{-1} - \gamma_2 > \left(\frac{1}{\bar{\gamma} + \gamma_y + \gamma_x} + \sigma_\theta^2\right)^{-1} - \bar{\gamma} = 0.\end{aligned}$$

Since $N^{\sigma^f}(\mu, \gamma) \underset{\sigma^f \to 0}{\longrightarrow} N^0(\mu, \gamma)$, for σ^f small enough, we will have: $\varphi_{\mu^*}^{N^{\sigma^f}}(\gamma_1) < 0$ and $\varphi_{\mu^*}^{N^{\sigma^f}}(\gamma_2) > 0$, which implies that there exists at least two locally stable steady-states γ_l^* and γ_h^* $\left(\varphi_{\mu^*}^{N^{\sigma^f}}(\gamma_l^*) = \varphi_{\mu^*}^{N^{\sigma^f}}(\gamma_h^*) = 0\right)$ with $\underline{\gamma} \leq \gamma_l^* < \gamma_1$ and $\gamma_2 < \gamma_h^* \leq \bar{\gamma}$ (one can pick at least 2 locally stable steady-states because $\varphi_{\mu^*}^{N^{\sigma^f}}$ must cross the x-axis from above at least twice). □

In what follows, we prove the technical lemma that establishes the continuity of the cutoff $f_c^{\sigma^f}$ in σ^f.

Lemma 3. *As $\sigma^f \to 0$, the equilibrium cutoff value $f_c^{\sigma^f}$ converges uniformly towards some limit f_c^0:*

$$\underset{(\mu,\gamma)\in\mathcal{S}}{sup} \left| f_c^{\sigma^f}(\mu, \gamma) - f_c^0(\mu, \gamma)\right| \underset{\sigma^f \to 0}{\longrightarrow} 0$$

and the number of investing firms converges pointwise to the following limit:

$$\forall (\mu, \gamma), \quad N^{\sigma^f}(\mu, \gamma) = \bar{N} \cdot F^{\sigma^f}\left(f_c^{\sigma^f}(\mu, \gamma)\right) \underset{\sigma^f \to 0}{\longrightarrow} N^0(\mu, \gamma) \equiv \bar{N} \cdot \mathbb{1}\left(\bar{f} \leq f_c^0(\mu, \gamma)\right).$$

Proof. This proof is similar to the proof of continuity of the mapping $\mathcal{T}^p$ in p. Since $N = \bar{N}p$, we use N and p interchangeably from now on and abuse notation in saying that $\mathcal{M}$ is a mapping for $N : \mathcal{N} \longrightarrow \mathcal{N}$. Pick two different variances for the fixed cost σ_1^f and σ_2^f. The notation $\mathcal{T}^{N,\sigma_i^f}$ denotes the mapping $\mathcal{T}$ for the value function G when N is the aggregate number of investing firms perceived by agents and the fixed costs are distributed according to $F^{\sigma_i^f}$.

Outline of the proof: Starting with the sane initial aggregate law N, we compare the objects f_c^{N,σ_1^f} and f_c^{N,σ_2^f} after the first iteration of the mappings $\mathcal{M}^{\sigma_1^f}$ and $\mathcal{M}^{\sigma_2^f}$. In a second step, we establish a recursive relationship to compare the same objects after an arbitrary number of iterations. We then conclude that the limits of both contractions $N^{\sigma_i^f} = \lim_{n\to\infty} \left(\mathcal{M}^{\sigma_i^f}\right)^n N$ produce equilibrium cutoffs that are close in the following sense:

$$\parallel f_c^{N^{\sigma_2^f}, \sigma_2^f} - f_c^{N^{\sigma_1^f}, \sigma_1^f} \parallel \leq \bar{A}\left|\sigma_2^f - \sigma_1^f\right|$$

for some strictly positive constant $\bar{A}$, which suffices to establish the result.

• Start with some functions G and N, identical for both mappings. Denote $G_n^{N,\sigma_i^f} \equiv \left(\mathcal{T}^{N,\sigma_i^f}\right)^n G$. Let me prove by recursion that:

$$\left|\left(G_n^{N,\sigma_2^f} - G_n^{N,\sigma_1^f}\right)(\mu,\gamma,f)\right| \leq \beta\frac{1-\beta^n}{1-\beta}\left|\sigma_2^f - \sigma_1^f\right|.$$

This is trivially true for $n=0$. Assume it is true for until $n \geq 0$, then:

$$\begin{aligned}
& \left|\left(G_{n+1}^{N,\sigma_2^f} - G_{n+1}^{N,\sigma_1^f}\right)(\mu,\gamma,f)\right| \\
\leq\ & \beta\left|E_{\sigma_2^f}\left[G_n^{N,\sigma_2^f}\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f'\right)\right] - E_{\sigma_1^f}\left[G_n^{N,\sigma_1^f}\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f'\right)\right]\right| \\
\leq\ & \beta\left|E_{\sigma_2^f}\left[G_n^{N,\sigma_2^f}\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f'\right)\right] - E_{\sigma_1^f}\left[G_n^{N,\sigma_2^f}\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f'\right)\right]\right| \\
& +\beta\left|E_{\sigma_1^f}\left[G_n^{N,\sigma_2^f}\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f'\right)\right] - E_{\sigma_1^f}\left[G_n^{N,\sigma_1^f}\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f'\right)\right]\right| \\
\leq\ & \beta\left|\int\left(G_n^{N,\sigma_2^f}\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),\bar{f}+\sigma_2^f\upsilon\right) - G_n^{N,\sigma_2^f}\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),\bar{f}+\sigma_1^f\upsilon\right)\right)d\Phi(\varepsilon)\,d\Phi(\upsilon)\right| \\
& +\beta\times\beta\frac{1-\beta^n}{1-\beta}\left|\sigma_2^f-\sigma_1^f\right| \\
\leq\ & \beta\int\left|\sigma_2^f-\sigma_1^f\right||\upsilon|\,d\Phi(\upsilon) + \beta^2\frac{1-\beta^n}{1-\beta}\left|\sigma_2^f-\sigma_1^f\right| \quad \text{(Lipschitz of constant 1 in } f\text{)} \\
\leq\ & \beta\left|\sigma_2^f-\sigma_1^f\right| + \beta^2\frac{1-\beta^n}{1-\beta}\left|\sigma_2^f-\sigma_1^f\right| = \beta\frac{1-\beta^{n+1}}{1-\beta}\left|\sigma_2^f-\sigma_1^f\right|
\end{aligned}$$

which proves the recursion. Taking the limit $G^{N,\sigma_i^f} = \lim_{n\to\infty}\left(\mathcal{T}^{N,\sigma_i^f}\right)^n G$:

$$\left|G^{N,\sigma_2^f} - G^{N,\sigma_1^f}\right| \leq \frac{\beta}{1-\beta}\left|\sigma_2^f-\sigma_1^f\right|. \tag{13}$$

Let us turn to the equilibrium cutoff rule:

$$\begin{aligned}
& \left|f_c^{N,\sigma_2^f}(\mu,\gamma) - f_c^{N,\sigma_1^f}(\mu,\gamma)\right| \\
=\ & \beta\left|E_{\sigma_2^f}\left[G^{N,\sigma_2^f}\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f'\right)\right] - E_{\sigma_1^f}\left[G^{N,\sigma_1^f}\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f'\right)\right]\right| \\
=\ & \beta\left|E_{\sigma_2^f}\left[G^{N,\sigma_2^f}\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f'\right)\right] - E_{\sigma_2^f}\left[G^{N,\sigma_1^f}\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f'\right)\right]\right. \\
& \left.+E_{\sigma_2^f}\left[G^{N,\sigma_1^f}\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f'\right)\right] - E_{\sigma_1^f}\left[G^{N,\sigma_1^f}\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),f'\right)\right]\right| \\
\leq\ & \beta\left(\frac{\beta}{1-\beta}\left|\sigma_2^f-\sigma_1^f\right|\right. \\
& \left.+\left|\int\left(G^{N,\sigma_1^f}\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),\bar{f}+\sigma_2^f\upsilon\right) - G^{N,\sigma_1^f}\left(\mu+s(N,\gamma)\varepsilon,\Gamma(N,\gamma),\bar{f}+\sigma_1^f\upsilon\right)\right)d\Phi(\varepsilon)\,d\Phi(\upsilon)\right|\right) \\
\leq\ & \beta\left(\frac{\beta}{1-\beta}\left|\sigma_2^f-\sigma_1^f\right| + \left|\sigma_2^f-\sigma_1^f\right|\right) \\
\leq\ & \frac{\beta}{1-\beta}\left|\sigma_2^f-\sigma_1^f\right|
\end{aligned} \tag{14}$$

Let us now turn to the number of investing firms N. Denote $N_n^{\sigma_i^f} \equiv \left(\mathcal{M}^{\sigma_i^f}\right)^n N$.

$$\left|\left(N_1^{\sigma_2^f} - N_1^{\sigma_1^f}\right)(\mu,\gamma)\right| \leq \bar{N}\left|F^{\sigma_2^f}\left(f_c^{N,\sigma_2^f}(\mu,\gamma)\right) - F^{\sigma_1^f}\left(f_c^{N,\sigma_1^f}(\mu,\gamma)\right)\right|$$
$$\leq \bar{N}\left|F^{\sigma_2^f}\left(f_c^{N,\sigma_2^f}(\mu,\gamma)\right) - F^{\sigma_2^f}\left(f_c^{N,\sigma_1^f}(\mu,\gamma)\right) + F^{\sigma_2^f}\left(f_c^{N,\sigma_1^f}(\mu,\gamma)\right) - F^{\sigma_1^f}\left(f_c^{N,\sigma_1^f}(\mu,\gamma)\right)\right|$$

where we see that $N^{\sigma_2^f}$ may not always be close to $N^{\sigma_1^f}$ under the sup norm. The problem is that the above expression could be close to 1 for a few of points if σ_i^f is low and $f_c^{N,\sigma_2^f} \neq f_c^{N,\sigma_1^f}$. However, we now show that this is not a problem as they will be close on *average*. The only thing we need for the final result is pointwise convergence for N.

• We will now establish a recursive relationship to compare the two objects $f_c^{N_n^{\sigma_1^f},\sigma_1^f}$ and $f_c^{N_n^{\sigma_2^f},\sigma_2^f}$. Assume that after n iterations of the mapping $\mathcal{M}$, we have two different functions $N_n^{\sigma_2^f}$ and $N_n^{\sigma_1^f}$ and that

$$\forall(\mu,\gamma), \quad \left|f_c^{N_n^{\sigma_2^f},\sigma_2^f}(\mu,\gamma) - f_c^{N_n^{\sigma_1^f},\sigma_1^f}(\mu,\gamma)\right| \leq A_n\left|\sigma_2^f - \sigma_1^f\right|.$$

Let us study the following term:

$$\begin{aligned}\left|\left(G^{N_{n+1}^{\sigma_2^f},\sigma_2^f} - G^{N_{n+1}^{\sigma_1^f},\sigma_1^f}\right)(\mu,\gamma,f)\right| &\leq \left|\left(G^{N_{n+1}^{\sigma_2^f},\sigma_2^f} - G^{N_{n+1}^{\sigma_2^f},\sigma_1^f}\right)(\mu,\gamma,f)\right| + \left|\left(G^{N_{n+1}^{\sigma_2^f},\sigma_1^f} - G^{N_1^{\sigma_1^f},\sigma_1^f}\right)(\mu,\gamma,f)\right| \\ &\leq \frac{\beta}{1-\beta}\left|\sigma_2^f - \sigma_1^f\right| + \left|\left(G^{N_{n+1}^{\sigma_2^f},\sigma_1^f} - G^{N_{n+1}^{\sigma_1^f},\sigma_1^f}\right)(\mu,\gamma,f)\right| \qquad (15)\end{aligned}$$

where we have controlled the first term by the same argument as in (13). We need to study the second term:

$$\begin{aligned}\left|\left(G^{N_{n+1}^{\sigma_2^f},\sigma_1^f} - G^{N_{n+1}^{\sigma_1^f},\sigma_1^f}\right)(\mu,\gamma,f)\right| &= \left|\left(\lim_{m\to\infty}\left(\mathcal{T}^{N_{n+1}^{\sigma_2^f},\sigma_1^f}\right)^m G - \lim_{m\to\infty}\left(\mathcal{T}^{N_{n+1}^{\sigma_1^f},\sigma_1^f}\right)^m G\right)(\mu,\gamma,f)\right| \\ &= \left|\left(\lim_{m\to\infty} G_m^{N_{n+1}^{\sigma_2^f},\sigma_1^f} - \lim_{m\to\infty} G_m^{N_{n+1}^{\sigma_1^f},\sigma_1^f}\right)(\mu,\gamma,f)\right|.\end{aligned}$$

Starting with the first iteration:

$$
\begin{aligned}
& \left|\left(G_1^{N_{n+1}^{\sigma_2^f},\sigma_1^f} - G_1^{N_{n+1}^{\sigma_1^f},\sigma_1^f}\right)(\mu,\gamma,f)\right| \\
\leq\ & \beta\left|\iint\left[G\left(\mu+s\left(N_{n+1}^{\sigma_2^f},\gamma\right)\varepsilon,\Gamma\left(N_{n+1}^{\sigma_2^f},\gamma\right),f'\right)-G\left(\mu+s\left(N_{n+1}^{\sigma_1^f},\gamma\right)\varepsilon,\Gamma\left(N_{n+1}^{\sigma_1^f},\gamma\right),f'\right)\right]d\Phi(\varepsilon)\,dF^{\sigma_1^f}(f')\right| \\
\leq\ & \beta\left[\iint\left[G\left(\mu+s\left(N_{n+1}^{\sigma_2^f},\gamma\right)\varepsilon,\Gamma\left(N_{n+1}^{\sigma_2^f},\gamma\right),f'\right)-G\left(\mu+s\left(N_{n+1}^{\sigma_1^f},\gamma\right)\varepsilon,\Gamma\left(N_{n+1}^{\sigma_2^f},\gamma\right),f'\right)\right]d\Phi(\varepsilon)\,dF^{\sigma_1^f}(f')\right. \\
& \left.+\iint\left[G\left(\mu+s\left(N_{n+1}^{\sigma_1^f},\gamma\right)\varepsilon,\Gamma\left(N_{n+1}^{\sigma_2^f},\gamma\right),f'\right)-G\left(\mu+s\left(N_{n+1}^{\sigma_1^f},\gamma\right)\varepsilon,\Gamma\left(N_{n+1}^{\sigma_1^f},\gamma\right),f'\right)\right]d\Phi(\varepsilon)\,dF^{\sigma_1^f}(f')\right| \\
\leq\ & \beta\left[\iint\bar{G}_\mu|\varepsilon|\left|s\left(N_{n+1}^{\sigma_2^f},\gamma\right)-s\left(N_{n+1}^{\sigma_1^f},\gamma\right)\right|d\Phi(\varepsilon)\,dF^{\sigma_1^f}(f')+\iint\bar{G}_\gamma\left|\Gamma\left(N_{n+1}^{\sigma_2^f},\gamma\right)-\Gamma\left(N_{n+1}^{\sigma_1^f},\gamma\right)\right|d\Phi(\varepsilon)\,dF^{\sigma_1^f}(f')\right] \\
\leq\ & \beta\left[\bar{G}_\mu\left|s\left(N_{n+1}^{\sigma_2^f},\gamma\right)-s\left(N_{n+1}^{\sigma_1^f},\gamma\right)\right|+\bar{G}_\gamma\left|\Gamma\left(N_{n+1}^{\sigma_2^f},\gamma\right)-\Gamma\left(N_{n+1}^{\sigma_1^f},\gamma\right)\right|\right] \\
\leq\ & \beta\left(\bar{G}_\mu\overline{\frac{\partial s}{\partial N}}+\bar{G}_\gamma\overline{\frac{\partial\Gamma}{\partial N}}\right)\left|\left(N_{n+1}^{\sigma_2^f}-N_{n+1}^{\sigma_1^f}\right)(\mu,\gamma)\right| \\
\leq\ & \beta B\gamma_x\left|\left(N_{n+1}^{\sigma_2^f}-N_{n+1}^{\sigma_1^f}\right)(\mu,\gamma)\right|
\end{aligned}
$$

where $B=\frac{1}{\underline{\gamma}+\gamma_y}\left(\left|\bar{G}_\mu\right|+\left|\bar{G}_\gamma\right|\gamma_\theta\right)$ is a constant similar to the one we used in proposition 2. We now establish recursively that for $m\geq 2$:

$$
\left|\left(G_m^{N_{n+1}^{\sigma_2^f},\sigma_1^f}-G_m^{N_{n+1}^{\sigma_1^f},\sigma_1^f}\right)(\mu,\gamma,f)\right|\leq\beta B\gamma_x\left|\left(N_{n+1}^{\sigma_2^f}-N_{n+1}^{\sigma_1^f}\right)(\mu,\gamma)\right|+\beta^2\frac{1-\beta^{m-1}}{1-\beta}B\gamma_x\left(A_nC+D\sigma_1^f\sigma_2^f\right)\left|\sigma_2^f-\sigma_1^f\right|
$$

where constants C and D are those coming from lemma 4 below. Assuming the relationship is true until $m\geq 2$, we have*:*

$$
\begin{aligned}
& \left|\left(G_{m+1}^{N_{n+1}^{\sigma_2^f},\sigma_1^f}-G_{m+1}^{N_{n+1}^{\sigma_1^f},\sigma_1^f}\right)(\mu,\gamma,f)\right|\leq\beta\left|E\left[G_m^{N_{n+1}^{\sigma_2^f},\sigma_1^f}-G_m^{N_{n+1}^{\sigma_1^f},\sigma_1^f}\right](\mu,\gamma,f)\right| \\
& \leq\beta\left|\int\left[G_m^{N_{n+1}^{\sigma_2^f},\sigma_1^f}\left(\mu+s\left(N_{n+1}^{\sigma_2^f},\gamma\right)\varepsilon,\Gamma\left(N_{n+1}^{\sigma_2^f},\gamma\right),f'\right)-G_m^{N_1^{\sigma_1^f},\sigma_1^f}\left(\mu+s\left(N_{n+1}^{\sigma_1^f},\gamma\right)\varepsilon,\Gamma\left(N_{n+1}^{\sigma_1^f},\gamma\right),f'\right)\right]d\Phi(\varepsilon)\,dF^{\sigma_1^f}(f')\right| \\
& \leq\beta\left|\int\left[G_m^{N_{n+1}^{\sigma_2^f},\sigma_1^f}\left(\mu+s\left(N_{n+1}^{\sigma_2^f},\gamma\right)\varepsilon,\Gamma\left(N_{n+1}^{\sigma_2^f},\gamma\right),f'\right)-G_m^{N_{n+1}^{\sigma_1^f},\sigma_1^f}\left(\mu+s\left(N_{n+1}^{\sigma_2^f},\gamma\right)\varepsilon,\Gamma\left(N_{n+1}^{\sigma_2^f},\gamma\right),f'\right)\right]d\Phi(\varepsilon)\,dF^{\sigma_1^f}(f')\right| \\
& +\beta\left|\int\left[G_m^{N_{n+1}^{\sigma_1^f},\sigma_1^f}\left(\mu+s\left(N_{n+1}^{\sigma_2^f},\gamma\right)\varepsilon,\Gamma\left(N_{n+1}^{\sigma_2^f},\gamma\right),f'\right)-G_m^{N_{n+1}^{\sigma_1^f},\sigma_1^f}\left(\mu+s\left(N_{n+1}^{\sigma_1^f},\gamma\right)\varepsilon,\Gamma\left(N_{n+1}^{\sigma_1^f},\gamma\right),f'\right)\right]d\Phi(\varepsilon)\,dF^{\sigma_1^f}(f')\right| \\
& \leq\beta\int\left|\left(G_m^{N_{n+1}^{\sigma_2^f},\sigma_1^f}-G_m^{N_{n+1}^{\sigma_1^f},\sigma_1^f}\right)\left(\mu+s\left(N_{n+1}^{\sigma_2^f},\gamma\right)\varepsilon,\Gamma\left(N_{n+1}^{\sigma_2^f},\gamma\right),f'\right)d\Phi(\varepsilon)\,dF^{\sigma_1^f}(f')\right| \\
& +\beta\int\left|\frac{\partial s}{\partial N}\varepsilon\frac{\partial G_1^{N_1^{\sigma_1^f},\sigma_1^f}}{\partial\mu}+\frac{\partial\Gamma}{\partial N}\frac{\partial G_1^{N_1^{\sigma_1^f},\sigma_1^f}}{\partial\gamma}\right|\left|\left(N_1^{\sigma_2^f}-N_1^{\sigma_1^f}\right)(\mu,\gamma)\right|d\Phi(\varepsilon)\,dF^{\sigma_1^f}(f') \\
& \leq\beta\left(\beta B\gamma_x\int\left|\left(N_{n+1}^{\sigma_2^f}-N_{n+1}^{\sigma_1^f}\right)\left(\mu+s\left(N_{n+1}^{\sigma_2^f},\gamma\right)\varepsilon,\Gamma\left(N_{n+1}^{\sigma_2^f},\gamma\right)\right)\right|d\Phi(\varepsilon)+\beta^2\frac{1-\beta^{m-1}}{1-\beta}B\gamma_x\left(A_nC+D\sigma_1^f\sigma_2^f\right)\left|\sigma_2^f-\sigma_1^f\right|\right) \\
& +\beta B\gamma_x\left|\left(N_{n+1}^{\sigma_2^f}-N_{n+1}^{\sigma_1^f}\right)(\mu,\gamma)\right|
\end{aligned}
$$

Using lemma 4, we can control the term:

$$\int \left| \left(N_{n+1}^{\sigma_2^f} - N_{n+1}^{\sigma_1^f} \right) \left(\mu + s\left(N_{n+1}^{\sigma_2^f}, \gamma \right) \varepsilon, \gamma' \left(N_{n+1}^{\sigma_2^f}, \gamma \right) \right) \right| d\Phi\left(\varepsilon\right) \leq \left(A_n C + D\sigma_1^f \sigma_2^f \right) \left| \sigma_2^f - \sigma_1^f \right|.$$

Therefore:

$$\left| \left(G_{m+1}^{N_{n+1}^{\sigma_2^f}, \sigma_1^f} - G_{m+1}^{N_{n+1}^{\sigma_1^f}, \sigma_1^f} \right) (\mu, \gamma, f) \right| \leq \beta B \left| \left(N_{m+1}^{\sigma_2^f} - N_{m+1}^{\sigma_1^f} \right) (\mu, \gamma) \right| + \beta^2 \frac{1-\beta^m}{1-\beta} B \left(A_n C + D\sigma_1^f \sigma_2^f \right) \left| \sigma_2^f - \sigma_1^f \right|$$

which establishes the recursion. Taking the limit as $m \to \infty$:

$$\begin{aligned} & \left| \left(G^{N_{n+1}^{\sigma_2^f}, \sigma_1^f} - G^{N_{n+1}^{\sigma_1^f}, \sigma_1^f} \right) (\mu, \gamma, f) \right| \\ \leq \ & \beta B \left| \left(N_{n+1}^{\sigma_2^f} - N_{n+1}^{\sigma_1^f} \right) (\mu, \gamma) \right| + \frac{\beta^2}{1-\beta} B \left(A_n C + D\sigma_1^f \sigma_2^f \right) \left| \sigma_2^f - \sigma_1^f \right|. \end{aligned} \tag{16}$$

We see that G may not converge pointwise. However, the expectation of G will, which is what we need for our final result. Going back to equation (15):

$$\begin{aligned} & \left| \left(G^{N_{n+1}^{\sigma_2^f}, \sigma_2^f} - G^{N_{n+1}^{\sigma_1^f}, \sigma_1^f} \right) (\mu, \gamma, f) \right| \\ \leq \ & \left| \left(G^{N_{n+1}^{\sigma_2^f}, \sigma_2^f} - G^{N_{n+1}^{\sigma_2^f}, \sigma_1^f} \right) (\mu, \gamma, f) \right| + \left| \left(G^{N_{n+1}^{\sigma_2^f}, \sigma_1^f} - G^{N_{n+1}^{\sigma_1^f}, \sigma_1^f} \right) (\mu, \gamma, f) \right| \\ \leq \ & \frac{\beta}{1-\beta} \left| \sigma_2^f - \sigma_1^f \right| + \beta B \gamma_x \left| \left(N_1^{\sigma_2^f} - N_1^{\sigma_1^f} \right) (\mu, \gamma) \right| + \frac{\beta^2}{1-\beta} B \gamma_x \left(A_n C + D\sigma_1^f \sigma_2^f \right) \left| \sigma_2^f - \sigma_1^f \right|. \end{aligned}$$

where I have used equations (13) and (16). Let us turn to the cutoff value:

$$\begin{aligned} & \left| f_c^{N_{n+1}^{\sigma_2^f}, \sigma_2^f} (\mu, \gamma) - f_c^{N_{n+1}^{\sigma_1^f}, \sigma_1^f} (\mu, \gamma) \right| \\ = \ & \beta \left| E_{\sigma_2^f} \left[G^{N_{n+1}^{\sigma_2^f}, \sigma_2^f} \left(\mu + s\left(N_{n+1}^{\sigma_2^f}, \gamma \right) \varepsilon, \Gamma\left(N_{n+1}^{\sigma_2^f}, \gamma \right), f' \right) \right] - E_{\sigma_1^f} \left[G^{N_{n+1}^{\sigma_1^f}, \sigma_1^f} \left(\mu + s\left(N_{n+1}^{\sigma_1^f}, \gamma \right) \varepsilon, \Gamma\left(N_{n+1}^{\sigma_1^f}, \gamma \right), f' \right) \right] \right| \\ \leq \ & \beta \left| E_{\sigma_2^f} \left[G^{N_{n+1}^{\sigma_2^f}, \sigma_2^f} \left(\mu + s\left(N_{n+1}^{\sigma_2^f}, \gamma \right) \varepsilon, \Gamma\left(N_{n+1}^{\sigma_2^f}, \gamma \right), f' \right) \right] - E_{\sigma_2^f} \left[G^{N_{n+1}^{\sigma_1^f}, \sigma_1^f} \left(\mu + s\left(N_{n+1}^{\sigma_1^f}, \gamma \right) \varepsilon, \Gamma\left(N_{n+1}^{\sigma_1^f}, \gamma \right), f' \right) \right] \right| \\ & + \beta \left| E_{\sigma_2^f} \left[G^{N_{n+1}^{\sigma_1^f}, \sigma_1^f} \left(\mu + s\left(N_{n+1}^{\sigma_1^f}, \gamma \right) \varepsilon, \Gamma\left(N_{n+1}^{\sigma_1^f}, \gamma \right), f' \right) \right] - E_{\sigma_1^f} \left[G^{N_{n+1}^{\sigma_1^f}, \sigma_1^f} \left(\mu + s\left(N_{n+1}^{\sigma_1^f}, \gamma \right) \varepsilon, \Gamma\left(N_{n+1}^{\sigma_1^f}, \gamma \right), f' \right) \right] \right| \\ \leq \ & \beta \left(\frac{\beta}{1-\beta} + \beta B \gamma_x \left(A_n C + D\sigma_1^f \sigma_2^f \right) + \frac{\beta^2}{1-\beta} B \gamma_x \left(A_n C + D\sigma_1^f \sigma_2^f \right) \right) \left| \sigma_2^f - \sigma_1^f \right| + \beta \left| \sigma_2^f - \sigma_1^f \right| \\ \leq \ & \beta \left(\frac{1}{1-\beta} + \frac{\beta}{1-\beta} B \gamma_x \left(A_n C + D\sigma_1^f \sigma_2^f \right) \right) \left| \sigma_2^f - \sigma_1^f \right| \\ \leq \ & \underbrace{\left[\frac{\beta}{1-\beta} \left(1 + \beta \gamma_x B D \sigma_1^f \sigma_2^f \right) + \frac{\beta^2}{1-\beta} \gamma_x B C A_n \right]}_{\equiv A_{n+1}} \left| \sigma_2^f - \sigma_1^f \right| \end{aligned}$$

This expression defines a recursive relationship:

$$A_{n+1} = \frac{\beta}{1-\beta} \left(1 + \beta \gamma_x B D \sigma_1^f \sigma_2^f \right) + \frac{\beta^2}{1-\beta} \gamma_x B C A_n$$

which converges to a unique limit $\bar{A}$ as long as $\frac{\beta^2}{1-\beta}\gamma_x BC < 1$ which is true if γ_x is chosen sufficiently small. Taking the limit as $n \to \infty$, we have:

$$\left| f_c^{N^{\sigma_2^f},\sigma_2^f}(\mu,\gamma) - f_c^{N^{\sigma_1^f},\sigma_1^f}(\mu,\gamma) \right| = \left| f_c^{\lim_{n\to\infty} N_n^{\sigma_2^f},\sigma_2^f}(\mu,\gamma) - f_c^{\lim_{n\to\infty} N_n^{\sigma_1^f},\sigma_1^f}(\mu,\gamma) \right| \leq \bar{A} \left| \sigma_2^f - \sigma_1^f \right|.$$

This tells us that as $\sigma^f \to 0$, the equilibrium cutoff converges uniformly to some limit:

$$\forall (\mu,\gamma), \quad f_c^{N^{\sigma^f},\sigma^f}(\mu,\gamma) \to f_c^0(\mu,\gamma).$$

Turning to the equilibrium entry schedule, N converges pointwise towards the limit:

$$\forall (\mu,\gamma), \quad N^{\sigma_n^f}(\mu,\gamma) = \bar{N} \cdot F^{\sigma_n^f}\left(f_c^{N^{\sigma_n^f},\sigma_n^f}(\mu,\gamma) \right) \to N^0(\mu,\gamma) = \bar{N} \cdot \mathbb{1}\left(\bar{f} \leq f_c^0(\mu,\gamma) \right).$$

□

Lemma 4. *Suppose two functions f_1 and f_2 are such that sup* $|f_2(\mu,\gamma) - f_1(\mu,\gamma)| \leq A \left| \sigma_2^f - \sigma_1^f \right|$ *for some strictly positive constant A. Assume also that both f_i's are continuous, differentiable and that $\frac{\partial f_i}{\partial \mu} > \underline{f}_\mu$. Then, for $N^i = \bar{N} F^{\sigma_i^f}(f_i)$, there exists two strictly positive constants C and D such that for $i = 1,2$:*

$$\int \left| \left(N^2 - N^1\right)\left(\mu + s\left(N^i,\gamma\right)\varepsilon, \Gamma\left(N^i,\gamma\right)\right) \right| d\Phi(\varepsilon) \leq \left(AC + D\sigma_1^f \sigma_2^f \right) \left| \sigma_2^f - \sigma_1^f \right|.$$

Proof. Abusing notation slightly with the convention $s \equiv s\left(N_1^{\sigma_i^f},\gamma\right)$ and $\gamma' \equiv \Gamma\left(N_1^{\sigma_i^f},\gamma\right)$:

$$\begin{aligned}
& \int \left| \left(N^2 - N^1\right)\left(\mu + s\left(N^i,\gamma\right)\varepsilon, \Gamma\left(N^i,\gamma\right)\right) \right| d\Phi(\varepsilon) \\
= \; & \bar{N} \int \left| F^{\sigma_2^f}\left(f_2\left(\mu + s\varepsilon, \gamma'\right)\right) - F^{\sigma_1^f}\left(f_1\left(\mu + s\varepsilon, \gamma'\right)\right) \right| d\Phi(\varepsilon) \\
\leq \; & \underbrace{\bar{N} \int \left| F^{\sigma_2^f}\left(f_2\left(\mu + s\varepsilon, \gamma'\right)\right) - F^{\sigma_2^f}\left(f_1\left(\mu + s\varepsilon, \gamma'\right)\right) \right| d\Phi(\varepsilon)}_{\equiv A_1} \\
& + \underbrace{\bar{N} \int \left| F^{\sigma_2^f}\left(f_1\left(\mu + s\varepsilon, \gamma'\right)\right) - F^{\sigma_1^f}\left(f_1\left(\mu + s\varepsilon, \gamma'\right)\right) \right| d\Phi(\varepsilon)}_{\equiv A_2}
\end{aligned}$$

Let us take care of the first term:

$$\begin{aligned}
A_1 \; = \; & \bar{N} \int \left| F^{\sigma_2^f}\left(f_2\left(\mu + s\varepsilon, \gamma'\right)\right) - F^{\sigma_2^f}\left(f_1\left(\mu + s\varepsilon, \gamma'\right)\right) \right| d\Phi(\varepsilon) \\
\leq \; & \bar{N} \int \left[F^{\sigma_2^f}\left(f_2\left(\mu + s\varepsilon, \gamma'\right) + A\left|\sigma_2^f - \sigma_1^f\right|\right) - F^{\sigma_2^f}\left(f_2\left(\mu + s\varepsilon, \gamma'\right) - A\left|\sigma_2^f - \sigma_1^f\right|\right) \right] d\Phi(\varepsilon)
\end{aligned}$$

using equation (14). f_2 is a nicely continuous, differentiable, strictly increasing function of μ, so we can proceed to the change of variable $x = f_2\left(\mu + s\varepsilon, \gamma'\right)$:

$$\begin{aligned}
A_1 &\leq \bar{N}\int\left[F^{\sigma_2^f}\left(x + A\left|\sigma_2^f - \sigma_1^f\right|\right) - F^{\sigma_2^f}\left(x - A\left|\sigma_2^f - \sigma_1^f\right|\right)\right]\underbrace{\frac{\Phi'\left(\left(f_2\right)^{-1}(x)\right)dx}{s\cdot\left(f_2\right)'\left(\left(f_2\right)^{-1}(x)\right)}}_{\equiv d\varphi(x)} \\
&\leq \bar{N}\int_{x=-\infty}^{\infty}\int_{f=x-A\left|\sigma_2^f-\sigma_1^f\right|}^{x+A\left|\sigma_2^f-\sigma_1^f\right|} dF^{\sigma_2^f}d\varphi(x) \leq \int_{f=-\infty}^{\infty}\int_{x=f-A\left|\sigma_2^f-\sigma_1^f\right|}^{f+A\left|\sigma_2^f-\sigma_1^f\right|} dF^{\sigma_2^f}d\varphi(x) \\
&\leq \bar{N}\int_{f=-\infty}^{\infty}\left[\varphi\left(f + A\left|\sigma_2^f - \sigma_1^f\right|\right) - \varphi\left(f - A\left|\sigma_2^f - \sigma_1^f\right|\right)\right]dF^{\sigma_2^f} \\
&\leq \bar{N}\int_{f=-\infty}^{\infty}\left[\varphi'\left(\tilde{f}\right)2A\left|\sigma_2^f - \sigma_1^f\right|\right]dF^{\sigma_2^f} \quad \text{(mean value theorem)} \\
&\leq 2A\bar{N}\left|\sigma_2^f - \sigma_1^f\right|\int_{f=-\infty}^{\infty}\left|\varphi'\left(\tilde{f}\right)\right|dF^{\sigma_2^f} \\
&\leq A\cdot 2\bar{N}\frac{\sup\left|\Phi'\right|}{\inf\left|s\cdot\left(f_2\right)'\right|}\left|\sigma_2^f - \sigma_1^f\right| \equiv AC\left|\sigma_2^f - \sigma_1^f\right|
\end{aligned}$$

where I have used the fact the cdf of a unit normal Φ is bounded, $s \equiv s\left(N_1^{\sigma_i^f}, \gamma\right)$ is uniformly bounded from below and away from 0, and the derivative of f_2 is strictly positive, uniformly bounded away from 0 for γ_x small enough. Notice that the upper bound we derived is uniform: it does not depend on μ, γ, γ_x, etc. Let us control the second term A_2:

$$\begin{aligned}
A_2 &= \bar{N}\int\left|F^{\sigma_2^f}\left(f_1\left(\mu + s\varepsilon, \gamma'\right)\right) - F^{\sigma_1^f}\left(f_1\left(\mu + s\varepsilon, \gamma'\right)\right)\right|d\Phi(\varepsilon) \\
&\leq \bar{N}\int\left|F^{\sigma_2^f}(x) - F^{\sigma_1^f}(x)\right|d\varphi(x) \quad \left(\text{change of variable } x = f_1\left(\mu + s\varepsilon, \gamma'\right)\right) \\
&\leq \bar{N}\int\left|\Phi\left(\frac{x - \bar{f}}{\sigma_2^f}\right) - \Phi\left(\frac{x - \bar{f}}{\sigma_1^f}\right)\right|d\varphi(x) \quad \left(\text{change of variable } x = \sigma_1^f\sigma_2^f\tilde{x} + \bar{f}\right) \\
&\leq \bar{N}\int\left|\Phi\left(\sigma_1^f\tilde{x}\right) - \Phi\left(\sigma_2^f\tilde{x}\right)\right|\sigma_1^f\sigma_2^f d\varphi\left(\sigma_1^f\sigma_2^f\tilde{x} + \bar{f}\right) \\
&\leq \left[\bar{N}\int\left|\Phi'(\hat{x})\tilde{x}\right|d\varphi\left(\sigma_1^f\sigma_2^f\tilde{x} + \bar{f}\right)\right]\sigma_1^f\sigma_2^f\left|\sigma_2^f - \sigma_1^f\right| \equiv D\sigma_1^f\sigma_2^f\left|\sigma_2^f - \sigma_1^f\right|.
\end{aligned}$$

□

Proposition 4. *The following results hold:*

1. *The decentralized competitive equilibrium is inefficient. The symmetric, socially efficient allocation can be implemented with positive investment subsidies $\tau(\mu, \gamma)$;*

2. *When γ_x and σ^f are small, the efficient allocation is still subject to uncertainty traps.*

Proof. 1. In the limit case where the number of firms is large enough that the approximation $N/\bar{N} = F(f_c)$ is valid, we can write the planner's decision as a choice over the optimal cutoff f_c^{eff}

under which firms should invest:

$$\begin{aligned} W(\mu,\gamma) &= \max_{f_c^{eff}} \bar{N}\int_{-\infty}^{f_c^{eff}} \left(E[u(x)]-\tilde{f}\right)dF\left(\tilde{f}\right)+\beta E\left[W\left(\mu',\gamma'\right)\right] \\ &\text{s.t}\quad \mu' = \frac{\gamma\mu+\gamma_y Y+N\gamma_x X}{\gamma+\gamma_y+N\gamma_x} \\ &\qquad\; \gamma' = \left(\frac{1}{\gamma+\gamma_y+N\gamma_x}+\frac{1}{\gamma_\theta}\right)^{-1} \\ &\qquad\; N = \bar{N}F\left(f_c^{eff}\right) \end{aligned}$$

The first order condition with respect to the cutoff is

$$\bar{N}F'\left(f_c^{eff}\right)\left(E[u(x)]-f_c^{eff}+\beta\frac{d}{dN}E[W(\mu+s(N,\gamma),\Gamma(N,\gamma))]\right)=0,$$

so that we can derive an expression for the efficient cutoff:

$$f_c^{eff}(\mu,\gamma)=E[u(x)]+\beta\frac{d}{dN}E[W(\mu+s(N,\gamma),\Gamma(N,\gamma))].$$

We show that this optimal cutoff is implementable using investment subsidies $\tau(\mu,\gamma)$. Let us write the problem of firms facing these subsidies:

$$V^\tau(\mu,\gamma,f)=\max\left\{E[u(x)]-f+\tau(\mu,\gamma),\beta E\left[V^\tau\left(\mu',\gamma',f'\right)\right]\right\}$$

which yields the individual cutoff rule f_c:

$$f_c^\tau(\mu,\gamma)=E[u(x)]+\tau(\mu,\gamma)-\beta E\left[V^\tau\left(\mu',\gamma',f'\right)\right].$$

To implement the efficient allocation, we must identify the two cutoffs

$$\begin{aligned} & f_c^\tau(\mu,\gamma)=f_c^{eff}(\mu,\gamma) \\ \Leftrightarrow\quad & \tau(\mu,\gamma)=\underbrace{\beta\frac{d}{dN}E[W(\mu+s(N,\gamma),\Gamma(N,\gamma))]}_{\text{information externality}}+\underbrace{\beta E\left[V^\tau\left(\mu',\gamma',f'\right)\right].}_{\text{option value of waiting}} \end{aligned} \tag{17}$$

Expression (17) is a functional equation in τ because of the dependence of V^τ in τ. We now prove that this functional equation defines a contraction and therefore that it has a solution, which is unique. Indeed, this mapping satisfies the Blackwell conditions:

1. Monotonicity: Pick $\tau_1 \leq \tau_2$, then it is easy to show that the contraction mapping that defines

V^{τ_i} preserves the ordering, $V^{\tau_1} \leq V^{\tau_2}$. Thus,

$$\begin{aligned} & \beta \frac{d}{dN} E\left[W\left(\mu + s\left(N,\gamma\right)\varepsilon, \Gamma\left(N,\gamma\right)\right)\right] + \beta E V^{\tau_1}\left(\mu + s\left(N,\gamma\right)\varepsilon, \Gamma\left(N,\gamma\right)\right) \\ \leq \; & \beta \frac{d}{dN} E\left[W\left(\mu + s\left(N,\gamma\right)\varepsilon, \Gamma\left(N,\gamma\right)\right)\right] + \beta E V^{\tau_2}\left(\mu + s\left(N,\gamma\right)\varepsilon, \Gamma\left(N,\gamma\right)\right). \end{aligned}$$

2. Discounting: it is easy to show that $V^{\tau+a} \leq V^{\tau} + a$, then

$$\begin{aligned} & \beta \frac{d}{dN} E\left[W\left(\mu + s\left(N,\gamma\right)\varepsilon, \Gamma\left(N,\gamma\right)\right)\right] + \beta E V^{\tau+a}\left(\mu + s\left(N,\gamma\right)\varepsilon, \Gamma\left(N,\gamma\right)\right) \\ \leq \; & \beta \frac{d}{dN} E\left[W\left(\mu + s\left(N,\gamma\right)\varepsilon, \Gamma\left(N,\gamma\right)\right)\right] + \beta E V^{\tau}\left(\mu + s\left(N,\gamma\right)\varepsilon, \Gamma\left(N,\gamma\right)\right) + \beta a. \end{aligned}$$

Therefore, we can conclude that equation (17) defines a unique set of transfers. By construction, these transfers implement the efficient allocation. To complete the proof, we are now going to show that these transfers are positive and non-zero in non-trivial cases. To be more precise, rewrite the mapping satisfied by these transfers:

$$\tau\left(\mu,\gamma\right) = \underbrace{\beta \frac{d}{dN} E\left[W\left(\mu',\gamma'\right)\right]}_{\equiv A(\mu,\gamma)} + \beta \underbrace{E\left[V^{\tau}\left(\mu',\gamma',f'\right)\right]}_{\equiv B(\mu,\gamma)}.$$

Term B is non-negative. In fact, as long as the efficient allocation is not trivial, i.e. that there exists some (μ,γ,f) at which firms invest (which is guaranteed since f has an unbounded support), term $B\left(\mu,\gamma\right)$ is strictly positive for some (μ,γ).

We now prove that A is non-negative. We want to understand the purely informational impact on welfare of an exogenous arrival of information. It is useful for our purpose to rewrite the planner's problem in a sequential way. A strategy for the planner is a collection of cutoff functions $\{f_0, f_1, \ldots, f_t, \ldots\}$ such that for each date t, f_t maps the set of all past histories of signals up to time t, $\{Y_s, X_s\}_{s=0}^{t}$, to the real line. Pick some date t_0. We are going to show that the exogenous arrival of a signal S of precision γ_S at date t_0 allows the planner to do at least as well as without it, because of the simple reason that it can simply ignore it. Denote $\mathcal{F}_t$ the information set $\{Y_s, X_s\}_{s=0}^{t}$ of the planner at each date without the exogenous signal, and $\mathcal{F}_t^S$ the information set $\left\{Y_s, X_s^S\right\}_{s=0}^{t}$ of the planner when the arrival of the exogenous signal is known and anticipated. Let $\{f_{c,t}\}$ any strategy considered by the planner without the exogenous signal. Construct the following strategy for the case with exogenous arrival of information:

$$\begin{aligned} \forall t < t_0, \qquad & f_{c,t}^{S}\left(\{Y_s, X_s\}_{s=0}^{t}\right) = f_{c,t}\left(\{Y_s, X_s\}_{s=0}^{t}\right) \\ \forall t \geq t_0, \qquad & f_{c,t}^{S}\left(\{Y_s, X_s\}_{s=0}^{t}, S\right) = f_{c,t}\left(\{Y_s, X_s\}_{s=0}^{t}\right) \end{aligned}$$

so that the two strategies and the information sets $\mathcal{F}_t$ and $\mathcal{F}_t^S$ coincide up to time $t_0 - 1$. After date t_0, strategy f_c^S deliberately ignores the new information. Therefore, by the law of iterated expectations, the two strategies have the same ex-ante payoffs. Welfare can only be increased with the arrival of new information, hence term $A(\mu,\gamma)$ is non-negative.

We can now safely conclude that the symmetric, efficient allocation can be implemented with positive transfers. In non-trivial cases, these transfers are strictly positive, which implies that the decentralized economy without transfers is in general inefficient.

2. The proof that the efficient allocation is subject to uncertainty traps follows closely that of the decentralized case. Thus, we only state the major steps of the proof:

- The optimal cutoff for the planner is defined by:

$$f_c^{eff}(\mu,\gamma) = E[u(x)] + \beta \frac{d}{dN} E[W(\mu + s(N,\gamma), \Gamma(N,\gamma))].$$

  The first step of the proof is to show that $\frac{d}{dN}E[W(\mu + s(N,\gamma), \Gamma(N,\gamma))]$ is a $O(\gamma_x)$, so that for γ_x low enough f_c^{eff} is strictly increasing in μ and γ with derivatives that can be bounded away from 0;

- In a second step, show that when $\sigma^f \to 0$, then f_c^{eff,σ^f} converges uniformly to some limit $f_c^{eff,0}$ that is bi-Lipschitz continuous, strictly increasing in μ and γ with derivatives bounded away from 0. Thus, we have the pointwise limit:

$$\forall(\mu,\gamma), \qquad N^{eff,\sigma^f}(\mu,\gamma) \to N^{eff,0}(\mu,\gamma) = \bar{N} \cdot \mathbb{1}\left(\bar{f} \leq f_c^{eff,0}(\mu,\gamma)\right);$$

- Conclude identically to proposition 3 that for σ^f sufficiently small there are at least two locally stable steady-states in the dynamics of γ.

□