

The Role of Default and Asset-Backed Securities for Sharing Idiosyncratic Risk

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Abstract

I develop a highly tractable, heterogeneous agent, incomplete markets dynamic stochastic general equilibrium model with collateralized lending and asset-backed securities. Agents who face aggregate and idiosyncratic investment risk can borrow by putting up their investments as collateral. Borrowers can default at any time, with the only penalty being the confiscation of collateral. The debt contracts are pooled and issued as asset-backed securities. Under a natural loan-to-value (LTV) constraint I prove the existence of equilibrium. The equilibrium allocation weakly Pareto dominates the autarky allocation, but strictly Pareto dominates only if there is active default. Assets in zero net supply or tranching are irrelevant for risk sharing when agents have a common relative risk aversion coefficient. In a numerical example with risk aversion 2, expected investment return 8%, and volatility 15%, as the collateral requirement loosens the welfare approaches that in complete markets, the borrowing rate sharply increases from -1% to 20% , while the risk-free rate stays below 2% . Therefore the model predicts realistic equity premium and risk-free rate.

Keywords: collateral equilibrium; costly state verification; efficiency; incomplete markets; power law; securitization.

JEL codes: C68, D31, D52, D53, D58, G11, G21.

1 Introduction

This paper studies the positive role of default and asset-backed securities created by pooling debt contracts for sharing idiosyncratic risk. Writing perfect insurance requires observing individual characteristics (to avoid adverse selection), individual actions (to avoid moral hazard), and individual states (to write contracts in the first place). Violation of any one condition leads to imperfect insurance, and therefore it should not be surprising that in the real world markets are incomplete and households are subject to uninsured idiosyncratic risk.¹ The fact that many insurance payments are contingent on easily verifiable states (auto

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¹One evidence of market incompleteness is the fact that household consumption is not perfectly correlated with aggregate consumption.

accident, death, fire, etc.) suggests that costly individual state verification—a particular form of asymmetric information—is an important source of market incompleteness. In this paper I show that collateralized lending serves as an insurance mechanism that does not require observing individual states.

In the model economy, agents can invest their wealth in either investment projects that are subject to aggregate and idiosyncratic risk or publicly traded assets. Due to costly individual state verification, insurance contracts on private investment projects cannot be written. In order to keep tractability and to make sharp predictions about to what extent risk sharing is possible under various financial structures, I assume that agents have a common relative risk aversion coefficient and have access to common technologies, but they may differ in patience, elasticity of intertemporal substitution, or initial wealth. In this setting I first prove a negative result: the equilibrium allocation with additional assets in zero net supply or with assets created by repackaging existing public investment projects (such as tranching) is the same as the allocation under autarky, and is constrained efficient (despite being Pareto inefficient). Therefore the government cannot improve upon the autarky allocation either by intervening in the existing asset markets or by creating such assets: in order to improve welfare over autarky, it is necessary to create assets that are in *nonzero* net supply which are *not* a repackaging of existing publicly traded assets.

As such assets I consider asset-backed securities (ABS)² created by collateralized lending: households borrow putting up their investment projects as collateral, and purchase securities created by pooling such debt contracts. When the collateral requirement is sufficiently tight (more precisely, the value of collateral must be at least as large as the amount of loan at the time of borrowing, which imposes a natural loan-to-value (LTV) constraint), an equilibrium exists. The equilibrium allocation weakly Pareto dominates the allocation without asset-backed securities, but strictly Pareto dominates only if there is active default.

To study the effect of market incompleteness, the macroeconomics literature on heterogeneous agent models has mostly considered models with idiosyncratic labor income risk (known as *Bewley models*).³ This paper, on the other hand, focuses on *entrepreneurial risk* or *capital income risk*, for three reasons. The first is its empirical relevance. For example, Bertaut and Starr-McCluer (2002) find that ownership of principal residence and private business equity account for 28.2% and 27% of household wealth, and Moskowitz and Vissing-Jørgensen (2002) find that 75% of all private equity is owned by households for which it constitutes at least 50% of their total net worth. The second is that uninsurable labor income risk cannot account for the power law behavior of the wealth distribution. As analytically shown in Benhabib et al. (2011), because labor income accumulates additively into wealth, its effect is dominated by the multiplicative process of wealth accumulation by capital income. The last reason is analytical tractability: as shown in Toda (2012c), in the presence of multiplicative shocks the single agent problem with homothetic preferences admits a closed-form solution even with many assets with arbitrary distributions of returns, which greatly simplifies the equilibrium analysis.

The rest of the paper is organized as follows. After a brief discussion of related literature, Section 2 treats the individual decision problem and the asset

²See Elul (2005) for an overview of ABS.

³See, for example, Bewley (1983, 1986), Huggett (1993, 1997), Aiyagari (1994), and Krusell and Smith (1998), among many others. See Krusell and Smith (2006) for a review.

pricing implications. Section 3 solves for the general equilibrium with complete and incomplete markets as benchmark economies. Section 4 introduces asset-backed securities in the benchmark incomplete market economy and proves existence of equilibrium and welfare properties. Section 5 discusses numerical examples.

1.1 Literature

In a seminal paper, Dubey et al. (2005) (the first draft of which circulated in 1988) introduced default and punishment in general equilibrium by thinking of assets as pools of promises. They find that despite the deadweight loss from punishment, default may be welfare-improving because it allows agents to tailor the security payoff according to their needs, and with many agents such tailoring greatly increases the dimension of the asset span. In a similar paper Zame (1993) notes the positive role of default by showing that when there are infinite states of nature, the equilibrium allocation does not necessarily approach that of complete markets by simply opening new markets without allowing default.

Kehoe and Levine (1993), Kocherlakota (1996), and Alvarez and Jermann (2000) study constrained efficient allocations in dynamic endowment economies where agents insure against idiosyncratic endowment shocks subject to participation constraints. The default penalty is exclusion from financial markets or confiscation of assets as in Chien and Lustig (2010). These papers are in contrast with Dubey et al. (2005) and Zame (1993) in that they assume complete markets (hence there is no cost in verifying individual states) and rule out default in equilibrium.

Geanakoplos and Zame (1995) and Geanakoplos (1997, 2003) introduced the concept of collateral equilibrium. An asset is viewed as a pair of a promise and a collateral, with actual delivery dependent on equilibrium default. In collateral equilibrium, default entails no penalty and hence no deadweight loss. Agents keep their promises if and only if the value of their collateral is higher than the value of their promises. In Araújo et al. (2000) and Araújo et al. (2005) borrowers can set their own collateral levels. Araújo et al. (2002) show that collateral imposes a natural borrowing constraint and we can dispense with the no Ponzi scheme condition in economies with infinite horizon. Steinert and Torres-Martínez (2007) introduce security pools and tranching into the basic two period collateral equilibrium model of Geanakoplos and Zame (1995) with extra-economic penalties as in Dubey et al. (2005). Due to complex budget and collateral constraints, computing the collateral equilibrium is a daunting task. Kubler and Schmedders (2003) prove the existence of a Markov collateral equilibrium in an infinite horizon endowment economy with symmetric information, provide a numerical algorithm for computing the equilibrium, and apply to a model with two agents. Fostel and Geanakoplos (2008, 2012) study the asset pricing implications of collateral equilibrium under heterogeneous beliefs. This paper is closest to Kubler and Schmedders (2003) and Steinert and Torres-Martínez (2007) in that it deals with an infinite horizon model with a focus on application (*i.e.*, not only proving existence of equilibrium but also computing the equilibrium), with the main contribution being the analytical tractability and the introduction of security pools which becomes meaningful due to asymmetric information. If we replace the endowment shocks in Kubler and Schmedders (2003) with investment shocks, with symmetric information (and hence no pooling) there will

be no risk sharing according to the no trade result (Theorem 3.3). Therefore asymmetric information and pooling are crucial for obtaining interesting results.

This paper is also related to the literature on tractable heterogeneous agent general equilibrium models with incomplete markets, which includes the following. In the little-known but pioneering work, Saito (1998) builds a continuous-time model with two linear technologies, one hit by aggregate shocks and the other by idiosyncratic shocks, and studies the implications to the equity premium and the risk-free rate. Calvet (2001) and Angeletos and Calvet (2005, 2006) use CARA-normal specifications to study the macroeconomic dynamics under idiosyncratic capital and labor income risk. Calvet (2001) shows that, somewhat surprisingly, even without aggregate shocks the system can exhibit a complicated dynamics (chaotic behavior). Krebs (2003a,b, 2006) studies the effect of idiosyncratic human capital risk on economic growth and business cycles when there is a firm employing physical and human capital. Angeletos (2007) studies the macroeconomic dynamics with constant relative risk aversion/constant elasticity of intertemporal substitution (CRRA/CEIS) preferences and entrepreneurial risk, which Angeletos and Panousi (2009, 2011) and Panousi (2010) apply to the analysis of fiscal policies and the global financial integration. Toda (2012c) studies the asset pricing implications and the wealth distribution under CRRA/CEIS preferences and idiosyncratic as well as aggregate risk with many investment projects, and shows that the ‘double power law’ (the power law holds not only in the upper tail but also in the lower tail) behavior of consumption and wealth distribution is a robust property.⁴

2 Individual decision and asset pricing

In this section I solve the individual decision problem of optimal consumption and portfolio, similar to the one in the classic paper of Samuelson (1969), and study the asset pricing implications. The additional aspects are that (i) I work with Kreps and Porteus (1978)-Epstein and Zin (1989) constant relative risk aversion (CRRA), constant elasticity of intertemporal substitution (CEIS) recursive utility, which generalizes the additive CRRA utility, (ii) I allow for an arbitrary number of assets with arbitrary return distributions (but i.i.d. overtime), (iii) I allow for arbitrary constraints on relative positions (such as margin requirements, but not absolute borrowing constraints), and (iv) I obtain an explicit formula for the optimal consumption/portfolio rule.

2.1 Individual decision

Given a consumption plan $\{c_t\}_{t=0}^{\infty}$,⁵ I define the one period utility (at time t) by $V_1(c_t) = c_t$, and given the T period utility $V_T(\{c_{t+s}\}_{s=0}^{T-1})$ I define the $T + 1$ period utility recursively by

$$V_{T+1}(\{c_{t+s}\}_{s=0}^T) = \left(c_t^{1-\sigma} + \beta \mathbb{E}_t[V_T(\{c_{t+1+s}\}_{s=0}^{T-1})^{1-\gamma}]^{\frac{1-\sigma}{1-\gamma}} \right)^{\frac{1}{1-\sigma}}, \quad (2.1)$$

⁴Toda and Walsh (2011) finds that the U.S. cross-sectional consumption distribution obeys the double power law, and more strongly, the double Pareto-lognormal distribution (Reed, 2001, 2003; Reed and Jorgensen, 2004). Toda (2011, 2012a) also finds the double power law in income distribution.

⁵Formally, a consumption plan is a nonnegative stochastic process on a probability space (Ω, \mathcal{F}, P) adapted to a filtration $\{\mathcal{F}_t\}_{t=0}^{\infty}$, but for ease of exposition I omit technical details.

where E_t denotes the expectation conditional on time t information, $1/\sigma > 0$ is the elasticity of intertemporal substitution, $\gamma > 0$ is the coefficient of relative risk aversion, and $\beta > 0$ is the subjective discount factor.⁶ If $\sigma = 1$ or $\gamma = 1$ we must interpret (2.1) using the logarithm and the exponential in the usual way, but for simplicity I shall ignore those cases henceforth. I define the recursive utility of the consumption stream from time t on by

$$V(\{c_{t+s}\}_{s=0}^{\infty}) = \lim_{T \rightarrow \infty} V_T(\{c_{t+s}\}_{s=0}^{T-1}). \quad (2.2)$$

This quantity exists in $[0, \infty]$ because we can easily see from (2.1) that the sequence $\left\{V_T(\{c_{t+s}\}_{s=0}^{T-1})\right\}_{T=1}^{\infty}$ is increasing in T . If $\sigma = \gamma$, the recursive utility defined by (2.1) and (2.2) reduces to $\left(E_t \sum_{s=0}^{\infty} \beta^s c_{t+s}^{1-\gamma}\right)^{\frac{1}{1-\gamma}}$, which is ordinarily equivalent to the standard additive CRRA utility $E_t \sum_{s=0}^{\infty} \beta^s \frac{c_{t+s}^{1-\gamma}}{1-\gamma}$.

There are J assets indexed by $j = 1, 2, \dots, J$. Let $\mathbf{R}_{t+1} = (R_{t+1}^1, \dots, R_{t+1}^J)$ be the vector of asset returns between periods t and $t+1$, assumed to be i.i.d. over time.⁷ Let $\theta = (\theta^1, \dots, \theta^J)$ be the portfolio shares of the consumer and $R_{t+1}(\theta) = \sum_j R_{t+1}^j \theta^j$ be the return on the portfolio. By definition, $\sum_j \theta^j = 1$. Of course, the consumer is short in asset j if $\theta^j < 0$ and long if $\theta^j > 0$. The consumer may or may not be constrained in the position he takes, but I assume that he can only be constrained in the relative size of the position (*i.e.*, the portfolio shares θ , such as margin requirements) and not in the absolute size. Let $\Theta \subset \mathbb{R}^J$ be the constraint set, assumed to be nonempty, compact, and convex. These assumptions are natural: the compactness of Θ implies limits to shortsales and the convexity of Θ implies that if two portfolios are feasible, so is a mixture of them.

Given initial wealth w_0 , the consumer's objective is to maximize the recursive utility (2.2) subject to the sequential budget constraints

$$(\forall t) w_{t+1} = R_{t+1}(\theta_t)(w_t - c_t) \geq 0 \quad (2.3)$$

and the portfolio constraint $\theta_t \in \Theta$. The following theorem shows the existence and essential uniqueness of the consumer's optimal consumption/portfolio rule.

Theorem 2.1. *Suppose that (i) the portfolio constraint $\Theta \subset \mathbb{R}^J$ is nonempty, compact, and convex, (ii) asset returns $\{\mathbf{R}_{t+1}\}_{t=0}^{\infty}$ are i.i.d. over time, and (iii) $E[\sup_{\theta \in \Theta} R(\theta)^{1-\gamma}] < \infty$. Then there exists an optimal consumption/portfolio rule if and only if*

$$\beta \left(\sup_{\theta \in \Theta} E[R(\theta)^{1-\gamma}]^{\frac{1}{1-\gamma}} \right)^{1-\sigma} < 1, \quad (2.4)$$

in which case $\theta^* \in \Theta$ is an optimal portfolio if and only if

$$\theta^* \in \arg \max_{\theta \in \Theta} E[R(\theta)^{1-\gamma}]^{\frac{1}{1-\gamma}}, \quad (2.5)$$

⁶Following Kocherlakota (1990) I do not restrict the analysis to discount factors smaller than 1. In fact, β can even be random. If the one period discount factor is stochastic, then the subsequent discussion goes through with almost no modifications except that β should be put inside the expectation in (2.1).

⁷The non-i.i.d. case can be treated as in Toda (2012c), but doing so makes the exposition more technical without much gain.

the optimal portfolio return $R(\theta^*)$ is unique almost surely, and the optimal consumption rule and the value function are given by

$$c(w) = \left(1 - \beta^{\frac{1}{\sigma}} \mathbb{E}[R(\theta^*)^{1-\gamma}]^{\frac{1-\sigma}{\sigma(1-\gamma)}}\right) w, \quad (2.6a)$$

$$V(w) = \left(1 - \beta^{\frac{1}{\sigma}} \mathbb{E}[R(\theta^*)^{1-\gamma}]^{\frac{1-\sigma}{\sigma(1-\gamma)}}\right)^{\frac{\sigma}{\sigma-1}} w. \quad (2.6b)$$

Furthermore, the optimal portfolio $\theta^* \in \Theta$ is unique if and only if there are no redundant assets, i.e., asset returns R^1, \dots, R^J are linearly independent.

Proof. See Appendix A.1. \square

The following comparative statics conform to intuition.

Proposition 2.2. *The saving rate (out of wealth) $\beta^{\frac{1}{\sigma}} \mathbb{E}[R(\theta^*)^{1-\gamma}]^{\frac{1-\sigma}{\sigma(1-\gamma)}}$ is higher with higher discount factor β (more patience). If the elasticity of intertemporal substitution $1/\sigma$ is smaller (larger) than 1, the saving rate is higher (lower) with higher risk.*

Proof. The first claim is trivial. To see that there is a precautionary saving motive when consumption is intertemporally inelastic, let \mathbf{R}' be a mean-preserving spread of asset returns \mathbf{R} (more precisely, $\mathbb{E}[\mathbf{R}' | \mathbf{R}] = \mathbf{R}$). Since $R^{1-\gamma}$ is convex (concave) in R when $\gamma > 1$ ($\gamma < 1$), conditioning on \mathbf{R} and invoking Jensen's inequality we obtain

$$\mathbb{E}[R(\theta)^{1-\gamma}]^{\frac{1}{1-\gamma}} < \mathbb{E}[R'(\theta)^{1-\gamma}]^{\frac{1}{1-\gamma}}.$$

Therefore for $\sigma > 1$ we obtain

$$\beta^{\frac{1}{\sigma}} \mathbb{E}[R(\theta^*)^{1-\gamma}]^{\frac{1-\sigma}{\sigma(1-\gamma)}} > \beta^{\frac{1}{\sigma}} \mathbb{E}[R'(\theta^*)^{1-\gamma}]^{\frac{1-\sigma}{\sigma(1-\gamma)}},$$

so the saving rate is higher with higher risk when $1/\sigma < 1$. \square

It is not surprising that the precautionary saving motive depends on the elasticity of intertemporal substitution. When the elasticity $1/\sigma$ is high, agents can tolerate low consumption while waiting for a big positive shocks. Hence more risk lowers the necessity of precautionary saving.

2.2 Asset pricing

Let $\theta^* \in \Theta$ be an optimal portfolio and $\phi \in \mathbb{R}^J$ be any portfolio (not necessarily $\phi \in \Theta$). What restrictions does the optimality of θ^* impose on $R(\phi)$, the return on the portfolio ϕ ? To answer this question let

$$A_\phi = \{\alpha \in \mathbb{R} \mid (1 - \alpha)\theta^* + \alpha\phi \in \Theta\}$$

be the set of feasible portfolio weights on the optimal portfolio θ^* and the portfolio ϕ . Since $\theta^* \in \Theta$ implies $0 \in A_\phi$, we have $A_\phi \neq \emptyset$. Since A_ϕ is homeomorphic to the intersection of Θ (a compact convex set) and a straight line, it is a compact interval. Let $A_\phi = [\underline{\alpha}_\phi, \bar{\alpha}_\phi]$ with $\underline{\alpha}_\phi \leq 0 \leq \bar{\alpha}_\phi$.

The following proposition summarizes the asset pricing implications of the CRRA/CEIS model.

Proposition 2.3. *Let everything be as in Theorem 2.1 and $\phi \in \mathbb{R}^J$, $\sum_j \phi^j = 1$ be any portfolio. If A_ϕ is not a single point, then*

$$\mathbb{E}[R(\theta^*)^{-\gamma}(R(\phi) - R(\theta^*))] \begin{cases} \leq 0, & (\underline{\alpha}_\phi = 0) \\ = 0, & (\underline{\alpha}_\phi < 0 < \bar{\alpha}_\phi) \\ \geq 0. & (\bar{\alpha}_\phi = 0) \end{cases} \quad (2.7)$$

If the risk-free asset is traded and unconstrained, the one period gross risk-free rate R_f is given by

$$R_f = \frac{\mathbb{E}[R(\theta^*)^{1-\gamma}]}{\mathbb{E}[R(\theta^*)^{-\gamma}]}, \quad (2.8)$$

and the risk premium of the portfolio ϕ is given by

$$\mathbb{E}[R(\phi)] - R_f = -\frac{\text{Cov}[R(\theta^*)^{-\gamma}, R(\phi)]}{\mathbb{E}[R(\theta^*)^{-\gamma}]} \quad (2.9)$$

provided that 0 is an interior point of A_ϕ . Furthermore, the risk premium on the optimal portfolio $\mathbb{E}[R(\theta^)] - R_f$ is positive.*

Proof. Since $\phi \in \mathbb{R}^J$ is fixed let us drop the subscript ϕ . For $\alpha \in A$ consider the return $(1 - \alpha)R(\theta^*) + \alpha R(\phi)$, which can be attained by investing the fraction of wealth $1 - \alpha$ in the optimal portfolio θ^* and α in portfolio ϕ . By (2.5) we obtain

$$0 \in \arg \max_{\alpha \in A} \mathbb{E}[(1 - \alpha)R(\theta^*) + \alpha R(\phi)]^{1-\gamma}.$$

(2.7) immediately follows by the first-order condition with respect to α at $\alpha = 0$.

Choosing ϕ to be the portfolio consisting entirely of the risk-free asset and applying (2.7), we obtain (2.8). To show (2.9), rewrite (2.7) as

$$\mathbb{E}[R(\theta^*)^{-\gamma}R(\phi)] = \mathbb{E}[R(\theta^*)^{1-\gamma}].$$

Using $\mathbb{E}[XY] = \text{Cov}[X, Y] + \mathbb{E}[X]\mathbb{E}[Y]$ for $X = R(\theta^*)^{-\gamma}$ and $Y = R(\phi)$, we obtain

$$\text{Cov}[R(\theta^*)^{-\gamma}, R(\phi)] + \mathbb{E}[R(\theta^*)^{-\gamma}]\mathbb{E}[R(\phi)] = \mathbb{E}[R(\theta^*)^{1-\gamma}].$$

Dividing both sides by $\mathbb{E}[R(\theta^*)^{-\gamma}]$ and invoking (2.8), we obtain (2.9).

If the risk premium of the optimal portfolio θ^* was nonpositive, since $R^{1-\gamma}$ is convex (concave) when $\gamma > 1$ ($\gamma < 1$), by Jensen's inequality we obtain

$$\mathbb{E}[R(\theta^*)^{1-\gamma}]^{\frac{1}{1-\gamma}} < \mathbb{E}[R(\theta^*)] \leq R_f = \mathbb{E}[R_f^{1-\gamma}]^{\frac{1}{1-\gamma}},$$

which contradicts (2.5). Therefore $\mathbb{E}[R(\theta^*)] - R_f > 0$. \square

Remark. Since all we need to prove Proposition 2.3 is (2.5), according to the proof of Theorem 2.1, Proposition 2.3 holds for finite period economies as well.

3 Benchmark economies

Having solved the single agent problem in Section 2, in this section I solve for the general equilibrium when the economy is populated by a continuum of agents under two extreme asset structures, with perfect risk sharing or with no risk sharing at all. These extreme cases serve as benchmarks to compare the case with some risk sharing, which is treated in Section 4. Because the portfolio choice depends only on returns and relative risk aversion, we can easily characterize the equilibrium analytically as long as agents have a common relative risk aversion coefficient, even if they differ in patience, elasticity of intertemporal substitution, and initial wealth.

3.1 Description of the economy and definition of equilibrium

The economy I consider in this section is similar to that of Section 2 except the agent characteristics and the asset structure. There are a continuum of agents with mass 1, indexed by $i \in I = [0, 1]$. Agent i has a recursive preference defined by (2.1) and (2.2) with discount factor β_i , coefficient of relative risk aversion γ_i , elasticity of intertemporal substitution $1/\sigma_i$, and is endowed with initial wealth w_{i0} . I assume that agents have a common relative risk aversion coefficient, so $\gamma_i = \gamma$ for all i . Therefore agent i is completely characterized by the vector $(\beta_i, \gamma, \sigma_i, w_{i0})$.

“Assets” in Section 2 are now in one of the following two categories: constant-returns-to-scale investment projects and publicly traded assets that are in zero net supply, such as risk-free bonds, Arrow securities, or financial derivatives. I interpret an investment project as a linear stochastic saving technology (hence the supply of “assets” is not fixed but automatically meets the demand): if agent i invests a unit of good in project j ($j = 1, \dots, J$) at the end of time t , he will receive $R_{i,t+1}^j$ at the beginning of time $t + 1$, where $R_{i,t+1}^j$ is the return on investment j . On the other hand, one share of asset k ($k = 1, \dots, K$) delivers D_{t+1}^k with no default (which does not depend on the owner of the asset).

The return of each investment project has an aggregate component R_a^j and an idiosyncratic component r_i^j , so $R_i^j = r_i^j R_a^j$,⁸ where I have dropped the time subscript. Of course assuming multiplicative investment shocks as the only idiosyncratic shocks (thus ignoring additive labor income shocks) is an oversimplification, but this modeling choice can be justified as follows. First, the time series of consumption inequality (measured by variance) for each cohort increases within a cohort almost linearly between the ages 20 and 80, as shown by Deaton and Paxson (1994) using U.S. household data from 1980 to 1990. Consumption inequality within a cohort for such a long period (60 years) is hard to explain by transitory idiosyncratic shocks, while it derives naturally from multiplicative (permanent) shocks. Second, the widely observed power law behavior of the wealth distribution cannot be explained by additive shocks (Benhabib et al., 2011) but can be explained by multiplicative shocks (Toda, 2012b). This is because as an agent gets richer, additive labor income shocks become negligible compared to wealth, so there is no longer an incentive to accumulate even more wealth. Third, in the presence of multiplicative shocks, the

⁸Such a multiplicative decomposition is always possible: apply the result of Al-Najjar (1995) to $\log R_i^j$.

single agent problem with homothetic preferences admits a closed-form solution, which greatly simplifies the equilibrium analysis.

Finally, the distributional assumptions are as follows. Let $\mathbf{R}_a = (R_a^1, \dots, R_a^J)$ and $\mathbf{r}_i = (r_i^1, \dots, r_i^J)$ be the vectors of aggregate and idiosyncratic components of investment returns and let $\mathbf{D} = (D^1, \dots, D^K)$ be the vector of asset deliveries. The aggregate shocks $\{(\mathbf{R}_{a,t+1}, \mathbf{D}_{t+1})\}_{t=0}^\infty$ are i.i.d. over time. The idiosyncratic shock $\mathbf{r}_{i,t+1}$ is Markovian in the sense that its distribution conditional on past and contemporaneous aggregate shocks depends only on the contemporaneous aggregate shock $(\mathbf{R}_{a,t+1}, \mathbf{D}_{t+1})$ with a time-invariant conditional distribution that satisfies

$$(\forall j = 1, \dots, J) \ E \left[r_i^j \mid \mathbf{R}_a, \mathbf{D} \right] = 1,$$

and is i.i.d. across individuals conditional aggregate shocks. The asset price P_t^k ($k = 1, \dots, K$) is to be determined in equilibrium and induces the asset return by $R_{t+1}^{J+k} = (P_{t+1}^k + D_{t+1}^k)/P_t^k$. We denote the portfolio share (relative position) in investments and asset holdings by a vector $(\theta, \phi) \in \mathbb{R}^J \times \mathbb{R}^K$. A consumer's portfolio share is constrained to be in the set $\Omega \subset \mathbb{R}^{J+K}$, which can be interpreted as a constraint on leverage or other institutional constraints (limits on shortsales, restrictions on access to certain capital markets, etc.). Letting

$$R_{t+1}(\theta_t, \phi_t) = \sum_{j=1}^J R_{t+1}^j \theta_t^j + \sum_{k=1}^K R_{t+1}^{J+k} \phi_t^k$$

be the return on portfolio of investments and assets (where I have suppressed the individual subscript), each consumer faces the budget constraint

$$w_{t+1} = R_{t+1}(\theta_t, \phi_t)(w_t - c_t) \geq 0, \quad (3.1)$$

which is similar to (2.3) in Section 2.

3.2 General equilibrium with incomplete markets

Let K be the set of assets in zero net supply. I define a sequential equilibrium as follows.

Definition 3.1 (Sequential GEI). $\{(c_{it}, w_{it}, \theta_{it}, \phi_{it})_{i \in I}, (P_t^k)_{k \in K}\}_{t=0}^\infty$ is a sequential general equilibrium with incomplete markets and heterogeneous agents if

1. given the asset returns $R_{t+1}^{J+k} = (P_{t+1}^k + D_{t+1}^k)/P_t^k$ for $k \in K$ individual consumption c_{it} and portfolio (θ_{it}, ϕ_{it}) are optimal subject to the budget constraint (3.1) and the portfolio constraint Ω ,
2. the markets for assets in zero net supply clears, *i.e.*, $\int_I \phi_{it}^k (w_{it} - c_{it}) di = 0$ for all $k \in K$,
3. individual wealth w_{it} evolves according to the budget constraint (3.1).

Proposition 3.2 below shows that the equilibrium is essentially unique and that we only need to look for a symmetric equilibrium (common portfolio choice) with no trade in assets in zero net supply.

Proposition 3.2. *If $\{(c_{it}, w_{it}, \theta_{it}, \phi_{it})_{i \in I}, (P_t^k)_{k \in K}\}_{t=0}^\infty$ is a sequential equilibrium, we can obtain a symmetric equilibrium by replacing θ_{it} with the (common) average portfolio $\theta_t = \int_I \theta_{it}(w_{it} - c_{it})di / \int_I (w_{it} - c_{it})di$ weighted by savings and setting $\phi_{it} = 0$ (i.e., no trade in assets). Furthermore, letting $\Theta = \{\theta \in \mathbb{R}^J \mid (\theta, 0) \in \Omega\}$ be the portfolio constraint for investment with holdings in assets in zero net supply restricted to be zero, we have $\Theta \neq \emptyset$,*

$$\theta_t \in \arg \max_{\theta \in \Theta} \mathbb{E}[R(\theta, 0)^{1-\gamma}]^{\frac{1}{1-\gamma}}, \quad (3.2a)$$

$$c_{it} = c_i(w_{it}) := \left(1 - \beta_i^{\frac{1}{\sigma}} \mathbb{E}[R(\theta_t, 0)^{1-\gamma}]^{\frac{1-\sigma_i}{\sigma_i(1-\gamma)}}\right) w_{it}. \quad (3.2b)$$

In particular, for a given realization of shocks, $(c_{it}, w_{it})_{t=0}^\infty$ is common across all equilibria.

Proof. Given asset prices $\{(P_t^k)_{k \in K}\}_{t=0}^\infty$ and the induced asset returns $R_{t+1}^{J+k} = (P_{t+1}^k + D_{t+1}^k)/P_t^k$, each individual faces the budget constraint (3.1). Since (θ_{it}, ϕ_{it}) is an optimal portfolio, it belongs to

$$\arg \max_{(\theta, \phi) \in \Omega} \mathbb{E}[R(\theta, \phi)^{1-\gamma}]^{\frac{1}{1-\gamma}}. \quad (3.3)$$

Since Ω is convex and $\mathbb{E}[R(\theta, \phi)^{1-\gamma}]^{\frac{1}{1-\gamma}}$ is quasi-concave in (θ, ϕ) , the weighted average portfolio

$$\begin{bmatrix} \theta_t \\ \phi_t \end{bmatrix} := \int_I \begin{bmatrix} \theta_{it} \\ \phi_{it} \end{bmatrix} (w_{it} - c_{it})di / \int_I (w_{it} - c_{it})di$$

also belongs to (3.3). Hence by Theorem 2.1, (θ_t, ϕ_t) is an optimal portfolio and we have $R(\theta_{it}, \phi_{it}) = R(\theta_t, \phi_t)$ almost surely. Then by market clearing and the definition of ϕ_t , we get

$$0 = \int_I \phi_{it}(w_{it} - c_{it})di = \phi_t \int_I (w_{it} - c_{it})di,$$

so $\phi_t = 0$ and hence $\theta_t \in \Theta \neq \emptyset$. (3.2b) follows from (2.6) and $\phi_t = 0$. Since $(\theta_t, 0)$ belongs to the set in (3.3), θ_t must also satisfy (3.2a).

Since on Θ the return on portfolio $R(\theta, 0)$ does not depend on asset prices $\{(P_t^k)_{k \in K}\}_{t=0}^\infty$, (3.2b) is also independent of asset prices. Therefore (c_{it}, w_{it}) is common across all equilibria. \square

The following theorem shows that an essentially unique sequential equilibrium⁹ exists, it can be computed, and that it is constrained efficient.

Theorem 3.3. *Let everything be as in Theorem 2.1 except that Θ is interpreted as $\Theta = \{\theta \in \mathbb{R}^J \mid (\theta, 0) \in \Omega\}$ (the portfolio constraint with holdings in assets in zero net supply restricted to be zero), $R(\theta)$ is replaced with $R(\theta, 0)$, and condition (2.4) holds for all i with (β, σ) replaced with (β_i, σ_i) . Then an essentially unique sequential equilibrium (i) can be constructed such that the individual consumption/portfolio rule is given by (3.2) and the asset prices are given by Proposition 2.3, and (ii) is constrained efficient.*

⁹The sequential equilibrium in Theorem 3.3 has a recursive structure and hence can be described as a recursive competitive equilibrium. See Krebs (2006) for details.

Proof. Let $c_i(w)$ and θ^* be the optimal consumption/portfolio rule given by (3.2). These rules exist and are essentially unique because by assumption (2.4) holds with $R(\theta)$ replaced by $R(\theta, 0)$. For $k \in K$, define the price of asset k such that the induced returns satisfy (2.7). Then by construction the first-order condition for the maximization (2.5) (with the constraint Ω) holds for every asset $k \in K$. By the definition of θ^* , the first-order condition for the maximization (2.5) holds for every investment project $j \in J$. Hence the first-order condition holds for every return $j \in J$. Since the first-order condition is sufficient for maximum because the objective function in (2.5) is quasi-concave, the consumption/portfolio rule $c_i(w)$ and θ^* are optimal even without the restriction $\phi = 0$. Since the individual holdings in asset k is zero by construction, the markets of assets in zero net supply clear. Therefore we obtain a sequential equilibrium.

The definition of constrained efficiency is that we cannot make everybody at least as well off and somebody better off by changing the asset holdings alone (Geanakoplos and Polemarchakis, 1986), which implies that individual budget constraints should be satisfied and the consumption decision is left to the agents after assigning the portfolio. Since by Theorem 2.1 θ^* solves the individual optimization problem regardless of other agents' portfolio choice, we cannot make anybody better off by changing only asset holdings. Hence the equilibrium is constrained efficient. \square

Remark. As is clear from the proof, since the portfolio choice depends only on relative risk aversion and there is no trade in assets in zero net supply in the symmetric equilibrium, the discount factor and the elasticity of intertemporal substitution play no essential role for characterizing the equilibrium. (They only determine individual consumption.) Therefore Theorem 3.3 can be generalized in a number of ways. For instance, the discount factor and the elasticity of intertemporal substitution can be time-dependent (but deterministic) and agents can have individual-specific and age-dependent death probability. In particular, overlapping generations can be studied in the same framework. These claims can be verified by slightly modifying Lemma A.1.

Theorem 3.3 gives us the first negative result: if an economy is populated by agents with a common relative risk aversion who face idiosyncratic investment shocks, introducing assets in zero net supply leaves the equilibrium allocation unchanged from the case with no such assets and is constrained efficient. Hence assets in zero net supply are of no use for risk sharing even if the government can intervene in the asset markets. Proposition 3.4 below gives the second negative result.

Proposition 3.4. *Suppose that there are investment projects with no idiosyncratic component (“public investment projects”) and there is a financial intermediary that sells a finite number of securities, invests the proceeds in public investment projects, and repackages the investment returns as dividends to the securities issued. Then the equilibrium allocation is the same as autarky.*

Proof. Let $l = 1, \dots, L$ be the securities the financial intermediary issues and let $(\theta, \phi, \psi) \in \Omega \times \mathbb{R}^L$ be the portfolio of an individual, including the new securities. If there exists an equilibrium, by the same argument as in Proposition 3.2, there is a symmetric equilibrium. Since the securities portfolio $\psi \in \mathbb{R}^L$ is common across individuals in the symmetric equilibrium and the dividends to the securities are a repackaging of public investment projects, holding ψ

is equivalent to directly investing in the public investment projects, which is individually feasible without the financial intermediary. \square

3.3 Consumption and wealth distributions

What does the cross-sectional consumption and wealth distribution look like in equilibrium? Since by Theorem 2.1 consumption is proportional to wealth, we only need to look at the wealth distribution. Although the wealth distribution is not a relevant state variable for describing the recursive equilibrium, characterizing the wealth distribution is interesting in its own right since it has drawn the attention of many researchers since the time of Pareto (1896).

The answer crucially depends on whether consumers are infinitely lived or not. Suppose that agents have a common preference with discount factor β , relative risk aversion γ , elasticity of intertemporal substitution $1/\sigma$, and they die with probability $\delta \in [0, 1]$ between two consecutive periods. I assume that when an agent die, he is reborn with some initial wealth. Before stating the result, I introduce some definitions.

The *Laplace* distribution has a density

$$f_L(x) = \begin{cases} \frac{\alpha\beta}{\alpha+\beta} e^{-\alpha|x-m|}, & (x \geq m) \\ \frac{\alpha\beta}{\alpha+\beta} e^{-\beta|x-m|}, & (x < m) \end{cases}$$

where m is the mode and $\alpha, \beta > 0$ are scale parameters.¹⁰ It is called *symmetric* if $\alpha = \beta$. If X is Laplace distributed, then $Y = \exp(X)$ is said to be *double Pareto* distributed.¹¹ The density of the double Pareto distribution is

$$f_{dP}(x) = \begin{cases} \frac{\alpha\beta}{\alpha+\beta} \frac{1}{M} \left(\frac{x}{M}\right)^{-\alpha-1}, & (x \geq M) \\ \frac{\alpha\beta}{\alpha+\beta} \frac{1}{M} \left(\frac{x}{M}\right)^{\beta-1}, & (0 \leq x < M) \end{cases}$$

where $M = e^m > 0$ is the mode and $\alpha, \beta > 0$ are power law exponents. The (discrete) double Pareto distribution first appeared in Champernowne (1953) but the above definition is due to Reed (2001).¹² If X_1, X_2 are independent normal and Laplace random variables, their convolution $Y = X_1 + X_2$ is said to be *normal-Laplace* distributed and e^Y *double Pareto-lognormal* (Reed, 2003; Reed and Jorgensen, 2004; Reed and Wu, 2008). See the appendix of Toda and Walsh (2011) for more information on the double Pareto-lognormal distribution.

Perhaps the most important property of the Laplace distribution is that it is the only limit distribution of geometric sums. Theorem 3.5 below, which is a counterpart of the Lindeberg-Feller central limit theorem for geometric sums, shows that it is a robust property that the limit of a geometric sum is a Laplace distribution.

Theorem 3.5. *Let $\{X_n\}_{n=1}^\infty$ be a sequence of zero mean random variables such that the central limit theorem holds, $N^{-1/2} \sum_{n=1}^N X_n \xrightarrow{d} \mathcal{N}(0, \sigma^2)$; $\{a_n\}_{n=1}^\infty$ be*

¹⁰See Kotz et al. (2001) for an exhaustive survey of the Laplace distribution.

¹¹Hence the Laplace and the double Pareto distributions have the same relation as the normal and the lognormal distributions.

¹²The double Pareto distribution has been shown to fit the conditional income distribution (Toda, 2011, 2012a).

a sequence such that $N^{-1} \sum_{n=1}^N a_n \rightarrow a$; and ν_p be a geometric random variable with mean $1/p$ independent from $\{X_n\}_{n=1}^{\infty}$. Then

$$p^{\frac{1}{2}} \sum_{n=1}^{\nu_p} (X_n + p^{\frac{1}{2}} a_n) \xrightarrow{d} \mathcal{AL}(0, a, \sigma)$$

as $p \rightarrow 0$.

Proof. See Toda (2012b). □

Theorem 3.6. *If agents are infinitely lived, the cross-sectional distribution of wealth relative to initial wealth is approximately lognormal. If agents die with positive probability $\delta > 0$ between periods, for small enough δ the cross-sectional distribution of wealth relative to initial wealth is approximately double Pareto.*

In particular, if the initial wealth distribution is lognormal, then the cross-sectional wealth distribution is approximately lognormal if agents are infinitely lived and double Pareto-lognormal if agents die.

Proof. First I prove the claim when agents die. Consider a consumer who starts with initial wealth w_{ini} . Let ν_δ be a geometric random variable with mean $1/\delta$ which describes the age of a consumer. Letting m be the (common) marginal propensity to consume and θ^* be an optimal portfolio, the consumer's wealth at the beginning of period T , w_T , is given by

$$\log \frac{w_T}{w_{\text{ini}}} = \sum_{t=1}^{\min\{\nu_\delta, T\}} \log[(1-m)R_{T-t+1}(\theta^*)] =: \sum_{t=1}^{\min\{\nu_\delta, T\}} X_{T-t}.$$

Conditioning on aggregate shocks, the random variables $\{X_t\}$ are independent across time and i.i.d. across individuals. Hence by Theorem 3.5, the random variable

$$\delta^{\frac{1}{2}} \left(\log \frac{w_T}{w_{\text{ini}}} - \mathbb{E}_T \left[\log \frac{w_T}{w_{\text{ini}}} \right] \right)$$

converges in distribution to a Laplace random variable as $T \rightarrow \infty$ and $\delta \rightarrow 0$. Therefore w_T/w_{ini} is approximately double Pareto distributed, which is the cross-sectional distribution of wealth relative to initial wealth if there are many agents. If the initial wealth distribution is lognormal, then the cross-sectional wealth distribution becomes double Pareto-lognormal (log wealth distribution becomes normal-Laplace).

For the infinitely lived case, replace ν_δ by ∞ and apply the Lindeberg-Feller central limit theorem. □

The argument for the infinitely lived case is identical to that of Gibrat (1931), who showed that multiplicative growth yields the lognormal distribution. To the best of my knowledge, Reed (2001) was the first to recognize that combining multiplicative growth and geometric age distribution (via constant birth/death probability or population growth) yields the double Pareto distribution. Proposition 3 of Gabaix (1999) contains similar results but does not explicitly mention the double Pareto distribution.¹³ However, the stochastic process in the models

¹³The double Pareto distribution is mentioned in the more recent review of Gabaix (2009), however.

of Reed and Gabaix is very special, namely geometric Brownian motion. As shown in this paper, the double Pareto property is robust in the sense that it depends only on multiplicative growth and a geometric age distribution and not on the details of the stochastic process governing growth. This mechanism might explain why there are so many *double power laws* (power laws in *both* the upper and lower tails and not just power laws in the upper tail¹⁴) in empirical data (Toda, 2011, 2012a; Toda and Walsh, 2011).

For example, Figure 1 shows the histogram of U.S. 1985 log real consumption together with the fitted normal-Laplace density using the data from the Consumer Expenditure Survey (CEX) prepared by Kocherlakota and Pistaferri (2009) and used in Toda and Walsh (2011), where we fitted the lognormal and the double Pareto-lognormal distributions to quarterly household consumption data in U.S. (fitted the normal and normal-Laplace distributions to log consumption data). There we found that (i) the lognormal distribution is rejected against the double Pareto-lognormal distribution in 97 out of 98 quarters by the likelihood ratio test at significance level 0.05, and (ii) the double Pareto-lognormal distribution is not rejected in 77 out of 96 quarters (79% of the time) by the Kolmogorov test at significance level 0.05. If we look at the data within age cohorts, similar to the case with no death the cross-sectional log consumption distribution should be approximately Gaussian, which is exactly what we observe (Battistin et al., 2007). These findings imply that the consumption distribution can be well explained by the present general equilibrium model with incomplete markets and heterogeneous agents, but only when we take death into account.

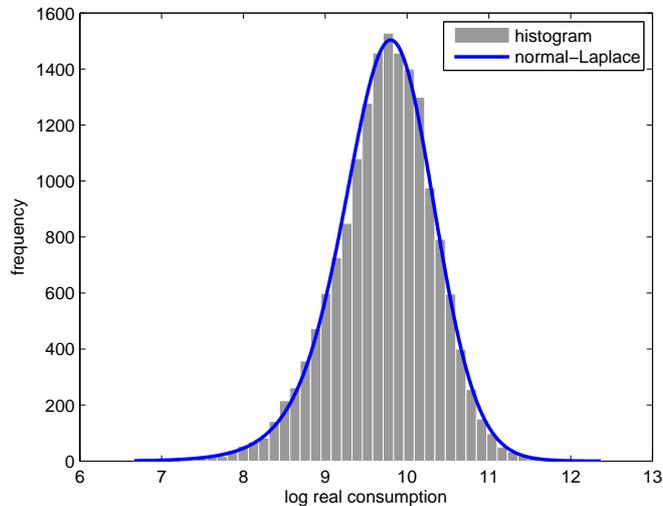


Figure 1. Log real consumption distribution and fitted normal-Laplace density. Data: Consumer Expenditure Survey (1985) used in Toda and Walsh (2011).

¹⁴See Gabaix (1999, 2008, 2009) for reviews of power laws.

3.4 General equilibrium with complete markets

If markets are complete, agents can perfectly insure against idiosyncratic risk. Since $E[r_i^j | \mathbf{R}_a, \mathbf{D}] = 1$ for all $j \in J$, assuming the law of large numbers for a continuum of random variables as in Uhlig (1996), in order to obtain the equilibrium all we need is to replace the idiosyncratic shock r_i^j by its conditional expectation, 1. Then we obtain an essentially unique equilibrium by Theorem 3.3.

4 Economy with asset-backed securities

In this section I consider an economy with asset-backed securities (ABS). As in Section 3, there are a continuum of agents indexed by $i \in I = [0, 1]$ with characteristics $(\beta_i, \gamma, \sigma_i, w_{i0})$. There are J types of investment projects with return $\mathbf{R}_i = (R_i^1, \dots, R_i^J)$. I ignore assets that are in zero net supply or repackaging of existing assets since they are irrelevant by Theorem 3.3 and Proposition 3.4.

4.1 Description of the economy and definition of equilibrium

Financial structure In addition to the setting in Section 3, there is a financial intermediary which makes loans of type $k = 1, \dots, K$. Loan k is characterized by a borrowing rate $R_b^k \in [0, \infty]$ (to be determined in equilibrium) and the (exogenously given) collateral requirement $\mathbf{c}^k = (c^{jk})_{j=1}^J \in \mathbb{R}_+^J$ (assumed to be constant over time for simplicity): for each dollar an agent borrows from loan k , he must invest c^{jk} dollars in investment project j and put up its return $R^j c^{jk}$ as collateral.¹⁵ Following Geanakoplos and Zame (1995), the sole penalty of default is the confiscation of collateral. Therefore in the next period agent i delivers $\min\{\mathbf{R}_i \cdot \mathbf{c}^k, R_b^k\}$ for each dollar taken from loan k . All the debt contracts of type k are pooled and issued as an ABS. The only role of the financial intermediary is to service the loans: it simply collects all payments of each loan and passes the proceeds to the holder of the corresponding ABS. In reality, collection of payments and securitization entails some costs.¹⁶ For this reason, I assume that the return on ABS k (the lending rate to loan k), R_l^k , is a function of the aggregate component of investment returns \mathbf{R}_a and the borrowing rate R_b^k and satisfies

$$(0 \leq) R_l^k(\mathbf{R}_a, R_b^k) \leq E[\min\{\mathbf{R}_i \cdot \mathbf{c}^k, R_b^k\} | \mathbf{R}_a], \quad (4.1)$$

where the right-hand side is the cross-sectional average of individual deliveries. Note that the lending rate is always finite even if $R_b^k = \infty$ because of the min function.

¹⁵In the original formulation of Geanakoplos and Zame (1995), there is an exogenous collateral requirement for each dollar promised (tomorrow) and the price of such a promise is determined in equilibrium. In our formulation the collateral requirement is for each dollar borrowed (today) and the interest rate is determined in equilibrium, which is essentially the same as in Kubler and Schmedders (2003).

¹⁶If the collateral is a house, there are legal costs and opportunity costs while transferring the property from the defaulter to the creditor. In addition, the property might be vandalized after the decision of default.

Budget constraint Let $\theta \in \mathbb{R}_+^J$, $\phi \in \mathbb{R}_+^K$, and $\psi \in \mathbb{R}_+^K$ be the portfolios of investment projects, purchase of ABS, and borrowing. By the collateral requirement, we have $\theta^j \geq \sum_k c^{jk} \psi^k$ for all j , or more compactly $\theta \geq C\psi$, where $C = (\mathbf{c}^k) = (c^{jk})$ is the nonnegative $J \times K$ collateral requirement matrix. Denote the portfolio constraint by

$$\Omega = \left\{ (\theta, \phi, \psi) \in \mathbb{R}_+^J \times \mathbb{R}_+^K \times \mathbb{R}_+^K \mid \sum \theta^j + \sum (\phi^k - \psi^k) = 1, \theta \geq C\psi \right\}, \quad (4.2)$$

which is clearly nonempty, closed, and convex. The return on portfolio $(\theta, \phi, \psi) \in \Omega$ is denoted by

$$R(\theta, \phi, \psi) = \sum_j R^j \theta^j + \sum_k R_l^k \phi^k - \sum_k \min \{ \mathbf{R} \cdot \mathbf{c}^k, R_b^k \} \psi^k, \quad (4.3)$$

where the lending rate R_l^k is given by (4.1) and I have suppressed the individual subscript. The portfolio return (4.3) is nonnegative and finite because of the min function. To see this, since $\theta \geq C\psi$, we obtain

$$\begin{aligned} R(\theta, \phi, \psi) &\geq \sum_k (\mathbf{R} \cdot \mathbf{c}^k) \psi^k + \sum_k R_l^k \phi^k - \sum_k \min \{ \mathbf{R} \cdot \mathbf{c}^k, R_b^k \} \psi^k \\ &\geq \sum_k R_l^k \phi^k \geq 0. \end{aligned}$$

The budget constraint is then

$$w' = R(\theta, \phi, \psi)(w - c) \geq 0, \quad (4.4)$$

where w is current period's starting wealth, c is current consumption, and w' is next period's wealth.

Equilibrium As usual, I define an equilibrium by individual optimization and market clearing. By Theorem 2.1 and its proof we can treat the optimal consumption problem and the optimal portfolio problem separately: the latter is

$$\max_{(\theta, \phi, \psi) \in \Omega} \frac{1}{1 - \gamma} E[R(\theta, \phi, \psi)^{1 - \gamma}] \quad (4.5)$$

and the optimal consumption rule is given by (the equivalent of) (2.6). By the same argument as in Proposition 3.2, if there is an equilibrium there is also a symmetric equilibrium. Finally, by the remark after the proof of Theorem 3.3, the discount factor and the elasticity of intertemporal substitution play no essential role. For these reasons I define a (symmetric) collateral equilibrium as follows.

Definition 4.1 (Collateral equilibrium). Borrowing rates $(R_b^k)_{k=1}^K \subset [0, \infty]^K$ and portfolio $(\theta^*, \phi^*, \psi^*) \in \Omega$ constitute a collateral equilibrium if

1. $(\theta^*, \phi^*, \psi^*)$ solves (4.5), and
2. lending and borrowing are matched: $\phi^* = \psi^*$.

4.2 Existence and computation of collateral equilibrium

As the following proposition shows, the collateral equilibrium fails to exist if the collateral requirement is too loose.

Proposition 4.2. *If $R^j > 0$ with positive probability for some j and $\sum_j c^{jk} < 1$ for some k , then the optimal portfolio problem (4.5) has no solution.*

Proof. Let j, k satisfy the assumption of the proposition. For any portfolio $(\theta, \phi, \psi) \in \Omega$ and $\epsilon > 0$, increase borrowing from loan k by ϵ , invest ϵc^{jk} in projects $j = 1, \dots, J$ to meet the collateral requirement, and invest the remaining $\epsilon' := \epsilon(1 - \sum_j c^{jk}) > 0$ in project j . Call this new portfolio (θ', ϕ', ψ') . Clearly $(\theta', \phi', \psi') \in \Omega$. Then by definition and (4.3) we obtain

$$\begin{aligned} R(\theta', \phi', \psi') - R(\theta, \phi, \psi) &= \epsilon' R^j + \epsilon(\mathbf{R} \cdot \mathbf{c}^k) - \epsilon \min\{\mathbf{R} \cdot \mathbf{c}^k, R_b^k\} \\ &\geq \epsilon' R^j \geq 0, \end{aligned}$$

and the last inequality is strict with positive probability. Since the function $x \mapsto \frac{x^{1-\gamma}}{1-\gamma}$ is strictly increasing, we obtain

$$\frac{1}{1-\gamma} \mathbb{E}[R(\theta', \phi', \psi')^{1-\gamma}] > \frac{1}{1-\gamma} \mathbb{E}[R(\theta, \phi, \psi)^{1-\gamma}],$$

so (θ, ϕ, ψ) cannot be optimal in Ω . \square

The assumption $R^j > 0$ with positive probability for some j is innocuous (otherwise $R^j = 0$ almost surely for all j , hence an agent has no incentive to save). Hence by Proposition 4.2, in studying collateral equilibrium, we may assume $\sum_j c^{jk} \geq 1$ for all k .¹⁷ This condition can be interpreted as a natural loan-to-value (LTV) constraint: whenever an agent borrows, he must put up at least as much as collateral, for otherwise the agent has an incentive to default right after taking the loan if confiscation of collateral is the only penalty from defaulting.

The following proposition provides a necessary condition for equilibrium, which is useful for computing the equilibrium as well as motivates the proof of the existence theorem.

Proposition 4.3. *Let*

$$\Omega' = \left\{ (\theta, \phi, \psi) \in \mathbb{R}_+^J \times \mathbb{R}_+^K \times \mathbb{R}_+^K \mid \sum \theta^j = 1, \phi = \psi, \theta \geq C\psi \right\}$$

be the portfolio constraint with matched borrowing and lending. If borrowing rates $(R_b^k)_{k=1}^K$ and portfolio $(\theta^, \phi^*, \psi^*)$ constitute a collateral equilibrium, then*

1. $(\theta^*, \phi^*, \psi^*)$ solves

$$\max_{(\theta, \phi, \psi) \in \Omega'} \frac{1}{1-\gamma} \mathbb{E}[R(\theta, \phi, \psi)^{1-\gamma}], \quad (4.6)$$

¹⁷Fortunately, this condition is also sufficient for the existence of collateral equilibrium. See Theorem 4.4.

2. for all k we have

$$\mathbb{E} [R(\theta^*, \phi^*, \psi^*)^{-\gamma} (R_l^k(\mathbf{R}_a, R_b^k) - R(\theta^*, \phi^*, \psi^*))] \leq 0, \quad (4.7)$$

with equality if $\phi^{*k} > 0$.

Proof. If $(\theta^*, \phi^*, \psi^*)$ is an equilibrium portfolio, by definition it is optimal in Ω . Since $\Omega' \subset \Omega$, it is also optimal in Ω' . Consider investing the fraction of wealth $1 - \alpha$ in the optimal portfolio $(\theta^*, \phi^*, \psi^*)$ and α in ABS k . Since $(\theta^*, \phi^*, \psi^*)$ is optimal in Ω and $\phi^{*k} \geq 0$, the new portfolio is feasible if and only if

$$1 - \alpha \geq 0 \text{ and } (1 - \alpha)\phi^{*k} + \alpha \geq 0 \iff \begin{cases} -\frac{\phi^{*k}}{1 - \phi^{*k}} \leq \alpha \leq 1, & (0 \leq \phi^{*k} < 1) \\ \alpha \leq 1. & (1 \leq \phi^{*k}) \end{cases}$$

Since the return on ABS k is $R_l^k(\mathbf{R}_a, R_b^k)$, by the asset pricing result (2.7) in Proposition 2.3 we obtain (4.7), with equality if $\phi^{*k} > 0$. \square

Under the natural LTV constraint $\sum_j c^{jk} \geq 1$, a collateral equilibrium exists.

Theorem 4.4 (Existence). *Suppose that (i) the collateral requirement matrix $C = (c^{jk})$ is nonnegative and $\sum_j c^{jk} \geq 1$ for all k , (ii) the lending rate $R_l^k(\mathbf{R}_a, R_b^k)$ is continuous and increasing in the borrowing rate R_b^k for all k , and (iii) for all borrowing rates $(R_b^k)_{k=1}^K \in [0, \infty]^K$ we have*

$$\mathbb{E} \left[\sup_{(\theta, \phi, \psi) \in \Omega'} R(\theta, \phi, \psi)^{1-\gamma} \right] < \infty.$$

Then a collateral equilibrium exists.

Proof. See Appendix A.2. \square

Using Proposition 4.3 and the proof of Theorem 4.4, computing the collateral equilibrium is straightforward.

Algorithm. Step 0. Select an error tolerance for the stopping criterion and a starting point $x \in [0, 1]^K$.

Step 1. Define the borrowing rate $R_b^k = \frac{x_k}{1-x_k} \in [0, \infty]$ for each $k = 1, \dots, K$ and compute the optimal portfolio $(\theta^*, \phi^*, \psi^*)$ with matched borrowing and lending by (4.6). (One can use any constrained optimization routine.)

Step 2. Update the borrowing rate by

$$R_b^{*k} = \sup \{ r \mid \mathbb{E} [R^{-\gamma} (R_l^k(\mathbf{R}_a, r) - R)] \leq 0 \}$$

for each $k = 1, \dots, K$, where $R = R(\theta^*, \phi^*, \psi^*)$ is the return on the optimal portfolio. (Since $R_l^k(\mathbf{R}_a, r)$ is increasing in r , R_b^{*k} obtains very quickly by the nested intervals algorithm.)

Step 3. Update x_k by $x_k^* = \frac{R_b^{*k}}{1+R_b^{*k}}$.

Step 4. Repeat Steps 1–3 until $(\theta^*, \phi^*, \psi^*)$ and $(R_b^{*k})_{k=1}^K$ converge and condition (4.7) holds for all k .

In general, we can say very little about the equilibrium portfolio except those properties proved in Proposition 4.3. However, Proposition 4.5 below shows that agents borrow to the maximum if default entails no cost and the idiosyncratic shocks are driven monotonically by a single factor.

Proposition 4.5. *Consider an economy with investment projects whose return $\mathbf{R} = (R^j)_{j=1}^J$ can be decomposed into the aggregate component $\mathbf{R}_a = (R_a^j)_{j=1}^J$ and the idiosyncratic component $\mathbf{r} = (r^j)_{j=1}^J$. Suppose that default entails no cost (equality holds in (4.1)) and there is a single idiosyncratic factor X such that $r^j = r^j(X)$, where the functions $(r^j(x))_{j=1}^J$ are increasing in $x \in \mathbb{R}$. Then agents borrow to the maximum from loans with active default.*

More precisely, if $(\theta^, \phi^*, \psi^*)$ is a collateral equilibrium portfolio and there is active default in loan k , then for any portfolio of loans ψ with $\psi^k > \psi^{*k}$ and $\psi^l = \psi^{*l}$ for all $l \neq k$, the collateral constraint $\theta^* \geq C\psi$ fails.*

Proof. See Appendix A.3. □

4.3 Welfare properties of collateral equilibrium

The collateral equilibrium has a number of desirable properties.

Theorem 4.6. *If $(\theta^*, \phi^*, \psi^*)$ is a collateral equilibrium portfolio, the recursive utility of every agent is a monotonic function of $E[R(\theta^*, \phi^*, \psi^*)^{1-\gamma}]^{\frac{1}{1-\gamma}}$, hence every agent agrees on policies. Furthermore, given an exogenous collateral requirement, the collateral equilibrium allocation in Theorem 4.4*

1. *is efficient among allocations that obtains under interventions that leave the borrowing rates unchanged,*
2. *weakly Pareto dominates the GEI allocation in Theorem 3.3, and*
3. *strictly Pareto dominates the GEI allocation only if active default occurs.*

Proof. By Claim 5 in Lemma A.1, the value function of any agent is a monotonic function of $E[R(\theta^*, \phi^*, \psi^*)^{1-\gamma}]^{\frac{1}{1-\gamma}}$. Therefore the welfare from a particular policy can be unambiguously evaluated by this quantity.

Efficiency in the above sense follows by the same argument as in the proof of Theorem 3.3 when borrowing rates remain unchanged. Collateral equilibrium weakly Pareto dominates the GEI allocation (autarky) because agents can always choose $\phi = \psi = 0$ (no borrowing or lending).

If there is no active default, by definition $\mathbf{R} \cdot \mathbf{c}^k \geq R_b^k$ for all k , so $R_b^k = \min \{ \mathbf{R} \cdot \mathbf{c}^k, R_b^k \}$. Hence by (4.1) we obtain

$$R_l^k(\mathbf{R}_a, R_b^k) \leq E[\min \{ \mathbf{R} \cdot \mathbf{c}^k, R_b^k \} | \mathbf{R}_a] = R_b^k.$$

Since the return on ABS k is less than the borrowing rate, choosing $\phi = \psi = 0$ (no borrowing or lending) weakly Pareto dominates the collateral equilibrium portfolio $\phi^* = \psi^*$. Therefore agents must be indifferent between the collateral equilibrium and the GEI. □

The first part of Theorem 4.6 is remarkable in that despite agent heterogeneity (as long as agents have a common relative risk aversion, they may differ in wealth, patience, elasticity of intertemporal substitution, death probability, and

these parameters may even be time-dependent¹⁸), the interest of every agent is aligned. One reason among many Kirman (1992) criticized the representative agent model is that individual interests are not necessarily aligned with that of the representative agent (even in the extremely restrictive case that a representative agent exist), but my model permits an unambiguous welfare analysis even though there is no representative agent.

Theorem 4.6 does not imply that government interventions are useless, for it says nothing about what happens when borrowing rates change. For instance, it may well be the case that welfare improves by closing some credit markets. Also, the government might be able to improve welfare by changing the collateral requirement.

5 Numerical examples

In this section I compute a few examples. Suppose that agents have relative risk aversion coefficient $\gamma = 2$, there is a single investment project with lognormally distributed return R (assumed to be purely idiosyncratic), $\log R \sim N(g, v)$, where the expected growth rate is $g = 0.08$ (8%) and the volatility is $v = 0.15$ (15%). I vary the down payment from 1% to 30% in increments of 1%. (The collateral requirement is computed by $1/(1 - \text{down payment})$.) In order to compute the equilibrium we must maximize $\frac{1}{1-\gamma} E[R(\theta, \phi, \psi)^{1-\gamma}]$ as in (4.6), which is computationally intensive unless the distribution of returns are discrete. For this reason I approximate the log return distribution (which is $N(g, v)$) as follows. First I take equally spaced points on $[-4, 4]$ with distance 0.1 (81 points in total), denoted by \mathbf{x} , and approximate the standard normal distribution using the method described in Tanaka and Toda (2012). Then I assign the computed probabilities on the points $\mathbf{y} = e^{(\mathbf{x}+g)v}$. I consider two cases for the returns on asset-backed securities (ABS), one case with no cost from default (hence the lending rate attains the upper bound in (4.1)) and the other with depreciation of the collateral by $\delta = 0.02$ (2%) when default occurs. In either case the return on ABS is given by

$$R_l(R_b) = \Pr(R \geq R_b)R_b + (1 - \delta) \Pr(R < R_b) E[R | R < R_b]. \quad (5.1)$$

For each fixed vector of parameters (γ, g, v, δ) , I perform the following three numerical experiments:

1. there is only one loan with down payment varying from 1% to 30% in 1% intervals,
2. starting from one loan with down payment 30%, adding new loans with down payment from 25% to 0% in 5% intervals, and
3. starting from one loan with down payment 0%, adding new loans with down payment from 5% to 30% in 5% intervals

The convergence criterion for portfolios is 10^{-6} .

Figure 2 shows the relation between the down payments of the single loan and the equilibrium interest rates for $\delta = 0, 0.02$. Since there are no aggregate shocks, ABS is risk-free and hence the return on ABS computed by (5.1) and

¹⁸See the remark after the proof of Theorem 3.3.

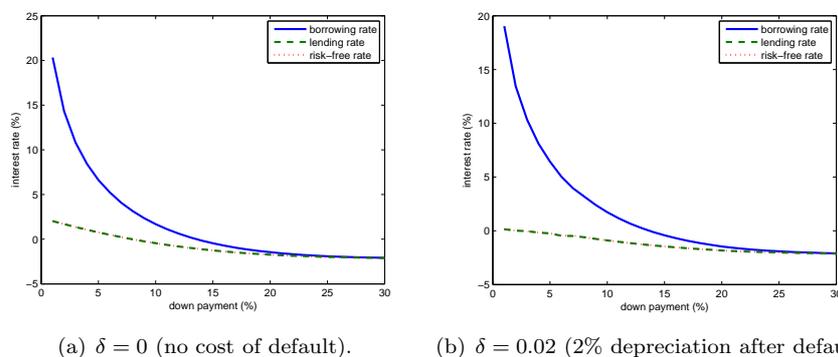


Figure 2. Relation between down payments and interest rates. Blue solid: borrowing rate, green dashed: lending rate (ABS return), red dotted: risk-free rate.

the risk-free rate computed by (2.8) coincide. In both cases with or without default cost, as the down payment decreases from 30% to 1% (hence the leverage increases from 3.3 to 100) both the borrowing rate and the lending rate increase. However, while the borrowing rate sharply increases up to 20%, the lending rate varies only from -1% to 2% . Since the expected return on investment is 8% , this implies that the private equity premium is $6-9\%$ and therefore neither the equity premium puzzle nor the risk-free rate puzzle happen. (Note that the risk aversion coefficient is only 2 and the expected return (8%) and volatility (15%) are realistic numbers.) It is also interesting that the borrowing rate of 20% with loose collateral payment (1% down payment) is comparable to the interest rate charged on credit cards in reality.

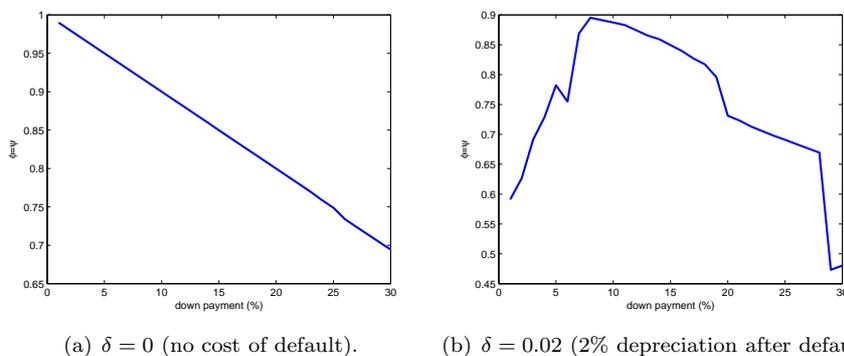
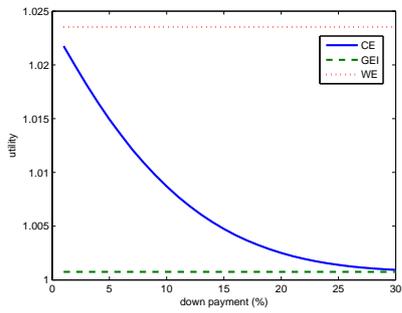


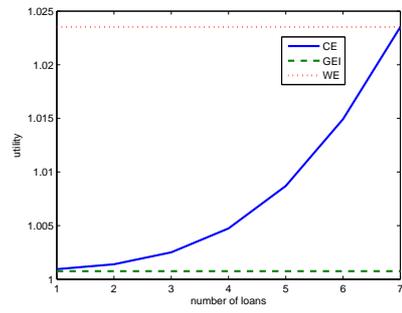
Figure 3. Relation between down payments and borrowing as a fraction of wealth.

Figure 3 shows the relation between down payments and borrowing as a fraction of wealth, which is equal to the portfolio $\phi = \psi$. With no cost of default (Figure 3(a)), agents borrow to the maximum ($\phi = \psi = 1 - \text{down payment}$) as shown in Proposition 4.5, but not necessarily so with default costs (Figure 3(b)).

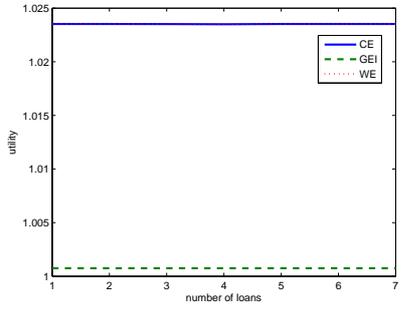
Figure 4 plots the welfare criterion $E[R(\theta^*, \phi^*, \psi^*)^{1-\gamma}]^{\frac{1}{1-\gamma}}$ of the collateral equilibrium (CE), equilibrium with incomplete markets (GEI), and equilibrium with complete markets (Walrasian equilibrium, WE) for the above experiments



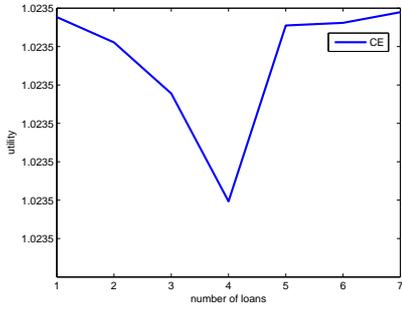
(a) Relation between welfare and down payment.



(b) Welfare when new loans become available from high to low down payments.



(c) Welfare when new loans become available from low to high down payments.

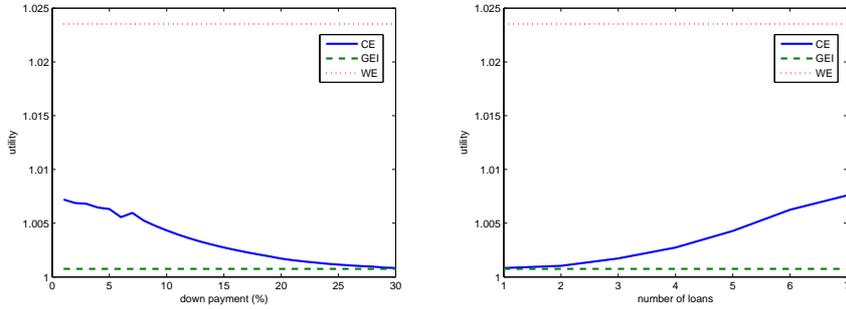


(d) Magnified view.

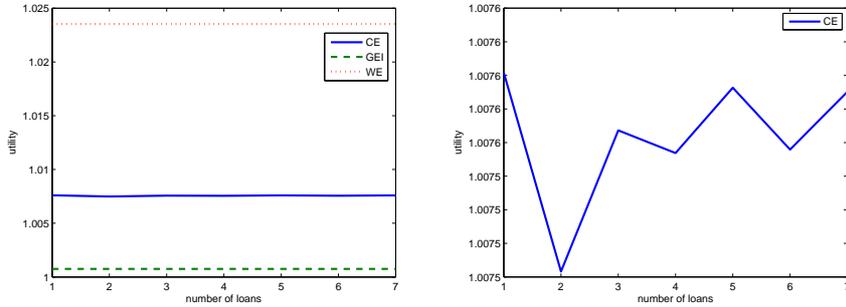
Figure 4. Welfare with no cost of default ($\delta = 0$). Blue solid: collateral equilibrium (CE), green dashed: incomplete markets (GEI), red dotted: complete markets (Walrasian equilibrium, WE).

1–3 with no default cost ($\delta = 0$). Figure 5 does the same for $\delta = 0.02$.

According to Figure 4(a), as the down payment is relaxed from 30% to 1%, the welfare monotonically approaches that in complete markets. Figure 4(b) plots the welfare when more and more ABS markets open, starting from down payment 30% and going all the way up to 0% in 5% increments. Again the welfare improves with opening of new markets. Figure 4(c) does the same but the down payment starts with 0% and loans with higher down payments become available. In this case the new markets are inactive and welfare does not change much. Figure 4(d) magnifies 4(c). The collateral equilibrium welfare appears to change, but the order of variability is 10^{-6} , which is the same as the convergence criterion.



(a) Relation between welfare and down payment. (b) Welfare when new loans become available from high to low down payments.



(c) Welfare when new loans become available from low to high down payments. (d) Magnified view.

Figure 5. Welfare with 2% depreciation after default ($\delta = 0.02$). Blue solid: collateral equilibrium (CE), green dashed: incomplete markets (GEI), red dotted: complete markets (Walrasian equilibrium, WE).

According to Figure 5(a), as the down payment is relaxed from 30% to 1%, the welfare does not approach that in complete markets because of default costs. Interestingly, relaxing the down payment does not necessarily improve welfare. (There is a dip in welfare at down payment=6%.) Figure 5(b) plots the welfare when more and more ABS markets open, starting from down payment 30% and going all the way up to 0% in 5% increments. Again the welfare improves with opening of new markets. Figure 5(c) does the same but the down payment starts with 0% and loans with higher down payments become available. Figure 5(d)

magnifies 5(c). This time the order of variability is 10^{-4} , much larger than the convergence criterion 10^{-6} . Therefore opening new markets does not necessarily improve welfare, or closing some markets might be welfare-improving.

6 Concluding remarks

This paper has introduced collateralized lending and asset-backed securities (ABS) in a heterogeneous agent, incomplete markets, dynamic stochastic general equilibrium model. The model is highly tractable despite heterogeneity in patience, elasticity of intertemporal substitution, and wealth. The major findings in the numerical examples are that the introduction of ABS greatly improves welfare (especially with relatively high leverage) and the model predicts realistic equity premium and risk-free rate. However, welfare does not necessarily improve monotonically by relaxing down payments or by opening new markets. An obvious limitation of the current model is that the idiosyncratic shocks are present only in investment in order to keep tractability. Therefore the validity of the model rests on the relative importance of idiosyncratic investment risk (multiplicative shocks) and idiosyncratic endowment risk (additive shocks), which is an empirical question not addressed in this paper.

A Proofs

A.1 Optimal consumption/portfolio problem

Theorem 2.1 is a special case of Toda (2012c), but to make the paper self-contained I provide a shorter proof that exploits the i.i.d. assumption.

Let us first solve the finite period optimal consumption/portfolio problem. Given current wealth w , define the T period value function (the supremum of recursive utility over budget-feasible consumption plans) $V_T(w)$ by

$$V_T(w) = \sup_{\{c_t, \theta_t\}} \left\{ V_T(\{c_t\}_{t=0}^{T-1}) \mid (\forall t) w_{t+1} = R_{t+1}(\theta_t)(w_t - c_t) \geq 0, \theta_t \in \Theta \right\}, \quad (\text{A.1})$$

where the recursive utility $V_T(\{c_t\}_{t=0}^{T-1})$ is defined by (2.1).

Lemma A.1. *Under the assumptions of Theorem 2.1,*

1. *the finite period value function has the form $V_T(w) = a_T w$,*
2. *the portfolio $\theta^* \in \Theta$ is optimal if and only if*

$$\theta^* \in \arg \max_{\theta \in \Theta} E[R(\theta)^{1-\gamma}]^{\frac{1}{1-\gamma}}, \quad (\text{A.2})$$

3. *the optimal return $R(\theta^*)$ is unique almost surely,*
4. *the optimal consumption when T periods are left is $c_T(w) = a_{T+1}^{\frac{\sigma-1}{\sigma}} w$, and*
5. *the coefficients $\{a_T\}_{T=1}^{\infty}$ satisfy $a_1 = 1$ and the recursive formula*

$$a_{T+1} = \left(1 + \beta^{\frac{1}{\sigma}} \left(a_T E[R(\theta^*)^{1-\gamma}]^{\frac{1-\sigma}{1-\gamma}} \right)^{\frac{1-\sigma}{\sigma}} \right)^{\frac{\sigma}{1-\sigma}}, \quad (\text{A.3})$$

and are increasing in $E[R(\theta^*)^{1-\gamma}]^{\frac{1}{1-\gamma}}$.

Proof. The proof is by induction. For $T = 1$, since $V_1(c) = c$ we have $V_1(w) = \sup\{c \mid 0 \leq c \leq w\} = w$, so $a_1 = 1$. (There is no need to choose a portfolio.) If the claim holds up to T , for $T + 1$ by (2.1) and (A.1) we obtain

$$\begin{aligned} V_{T+1}(w) &= \sup_{\substack{0 \leq c \leq w \\ \theta \in \Theta}} \left[c^{1-\sigma} + \beta E[(a_T R(\theta)(w-c))^{1-\gamma}]^{\frac{1-\sigma}{1-\gamma}} \right]^{\frac{1}{1-\sigma}} \\ &= \sup_{0 \leq c \leq w} \left[c^{1-\sigma} + \beta a_T^{1-\sigma} (w-c)^{1-\sigma} \left(\sup_{\theta \in \Theta} E[R(\theta)^{1-\gamma}]^{\frac{1}{1-\gamma}} \right)^{1-\sigma} \right]^{\frac{1}{1-\sigma}}. \end{aligned} \quad (\text{A.4})$$

Since $R(\theta)$ is continuous in θ and $E[\sup_{\theta \in \Theta} R(\theta)^{1-\gamma}] < \infty$, by the Lebesgue convergence theorem $E[R(\theta)^{1-\gamma}]^{\frac{1}{1-\gamma}}$ is finite and continuous in θ . Since Θ is compact, the maximum in (A.2) is attained. Since $R \mapsto E[R^{1-\gamma}]^{\frac{1}{1-\gamma}}$ is strictly quasi-concave in the return R , $R(\theta)$ is linear in θ , and Θ is convex, if θ^*, θ^\dagger are both optimal it must be the case that $R(\theta^*) = R(\theta^\dagger)$ almost surely. Therefore $\theta^* = \theta^\dagger$ if and only if R^1, \dots, R^J are linearly independent. Let θ^* be an optimal portfolio and define

$$k_T = \beta a_T^{1-\sigma} E[R(\theta^*)^{1-\gamma}]^{\frac{1-\sigma}{1-\gamma}} > 0.$$

Then the objective function with respect to c in (A.4) becomes

$$(c^{1-\sigma} + k_T(w-c)^{1-\sigma})^{\frac{1}{1-\sigma}},$$

which is strictly quasi-concave. The first-order condition for the maximization is

$$c^{-\sigma} - k_T(w-c)^{-\sigma} = 0 \iff c = \frac{1}{1 + k_T^{\frac{1}{\sigma}}} w,$$

so the maximum is attained. Substituting the optimal consumption into (A.4), after some algebra we obtain $V_{T+1}(w) = (1 + k_T^{\frac{1}{\sigma}})^{\frac{\sigma}{1-\sigma}} w =: a_{T+1} w$. Hence the optimal consumption rule when T periods are left is $c_T(w) = a_{T+1}^{\frac{\sigma-1}{\sigma}} w$ and the coefficients of the value function satisfy the recursive formula (A.3). That a_T is increasing in $E[R(\theta^*)^{1-\gamma}]^{\frac{1}{1-\gamma}}$ is immediate by induction using $a_1 = 1$ and the recursive formula (A.3). \square

Proof of Theorem 2.1.

Step 1. Sufficiency of (2.4).

Let $V_T(w) = a_T w$ be the T period value function in Lemma A.1. Let us show that $\{a_T\}_{T=1}^\infty$ has a finite positive limit. Let $x_T = a_T^{\frac{1-\sigma}{\sigma}}$. Then by (A.3) we obtain

$$x_{T+1} = f(x_T) := 1 + \left(\beta E[R(\theta^*)^{1-\gamma}]^{\frac{1-\sigma}{1-\gamma}} \right)^{\frac{1}{\sigma}} x_T.$$

By (2.4) $f: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a contraction mapping. Therefore $\{x_T\}_{T=1}^\infty$ converges to the unique fixed point of f , which is $x = \left(1 - \beta^{\frac{1}{\sigma}} E[R(\theta^*)^{1-\gamma}]^{\frac{1-\sigma}{\sigma(1-\gamma)}} \right)^{-1}$. Let $a = x^{\frac{\sigma}{1-\sigma}}$. Then $V_T(w) = a_T w \rightarrow aw$ as $T \rightarrow \infty$.

Define the infinite period value function by

$$V(w) = \sup_{\{c_t, \theta_t\}} \{V(\{c_t\}_{t=0}^\infty) \mid (\forall t) w_{t+1} = R_{t+1}(\theta_t)(w_t - c_t) \geq 0, \theta_t \in \Theta\},$$

where the infinite period recursive utility is defined (in $[0, \infty]$) by (2.2). By the definition of the finite period value function, we obtain

$$V(w) = \sup_{\{c_t, \theta_t\}} \lim_{T \rightarrow \infty} V_T(\{c_t\}_{t=0}^{T-1}) \leq \sup_{\{c_t, \theta_t\}} \lim_{T \rightarrow \infty} V_T(w) = aw.$$

Since by (2.1) and (2.2) for any consumption plan $\{c_t\}_{t=0}^\infty$ we have $V(\{c_t\}_{t=0}^\infty) \geq V_T(\{c_t\}_{t=0}^{T-1})$, by taking the supremum over feasible consumption plans we obtain $V(w) \geq V_T(w)$ for all T . Therefore

$$V(w) \geq \lim_{T \rightarrow \infty} V_T(w) = aw,$$

hence $V(w) = aw$ and (2.6b) holds. Consider the portfolio rule θ^* given by (A.2) and the consumption rule

$$c(w) = a^{\frac{\sigma-1}{\sigma}} w = \left(1 - \beta^{\frac{1}{\sigma}} \mathbb{E}[R(\theta^*)^{1-\gamma}]^{\frac{1-\sigma}{\sigma(1-\gamma)}}\right) w,$$

which is precisely (2.6a). By repeating the argument of Lemma A.1, it follows that the infinite period value function $V(w)$ is attained by these consumption/portfolio rules. Uniqueness follows by strict quasi-concavity.

Step 2. Necessity of (2.4).

The above proof that $V(w) = \lim_{T \rightarrow \infty} V_T(w)$ does not require condition (2.4). If

$$\beta \left(\sup_{\theta \in \Theta} \mathbb{E}[R(\theta)^{1-\gamma}]^{\frac{1}{1-\gamma}} \right)^{1-\sigma} \geq 1,$$

then by (A.3) we obtain $a_T^{\frac{1-\sigma}{\sigma}} \rightarrow \infty$ and hence $V_T(w) \rightarrow 0$ or $V_T(w) \rightarrow \infty$ depending on the value of σ . Since $V(\{c_t\}_{t=0}^\infty) \geq 0$, in the former case we have $V(\{c_t\}_{t=0}^\infty) = 0$ for any feasible consumption plan, so any consumption plan is optimal. In the latter case there exists no maximum. \square

A.2 Existence of collateral equilibrium

Proof of Theorem 4.4. Let

$$\Omega' = \left\{ (\theta, \phi, \psi) \in \mathbb{R}_+^J \times \mathbb{R}_+^K \times \mathbb{R}_+^K \mid \sum \theta^j = 1, \phi = \psi, \theta \geq C\psi \right\}$$

be the portfolio constraint with matched borrowing and lending.

Step 1. Ω' is nonempty, compact, and convex.

Ω' is clearly nonempty, closed, and convex. Since $\sum_j c^{jk} \geq 1$ for all k , there exists j with $c^{jk} > 0$. Hence $(\theta, \phi, \psi) \in \Omega'$ implies $1 \geq \theta^j \geq \sum_k c^{jk} \psi^k \geq c^{jk} \psi^k$, so $0 \leq \phi^k = \psi^k \leq 1/c^{jk}$ is bounded.

Step 2. Construction of borrowing rates with a fixed point property.

Given $x \in [0, 1]^K$, define $R_b^k = \frac{x_k}{1-x_k} \in [0, \infty]$. As noted right after (4.3), the portfolio return $R(\theta, \phi, \psi)$ is nonnegative and finite even if $R_b^k = \infty$ because of the min function. Since $\mathbb{E}[R(\theta, \phi, \psi)^{1-\gamma}]$ is continuous in (θ, ϕ) by the Lebesgue convergence theorem and Ω' is compact by Step 1,

$$\max_{\theta, \phi} \frac{1}{1-\gamma} \mathbb{E}[R(\theta, \phi, \psi)^{1-\gamma}] \text{ subject to } \theta \in \Delta^{J-1}, \phi \in \mathbb{R}_+^K, \theta \geq C\phi$$

has a solution (θ^*, ϕ^*) . By Theorem 2.1, $R(\theta^*, \phi^*, \psi^*)$ is unique almost surely. Define

$$R_b^{*k} = \sup \{ r \mid \mathbb{E} [R(\theta^*, \phi^*, \psi^*)^{-\gamma} (R_l^k(\mathbf{R}_a, r) - R(\theta^*, \phi^*, \psi^*))] \leq 0 \},$$

which is well-defined and satisfies $R_b^{*k} \in [0, \infty]$. By Berge's maximum theorem (Berge, 1959), $(R_b^k) \mapsto (R_b^{*k})$ is continuous. Finally, define $x^* \in [0, 1]^K$ by $x_k^* = \frac{R_b^{*k}}{1+R_b^{*k}}$. Since $[0, 1]^K$ is nonempty, compact, convex, and $x \mapsto x^*$ is continuous, by the Brouwer fixed point theorem there is a fixed point x . Let $(R_b^k)_{k=1}^K$ and (θ^*, ϕ^*) be the corresponding borrowing rate and portfolio and define $\psi^* = \phi^*$.

Step 3. Proof that $(\theta^, \phi^*, \psi^*)$ is an equilibrium portfolio.*

Since $\phi^* = \psi^*$ by construction, to prove that $(\theta^*, \phi^*, \psi^*)$ is an equilibrium portfolio it suffices to show that $(\theta^*, \phi^*, \psi^*)$ solves

$$\max_{(\theta, \phi, \psi) \in \Omega} \frac{1}{1-\gamma} \mathbb{E}[R(\theta, \phi, \psi)^{1-\gamma}].$$

The Lagrangian of this maximization can be written as

$$\begin{aligned} L = \frac{1}{1-\gamma} \mathbb{E}[R(\theta, \phi, \psi)^{1-\gamma}] + \lambda_b \left(1 - \sum \theta^j - \sum \phi^k + \sum \psi^k \right) \\ + \lambda'_\theta \theta + \lambda'_\phi \phi + \lambda'_\psi \psi + \lambda'_c (\theta - C\psi), \end{aligned}$$

where $\lambda_b \in \mathbb{R}$, $\lambda_\theta, \lambda_c \in \mathbb{R}_+^J$, and $\lambda_\phi, \lambda_\psi \in \mathbb{R}_+^K$ are Lagrange multipliers. Let $J^+ \subset J$ be the set of j for which $c^{jk} > 0$ for some k . Then the constraints $\theta \geq C\psi$ and $\psi \geq 0$ imply $\theta^j \geq 0$ for $j \in J^+$, so we may assume $\lambda_\theta^j = 0$ for such j . Since the objective function is concave with linear constraints, the first order condition is necessary and sufficient for maximum. These conditions are

$$\frac{\partial}{\partial \theta^j} : \quad \mathbb{E}[R^{-\gamma} R^j] - \lambda_b + \lambda_\theta^j + \lambda_c^j = 0, \quad (\text{A.5a})$$

$$\frac{\partial}{\partial \phi^k} : \quad \mathbb{E}[R^{-\gamma} R_l^k] - \lambda_b + \lambda_\phi^k = 0, \quad (\text{A.5b})$$

$$\frac{\partial}{\partial \psi^k} : \quad -\mathbb{E}[R^{-\gamma} \min \{ \mathbf{R} \cdot \mathbf{c}^k, R_b^k \}] + \lambda_b + \lambda_\psi^k - \lambda'_c \mathbf{c}^k = 0, \quad (\text{A.5c})$$

where we have used the shorthand $R = R(\theta, \phi, \psi)$. Computing

$$\sum_j \theta^j \times (\text{A.5a}) + \sum_k \phi^k \times (\text{A.5b}) - \sum_k \psi^k \times (\text{A.5c})$$

and using the definition of the portfolio return, we obtain $\lambda_b = \mathbb{E}[R^{1-\gamma}]$.

Now consider the maximization problem

$$\max_{(\theta, \phi, \psi) \in \Omega'} \frac{1}{1-\gamma} \mathbb{E}[R(\theta, \phi, \psi)^{1-\gamma}],$$

with Lagrangian

$$L' = \frac{1}{1-\gamma} \mathbb{E}[R(\theta, \phi, \psi)^{1-\gamma}] + \mu_b \left(1 - \sum \theta^j\right) + \mu'_\theta \theta + \mu'_\phi \phi + \mu'_c (\theta - C\phi),$$

where $\mu_b \in \mathbb{R}$, $\mu_\theta, \mu_c \in \mathbb{R}_+^J$, and $\mu_\phi \in \mathbb{R}_+^K$ are Lagrange multipliers. Again we may assume $\mu_\theta^j = 0$ for $j \in J^+$. The first-order conditions are

$$\frac{\partial}{\partial \theta^j} : \quad \mathbb{E}[R^{-\gamma} R^j] - \mu_b + \mu_\theta^j + \mu_c^j = 0, \quad (\text{A.6a})$$

$$\frac{\partial}{\partial \phi^k} : \quad \mathbb{E}[R^{-\gamma} (R_l^k - \min\{\mathbf{R} \cdot \mathbf{c}^k, R_b^k\})] + \mu_\phi^k - \mu_c^k = 0. \quad (\text{A.6b})$$

Let (θ^*, ϕ^*) and (R_b^k) be the portfolio and borrowing rates found in Step 2 and define $\psi^* = \phi^*$. By computing

$$\sum_j \theta^{*j} \times (\text{A.5a}) + \sum_k \phi^{*k} \times (\text{A.5b})$$

we obtain $\mu_b = \mathbb{E}[R^{1-\gamma}]$, where $R := R(\theta^*, \phi^*, \psi^*)$. Define

$$\begin{aligned} (\lambda_b, \lambda_\theta, \lambda_c) &= (\mu_b, \mu_\theta, \mu_c) \in \mathbb{R} \times \mathbb{R}_+^J \times \mathbb{R}_+^J, \\ (\forall k) \lambda_\phi^k &= -\mathbb{E}[R^{-\gamma} (R_l^k - R)], \\ (\forall k) \lambda_\psi^k &= \mu_\phi^k - \lambda_c^k. \end{aligned}$$

By (A.6a) and the definition of $\lambda_b, \lambda_\theta, \lambda_c$, (A.5a) holds. By the definition of λ_b and λ_ϕ^k , (A.5b) holds and $\lambda_\phi^k \geq 0$. By (A.6b) and the definition of λ 's, (A.5c) holds. Therefore all we need to show is $\lambda_\phi^k \phi^{*k} = 0$, $\lambda_\psi^k \geq 0$, and $\lambda_\psi^k \psi^{*k} = 0$.

If $\lambda_\phi^k = 0$, then $\lambda_\psi^k \phi^{*k} = 0$, $\lambda_\psi^k = \mu_\phi^k \geq 0$, and $\lambda_\psi^k \psi^{*k} = \mu_\phi^k \phi^{*k} = 0$.

If $\lambda_\phi^k > 0$, by Step 2 we have $R_b^k = \infty$. Therefore (A.5c) becomes

$$-\mathbb{E}[R^{-\gamma} (\mathbf{R} \cdot \mathbf{c}^k)] + \lambda_b + \lambda_\psi^k - \lambda_c^k = 0. \quad (\text{A.7})$$

Since $\lambda_\theta^j = \mu_\theta^j = 0$ for $j \in J^+$, that is, $\lambda_\theta^j > 0$ only if $c^{jk} = 0$ for all k , it follows that $\lambda_\theta^j c^{jk} = 0$ for all j, k . Using this fact and computing $\sum_j c^{jk} \times (\text{A.5a})$, we obtain

$$\mathbb{E}[R^{-\gamma} (\mathbf{R} \cdot \mathbf{c}^k)] - \lambda_b \sum_j c^{jk} + \lambda_c^k = 0. \quad (\text{A.8})$$

Since $\lambda_b = \mu_b = \mathbb{E}[R^{1-\gamma}] > 0$ and $\sum_j c^{jk} \geq 1$ by assumption, adding (A.7) and (A.8) we obtain

$$\lambda_\psi^k = \mathbb{E}[R^{1-\gamma}] \left(\sum_j c^{jk} - 1 \right) \geq 0.$$

Since $0 \leq \lambda_\psi^k = \mu_\phi^k - \lambda_c^k$ by definition and $\lambda_\phi^k > 0$ by assumption, it must be $\mu_\phi^k > 0$ and hence $\phi^{*k} = 0$, thus $\lambda_\phi^k \phi^{*k} = 0$ and $\lambda_\psi^k \psi^{*k} = 0$. \square

A.3 Proof of Proposition 4.5

First we prove a lemma.

Lemma A.2. *Let $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be increasing (decreasing) functions, g positive, and X be a random variable such that $\mathbb{E}[f(X)] = 0$. Then $\mathbb{E}[f(X)g(X)] \geq 0$. If $f(X) \neq 0$ with positive probability and $g(X)$ is not almost surely constant either on the set $\{f(X) < 0\}$ or $\{f(X) > 0\}$, then the inequality is strict.*

Proof. Suppose that f, g are increasing. (The proof when f, g are decreasing is similar.) Since $\mathbb{E}[f(X)] = 0$, f cannot be always positive or always negative. Since f is increasing, there exists x_0 such that $f \leq 0$ on $(-\infty, x_0)$ and $f \geq 0$ on (x_0, ∞) . If $X < x_0$, since f, g are increasing and g is positive, by the definition of x_0 we have $f(X) \leq 0$ and $0 < g(X) \leq g(x_0)$, so $f(X)g(X) \geq f(X)g(x_0)$. If $X > x_0$, we have $f(X) \geq 0$ and $0 < g(x_0) \leq g(X)$, so $f(X)g(X) \geq f(X)g(x_0)$. If $X = x_0$, clearly $f(X)g(X) = f(X)g(x_0)$. Therefore

$$\mathbb{E}[f(X)g(X)] \geq \mathbb{E}[f(X)g(x_0)] = g(x_0) \mathbb{E}[f(X)] = 0.$$

If $f(X) \neq 0$ with positive probability, then both the sets $A^- = \{f(X) < 0\}$ and $A^+ = \{f(X) > 0\}$ have positive probability because $\mathbb{E}[f(X)] = 0$. Then

$$\mathbb{E}[f(X)g(X) | A^\pm] > \mathbb{E}[f(X)g(x_0) | A^\pm]$$

unless $g(X) = g(x_0)$ almost surely on A^\pm . \square

Proof of Proposition 4.5. Let $\Omega \subset \mathbb{R}_+^J \times \mathbb{R}_+^K \times \mathbb{R}_+^K$ be the portfolio constraint in (4.2), $R(\theta, \phi, \psi)$ be the return on portfolio defined by (4.3), and $(\theta^*, \phi^*, \psi^*)$ be the equilibrium portfolio. Assume that there is active default in loan k . Since by definition $\phi^* = \psi^*$ and the equilibrium portfolio solves (4.5), in particular we have

$$0 \in \arg \max_{\alpha \in A} \frac{1}{1-\gamma} \mathbb{E}[R(\theta^*, \psi^* + \alpha \mathbf{e}_k, \psi^* + \alpha \mathbf{e}_k)^{1-\gamma}], \quad (\text{A.9})$$

where \mathbf{e}_k is the k -th unit vector in \mathbb{R}^K and

$$A = \{\alpha \in \mathbb{R} \mid (\theta^*, \psi^* + \alpha \mathbf{e}_k, \psi^* + \alpha \mathbf{e}_k) \in \Omega\}.$$

If agents do not borrow to the maximum from loan k in equilibrium, there exists $\epsilon > 0$ such that $[0, \epsilon] \subset A$. Differentiating the objective function in (A.9) with respect to α , by the optimality of $\alpha = 0$ on $[0, \epsilon]$ we obtain

$$\mathbb{E}[R(\theta^*, \psi^*, \psi^*)^{-\gamma} (R_l^k - \min\{\mathbf{R} \cdot \mathbf{c}^k, R_b^{*k}\})] \leq 0, \quad (\text{A.10})$$

where $\mathbf{R} = (R^j)_{j=1}^J$ is the investment return, $(R_b^{*k})_{k=1}^K$ are equilibrium borrowing rates, and $R_l^k = \mathbb{E}[\min\{\mathbf{R} \cdot \mathbf{c}^k, R_b^{*k}\} \mid \mathbf{R}_a]$ is the return on ABS k conditional on the aggregate shock \mathbf{R}_a . Let X be the single idiosyncratic factor and given $X = x$ define

$$\begin{aligned} f(x; \mathbf{R}_a) &= R_l^k - \min\{\mathbf{R} \cdot \mathbf{c}^k, R_b^{*k}\}, \\ g(x; \mathbf{R}_a) &= R(\theta^*, \psi^*, \psi^*)^{-\gamma}. \end{aligned}$$

Fixing \mathbf{R}_a , since $R^j = r^j(X)R_a^j$ and $r^j(x)$ is increasing in x , clearly f is decreasing and $\mathbb{E}[f(X; \mathbf{R}_a) \mid \mathbf{R}_a] = 0$ by the definition of the return on ABS. Since

there is active default, we have $f(X; \mathbf{R}_a) \neq 0$ with positive probability. Finally, by (4.3) the return on portfolio

$$\begin{aligned} R(\theta, \phi, \psi) &= \sum_j R^j \theta^j + \sum_k R_l^k \phi^k - \sum_k \min \{ \mathbf{R} \cdot \mathbf{c}^k, R_b^k \} \psi^k \\ &= \sum_j R^j \left(\theta^j - \sum_k c^{jk} \psi^k \right) + \sum_k R_l^k \phi^k + \sum_k \max \{ 0, \mathbf{R} \cdot \mathbf{c}^k - R_b^k \} \psi^k \end{aligned}$$

is increasing in X , so g is decreasing and clearly nonconstant. Therefore by the conditional version of Lemma A.2 we obtain

$$\mathbb{E}[f(X; \mathbf{R}_a)g(X; \mathbf{R}_a)] = \mathbb{E}[\mathbb{E}[f(X; \mathbf{R}_a)g(X; \mathbf{R}_a) | \mathbf{R}_a]] > 0.$$

However, (A.10) implies $\mathbb{E}[f(X; \mathbf{R}_a)g(X; \mathbf{R}_a)] \leq 0$, a contradiction. \square

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