

# A Law of the Iterated Logarithm for Grenander's Estimator



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# Outline

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- 1: Introduction: classical LIL's.
- 2: Grenander's estimator: the MLE of a decreasing density.
- 3: Chernoff's distribution and tail behavior.
- 4: A LIL for Grenander's estimator.
- 5: Summary; further questions and open problems.

# 1. Introduction: Classical LIL's

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- Setting:  $X_1, X_2, \dots, X_n, \dots$  i.i.d. random variables with  $\mu = E(X_1) = 0$ ,  $Var(X_1) = \sigma^2$ .
- $S_n \equiv \sum_{i=1}^n X_i = n\bar{X}_n$ .
- Then, with  $Z \sim N(0, 1)$ ,

$$\frac{S_n}{\sqrt{n}} = \sqrt{n}(\bar{X}_n - \mu) \rightarrow_d \sigma Z$$

**Theorem.** (Hartman and Wintner (1941)).

$$\limsup_{n \rightarrow \infty} \frac{S_n}{\sqrt{2n \log \log n}} = +\sigma \quad \text{a.s.},$$
$$\liminf_{n \rightarrow \infty} \frac{S_n}{\sqrt{2n \log \log n}} = -\sigma \quad \text{a.s..}$$

Moreover, Strassen (1964) showed that

$$\frac{S_n}{\sqrt{2n \log \log n}} \rightsquigarrow [-\sigma, \sigma] \quad \text{a.s.} \dots \text{and} \dots$$

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with

$$S_n(t) \equiv \begin{cases} \frac{S_k}{\sqrt{2n \log \log n}} & \text{at } t = k/n, \\ \text{linearly interp.} & \text{on } k/n \leq t \leq (k+1)/n, \quad k = 0, 1, \dots, n-1, \end{cases}$$

we have

$$S_n \rightsquigarrow \sigma \mathcal{K} \quad \text{a.s.}$$

where

$$\mathcal{K} \equiv \left\{ f \in C[0, 1] : f(t) = \int_0^t \dot{f}(s) ds, \text{ some } \dot{f}, \int_0^1 \dot{f}^2(s) ds \leq 1 \right\}.$$

**Corollary.**

$$\limsup_{n \rightarrow \infty} \frac{S_n}{\sqrt{2n \log \log n}} = \limsup_{n \rightarrow \infty} S_n(1) = \sigma \sup_{f \in \mathcal{K}} f(1) = \sigma \cdot 1$$

where the supremum in the last line is achieved at  $f(t) = t$ ,  $0 \leq t \leq 1$ .

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## Connection with exponential bounds:

Suppose that  $X_i \sim N(0, 1)$  for  $1 \leq i \leq n$ .

- $n^{-1/2}S_n \sim N(0, 1)$ ,  $\sigma = 1$ , and

$$P\left(n^{-1/2}S_n > x\right) = 1 - \Phi(x) \leq x^{-1}\phi(x) = \frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{-x^2/2}.$$

- Thus with  $b_n \equiv (2\log\log n)^{1/2}$ , and  $\delta > 0$ ,

$$\begin{aligned} & P(n^{-1/2}S_n > (1 + \delta)(2\log\log n)^{1/2}) \\ & \leq \frac{1}{\sqrt{2\pi}} \frac{1}{(1 + \delta)b_n} \exp(-(1 + \delta)^2 \log\log n) \\ & \equiv \frac{C}{b_n} (\log n)^{-(1+\delta)^2} \\ & \sim \frac{C}{b_{n_k}} (k \log \alpha)^{-(1+\delta)^2}, \quad n_k \equiv \lfloor \alpha^k \rfloor, \quad \alpha > 1. \end{aligned}$$

- Therefore, by the Borell-Cantelli lemma,

$$P(n_k^{-1/2}S_{n_k} > (1 + \delta)(2\log\log n_k)^{1/2} \text{ i.o.}) = 0.$$

## 2. Grenander's estimator

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- $X_1, X_2, \dots, X_n$  are i.i.d. with density  $f$  on  $\mathbb{R}^+ \equiv [0, \infty)$ .
- Then Grenander's estimator  $\hat{f}_n$  of  $f$  is the Maximum Likelihood Estimator of  $f_0$  over the class of decreasing densities on  $\mathbb{R}^+$ : if

$$L_n(g) \equiv n^{-1} \sum_{i=1}^n \log\{g(X_i)\} = \mathbb{P}_n(\log g)$$

then  $\hat{f}_n$  satisfies

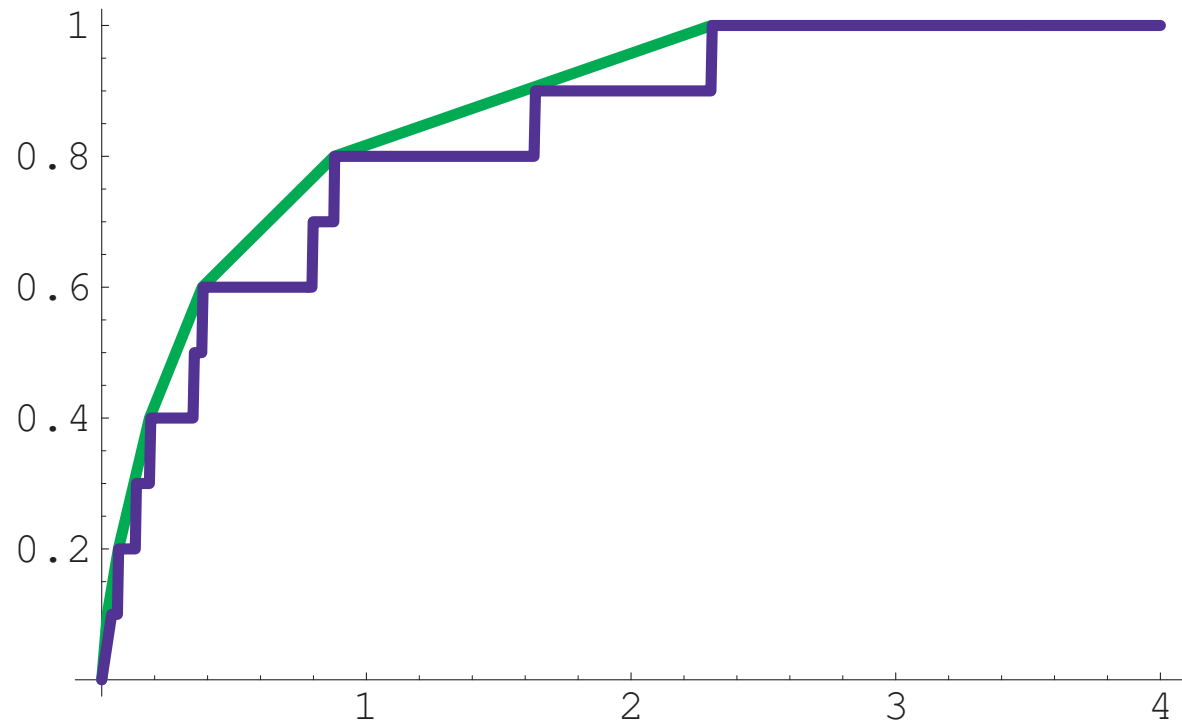
$$L_n(\hat{f}_n) = \max\{L_n(g) : g \text{ a monotone decreasing density on } \mathbb{R}^+\}.$$

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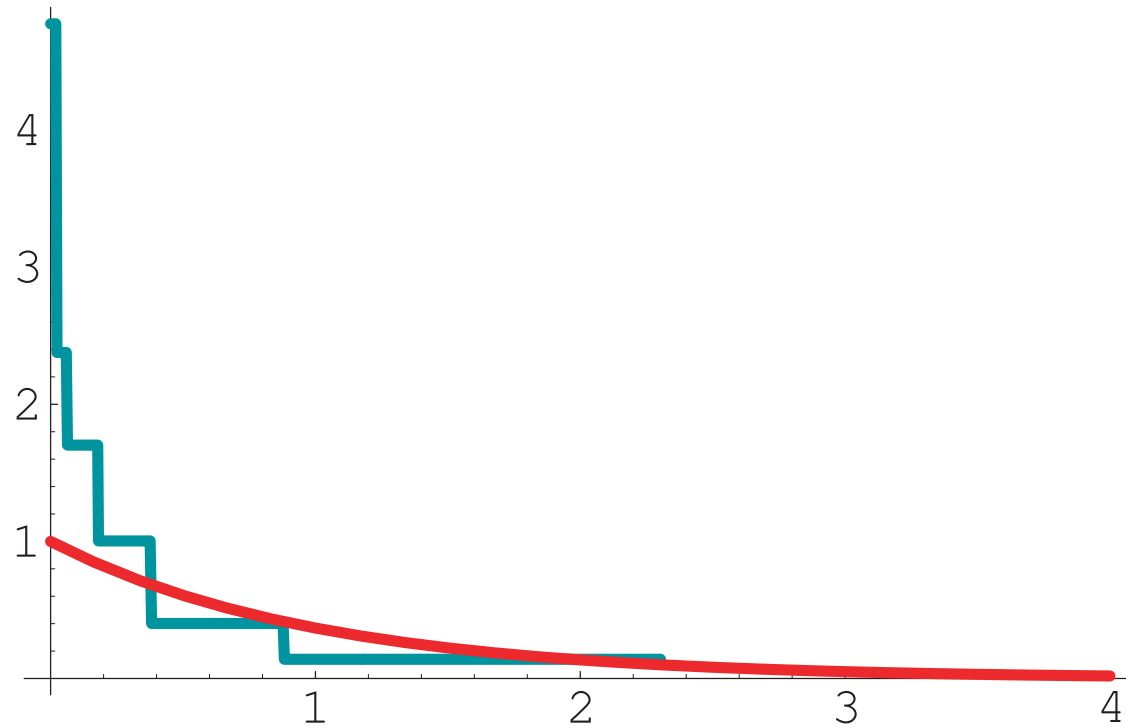
Grenander (1956) showed that the maximum likelihood estimator  $\hat{f}_n$  of  $f$  is the (left-) derivative of the least concave majorant of the empirical distribution function  $\mathbb{F}_n$

$\hat{f}_n =$  left derivative of the least concave majorant of  $\mathbb{F}_n$ ,  
the empirical distribution of  $X_1, \dots, X_n$  i.i.d.  $F$

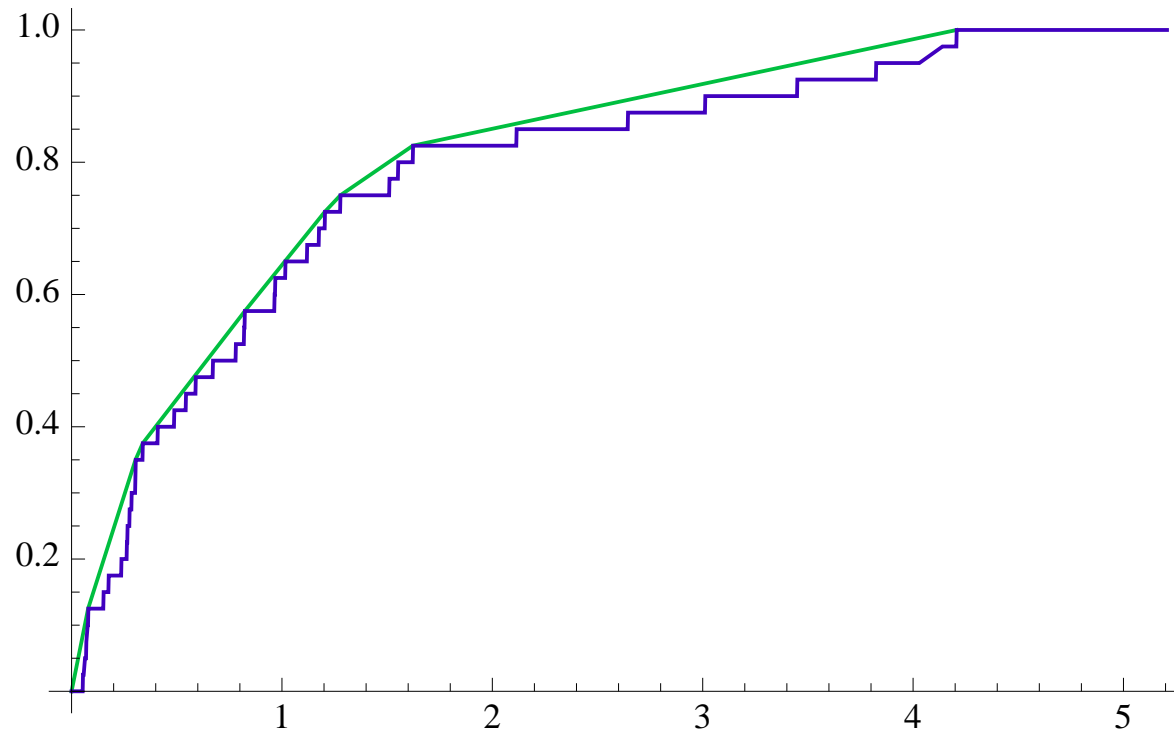




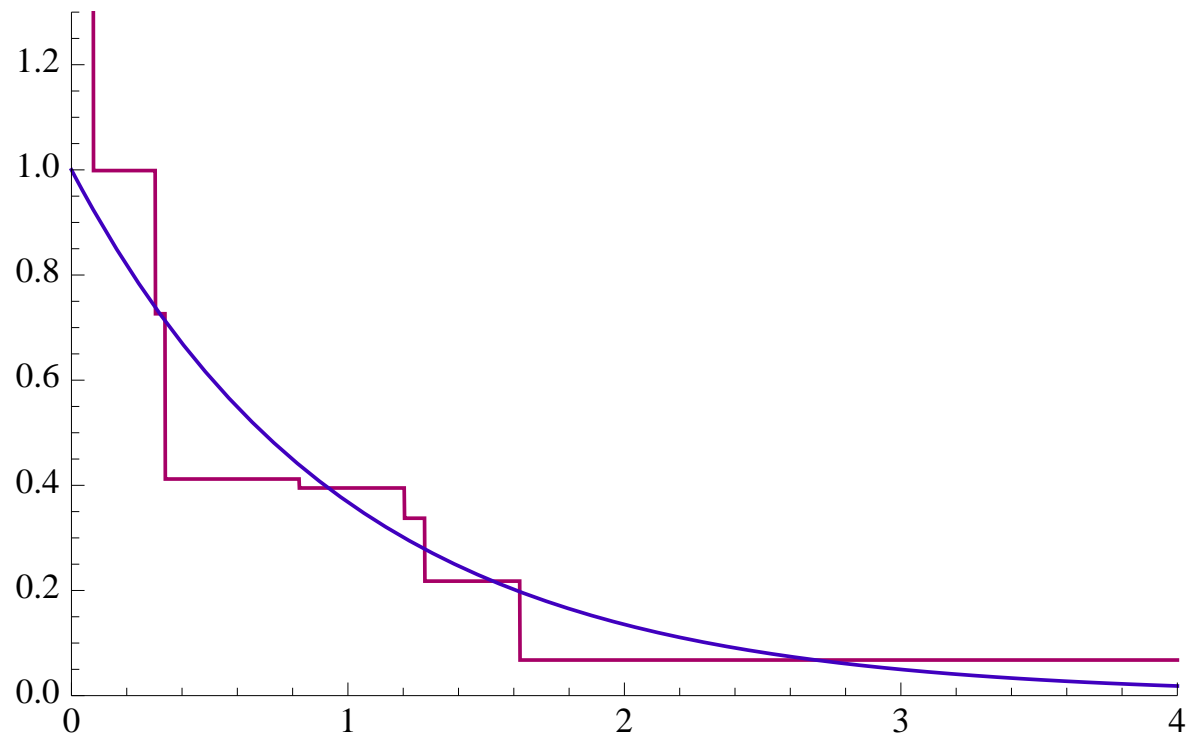
Least Concave Majorant and Empirical  $n = 10$



Grenander Estimator and Exp(1) density,  $n = 10$



Least Concave Majorant and Empirical  $n = 40$



Grenander Estimator and  $\text{Exp}(1)$  density,  $n = 40$

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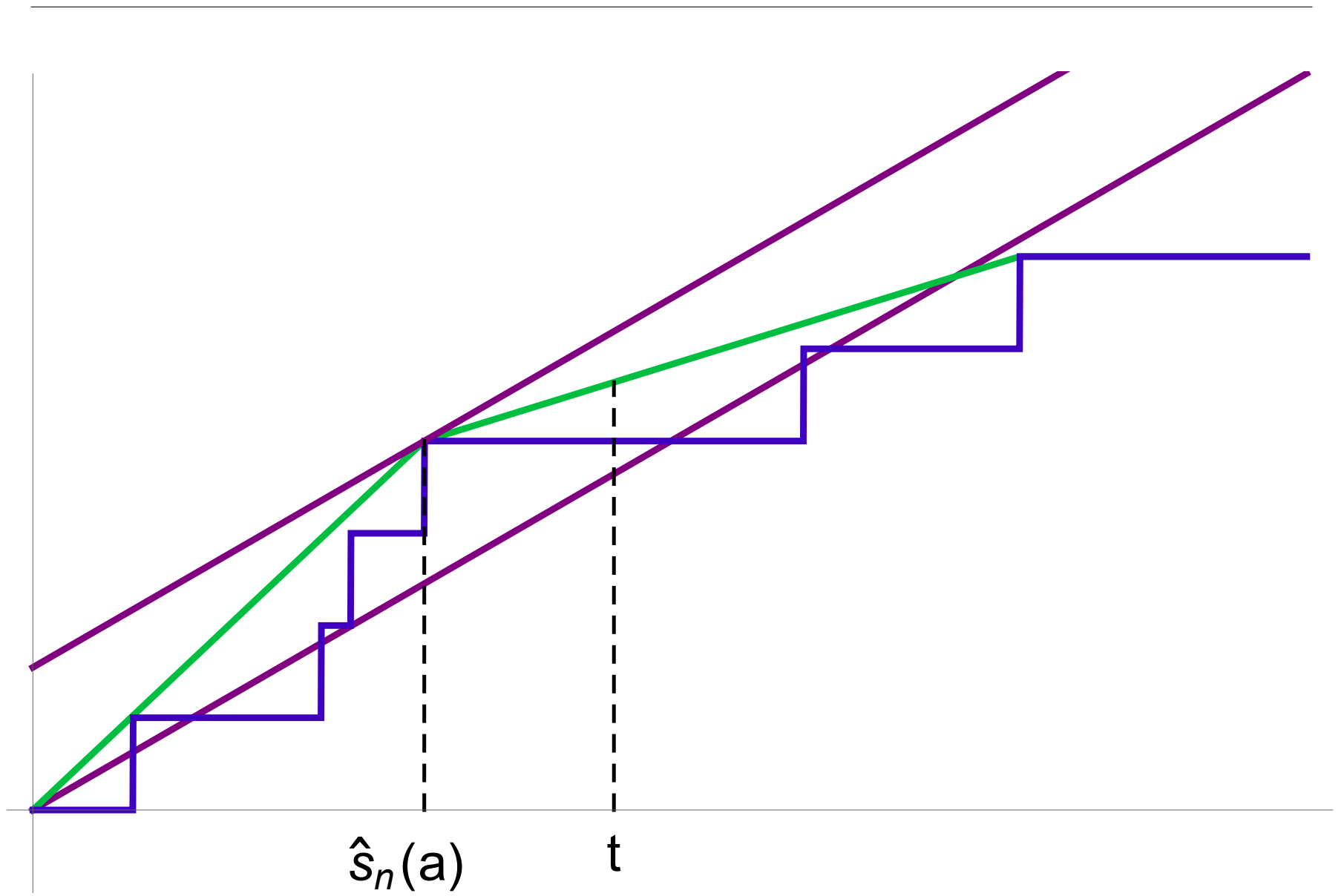
How to study  $\hat{f}_n$ ? Groeneboom's switching relation!

**Switching for  $\hat{f}_n$ :** Define

$$\begin{aligned}\hat{s}_n(a) &\equiv \operatorname{argmax}_{s \geq 0} \{\mathbb{F}_n(s) - as\}, \quad a > 0 \\ &\equiv \sup\{s \geq 0 : \mathbb{F}_n(s) - as = \sup_{z \geq 0} (\mathbb{F}_n(z) - az)\}.\end{aligned}$$

Then for each fixed  $t \in (0, \infty)$  and  $a > 0$

$$\{\hat{f}_n(t) < a\} = \{\hat{s}_n(a) < t\}.$$



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**Theorem.** (Prakasa Rao (1969), Groeneboom (1985)): If  $f(x) > 0$ ,  $f'(x) < 0$ , and  $f'$  is continuous at  $x$ , then with  $c \equiv (4f(x)/(f'(x))^2)^{1/3}$  and  $C \equiv (2^{-1}f(x)|f'(x)|)^{1/3}$ ,

$$\begin{aligned} S_n(x, t) &\equiv n^{1/3} \left( \hat{f}_n(x + n^{-1/3}ct) - f(x) \right) \\ &\rightarrow_d C \cdot S(t) \end{aligned}$$

where

$W$  is a standard two-sided Brownian motion starting at 0, and

$S \equiv$  the left - derivative of the least concave majorant  $\mathbb{C}$  of  $W(t) - t^2$ .

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In particular, with  $t = 0$ ,

$$n^{1/3} (\hat{f}_n(x) - f(x)) \rightarrow_d CS(0) \stackrel{d}{=} C2Z$$

where the distribution of  $Z \equiv \operatorname{argmax}_h \{W(h) - h^2\}$  is  
Chernoff's distribution .

**Proof:** By the switching relation  $\{\hat{f}_n(t) < a\} = \{\hat{s}_n(a) < t\}$ :

$$\begin{aligned} & P(n^{1/3}(\hat{f}_n(x_0 + n^{-1/3}t) - f(x_0)) < y) \\ &= P(\hat{f}_n(x_0 + n^{-1/3}t) < f(x_0) + yn^{-1/3}), \\ &= P(\hat{s}_n(f(x_0) + yn^{-1/3}) < x_0 + n^{-1/3}t) \\ &= P(\operatorname{argmax}_v \{\mathbb{F}_n(v) - (f(x_0) + n^{-1/3}y)v\} < x_0 + n^{-1/3}t) \end{aligned}$$



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Now we change variables  $v = x_0 + n^{-1/3}h$  in the argument of  $\mathbb{F}_n$ , center and scale to find: the right side in the last display equals

$$\begin{aligned}
& P(\operatorname{argmax}_h \{ \mathbb{F}_n(x_0 + n^{-1/3}h) - (f(x_0) + n^{-1/3}y)(x_0 + n^{-1/3}h) \} < t) \\
&= P\left(\operatorname{argmax}_h \{ \mathbb{F}_n(x_0 + n^{-1/3}h) - \mathbb{F}_n(x_0) \right. \\
&\quad \left. - (F(x_0 + n^{-1/3}h) - F(x_0)) \right. \\
&\quad \left. + F(x_0 + n^{-1/3}h) - F(x_0) - f(x_0)n^{-1/3}h - n^{-2/3}yh \} < t\right). \tag{1}
\end{aligned}$$

Let  $\mathbb{G}_n(t) \equiv n^{-1} \sum_{i=1}^n \mathbf{1}\{\xi_i \leq t\}$ ,  $\mathbb{U}_n(t) \equiv \sqrt{n}(\mathbb{G}_n(t) - t)$  with  $\xi_i$  i.i.d. Uniform $[0, 1]$ . Thus  $\mathbb{F}_n \stackrel{d}{=} \mathbb{G}_n(F)$ . The the stochastic term in (1) satisfies, with  $W \equiv$  two-sided standard Brownian motion,

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$$\begin{aligned}
& n^{2/3} \left\{ \mathbb{F}_n(x_0 + n^{-1/3}h) - \mathbb{F}_n(x_0) - (F(x_0 + n^{-1/3}h) - F(x_0)) \right\} \\
& \stackrel{d}{=} n^{2/3-1/2} \left\{ \mathbb{U}_n(F(x_0 + n^{-1/3}h)) - \mathbb{U}_n(F(x_0)) \right\} \\
& = n^{1/6} \left\{ \mathbb{U}(F(x_0 + n^{-1/3}h)) - \mathbb{U}(F(x_0)) \right\} + o_p(1) \quad \text{by KMT} \\
& \stackrel{d}{=} n^{1/6} W(f(x_0)n^{-1/3}h) + o_p(1) \\
& \stackrel{d}{=} \sqrt{f(x_0)} W(h) + o_p(1)
\end{aligned}$$

where  $W$  is a standard two-sided Brownian motion process starting from 0. On the other hand, with  $\delta_n \equiv n^{-1/3}$ ,

$$\begin{aligned}
& n^{2/3} \left( F(x_0 + n^{-1/3}h) - F(x_0) - f(x_0)n^{-1/3}h \right) \\
& = \delta_n^{-2} \left( F(x_0 + \delta_n h) - F(x_0) - f(x_0)\delta_n h \right) \\
& \rightarrow -b|h|^2 \quad \text{with } b = |f'(x_0)|/2
\end{aligned}$$

by our hypotheses, while  $n^{2/3}n^{-1/3}n^{-1/3}h = n^0h = h$ .

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Thus it follows that the last probability above converges to

$$\begin{aligned}
 & P\left(\operatorname{argmax}_h \{aW(h) - b|h|^2 - yh\} < t\right) \\
 &= \begin{cases} P(\mathbb{S}_{a,b}(t) < y) & \text{by switching again} \\ P\left(\left(\frac{a}{b}\right)^{2/3} \operatorname{argmax}_h \{W(h) - h^2\} - (2b)^{-1}y < t\right) & \text{, by (2) below} \end{cases}
 \end{aligned}$$

where

$$\begin{aligned}
 \mathbb{S}_{a,b}(t) &= \text{slope at } t \text{ of the least concave majorant of} \\
 & aW(h) - bh^2 \equiv \sqrt{f_0(x_0)}W(h) - |f'_0(x_0)||h|^2/2 \\
 & \stackrel{d}{=} |2^{-1}f_0(x_0)f'_0(x_0)|\mathbb{S}(t/c_0).
 \end{aligned}$$

and where we used

$$\operatorname{argmax}\{aW(h) - bh^2\} \stackrel{d}{=} \left(\frac{a}{b}\right)^{2/3} \operatorname{argmax}\{W(h) - h^2\} - \frac{1}{2b}y. \quad (2)$$

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First appearance of  $Z$ :

Chernoff (1964), *Estimation of the mode*:

- $X_1, \dots, X_n$  i.i.d. with density  $f$  and distribution function  $F$ .
- Fix  $a > 0$ ;  $\hat{x}_a \equiv$  center of the interval of length  $2a$  containing the most observations.
- $x_a \equiv$  center of the interval of length  $2a$  maximizing  $F(x + a) - F(x - a)$ .
- Chernoff shows:
  - ▶  $n^{1/3}(\hat{x}_a - x_a) \rightarrow_d \left( \frac{8f(x_a + a)}{c} \right)^{1/3} Z$   
where  $c \equiv f'(x_a - a) - f'(x_a + a)$ .
  - ▶  $f_Z(z) = \frac{1}{2}g(z)g(-z)$  where

$$g(t) \equiv \lim_{x \nearrow t^2} \frac{\partial}{\partial x} u(t, x) = \lim_{x \nearrow t^2} u_x(t, x),$$

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- ▷  $u(t, x) \equiv P^{(t,x)}(W(z) > z^2, \text{ for some } z \geq t)$  is a solution to the backward heat equation

$$\frac{\partial}{\partial t} u(t, x) = -\frac{1}{2} \frac{\partial^2}{\partial x^2} u(t, x)$$

under the boundary conditions

$$u(t, t^2) = \lim_{x \nearrow t^2} u(t, x) = 1, \quad \lim_{x \rightarrow -\infty} u(t, x) = 0.$$

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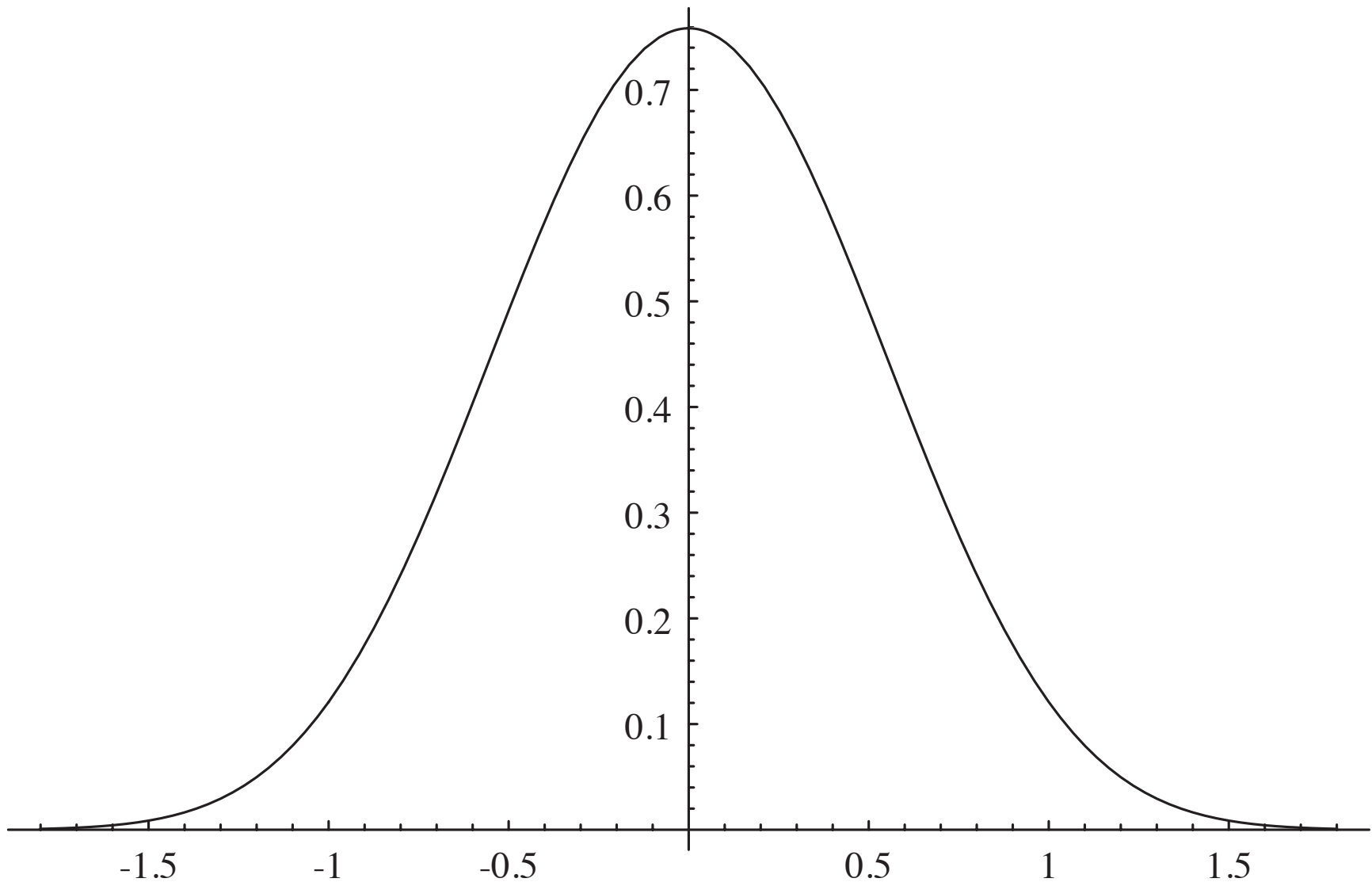
Groeneboom (1989) showed that  $g$  has Fourier transform given by

$$\hat{g}(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda s} g(s) ds = \frac{2^{1/3}}{Ai(i(2)^{-1/3}\lambda)}; \quad (3)$$

Groeneboom (1989) also showed that

$$f_Z(z) \sim \frac{1}{2Ai'(a_1)} 4^{4/3} z \exp\left(-\frac{2}{3}z^3 + 3^{1/3}a_1z\right),$$
$$P(Z > z) \sim \frac{1}{2Ai'(a_1)} 4^{4/3} \frac{1}{z} \exp\left(-\frac{2}{3}z^3\right) \quad \text{as } z \rightarrow \infty$$

where  $a_1 \approx -2.3381\dots$  is the largest zero of the Airy function  $Ai$  on the negative real axis and  $Ai'(a_1) \approx 0.7022$ .



## 4. A LIL for Grenander's estimator

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The tail behavior of  $Z$  given in the last display leads naturally to the following conjecture concerning a LIL for the Grenander estimator:

$$\limsup_{n \rightarrow \infty} \frac{n^{1/3}(\hat{f}_n(x_0) - f(x_0))}{((3/2)\log\log n)^{1/3}} =_{a.s.} \left| \frac{1}{2}f(x_0)f'(x_0) \right|^{1/3} \cdot 2;$$

note that if  $\delta > 0$  and  $n_k \equiv \lfloor \alpha^k \rfloor$  with  $\alpha > 1$ , then

$$\begin{aligned} & \exp\left(-\frac{2}{3}[(1+\delta)((3/2)\log\log n_k)^{1/3}]^3\right) \\ &= \exp\left(-(1+\delta)^3 \log\log n_k\right) \\ &= (\log n_k)^{-(1+\delta)^3} \sim \left(k \log \alpha\right)^{-(1+\delta)^3}. \end{aligned}$$

Equivalently

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{n^{1/3}(\hat{f}_n(x_0) - f(x_0))}{(2\log\log n)^{1/3}} &=_{a.s.} \left| \frac{1}{2}f(x_0)f'(x_0) \right|^{1/3} \cdot 2 \cdot \frac{1}{2^{1/3}} \cdot \left(\frac{3}{2}\right)^{1/3} \\ &=_{a.s.} \left| \frac{1}{2}f(x_0)f'(x_0) \right|^{1/3} \cdot 2 \cdot \left(\frac{3}{4}\right)^{1/3}. \end{aligned}$$



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How to prove this conjecture?

- Step 1: Switching!
- Step 2: localize; use a functional LIL for the local empirical process: Deheuvels and Mason (1994); Mason (2004).
- Step 3: Find the set of limit points via 1 and 2; study these via properties of the natural Strassen limit sets analogous to the distributional equivalents for Brownian motion.
- Step 4: Solve the resulting variational problem over the set of limit points expressed in terms of the Strassen limit points.

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**Step 1: Switching.** Let  $r_n \equiv (n^{-1}2\log\log n)^{1/3}$ .

We want to find a number  $y_0$  such that

$$P(r_n^{-1}(\hat{f}_n(x_0) - f(x_0)) > y \text{ i.o.}) = \begin{cases} 0, & \text{if } y > y_0, \\ 1, & \text{if } y < y_0. \end{cases}$$

Now  $\{\hat{f}_n(x_0) > a\} = \{\hat{s}_n(a) > x_0\}$ , so

$$\{\hat{f}_n(x_0) > f(x_0) + r_n y \text{ i.o.}\} = \{\hat{s}_n(f(x_0) + r_n y) > x_0 \text{ i.o.}\}. \quad (4)$$

By letting  $s = x_0 + r_n h$  in the definition of  $\hat{s}_n$  we see that

$$\begin{aligned} & \hat{s}_n(f(x_0) + r_n y) - x_0 \\ &= r_n \operatorname{argmax}_h \{\mathbb{F}_n(x_0 + r_n h) - f(x_0 + r_n y)(x_0 + r_n h)\} \end{aligned}$$

and hence the right side of (4) can be rewritten as  $\{\hat{h}_n > 0 \text{ i.o.}\}$  where

$$\begin{aligned} \hat{h}_n &= \operatorname{argmax}_h \{\mathbb{F}_n(x_0 + r_n h) - (f(x_0) + r_n y)(x_0 + r_n h)\} \\ &= \operatorname{argmax}_h \left\{ r_n^{-2} [\mathbb{F}_n(x_0 + r_n h) - \mathbb{F}_n(x_0) - (F(x_0 + r_n h) - F(x_0))] \right. \\ &\quad \left. + r_n^{-2} [F(x_0 + r_n h) - F(x_0) - f(x_0)r_n h] - y h \right\}. \quad (5) \end{aligned}$$

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The deterministic (drift) term on the right side converges to  $f'(x_0)h^2/2$  as  $n \rightarrow \infty$ .

**Step 2: LIL for the local empirical process.** By a theorem of Mason (1988), the sequence of functions

$$\left\{ r_n^{-2} [\mathbb{F}_n(x_0 + r_n h) - \mathbb{F}_n(x_0) - (F(x_0 + r_n h) - F(x_0))] : h \in R \right\}$$

is almost surely relatively compact with limit set

$$\{g(f(x_0)\cdot) : g \in \mathcal{G}\}$$

where the two-sided Strassen limit set  $\mathcal{G}$  is given by

$$\mathcal{G} = \left\{ g : \mathbb{R} \rightarrow \mathbb{R} \mid g(t) = \int_0^t \dot{g}(s) ds, t \in \mathbb{R}, \int_{-\infty}^{\infty} \dot{g}^2(s) ds \leq 1 \right\}.$$

**Proof:** Reduce to the local empirical process of  $\xi_1, \dots, \xi_n$  i.i.d.  $\text{Uniform}(0, 1)$ . As in Mason (1988), take  $k_n \equiv nr_n = n^{2/3}(2\log\log n)^{1/3} \nearrow \infty$  and  $n^{-1}k_n = r_n \searrow 0$ .

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Thus the processes involved in the argmax in (5) are almost surely relatively compact with limit set

$$\{g(f(x_0)h) + 2^{-1}f'(x_0)h^2 - yh : g \in \mathcal{G}\} \quad (6)$$

### Step 3: Strassen set equivalences.

**Lemma 1.** Let  $c > 0$  and  $d \in R$ . then

$$\{t \mapsto g(ct + d) - g(d) : g \in \mathcal{G}\} = \sqrt{c}\mathcal{G}.$$

[This is analogous to  $W(ct + d) - W(d) \stackrel{d}{=} \sqrt{c}W(t)$ .]

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**Lemma 2.** Let  $\alpha, \beta$  be positive constants and  $\gamma \in R$ . Then

$$\begin{aligned} & \{\operatorname{argmax}\{\alpha g(h) - \beta h^2 - \gamma h\} : g \in \mathcal{G}\} \\ &= \{(\alpha/\beta)^{2/3} \operatorname{argmax}_h\{g(h) - h^2\} - \gamma/(2\beta) : g \in \mathcal{G}\}. \end{aligned}$$

[This is analogous to

$$\operatorname{argmax}(\alpha W(h) - \beta h^2 - \gamma h) \stackrel{d}{=} (\alpha/\beta)^{2/3}\{W(h) - h^2\} - \gamma/(2\beta).]$$

By Lemmas 1 and 2, with  $a = \sqrt{f(x_0)}$ ,  $b = |f'(x_0)|/2$ ,

$$\begin{aligned} & \{g(f(x_0)h) + 2^{-1}f'(x_0)h^2 - yh : g \in \mathcal{G}\} \\ &= \{ag(h) - bh^2 - yh : g \in \mathcal{G}\} \\ &= \{(a/b)^{2/3} \operatorname{argmax}\{g(h) - h^2\} - y/(2b) : g \in \mathcal{G}\} \end{aligned}$$

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Hence with  $T_g \equiv \operatorname{argmax}_h \{g(h) - h^2\}$ ,

$$\begin{aligned} \{\widehat{h}_n > 0 \text{ i.o.}\} &\stackrel{a.s.}{=} \left\{ \left(\frac{a}{b}\right)^{2/3} \sup_{g \in \mathcal{G}} T_g > \frac{y}{2b} \right\} \\ &= \left\{ 2b \left(\frac{a}{b}\right)^{2/3} \sup_{g \in \mathcal{G}} T_g > y \right\} \\ &= \emptyset \end{aligned}$$

if

$$y > y_0 \equiv 2b \left(\frac{a}{b}\right)^{2/3} \sup_{g \in \mathcal{G}} T_g = |2^{-1} f(x_0) f'(x_0)|^{1/3} 2 \sup_{g \in \mathcal{G}} T_g.$$

It remains only to show that

$$\sup_{g \in \mathcal{G}} T_g = (3/4)^{1/3} \approx 0.90856 \dots$$

This follows from Lemma 3:

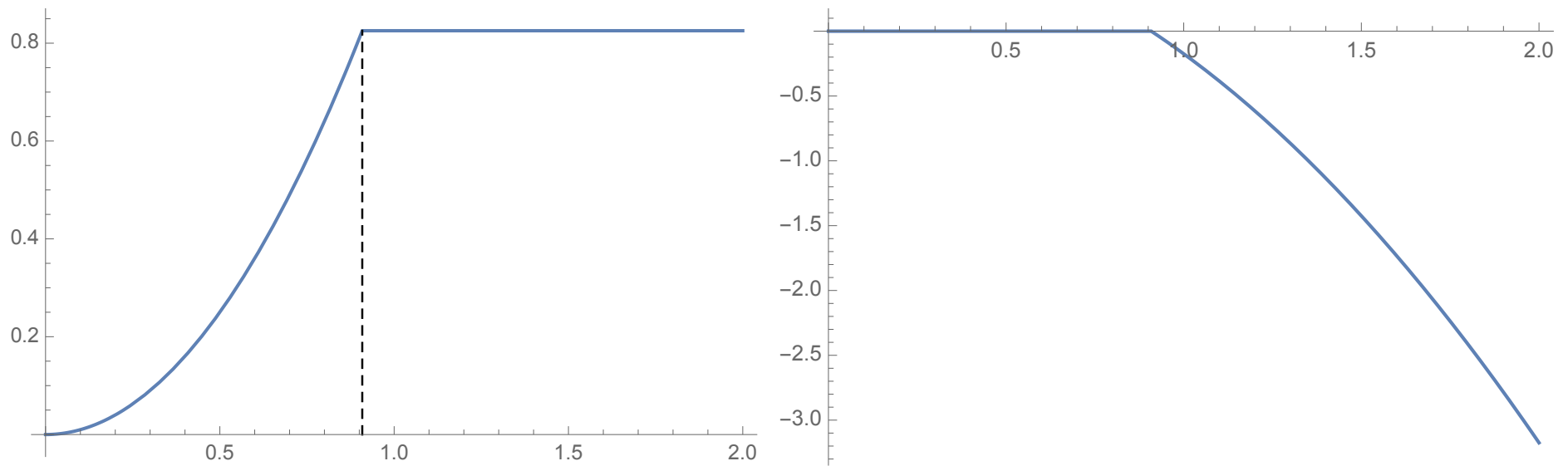
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**Lemma 3.** Let  $t_0$  be an arbitrary positive number and let  $\dot{g} \in L_1([0, t_0])$  be an arbitrary function satisfying

$$\int_0^{t_0} \dot{g}(s) ds - t_0^2 \geq \int_0^t \dot{g}(s) ds - t^2 \quad \text{for } 0 \leq t \leq t_0.$$

Then, with  $\dot{g}_0(u) \equiv 2u$ ,  $0 \leq t \leq u$ ,

$$\int_0^{t_0} \dot{g}(u)^2 du \geq \int_0^{t_0} \dot{g}_0(u)^2 du = \int_0^{t_0} (2u)^2 du = \frac{4t_0^3}{3}.$$



$g_0(t)$  and  $g_0(t) - t^2$



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**Intuition:** A few pictures and trial functions  $\dot{g}$  quickly lead to

$$\dot{g}_0(s) \equiv \begin{cases} 2s, & 0 \leq s \leq t_0, \\ 0, & t_0 < s < \infty, \\ 0, & t < 0. \end{cases}$$

For this  $\dot{g}_0$  we have

$$g_0(t) = t^2 1_{[0,t_0]}(t) + t_0^2 1_{(t_0,\infty)}(t) + 0 \cdot 1_{(-\infty,0)}(t).$$

Note that  $\int_{-\infty}^{\infty} \dot{g}_0^2(s) ds = \int_0^{t_0} (2s)^2 ds = 4t_0^3/3$ . and that

$$g_0(t) - t^2 = \begin{cases} 0, & t \leq t_0, \\ t_0^2 - t^2, & t > t_0. \end{cases}$$

Thus  $\operatorname{argmax}(g_0(t) - t^2) = t_0$  while  $\sup_{t \geq 0} (g_0(t) - t^2) = 0$  is achieved for all  $0 \leq t \leq t_0$ . To force  $g_0 \in \mathcal{G}$  we simply require that  $\int_0^{t_0} \dot{g}_0^2(s) ds = 1$  and this yields  $4t_0^3/3 = 1$  or  $t_0 = (3/4)^{1/3}$ .

**Proof.**

## 6. Open questions and further problems:

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- For the mode estimator  $M(\hat{f}_n)$  of a log-concave density with mode  $m \equiv M(f)$  and  $f''(m) < 0$ , we know that

$$n^{1/5}(M(\hat{f}_n) - M(f)) \rightarrow_d C_2 M(H^{(2)})$$

where  $H^{(2)}$  is the convex function given by the second derivative of the “invelope process”

$$H \text{ of } Y(t) \equiv t^4 + \int_0^t W(s)ds.$$

Do we have a **LIL** for  $M(\hat{f}_n)$ :

$$\limsup_{n \rightarrow \infty} \frac{n^{1/5}(M(\hat{f}_n) - M(f))}{(2 \log \log n)^{1/5}} \stackrel{a.s.}{=} \tilde{C}_2 < \infty?$$

- With definitions as in the last problem, do we have

$$P(M(H^{(2)}) \geq t) \leq K_1 \exp(-K_2 t^5) \text{ for all } t \geq 0$$

for some  $K_1, K_2 < \infty$ ? An answer to the previous problem would start to give information about  $K_2$ !

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- Are the touch points of  $H$  and  $Y$  isolated? (See Groeneboom, Jongbloed, & W (2001), Groeneboom and Jongbloed (2015), page 320.)
  - Balabdaoui and W (2014) show that Chernoff's distribution is log-concave. Is it strongly log-concave?
  - Is there a Berry-Esseen type result for Grenander's estimator? That is, for what sequences  $r_n \rightarrow \infty$  do we have

$$\sup_{t \in \mathbb{R}} \left| P \left( n^{1/3} (\hat{f}_n(x) - f(x)) / C \leq t \right) - P(2Z \leq t) \right| \leq \frac{K}{r_n}$$

for some  $K$  finite.

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