Settled equilibria

Roger Myerson*† and Jörgen Weibull‡

September 3, 2012. This version: October 29, 2013.

Abstract. When people interact in familiar settings, social conventions usually develop so that people tend to disregard alternatives outside the convention. For rational players to usually restrict attention to a block of conventional strategies, no player should prefer to deviate from the block when others are likely to act rationally and conventionally inside it. We explore concepts that formalize this idea for finite normal-form games. Tenable blocks are product sets of pure strategies that have the above-mentioned robustness property. We call Nash equilibria with support in minimal such blocks settled. This approach has substantial cutting power and differs significantly from established solution concepts. We provide a psychological/behavioral micro foundation for our approach and show that it implies properness. We establish existence of settled equilibria in all finite games. Being proper, they induce sequential equilibria in all extensive-form games with the given normal form.

Keywords: Settled equilibrium, tenable block, convention, norm.
JEL-codes: C70, C72, C73, D01, D02, D03.

*Department of Economics, University of Chicago.

†Earlier versions of this manuscript have been presented in seminars at Chicago University, Ecole Polytechnique, at the GAMES 2012 world congress in Istanbul and at the conference in Jerusalem, June 2013, in honor of Jean-Francois Mertens. We thank Erik Mohlin, Olivier Tercieux and Troels Sørensen for helpful comments.

‡Department of Economics, Stockholm School of Economics; Institute for Advanced Studies in Toulouse; Department of Mathematics, KTH Royal Institute of Technology, Stockholm.

1. INTRODUCTION

Schelling (1960) pointed out the importance of pure coordination problems. In such problems the participants have common interests but there are multiple ways to coordinate. Sometimes one of the solutions may be "salient" (Schelling, 1960). However, in most situations we must in practice rely on what Lewis (1969) calls "precedent" in order to solve coordination problems. If all participants know that a particular coordination problem has been solved in a particular way numerous times before, and this is common knowledge among them, then this may help them solve a current coordination problem. That they have solved the problem successfully will over time usually be seen by others and thus the convention may spread in society.

More generally, consider a large population that plays familiar games, not necessarily coordination games, in historical and cultural contexts where individuals know how similar games have been played in the past. When people interact in familiar settings, social norms or conventions usually develop, specifying which actions or decision alternatives individuals are expected to, or should, consider.[1] Such informal institutions, norms or conventions develop over time and people tend to disregard alternatives that are physically available to them but fall outside the norm or convention. In order for such a conventions to persist, it should not be advantageous for an individual to take an unconventional action if others are very likely to take conventional actions. This setting is close to that in Nash's (1950) so-called mass-action interpretation:

> "It is unnecessary to assume that the participants in a game have full knowledge of the total structure of the game, or the ability and inclination to go through any complex reasoning processes. But the participants are supposed to accumulate empirical information on the relative advantages of the various pure strategies at their disposal. ... To be more detailed, we assume that there is a population (in the sense of statistics) of participants for each position of the game. Let us also assume that the 'average playing' of the game involves n participants selected at random from the n populations, and that there is a stable average frequency with which each pure strategy is employed ... The assumptions we made in this 'mass action' interpretation lead to the conclusion that the mixed strategies representing the average behavior in each of the populations form an equilibrium."

We here represent the interaction at hand as a finite game in normal form. A *block* in such a game is a non-empty set of pure strategies for each player role. We view a block as a potential norm or convention, a candidate for what strategies individuals

[1] A convention is a pattern of behavior that is customary, expected, and self-enforcing, see Lewis (1969). See also Young (1993,1998).

are likely to seriously consider when called upon to play the game in their player role. The associated restricted game, or *block game*, is the game in which all players are confined to the block strategies. A block is *curb* (Basu and Weibull, 1991) if every strategy profile with support in the block has all best replies in the block. A block is *absorbing* (Kalai and Samet, 1984) if, for every strategy profile with support in the block, every nearby strategy profile (in the full strategy space of the game) has some best reply in the block.

We call a block *coarsely tenable* if nobody could do better by choosing a strategy outside the block when all other players are very likely to only use strategies in the block and play rationally in the block. We formalize this notion by allowing for the possibility that not all players always consider all strategies available to them, and that a player's consideration set is a random subset of his or her pure strategy set, but most likely it is the player's set of block strategies. More exactly, a block is coarsely tenable if it contains a best reply to every Nash equilibrium of such a random consideration-set game, when other consideration sets than those constituting the block are sufficiently unlikely. This block property is arguably a necessary condition for a convention to be sustainable. A *coarsely settled equilibrium* is any Nash equilibrium (of the whole game) with support in a minimal coarsely tenable block. Minimality is invoked because people tend to forget or disregard unused strategies, and minimality allows players to disregard as many unused pure strategies as possible without upsetting the tenability of the block. This simplifies the convention at hand and saves on players' cognitive costs.[2] Another reason why minimality is desirable is that it comes as a result from certain dynamic models of the formation and stability of conventions (Young (1993, 1998), Hurkens (1995) and Sanchirico (1996)), a topic we will briefly discuss at the very end of the paper.

Although minimal coarsely tenable blocks exclude many unused strategies, some coarsely settled equilibria do not use all pure strategies in their minimal coarsely tenable block. Can the block and equilibrium then still be thought of as a convention or social norm, or can neglect of all or some of the unused pure strategies in the block upset the convention? This concern motivated us to formulate a finer concept of tenability, one that allows tenable blocks to be smaller. We do this by assuming that all consideration sets have positive probability and that consideration sets other than those in the conventional block are much more likely to be large than small. We call a block *finely tenable* if it contains some best reply to each Nash equilibrium in any random consideration-set game of this variety, when (as under coarse tenability) other consideration sets than those constituting the block are sufficiently unlikely. Since we impose additional assumptions on what players are likely to consider, every coarsely tenable block is *a fortiori* also finely tenable. As candidates for conventions

[2] See e.g. Halpern and Pass (2009) and their references.

there are thus more finely than coarsely tenable blocks to choose from. We show that all limits of Nash equilibria of random consideration-set games of the "fine" variety, as the probability for players to look outside the conventional block tends to zero, are proper equilibria. A *finely settled* equilibrium is any equilibrium (of the whole game) that has support in a minimal finely tenable block.

The four mentioned block properties are nested: a curb block is absorbing, an absorbing block is coarsely tenable, and a coarsely tenable block is finely tenable. We define a *fully settled equilibrium* as a proper equilibrium that has support in a minimal finely tenable block that is a subset of a minimal coarsely tenable block that is a subset of a minimal absorbing block that is a subset of a minimal curb block. We show that every finite game has such an equilibrium.

While the notions of coarse and fine tenability in general may differ, for generic normal-form games they actually coincide: a block is coarsely tenable if and only if it finely tenable. While Nash equilibria are generically perfect and proper, this is not true for coarsely and finely settled equilibria. The latter, while being generically identical with each other, constitute a strict subset of the Nash equilibria in an open set of normal-form games. They also constitute a distinct subset from the persistent equilibria (Kalai and Samet, 1984) in an open set of normal-form games.

We illustrate the above reasoning by means of a very simple example, driving on the right or left side of the road:

Game 1:

	a_2	b_2
a_1	$1,1$	$0,0$
b_1	$0,0$	$1,1$

This game has two pure and strict and one mixed Nash equilibrium. All three are proper equilibria that, when viewed as singleton sets, are strategically stable in the sense of Kohlberg and Mertens (1986). If this game is played only once by rational players, in the absence of a cultural, historical or social context, the mixed equilibrium may be a reasonable prediction. Indeed, any strategy profile is rationalizable and thus compatible with common knowledge of the game and the players' rationality (Bernheim,1984; Pearce, 1984). However, if such a game is often played in a large population, in culturally familiar settings, the mixed equilibrium appears unlikely. One would rather expect individuals to develop an understanding that coordinates their expectations at one of the strict equilibria. This intuition is captured by the solution concepts developed here. The intuition is also captured by the notion of persistent equilibrium, and this is true for an open set of 2×2 games like Game 1. However, in other games, persistence and our solution concepts differ, and, as will be detailed below, this is true even under arguably minor elaborations of Game 1.

The rest of the paper is organized as follows. Basic notation and definitions are given in Section 2. Our model of random consideration sets is developed in Section

3. Notions of coarsely tenable blocks and coarsely settled equilibria are introduced and analyzed in Section 4. Section 5 defines finely tenable blocks and finely settled equilibria, as well as the concept of a fully settled equilibrium, and we there also establish the existence of these varieties of blocks and equilibria in all finite games. Section 6 shows that the two varieties of blocks defined here, coarsely and finely tenable blocks, generically coincide. Section 7 concludes.

2. PRELIMINARIES

We consider finite normal-form games $G = \langle N, S, u \rangle$, where $N = \{1, \ldots, n\}$ is the set of players, $S = \times_{i \in N} S_i$ is the non-empty and finite set of pure-strategy profiles, $u : S \to \mathbb{R}^n$ is the combined payoff function, where $u_i(s)$ is i's payoff under pure-strategy profile s. Let m_i be the number of elements of S_i and let $\Delta(S_i)$ denote the set of mixed strategies available to player i:

$$\Delta(S_i) = \left\{ \sigma_i \in \mathbb{R}_+^{m_i} : \sum_{s_i \in S_i} \sigma_i(s_i) = 1 \right\}.$$

A strategy $\sigma_i \in \Delta(S_i)$ is *totally mixed* if it assigns positive probability to all pure strategies. Write $\Delta^o(S_i)$ for this subset. Likewise, a strategy profile is totally mixed if all strategies are totally mixed. Let $M(S) = \times_{i \in N} \Delta(S_i)$ denote the set of mixed-strategy profiles on S and let $M^o(S) = \times_{i \in N} \Delta^o(S_i)$. We extend the domain of each payoff function u_i in the usual way from the finite set S to the polyhedron $M(S)$ by

$$u_i(\sigma) = \sum_{s \in S} \left[\Pi_{j \in N} \sigma_j(s_j) \right] \cdot u_i(s).$$

We use $u_i(s_{-i}, s_i)$ to denote the payoff that player i obtains from pure strategy $s_i \in S_i$ when everyone else plays according to $s \in S$, and likewise for mixed strategies. Let $u_i(\sigma_{-i}, [s_i])$ be the (expected) payoff that player i obtains from pure strategy $s_i \in S_i$ when everyone else plays according to $\sigma \in M(S)$. Two pure strategies $s_i', s_i'' \in S_i$ are *payoff equivalent* if $u_j(s_{-i}, s_i') = u_j(s_{-i}, s_i'')$ for all $s \in S$ and $j \in N$. A *purely reduced* normal form game is a game in which no pure strategies are payoff equivalent.[3] A pure strategy $s_i \in S_i$ is *weakly dominated* if there exists a $\sigma_i' \in \Delta(S_i)$ such that $u_i(\sigma_{-i}, \sigma_i') \geq u_i(\sigma_{-i}, [s_i])$ for all $\sigma \in M(S)$ with strict inequality for some $\sigma \in M(S)$.

Definition 1. *A* ***Nash equilibrium*** *is any strategy profile $\sigma \in M(S)$ such that*

$$u_i(\sigma_{-i}, [s_i]) < \max_{r_i \in S_i} u_i(\sigma_{-i}, [r_i]) \quad \Rightarrow \quad \sigma_i(s_i) = 0.$$

[3] This is also called the *semi-reduced* normal form representation, see e.g. van Damme (1991).

A Nash equilibrium is *strict* if any unilateral deviation incurs a payoff loss.

Definition 2 [Myerson, 1978]**.** *For any $\varepsilon > 0$, a strategy profile $\sigma \in M^o(S)$ is ε-**proper** if*

$$u_i(\sigma_{-i}, [s_i]) < u_i(\sigma_{-i}, [r_i]) \quad \Rightarrow \quad \sigma_i(s_i) \leq \varepsilon \cdot \sigma_i(r_i).$$

*A **proper equilibrium** is any limit of ε-proper strategy profiles as $\varepsilon \to 0$.*

The proper equilibria constitute a non-empty subset of the set of Nash equilibria. We next turn to the concepts of persistent retracts and persistent equilibrium. Every finite game has a persistent retract and a persistent equilibrium.

Definition 3 [Kalai and Samet, 1984]**.** *A **retract** is any set $X = \times_{i \in N} X_i$ such that $\varnothing \neq X_i \subseteq \Delta(S_i)$ is closed and convex $\forall i \in N$. A retract X is **absorbing** if it has a neighborhood $U \subseteq M(S)$ such that for all $\sigma' \in U$:*

$$\max_{\sigma_i \in X_i} u_i(\sigma'_{-i}, \sigma_i) = \max_{s_i \in S_i} u_i(\sigma'_{-i}, [s_i]) \qquad \forall i \in N.$$

*A **persistent set** is any minimal absorbing retract. A **persistent equilibrium** is any Nash equilibrium belonging to a persistent set.*

We will use the following terminology and notation: a *block* is any set $T = \times_{i \in N} T_i$ such that $\varnothing \neq T_i \subseteq S_i \, \forall i \in N$. The associated *block game* is the game $G_T = \langle N, T, u \rangle$, with u restricted to T. We embed its mixed strategies in the strategy space of the whole game G: $M(T) = \{\sigma \in M(S) : \sigma_i(s_i) = 0 \; \forall s_i \notin T_i, \forall i \in N\}$. If T is a block, then clearly $M(T)$ is a retract. By a slight abuse of language, we will call a block T absorbing if $M(T)$ is absorbing. A strategy profile σ has *support in a block T* if $\sigma_i(s_i) = 0$ for all players $i \in N$ and strategies $s_i \notin T_i$. Write $\sigma(T)$ for the probability that a mixed-strategy profile $\sigma \in M(S)$ puts on a block: $\sigma(T) = \sum_{s \in T} [\Pi_{i \in N} \sigma_i(s_i)]$. Thus $\sigma \in M(T)$ iff $\sigma(T) = 1$.

Definition 4. *A **Nash equilibrium of a block game** G_T is any strategy profile $\sigma \in M(T)$ such that*

$$\sigma_i(t_i) > 0 \quad \Rightarrow \quad t_i \in \arg\max_{s_i \in T_i} u_i(\sigma_{-i}, [s_i]).$$

Clearly every block game has at least one Nash equilibrium. For $T = S$, this is nothing but the usual definition of Nash equilibrium.

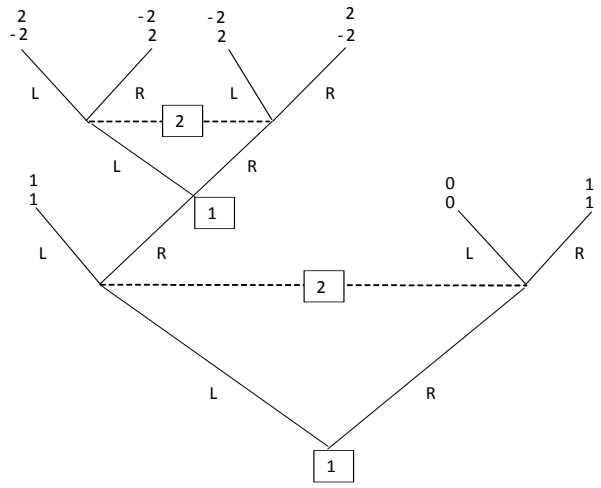
Definition 5 [Basu and Weibull, 1991]**.** *A block T is **curb** ("closed under rational behavior") if*

$$\arg\max_{s_i \in S_i} u_i(\sigma_{-i}, [s_i]) \subseteq T_i$$

for every strategy profile $\sigma \in M(T)$ and every player $i \in N$.

Every finite game has a minimal curb block. We also note that every curb block is absorbing.[4] Game 1 in the introduction has two minimal curb blocks, the supports of its two strict equilibria. Hence, the mixed equilibrium is not persistent. However, in a slight elaboration of that game, the only absorbing block is the full pure-strategy space. Hence, persistence then has no cutting power.

Example 1. *Consider the extensive-form game*



Arguably, this elaboration of Game 1 should not alter our prediction, since the added subgame is a zero-sum game with value zero. In particular, backward induction requires the players to attach value zero to the subgame, which renders it equivalent with the original game. The purely reduced normal-form representation of the elaboration is

Game 2:

	L	RL	RR
LL	$1,1$	$2,-2$	$-2,2$
LR	$1,1$	$-2,2$	$2,-2$
R	$0,0$	$1,1$	$1,1$

This game has three Nash equilibrium components: A, B and $C = \{\sigma^m\}$, where A consists of all strategy profiles of the form $\sigma = (p[LL] + (1-p)[LR]\ ,\ [L])$ for $1/4 \leq p \leq 3/4$, B consists of all strategy profiles of the form $\sigma = ([R]\ ,\ q[RL] + (1-q)[RR])$ for $1/4 \leq q \leq 3/4$, and

$$\sigma^m = (\ \frac{1}{2}[R] + \frac{1}{4}[LL] + \frac{1}{4}[LR])\ ,\ \frac{1}{2}[L] + \frac{1}{4}[RL] + \frac{1}{4}[RR]).$$

[4] To see this, suppose that T is curb. By continuity of payoff functions and the finiteness of the game all best replies to all mixed-strategy profiles in a neighborhood of $M(T)$ will have all best replies in the block.

There are three proper equilibria: $\sigma^a = (\frac{1}{2}[LL] + \frac{1}{2}[LR], [L]) \in A$, $\sigma^b = ([R], \frac{1}{2}[RL] + \frac{1}{2}[RR]) \in B$, *and* σ^m. *The only curb or absorbing block is the whole pure-strategy space* S, *so all Nash equilibria are persistent.*

The mixed equilibrium σ^m in the last example would appear to be non-robust as a convention, since players would probably learn to avoid entering the zero-sum subgame and instead be likely to end up playing in one of the equilibrium components A or B. The solution concepts to be developed here formalize such intuitions. We relax the definitions of curb and absorbing blocks by applying their best-response conditions only at Nash equilibria of the block game in question. This formalizes the notion that unconventional alternatives should not be advantageous when others are expected to behave rationally within the norm.

Remark 1. *Other related ideas in the recent literature are so-called prep sets (Voorneveld 2004, 2005) and p-best response sets (Tercieux, 2006 a,b). A prep set (or preparation) is a block* T *that contains at least one best reply for each player to every mixed strategy on the block. Every pure Nash equilibrium (viewed as a singleton block) is thus a prep set and every curb set is a prep set. Voorneveld (2004) shows that minimal prep sets generically coincide with minimal curb sets and Voorneveld (2005) establishes that prep sets also generically coincide with persistent retracts (*T *being a prep set and* $M(T)$ *an absorbing retract). Tercieux (2006a) defines a p-best response set as a block that contains all best replies to all beliefs that put at least probability* p *on the block, where beliefs are not constrained to treat other players' strategy choices as statistically independent (a constraint we here impose). Tercieux (2006b) weakens the requirement "all best replies" to "some best reply," and calls the first notion strict p-best response sets. For all finite two-player games: (a) any strict p-best response set with* $p < 1$ *is curb, and every curb set is a strict p-best response set for some* $p < 1$ *(see Lemma 2 in Ritzberger and Weibull, 1995), and (b) if a block* T *is a (weak) p-best response set with* $p < 1$, *then* $M(T)$ *is an absorbing retract, and if* $M(T)$ *is an absorbing retract, then* T *is a (weak) p-best response set for some* $p < 1$. *It follows that* T *is a minimal (weak) p-best response set for some* $p < 1$ *if and only if* $X = M(T)$ *is a persistent set.*

3. RANDOM CONSIDERATION SETS

In familiar games, people may come to view some strategies as "conventional" or "normal", and then may generally disregards other strategic alternatives. Arguably, this is a pervasive phenomenon in all societies. Social institutions are sustained in a larger natural interaction by viewing such unconventional actions as "illegal" (Hurwicz, 2007). Conformity with such social norms helps simplify people's decision-making and coordination. When people are generally expected to act rationally within

the conventional norms, unconventional alternatives should not be advantageous. We proceed to formalize this idea in terms of a situation in which players are very likely to consider only the strategies in some conventional block, but allowing for the possibility that some individuals may sometimes, perhaps rarely, also consider strategies outside the block. A player's effective strategy set, to be called the player's *consideration set*, will be treated as a random non-empty subset of the player's full strategy set. A player's actual consideration set, the realization $X_i \subseteq S_i$, is taken to be the player's private information. Players' consideration sets are given to them by some psychological mechanism outside their control, but they act rationally within their given consideration sets.

More precisely, let $G = \langle N, S, u \rangle$ be a finite game. For each player $i \in N$, let $\mathcal{C}(S_i)$ be the collection of non-empty subsets $C_i \subseteq S_i$. A *consideration (probability) profile* is any $\mu = (\mu_1, .., \mu_n)$ such that each μ_i is a probability distribution over $\mathcal{C}(S_i)$. For each player $i \in N$ and $C_i \in \mathcal{C}(S_i)$, $\mu_i(C_i) \in [0,1]$ is the probability that C_i will be i's consideration set; $\mu_i(C_i) = \Pr[X_i = C_i]$.[5] These random draws of consideration sets are statistically independent across players. For any block $T = \times_{i \in N} T_i$ we write $\mu(T)$ for the probability that T will be the actual consideration block; $\mu(T) = \mu_1(T_1) \cdot \cdot \cdot \mu_n(T_n)$.

Each consideration profile μ defines a game $G^\mu = \langle N, F, \tilde{u} \rangle$ in which a pure strategy for each player $i \in N$ is a function $f_i : \mathcal{C}(S_i) \to S_i$ such that $f_i(C_i) \in C_i$ for all $C_i \in \mathcal{C}(S_i)$. In other words, a pure strategy f_i is a deterministic rule that from each consideration set selects an element. Denote the set of such functions F_i and write $F = \times_{i \in N} F_i$. Each pure-strategy profile $f \in F$ in G^μ induces a mixed-strategy profile $\sigma^{f,\mu} \in M(S)$ in G, where the probability that player $i \in N$ will use pure strategy $s_i \in S_i$ is

$$\sigma_i^{f,\mu}(s_i) = \sum_{C_i \in \mathcal{C}(S_i)} \mu_i(C_i) \cdot \mathbf{1}_{f_i(C_i) = s_i}.$$

The resulting payoff to each player i is $u_i^\mu(f) = u_i\left(\sigma^{f,\mu}\right)$. This defines the pure-strategy payoff functions $u_i^\mu : F \to \mathbb{R}$ for all players $i \in N$ in G^μ. The *random consideration-set game* G^μ, so defined, is finite, and payoffs to mixed-strategy profiles are defined in the usual way. By Nash's existence theorem, each random consideration-set game has at least one Nash equilibrium in pure or mixed strategies.[6]

[5] The term "consideration set" is borrowed from Eliaz and Spiegler (2011), who use it in an entirely different context. The basic idea, originating in the marketing literature, is that consumers may be unaware of some of the products available to them, see also Manzini and Mariotti (2007) and Salant and Rubinstein (2008).

[6] A special case of this set-up is when $\mu(S_i) = 1$ for all players $i \in N$. Then G^μ is effectively the same as G; the probability is then one that all players will consider all pure strategies at their disposal in the underlying game G.

For any mixed-strategy profile $\tau \in M(F)$, any player $i \in N$ and consideration set $C_i \in \mathcal{C}(S_i)$, let $\tau_{i|C_i} \in \Delta(S_i)$ be the induced conditional probability distribution over the strategy set S_i of player i, given that C_i is i's consideration set (so $\tau_{i|C_i}(s_i) = 0$ if $s_i \notin C_i$).[7] When a mixed-strategy profile $\tau \in M(F)$ is played in G^μ, player $i \in N$ will use pure strategy $s_i \in S_i$ with probability

$$\tau_i^\mu(s_i) = \sum_{C_i \in \mathcal{C}(S_i)} \mu_i(C_i) \cdot \tau_{i|C_i}(s_i). \tag{1}$$

This defines the mixed-strategy profile $\tau^\mu \in M(S)$ induced by τ in the underlying game G. We will sometimes refer to τ^μ as the *projection* of $\tau \in M(F)$ to $M(S)$ under consideration profile μ.

A strategy profile $\tau \in M(F)$ is a Nash equilibrium of the random consideration-set game G^μ if and only if, for all players $i \in N$ and consideration sets $C_i \in \mathcal{C}(S_i)$,

$$\mu_i(C_i) > 0 \quad \Rightarrow \quad u_i(\tau_{-i}^\mu, \tau_{i|C_i}) = \max_{s_i \in C_i} u_i(\tau_{-i}^\mu, [s_i]). \tag{2}$$

The next result establish a characterization of the set of Nash equilibria of any given block game as the limit points of projections of Nash equilibria of random consideration-set games when the consideration probability for the block tends to one.

Lemma 1. *Let T be any block in $G = \langle N, S, u \rangle$, and let $\langle \mu^k, \tau^k \rangle_{k \in \mathbb{N}}$ be any sequence where each μ^k is a consideration probability profile, each τ^k a Nash equilibrium of G^{μ^k}, and $\mu^k(T) \to 1$ as $k \to \infty$. Any limit of projections of these Nash equilibria to $M(S)$ is a Nash equilibrium of the block game G_T. Conversely, any Nash equilibrium of the block game G_T can be expressed as such a limit.*

Proof: The second claim is immediate: If $\sigma \in M(S)$ is a Nash equilibrium of G_T then it is the projection of some Nash equilibrium of G^μ with $\mu(T) = 1$. Hence, σ has the claimed property with respect to the constant sequence $\mu^k = \mu$ $\forall k \in \mathbb{N}$. For the first claim, suppose that $\mu^k(T) \to 1$ as $k \to \infty$ and let $\langle \tau^k \rangle_{k \in \mathbb{N}}$ be a convergent (sub)sequence of Nash equilibria from the associated games G^{μ^k}, with limit $\tau^* \in M(F)$. Without loss of generality assume $\mu^k(T) > 0$ $\forall k$. Let σ^* be the projection of τ^* to $M(S)$ and suppose that σ^* is not a Nash equilibrium of G_T. Then there exists a player $i \in N$ and pure strategy $t_i \in T_i$ such that $u_i(\sigma_{-i}^*, [t_i]) > u_i(\sigma^*)$.

[7] If one views the random consideration-set game as an extensive-form game, where players are first informed of their consideration set and then chooses randomly or deterministically from this consideration set, then τ_i is player i's behavior strategy and $\tau_{i|C_i}$ is his local strategy at the information set where he is informed that his consideration set is C_i.

By continuity, $u_i(\sigma^k_{-i}, [t_i]) > u_i(\sigma^k)$ for all $k \in \mathbb{N}$ sufficiently large, where σ^k is the projection of τ^k. Hence,

$$u_i(\sigma^k_{-i}, \tau^k_{i|T_i}) < \max_{t_i \in T_i} u_i(\sigma^k_{-i}, [t_i]),$$

so $\mu^k_i(T_i) = 0$ by (2), a contraction. **Q. E. D.**

4. Coarsely tenable blocks and coarsely settled equilibria

Let $G = \langle N, S, u \rangle$ be any finite game and let T be any block, interpreted as a potential convention. The following definition formalizes an arguably weak robustness requirement on such a convention, namely, that if players' consideration sets are very likely to constitute the block and play in the random consideration-set game is in equilibrium, then nobody could do better by choosing a strategy outside the block.

Definition 6. *A block T is **coarsely tenable** if there exists an $\varepsilon \in (0,1)$ such that if $\tau \in M(F)$ is a Nash equilibrium of a random consideration-set game G^μ with $\mu_i(T_i) > 1 - \varepsilon$ $\forall i \in N$, then*

$$u_i(\tau^\mu_{-i}, \tau_{i|T_i}) = \max_{s_i \in S_i} u_i(\tau^\mu_{-i}, [s_i]) \ \forall i \in N. \tag{3}$$

The full block $T = S$ is coarsely tenable in this sense. Also, a singleton block that is the support of any pure strict equilibrium is coarsely tenable, and so is any curb block and indeed any absorbing block. To see why the last (and strongest) claim holds, let T be any absorbing block. By definition, there then exists an $\varepsilon \in (0,1)$ such that if $\sigma \in M(S)$ is such that $\sigma_i(T_i) > 1 - \varepsilon$ $\forall i \in N$, then

$$\max_{t_i \in T_i} u_i(\sigma_{-i}, [t_i]) = \max_{s_i \in S_i} u_i(\sigma_{-i}, [s_i]) \quad \forall i \in N.$$

Let $\tau \in M(F)$ be a Nash equilibrium of any random consideration-set game G^μ such that $\mu_i(T_i) > 1 - \varepsilon$ for all players i. Then $\mu_i(T_i) > 0$ and thus $u_i(\tau^\mu_{-i}, \tau_{i|T_i}) = \max_{t_i \in T_i} u_i(\tau^\mu_{-i}, [t_i])$ by (2). Moreover, $\tau^\mu_i(T_i) > 1 - \varepsilon$ $\forall i \in N$.[8] Thus

$$u_i(\tau^\mu_{-i}, \tau_{i|T_i}) = \max_{t_i \in T_i} u_i(\tau^\mu_{-i}, [t_i]) = \max_{s_i \in S_i} u_i(\tau^\mu_{-i}, [s_i]) \quad \forall i \in N.$$

[8] By definition,

$$\begin{aligned} \tau^\mu_i(T_i) &= \sum_{t_i \in T_i} \tau^\mu_i(t_i) = \sum_{t_i \in T_i} \sum_{C_i \in \mathcal{C}(S_i)} \mu_i(C_i) \cdot \tau_{i|C_i}(t_i) \\ &\geq \mu_i(T_i) \cdot \sum_{t_i \in T_i} \tau_{i|T_i}(t_i) = \mu_i(T_i) > 1 - \varepsilon. \end{aligned}$$

Another feature of coarsely tenable blocks is that elimination of weakly dominated pure strategies leaves a coarsely tenable block. More precisely, for each player $i \in N$, let $T_i \subseteq S_i$ be such that every pure strategy that is not in T_i is weakly dominated by some mixed strategy with support in T_i. Then T is coarsely tenable, since each player i will have some (globally) best reply in T_i to the projection of any mixed-strategy profile $\tau \in M(F)$ in any random consideration-set game G^μ. By the same token, elimination of payoff-equivalent pure strategies leaves a coarsely tenable block; for each player $i \in N$, let $T_i \subseteq S_i$ be such that every pure strategy that is not in T_i has an equivalent strategy in T_i.

We finally note that the equilibria of the block game associated with a coarsely tenable block coincide with the equilibria of the original game that have support in the block:

Proposition 1. *If T is coarsely tenable, then the Nash equilibria of the block game G_T are the Nash equilibria of the original game $G = \langle N, S, u \rangle$ that have support in T.*

Proof: Let $G = \langle N, S, u \rangle$ be a finite game and let T be any block in S. First, if $\sigma \in M(T)$ is a Nash equilibrium of G, it is, *a fortiori*, a Nash equilibrium of the block game G_T. Secondly, suppose that $\sigma \in M(T)$ is a Nash equilibrium of G_T, and let $\mu(T) = 1$. Let $\tau \in M(F)$ be such that $\tau_{i|T_i} = \sigma_i$. Then $\tau^\mu = \sigma$, and by (2), τ is a Nash equilibrium of G^μ. If T is coarsely tenable:

$$u_i(\sigma) = u_i(\tau^\mu_{-i}, \tau_{i|T_i}) = \max_{s_i \in S_i} u_i(\tau^\mu_{-i}, [s_i]) = \max_{s_i \in S_i} u_i(\sigma_{-i}, [s_i]) \quad \forall i \in N,$$

so $\sigma \in M(T)$ is a Nash equilibrium of G. **Q.E.D.**

Conventional norms tend to simplify the interaction at hand by excluding as many strategies as possible, subject to the rationality restriction that it should not be advantageous to use an unconventional strategy. This suggests that minimal coarsely tenable blocks are particularly relevant.[9] The games we study are finite and hence admit at least one minimal coarsely tenable block. It follows from the above that a minimal coarsely tenable block will not contain any weakly dominated pure strategies, nor any pair of strategies that are payoff equivalent.

The following definition formalizes an equilibrium notion that combines the (simplicity) requirement of minimality of the set of conventional strategies — the strategy block in question — with the (rationality) requirement that individual players should not be able to benefit by using unconventional strategies when other players are likely to use conventional strategies.

[9] A coarsely tenable block is *minimal* if it does not properly contain any coarsely tenable block.

Definition 7. *A* ***coarsely settled equilibrium*** *is any Nash equilibrium of G that has support in some minimal coarsely tenable block T.*

Evidently, any pure strict equilibrium is coarsely settled. By contrast, the mixed equilibrium in Game 1 in the introduction is not coarsely settled, since it does not have support in a minimal coarsely tenable block. In that example, also the notion of persistent equilibrium rejects the mixed equilibrium. By contrast, in the elaborated version of this game, Game 2 in Example 1, the totally mixed Nash equilibrium, that was seen to be persistent, is not coarsely settled. Indeed, Game 2 has two minimal coarsely tenable blocks, associated with the two continuum Nash equilibrium components, A and B. These blocks are $T^A = \{LL, LR\} \times S_2$ and $T^B = S_1 \times \{L\}$, and the coarsely settled equilibria in this game are precisely the Nash equilibria in these two components, in line with the result for the original Game 1.

5. Finely tenable blocks and finely settled equilibria

In Game 2, we note that while player 2's pure strategies RL and RR are not used in any equilibrium of the block game defined on the coarsely tenable block T^A, they are nevertheless needed in order to render the block coarsely tenable. For if they are removed from the block, the remaining block $T^* = \{LL, LR\} \times \{L\}$ is not coarsely tenable; the block game associated with this subblock has equilibria, such as $s = (LL, L)$, that are not equilibria of the whole game on S. This suggests that one might want to look for a weaker block property that to a lesser extent needs to incorporate unused pure strategies within the block of conventional strategies. Indeed, this need to include some unused strategies in coarsely tenable blocks may, in some games, lead to that all pure strategies have to be incorporated in the block, and so make the concept blunt. This is the case if *both* zero payoff vectors in the Game 1 are replaced by zero-sum subgames with value zero. Then the only coarsely tenable block is the full pure-strategy space, and both persistence and coarse settledness accepts all Nash equilibria (see Example 3).

Hence, something "finer" than coarse tenability is needed! More specifically, in the above example we want to identify a narrower block in Game 2 that generalizes the pure strict equilibrium $s = (a_1, a_2)$ of Game 1. Coarse tenability leaves us no way to exclude the extreme pure equilibria in the Nash equilibrium component A in the elaborated version without adding in all 2's strategies. Imposing some rationality on the out-of-block deviations could narrow the scope to the proper equilibrium in the component A without need to enlarge the block. We achieve this by testing blocks against a smaller class of consideration-probability profiles. The following definition formalizes the notion that (a) players' consideration sets are very likely to constitute the candidate block, (b) all consideration sets have positive probability,

and (c) consideration sets other than those in the conventional block are much more likely to be large than small.

Definition 8. *For any block T and any $\varepsilon > 0$, a consideration-probability profile μ is ε-**proper on** T if for every player $i \in N$:*

$$\begin{cases} \mu_i(T_i) > 1 - \varepsilon \\ \mu_i(C_i) > 0 \quad \forall C_i \in \mathcal{C}(S_i) \\ T_i \neq C_i \subset D_i \quad \Rightarrow \quad \mu_i(C_i) \leq \varepsilon \cdot \mu_i(D_i) \end{cases}.$$

The following remark shows that a consideration-probability profile does have this property if a player's inattention to individual pure strategies, when the player is not playing conventionally, are (conditionally) statistically independent.

Remark 2. *Let $G = \langle N, S, u \rangle$ be any finite game and let T be any block, interpreted as a potential convention. For all players $i \in N$ and all consideration sets $C_i \in \mathcal{C}(S_i)$ other than T_i, let*

$$\mu_i(C_i) = \varepsilon \cdot \Pi_{s_i \in C_i}(1 - \delta_i(s_i)) \cdot \Pi_{s_i \notin C_i} \delta_i(s_i) \tag{4}$$

(with the last product defined as unity in case $C_i = S_i$). This can be interpreted as follows. For each player role $i \in N$ in the game, there is a large population of individual who are now and then called to play the game, just as in Nash's mass-action interpretation. An individual who is about to play the game is in one of two states of mind, and all players' (individuals called to play) states of mind are i.i.d. draws of nature. With probability $1-\varepsilon$, a player is in the conventional state of mind. With the complementary probability, $\varepsilon \in (0,1)$, she is in the unconventional state of mind. When in the conventional state in player role i of the game, her consideration set is T_i. In the unconventional state of mind, each pure strategy $s_i \in S_i$ is ignored with some probability $\delta_i(s_i) \in (0,1)$, and these are statistically independent draws for all pure strategies and players. If it happens that a player's all pure strategies would be so ignored, then the player returns to the conventional state of mind.[10] *Such a consideration profile μ is ε-proper on T if all neglection probabilities $\delta_i(s_i)$ are sufficiently small. To see this, let $\|\delta_i\| = \max_{s_i \in S_i} \delta_i(s_i)$. Clearly $\mu_i(T_i) > 1 - \varepsilon$ and $\mu_i(C_i) > 0$ for all $D_i \in \mathcal{C}(S_i)$. Suppose that $C_i, D_i \in \mathcal{C}(S_i)$, $C_i \subset D_i$ and $C_i \neq T_i$. Then*

$$\mu_i(C_i) \leq \mu_i(D_i) \cdot \prod_{s_i \in D_i \setminus C_i} \frac{\delta_i(s_i)}{1 - \delta_i(s_i)} \leq \frac{\|\delta_i\|}{1 - \|\delta_i\|} \cdot \mu_i(D_i).$$

[10] It follows from (4) that

$$\mu_i(T_i) = (1 - \varepsilon) + \varepsilon \cdot [\Pi_{s_i \in T_i}(1 - \delta_i(s_i)) \cdot \Pi_{s_i \notin T_i} \delta_i(s_i) + \Pi_{s_i \in S_i} \delta_i(s_i)]$$

The factor in front of $\mu_i(D_i)$ *is less than* ε *if* $\|\delta_i\| < \varepsilon/(1+\varepsilon)$.

By requiring robustness only to consideration profiles that are ε-proper on the conventional block, we obtain the following definition:

Definition 9. *A block* T *is* ***finely tenable*** *in* $G = \langle N, S, u\rangle$ *if there exists an* $\varepsilon \in (0,1)$ *such that if* $\tau \in M(F)$ *is a Nash equilibrium of a random consideration-set game* G^μ *where* μ *is* ε*-proper on* T*, then*

$$u_i(\tau^\mu_{-i}, \tau_{i|T_i}) = \max_{s_i \in S_i} u_i(\tau^\mu_{-i}, [s_i]) \ \forall i \in N. \tag{5}$$

Every coarsely tenable block is, *a fortiori*, also finely tenable. Hence, in general there are more finely tenable blocks than there are coarsely tenable blocks, and minimal finely tenable blocks may be smaller than the minimal coarsely tenable blocks.

Example 2. *We noted above that the block* $T^* = \{LL, LR\} \times \{L\}$ *in Game 2 is not coarsely tenable. However, it is finely tenable. To see this, let* $\varepsilon \in (0,1)$ *and let* μ *be any* ε*-proper consideration-probability profile on* T^**. Consider any Nash equilibrium* $\tau \in M(F)$ *in the associated random consideration game* G^μ*. Then* $\tau^\mu_1(LL) = \tau^\mu_1(LR)$*. For suppose that* $\tau^\mu_1(LL) > \tau^\mu_1(LR)$*. By (1) we then have, for* ε *sufficiently small,* $\tau^\mu_2(RL) = \mu_2(\{RL\})$ *and* $\tau^\mu_2(RR) \geq \mu_2(\{RL, RR\})$*. Since* μ *is* ε*-proper on* T^**, we also have*

$$\mu_2(\{RL\}) \leq \varepsilon \cdot \mu_2(\{RL, RR\}),$$

so $\tau^\mu_2(RL) < \varepsilon \cdot \tau^\mu_2(RR)$*. Then* LR *is a best reply for player 1, and* $\tau^\mu_1(LL) < \tau^\mu_1(LR)$*, a contradiction. By the same token,* $\tau^\mu_1(LL) > \tau^\mu_1(LR)$ *is not possible. A similar argument establishes* $\tau^\mu_2(RL) = \tau^\mu_2(RR)$*. These two equations together imply that each player* i *has a best reply to* τ^μ *in* T^*_i*, that is (5) holds and* T^* *is finely tenable.*

Moreover, when a block is finely tenable, then the projection of any Nash equilibrium in any random consideration-set game G^μ where μ is ε-proper on T, is an ε-proper strategy profile in the original game G.

Proposition 2. *Let* T *be a finely tenable block and let* ε *be as in Definition 9. If* μ *is any consideration probability profile that is* ε*-proper on* T*, and if* $\tau \in M(F)$ *is a Nash equilibrium of* G^μ*, then the projection* $\tau^\mu \in M(S)$ *is an* ε*-proper strategy profile in* G*.*

Proof: Let T be any finely tenable block and let $\varepsilon > 0$ be as in Definition 9. Let $\tau \in M(F)$ be any Nash equilibrium of G^{μ}, and let $\tau^{\mu} \in M(S)$ be its projection. We proceed to show that τ^{μ} is an ε-proper strategy profile in G. Since each $C_i \in \mathcal{C}(S_i)$ has positive probability of being the consideration set under μ, $\tau_i(f_i) = 0$ for all pure strategies $f_i \in F_i$ such that $f_i(C_i) \notin \arg\max_{s_i \in C_i} u_i(\sigma^k_{-i}, [s_i])$ for some $C_i \in \mathcal{C}(S_i)$. Let $r_i, s_i \in S_i$ where $u_i(\tau^{\mu}_{-i}, [r_i]) < u_i(\tau^{\mu}_{-i}, [s_i])$. Let $\mathcal{R}_i \subseteq \mathcal{C}(S_i)$ be the collection of sets $C_i \in \mathcal{C}(S_i)$ such that $r_i \in \arg\max_{s'_i \in C_i} u_i(\tau^{\mu}_{-i}, [s'_i])$. Clearly $C_i \in \mathcal{R}_i \Rightarrow s_i \notin C_i$. Moreover, $T_i \notin \mathcal{R}_i$, since T is finely tenable and thus contains a pure best reply to τ^{μ}. For each $C_i \in \mathcal{R}_i$:

(i) $\{s_i\} = \arg\max_{s'_i \in C_i \cup \{s_i\}} u_i(\tau^{\mu}_{-i}, [s'_i])$,

(ii) $\mu_i(C_i) \leq \varepsilon \cdot \mu_i(C_i \cup \{s_i\})$ and

(iii) $\sum_{C_i \in \mathcal{R}_i} \mu_i(C_i \cup \{s_i\}) \leq \tau^{\mu}_{-i}(s_i)$.

Hence,

$$\tau^{\mu}_{-i}(r_i) \leq \sum_{C_i \in \mathcal{R}_i} \mu_i(C_i) \leq \varepsilon \cdot \sum_{C_i \in \mathcal{R}_i} \mu_i(C_i \cup \{s_i\}) \leq \varepsilon \cdot \tau^{\mu}_{-i}(s_i).$$

This establishes that $\tau^{\mu}_{-i} \in M^o(S)$ is an ε-proper strategy profile in G. **Q.E.D.**

Remark 3. *The above proof goes through also for a weaker versions of Definition 8. One such version is obtained when the hypothesis $C_i \subset D_i$ in condition (c) is strengthened to also require that D_i contains some strategy that is not payoff equivalent to any strategy in C_i Another, even weaker version is obtained when the hypothesis $C_i \subset D_i$ in condition (c) is strengthened to also require that D_i contains a strategy that is a strictly better reply than those in C_i to some Nash equilibrium in G^{μ}.*

Given T finely tenable, let τ^* be a limit point of a sequence $\langle \tau^k \rangle_{k \in \mathbb{N}}$ of Nash equilibria $\tau^k \in M(F)$ of random-consideration games G^{μ_k} where each μ_k is an ε_k-proper consideration-probability profile on T and $\varepsilon_k \to 0$. Let σ^k and $\sigma^* \in M(S)$ be the projections of τ^k and τ^* in G. By construction, σ^* is a proper equilibrium of G. Moreover, σ^* has support in T, because $\mu^k(T) \to 1$ and thus $\forall s_i \in S_i$,

$$\sigma^k_i(s_i) = \sum_{C_i \in \mathcal{C}(S_i)} \mu^k_i(C_i) \cdot \tau^k_{i|C_i}(s_i) \quad \to \quad \sigma^*_i(s_i) = \tau^*_{i|T_i}(s_i),$$

so $\sigma^*_i(s_i) = 0$ if $s_i \notin T_i$. By the Bolzano-Weierstrass theorem, every sequence from a compact set has a convergent subsequence, so we have established the following result:

Corollary 1. *Each finely tenable block contains the support of a proper equilibrium.*

Remark 4. *These observations provide a micro foundation for proper equilibrium when applied to the maximal finely tenable block $T = S$. By Proposition 2, all limit points (as $\varepsilon \to 0$) to projections of sequences of Nash equilibria in the associated random consideration-set games are proper equilibria.*[11]

We define settledness with respect to finely tenable blocks in the same way as for coarsely tenable blocks:

Definition 10. *A **finely settled equilibrium** is any proper equilibrium that has support in some minimal finely tenable block.*

It follows immediately that every finite game has at least one proper equilibrium that is both finely and coarsely tenable. Because if T is any minimal coarsely tenable block (and such exist), then T is also finely tenable. If T is not a minimal finely tenable block, it will ontain such a block, T^*, since S is finite. According to the above corollary, there exist a proper equilibrium with support in T^*.

Indeed this requirement can be taken further, leading to our most refined solution concept, that of a *fully settled equilibrium*, by which we mean a proper equilibrium that has support in a minimal finely tenable block in a minimal coarsely tenable block in a minimal absorbing block in a minimal curb block. Every finite game clearly admits such an equilibrium (since curb $\Rightarrow$ absorbing $\Rightarrow$ coarsely tenable $\Rightarrow$ finely tenable).

In the following elaboration of Game 1, the only coarsely tenable block is the whole strategy space S, so all Nash equilibria are coarsely tenable. By contrast, there are smaller finely tenable blocks and only two finely settled equilibria, and they correspond to the two strict equilibria of Game 1.

Example 3. *Reconsider the extensive-form game in Example 1. If one would replace the $(0,0)$ end-node by a zero-sum subgame like the other zero-sum subgame, the purely reduced normal form (with abstract labeling of pure strategies) would be*

Game 3:

	ax_2	ay_2	bx_2	by_2
ax_1	$1,1$	$1,1$	$2,-2$	$-2,2$
ay_1	$1,1$	$1,1$	$-2,2$	$2,-2$
bx_1	$2,-2$	$-2,2$	$1,1$	$1,1$
by_1	$-2,2$	$2,-2$	$1,1$	$1,1$

[11] A micro foundation in the same vein can also be given for perfect equilibrium. Call a consideration-probability profile μ *ε-perfect* if it meets the first two conditions for ε-properness ($\mu_i(S_i) > 1-\varepsilon$ and $\mu_i(C_i) > 0$ for all players $i \in N$ and consideration sets $C_i \in \mathcal{C}(S_i)$). It is not difficult to verify that all limit points, as $\varepsilon \to 0$, to projections of sequences of Nash equilibria in such random consideration-set games are perfect equilibria.

Subgame perfection in the extensive form requires 50/50 randomization in the off-diagonal subgames. Also this elaboration of Game 1 has three Nash equilibrium components:

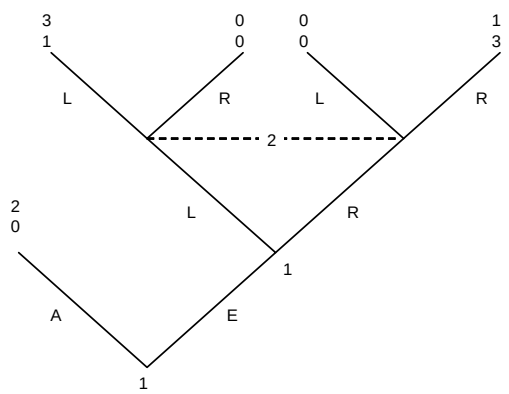
$$A = \left\{ \sigma = (p[ax_1] + (1-p)[ay_1]\ ,\ q[ax_2] + (1-q)[ax_2]))\ \textit{for}\ p, q \in \left(\frac{1}{4}, \frac{3}{4}\right) \right\}$$

$$B = \left\{ \sigma = (p[bx_1] + (1-p)[by_1]\ ,\ q[bx_2] + (1-q)[bx_2]))\ \textit{for}\ p, q \in \left(\frac{1}{4}, \frac{3}{4}\right) \right\}$$

and $C = \{\sigma^m\}$*, where* σ^m *is uniform randomization over each strategy set. The proper equilibria are* $\sigma^a = (\frac{1}{2}[ax_1] + \frac{1}{2}[ay_1], \frac{1}{2}[ax_2] + \frac{1}{2}[ay_2]) \in A$*,* $\sigma^b = (\frac{1}{2}[bx_1] + \frac{1}{2}[by_1], \frac{1}{2}[bx_2] + \frac{1}{2}[by_2]) \in B$*, and* σ^m*. The only curb, absorbing or coarsely tenable block is the whole pure-strategy space* S*, so all Nash equilibria are persistent and coarsely settled. However,* $T^a = \{ax_1, ay_1\} \times \{ax_2, ay_2\}$ *and* $T^b = \{bx_1, by_1\} \times \{bx_2, by_2\}$ *are finely tenable blocks by similar arguments to those given in Example 2. The game thus has many persistent and coarsely settled equilibria but only two finely and indeed fully settled equilibria,* σ^a *and* σ^b*, corresponding to (and behaviorally indistinguishable from) the two strict equilibria in the original game (**??**).*

Both finely settled equilibria in the preceding example are also coarsely settled. The next example provides a finely settled equilibrium that is not coarsely settled; the "non-forward inductive" sequential equilibrium of a standard outside-option game. Only the "forward inductive" sequential equilibrium is both coarsely and finely settled, and, indeed, fully settled.

Example 4. *Consider a version of the battle-of-the-sexes game where player 1 has an outside option:*



Its purely reduced normal form (with A representing the two payoff-equivalent strategies AL and AR) is

Game 4:

	L	R
EL	$3,1$	$0,0$
ER	$0,0$	$1,3$
A	$2,0$	$2,0$

This normal-form game has two Nash equilibrium components, the singleton set $T^* = \{EL\} \times \{L\}$, consisting of the strict pure equilibrium $s^* = (EL, L)$, the "forward-induction" solution, and a continuum component in which player 1 plays A for sure while player 2 plays L with probability at most $2/3$. The block T^* is minimal curb and s^* is thus fully settled. Another proper equilibrium of this game is $s^o = (A, R)$. (To see this note that for all $\varepsilon > 0$ small enough, $\sigma_1^\varepsilon = (\varepsilon^2, \varepsilon, 1 - \varepsilon - \varepsilon^2)$ and $\sigma_2^\varepsilon = (\varepsilon, 1 - \varepsilon)$ make up an ε-proper strategy profile σ^ε.) What about its supporting block, $T^o = \{A\} \times \{R\}$? Clearly, T^o is not coarsely tenable, since 2 will need to use strategy L if there is a positive probability that 1 plays EL, which indeed is the case under some consideration profiles that attach arbitrary little, but positive probability to other blocks than those in T^o. However, T^o is finely tenable. To see this, let μ be an ε-proper consideration-probability profile on T^o and let $\tau \in M(F)$ be any Nash equilibrium of G^μ. Then $\tau_2^\mu(L) < \varepsilon$, so for $\varepsilon > 0$ small enough $\tau_1^\mu(EL) \leq \mu_1(\{EL\}) < \varepsilon \cdot \mu_1(\{EL, ER\}) \leq \tau_1^\mu(ER)$, which implies that A and R are best replies to τ^μ, so T^o is indeed finely settled. In sum, s^* is fully settled while s^o is finely but not coarsely settled. (Note that s^o corresponds to the sequential equilibrium of the extensive-form game in which play of (R, R) would be expected in the battle-of-sexes subgame.)

We proceed to establish that coarsely and finely tenable blocks, and thus also coarsely and finely settled equilibria, generically coincide.

6. GENERIC NORMAL-FORM GAMES

The concept of regular equilibrium was introduced by Harsanyi (1973) and slightly modified by van Damme (1987), who defines a Nash equilibrium of a finite normal-form game to be regular (op. cit. Definition 2.5.1) if the Jacobian associated with a certain system of equations (op.cit. 2.5.4 and 2.5.5), closely related to those characterizing Nash equilibrium, is non-singular.

Definition 11. *A game $G = \langle N, S, u \rangle$ is* ***hyper-regular*** *if, for every block $T \subseteq S$, all Nash equilibria of the associated block game G_T are regular in the sense of van Damme (1987).*

In a well-defined sense, almost all normal-form games are hyper-regular:

Lemma 2. *For any (finite) set of players N and (finite) sets of strategies S_i for each player $i \in N$, the set of utility functions u in $\mathbb{R}^{|N|\cdot|S|}$ such that $G = \langle N, S, u\rangle$ is not hyper-regular is contained in a closed set of Lebesgue measure 0 in $\mathbb{R}^{|N|\cdot|S|}$.*

Proof: The property of regularity of a block game G_T depends only on the payoffs on T, and this property will fail only for utility functions in a closed set of Lebesgue measure zero (van Damme, 1987, Theorem 2.6.1). There are only finitely many blocks $T \subseteq S$, and the union of finitely many such sets is still a closed set of measure 0. **Q.E.D.**

In a hyper-regular game, the collection of finely tenable blocks is identical with the collection of coarsely tenable blocks, and the sets of finely and coarsely settled equilibria coincide:

Proposition 3. *If a game $G = \langle N, S, u\rangle$ is hyper-regular, then any block $T \subseteq S$ is finely tenable if and only if it is coarsely tenable. Any equilibrium of G is finely settled if and only if it is coarsely settled.*

Proof: We first establish that for a hyper-regular game there cannot exist any Nash equilibrium τ of any block game G_T such that a player i has an alternative $s_i \in S_i \backslash T_i$ with $u_i(\tau_{-i}, [s_i]) = u_i(\tau)$. If this equality would hold, and if we added s_i to T_i (obtaining $T_i' = T_i \cup \{s_i\}$), then we would obtain a block game $G_{T'}$ in which τ would still be a Nash equilibrium but, having an alternative best reply in T' that gets zero probability in τ, τ would not be quasi-strict in this new block game $G_{T'}$. By hyper-regularity of G, τ is a regular equilibrium of $G_{T'}$ and hence τ is quasi-strict (Corollary 2.5.3 in van Damme, 1987), a contradiction. So if τ is a Nash equilibrium of some block game G_T, then either $u_i(\tau_{-i}, [s_i]) > u_i(\tau)$ or $u_i(\tau_{-i}, [s_i]) < u_i(\tau)$. The first of these inequalities, for any i and s_i, would imply that T is not coarsely tenable. The second inequality, for all i and s_i, would imply that T is coarsely tenable. Thus, in the given hyper-regular game G, a block T is coarsely tenable if and only if $u_i(\tau_{-i}, [s_i]) < u_i(\tau)$ for all i and $s_i \in S_i \backslash T_i$, at all equilibria τ of the block game. Coarsely tenable blocks are always finely tenable, so it remains to prove that, for our hyper-regular game G, any block T that is not coarsely tenable is not finely tenable.

In order to establish this, consider any Nash equilibrium τ of any block game G_T. The utility function of G_T can be viewed as a vector u in $\mathbb{R}^{|N|\cdot|T|}$. By hyper-regularity of G, the equilibrium τ is regular, and thus also strongly stable, in G_T (van Damme, 1987, Definition 2.4.4 and Theorem 2.5.5). This means that there is some open neighborhood V of $\tau \in M(T)$ and some open neighborhood U of $u \in \mathbb{R}^{|N|\cdot|T|}$ such that, for any perturbation of G_T that has a utility function $\tilde{u}$ in U, we obtain a game $\tilde{G}_T = \langle N, T, \tilde{u}\rangle$ that has exactly one equilibrium $\tilde{\tau}$ in V, and this equilibrium depends continuously on the utility function $\tilde{u}$.

Now let's think about a random consideration-set game G^{μ}. Let ρ be a partial behavior-strategy profile in the extensive form of G^{μ} (described in Section 3) that defines a mixed strategy $\rho_{i|C_i} \in \Delta(C_i)$ for every player i, the player's local strategy at that information set, and for every consideration set C_i other than T_i. Let $B(T)$ be the set of all such partial behavior-strategy profiles. When ρ defines the behavior of players at all consideration sets other than those of T, then the only question remaining in G^{μ} is what each player i would do when considering T_i, which will happen with probability at least $1-\varepsilon$. So with any given ε, the random consideration-set game G^{μ} becomes a perturbation of G_T, and its utility function in $\mathbb{R}^{|N|\cdot|T|}$ will be in the open set U for all $\varepsilon > 0$ sufficiently small, given ρ. In fact, there exists an $\bar{\varepsilon} > 0$ such that $\tilde{u} \in U$ for all ρ. Now, given any $\varepsilon \in (0, \bar{\varepsilon})$, consider the correspondence that sends any profile $\rho \in B(T)$ to each player i's (non-empty, compact and convex) best local replies at every consideration set $C_i \neq T_i$ to the ρ and $\tilde{\tau}$ strategies, where $\tilde{\tau}$ is the (continuously defined) equilibrium in V for this perturbation of G_T This correspondence is upper hemi-continuous in ρ, so, by Kakutani's fixed-point theorem, for any such ε, there exists a fixed point ρ^*. This fixed-point ρ^*, together with its corresponding $\tilde{\tau}$ at T, will constitute an equilibrium of the random consideration-set game G^{μ}. The projection of this equilibrium to $M(S)$ will be ε-proper with respect to the block T (along the lines given in Section 3) and these strategy profiles converge to the given block-game equilibrium τ as $\varepsilon \to 0$. But then this sequence would yield a contradiction of T being tenable if we had $u_i(\tau_{-i}, [s_i]) > u_i(\tau)$ for some player i and some strategy $s_i \in S_i \backslash T_i$. Thus, if T is not coarsely tenable, then T is not finely tenable either.

So for a hyper-regular game, a block is coarsely tenable if and only if it is finely tenable. Since all regular equilibria are proper (van Damme, 1987, Theorems 2.5.5, 2.4.7, 2.3.8), an equilibrium in a hyper-regular game is coarsely settled if and only if it is finely settled. **Q.E.D.**

By slightly perturbing the payoffs in example (**??**), one sees that there are generic games in which the set of settled equilibria is strictly smaller than the sets of Nash equilibria. The game in the next example, taken from Table 7 in Myerson (1996), has an open neighborhood (in the space of 2×4 normal-form games) in which there is always a Nash equilibrium (near σ^{bc}, see below) which is persistent but not settled in any of our senses. This shows that our concepts are not generically equivalent to Nash or persistent equilibrium.

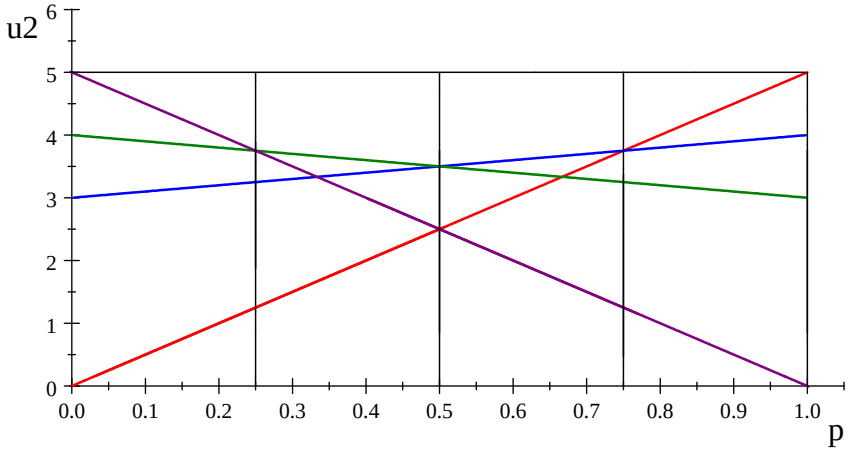
Example 5. *Consider the game*

Game 5:

	a_2	b_2	c_2	d_2
a_1	$0,2$	$1,1$	$0,0$	$1,-3$
b_1	$1,-3$	$0,0$	$1,1$	$0,2$

(6)

and note that the "middle block" $T^{bc} = \{a_1, b_1\} \times \{b_2, c_2\}$ *is identical with our first example, Game 1. The diagram below shows the payoffs to player 2's pure strategies as functions of the probability* p *by which player 1 uses her first pure strategy.*



This game has three Nash equilibria, all mixed. In each equilibrium, player 1 uses both her pure strategies while player 2 uses only two of his four pure strategies; either the two left-most, $\{a_2, b_2\}$*, the two middle ones,* $\{b_2, c_2\}$*, or the two right-most,*$\{c_2, d_2\}$*, each equilibrium corresponding to an intersection in the above diagram.*

This game is hyper-regular. This follows from Theorem 7.4 in Jansen (1981) (see also Theorem 3.4.5 in van Damme, 1987), according to which a Nash equilibrium of a finite two-player game is regular if and only if it is essential and quasi-strict.[12] *It is not difficult to verify that all block equilibria, of all blocks in this game, have both properties. Moreover, the game has only one curb block, the whole set* S*, and it has only one absorbing retract, the whole set* $M(S)$*. Hence, all three equilibria are persistent. However, the "middle" equilibrium is not settled.*

More exactly, the three Nash equilibria of this game are

$$\sigma^{ab} = \left(\frac{3}{4}[a_1] + \frac{1}{4}[b_1], \frac{1}{2}[a_2] + \frac{1}{2}[b_2]\right)$$

$$\sigma^{bc} = \left(\frac{1}{2}[a_1] + \frac{1}{2}[b_1], \frac{1}{2}[b_2] + \frac{1}{2}[c_2]\right)$$

$$\sigma^{cd} = \left(\frac{1}{4}[a_1] + \frac{3}{4}[b_1], \frac{1}{2}[c_2] + \frac{1}{2}[d_2]\right)$$

[12] An *essential equilibrium* (Wu and Jiang, 1962) is any Nash equilibrium such that every nearby game, in terms of payoffs, has some nearby Nash equilibrium. A *quasi-strict equilibrium* (Harsanyi, 1973) is any Nash equilibrium in which all players use all their pure best replies.

Consider first the "middle" block $T^{bc} = \{a_1, b_1\} \times \{b_2, c_2\}$, *the support of* σ^{bc}. *The associated block contains, in addition to* σ^{bc}, *two (strict pure) block-game equilibria, each, however, with better replies outside the block. Hence, by Proposition 1, this block is not coarsely tenable. By contrast, the supports of each of the two other equilibria, the "side" blocks* $T^{ab} = \{a_1, b_1\} \times \{a_2, b_2\}$ *and* $T^{cd} = \{a_1, b_1\} \times \{c_2, d_2\}$, *do not contain any other block equilibria and are coarsely tenable. The only coarsely tenable block that contains* σ^{bc} *is* S, *which, however, is not minimal. Hence, while all three equilibria are persistent, only* σ^{ab} *and* σ^{cd} *are coarsely settled. These claims hold for an open set of payoff perturbations of the game. Thus, the property of being coarsely settled is not generically equivalent to persistence.*

The two minimal coarsely tenable blocks, T^{ab} *and* T^{cd}, *are, a fortiori, also finely tenable. Since they contain no other finely tenable block, they are minimal and hence* σ^{ab} *and* σ^{cd} *are also finely settled.*

We finally re-consider the middle block T^{bc}, *this time investigating if it is finely tenable. As we will see, this is not the case although* T^{bc} *is the support of a proper equilibrium,* σ^{bc}. *First, to see that* σ^{bc} *is proper, let* $\sigma_1^{\varepsilon} = (1/2, 1/2)$ *and* $\sigma_2^{\varepsilon} = (\varepsilon, 1/2 - \varepsilon, 1/2 - \varepsilon, \varepsilon)$. *Clearly* σ^{ε} *is* ε*-proper for all* $\varepsilon \in (0, 1/2)$, *and* $\sigma^{\varepsilon} \to \sigma^{bc}$ *as* $\varepsilon \to 0$. *Second, to see that* T^{bc} *is not finely tenable, let* $\varepsilon > 0$ *and let* μ *be as in Remark 2 , with* $\delta_2(b_2) = \varepsilon$, $\delta_2(d_2) = \varepsilon \cdot \delta_2(c_2) = \varepsilon^2 \cdot \delta_2(a_2)$ *and* $\|\delta_i\| < \varepsilon/(1+\varepsilon)$ *for both players* i. *Then* μ *is* ε*-proper on* T^{bc} *for all* $\varepsilon > 0$. *However, for all* $\varepsilon > 0$ *sufficiently small and all Nash equilibria* $\tau \in M(F)$ *of* G^{μ} *we have*

$$u_2(\tau_1^{\mu}, \tau_{2|T_2}) < u_2(\tau_1^{\mu}, [a_2]), \tag{7}$$

so T^{bc} *is not finely tenable. In sum:* σ^{ab} *and* σ^{cd} *are the only fully settled equilibria of this game.*

We conclude by giving an example that shows that, unlike minimal curb blocks, minimal tenable blocks may overlap.

Example 6. *Consider the case of two "overlapping" matching-pennies games:*

Game 6:

	a_2	b_2	c_2
a_1	$3,1$	$1,3$	$0,0$
b_1	$1,3$	$3,1$	$1,3$
c_1	$0,0$	$1,3$	$3,1$

This game has three Nash equilibria: σ^*, *in which each player randomizes uniformly across his or her two first pure strategies,* σ^{**}, *in which they randomize uniformly across their last two pure strategies, and* σ^m, *which is totally mixed,*

$$\sigma^m = \left(\frac{2}{9}[a_1] + \frac{5}{9}[b_1] + \frac{2}{9}[c_1], \frac{2}{5}[a_2] + \frac{1}{5}[b_2] + \frac{2}{5}[c_2]\right)$$

We note that the supporting blocks for σ^* and σ^{**} overlap; they both contain (b_1, b_2). Moreover, they are not absorbing, since player 2 has no best reply in the block for certain mixed-strategy profiles near the profile that puts unit probability on b_1 and b_2. The only absorbing block is the full strategy space S, so all equilibria are persistent. Nevertheless, σ^m is not coarsely settled since the supporting blocks of σ^* and σ^{**} are coarsely tenable subblocks. These two equilibria are fully settled, a conclusion that seems to agree with intuition for what conventions would be likely in interactions such as this. People would arguably learn to either use only their first two strategies or their last two strategies.

7. CONCLUSION

For any finite normal-form game G, let $B^o(G)$ be its collection of coarsely tenable blocks, $B^+(G)$ its collection of finely tenable blocks and $B^*(G)$ its collection of minimal finely tenable blocks contained in minimal coarsely tenable blocks contained in minimal absorbing blocks contained in minimal curb sets. Likewise, let $E^o(G)$ be the set of coarsely settled equilibria, $E^+(G)$ those that are finely settled, and $E^*(G)$ those that are fully settled. We have seen that all sets are non-empty. Moreover, $B^o(G) \subseteq B^+(G)$, $E^*(G) \subseteq E^o(G) \cap E^+(G)$, and $E^+(G)$ consists of proper equilibria, equilibria that induce a sequential equilibrium in every extensive-form game with the normal form G. We have established that for generic normal-form games $B^o(G) = B^+(G)$ and $E^o(G) = E^+(G)$.

As mentioned in the introduction, minimality comes as a result in certain models of the formation of conventions. We will here briefly refer to one of these, the model in Young (1993, 1998). The setting in this model is very close to that of Nash's mass action interpreation. For each player role there is a finite population, and the (finite) game in question is played perpetually by randomly drawn members of these populations, one from each player population. When called upon to play in his or her role, the individual is given a random sample of size k from the m last rounds of play (without replacement and statistically independent across individuals). Each item in the sample is an action-profile, the pure strategies used in the different player roles in a play of the game. In the unperturbed process in Young's model, the individuals (one from each player population) faced with such an empirical sample plays a pure best reply against the associated empirical mixed-strategy profile. (In case there are multiple best replies, the individual randomizes uniformly across these.) This results in a new action-profile that will replace the oldest of the m action profiles from which the samples were drawn. This defines a finite Markov chain where its state is the list of the m most recent action profiles. Young (1998, Theorem 7.2) shows for a generic class of finite normal-form games that if k is sufficiently large and k/m sufficiently small, then the chain will converge from any initial state to some minimal curb set. It is also known from the learning literature (see e.g. Nachbar, 1990) that if a learning process

meets some fairly weak regularity condition and converges from some initial state that contains "a grain of truth", then the limit state will constitute a Nash equilibrium. It thus seems that the solution concepts developed and discussed here, tenable blocks and settled equilibria, could be related to the outcomes of explicitly dynamic models of social learning in populations, an avenue for future research. Another avenue would be to apply our solutions to models in economics and political science. Evidently the value of these solution concepts depends to a high degree on their usefulness in applications. A third avenue is to study the solutions' predictive power in controlled laboratory experiments, to investigate if human subjects, playing a game over and over again under random rematching, with some information about past play, will tend to play according to some settled equilibrium.

References

[1] Basu, K. and J. Weibull (1991), Strategy subsets closed under rational behavior, *Economics Letters* 36, 141–146.

[2] Bernheim, D. (1984), Rationalizable strategic behavior, *Econometrica* 52, 1007–1028.

[3] Cho, I.-K. and D. Kreps (1987), Signalling games and stable equilibria, *Quarterly Journal of Economics* 102, 179-221.

[4] van Damme E. (1984), A relation between perfect equilibria in extensive-form games and proper equilibria in normal-form games, *International Journal of Game Theory* 13, 1-13.

[5] van Damme E. (1987), *Stability and Perfection of Nash Equilibrium*, Springer Verlag, Berlin.

[6] Eliaz K. and R. Spiegler (2011), Consideration sets and competitive marketing, *Review of Economic Studies* 78, 235-262.

[7] Halpern, J. and R. Pass (2009), Algorithmic rationality: game theory with costly computation, Working Paper, Department of Computer Science, Cornell University.

[8] Harsanyi, J. (1973), Oddness of the number of equilibrium points: a new proof, *International Journal of Game Theory* 2, 235-250.

[9] Hurkens, S. (1995), Learning by forgetful players, *Games and Economic Behavior* 11, 304–329.

[10] Hurwicz, L. (2008), But who will guard the guardians?, *American Economic Review* 98, 577-585.

[11] Jansen, (1981), Regularity and stability of equilibrium points of bimatrix games, *Mathematics of Operations Research* 6, 530-550.

[12] Kalai, E. and D. Samet (1984), Persistent equilibria in strategic games, *International Journal of Game Theory* 13, 129-144.

[13] Kets, W. and M. Voorneveld (2008), Learning to be prepared, *International Journal of Game Theory* 37, 333-352.

[14] Kohlberg, E. and J.-F. Mertens (1986), On the strategic stability of equilibria, *Econometrica* 54, 1003-1037.

[15] Lewis D. (1969), *Convention: A Philosophical Study.* Harvard University Press, Cambridge MA, USA.

[16] Manzini, P. and M. Mariotti (2007), Sequentially rationalizable choice, *American Economic Review* 97, 1824-1839.

[17] Myerson, R. (1978), Refinements of the Nash equilibrium concept, *International Journal of Game Theory* 7, 73-80.

[18] Myerson, R. (1996), Sustainable equilibria in culturally familiar games, in W. Albers et al. (eds.), *Understanding Strategic Interaction: Essays in Honor of Reinhard Selten.* Springer Verlag, Berlin.

[19] Nash, J.F. (1950), *Non-cooperative games*, Ph.D. thesis, Princeton University. Reprinted in H. Kuhn and S. Nasar (2002), *The Essential John Nash*, Princeton University Press, 53-84.

[20] Pearce, D. (1984), Rationalizable strategic behavior and the problem of perfection, *Econometrica* 52, 1029–1050.

[21] Ritzberger, K. and J. Weibull (1995), Evolutionary selection in normal-form games, *Econometrica* 63, 1371-1399.

[22] Salant, Y. and A. Rubinstein (2008), Choice with frames, *Review of Economic Studies* 75, 1287-1296.

[23] Sanchirico, C. (1996), A probabilistic model of learning in games, *Econometrica* 64, 1375-1394.

[24] Schelling T. (1960), *The Strategy of Conflict*. Harvard University Press, Cambridge MA, USA.

[25] Tercieux O. (2006a), p-Best response set, *Journal of Economic Theory* 131, 45–70.

[26] Tercieux O. (2006b), p-Best response set and the robustness of equilibria to incomplete information, *Games and Economic Behavior* 56, 371–384.

[27] Voorneveld M. (2004), Preparation, *Games and Economic Behavior* 48, 403-414.

[28] Voorneveld M. (2005), Persistent retracts and preparation, *Games and Economic Behavior* 51, 228-232.

[29] Wu W. and J. Jiang (1962), Essential equilibrium points of n-person non-cooperative games, *Scientia Sinica* 11, 1307-1322.

[30] Young P. (1993), The evolution of conventions, *Econometrica* 61, 57-84.

[31] Young P. (1998), *Individual Strategy and Social Structure*. Princeton University Press, Princeton, NJ.