I now derive some properties of $\int_0^T g(t) \, dW_t$.

**Theorem 15.1:** $\left\| \int_0^T g(t) \, dW_t \right\|_2^2 = \int_0^T \left\| g(t) \right\|_2^2 \, dt$.

**Proof:** Let $V_N(s)$ be defined by equation 14.2. By equation 14.5,

$$\lim_{N \to \infty} \left\| V_N \right\|_2^2 = \int_0^T \left\| g(t) \right\|_2^2 \, dt.$$  

By the definition of $\int_0^T g(t) \, dW_t$,

$$\lim_{N \to \infty} \left\| V_N - \int_0^T g(t) \, dW_t \right\|_2 = 0.$$  

By the triangle inequality for the norm $\left\| . \right\|_2$,

$$\left\| \left\| V_N \right\|_2^2 - \left\| \int_0^T g(t) \, dW_t \right\|_2^2 \right\| \leq \left\| V_N - \int_0^T g(t) \, dW_t \right\|_2.$$  

Hence

$$\int_0^T \left\| g(t) \right\|_2^2 \, dt = \lim_{N \to \infty} \left\| V_N \right\|_2^2 = \left\| \int_0^T g(t) \, dW_t \right\|_2^2.$$

The next theorem should be obvious.

**Theorem 15.2:** If $a$ and $b$ are numbers and $f(t) : S \to R$ and $g(t) : S \to R$ are stochastic processes that satisfy assumptions 14.6, then

$$\int_0^T [a f(t) + b g(t)] \, dW_t = a \int_0^T f(t) \, dW_t + b \int_0^T g(t) \, dW_t.$$  

If for all $t$, $f(t, s) = 0$ with probability 1, then

$$\int_0^T f(t) \, dW_t = 0.$$
I now present two basic theorems on integration, which I need to prove another property of \( \int_0^T g(t) \, dW_t \).

**Theorem 15.3:** If \((S, \mathcal{F}, P)\) is a probability space and \(x : S \to \mathbb{R}\) is a random variable such that \(E x^2 < \infty\), then \(|x|\) is integrable and \(E|x| \leq ||x||_2\).

**Proof:** If \(x^2\) is integrable, so is \(x^2 + 1\), since the integral of 1 is simply 1. Since \(0 \leq |x| \leq x^2 + 1\) for all \(x\), it follows that \(|x|\) is integrable if \(x^2\) is integrable.

Since the expected value of \(|x|\) is a limit of convex combinations of possibly infinitely many values \(|x(s)|\), it follows that if \(f : \mathbb{R} \to \mathbb{R}\) is a convex and continuous function, then

\[
E[f(|x|)] = f\left(\int_S |x(s)| \, P(\, ds)\right) \leq \int_S f(|x(s)|) \, P(\, ds) = Ef(|x|).
\]

Therefore because the square function is convex and continuous,

\[
(E|x|)^2 \leq E|x|^2 = Ex^2.
\]

Taking the square root of both sides of this inequality, we find that

\[
E|x| \leq \sqrt{Ex^2} = ||x||_2.
\]

**Theorem 15.4:** If \((S, \mathcal{F}, P)\) is a probability space and \(x : S \to \mathbb{R}\) is an integrable random variable, then \(|x|\) is integrable and \(|Ex| \leq E|x|\).

**Proof:** Let \(S^+ = \{s \in S \mid x(s) > 0\}\) and let \(S^- = \{s \in S \mid x(s) \leq 0\}\). Then \(S^+\) and \(S^-\) belong to \(\mathcal{F}\). I hope it is clear that if \(x\) is integrable, then \(\mathcal{F}_{S^+} x\) and \(\mathcal{F}_{S^-} x\) are integrable. Since

\[
|x(s)| = \mathcal{F}_{S^+} x(s) - \mathcal{F}_{S^-} x(s),
\]

for all \(s\), it follows that \(|x|\) is integrable, if \(x\) is integrable.

Since the absolute value function is convex and continuous, an argument similar to that in the second paragraph of the previous proof implies that

\[
|Ex| \leq E|x|.
\]

I may now prove the following theorem, which says that because the Wiener process goes as much up as it goes down, the integral of any function with respect to it has expected value 0.

**Theorem 15.5:** \(E\left[\int_0^T g(t) \, dW_t\right] = 0\).
Proof: By definition, \[ \int_0^T g(t) \, dW = \lim_{N \to \infty} V_N \] where \( V_N \) is defined by equation 14.2. By equation 14.3, \( \mathbb{E}[V_N] = 0 \). Let \( V = \int_0^T g(t) \, dW \). Then
\[
0 \leq |EV| = |EV - \mathbb{E}[V_N]| = |\mathbb{E}[V - V_N]| \leq |V - V_N|_2,
\]
where the second inequality follows from theorem 15.4 and the third inequality follows from theorem 15.3. Since \( \lim_{N \to \infty} |V - V_N|_2 = 0 \), it follows that \( EV = 0 \).

I show that the stochastic process \( g(t) \) satisfies assumptions 14.6, then the stochastic integral \( \int_0^T g(t) \, dW \) does so as well. It should be clear that the random variable \( \int_0^T g(t) \, dW \) is measurable with respect to \( \mathcal{F}_T \), because the functions \( V_N \) are measurable with respect to \( \mathcal{F}_T \) and hence belong to \( L^2(S, \mathcal{F}_T, \mathbb{P}) \). Since \( L^2(S, \mathcal{F}_T, \mathbb{P}) \) is complete and \( V_N \) converges to \( \int_0^T g(t) \, dW \) with respect to the \( L^2 \) norm \( ||.||_2 \), it follows that \( \int_0^T g(t) \, dW \) belongs to \( L^2(S, \mathcal{F}_T, \mathbb{P}) \) and hence is measurable with respect to \( \mathcal{F}_T \). Since \( L^2(S, \mathcal{F}_T, \mathbb{P}) \) is a subset of \( L^2(S, \mathcal{F}_T, \mathbb{P}) \), \( \int_0^T g(t) \, dW \) is a function from \([0, \infty)\) to \( L^2(S, \mathcal{F}_T, \mathbb{P}) \). As such a function, \( \int_0^T g(t) \, dW \) is continuous with respect to \( t \), for suppose that \( T_n \) is a sequence in \([0, \infty)\) converging to \( T \). Then
\[
\lim_{n \to \infty} \left| \int_0^T g(t) \, dW - \int_0^T g(t) \, dW \right|_2^2 = \lim_{n \to \infty} \left| \int_0^T g(t) \, dW \right|_2^2
\]
\[= \lim_{n \to \infty} \left| \int_0^T \left| g(t) \right|^2 \, dt \right|_2^2 = 0,
\]
where the second equation follows from theorem 15.1 and last equation applies because \( \left| g(t) \right|^2_2 \) is a continuous function of \( t \) by assumption 3 of assumptions 14.6.
Stochastic Differential Equations

We now turn to the study of stochastic processes $x(T, s)$ that satisfy the equation

$$x(T, s) = x(0, s) + \int_0^T \mu(t, x(t, s)) \, dt + \int_0^T \sigma(t, x(t, s)) \, dW(t, s),$$

(15.1)

for all $T$ and $s$, where $T$ varies over $[0, \infty)$, $\mu : [0, \infty) \times (-\infty, \infty) \rightarrow (-\infty, \infty)$, and $\sigma : [0, \infty) \times (-\infty, \infty) \rightarrow [0, \infty)$. Such a process is called an Ito process. Equation 15.1 is called a stochastic differential equation or an Ito differential equation. Notice that if $\sigma = 0$, then equation 15.1 becomes

$$x(T, s) = x(0, s) + \int_0^T \mu(t, x(t, s)) \, dt,$$

which, for each $s$, is the same as the ordinary differential equation

$$\frac{dx(t, s)}{dt} = \mu(t, x(t, s))$$

with the initial condition $x(0, s)$. For this equation to have a unique solution, we need to assume that $\mu$ is continuous and satisfies what is known as a Lipschitz condition with respect to $x$. The Lipschitz condition is that there is a positive number $K$ such that

$$|\mu(t, x) - \mu(t, y)| \leq K|x - y|,$$

for all $x$ and $y$. The Lipschitz condition is stronger than continuity and applies if $\mu$ is differentiable with respect to $x$ and the derivative is uniformly bounded, which means that there is some $b > 0$ such that

$$\left|\frac{\partial\mu(t, x)}{\partial x}\right| < b,$$

for all $t$ and $x$. I make the same assumptions about $\sigma$.

More formally,

Assumption 15.6: Both $\mu$ and $\sigma$ are continuous with respect to $t$ and $x$ and both satisfy a Lipschitz condition with respect to $x$.

Theorem 15.7: Under assumption 15.6, the stochastic differential equation 15.1 has a unique solution $x(t, s)$ with initial condition $x(0, s)$, provided the function of $s$, $x(0, s)$, belongs to $L_2(S, \mathcal{F}_0, P)$, (which implies that $x(0, s)$ is measurable with respect to $\mathcal{F}_0$).

Furthermore $x(t, s) \in L_2(S, \mathcal{F}_t, P)$, for every $t$, and $x(t, s)$ is a continuous function of $t$ with probability 1.

I do not attempt a proof of this important theorem.

We can imagine finding an approximate solution to a stochastic differential equation by
dividing time into tiny intervals of length $\Delta$. If the non-negative integer $n = n(t, \Delta)$ is such that $n\Delta < t \leq (n+1)\Delta$, we could form the function

$$x_n(t, s) = x(0, s) + \sum_{k=0}^{n-1} \mu(k\Delta, x_n(k\Delta, s)) \Delta + \mu(n\Delta, x_n(n\Delta, s))(t - n\Delta)$$

$$+ \sum_{k=0}^{n-1} \sigma(k\Delta, x_n(k\Delta, s)) \left( W_{(k+1)\Delta}^1(s) - W_{k\Delta}^1(s) \right) + \sigma(n\Delta, x_n(n\Delta, s)) \left( W_t^1(s) - W_{n\Delta}^1(s) \right),$$

where the functions $x_n(k\Delta, s)$ are defined by induction on $k$ beginning with $x_n(0\Delta, s) = x(0, s)$, for $k = 0$, and by the formula

$$x_n((k+1)\Delta, s) = x_n(k\Delta, s) + \mu(k\Delta, x_n(k\Delta, s)) \Delta$$

$$+ \sigma(k\Delta, x_n(k\Delta, s)) \left( W_{(k+1)\Delta}^1(s) - W_{k\Delta}^1(s) \right).$$

The functions $x_n(t, s)$ are approximate solutions to the stochastic differential equation that converge in $L^2(S, \mathbb{S}, \mathbb{P})$ to the actual solution.

Imagine now that the exogenous stochastic influence is not the Wiener process $W_t^1(s)$ but some other stochastic process $Z_t^1(s)$, where $Z_t^1$ is measurable with respect to $\mathbb{S}_t$ and where

$\mathbb{E}[Z_T^1 - Z_t^1] = 0$, for all $t$ and $T$ such that $T > t$. Finally suppose that

$\mathbb{V}ar[Z_T^1 - Z_t^1] = \mathbb{E}[Z_T^1 - Z_t^1]^2 = (T - t)^a$, where $a > 1$. Let $x(t, s)$ be a solution to the stochastic differential equation

$$x(T, s) = x(0, s) + \int_0^T \mu(t, x(t, s)) \, dt + \int_0^T \sigma(t, x(t, s)) \, dZ_t^1(s),$$

assuming that such a solution exists. I argue intuitively that

$$\int_0^T \sigma(t, x(t, s)) \, dZ_t^1(s) = 0,$$

so that the solution $x(t, s)$ depends on $s$ only through $x(0, s)$, and the exogenous stochastic process $Z_t^1$ has no affect on the solution $x(t, s)$. The integral

$$\int_0^T \sigma(t, x(t, s)) \, dZ_t^1(s)$$

may be approximated by the finite sum

$$V_N^1(s) = \sum_{n=0}^{N-1} \sigma\left(\frac{n}{N}, x\left(\frac{n}{N}, T, s\right)\right) Z_{\frac{n+1}{N}}^1(s) - Z_{\frac{n}{N}}^1(s).$$

Assume that $V_N^1(s)$ converges in the $L^2(S, \mathbb{S}, \mathbb{P}, \|\cdot\|_2)$, to

$$\int_0^T \sigma(t, x(t, s)) \, dZ_t^1(s).$$

We can use the argument in the proof of equation 14.4 to show that
\[ E[V^2(s)] = \sum_{n=0}^{N-1} E[\sigma^2(\frac{n}{N}, x(\frac{n}{N}, s))] E[Z_{\frac{n+1}{N}} - Z_{\frac{n}{N}}]^2 \]

\[ = \left(\frac{T}{N}\right)^{n-1} \sum_{n=0}^{N-1} E[\sigma^2(\frac{n}{N}, x(\frac{n}{N}, s))] \]

As \(N\) goes to infinity, \(\left(\frac{T}{N}\right)^{n-1}\) converges to 0, and it is reasonable to assume that

\[ \left(\frac{T}{N}\right)^{n-1} \sum_{n=0}^{N-1} E[\sigma^2(\frac{n}{N}, x(\frac{n}{N}, s))] \]

converges to \(\int_0^T E[\sigma^2(t, x(t, s))] dt\), which I assume to be finite. Then

\[ \left(\frac{T}{N}\right)^{n-1} \sum_{n=0}^{N-1} E[\sigma^2(\frac{n}{N}, x(\frac{n}{N}, s))] \]

converges to 0 and so \(E[V^2(s)] = \|V\|_2^2\) converges to 0.

Since \(\lim_{N \to \infty} \|V\|_2 = 0\) and by assumption \(\lim_{N \to \infty} \|V - \int_0^T \sigma(t, x(t, s)) dZ_t(s)\|_2 = 0\), we see that

\[ \int_0^T \sigma(t, x(t, s)) dZ_t(s) = 0. \]

This rough argument can be made rigorous and shows that exogenous stochastic influences, \(Z_t\), of the sort just described can be ignored if \(E[Z_{t+\delta} - Z_t]^2 = (\delta t)^a\), where \(a > 1\).

As an example of the application of the idea just explained, suppose we have a stochastic differential equation of the form

\[ dx(t, s) = \mu(t, x(t, s)) dt + \sigma(t, x(t, s)) dZ_t(s), \]

where \(Z_t(s)\) is a stochastic process such that, for all \(t\), \(Z_t(s)\) is measurable with respect to \(\mathcal{F}_t\), \(Z_{t+\delta} - Z_t(s)\) is independent of \(\mathcal{F}_t\), \(E[Z_{t+\delta} - Z_t] = 0\), and \(\text{Var}[Z_{t+\delta} - Z_t] = \Delta^2\), for any positive number \(\Delta\). If \(\Delta\) is small, then \(\Delta^2\) is another order of small than \(\Delta\), so that we can ignore \(Z_t\) and the stochastic differential equation has the same solutions as does the ordinary differential equation

\[ \frac{dx(t, s)}{dt} = \mu(t, x(t, s)), \]

and the solution depends on \(s\) only through the dependence of the initial condition, \(x(0, s)\), on \(s\).

To make another application of the same idea, consider the stochastic process defined by the equation

\[ dx(t, s) = \mu(t, x(t, s)) dt + \sigma(t, x(t, s)) dW_t(s) + \gamma(t, x(t, s)) dW_t(s)^y, \]
where $W_t(s)$ is the Wiener process. Since the mean of $(dW_t)^2$ is $dt$, we can rewrite this equation as

$$dx(t, s) = [\mu(t, x(t, s)) + \gamma(t, x(t, s))]dt + \sigma(t, x(t, s))dW_t(s)$$

$$+ \gamma(t, x(t, s))[(dW_t(s))^2 - dt].$$

The term $[(dW_t(s))^2 - dt]$ has mean zero and variance $3(dt)^2$, since $dW_t(s)$ is normally distributed with mean $0$ and variance $dt$. Hence we may ignore the term $\gamma(t, x(t, s))[(dW_t(s))^2 - dt]$ and write the equation as

$$dx(t, s) = [\mu(t, x(t, s)) + \gamma(t, x(t, s))]dt + \sigma(t, x(t, s))dW_t(s).$$

The next topic is Itô's lemma. Suppose that $x(t, s)$ is an Itô process satisfying the equation

$$dx(t, s) = \mu(t, x(t, s))dt + \sigma(t, x(t, s))dW_t(s).$$

Let $h(t, x)$ be a twice continuously differentiable function of $t$ and $x$ and let $y(t, s) = h(t, x(t, s))$. Itô's lemma shows that $y(t, s)$ is also an Itô process. In order to see how the argument works, we apply Taylor's expansion to the function $h$ to obtain

$$dy(t, s) = \frac{\partial h(t, x(t, s))}{\partial t} dt + \frac{\partial h(t, x(t, s))}{\partial x} dx(t, s)$$

$$+ \frac{1}{2} \frac{\partial^2 h(t, x(t, s))}{\partial t^2} dt^2 + \frac{\partial^2 h(t, x(t, s))}{\partial x \partial t} dx(t, s) dt$$

$$+ \frac{1}{2} \frac{\partial^2 h(t, x(t, s))}{\partial x^2} (dx(t, s))^2$$

$$= \frac{\partial h}{\partial t} dt + \frac{\partial h}{\partial x} [\mu(t, x(t, s)) dt + \sigma(t, x(t, s)) dW_t(s)]$$

$$+ \frac{1}{2} \frac{\partial^2 h}{\partial t^2} dt^2 + \frac{\partial^2 h}{\partial x \partial t} \left[ \mu(t, x(t, s)) dt + \sigma(t, x(t, s)) dW_t(s) \right]$$

$$+ \frac{1}{2} \frac{\partial^2 h}{\partial x^2} \left[ \mu^2(t, x(t, s)) dt^2 + 2 \mu(t, x(t, s)) \sigma(t, x(t, s)) dt dW_t(s) + \sigma^2(t, x(t, s)) (dW_t(s))^2 \right].$$
\[
\begin{aligned}
&= \left[ \frac{\partial h}{\partial t} + \frac{\partial h}{\partial x} \mu + \frac{1}{2} \frac{\partial^2 h}{\partial x^2} \sigma^2 \right] dt \\
&\quad + \left[ \frac{1}{2} \frac{\partial^2 h}{\partial t^2} + \frac{\partial^2 h}{\partial x^2} \frac{\mu}{\partial t} + \frac{1}{2} \frac{\partial^2 h}{\partial x^2} \mu^2 \right] dt^2 + \frac{\partial h}{\partial x} \sigma dW_t \\
&\quad + \left[ \frac{\partial^2 h}{\partial x \partial t} \mu + \frac{\partial^2 h}{\partial x^2} \mu \sigma \right] dtdW_t + \frac{1}{2} \frac{\partial^2 h}{\partial x^2} \left[ (dW_t)^2 - dt \right].
\end{aligned}
\]

The stochastic terms are multiples of \(dW_t\), \(dtdW_t\), and \((dW_t)^2 - dt\), all of which have mean 0. The variance of \(dW_t\) is \(dt\). That of \(dtdW_t\) is \(dt^2\), and that of \((dW_t)^2 - dt\) is \(3dt^2\). As argued earlier, the stochastic terms with variance of order \(dt^2\) and \(dt^3\) can be ignored. We can also ignore terms with a deterministic differential of order \(dt^3\). After dropping the terms of order \(dt^2\) and \(dt^3\), we are left with the stochastic differential equation

\[
dy(t, s) = \left[ \frac{\partial h(t, x(t, s))}{\partial t} + \frac{\partial h(t, x(t, s))}{\partial x} \mu(t, x(t, s)) \right] dt + \frac{\partial h(t, x(t, s))}{\partial x} \sigma(t, x(t, s)) dW_t(s).
\]

That \(y(t, s)\) satisfies this stochastic differential equation is known as Itô’s lemma.

**Example:** What is known as geometric Brownian motion satisfies the stochastic differential equation

\[
dx(t, s) = \mu x(t, s) dt + \sigma x(t, s) dW_t(s),
\]

where \(\mu\) is a constant and \(\sigma\) is a positive constant. The non-stochastic version of this equation is \(dx(t)/dt = \mu x(t)\), which has the solution \(x(t) = e^{\mu t}\), where \(a\) is an arbitrary constant. This solution may be found by considering the corresponding differential equation for \(y(t) = \ln(x(t))\), which is \(dy(t)/dt = \mu\). The solution of this differential equation is \(y(t) = a + \mu t\).

Taking the exponential of this equation, we obtain the equation \(x(t) = e^{\mu t}\). Let us do the same thing in the stochastic case and let \(y(t, s) = \ln(x(t, s))\) and apply Itô’s lemma with \(h(t, x) = \ln(x)\). Notice that the function \(\mu(t, x) = \mu x\) and the function \(\sigma(t, x) = \sigma x\). Then

\[
dy(t, s) = \left[ \frac{1}{x(t, s)} \mu x(t, s) + \frac{1}{2} \sigma^2 x^2(t, s) \right] dt + \frac{1}{x(t, s)} \sigma x(t, s) dW_t(s)
\]

\[
= \left[ \mu - \frac{\sigma^2}{2} \right] dt + \sigma dW_t(s).
\]

The solution of this equation is
\[ y(t, s) = a + \left( \mu - \frac{1}{2} \sigma^2 \right) t + \sigma W_t(s). \]

The corresponding solution for \( x(t, s) \) is

\[ x(t, s) = e^{\int_0^t \left( \mu - \frac{1}{2} \sigma^2 \right) \theta \, d\theta} e^{\sigma W_t(s)}. \]

**Stochastic Optimal Control Theory**

I now discuss briefly optimal control theory, where the differential equation to be controlled is stochastic rather than deterministic. Consider the problem

\[
\max_{u \in \mathcal{A}} \mathbb{E} \left[ \int_0^T f(x(t, s), u(t, s)) \, dt \right]
\]

s.t. \( dx(t, s) = \mu(x(t, s), u(t, s)) \, dt + \sigma(x(t, s), u(t, s)) \, dW_t(s), \tag{15.2} \)

for all \( s \) and \( t \), and \( x(0, s) \) is given.

The set \( \mathcal{A} \) is the set of admissible controls, which is defined to be all stochastic processes \( u(t, s) \) such that \( u(t, s) \) is measurable with respect to \( \mathcal{F}_t \), for all \( t, u(t, s) \in L^2(S, \mathcal{F}, P) \), for all \( t \), and the mapping from \( t \) to \( u(t, s) \) is a continuous function from \( [0, T] \) to \( L^2(S, \mathcal{F}, P) \). Notice that the control at time \( t \) depends only on information available at time \( t \). It follows that the functions \( x(t, s) \) are measurable with respect to \( \mathcal{F}_t \), for all \( t \). Note that the expectations in the objective function is over all states. It is not conditional on the information available at time \( 0 \). One can add a terminal condition at time \( T \), but I ignore this possibility.

Assume the following. (These assumptions are stronger than what is needed.)

**Assumptions 15.8:**

1) \( f(x, u) \) is a continuous function.

2) Both \( \mu(x, u) \) and \( \sigma(x, u) \) satisfy a Lipschitz condition with respect to \( x \) and \( u \) and hence are continuous.

3) The function of \( s, x(0, s) \) belongs to \( L^2(S, \mathcal{F}_0, P) \) and so is measurable with respect to \( \mathcal{F}_0 \).

Notice that if \( \sigma(x, u) \) is Lipschitz and \( x(t, s) \) and \( u(t, s) \) are continuous functions from \( t \)
to $L^2(S, \mathcal{F}, P)$, then $\sigma(x(t, s), u(t, s))$ is a continuous function from $t$ to $L^2(S, \mathcal{F}, P)$, so that the stochastic integral $\int_0^T \sigma(x(t, s), u(t, s)) \, dW(s)$ exists and the stochastic differential equation

$$dx(t, s) = \mu(x(t, s), u(t, s)) \, dt + \sigma(x(t, s), u(t, s)) \, dW(s)$$

should have a solution.

The value function $V$ is defined as

$$V(t, x(t, s), s) = \max_u \mathbb{E}\left[ \int_t^T \{f(x(\tau, s), u(\tau, s)) \, d\tau \mid \mathcal{F}_t \} \right]$$

s.t.

$$dx(\tau, s) = \mu(x(\tau, s), u(\tau, s)) \, d\tau + \sigma(x(\tau, s), u(\tau, s)) \, dW(\tau, s),$$

for all $\tau$ and $s$, and $x(t, s)$ is given,

where the expectation $\mathbb{E}[\cdot \mid \mathcal{F}_t]$ is the expectation conditional on information available at time $t$, that is, conditional on $\mathcal{F}_t$.

I have not defined expectation conditional on a $\sigma$-field, but I will try to convey what it means. Let $\mathcal{A}$ be a $\sigma$-field that is a subfield of $\mathcal{F}$. If $y : S \rightarrow R$ is a random variable measurable with respect to $\mathcal{F}$ and if $A \in \mathcal{A}$ is such that $\text{Prob}(A) > 0$, then the number $\mathbb{E}[y \mid A]$ is the expected value of $y$ given knowledge that the true state belongs to $A$. That is, $P(A)\mathbb{E}[y \mid A] = \int y(s) \, P(ds)$.

The conditional expectation $\mathbb{E}[y \mid \mathcal{A}]$ is a random variable $\mathbb{E}[y \mid \mathcal{A}] : S \rightarrow R$ that is measurable with respect to $\mathcal{A}$ and is such that $P(A)\mathbb{E}[y \mid A] = \int \mathbb{E}[y \mid \mathcal{A}](s) \, P(ds)$, for all $A \in \mathcal{A}$ such that $P(A) > 0$. That is, $P(A)\mathbb{E}[y \mid A]$ is the integral over $A$ of the random variable $\mathbb{E}[y \mid \mathcal{A}](s)$ that is measurable with respect to $\mathcal{A}$. For this reason, as a function of $s$, $V(t, x(t, s), s)$ is measurable with respect to $\mathcal{F}_t$.

An important fact about conditional expectations is that if $y : S \rightarrow R$ is measurable with respect to $\mathcal{A}$, then $\mathbb{E}[y \mid \mathcal{A}] = y$. This is so simply because if $y$ is measurable with respect to $\mathcal{A}$ then it satisfies the definition of $\mathbb{E}[y \mid \mathcal{A}]$. Also if $y : S \rightarrow R$ and $z : S \rightarrow R$ are stochastically independent random variables that are measurable with respect to $\mathcal{F}$, then $\mathbb{E}[yz \mid \mathcal{A}] = \mathbb{E}[y \mid \mathcal{A}] \mathbb{E}[z \mid \mathcal{A}]$. Finally if $y : S \rightarrow R$ is independent of $\mathcal{A}$, then $\mathbb{E}[y \mid \mathcal{A}] = \mathbb{E}(y)$. This is so because if $A \in \mathcal{A}$ and $P(A) > 0$, then $P(A)\mathbb{E}[y \mid A] = \int y(s) \, P(ds) = P(A) \int y(s) \, P(ds)$

$= P(A)\mathbb{E}(y)$, so that $\mathbb{E}[y \mid A] = \mathbb{E}(y)$. Therefore the constant random variable $\mathbb{E}(y)$ satisfies the equation $\mathbb{E}[y \mid A] = \int \mathbb{E}(y) \, P(ds)$, for every $A \in \mathcal{A}$ is such that $\text{Prob}(A) > 0$. Since a constant function is clearly measurable with respect to $\mathcal{A}$ (or any other $\sigma$-field), $\mathbb{E}(y)$ satisfies the definition of $\mathbb{E}[y \mid \mathcal{A}]$.
It would seem that the value function \( V(t, x(t), s) \) should depend on \( s \) directly as well as on \( t \) and \( x(t) \). This dependence does not exist, however, and I now try to explain why. Recall that the states \( s \) in \( S \) should be thought of as functions \( s(t) \) from \([0, \infty) \) to \( R \). To say that a function \( g(s) \) is measurable with respect to \( \tilde{S} \) means that \( g(s) \) depends only on the values of \( s(\tau) \), for \( \tau \leq t \). Now imagine that you control the variable \( u \), that you are in state \( s \) and know \( s(\tau) \), for \( \tau \leq t \), and that you wish to solve the optimal control problem. You must determine the value of \( u \) over a small interval of time from \( t \) to \( t + dt \). The problem you must solve is

\[
V(t, x(t), s) = \max_u \left\{ f(x(t), s), u)dt + \mathbb{E}\left[ V(t+dt, x(t+dt), s) \mid S_t \right](s) \right\}
\]

\[
= \max_u \left\{ f(x(t), s), u)dt + \mathbb{E}\left[ V(t+dt, x(t), s) + \mu(x(t), s), u)dt + \sigma(x(t), s), u) \left\{ W_{t+dt}(s) - W_t(s) \right\} \right\mid S_t \right\}(s)
\]

Let us look at this problem carefully. The term \( f(x(t), s), u)dt \) depends on \( s \) only through \( x(t, s) \). Within the term

\[
\mathbb{E}\left[ V(t+dt, x(t), s) + \mu(x(t), s), u)dt + \sigma(x(t), s), u) \left\{ W_{t+dt}(s) - W_t(s) \right\} \right\mid S_t \right\}(s)
\]

\( \mu(x(t), s), u) \) and \( \sigma(x(t), s), u) \) depend on \( s \) only through \( x(t, s) \), and the random variation \( W_{t+dt}(s) - W_t(s) \) is stochastically independent of \( S_t \), so that knowledge of \( s(\tau) \), for \( \tau \leq t \), does not help predict it. If \( V(t+dt, x(t+dt), s) \) does not depend directly on \( s \), then given knowledge of \( x(t, s) \), knowing \( s(\tau) \), for \( \tau \leq t \), will not improve your choice of \( u \). So if we assume that \( V(t+dt, x(t+dt), s) \) does not depend directly on \( s \), then \( V(t, x(t), s) \) does not depend directly on \( s \) either and furthermore the optimal choice of \( u \) may be made to depend only on \( x(t, s) \). (If, however, there was more than one optimal choice of \( u \), you could make the choice among these optimal choices depend artificially on \( s \).) Now imagine that we divide the time interval \([t, T]\) into great many little intervals of length \( dt \) and make the argument just made by backward induction on the intervals, \([t, t + dt], [t + dt, t + 2dt], \ldots, [T - dt, T]\). Since \( V(T, x(T), s) = 0 \), it does not depend on \( s \). By the above argument,

\[
V(T - dt, x(T - dt), s) = \max_u f(x(T - dt), s), u) dt
\]

does not depend directly on \( s \) and the optimal value of \( u \) and hence of \( V(T - dt, x(T - dt), s) \) depends only on \( x(T - dt), s \). By continuing by backward induction on the intervals, we see that \( V(t, x(t), s) \) does not depend directly on \( s \). I hope that this informal argument convinces you that we may replace \( V(t, x(t), s) \) with \( V(t, x(t)) \) or \( V(t, x) \) and we may assume that the optimal choice of \( u \) at time \( t \) is a function of \( x(t) \) alone. We should not, however, dispense with the conditional expected value in the definition of the value function \( V \), for if we did \( V(t, x(t), s) = V(t, x(t, s)) \) would be a constant function of \( s \), which would make no sense if \( x(t, s) \) depended on \( s \). If \( x(t, s) \) is a constant \( x(t) \) independent of \( s \), then we may replace the conditional expectation in the definition of \( V \) with the ordinary expectation.
I now derive in a loose way a version of the Hamilton Jacobi Bellman equation that applies to problem 15.2. Assume that the function \( V(t, x) \) is twice continuously differentiable. Applying the Bellman principle and then applying Ito's lemma to the stochastic process \( y(t, s) = V(t, x(t, s)) \), we have that

\[
V(t, x(t, s)) = \max_u \left\{ f(x(t, s), u) dt + E \left[ V(t+dt, x(t+dt, s)) \bigg| \mathcal{F}_t \right] \right\}
\]

\[
= \max_u \left\{ f(x(t, s), u) dt + E \left[ V(t, x(t, s)) + dV(t, x(t, s)) \bigg| \mathcal{F}_t \right] \right\}
\]

\[
= \max_u \left\{ f(x(t, s), u) dt + E \left[ V(t, x(t, s)) + \left( \frac{\partial V(t, x(t, s))}{\partial t} + \frac{1}{2} \frac{\partial^2 V(t, x(t, s))}{\partial x^2} \sigma^2(x(t, s), u) \right) dt \right. \right. 
\]

\[
\left. \left. + \frac{\partial V(t, x(t, s))}{\partial x} \sigma(x(t, s), u) dW(s) \bigg| \mathcal{F}_t \right] \right\}
\]

\[(15.3)\]

\[
= V(t, x(t, s)) + \frac{\partial V(t, x(t, s))}{\partial t} dt 
\]

\[
+ \max_u \left\{ f(x(t, s), u) + \frac{\partial V(t, x(t, s))}{\partial x} \mu(x(t, s), u) 
\right. 
\]

\[
\left. + \frac{1}{2} \frac{\partial^2 V(t, x(t, s))}{\partial x^2} \sigma^2(x(t, s), u) \right\} dt ,
\]

where I have used facts (1) - (5) listed below to arrive at the last equation.

1) The function of \( s, x(t, s) \), is measurable with respect to \( \mathcal{F}_t \), so that functions of \( s \), such as \( V(t, x(t, s)) \), that depend on \( x(t, s) \) are measurable with respect to \( \mathcal{F}_t \) as well.

2) \( E \left[ V(t, x(t, s)) + \frac{\partial V(t, x(t, s))}{\partial t} \bigg| \mathcal{F}_t \right] = V(t, x(t, s)) + \frac{\partial V(t, x(t, s))}{\partial t} \),

because \( V(t, x(t, s)) + \frac{\partial V(t, x(t, s))}{\partial t} \) is measurable with respect to \( \mathcal{F}_t \).

Similarly
\[ E \left[ f(x(t, s), u) + \frac{\partial V(t, x(t, s))}{\partial x} \mu(x(t, s), u) + \frac{1}{2} \frac{\partial^2 V(t, x(t, s))}{\partial x^2} \sigma^2(x(t, s), u) \bigg| \mathcal{F}_t \right] \]

\[ = f(x(t, s), u) + \frac{\partial V(t, x(t, s))}{\partial x} \mu(x(t, s), u) + \frac{1}{2} \frac{\partial^2 V(t, x(t, s))}{\partial x^2} \sigma^2(x(t, s), u), \]

because \( f(x(t, s), u) + \frac{\partial V(t, x(t, s))}{\partial x} \mu(x(t, s), u) + \frac{1}{2} \frac{\partial^2 V(t, x(t, s))}{\partial x^2} \sigma^2(x(t, s), u) \) is measurable with respect to \( \mathcal{F}_t \).

3 ) \[ E \left[ \frac{\partial V(t, x(t, s))}{\partial x} \sigma(x(t, s), u) dW_t(s) \bigg| \mathcal{F}_t \right] \]

\[ = E \left[ \frac{\partial V(t, x(t, s))}{\partial x} \sigma(x(t, s), u) \bigg| \mathcal{F}_t \right] E \left[ dW_t(s) \bigg| \mathcal{F}_t \right], \]

since \( \frac{\partial V(t, x(t, s))}{\partial x} \sigma(x(t, s), u) \) and \( dW_t(s) \) are stochastically independent.

4 ) \[ E \left[ dW_t(s) \big| \mathcal{F}_t \right] = E[ dW_t ] = 0, \] where the first equation applies because \( dW_t \) is independent of \( \mathcal{F}_t \).

Facts (3) and (4) imply that

5 ) \[ E \left[ \frac{\partial V(t, x(t, s))}{\partial x} \sigma(x(t, s), u) dW_t(s) \bigg| \mathcal{F}_t \right] \]

\[ = E \left[ \frac{\partial V(t, x(t, s))}{\partial x} \sigma(x(t, s), u) \bigg| \mathcal{F}_t \right] E \left[ dW_t(s) \bigg| \mathcal{F}_t \right] \]

\[ = E \left[ \frac{\partial V(t, x(t, s))}{\partial x} \sigma(x(t, s), u) \bigg| \mathcal{F}_t \right](0) = 0. \]

Canceling \( V(t, x(t, s)) \) from both of the extreme sides of equation 15.3 and dividing these two sides by \( dt \), we see that

\[ \frac{\partial V(t, x(t, s))}{\partial t} = - \max_u \left\{ f(x(t, s), u) + \frac{\partial V(t, x(t, s))}{\partial x} \mu(x(t, s), u) \right. \]

\[ + \left. \frac{1}{2} \frac{\partial^2 V(t, x(t, s))}{\partial x^2} \sigma^2(x(t, s), u) \right\}. \]  

(15.4)

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This equation is another Hamilton Jacobi Bellman equation.

It should be clear from equation 15.3 that the maximizing value of \( u \) in equation 15.4 is the optimal control \( u(t, s) \), so that the derivation of the Hamilton Jacobi Bellman equation includes an informal verification of a maximum principle for the stochastic optimal control problem. Notice that the maximization problem in equation 15.4 is deterministic for each value of \( s \).

A maximum principle and a Hamilton Jacobi Bellman equation apply also to stochastic optimal control problems where the gain is discounted and integrated over an infinite horizon. Consider the problem

\[
\max_u E \left[ \int_0^T e^{-\gamma t} f(x(t, s), u(t, s)) \, dt \right]
\]
\[
s.t. \quad dx(t, s) = \mu(x(t, s), u(t, s)) \, dt + \sigma(x(t, s), u(t, s)) \, dW_t(s),
\]
\[
\text{for all } s \text{ and } t, \text{ and } \quad x(0, s) \text{ is given.}
\]

Assume that assumptions 15.8 apply. Define the value function \( V \) by the equation

\[
V(t, x(t, s), s) = E \left[ \max_u \int_0^T e^{-\gamma \tau} f(x(\tau, s), u(\tau, s)) \, d\tau \right] |_{\tau=t}(s)
\]
\[
s.t. \quad dx(\tau, s) = \mu(x(\tau, s), u(\tau, s)) \, d\tau + \sigma(x(\tau, s), u(\tau, s)) \, dW_t(s),
\]
\[
\text{for all } s \text{ and } \tau, \text{ and } \quad x(t, s) \text{ is given.}
\]

Just as in the case of the undiscounted problem, the value function does not depend directly on \( s \), so that we may write \( V(t, x(t, s), s) = V(t, x(t, s)) \), and if the initial condition \( x(t, s) \) does not depend on \( s \), we may replace the conditional expectation with the ordinary expectation in the equation defining \( V \). Also we may assume that the optimal value of the control at time \( t \), \( u(t, s) \), depends only on \( x(t, s) \). The value function does not depend on \( t \) either, for suppose that you control \( u \) and move with the system through time. The only thing about the optimal control problem that changes when you move from time \( t \) to time \( T > t \) is that you know more about the state \( s \). In the time interval \( (t, T] \), you will have learned \( s(\tau) \), for \( t < \tau \leq T \). But we know that the value function does not depend directly on the state, but only on the variable \( x(t, s) \) and perhaps on \( t \). Similarly the optimal control need not depend directly on the state and may be made to depend only on \( x(t, s) \). Hence the optimization problem is the same at time \( T \) as at time \( t \), and so we can write the value function as \( V(x(t, s)) \) or \( V(x) \).

Assume that the function \( V(x) \) is twice continuously differentiable. Again applying the Bellman principle and then applying Ito’s lemma to the stochastic process \( y(t, s) = V(x(t, s)) \).
we have that

\[ V(x(t, s)) = \max_u \left\{ f(x(t, s), u) dt + e^{-rt} \mathbb{E} \left[ V(x(t + dt, s)) \Big|\Big| S_t \right] \right\} \]

\[ = \max_u \left\{ f(x(t, s), u) dt + e^{-rt} \mathbb{E} \left[ V(x(t, s)) + \sigma V(x(t, s)) \right| S_t \right\} \]

\[ = \max_u \left\{ f(x(t, s), u) dt + \frac{1}{1 + rd} \mathbb{E} V(x(t, s)) \right\} \]

\[ + \left( \frac{dV(x(t, s))}{dx} \mu(x(t, s), u) + \frac{1}{2} \frac{d^2V(x(t, s))}{dx^2} \sigma^2(x(t, s), u) \right) dt \]

\[ + \left( \frac{dV(x(t, s))}{dx} \sigma(x(t, s), u) dW(s) \right| S_t \right\} \]

so that

\[ V(x(t, s)) + rV(x(t, s)) dt = \max_u \left\{ f(x(t, s), u) dt + rf(x(t, s), u) dt^2 + V(x(t, s)) \right\} \]

\[ + \left( \frac{dV(x(t, s))}{dx} \mu(x(t, s), u) + \frac{1}{2} \frac{d^2V(x(t, s))}{dx^2} \sigma^2(x(t, s), u) \right) dt \]

\[ + \mathbb{E} \left[ \frac{dV(x(t, s))}{dx} \sigma(x(t, s), u) dW(s) \right| S_t \right\} \}

\[ = V(x(t, s)) + \max_u \left\{ f(x(t, s), u) dt + rf(x(t, s), u) dt^2 \right\} \]

\[ + \left( \frac{dV(x(t, s))}{dx} \mu(x(t, s), u) + \frac{1}{2} \frac{d^2V(x(t, s))}{dx^2} \sigma^2(x(t, s), u) \right) dt \]

The steps in each of these equations follow for the same reasons as the corresponding steps did in the undiscounted case. If we cancel \( V(x(t, s)) \) from both sides of this last equation, divide the result by \( dt \), and then let \( dt \) converge to 0, we see that

\[ rV(x(t, s)) \]

\[ = \max_u \left\{ f(x(t, s), u) + \frac{dV(x(t, s))}{dx} \mu(x(t, s), u) + \frac{1}{2} \frac{d^2V(x(t, s))}{dx^2} \sigma^2(x(t, s), u) \right\} \]

This is still another Hamilton Jacobi Bellman equation. In the stochastic case, its solution is not sufficient for optimality. We have also come up with a maximum principle for problem 15.5,
since the maximizing value of $u$ in the Hamilton Jacobi Bellman equation is the optimal value $u(t, s)$.

**Example:** This is a stochastic version of example 13.1. Consider the problem

$$\max_u \mathbb{E}\left[ e^{-\gamma t}\left( -x^2(t, s) - u^2(t, s) \right) \right]$$

subject to

$$\frac{dx(t, s)}{dt} = u(t, s) dt + \sigma dW(t)$$

and $x(0, s)$ is given,

where $\sigma > 0$.

In applying the Hamilton Jacobi Bellman equation, let $f(x, u) = -x^2 - u^2$, $\mu(x, u) = u$, and $\sigma(x, u) = \sigma$. Guess that $V(x) = ax^2 + b$, where $a < 0$ and $b < 0$. Then

$$\frac{dV(x)}{dx} = 2ax \quad \text{and} \quad \frac{d^2V(x)}{dx^2} = 2a.$$ 

By the Hamilton Jacobi Bellman equation,

$$r(ax^2 + b) = \max_u \left[ -x^2 - u^2 + 2axu + a\sigma^2 \right]. \quad (15.6)$$

The maximizing value of $u$ is $u = ax$ and

$$\max_u \left[ -x^2 - u^2 + 2axu + a\sigma^2 \right] = -x^2 + a^2x^2 + a\sigma^2,$$

so that equation 15.6 becomes

$$rax^2 + rb = -x^2 + a^2x^2 + a\sigma^2.$$ 

Since this equation holds for all $x$, we have that

$$ra = -1 + a^2 \quad \text{and} \quad b = \frac{a\sigma^2}{r}.$$ 

Since $a < 0$, the solution of the quadratic equation $a^2 - ra - 1 = 0$ is

$$a = \frac{r - \sqrt{r^2 + 4}}{2}.$$ 

The optimal control is $u(t, s) = ax(t, s)$. If this control is used, $x(t, s)$ evolves according to the stochastic differential equation.
\[ dx(t, s) = ax(t, s)dt + \sigma dW_t(s). \]

This is only a candidate optimal control, since the Hamilton Jacobi Bellman equation does not provide a sufficient condition for optimality.