Some Notes on Cournot and the Bargaining Problem

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WITH A FOREWORD BY OLAV BJERKHOLT^

ABSTRACT

The basic ideas of Cournot and those who came after him are related to the recent work of Nash and his notion of an "equilibrium point." It is shown that the Nash equilibrium point incorporates the main contribution of Cournot to the solution of the duopoly problem and that the major criticism that may be made against the Cournot equilibrium may also be made against the Nash equilibrium. It is then indicated to what use this weakness might be put in the study of bargaining.

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* I am indebted to Olav Bjerkholt for unearthing this paper, which I wrote during the summer of 1953, in the course of his research on the history of the Cowles Foundation. The Foreword to this paper explaining the significance of the paper to the history and development of two-person bargaining games was written by Olav Bjerkholt.

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Foreword

The paper *Some Notes on Cournot and the Bargaining Problem* by Marc Nerlove was written in August 1953 and is remarkable in its grasp of essentials and in its maturity, given that Nerlove was only 19 years of age at the time. Nerlove had been an undergraduate at the University of Chicago. He took his first year of graduate study at the Johns Hopkins University, 1952-53, but returned to Chicago for the summer of 1953. He was fortunate enough to obtain a summer job as research assistant to Professors Jacob Marschak and Tjalling Koopmans at the Cowles Commission. That summer, in partial fulfilment of his joint commitment to Marschak, he wrote the afore mentioned paper. Marschak did not pay attention to it and the paper was forgotten by Nerlove, lost from his files and does not appear on his CV. It has recently been retrieved from the Jacob Marschak Papers at UCLA.

The 1953 paper pointed out that in the Cournot-Nash equilibrium each agent makes a particular incorrect assumption about the other's out-of-equilibrium behavior. This is noteworthy, given its date. Nerlove recapitulated Nash's proof of the existence of a mixed strategy Nash equilibrium and provided a simple example of how to calculate it.

But Nerlove probed deeper. He knew about "conjectural variation" from Frisch (1933), rephrasing it as "each producer believes that the output of his rival depends on his own output <u>in some definite fashion</u>," and proceeded to discuss the appropriateness of the Nash equilibrium in the Cournot model, leading to a suggested definition of a "strong equilibrium."

This line of reasoning in Nerlove's brief paper is very interesting, as it predated Bresnahan (1981) by more than 25 years. Nerlove's "strong equilibrium" captured the same idea as Bresnahan's "consistent conjectures equilibrium". The "strong equilibrium," if it exists, is a "consistent conjectures equilibrium." No further analysis was offered by Nerlove. Bresnahan's "consistent conjectures equilibrium" received criticism, which would also apply to Nerlove's suggested definition of a "strong equilibrium." The essentially dynamic character of the argument was later captured in a dynamic model by Maskin and Tirole (1987). Later literature on learning and evolution puts more emphasis on Cournot's fictitious play, thereby enhancing the relevance of this process. On the other hand, it might be argued that evolution leads to conjectural variations (Dixon and Somma 2003).

Nerlove's proof of the existence of equilibrium is not for games in extensive form. Each player is assumed to choose strategies for the whole of the play simultaneously. Understanding of games in extensive form and the complications thereof came later, with Schelling (1958).

Nerlove had indeed taken an interest in game theory even earlier; his undergraduate honors thesis in mathematics was titled *On the Theory of Games*. It had won him a prize and was published in the *Student Essay Annual* in 1952.

Some Notes on Cournot and the Bargaining Problem

In 1838 Augustin Cournot showed, in his famous discussion of the two mineral springs, that on certain assumptions a determinate equilibrium solution is obtained for the duopoly problem and that this solution can be extended from duopoly to oligopoly. It is not consonant with our purpose here to discuss Cournot's extension to problems of oligopoly, but to attempt an extension of Cournot's basic assumptions and concept of a solution to the general two-person bargaining situation.

The Cournot solution for undifferentiated duopoly is based upon the assumption on the part of each rival that the other will go on producing a definite quantity irrespective of what he himself produces. By substituting this constant quantity into the demand function, the duopolist can now proceed to equate his individual marginal revenue with marginal cost and thus maximize his own profits. Since initially neither duopolist is likely to guess the output of his rival correctly, outputs will be altered subsequently until, in fact, the assumption on the part of each rival that the other will not alter his output becomes correct. The characteristic feature of the Cournot model is that if each duopolist continues to assume that the other will not change his rate of output, then ultimately they will prove to be correct although during the approach to equilibrium they will be wrong.

The Cournot model may be extended by dropping the assumption that each rival assumes that the other's rate of output is fixed and replacing it by the assumption that each producer believes that the output of his rival depends on his own output <u>in some definite fashion</u>. The assumed change in one's rival's output due to an incremental change in one's own output is usually termed "conjectural variation", see Frisch [1933]. Thus the original Cournot model may be thought of as a species of the extended model in which "conjectural variation" is zero.

The extended Cournot model may be illustrated by the following schematic diagram (Fig. 1). φ_1 is a function which maps the set of present outputs of duopolist 2,

$\{q_2\}$, into the set of next period outputs of duopolist 1, $\{q_1\}$. Similarly, φ_2 maps $\{q_1\}$ into $\{q_2\}$. The mapping is carried out in the following way: Given the present output of one's rival, a function relating changes in one's own output to changes in his own marginal cost curve, and the demand for the product, one calculates that output, say q_1', which will maximize one's own profit, taking account of the belief, of course, that the rival's output will vary with one's own in some definite fashion. Outputs will be altered until each producer no longer has the desire to alter his output, i.e., cannot increase his profit, given his rival's output and his conjectured relation between that output and his own. Thus the point of equilibrium is that point at which φ_1 and φ_2 cross.

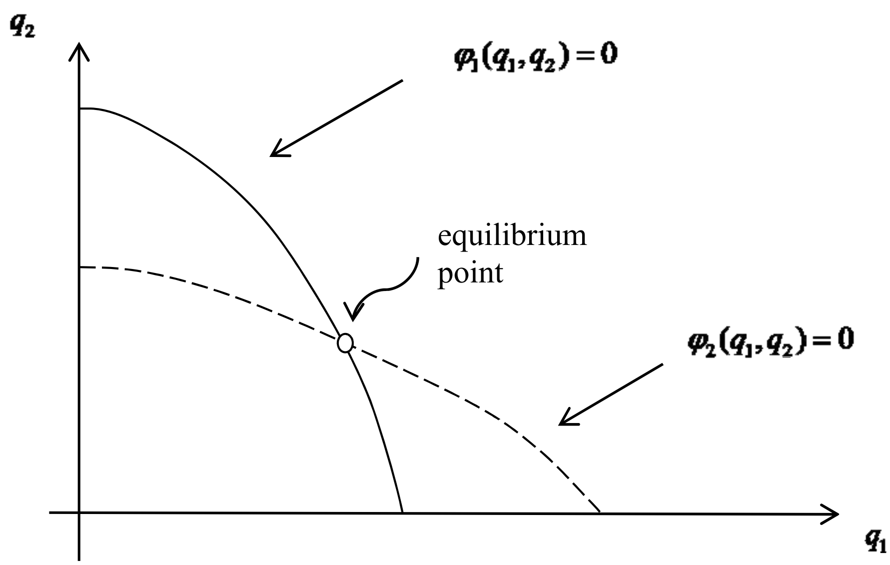

Figure 1

There is, of course, no necessity that such a point exist or that it be unique. If φ_1 crosses φ_2 from below, the point of intersection is clearly not an equilibrium, for to the right of the intersection 1's output tends to rise and 2's output tends to fall. The functions φ_1 and φ_2 need not be monotonic; and, hence, φ_1 may intersect φ_2 at many points, more than one of which may be an equilibrium. For our purposes it does not seem worthwhile to derive conditions on the marginal cost and revenue curves and the conjectured relations of output of the two producers that would insure the existence and unicity of an equilibrium. [Should the reader be interested in such a discussion, the best, to my knowledge, may be found in Stackelberg [1934], especially chapter 4 and the mathematical appendix. Fellner [1949] gives an excellent English discussion of Stackelberg's work, see chapter 3, and Lewis [1948] uses Stackelberg's indifference maps to show that certain traditional duopoly equilibria (Cournot and others) do not lie on the Edgeworth Contract Curve, i.e., possess Pareto optimum properties.]

The basic ideas involved in the Cournot solution are, then: a) that each of the two rivals believes that his opponent will react in a certain definite way to his own action (including the possibility that he will not act at all), and b) that there exists at least one set of two actions, one on the part of each rival, such that profit, utility, or some other measure of satisfaction, will be maximized for each action separately given the other action as some function of the first. It is clearly not necessary to assume, with Cournot, that the only course of action a producer may take is to vary the quantity he produces. Had Cournot been writing in 1953 instead of 1838, he might have formulated his problem in terms of strategies and the maximization of utility rather than quantity produced and the maximization profit.

In a recent communication Nash [1950a] has formulated the Cournot solution in a theory of games fashion. In two-person constant sum games "minimax" rationality and "saddle point" equilibrium can be attained simultaneously. When the number of players is increased beyond two and/or the game is made variable-sum, a specific von Neumann solution does not, in general, exist. By sacrificing "minimax" rationality and defining a solution in terms of the Cournot type equilibrium property, Nash has arrived at an uneasy compromise.

For the sake of exposition, and following Samuelson [1950], let us give the Cournot-Nash definition for the two-person case only. Let there be two rival bargainers, *A* and *B*, and denote by s_1 and s_2 their respective pure or mixed strategies, which they vary directly so as to maximize $u_1(s_1, s_2)$ and $u_2(s_1, s_2)$. *A*'s Cournot reaction locus (corresponding to φ_1 in Fig. 1) is defined by the value of s_1 which maximizes u_1 for a given s_2 or s_2 as a function of s_1. This value is not necessarily unique, but we shall denote any such value by $s_1(s_2)$. We may similarly define *B*'s reaction locus by the value $s_2(s_1)$ which maximizes u_2 for a given s_1, and this corresponds to φ_2 in Fig. 1. The solution defined by Cournot-Nash is, thus, the intersection of $s_1(s_2)$ and $s_2(s_1)$, call the point $\left(s_1^0, s_2^0\right)$, such that,

$$u_1(s_1^0, s_2^0) \geq u_1(s_1, s_2^0); \tag{1}$$

$$u_2(s_1^0, s_2^0) \geq u_2(s_1^0, s_2). \tag{2}$$

So much for the definition of equilibrium. Need such an equilibrium point always exist? Nash [1950a] shows that the answer is yes, if we do not confine ourselves to simple-discrete pay-off matrices and to pure strategies, but admit mixed strategies and/or continuous decisions, and if we assume u_1 and u_2 to have simple convexity properties in terms of the variables $\underline{s_1}$ and $\underline{s_2}$. Nash's proof of the existence of the equilibrium is roughly as follows (See Nash [1950a] and [1951]):

Let us assume the von Neumann-Morgenstern utility postulates for each individual. These assumptions suffice to show the existence of a utility function, assigning a real number to each anticipation of an individual. This utility function is not unique, i.e., the origin and the unit of measurement are arbitrary. Thus, every linear transformation of the utility function also satisfies the postulates; i.e., if u is a utility function then so is $a \cdot u + b$, with $a > 0$. Let *A* and *B* be anticipations of an individual, then such a utility function will have the following properties:

1) $u(A) > u(B)$ is equivalent to the statement that *A* is preferable to *B* for the individual in question; similarly, $u(A) = u(B)$ or $u(A) < u(B)$ is equivalent to saying that the individual is indifferent as between *A* and *B* or that he prefers *B* to *A*, respectively.

2) If $0 \leq p \leq 1$ then $u(p \cdot A + (1-p) \cdot B) = p \cdot u(A) + (1-p) \cdot u(B)$; i.e., the utility function is linear in its arguments.[1]

Nash defines a two-person anticipation as a combination of two one-person anticipations. We have two bargainers, each with a certain expectation corresponding to each possible outcome of the bargaining situation. Each two-person anticipation is defined as an ordered pair of one-person anticipations, to which the assumed one-person utility functions are applicable if applied component-wise. I.e., if (*A, B*) is a two-person anticipation and u_1 and u_2 are the utility functions of the two bargainers, then

[1] Utility and its relationship to game theory and the bargaining problem, and particularly the problem of the transferability of utility will be discussed in a later paper. Suffice it to say that Nash's formulation of the bargaining problem avoids problems of transferability.

1) $u_1(A,B) = u_1(A)$, and

2) $u_2(A,B) = u_2(B)$.

Furthermore, if (*A, B*) and (*C, D*) are two two-person anticipations and $0 \le p \le 1$, then $p \cdot (A,B) + (1-p) \cdot (C,D)$ is defined as $(p \cdot A + (1-p) \cdot C, p \cdot B + (1-p) \cdot D)$. Thus, for example

$$\begin{aligned} p \cdot u_1(A,B) + (1-p) \cdot u_1(C,D) &= u_1(p \cdot A + (1-p) \cdot C, p \cdot B + (1-p) \cdot D) \\ &= u_1(p \cdot A + (1-p) \cdot C) \\ &= p \cdot u_1(A) + (1-p) \cdot u_1(C) \end{aligned}$$

and so on.

Each possible outcome of the bargaining situation is a two-person-anticipation of the form (*A, B*), or, allowing for the possibility of mixed strategies on the part of the bargainers, a probability combination of such anticipations. Given the utility functions u_1 and u_2 of the two bargainers, the set of all possible outcomes can be transformed into a set of points of the form (u_1, u_2), where u_1 and u_2 are real numbers. The set of points lies in the $u_1 \times u_2$ plane and is arbitrary with respect to the position of the origin and the determination of scale.

Since every possible outcome, (*A, B*), is the result of an action or series of actions performed by the bargainers, u_1 and u_2 in (u_1, u_2) may be regarded as functions of two pure strategies π_1 and π_2, chosen by the first and second bargainers, respectively. Because of the way a two-person anticipation has been defined, u_1 and u_2 are bilinear forms in π_1 and π_2. Thus, for mixed strategies, u_1 and u_2 are mathematical expectations of the two bargainers.

Let s_1 and s_2 represent possible mixed strategies of the first and second bargainers, respectively, then the existence of a Cournot-Nash equilibrium resolves into the question of whether there exists a pair of strategies (s_1^0, s_2^0) or a point (u_1^0, u_2^0) such that:

1) For all s_1, $u_1^0 = u_1(s_1^0, s_2^0) \geq u_1(s_1, s_2^0)$.

2) For all s_2, $u_2^0 = (s_1^0, s_2^0) \geq u_2(s_1^0, s_2)$.

By defining a suitable transformation on the set of ordered pairs of mixed strategies $\{(s_1, s_2)\}$, it is possible to show that fixed points under this transformation are equilibrium points and vice versa, provided only that the mapping $(s_1, s_2) \to (u_1, u_2)$ is continuous and provided the image set $\{(u_1, u_2)\}$ is closed, bounded and convex. It is then possible to apply the Kakutani extension of the Brouwer fixed-point theorem to show the existence of such a point, see Kakutani [1941].

Clearly the mapping $(s_1, s_2) \to (u_1, u_2)$ is continuous by virtue of the assumption of the von-Neumann-Morgenstern utility postulates. Furthermore, it does not seem unreasonable to assume that the set $\{(u_1, u_2)\}$ is bounded. By virtue of the continuity of the mapping the set $\{(u_1, u_2)\}$ is then closed, so that any continuous function of the utilities assumes a maximum value for the set at some point of the set. There is, however, no necessity to assume the convexity of the set $\{(u_1, u_2)\}$, as Nash does, as shown by the theorem below.

Let $\mathfrak{T} = \{(u_1, u_2)\}$ and let any two-person expectation be of the form (*A*, *B*) and be the result of either two pure strategies, (π_1, π_2), or two mixed strategies (s_1, s_2). The convexity of $\mathscr{S}$ means simply that given any two points in $\mathscr{S}$, say (u_1^0, u_2^0) and (u_1^1, u_2^1), then any convex linear combination of them, such as a point $(u_1^p, u_2^p) = p \cdot (u_1^0, u_2^0) + (1-p) \cdot (u_1^1, u_2^1)$ where $0 \le p \le 1$, is also in $\mathscr{S}$. Now the points (u_1^0, u_2^0) and (u_1^1, u_2^1) were the result of applying the utility operators of the first and second bargainers to anticipations (A^0, B^0) and (A^1, B^1), which, without loss of generality, we can assume to be the result of the two pairs of pure strategies, (π_1^0, π_2^0) and (π_1^1, π_2^1). We want to prove the following theorem:

<u>Theorem</u>: Given any *p*, $0 \le p \le 1$, and any $(u_1^0, u_2^0) \in \mathscr{S}$ and $(u_1^1, u_2^1) \in \mathscr{S}$, and utility functions u_1 and u_2 such that (*A*, *B*) $\rightarrow \left(u_1, u_2\right)$ and u_1 and u_2 satisfy the von Neumann-Morgenstern utility postulates, then there exists an (A^p, B^p) such that $(u_1^p, u_2^p) = p \cdot (u_1^0, u_2^0) + (1-p) \cdot (u_1^1, u_2^1)$.

<u>Proof</u>: Let $(A^p, B^p) = (p \cdot A^0 + (1-p)A^1, p \cdot B^0 + (1-p) \cdot B^1)$; i.e., (A^p, B^p) is the result of the pair of mixed strategies $(p \cdot \pi_1^0 + (1-p) \cdot \pi_1^1, p \cdot \pi_2^0 + (1-p) \cdot \pi_2^1)$.

Then

$$(u_1^p, u_2^p) = (u_1(A^p), u_2(B^p)) = (p \cdot u_1(A^0) + (1-p) \cdot u_1(A^1), p \cdot u_2(B^0) + (1-p) \cdot u_2(B^1))$$

because, since we have assumed the von Neumann-Morgenstern utility postulates, *u* is such that $u(p \cdot a + (1-p) \cdot b) = p \cdot u(a) + (1-p) \cdot u(b)$. By the definition of u_1^0 etc. we have, $(u_1^p, u_2^p) = (p \cdot u_1^0 + (1-p) \cdot u_1^1, p \cdot u_2^0 + (1-p) \cdot u_2^1) = p \cdot (u_1^0, u_2^0) + (1-p) \cdot (u_1^1, u_2^1)$.

Therefore $\mathscr{S}$ is convex.

Before we discuss the actual process of locating the Cournot-Nash equilibrium points for games, we might at this point go into some of the objections to the basic concept underlying this solution to the bargaining problem and the question of its relevance to experimental work in bargaining.

It is essential to realize that, as long as rivals make Cournot assumptions about each other's behavior, the analysis cannot be adjusted in such a way to make the rivals right for the right reasons, instead of describing a situation in which they turn out to be right for the wrong reason. If an individual knew that his rival was reacting along one of the reaction functions, say $s_2(s_1)$, then he would <u>not</u> be reacting along $s_1(s_2)$. Instead he would try to select the point along $s_2(s_1)$ which was optimal from his own point of view. Normally the point so selected would not be the intersection point. Therefore, if we make the assumption that the two rivals are aware of the "true" reaction functions, then the "true" reaction is no longer true! Hence, the assumption that the two rivals know of each other that they react <u>à la</u> Cournot is an inconsistent assumption. A determinate equilibrium of the Nash-Cournot type is not what might be called a "rational" solution. Generally speaking, if the two rivals change their rather arbitrary and incorrect assumption about each other's behavior, then there will be no tendency towards a restoration of the <u>Cournot-Nash</u> equilibrium. The kind of stability which exists in the Cournot-Nash equilibrium is of very little importance because the equilibrium proves unstable as soon as one rival becomes doubtful about his rival's behavior.

It is possible to define what might be termed a <u>strong equilibrium point</u> which would be immune to the above criticism; i.e., to the unilateral waking up by one of the passive Cournot-Nash rivals. All that is required is that in addition to the condition on (s_1^0, s_2^0) given above in (1) and (2) we add the condition:

$$u_1^0 \geq u_1(s_1, s_2(s_1)); \tag{3}$$

$$u_2^0 \geq u_2(s_1(s_2), s_2). \tag{4}$$

That is, the equilibrium strategy for each rival must be optimum, not only against the other's equilibrium strategy, but also with respect to any other of his own when the rival plays optimally against that other strategy. I strongly suspect that this definition of a <u>strong equilibrium</u> point restricts, to the class of game with ordinary von Neumann-Morgenstern saddle points in pure strategies, the existence of games with strong equilibrium points.

The Cournot-Nash weak equilibrium has the advantage of being easily calculable for normalized games of the non-constant-sum type. I believe quite definitely that deviations from a Cournot-Nash equilibrium in an experimental game will be extremely useful in measuring the effect of the learning process upon the ultimate outcome of a bargaining situation; for, as indicated above, the Cournot-Nash equilibrium is an "irrational" or non-optimum equilibrium which nonetheless possesses from the standpoint of the bargainer some important characteristics of an optimum. Other factors will undoubtedly play a role in producing outcomes significantly different from Cournot-Nash equilibrium, and the relationship of the equilibrium to other factors needs more looking into.

It is worth noting that both the initial experiment and the Cournot-Nash equilibrium concept presuppose an attempt to exclude collusion. To what degree this may be achieved and relevance of the Cournot-Nash equilibrium to situations where it cannot be excluded is an important consideration.

To illustrate how a Cournot-Nash equilibrium is actually found, let us consider a simple game. Let *A* and *B* be the two players, who have control over the sets of pure strategies $\{a_1, a_2\}$ and $\{b_1, b_2\}$ respectively. The pay-off matrices for *A* and *B* are as follows:

Table 1

<u>*A*</u>

	b_1	b_2
a_1	5	-4
a_2	-5	3

<u>*B*</u>

	a_1	a_2
b_1	-3	5
b_2	4	-4

Let *A* play a_1 with probability *p* and a_2 with probability (1-*p*), and let *B* play b_1 with probability *q* and b_2 with probability (1-*q*). *A*'s mathematical expectation, which we assume he wished to maximize is

$$\begin{aligned} V_a &= 5 \cdot p \cdot q - 4 \cdot p \cdot (1-q) - 5 \cdot (1-p) \cdot q + 3 \cdot (1-p) \cdot (1-q) \\ &= 17 \cdot p \cdot q - 7 \cdot p - 8 \cdot q + 3 \end{aligned}$$

B's mathematical expectation is

$$\begin{aligned} V_b &= -3 \cdot p \cdot q + 5 \cdot q \cdot (1-p) + 4 \cdot (1-q) \cdot p - 4 \cdot (1-p) \cdot (1-q) \\ &= -16 \cdot p \cdot q + 8 \cdot p + 9 \cdot q - 4 \end{aligned}$$

Taking partial derivatives and setting them equal to zero we have two equations which may be solved for equilibrium *p* and *q*:

$$\frac{\partial V_a}{\partial p} = 17q - 7 = 0;$$
$$\frac{\partial V_b}{\partial q} = -16p + 9 = 0.$$

Hence, the equilibrium point is

$$(\frac{9}{16}a_1 + \frac{7}{16}a_2, \frac{7}{17}b_1 + \frac{10}{17}b_2)$$

and the expected equilibrium pay-off to *A* is $\frac{5}{17}$ and to *B* is $\frac{1}{2}$.

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