DYNAMIC ECONOMICS

THEORETICAL AND STATISTICAL STUDIES OF DEMAND,

PRODUCTION AND PRICES

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DYNAMIC ECONOMICS

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THE COWLES COMMISSION FOR RESEARCH IN ECONOMICS is a not-for-profit corporation, founded in 1932 for the purpose of conducting investigations into economic problems. A function of the COMMISSION is to issue from time to time significant monographs of an econometric or economic-statistical nature without, however, assuming responsibility for theories or opinions expressed therein. The COMMISSION is affiliated with the ECONOMETRIC SOCIETY, an international society for the advancement of economic theory in its relation to statistics and mathematics.

TO MY MOTHER,
MARY ISABEL HOLDSWORTH ROOS,
WHOSE THRIFT MADE MY EDUCATION POSSIBLE AND
TO MY WIFE
MARY MAE BARKULOO ROOS,
WHOSE ENCOURAGEMENT AND DEVOTION HAVE CONSTANTLY INSPIRED ME IN MY WORK

TABLE OF CONTENTS

CHAPTER I

STATIC VERSUS DYNAMIC ECONOMICS

CHAPTER II

DEMAND FOR CONSUMER GOODS

CHAPTER III

AUTOMOTIVE DEMAND FOR GASOLINE

CHAPTER IV

DEMAND FOR AGRICULTURAL PRODUCTS

CHAPTER V

DEMAND FOR CAPITAL GOODS

CHAPTER VI

FACTORS INFLUENCING RESIDENTIAL BUILDING

CHAPTER VII

GROWTH AND DECLINE OF INDUSTRY

CHAPTER VIII

JOINT DEMAND AND LOSS LEADERS

CHAPTER IX
PRODUCTION, COST AND PROFIT

CHAPTER X
ADJUSTMENTS OF COST

CHAPTER XI
PRODUCTION INCENTIVES

CHAPTER XII
BEHAVIOR OF FREE AND RESTRAINED PRICES

CHAPTER XIII
EXCHANGE IN A CAPITALISTIC ECONOMY

LIST OF CHARTS

PREFACE

In 1929 I decided to refrain from publishing research papers in economics until I had a sufficient number ready to justify the publication of a book which would relate these researches to each other and to my earlier papers. The basic reason for this decision was the fact that each paper required so much of the background of previous papers that an essay of great length was required to explain a simple new idea.

I originally wrote up ideas associated with my paper "A Mathematical Theory of Price and Production Fluctuations and Economic Crises"* in a sixty-five page paper that explained a few of the concepts. I was at a loss to know where to send the paper for publication. Finally, I sent it to a mathematical journal. The editor wrote me that he would be glad to publish the mathematical parts of the paper if I would rewrite it and delete the economics. Instead of following this suggestion I sent the paper to a statistical journal and was told that the journal would publish the parts dealing with statistics and economics if I deleted the mathematics! I might have complied with this suggestion and the one of the mathematical journal and published two papers instead of one, but I still felt that the paper ought to be published in full in one journal. I therefore sent the manuscript to the *Journal of Political Economy.* It was accepted but publication was delayed. After waiting for a year an editor suggested that I write an abstract about a dozen pages in length. I did this and as a consequence produced a paper that I myself have great difficulty in reading now.

Following my experience with the paper on crises I set about to write a treatise on dynamic economics. While holding a *Social Science Research Council Fellowship* in the summer of 1930, I wrote Chapters I, II, IX and X almost in the form in which they appear in this book. In fact, the sections on the relation of maximum profits to employment (Chapter IX) and on technological unemployment (Chapter X) are the chief additions to the work done on this fellowship. Parts of Chapters II and V were published recently in my paper "Theoretical Studies of Demand."†

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* *Journal of Political Economy,* Vol. XXXVIII, October, 1930.

†*Econometrica,* January, 1934.

After I became permanent secretary of the American Association for the Advancement of Science in February, 1931, I was unable to carry forward the job of writing a book. In the spring of 1933 the award of a *Guggenheim Memorial Foundation Fellowship* made it possible for me to take up my work where I had laid it aside two years previously. During the three months that I spent in London on this fellowship I was able to write most of Chapter V, the section on technological unemployment in Chapter IX, most of Chapter XI, Sections 3, 4 and 5 of Chapter XII, and most of Chapter XIII.

On July 10, I received a cablegram from Alexander Sachs, then Director of the Division of Research and Planning of the National Recovery Administration, requesting me to return to the United States to do econometric research work for the Recovery Administration. I owe a large debt of gratitude to the Guggenheim Foundation for granting me indefinite leave so that I could accept this position. One of my earliest assignments at the National Recovery Administration called for an analysis of Roy Wenzlick's data on factors influencing residential building activity in Greater St. Louis. While this study was in progress, the Recovery Administration asked for a theory of joint demand and loss leaders. The next request was for "an equation of exchange in a capitalistic economy by tomorrow." By drawing on the unfinished work of Chapter XIII, Max Sasuly and I were able to produce "an equation of exchange in a capitalistic economy" in the time allowed. The Administration next suggested a study of automotive demand for gasoline in order to determine whether public works expenditures for highways could be expected to bring increased demand for gasoline, automobiles, steel, etc., so that acceleration of recovery could be attained by such expenditures.

Several of the above-mentioned studies were presented as papers at the 1933 winter meetings of the Econometric Society. Professor Harold T. Davis of Indiana University suggested that the papers be published together in a monograph. I countered with a suggestion that they be made part of a book on *Dynamic Economics*. I felt that the new material together with what I had prepared at Cornell University and in London could be put together very quickly in a book which in some respects would go beyond what I had intended to accomplish on my fellowship. I felt that even though chapters on banking, wages, value and foreign exchange, originally intended to be included in my book *Dynamic Economics,* were omitted, the material that I had was sufficiently cohesive and complete to be brought out at this time, especially if certain additions—Chapters IV, VII* and the first part of XII—were made. Accordingly, I am able at this time to publish the first part of what I hope to be a three volume presentation of *Dynamic Economics.*

Concomitant with the writing of this book much important economic legislation has been enacted in the United States. The National Industry Recovery Act, which was passed by Congress, June 16, 1933, has led to important fundamental changes in methods of doing business. It has been my privilege to follow closely the course of these changes as Director of Research for the National Recovery Administration, since July 27, 1933.

On September 15, 1934, I shall become Director of Research for the Cowles Commission for Research in Economics and will then begin to write an economic and statistical appraisal of this New Deal legislation. I hope to publish this Volume some time in 1935. The third part of my work will deal with such subjects as banking, wages, foreign exchange, and value.

While research material developed at the N.R.A. appears in this work, theories presented are my own personal ones and are in no sense to be taken to represent the official views of the National Recovery Administration. This work is presented as research material and not as something to uphold or criticize the economic policies of the present Administration. I have tried to be an econometrist; that is, I have tried to present my material without political or nationalistic bias. I make no claim of having proved the various theses offered. In a science that is developing as rapidly as is econometrics it must be expected that there will be false leads which will have to be corrected. If my effort is stimulating to economic thought and if it is provocative of further research, I shall feel happy even though some of the theses are disproved.

I have not attempted to provide an extensive bibliography of econometric material pertaining to the work presented here, nor have I attempted to acknowledge all ideas that I might have borrowed from others, or had independently of them. I have naturally done a large amount of reading of economic works by Pareto, Cournot,

*Chapter VII grew out of a suggestion of Victor von Szeliski that I ought to include something on the economic structure of an expanding economy.

Marshall, Bowley, Fisher, Edgeworth, Wicksell, Amoroso, Divisia, Schumpeter, Davenport, Moore, Young, Keynes, Pigou, Mitchell, Frisch, Evans, Schultz, Hotelling and others.

I am deeply indebted to my secretary, Mrs. Mildred Chisholm, who spent much time at night after working hours typing the manuscript. Her painstaking care in checking statements and equations has made it possible for me to present an accurate manuscript. I am also greatly indebted to Andrew Court for reading the manuscript and suggesting pertinent changes. I, of course, assume full responsibility for any errors that may appear.

Emily Pixley, Victor Perlo, Max Sasuly, Clement Winston, Jack Biscoe, Goldie Back, Anne Golden and Elizabeth Wilcox helped with the computations for Chapters III and VI. Other acknowledgments to Mrs. Pixley, Victor Perlo and Max Sasuly appear in the text.

Professor H. T. Davis of Indiana University and the Principia Press offered valuable suggestions and arranged for the printing of the manuscript. Alfred Cowles III, Director of the Cowles Commission for Research in Economics, had the charts redrawn so that lettering etc., would be uniform, and offered helpful advice as to arrangement and format.

The Dentan Printing Company has patiently incorporated changes in the text and in general extended every possible courtesy to me.

To all who have helped in the preparation of this volume, I am deeply grateful.

C. F. ROOS.

CHAPTER I.

STATIC *Versus* DYNAMIC ECONOMICS

1. Static Economic Theory as an Approximation. In 1837 Lord Overstone described changing business conditions as follows:

> "The history of what we are in the habit of calling the 'state of trade' is an instructive lesson. We find it subject to various conditions which are periodically returning; it revolves apparently in an established cycle. First we find it in a state of quiescence, — next improvement, — growing confidence, — prosperity, — excitement, — overtrading, — convulsions, — pressure, — stagnation, — distress, — ending again in quiescence."*

In 1838 Augustin Cournot produced his now classical contribution to economic theory, *Researches into the Mathematical Principles of the Theory of Wealth.* He too realized that economic phenomena are dynamic; that is, that there are forces at work which are almost constantly bringing about changes. In fact, he wrote

> "The rise and fall of exchange show perpetual oscillations in values, or in the abstract wealth in circulation, without intervention of actual production or destruction of the physical objects to which, in the concrete sense, the term wealth is applicable . . . Just as we can only assign situation to a point by reference to other points, so we can only assign value to a commodity by reference to other commodities. In this sense there are only relative values."†

Several additional quotations from Cournot seem to be particularly appropriate here because they show how, in spite of the above dynamic conceptions of economic phenomena, static theories have occupied the stage. This is shown clearly by the following set of quotations:

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*Lord Overstone, *Reflections suggested by a perusal of M. J. Horsley Palmer's pamphlet on the causes and consequences of the pressure on the money market*, London, 1837, p. 44.

†Augustin Cournot, *Researches into the Mathematical Principles of the Theory of Wealth* (Bacon's translation), New York, 1929.

"For instance, an observer who should see by inspection of a table of statistics of values (prices) from century to century, that the value of money fell about four-fifths towards the end of the sixteenth century, while other commodities preserved practically the same relative values, would consider it very probable that an absolute change had taken place in the value of money, even if he were ignorant of the discovery of mines in America . . . . The monetary metals are among the things which under ordinary circumstances and provided too long a period is not considered, only experience slight absolute variations in their values . . . On the other hand articles such as wheat, which form the basis of food supply, are subject to violent disturbances; but if a sufficient period is considered, these disturbances balance each other and the average value approaches fixed conditions, perhaps even more closely than the monetary metals. Here as in astronomy, it is necessary to recognize *secular* variations, which are independent of *periodic* variations. . .

"In fact, the year is the natural unit of time, especially for researches having any connection with social economy. All the wants of mankind are reproduced during this term, and all the resources which mankind obtains from nature and by labor. Nevertheless, the price of an article may vary notably in the course of a year, and, strictly speaking, the law of demand may also vary in the same interval, if the country experiences a movement of progress or decadence. For greater accuracy, therefore, in the expression $F(p)$ (denoting demand), p must be held to denote the annual average price, and the curve which represents the function F to be in itself an average of all the curves which would represent this function at different times of the year. But this extreme accuracy is only necessary in case it is proposed to go on to numerical applications, and it is superfluous for researches which only seek to obtain a general expression of average results, independent of periodical oscillations."

It is quite clear that Cournot had a dynamic conception of economic phenomena and that he deliberately simplified the problem by considering only average results. Thus, he developed a static theory of economics as a first approximation to a description of economic relationships. Following Cournot, the mathematical economist, and Ricardo, the literary economist, economists have developed economic theory from the static point of view. Few economic studies have been successful in going on to the numerical application.

The principle economic theories, both old and new, have been based mainly upon a static view of economic phenomena, that is, upon a view which, as a matter of principle, does not take into account variations of economic situations with time. Such theories are concerned chiefly with a hypothetical state of equilibrium and the inter-relations of prices, demand, supply, and so forth, when equilibrium has been attained. As a result economic theory is today essentially an analysis of invariability and identity and of a fixed standard of elements.

It is, of course, true that the static theories in a sense do operate with variations and fluctuations in economic quantities in order to interpret how these quantities are related at equilibrium. The purpose of doing this, however, is inevitably to show that all the fluctuations of elements, under certain assumed conditions, tend unavoidably toward a state of equilibrium, and it is this equilibrium alone that is then investigated. Static theory does not propose to investigate the dynamic processes by which this hypothetical equilibrium is reached nor to discover the laws or rules governing fluctuations. From the static point of view these fluctuations are merely incidental.

2. The Trend of Economics. Today Cournot's exception, "if the country experiences a movement of progress or decadence," seems to be assuming more and more importance. Economists are rather generally beginning to feel that economic theories based on static conceptions are inadequate. Statistical studies of demand and supply phenomena have emphasized the importance of secular and cyclical changes. Thus, not long before his untimely death Allyn Young wrote:

> "The growing use of quantitative methods is the most promising development in contemporary economics. But it will prove relatively sterile if it does not lead to a renaissance of theory."*

Much of the quantitative method to which Allyn Young referred has resulted from the attempts of H. L. Moore and his students to determine statistical laws of demand. Every investigator of economic statistics has had impressed upon him the importance of trends and periodical movements. Attempts to analyze time series

*Allyn A. Young, "The Trend of Economics," *Quarterly Journal of Economics*, 1925, p. 167.

that large and growing field of statistics known as the analysis of of prices, production and so forth have largely been responsible for time series.*

Professor Moore long ago recognized the importance of a dynamic theory. In fact he recognized this in his classes and in 1926 published a paper purporting "to pass from the statical, hypothetical equilibrium to a realistic treatment of an actual, moving equilibrium."† Professor Moore passed from the static equilibrium theory to a dynamic equilibrium theory by means of the trend ratio; that is, he proposed to fit an empirical trend curve to the data and substitute for actual prices and actual quantities the ratios of actual prices and actual quantities to their respective trends. As a first approximation to a dynamic theory of economics, this theory cannot be criticized, but it must not be thought that Moore's theory is the general dynamic theory of economics that will reconcile much of the apparent contradiction in economic theory.‡

Although the method of trend analysis developed by Professor Moore and his students occupies an important historical position, statisticians are more and more beginning to question the significance of results obtained by those methods and are asking for increasingly realistic treatments. It is more and more being recognized that the use of a trend in statistical analysis of economic relationships is essentially a confession of ignorance of some of the important factors involved or is the result of a desire to discuss these factors without identifying them. This latter is the attitude taken in this work.

3. The Need for a Dynamic Theory of Economics. It is quite clear that economists rather generally recognize the need for a dynamic theory of economics. The reasons for this attitude are to be found in the nature of economic phenomena, which are always changing, ever in a state of flux.

Near the end of the eighteenth century a new era was ushered

*For a discussion of the validity of correlating time series see Appendix I.

†H. L. Moore, "A Theory of Economic Oscillations," *Quarterly Journal of Economics*, November, 1926, p. 28.

‡On page 104 of Irving Fisher's doctoral dissertation the following statement will be found: "The dynamical side of economics has never yet received systematic treatment. When it has, it will reconcile much of the apparent contradiction; e.g., if a market is out of equilibrium, things may sell for 'more than they are worth,' as every practical man knows; that is, the proper ratios of marginal utilities and prices are not preserved." See Irving Fisher, *Mathematical Investigations in the Theory of Value and Prices*, New Haven (1926).

in by James Watt and his steam power. The power-driven cotton gin which made the gigantic cotton industry, the sewing machine, steam power transportation, electric telegraphy, Bessemer steel and open hearth steel, and the agricultural reaper, all had their origin in the last quarter of the eighteenth century and the first half of the nineteenth. These were followed by machinery and equipment for generating, transmitting and utilizing electric power, modern methods of milling grain, the telephone, the automobile, refrigeration and so forth. At the present time all these have become intimate parts of economic life in the United States. Infant industries of a century ago have grown to maturity.

The consumer goods economy of the early theorists has been changed into an economy of capital goods. The prime costs of manufacturing have become relatively much less important. In fact, for many industries prime cost is much less than one-fifth of the retail selling price. It must not be inferred, however, that the four-fifths represent profit. Today, advertising, research or direction of production and distribution may mean the difference between success and failure for an industry. Thus, in highly mechanized industries the direct labor cost of producing a commodity — that is, the cost which in the days of Cournot, Ricardo and others was the chief determinant of price or value — is now a minor factor. In non-mechanized industries and in partially mechanized industries this labor cost may be quickly changed by the introduction of machinery.

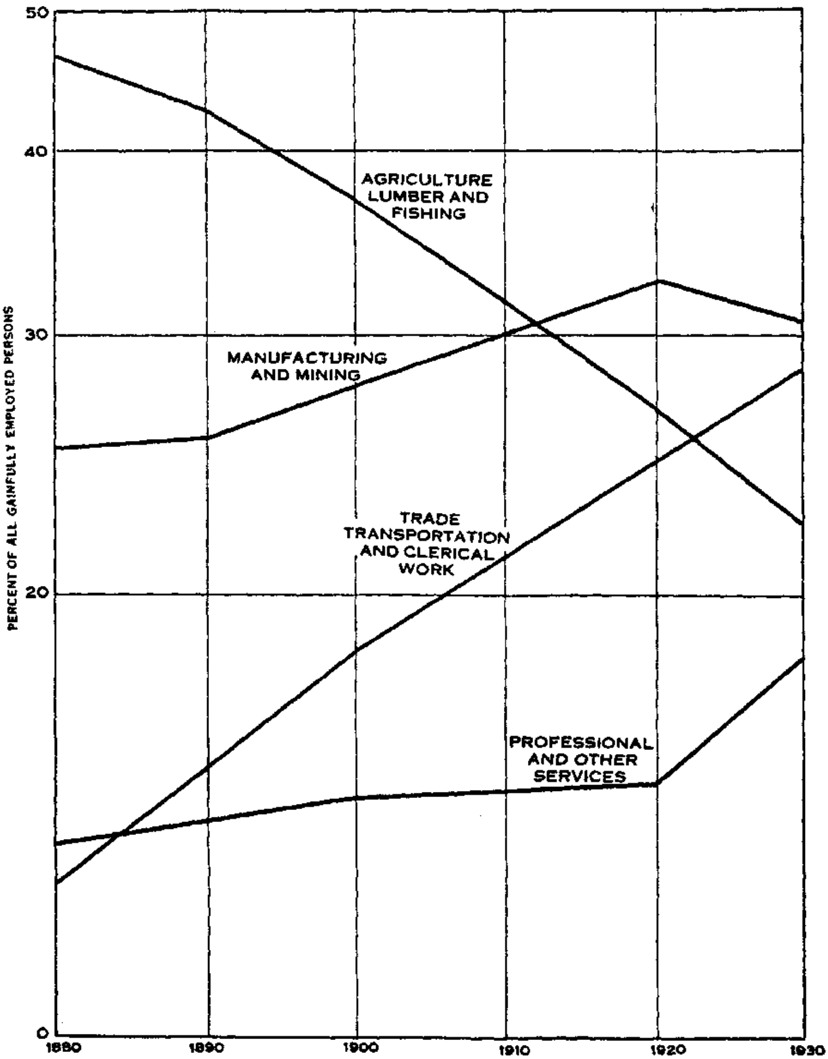
By the use of gang plows pulled by tractors, and combines for harvesting, it is now possible to produce wheat on some of the large farms in Nebraska for something like twenty-five cents per bushel exclusive of land rents. If land is figured at $100.00 per acre (the capitalized value of the land depends upon the yield—see Chapter XI, Section 7), interest at six per cent per annum, taxes at $2.00 per acre per annum, and the average yield per acre is assumed to be twenty bushels, a gross profit will result whenever the price of wheat at the field exceeds sixty-five cents per bushel. To this should be added insurance for bad years.

Furthermore, the introduction of labor saving machinery into agriculture has released great numbers of workers who have gone into industry and transportation. Carl Snyder estimates that the percentage engaged in agriculture in the United States has decreased from something like fifty per cent in 1870 to slightly over twenty per cent in 1930.* In the same period the number engaged in

***Stabilization of Employment*, edited by C. F. Roos, Chapter IV, Bloomington, 1933.

OCCUPATIONAL TRENDS IN U. S.
BY MAJOR ECONOMIC GROUPS

CHART I



Source: Census of Manufactures. Prepared by Carl Snyder.

trade, transportation and clerical work has increased from something like twelve per cent to nearly thirty per cent.

In other words, the problems of producing goods in the United States have been solved more rapidly than the problems of distribution. Instead of an economy of basic scarcity there is now presented an economy of basic surpluses and clogged distributive mechanism. The chief problem today is therefore the problem of distributing goods and not that of producing them. This does not mean, however, that all the problems of economics have changed. It merely means that the emphasis has changed and hence that some of the postulates must be changed. The new postulates must recognize the importance of change — change in the cost of producing goods, change in the cost and method of distributing goods, risk of obsolescence, and so forth. In reality, therefore, it is essential to revise the old static theories of economics and to construct a more adequate theory based on dynamic conceptions.*

4. Dynamic Economics. An accurate dynamic conception should consider economic phenomena and functions in the process of change and the interrelations of these changes in the course of time. It should seek to discover laws or rules governing these variations, but it must recognize that any law or rule may be completely or partially invalidated by invention, by change in social taste, by law, by disease or by other accident or disaster. Thus, risk must play an important role in the construction of a dynamic theory.

The dynamic theory must recognize that it is in the nature of economic phenomena to change. In fact, static equilibrium conditions have never completely prevailed. When any line of industry becomes notably profitable, capital and labor rush to it and develop it. New industries diminish or destroy old ones. Wars disrupt conditions. Tariffs modify them. Even change of taste and fashion seriously disturb the *status quo ante.* In other words, modifying forces are always at work and must be taken into account by economic theory. Such theory, if realistic, must be essentially dynamic. Some parts of static theory may be carried over into dynamic theory, but only those parts that are invariants of time or that have introduced the time element in an auxiliary role. This, of course, does not mean that time *per se* has any effect.

*J. M. Keynes, V. Pareto, G. C. Evans, F. Divisia, J. Schumpeter, J. M. Clark, H. L. Moore, S. Tinbergen, R. Frisch, Alvin Hansen, J. Viner, Irving Fisher, Allyn Young and others have already made important contributions to dynamic theory.

The dynamic theory may, of course, also make use of the conception of economic equilibrium, but it cannot be the same as the static conception. Dynamic equilibrium must be an equilibrium of variable elements. Thus, static equilibrium may be compared to the equilibrium of grains of sand on the seashore and dynamic equilibrium to that of gas molecules at constant temperature, pressure and volume. More accurately, perhaps, dynamic equilibrium might be compared to that of the various parts of the human body: individual cells may live, grow, stagnate and then die, and yet the general appearance and condition of the body remain the same, at least over short periods of time and provided too many cells do not undergo metamorphoses simultaneously.

In a workable dynamic theory, speculation, debts, interest, depreciation, obsolescence and other familiar economic functions of time, must assuredly be assigned important roles. Thus, it is not enough to lump all these quantities together in a trend curve and develop a so-called dynamic theory as H. L. Moore has done.

In the present work the task of building up a dynamic theory is begun from the concept of demand for goods. The influences on demand functions of habit or custom, budgetary restrictions, time, usability of goods, advertising and so forth are considered at some length. So from the beginning the theory presented is dynamic. There is necessarily little use made of the static theory of utility and its relation to demand. This does not mean that utility has *nothing* to do with the determination of demand. It does imply, however, that too much emphasis has been placed on utility and that it might be time to remove some of this.

It is probably true that a good deal of economic theory has gone on the rocks because of a more or less general tendency of economists to emphasize the role of the individual. As long as physicists dealt with composites of molecules and atoms they were able to discover useful laws. When they attempted the problem of analyzing the atom, they soon came upon the problem of indeterminacy. In economics the individual occupies a role closely analogous to that of the atom in physics.* Just as it is impossible to state where a gas atom or molecule will be at any particular time aside from such general statements as that "it will be confined in the test tube," so it is impossible to state anything regarding the economic behavior of an individual except within broad limits.

*In some cases the unit might be a group, such as a corporation, instead of an individual.

This comparison of economics to physics requires further analysis. There is, in fact, a very great difference between physics and economics. For one thing, the physicist knows the laws of motion of individual bodies under the influences of forces, whereas the economist has only a rough notion of the laws of human behavior. If a group of ten individuals were confronted with the same social situation, ten different reactions might possibly result. Furthermore, the same individual might react differently at different times. Thus, the individual is somewhat capricious, but although individuals often appear to react capriciously, nevertheless, for a large enough group it is possible to establish the concept of an average individual who characterizes the group. This average individual might be expected to react almost identically to identical situations, for otherwise there could be no norm of rationality.

Economic reactions of individuals may differ as much as social reactions. In the consummation of a sale such elusive factors as the personality of the salesman, individual prejudices, misinformation or, perhaps more often, lack of information, etc., play such important roles that it may never be possible to determine the demand of any particular individual for a given product. Nevertheless there might reasonably be an average buyer, and an average seller, and it might be possible to determine average demand, average sales and so forth. Again, although it may not be possible to predict how many telephone calls John Smith will make at five or ten cents per call, it may be possible to predict how many calls 100,000 John Smiths will make. From this point of view mathematical economics is not unlike what mathematical physics is becoming. For example, the following quotation is taken from *Engineering,* July, 1927:

> "To-day the mathematical physicist seems to be more and more inclined to the opinion that each of the so-called laws of nature is essentially statistical, and that all our equations and theories can do is to provide us with a series of orbits of varying probabilities."

Since averages conceal as well as reveal, it becomes necessary to decide just what averages may be used. This is a difficult question and the answer must depend upon the problem under consideration. Unquestionably demand must be the average demand of individuals in the market over the interval of time considered, and in demand analysis, price must be the average price over this same time interval.

The demand for capital goods arising out of savings must be essentially different from the demand for consumer goods. Surely money income or money incentive plays an important role in determining desirability of capital goods, and hence in determining their money value. Furthermore, for these goods, long established customs regarding incomes, etc., are of tremendous importance in determining demand.

A dynamic consideration of demand leads quite naturally to a dynamic conception of value of a piece of goods as the present amount of discounted expected future income or enjoyment received from the goods. Thus, risk and personal estimate play important roles in determining value. When these concepts are given consideration it then becomes appropriate to consider the growth and decline of industries, the mechanism of production, exchange and so forth. These invaribly lead to conditions of maxima and minima but not to the classical conditions. Thus, at this point prices and quantities become determinable as functions of time. It is then a relatively simple matter to pass to a consideration of price and production fluctuations and economic crises.

5. The Mathematical Method. As already pointed out, a study of dynamic economics is necessarily a study of changing relationships. Even the layman is now familiar with the fact that prices, production rates, taxes, etc., rise and fall. There is, therefore, change in the relationships of economic factors to one another. The process of determining how variations in certain elements influence other elements must be fundamental to economic theory and the method used for developing the theory must be able to deal with changes. Clear statements of hypotheses are absolutely essential for progress.

In order to state hyphotheses so clearly that they will not be misunderstood, it is necessary to choose language carefully. In order to reach valid conclusions it is necessary to make certain that the hypotheses are self-consistent and that all possible variations are taken into account. As G. C. Evans has so aptly remarked, "When we find this feeling for hypotheses and definition and, in addition, become involved in chains of deductive reasoning, we are driven to a characteristic method of construction and analysis which we may call the mathematical method. It is not a question as to whether mathematics is desirable or not in such a subject.

We are in fact forced to adopt the mathematical method as a condition of further progress."*

Some economists have said that there are so many variables involved in the study of human behavior that it will never be possible to develop a science of economics. Perhaps it is true that the entire field of economics is too vast to be covered by a single theory. If such is the case, then it is all the more important that each hypothesis be brought out fully. In fact, the more variables there are the more necessary it becomes to have a symbolic language to keep track of them. No one would advise an astronomer to develop a science of the stars without using mathematics.

Every financial transaction is a quantitative one. From this point of view alone, it would seem that mathematics is indispensable to economic theory. One might well venture the opinion that the chief reason why so much has been written on economic theory and so little advancement made is that economists know so little of mathematics. Almost without exception those economists who have made lasting contributions to economic theory have been mathematicians or economists who have known considerable mathematics. Consider, for example, the following extract taken from a letter of Alfred Marshall to C. Colson, written one year after Marshall had retired.

> "Briefly — I read Mill's *Political Economy* in 1866 or '7, while I was teaching advanced mathematics: and, as I thought much more easily in mathematics at that time than in English, I tried to translate him into mathematics before forming an opinion as to the validity of his work. I found much amiss in his analysis, and especially in two matters. He did not seem to have assimilated the notion of gradual growth by imperceptible increments; and he did not seem to have a sufficient responsibility — I know I am speaking to a mathematician — for keeping the number of his equations equal to the number of his variables, neither more nor less. Since then I have found similar matters not quite to my taste in the economic work of nearly all those who have had no definite scientific training.
>
> "At that time and for long after I knew very little of the realities of economic life. But I worked at what I regard as the central problem of distribution and exchange.

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*G. C. Evans, *Mathematical Introduction to Economics*, New York, 1930, p. 113.

> Before 1871 when Jevons' very important *Theory of Political Economy* appeared, I had worked out the whole skeleton of my present system in mathematics though not in English."*

Other economists who have found the mathematical method invaluable include W. S. Jevons, Léon Walras, Vilfredo Pareto, F. Y. Edgeworth, J. B. Clark, Knut Wicksell, J. M. Keynes, Irving Fisher, A. L. Bowley, L. Amoroso, H. L. Moore, and Joseph Schumpeter.†

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*"Alfred Marshall, the Mathematician, as Seen by Himself," *Econometrica,* Vol. I, No. 2, April, 1933, p. 221.

†In 1838 Augustin Cournot succeeded in driving the entering wedge of mathematics into economics when he published his *Researches into the Mathematical Principles of the Theory of Wealth.* Prior to Cournot a Frenchman by the name of Canard had presented a paper on mathematical economics which received the approval of the French Institute, but his principles were so radically at fault and his applications of them so erroneous that he did little more than bring scathing criticism upon the mathematical method. It was Canard's paper that inspired J. B. Say to criticise so severely the use of mathematics in economics.

CHAPTER II

Demand for Consumer Goods

1. The Nature of Demand. Almost every student of elementary economic text books is familiar with a chart showing two intersecting curves sloping in opposite directions. These curves purporting to show demand and supply as functions of price intersect in a point which is usually labeled the equilibrium point. This label means that the supply is equal to the demand at the point of intersection. Good text books mention that demand and supply shift and that instead of one downward sloping curve and one upward sloping curve there are in reality many curves of each kind; i.e., there are many demand curves and many supply curves for each commodity. A few books state that factors which are capable of causing shifts in demand are advertising, introduction of new products, etc., but usually they do not give a great deal of attention to these elements.* The position taken in this work is that there are infinitely many demand and supply curves; that is, there are factors other than price which are constantly shifting.

Price is an important factor, but it is not enough to say that demand depends upon price. There are many other determining factors, some of them often ranking in importance with price and advertising; e.g., consumer income, prices of competing or substituting goods, monetary and credit conditions, the influence of past prices, speculation and so forth.

If there were no highways, the demand for gasoline for motor cars would be very small indeed. Again, in a community where there is little consumer income other than that needed to buy food and some clothing, there is necessarily little demand for gasoline to run automobiles. As indicated in the next chapter, imposition of a sales tax might decrease consumption more than would an equal increase in price, without a tax. Thus it is seen that some factors influencing demand are of a physical nature, some are economic, and some purely psychological.

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*For diagrammatic studies of demand and some mention of shifts see E. H. Chamberlin, *The Theory of Monopolistic Competition*, Harvard Press, 1933, Chapter V.

2. Factors Other Than Price Influencing Demand. When an attempt is made to take all influencing factors into account, or even a few of the more important ones, the tremendous complexity of economics becomes apparent. Actually, of course, all economic quantities in the same market are related, but fortunately only a few seem to have major influences on the demand for any particular commodity.

In the older type of statistical study of demand, the chief purpose was the determination of "the demand curve," or in some cases the more limited purpose of determining the percentage decrease in demand corresponding to a one per cent increase in price (elasticity of demand). From the standpoint of such studies all factors other than price were regarded as "disturbing factors" whose effect should be eliminated. Various devices invented for performing this elimination included the method of trend ratios, of link relatives, and of first differences. On the whole this older type of study proceeded on the assumption that changes in demand due to factors other than price were of a gradual nature due to changes in habit, customs and the growth of population. Under such assumptions it appeared to be desirable to remove the effects of trends.

In the past four years it has become apparent that the demand studies of the old type which had been more or less successful prior to 1930 were faulty. In many cases the difficulty was that the only changes in the demand function of which specific account had been taken were those which could be explained by algebraic trends. Apparently these studies were empirically satisfactory prior to 1930 chiefly because shifts in demand which occurred during the period covered by the analysis were small, or were closely correlated with other factors included in the study for other purposes. An example of the latter type of factor commonly used as an omnibus was the index of wholesale prices used ostensibly to reduce "money prices" to "real prices," but also of value in explaining shifts of demand. Thorne and Beane have shown how what was formerly treated as merely an upward trend in demand for beef and pork was primarily the result of increasing business activity and payrolls.*

3. Effects of Past Factors on Demand. Past prices and sales

*G. B. Thorne and L. H. Bean, "The Use of 'Trends in Residuals' in Constructing Demand Curves," *Journal of the American Statistical Association*, Vol. XXVII, No. 177, March, 1932, pp. 61-67.

policies undoubtedly have important effects on current demand. As a result of continuous advertising of a product such as chewing gum at five cents per package, a demand can be built up that is almost independent of present price, consumer income, etc. Again, certain products like coffee become such integral parts of the accepted standard of living that present price must change very much before demand is appreciably affected. In such instances the effects of price are effects of past prices and policies rather than of present price. Thus, at least in many cases, the demand can be taken to be the sum of two functions, one depending upon past prices of the good and certain other pertinent past factors, the other depending upon the present price and present values of pertinent factors. A mathematical expression relating these factors can be obtained by using the limiting processes of the calculus.

Let $p(t)$ represent the average retail price of a commodity U during the time t to $t + \Delta t$ and suppose that the commodity has been offered continuously in a *particular* market for some time. Let $y(t)$ represent the amount purchased for consumption or speculation during the time Δt. In a general theory of demand the time required for consumption of the goods is undoubtedly an important factor which should be taken into account. It is easier and less confusing, however, to build a theory without taking this factor into account and then to extend the theory. Hence, the theory as first developed applies to demand for goods consumed immediately upon purchase, e.g., electric energy. However, it is equally valid if Δt is sufficiently long to extend over the average time required for consumption.

Consider, first of all, goods for which there is no buying for speculative purposes. This is practically equivalent to specifying that the problem be limited to retail buying and selling of perishables, since considerable speculative activity is associated with nearly all, if not all, commodities in the wholesale markets and durable goods at retail. After the basic theory has been developed the speculative factor can be easily taken into account.

Non-speculative demand for goods certainly depends upon the price of the goods as well as upon the prices of competing or substitute goods; that is, it is a function of the present price $p(t)$, the past prices $p(t_i)$, $i = 0, 1, \cdots, n-1$, $t_0 = 0$, $t_n = t$, and the prices of m competing goods, $p_1, \cdots, p_m$. Theoretically, it is also a function of the past prices of competing or substitute goods. There is, however, no serious objection to simplifying the problem by assuming that each $p_k(t_i)$, $k = 1, \cdots, m$, is equal to p_k. The demand, in general, also varies with the seasons and changes in social taste which may be assumed to include obsolescence, etc.; that is, demand may depend explicity upon time t. The above may be summarized by taking the demand as a function

$$y(t) = f[p(t_0), p(t_1), \cdots, p(t_{n-1}), p(t), p_1, \cdots, p_m, t]$$

As a first approximation it can be assumed that $f(t)$ is a linear function of each $p(t_i)$, so that

$$y(t) = \varphi(t) + [K(x_0, t)\,p(x_0) + \cdots + K(x_{n-1}, t)p(x_{n-1})] + K(x_n, t)\,p(x_n)$$

where, for simplicity in writing, the arguments, $p_1, \cdots, p_m$, of φ, and K have not been written. To make the theory perfectly general one need only enlarge the definition of the p_i so that they include factors other than price· Thus, in particular p_1 might be a physical factor such as number of miles of highway, etc.

Here $K(x,t)$ is a weight function and the expression in brackets is the weighted average of the prices $p(t_0), \cdots, p(t_{n-1})$, in which, for the sake of clarity in the work that follows, x_i has been used in place of t_i. That is, x refers to past time and t to present time. It is understood, of course, that time is used to express certain group effects due to numerous small factors or it is used implicitly to denote some important factor changing with time; for example,

$$f(t) = f[p_1(t), p_2(t), \cdots, p_m(t)].$$

There is no loss of generality in assuming $x_i - x_{i-1} = 1$. If this assumption is made, it follows readily that the bracketed expression is the sum of rectangles of unit width and height $K(x_i, t)p(x_i)$. This sum may be replaced by a definite integral; that is,

$$(3.1) \qquad y(t) = \varphi(t) + \int_0^t K(x,t)p(x)dx + K(t,t)p(t),$$

where $K(x,t)$ is a discontinuous step function. If price $p(x)$ is also discontinuous, the infinite sum or integral of $K(x,t)p(x)$ can be taken in the sense of Stieltjes — a finite sum. This is, however, a refinement that need not be given serious attention here since there is no important objection to replacing $K(x,t)$ by a continuous function yielding the same integral, or for that matter, by one yielding approximately the same integral.

It should hardly be necessary to point out that physicists and astronomers have been using continuous functions to represent discontinuous phenomena and have thereby achieved considerable success in discovering laws of nature. There is considerable doubt that economic phenomena are more discontinuous than physical phenomena; at least they are not more discontinuous from the mathematical point of view. Of course, the econometrist should be careful to consider only a sufficiently large number of elements (see Chapter I, Section 4). In the work that follows, $K(x,t)p(x)$ and other quantities are assumed to be continuous.

There are some examples for which $K(t,t) \equiv 0$. Thus, the demand for goods bought through a broker "at market" depends upon past prices, but not at all or only very slightly upon the price at the time t. In general, however, demand depends upon the present price as well as upon past prices and $K(t,t) \neq 0$. In fact, $K(t,t)$ is, in general, not zero on the interval 0 to t. Mathematically there is an important distinction between the two cases $K \equiv 0$ and $K \neq 0$. For the first case, equation (3.1) becomes a Volterra integral equation of the *first kind*,* whereas for the second case it becomes a Volterra integral equation of the *second kind*.†

It is quite natural to ask if, conversely, price depends upon past demand. The answer is that *whenever present demand depends upon past prices, present price depends upon past demands and the two statements are equivalent.* In fact, those familiar with the theory of integral equations will recognize that equation (3.1) yields this dual theory. As is well known, if $K(t,t) \neq 0$ in the interval t_0 to t, equation (3.1) can be written in the completely equivalent form

$$K(t,\, t)p(t) = y(t) + \varphi(t) + \int_0^t k(x,\, t)\varphi(x)dx + \int_0^t k(x,t)y(x)dx\;,$$

where $k(x,t)$ is the resolvent kernel of $K(x,t)/K(t,t)$ and as such is completely determined as a function of x and t. ‡

*Vito Volterra, *Leçons sur les Équations Integrals*, Paris, 1913, p. 56.

†*Ibid.* p. 40.

‡Vito Volterra, *loc. cit.*, p. 45.

If the dependence of demand upon past prices is not linear, some other form of functional equation may be used. In fact very few demand equations are linear functions of price (other factors held constant) throughout the entire range of values of price. Most statistical studies indicate that a better approximation to the demand price relationship is given by a curve of the type p^{-a} where a is a positive fraction usually less than one. Thus, it is better to write

$$y(t) = \varphi(t) + K(t, t)F(p) + \int_0^t K(x,t)F_1[p(x)]dx \ ,$$

where F and F_1 are in general non-linear functions of price. In many cases it may be expected that $F \equiv F_1$. Unfortunately for the use of such general equations by econometrists, mathematicians have not yet been able to find out a great deal about them.

For consumer goods, p can generally be taken to be price. By this it is meant that, in general for consumer goods, price measures the marginal desirability of the goods with respect to other goods whose marginal desirability is also measured by prices. In the case of capital goods, p is more likely to be an income factor, as is explained in a later chapter.

4. Demand Approximations. The sum (integral) of $K(x,t)F(p(x))$ may vary little from an average value. In some cases, then, it is possible to replace the effects of past events by a constant or at least by a quantity that differs little from a constant. This should be especially true of demand for some consumer goods. For such goods, therefore,

$$y(t) = \varphi_1(t) + K(t, t)F(p) \tag{4.1}$$

at least to a good approximation, provided $y(t)$ is taken over a period of time sufficiently long to average out some of the effects of immediate past prices, etc.

It is perhaps not amiss to point out again that both $\varphi_1(t)$ and $K(t, t)$ are denoted as functions of time to indicate that both may depend upon economic, physical and psychological factors that change with time. Over short periods of time both quantities may appear to be constant, but it need not follow that they *are* constants.

It is possible that only two or three factors, other than price p, say p_1, p_2, p_3, affect the demand y. In such an instance, it may be possible to write

$$y(t) = A_1F_1(p_1) + A_2F_2(p_2) + A_3F_3(p_3) + K(t,t)F(p) \ ,$$

$$y(t) = A_1F_0(p_1, p_2, p_3) + K(t,t)\ F(p)$$

$$\text{or} \quad y(t) = A_1F_0(p_1, p_2, p_3) + af_0(p_1, p_2, p_3)F\ (p) \ , \text{etc.},$$

in which F_0, F_1, F_2, F_3 and f_0 are known functions of the given arguments and A_1, A_2, A_3 and a appear to be constants. Actually each of the quantities A_1, A_2, A_3 and a depend upon other factors in the economy, but by hyphothesis the effect of any single factor is small and since some of the small factors increase while others decrease, it may be assumed that A_1, A_2, A_3 and a are constants even though the elements of which they are made are constantly changing. There are many physical analogies. For example, the shape of a piece of matter may appear unchanged while the atoms of which it is composed undergo constant motion. Of course, if some new element is introduced or if some small factor becomes large, the original hypothesis is no longer valid.

Several studies of demand which appeared to be successful before 1929 failed to give satisfactory results in the subsequent years because the balance of neglected factors was upset. Correlation studies cannot be expected to give any clue to the nature of the causal relationship which exists between price and some other variable that has not shown significant variation in the period studied. Averages often cover up as much as they reveal, and must always be used with caution. Nevertheless, information gained through their use is distinctly valuable, if care is taken to allow for the introduction of new major factors. During the World War, price-fixing, propaganda, shifts in production emphasis, etc., must surely have had effects in modifying demand. It becomes desirable, therefore, to distinguish between what might be called normal economic demand and demand due to specific factors that have not normally existed or have been of no more importance than many of the other small factors.

Closest attention must be given to *a priori* reasoning concerning the causal relations that may be involved and to all inductive evidence that may bear upon the problem. Close relation between two series of data cannot of itself be considered to be of much significance as empirical evidence. Even where data are not of a cyclical nature, statistical methods alone are incapable of yielding definite evidence of causation. It is only when statistical evidence is combined with *a priori* reasoning in a closely knit argument that confidence can be felt in statistical relationships.*

Several special cases of *(4.1)* that have been used in statistical studies of demand are

(1)	$y = ap + b$	$a<0, b>0$
(2)	$y = b/p^a + c$	$b > 0, c > 0$
(3)	$y = b - (c - ap)^{1/2}$	$b > 0, c > 0, a > 0$
(4)	$y = ap^2 - bp + c$	$a > 0, b > 0, c > 0$
(5)	$y = p^a e^{b(p-c)}$	$a > 0, b < 0, c > 0, e = 2.718$

In these equations, a, b and c, although written as constants, should not be strictly interpreted as such. They must be interpreted to be composites of other factors as already explained. Thus, in the case of equation (1), b might be $b_1p_1 + b_2p_2 + \cdots + b_mp_m$, where $p_1, \cdots, p_m$ are m additional factors influencing demand. If the quantity Σb_ip_i remains constant over a period of time, then b can be taken to be constant. Once more for emphasis it might be pointed out that if Σb_ip_i is constant, it does not follow that each p_i or, for that matter, that any one p_i is constant. The sum $3 + 5 + 6 - 7$ is equal to the sum $4 + 9 - 8 + 2$ which is equal to 7. If Σb_ip_i is not constant, then it may for simplicity be regarded as a function of time unless, of course, one wishes to study the effects of particular factors. In the case of the gasoline study described in Chapter III, b depends upon highway mileage, taxes and other quantities. An important point to notice, however, is that for the equation (1) as written, a and b are independent of price p.

In the discussion that follows, a, b, and c will be regarded as constants. This is really no restriction on their generality since the arguments used can be carried over into n-dimensional space by substituting the language of the general geometry for the language used here. It seems to be undesirable, however, to complicate the description here since nothing is gained by doing it.

Formula (1) represents a straight line with negative slope. H. L. Moore† used this form to obtain a statistical law of demand for cotton. Henry Schultz‡ used it to obtain a statistical law of

*This is essentially the scientific method familiar to physicists, chemists and other scientists. The argument as presented here follows closely that given by E. J. Working in his paper "Demand Studies During Times of Rapid Economic Changes," *Econometrica*, April, 1934.

†H. L. Moore, *Forecasting the Yield and Price of Cotton*, New York, 1917, p. 143.

‡Henry Schultz, *Statistical Laws of Demand and Supply*, Chicago, 1928, p. 61.

demand for sugar in terms of link relatives and trend ratios. G. P. Scoville* used it in studies of potatoes and hay. For small price variations this form must be considered to be a fair approximation for almost any type of demand law.

Formula (2) defines an hyperbola which approaches the demand axis and the price $p = c$, asymptotically. For $c = 0$ this becomes the law $y = b/p^a$, which was popularized by Alfred Marshall† and used in statistical work by W. P. Heddon, J. Bloxom, H. H. Holland and others. It has undoubtedly led to occasional statements that the amount demanded is infinite when the price is zero and the demand is zero when the price is infinite. These statements must surely have been brought about by attempts to popularize a mathematical formula or by misconceptions of demand. G. F. Warren and F. A. Pearson‡ have used formula (2) with $c \neq 0$. Of course, it must be admitted that for values of p and y not close to zero a law of the type (2) with c zero or not zero may give a better approximation of a law of demand than equation (1), but loose statements regarding infinite demand for zero price should not be tolerated.

Equation (3) represents the *lower branch* of a parabola open to the right, whose vertex is at $y = b$, $p = c/a$. Thus, y is not defined for $p < c/a$. This curve therefore typifies the situation for which there is a finite demand $y = b$ at a minimum price $p = c/a$, which price may be taken to be the price of distribution. Furthermore, the demand is zero when the price $p = (c—b^2)/a$. Theoretically the demand should remain zero for $p > (c—b^2)/a$, so that formula (3) has no significance for p greater than this value.

Formula (4) represents a parabola open above with vertex at $p = +b/2a$, $y = c—b^2/2a$. This curve crosses the y-axis at $y = c$ and may or may not cross the p-axis depending upon the values of a, b and c.

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*G. P. Scoville, Per Cent of the Expected Crop Correlated with Purchasing Power of the Price of Potatoes for Each of the Last Fifty-four years, Cornell University, Dept. Farm Marketing, 1919.

†Alfred Marshall, *Principles of Economics*, London, 1920.
For a theoretical derivation of the law $y = b/p^a$, see V. Pareto, Mathématique Économie, *Encyclopedia des Sciences Mathématique*, Tome I, p. 59.
W. P. Heddon, "Measuring the Melon Market," *U. S. Dept. of Agriculture and Port of New York Authority*, August, 1924.
J. Bloxom, "Some Determining Factors in Apple Prices," New York Food Marketing Research Council, *Food Marketing Studies*, I. 1926.
H. H. Holland, "The Demand for Peaches in New York," *ibid*, III, 1926.

‡G. F. Warren and F. A. Pearson, *The Interrelationships of Supply and Price*, Cornell University, 1927.

Formula (5) defines a curve passing through the origin and approaching the p-axis asymptotically on the right. Thus, the curve rises from the origin, reaches a maximum (but the slope is not zero at the maximum) and then drops again to zero. H. L. Moore, Henry Schultz and H. B. Killough have used this curve in statistical studies.*

In the above discussion the quantities a, b and c have been regarded as constants. As previously indicated, for greater generality one might suppose them to be functions of time t, and economic factors. An important point to remember is that for typical economic situations dy/dp will be negative, so that curves that satisfy this condition should be used.

5. *Demand and Consumer Income.* It has been intimated that consumer demand depends upon consumer income, but no definite formulation has been given. Most demand studies have used a "deflating" series; that is, in these studies prices have been divided by an index of wholesale prices or something of the sort. In other words, statisticians have learned that adjustments must be made for changes in the price level.

In general, purchasing power theories of economics have been condemned by orthodox economists. These condemnations may be in part warranted, but it must be admitted that consumer purchases are high when consumer income is high and they are low when it is low.

It may be assumed that when consumer income, I (a particular part of it if only part of the economy is being considered), is large, the demand will not be so much affected by increase in p as in the case of small I, hence $|\ dy/dp\ |$ is smaller for larger I; that is,

$$(dy/dy)^2 = f(I,p) \ ,$$

where $f(I,p)$ decreases as I increases for a fixed p. A simple function of I,p satisfying the given conditions is $I^{-2\beta} f^2(p)$ so that

$$y = I^{-\beta} \int f(p)dp + C(t) \ ,$$

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*H. L. Moore, "The Elasticity of Demand and Flexibility of Prices," *Jour. Amer. Stat. Ass'n*, Vol. 23, 1922, pp. 8-19.

Henry Schultz, "The Statistical Measurement of the Elasticity of Beef," *Jour. Farm. Econ.*, July, 1924.

H. B. Killougn, "The Price of Oats," *U. S. Dept. Agr. Dept. Bull. 1351*, pp. 8-9, September, 1925.

where $C(t)$ is a constant or function of t. If $f(p)$ is equal to a, a negative constant or function of t, then

$$y = ap/I^{-\beta} + C(t) \ .$$

Other formulas can, of course, be derived. A more penetrating discussion of purchasing power and demand is given in the next chapter.

CHAPTER III

AUTOMOTIVE DEMAND FOR GASOLINE*

1. Introduction. The demand for a commodity (quantity sold) is dependent upon many factors other than price. Nevertheless, for many products there are a few major factors which account for the greatest part of the demand, and in general the effects of the various small factors can be assumed to be compensating. In other words, in statistical studies it is possible to isolate the major factors and study their effects on the assumption that the net effects of the neglected factors are constant. The relationships obtained in this way will be valid so long as the neglected factors give a constant effect. Whenever a minor neglected factor becomes a major one it must, of course, be taken into account. However, introduction of a new major factor or an increase in importance of a neglected minor factor, should not, in general, nullify the effects of the other major factors. It is merely necessary to add something additional to the formula obtained by neglecting such factors.

In general there are physical, economic and psychological factors entering into the determination of the demand for a product. In the general case it may not be possible to separate completely these factors, and, of course, it may not be necessary or desirable to do this even if it could be done. Thus, if it is possible to write demand, y (quantity sold per unit time), in the form

$$y = A_1F + A_2E + A_3S$$

where A_1, A_2 and A_3 are constants (or functions of time if they are aggregates of changing factors), F is the physical factor, E is the economic factor, and S, the psychological one, it almost inevitably happens that A_1, A_2 and A_3 are each functions of physical, psychological and economic factors. Furthermore, it is often impossible to decide whether a factor is physical, psychological or economic,

*This study was undertaken at the suggestion of Stephen Du Brul, Acting Director of the Division of Research and Planning of the National Recovery Administration. The study, which is a joint effort of Victor Perlo and C. F. Roos, was presented to the Econometric Society at Philadelphia, Pennsylvania, December 28, 1933; see "The Meetings of the Econometric Society in Philadelphia and Boston, December, 1933," *Econometrica,* Vol. 2, 1934, pp. 210-211.

but this is immaterial since it is unnecessary to make the distinction. The equation above is merely presented to indicate one kind of law. Some other kinds are:

$$y = A_1 FE + A_3 S \ ,$$
$$y = A_1 (F + E) + A_3 S \ ,$$
$$y = A_1 FS + A_2 E + A_3 S$$

and so forth. A careful analysis of the factors involved, with much emphasis placed on dimensional analysis,* is desirable before any particular form of law is selected. An arbitrary selection can then be made.

Demand for consumer goods is somewhat different from demand for capital goods. There are, of course, almost all degrees of consumer goods and capital goods, and some goods are either consumer or capital goods, depending upon their use. An automobile used for pleasure is a consumer good, whereas the same automobile used for taxicab service is a capital good. A differentiation can be made by saying *arbitrarily* that a capital good is one expected to render service which is to be exchanged for money income or its equivalent.

Statistical analysis of demand for consumer goods and capital goods is essentially different chiefly because expected income (which largely determines the demand for and price of a capital good) is a quantity that must be taken into account over a considerable period of time, whereas its equivalent for consumer goods—namely, price—has little future influence.†

2. Form of Demand Law for Gasoline. Factors that might be expected to be of major importance in determining the automotive demand for gasoline include the number of miles of state surfaced highway, new automobile registrations, price of gasoline including tax, the total of license and registration fees and the increase in tax rate (state and federal). Each of these factors depends upon many other factors and, furthermore, some of the factors, such as miles of highway and tax rate, are not independent. In special localities some of the factors mentioned may be of minor importance

*See G. C. Evans, *Mathematical Introduction to Economics* (1930), Chapter II. Wherever feasible, constants used should be of dimension one. Quantity is of dimension M (mass). Quantity per unit time is of dimension M/T. Price is of dimension one since it is a ratio of quantity of a good to a quantity of money.

†No influence if a year is chosen as the unit of time.

and other factors not mentioned may be of major importance. New automobile registrations are proportional to automotive consumer purchasing power.*

The following notation is used:

G = Gallons of gasoline consumed per year, in millions
R = Average motor vehicle registrations per year, in thousands
$K = (1000\, G/R)$
H = Miles of state surfaced highway
N = New passenger car registrations, in thousands
P = Price, in cents per gallon (including tax)
F = Total of license and registration fees, in thousands of dollars
I = Cents increase in tax rate (state and federal)

The quantities R, H, N, P, F, I, are the factors which, it may be assumed, determine K, the per capita consumption. An estimate of K, called $\overline{K}$, can be obtained by the formula

$$(2.1) \qquad \overline{K} = A_0 + H(A_1N/P + A_2F/RN) - A_3I, \dagger$$

where A_0, A_1, A_2 are constants determined separately for each state.

The dimensions of the factors used are as follows:

$$G = MT^{-1} \qquad N = MT^{-1}$$
$$R = MT^{-1} \qquad P = 1$$
$$K = T^{-1} \qquad F = MT^{-1}$$
$$H = M \qquad I = 1$$

so that $A_0 = 1$, $A_1 = M^{-2}T$, $A_2 = T^{-1}$ and $A_3 = 1$. The factor A_0 is an aggregate of factors of dimension 1. In other words, A_0 can be expected to be constant only so long as these factors give a constant aggregate effect. A_1 and A_2 have dimension $M^{-2}T$ and T^{-1} respectively. Hence none of the constants should be the same from state to state.

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*See Chapter VII.

†Economic theory rarely indicates a particular mathematical function. Often nothing more is determined *a priori* than, for instance, the sign of the derivative. This is true also in the case of the physical sciences. All that science, either economic or physical, can do is to make postulates, follow their implications through logically, and statistically test the whole structure or parts of it.

As will be seen later on, formula (2.1) is adequate to represent automotive gasoline consumption in any of the states of the United States. Statistical tests clearly indicate that the accuracy of this estimate is not due to mathematical manipulation. The formula adequately describes the interplay of physical, economic and psychological factors which determine consumption. It follows that an analysis of the working out of this formula may be used to determine historically and theoretically the effects of these factors.

3. The Data. Before a statistical study of significance can be made, adequate and reliable data must be obtained. Furthermore, as already pointed out, the population must be sufficiently large for the measurement of mass effects. Before great reliance can be placed upon apparent verification of hypotheses, studies of several similar populations must lead to essentially the same verifications.

For a study of demand for gasoline a state population is a natural unit of sufficient size to reflect mass reactions. To find several populations that might yield essentially the same verifications it is merely necessary to select several states where conditions are similar, i.e., where the major factors should be of the same importance and where the neglected minor factors should give net constant effects.

In this study Government data, Bureau of Public Roads, are used except for prices. For gasoline consumption, total gallons taxed for each state are used. This figure, G, is actually but a crude estimate of gasoline consumed by motor vehicles. Bootleg gasoline is consumed in sizable but unknown quantities. Exemptions for farm cooperatives are large in many states. In this study, G is used as an *index* of motor-vehicle gasoline consumption, and it is assumed that it is linearly related to actual consumption; that is, it comes sufficiently close to being an index of actual consumption, so as not to invalidate the demand formulation.

In a study of demand for a product such as gasoline, the year is a natural unit of time. Using a year as the unit automatically eliminates many of the random and seasonal variations that would be present if a month or week were used.*

The figure R, for total registered motor vehicles, apparently neglects the fact that trucks consume several times as much gasoline as do passenger automobiles. But it would be wrong to weight

*See Appendix II for a discussion of choice of the unit of time.

the number of trucks more heavily, because truck consumption is relatively far more stable than passenger consumption. Year-to-year variations in per capita truck consumption differ by less than one-third from variations in passenger consumption. Since it is intended as an explanation of year-to-year variations in consumption, the use of a simple total is far more representative than would be the use of a weighted total. It is known that registrations may differ by as much as ten per cent from cars in actual use, but while estimates of actual cars in use have been made on a national basis, they cannot be applied to individual states, for the differences between registered cars and cars in use vary so widely in the various states as to make useless any correction based upon a national index of cars in use.

State surfaced highways are only a small percentage of total roads, but it is probable that a majority of the miles traveled by motor vehicles are upon such roads. Since it is impossible correctly to weight the vast mileage of country and dirt roads, these are left out of consideration, and the number, H, is used as an approximate *index* of physical opportunity for travel.

The National Petroleum News gives weekly price figures at important distribution centers throughout the country, for standard grades of gasoline. An index for a state in a given year is obtained by averaging the 52 weekly price figures given that year for the various centers in the state. The total of license and registration receipts is the best available indication of the per annum overhead governmental cost. It does not include *ad valorem* tax, a highly important factor in some states for which no collection figures are available. But rates of such taxes have followed much the same course as fees, and so fees furnish a satisfactory index.

The yearly figures by states for purchases of new passenger automobiles were obtained from the automotive bureau of the United States Department of Commerce.

An important point to notice is that all the variables used are indices of quantities which cannot be directly ascertained. These indices are to a different base for each state, and the relations between the bases cannot be determined. As a result, the constants in the formula have to be determined separately for each state.

4. Nature of Factors in Formula Used. Highway mileage, H, represents the incentive to consume gasoline contained in the current mechanical and geographical facilities for automotive consumption. It serves as an *index* of the net effect upon consumption of such facilities, which effect is called the physical factor.

That part of money cost of automotive consumption which varies with the mileage driven is represented by P. Since N may be regarded as an index of purchasing power, P/N may be taken to be *real variable cost or price.* As set up in the equation, in the form $A_1\ N/P$, it is an index of the effect upon consumption of changes in real variable cost of operation, which effect is here defined as the *variable cost factor.*

The ratio F/R is an index of those money costs to the motor-vehicle owner which are fixed within a given year. Then F/RN, i.e., fixed costs deflated for purchasing power, is an index of real fixed costs, and the term $A_2\ F/RN$ approximates the effect of such annually fixed costs, which effect is defined as the *fixed cost factor.*

The sum of the variable cost factor and of the fixed cost factor is taken as a measure of the effect upon consumption of the economy and the moving economic situation; this effect is called the economic factor, and is treated at a given time as a factor modifying the consumption consistent with the physical factor existent at that time.

The factor $A_3 I$ is a measure of an important psychological deterrent to consumption caused by a tax increase. When a tax increase goes into effect every motorist knows about it, and a tax increase occurs at all places and in all brands. This study indicates that there is a resultant economizing in consumption considerably larger than that which would follow an equal rise in price without any change in the tax rate.

5. *Statistical Law of Demand for Automotive Gasoline.* Each of the series described in the preceding section is a time series. If it is desired to combine them into a formula, the question of the validity of correlating time series must be raised. Nonsense correlations are often obtained by correlating time series with trends.* This emphasizes the importance of a rational theoretical analysis based on broad acquaintance with the economic situation to be analyzed. If the analysis is complete, and it is believed that it is in this study of gasoline, the influence of the neglected but minor factors (all economic quantities are related) will give a net constant

*See, for example, G. Udney Yule, "Why Do We Sometimes Get Nonsense Correlations Between Time Series," *Journal of the Royal Statistical Society,* Vol. 89, 1926, pp. 1-84.

effect (systematic error in the theory of adjustment of observations). Care must be taken, however, to use an interval of time sufficiently long to insure that random series are not correlated. The question of the validity of correlating time series is discussed more fully in Appendix I.

By using the method of least squares,* the coefficients A_0, A_1, A_2 and A_3 are readily obtained. Only the calculations made for the states of Pennsylvania, Virginia, Mississippi and Kansas, are exhibited here. (See Section 5).

The relative importance of the various factors is shown in *Table I*. By this is meant the percentage of the total change in consumption over the period 1925-1932 due to each of the various factors as computed from formula (2.1) by means of regression coefficients. The psychological factor is omitted in *Table I* since it has been assumed as equal to the average value obtained for it in the four states, namely $A_3 = -7$. This seemed to be preferable to using a degree of freedom in fitting.

TABLE I

	Pennsylvania	Virginia	Mississippi	Kansas	Average
H	47	44	21	52	41
P	27	28	15	15	21
F/R	6	5	7	6	6
N	20	23	57	27	32
A_0	101.6	267.9	436.2	451.6	314.3
A_1	0.002740	0.01810	0.01089	0.000810	0.008135
A_2	0.1558	0.04583	0.02261	0.0165	0.0601875

The physical factor, H, is definitely the most important. It is far larger than any other factor except in Mississippi. Here the depression was very severe, decline in purchasing power most sharp, and so purchasing power played the major role. Generally, it appears that price and purchasing power are of nearly equal importance, and fees a minor, though noticeable, factor. For explanation of the difference in values of A_0, A_1 and A_2, see Section 2.

The historical relation between the variable cost factor, $A_1\,N/P$, and the fixed cost factor, $A_2\,F/RN$, is seen from Chart II,

*Whittaker and Robinson, *Calculus of Observations*, London 1929, pp. 209-211, 231-234.

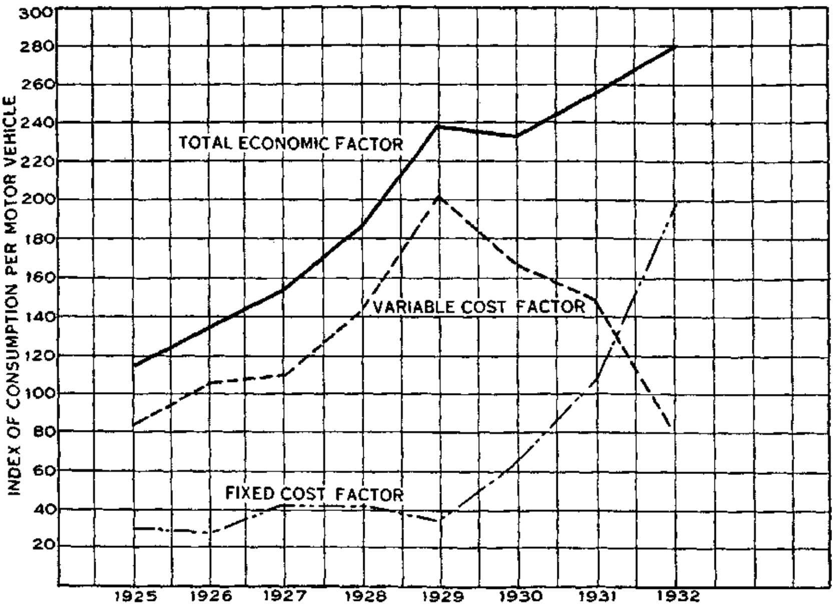
TABLE II

	(1) Taxed Gasoline	(2) Regis-tration	(3) Highways	(4) Price	(5) Fees	(6) New Pas-senger Auto-mobiles	(7) Tax Increase	(8) H × Vari-able cost Factor	(9) H × Fixed Cost Factor	(10) Estimated Consump-tion	(11) Actual Con-sumption	(12)
	G	R	H	P	F	N	I	$HA_1\frac{N}{P}$	$HA_2\frac{F}{RN}$	$\bar{K}$*	K	$K-\bar{K}$
PENNSYLVANIA												
1925	525	1,330	7,540	23.2	21,927	211.1	0	187.7	91.8	381*	395	14
1926	588	1,455	8,440	23.7	24,045	247.2	0	240.8	88.0	430	404	—26
1927	692	1,555	8,830	21.7	26,017	209.7	1	232.6	109.8	437	445	8
1928	759	1,642	9,170	21.6	27,114	219.0	0	253.7	107.7	463	462	— 1
1929	900	1,733	9,530	21.3	29,265	268.7	1	330.4	93.3	518	519	1
1930	929	1,754	10,040	20.7	33,112	199.3	0(b)	264.4	148.2	514	500	—14
1931	1,063	1,742	10,990	16.1	31,607	150.9	0	281.4	205.8	589	610	21
1932	1,041	1,664	11,980(a)	16.8	29,816	95.3	1(c)	185.5	350.9	631	626	—5
									*$\bar{K}$ = 101.6 + col.(8) + col.(9) — 7 col.(7)			
VIRGINIA												
1925	129	283	3,560	22.8	4,301	45.3	0	127.8	55.0	451*	456	5
1926	143	323	3,840	25.0	4,624	52.0	1.5	144.4	48.6	451	443	— 8
1927	167	338	4,160	23.7	5,236	41.8	0	133.0	70.6	471	494	23
1928	183	361	4,720	22.7	5,572	48.6	0.5	183.0	68.7	516	507	— 9
1929	208	387	4,980	22.5	6,145	59.7	0	239.1	60.5	567	537	—30
1930	228	376	5,600	20.0	6,494	44.7	0	226.2	99.0	593	606	13
1931	244	379	6,190	17.2	6,159	37.5	0	244.2	122.8	635	644	9
1932	229	371	6,350	18.9	6,241	20.8	1(c)	126.5	235.1	622	617	— 5
									*$\bar{K}$ = 267.9 + col. (8) + col. (9) — 7 col. (7)			
MISSISSIPPI												
1925	86	177	3,410(a)	23.0	1,530	24.5(e)	0	39.4	27.4	503*	486	—17
1926	106	205	3,840	24.2	1,973	35.2	1	60.7	23.7	514	517	3
1927	118	218	4,130	20.6	2,557	25.3	0	55.1	43.4	535	541	6
1928	129	246	4,540	21.0	2,814	28.7	1	67.5	40.9	538	524	—14
1929	141	250	5,110	22.5	2,963	36.9	0	91.3	37.1	565	564	— 1
1930	135	237	5,330	20.4	3,046	20.6	0	58.7	75.3	570	570	0
1931	123	184	5,240	16.5	2,421	9.5	0.3(d)	32.8	164.1	631	668	37
1932	107	149	5,220	18.8	2,138	5.9	1.7(c)	17.9	287.1	729	718	—11
									*$\bar{K}$ = 436.2 + col. (8) + col. (9) — 7 col. (7)			
KANSAS												
1925	212	457	960	18.0	4,610	59.6	2	25.8	2.7	466*	464	— 2
1926	241	491	1,340	21.0	4,803	57.9	0	29.9	3.7	485	478	— 7
1927	271	502	1,980	17.6	6,519	50.7	0	46.3	8.3	506	504	— 2
1928	316	534	2,730	16.8	5,394	54.0	0	71.1	8.4	531	560	29
1929	361	581	3,460	17.8	5,697	73.5	1	115.3	5.0	565	559	— 6
1930	387	595	4,240	15.0	6,084	44.8	0	102.5	15.9	570	571	1
1931	388	559	4,810	13.2	6,060	28.3	0	83.4	30.4	565	541	—24
1932	351	504	5,310	14.6	5,439	14.8	1(c)	43.5	63.5	552	560	8
									*$\bar{K}$ = 451.6 + col. (8) + col. (9) — 7 col. (7)			

(a) Adjusted for transfer of mileage from county to state.
(b) The tax rate was decreased 1 cent.
(c) Includes 1 cent federal tax.
(d) 0.5 increase November 5, 1931. Effect apportioned 60% to 1931, 40% to 1932.
(e) 1925 figure estimated from Alabama figure and later Mississippi figures.

DEPENDENCE OF GASOLINE CONSUMPTION ON VARIABLE AND FIXED COSTS

CHART II



Per capita consumption varies inversely with variable costs and directly with fixed costs. From 1925 to 1929 variable cost was the major economic factor, with fixed costs acting as a brake and smoothing the trend of the economic factor. From 1930 the fixed cost factor rapidly increased in importance and by 1931 outweighed the variable cost factor.

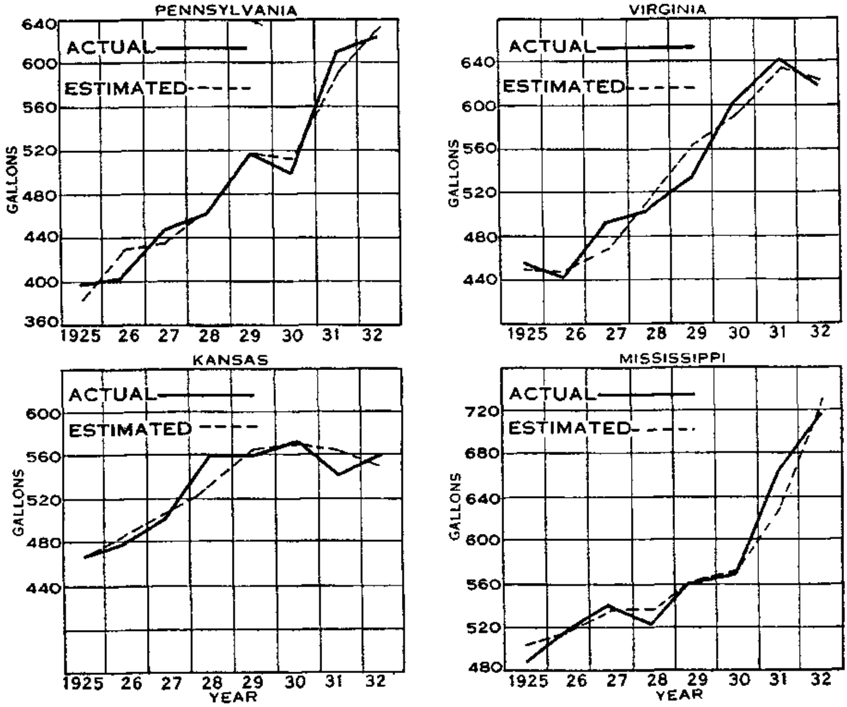
Here and throughout the text, per capita means per consumer i.e., per motor vehicle.

which averages these factors for Pennsylvania, Virginia, Mississippi, and Kansas, for the years 1925-1932. From the year 1925 through 1929 the fixed cost factor acted as a brake upon the major variable cost factor, and smoothed the trend of the total economic factor. After 1929 the fixed cost factor rapidly increased in importance and by 1931 outweighed the variable cost factor.

If the values of H, P, etc., actually occurring in a given state in a given year are substituted in the formula (2.1) for that state, an estimate of per capita consumption is obtained which is very close to actual consumption, as can be seen from Chart III. Thus, the fundamental assumption that the important major factors are number of miles of surfaced highway, new automobile registra-

ACTUAL AND ESTIMATED CONSUMPTION OF GASOLINE PER MOTOR VEHICLE

CHART III



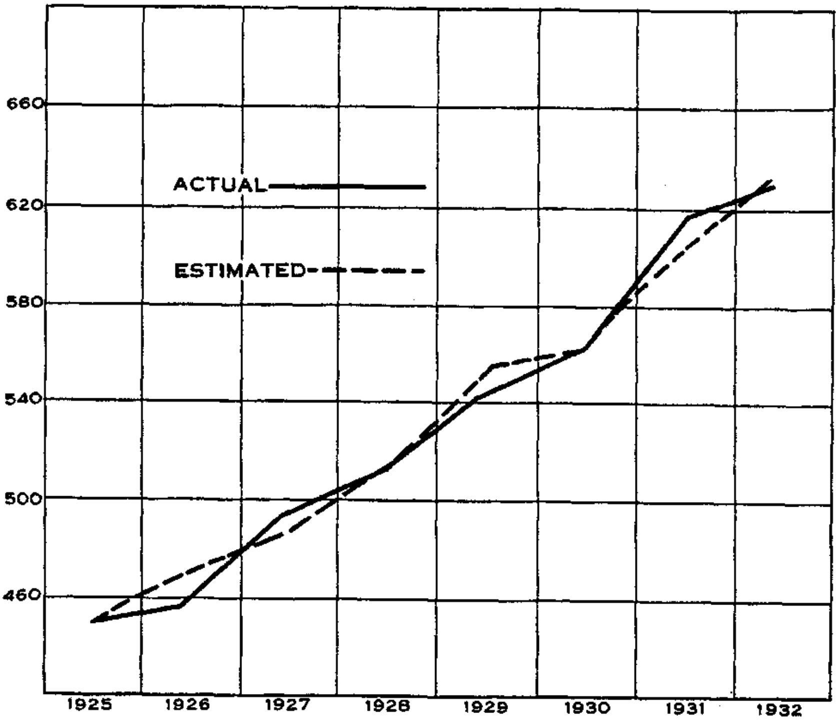
While total gasoline consumption reached a peak in 1931 and declined in 1932, consumption per registered motor vehicle generally continued to increase. Consumption as estimated from the formula is much closer to actual consumption in the average than in any of the individual states. This suggests that the deviations for the individual states are due to chance, or to special local circumstances, and that the formula left out no essential and universal factor.

tions, price of gasoline including tax, the total of license and registration fees and the increase in tax rate, seems to be verified to an unusual degree. The other minor factors give net constant effects except where the residuals are large.

A study based on the years 1925-1929 might well overlook the fixed cost factor, and if the existence of this factor were noted, its effect would probably be incorrectly formulated. To determine demand relationships, it is necessary to take into account specific his-

AVERAGE ACTUAL AND ESTIMATED GASOLINE CONSUMPTION PER CAPITA FOR VIRGINIA, MISSISSIPPI, PENNSYLVANIA AND KANSAS

CHART IV



torical economic changes. A formula based upon a period of economic stability will break down in times of rapid change. The formula (2.1) should apply to any state east of the Rocky Mountains, but it is not satisfactory for Oregon, and should not be for far western states, because the pace and character of economic and physical development in the far west greatly differ from those in the rest of the country.

6. Uses of Demand Law for Automotive Gasoline. The results of this study may be used to determine the effects upon gasoline consumption of road building, tax increases, price fixing, or any other change in the factors determining consumption. Its utility is increased by a parallel study of the demand relations for automobiles (see Chapter VII). This combination with the gasoline study can be used to predict with a high degree of accuracy changes in total gasoline consumption, income to gasoline distributors, tax income to government, which would follow proposed governmental or industrial changes or regulations, etc.

An example of how an increase in fixed costs combined with the decrease in already low purchasing power acts to increase consumption per registered motor vehicle is seen from a comparison of Mississippi figures for the years 1930 and 1931, as presented in Table III.

TABLE III

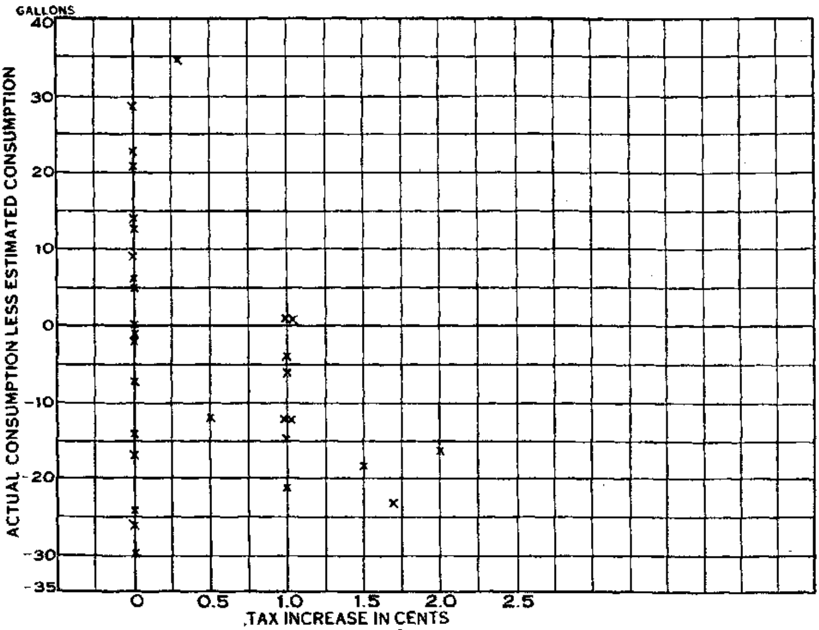
	H	*P/N*	*K*	*F/R*	*N*	*F/RN*	*R*
1930	5,330	989	570	$12.85	20,600	624	237
1931	5,240	1,741	668	13.16	9,500	1,385	184

State surfaced highway mileage decreased from 5,330 to 5,240. The index of price deflated for purchasing power (real variable costs) increased from 989 to 1,741. These two facts alone should cause a decrease in *K*, per capita consumption. Instead, *K* increased from 570 gallons per registered motor vehicle to 668. The reasons may be seen from the remaining figures. License and registration fees per vehicle increased from $12.85 to $13.16. Registration of new passenger automobiles decreased from 20,600 to 9,500, indicating a drop of over 50 per cent in purchasing power available for automobile consumption. The index of per capita real fixed costs obtained by dividing fees by new passenger automobiles increased from 624 to 1,385. This sharp rise resulted in an unusually severe decrease in registrations, from 237,000 to 184,000. The 184,000 active consumers remaining represented those who could afford to use their vehicles sufficiently to cover the increased real fixed cost. The 53,000 dropping out of the market were mainly small and marginal consumers. As a result the percentage of large consumers was far higher in 1931 than in 1930, accounting for the sharp increase in consumption per registered motor vehicle.

The measurable effect of the influence of an increase in taxes in bringing about a decrease in consumption greater than the increased price due to the sales tax has a very important bearing on sales taxes since a decrease in consumption must ultimately mean,

SCATTER DIAGRAM SHOWING PSYCHOLOGICAL EFFECT OF TAX INCREASE

CHART V



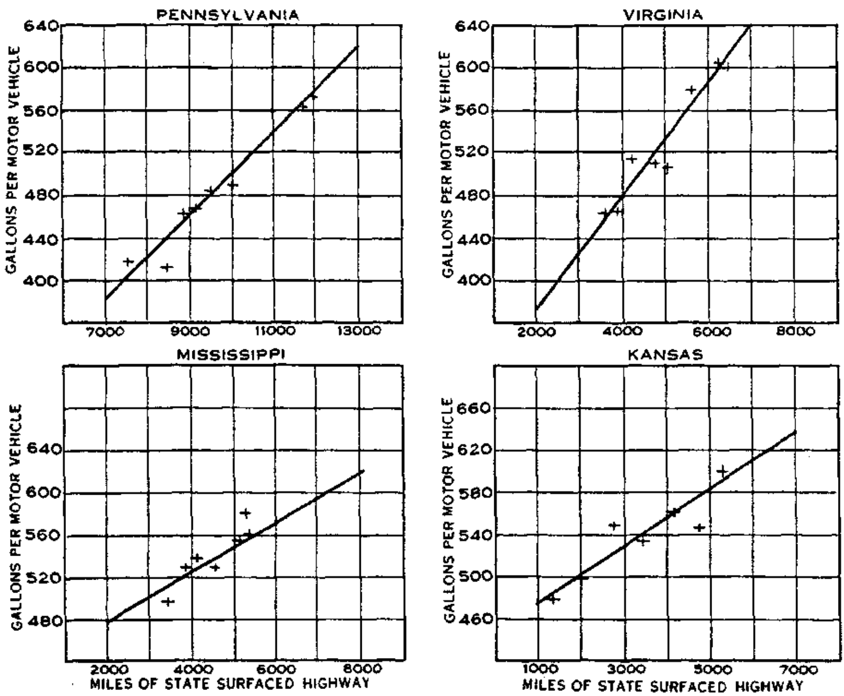
The plotted deviations (actual consumption less estimated consumption) omit the correction for the psychological deterrent resulting from a tax increase. They show clearly a systematic diminution in consumption associated with a tax increase (beyond the effect of the price increase which accompanies the tax levy) and decisively indicate the existence of a measurable psychological factor.

at least for the short period of time, a decrease in employment and some retrogression in standards of living. For a year following imposition of a sales tax, there was observed, in every state studied, a decrease in consumption greater than could be accounted for by price or other factors. If this decrease occurred only once or twice it might be explained as an accident, but when it occurs as often as is indicated in this study it cannot be dismissed lightly.

In 1926 the tax rate was increased in Virginia by 1.5 cents. The decline in consumption to be expected from a 1.5 cent rise in price was nine gallons per capita; the actual decline was twenty-three gallons per capita. The amount by which this abnormal de-

EFFECT OF HIGHWAY MILEAGE UPON CONSUMPTION

CHART VI



Consumption varies directly with highway mileage, the rate of change varying considerably from state to state. In Virginia 1000 miles of added highway will increase consumption by 56 gallons per motor vehicle per year, all other factors remaining unchanged. In Mississippi a corresponding mileage increase will increase per capita consumption by only 24 gallons. One reason is that good roads in Virginia and Pennsylvania provide additional stimulus to tourist travel, which is largely absent in Kansas and Mississippi.

The crosses show actual consumption with the estimated effects of all factors except highways eliminated, plotted against highway mileage.

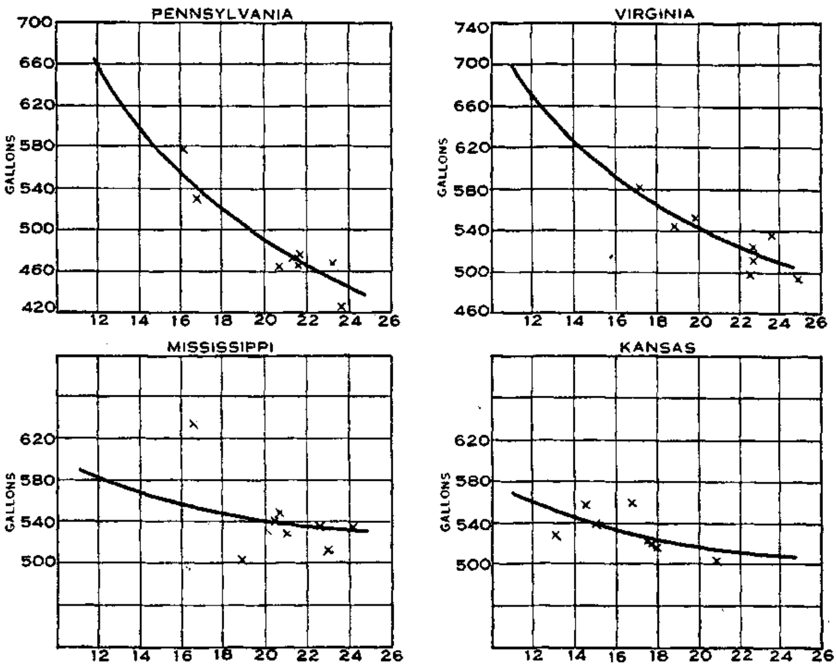
crease in consumption exceeds the ordinary decrease may be called the psychological factor, and may be represented by A_3I. Since its existence might be questioned, statistical tests were applied, which definitely indicated the significance of this factor. The effect of a tax increase is shown in Chart V.

Consumption increases directly with highway mileage as illustrated in Chart VI. There is no indication that a limit to this factor exists. However, it is obvious from *a priori* consideration that an asymptote or limit does exist even though the study indicates that it has not nearly been reached as yet (see Chapter VII).

Per capita consumption of automobile gasoline decreases as price P increases, but the rate of decrease becomes smaller as P becomes larger. The effect of price on consumption is shown in Chart VII.

EFFECT OF PRICE UPON CONSUMPTION

CHART VII



The amount of decrease in consumption caused by a one cent increase in price decreases as the price advances, i.e., the rate of decrease diminishes. A price change in Pennsylvania or Virginia has three times as much effect as one in Mississippi or Kansas. This is probably because the former states have large urban centers, where cheap substitutes for motor transportation are readily available, and utilized when the cost of motor transportation increases.

The crosses show actual consumption, with the effect of all factors except prices eliminated, plotted against prices.

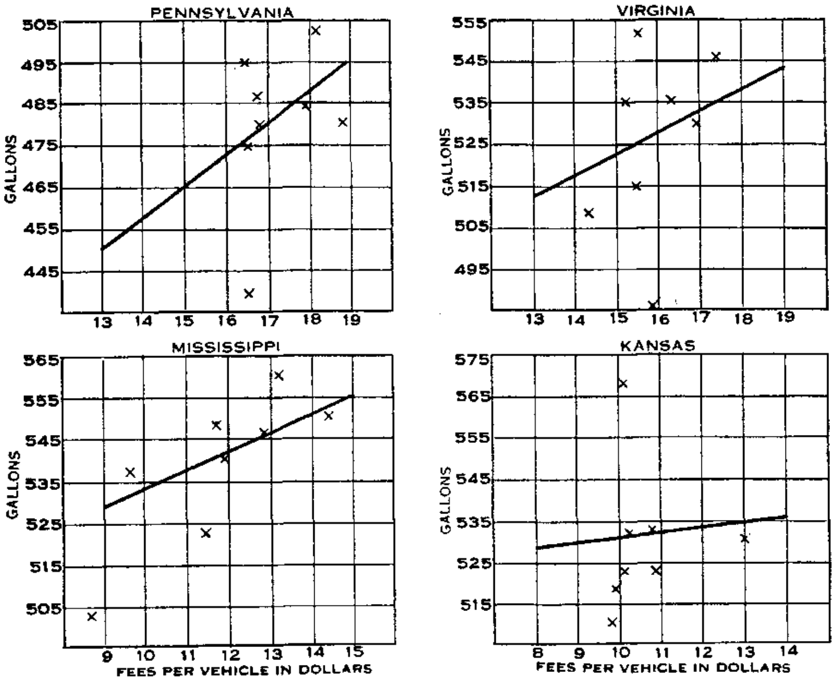
The rate of change of consumption with respect to price is the partial derivative of K with respect to P, that is, $\partial K/\partial P$. From the formula for gasoline consumption, the change in gasoline consumption resulting from a change in price ΔP is given approximately by

$$(\partial K/\partial P)\Delta P = -(A_1HN/P^2)\Delta P.$$

This formula is, of course, approximately correct only for ΔP small. If P is also small and ΔP is taken equal to P, i.e., if a small price P is doubled, the decrement in demand is $-(A_1HN/P)$.

EFFECT OF REGISTRATION FEES UPON CONSUMPTION

CHART VIII



An increase in registration or license fees compels the motorist to use more gasoline per annum to derive enough benefit from the vehicle to cover the increased fixed cost. The little fellow who uses his car occasionally does not run it when the fees are high; i.e., he junks his car.

The crosses show actual consumption, with the effects of all factors except fees eliminated, plotted against fees. The money expense of fees is a comparatively minor factor (although, when expressed in terms of purchasing power, it is extremely important). As a result random causes have more weight than fees, and the crosses do not adhere to the lines except in Mississippi, where fees have steadily increased.

The elasticity of demand, η, is defined as the percentage increase in demand due to a one per cent increase in price; that is, in general

$$\eta = \frac{\Delta K}{K} \bigg/ \frac{\Delta P}{P} = \frac{(\partial K/\partial P)}{K} P ,$$

so that the elasticity of demand for gasoline is given by

$$\eta = -(A_1HN/P^2)P/[A_0 + A_1HN/P + A_2HF/RN - 7I]$$
$$= -1/[1 + A_0P/A_1HN + A_2FP/A_1RN^2 - 7IP/A_1HN]$$

Thus, the elasticity of demand for automotive gasoline depends upon the character of competition involved. It would be -1, if $A_0 = A_2 = I = 0$, but, as this study shows, A_0, A_2 and I are important quantities which are not zero. Given all the factors in the equation, it is a simple matter of arithmetic to compute the percentage decrease in demand accompanying a one per cent price increase.

Higher fixed costs make it imperative to get more use out of a car to keep it in operation. High fixed costs also eliminate little fellows who use their cars only occasionally. Hence while total consumption may decline, due to the elimination of small consumers, per capita consumption increases with higher fixed costs, as shown in Chart VIII.

The effect of changes in purchasing power is shown in Chart IX. If N is started at a "normal" level and increased, the average consumption of gasoline steadily increases. But if purchasing power decreases, this effect is soon counteracted by the elimination of small consumers driven completely out of the market, which tends to increase average per capita consumption, since many small consumers no longer enter into the calculation. After a certain stage in the decline of purchasing power, this effect counteracts the tendency to decrease per capita consumption, and average consumption increases even more rapidly, while total consumption is decreasing at an accelerating pace.

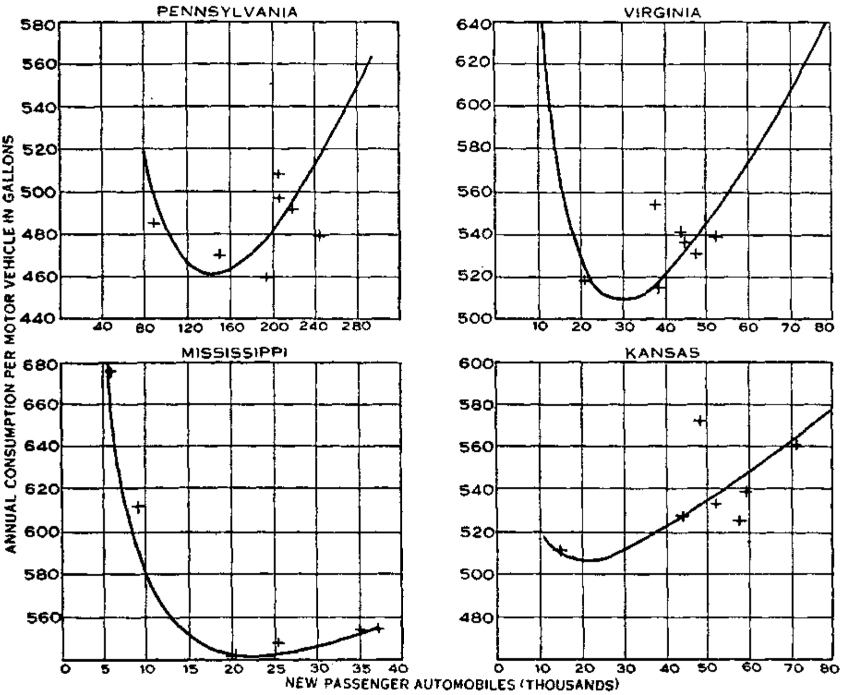
7. *Significance of Formulation.* The validity of the assumptions, (1) that the data used truly approximate the factors they represent, and (2) that the mathematical formulation is satisfactory, may now be tested by four criteria. These criteria follow:

1. The signs of the fitted constants for each state

EFFECT OF PURCHASING POWER UPON CONSUMPTION

New Passenger Automobiles (thousands)

CHART IX



New passenger automobiles serve as an index of purchasing power. All other things remaining constant, increased purchasing power increases per capita (i.e., per motor vehicle) consumption, and vice versa. But when purchasing power declines to abnormally low levels small consumers are driven out of the market, so that the *average* consumption of those remaining increases while total consumption is decreasing.

The crosses show actual consumption, with the estimated effects of all factors except new passenger registrations eliminated, plotted against new passenger registrations.

should be in accord with the theory as developed.

2. The results for the four states should be roughly comparable (fairly wide variations must be expected due to the wide differences in the physical and economic conditions).

3. There should be no systematic deviations $(K—\overline{K})$ of actual consumption from estimated consumption; i.e., the deviations should be distributed normally.

4. The "fit" of the estimated consumption to the actual consumption should be "good."

If these four criteria are satisfied it may be said that the choice of the data and the mathematical formulation of the interrelations of the physical, economic and psychological factors, are *satisfactory*. It should be emphasized, however, that in no case can it be said that these are the "best" possible.

For the first criterion, A_1 and A_2 should certainly be positive. Increase in variable costs decreases consumption per motor vehicle, and since variable cost is given inversely, its coefficient should be positive. Increase in fixed costs increases per capita consumption (although it decreases total consumption by eliminating small consumers), and hence its coefficient should be positive. The constant term should be positive, since the constant unmeasured factors represent a situation in which some gasoline would be consumed; e.g., if there were no state surfaced highways and no gasoline taxes, A_0 would be the only term left. Automobiles would, however, be driven even if there were only dirt roads. Since the theoretically correct negative sign has been arbitrarily assigned to the coefficient of I, the criterion here is that the fit with —7 I be better in each case than the fit without it. In each case the signs of A_0, A_1, and A_2 are positive as required by theory. In each state the fit is better when —7 I is included.

For the second criterion, the changes in $\overline{K}$ which would be caused by changes in a unit of each of the variables F/R, H, and P at values of variables in (2.1) which are the averages of these variables for the period of time taken are significant. These changes are presented in Table I. Generally, the relative weights of the factors do not vary greatly from state to state. When they do vary, the variation can easily be explained by the conditions within the specific state, e.g., fees are very small in Kansas, and changes would not be greatly noticed except in very hard times. The above treatment is the most satisfactory for applying the second criterion. Comparison of constants would have no value because of the differences in scaling mentioned previously. One more treatment may be used; namely, the comparison of the ratios of the average weights of A_1N/P and A_2F/RN. It is found, roughly, for Pennsylvania that the difference is given by 6-4; for Virginia, 7-3; for Mississippi, 4-6; and for Kansas, 8-2. Again the differences are only those consistent with the varying conditions in the states. Mississippi has one of the highest sets of fixed costs in the country, whereas fixed costs in Kansas are relatively low. It may be said that the second criterion is reasonably well satisfied.

For the third criterion, if the formula (2.1) is correct, its application will leave no systematic deviations running throughout the states in certain years. The deviations are summed over the four states for each year and the sum of squares calculated. If the formulation is correct, the deviations in any state can be regarded as being independent and normally distributed, and the sums should be normally distributed with variance equal to the sums of the variances for the individual states, or sum of squares equal to the sum of squared deviations of all the states individually. The sums of squares for the individual states are 1600, 1874, 1575, 2021, totaling 7070. The sum of the squared summed deviations is 6008. If there were persistent deviations the last figure would be far greater than 7070, so the third criterion is satisfied.*

For the fourth criterion the estimated standard deviations of the deviations $K - \bar{K}$, that is, $\Sigma(K - \bar{K})^2/5$ (5 represents the number of degrees of freedom), are 17.9 for Pennsylvania, 19.4 for Virginia, 20.1 for Mississippi, 17.7 for Kansas, and 8.94 for the average K's and $\bar{K}$'s of the four states. As percentages of average actual consumption, these are

Pennsylvania	Virginia	Mississippi	Kansas	Average
3.61	3.60	3.50	3.35	1.67

Empirically these figures indicate a high degree of goodness of fit. There is no mathematically valid technique for testing the significance of least square coefficients fitted to time series. The ratio of a coefficient to its estimated standard deviation (Student's ratio) may be computed and its significance determined by the use of Student's distribution, but this test must be used with care since it implicitly assumes independence of successive observations. This independence, of course, does not strictly exist for dynamic time series.† Student's ratio is computed here for whatever it may be worth as the best objective test available, and the results are given in the table below:

*For theoretical validity of criteria used in this paragraph, see any elementary text on probabilities. For example, see R. A. Fisher, *Statistical Methods for Research Workers*, Chapter 5, pp. 123-127, Fourth Edition, London 1932.

†See Appendix II.

TABLE IV

SIGNIFICANCE OF COEFFICIENTS

	Pennsylvania	Virginia	Mississippi	Kansas
A_0	101.62	267.92	436.20	451.63
σA_0	43.83	29.53	36.19	15.67
$t A_0$	2.32	9.07	12.05	28.82
$p A_0$	.07	<.01	<.01	<.01
A_1	.002740	.01810	.01089	.000810
σA_1	.0004161	.002527	.005489	.00001619
$t A_1$	6.58	7.16	1.98	5.00
$p A_1$	<.01	<.01	.10	<.01
A_2	0.1558	.04583	.02261	0.0165
σA_2	.01251	.005478	.002763	0.005224
$t A_2$	12.45	8.37	8.18	3.16
$p A_2$	<.01	<.01	<.01	.02

For nine out of twelve coefficients there is less than one chance out of one hundred that values of the ratios as large as or larger than those obtained would have been obtained if there were no statistically significant relationship. The other three coefficients also have small probabilities associated with them. Hence, on any customary basis of judgment these figures indicate decidedly significant coefficients; that is, a great degree of "goodness of fit." Within the limitations of current mathematical statistics, the fourth criterion is satisfied. It is thus seen that the four criteria listed are all satisfied.

CHAPTER IV

DEMAND FOR AGRICULTURAL PRODUCT

1. Demand for Pork. As already mentioned, the abrupt decline in demand levels for most products during the past four years has emphasized the need for studying shifts in demand of products such as agricultural ones for which "laws" of demand were formerly known, as well as the relation between prices and quantities demanded. The precipitous decline has also provided a much better opportunity for studying these shifts.

Chart X presents a study of demand for pork made by G. B. Thorne and Preston Richards of the United States Bureau of Agricultural Economics. Correlations are based on data for the period 1921 to 1931 but the curve is satisfactory for the years 1932 and 1933 if the regression of price deviations on consumer incomes is extended. The data of the chart are percentages of the 1921-30 averages, not of trends, so that by multiplying by these averages, the ratios can be quickly converted to actual prices and quantities.* The prices used are money prices and not prices adjusted for changes in the price level. The retail price of pork is here shown to be very highly correlated with the consumption of pork and an index of consumer incomes.

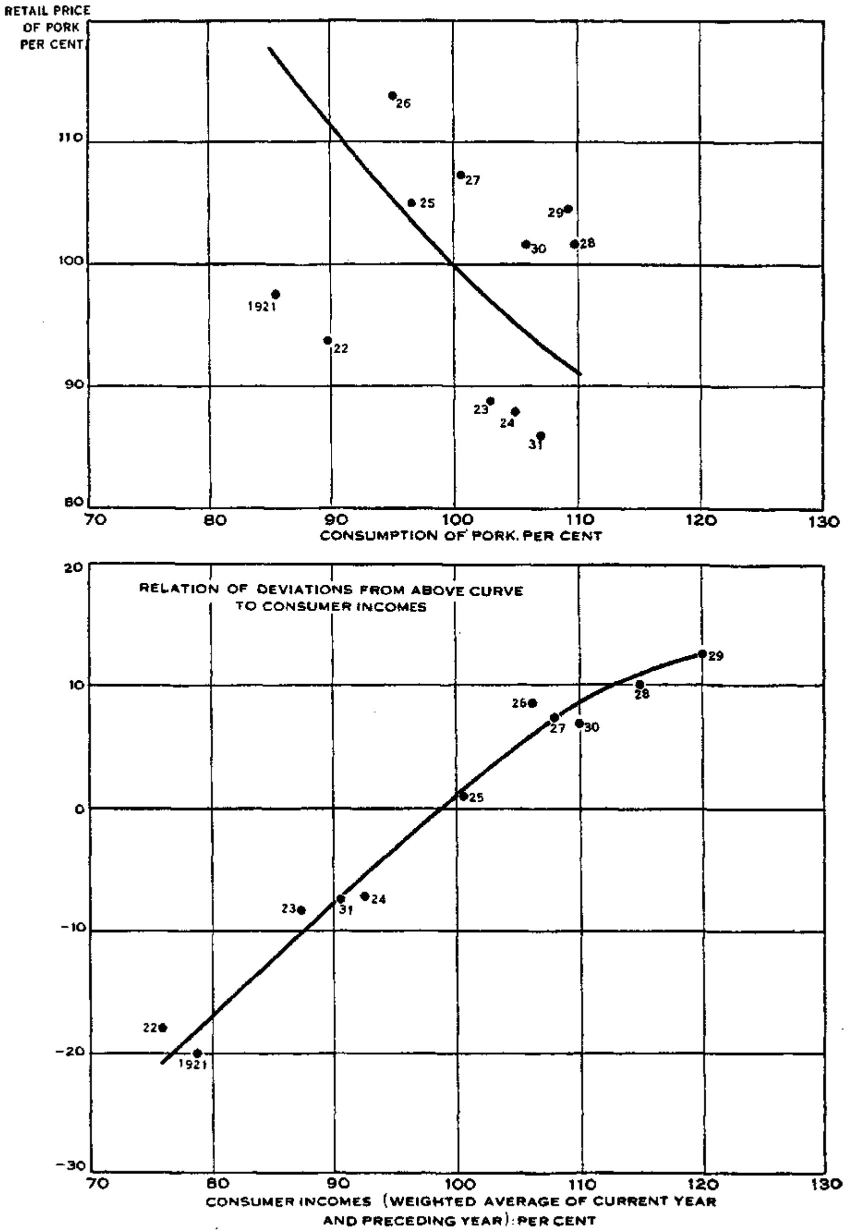
The term "money demand" may be used to mean the relation between quantities demanded and money prices. The term real demand may be used to indicate the relation between quantities demanded and real prices; that is, prices that have been adjusted for changes in the price level. As E. J. Working points out, in times of changing price levels it means nothing to say that demand has increased or that demand has decreased unless it is clear whether "money demand" or "real demand" is meant.

There is, of course, no danger of confusing terminology if one uses a mathematical analysis such as that given in the preceding chapters. Again it might be emphasized that an important advantage of the mathematical method over the literary one is that the mathematical method is a precise one and concepts can therefore

*The use of ratios of this nature is to be commended for purposes of curve fitting. Errors due to weighting are avoided in this way.

RELATION OF RETAIL PRICE OF PORK TO CONSUMPTION OF PORK AND TO CONSUMER INCOMES IN THE UNITED STATES, 1921-1931

CHART X



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Neg. 25301 Bureau of Agricultural Economics be defined without ambiguity. Thus, if p represents money price, the equation of demand can be taken as

$$y = ap + b$$

and a depends upon an index of price levels, I, whereas if the equation is desired to express real demand it may be written in the form

$$y = a_1 P + b$$

where $P = F(p,I)$ or more simply if $P = p/I$, $a_1 = aI$. A similar analysis can be applied to b. Just how the correction for price levels is to be made is discussed briefly in the next section.

2. Graphical Curvilinear Correlations. In correlations made by the Bureau of Agricultural Economics a graphical method developed by Mordecai Ezekiel is used.* The method is essentially that of drawing a free hand curve to approximate a relationship between two variables such as cotton consumption and prices of cotton, plotting the residuals (actual consumption minus consumption represented by the free hand curve) against a third variable such as industrial activity and drawing a free hand curve to represent a relationship between the residuals described above and industrial activity. In the case of cotton it is then found that the residuals from the industrial production curve have a secular downward trend, due to factors not taken into account. This secular trend is then removed by plotting the residuals against time and drawing a line. To obtain a second approximation, consumption "due" to industrial activity and trend are subtracted from actual consumption and the residuals plotted against price. A slightly different curve is suggested for the regression of consumption on price. From this point the procedure just stated may be carried through to obtain second approximations to express the effects of industrial production, etc.

In the method just outlined it is implicitly assumed that the effect of each factor, price, industrial activity and trend is additive. This, of course, may or may not be true. In the case of automotive demand previously discussed it was not considered desirable to introduce each factor in an additive way. In Chapter VI a modification of the Ezekiel method for use in approximating multiplicative

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*Mordecai Ezekiel, *Methods of Correlation Analysis*, New York, 1930, pp. 235-241.

factors is given. The problem of determining in advance whether effects are additive (addition or subtraction) or multiplicative (multiplication or division) is a theoretical one that will probably not be completely solved. However, prices are ratios and from a mathematical point of view it would seem desirable to correct prices for changes in price level by multiplicative processes. General rules cannot be laid down for physical factors and other economic ones. Individual judgments must be made here even as they are in the physical sciences. Any judgment regarding form is better than no judgment.

3. Demand for Wheat — An International Commodity. As E. J. Working has pointed out,* "it is not always possible to explain cyclical shifts in demand on the basis of changes in industrial production, consumer income or similar indexes of United States conditions".

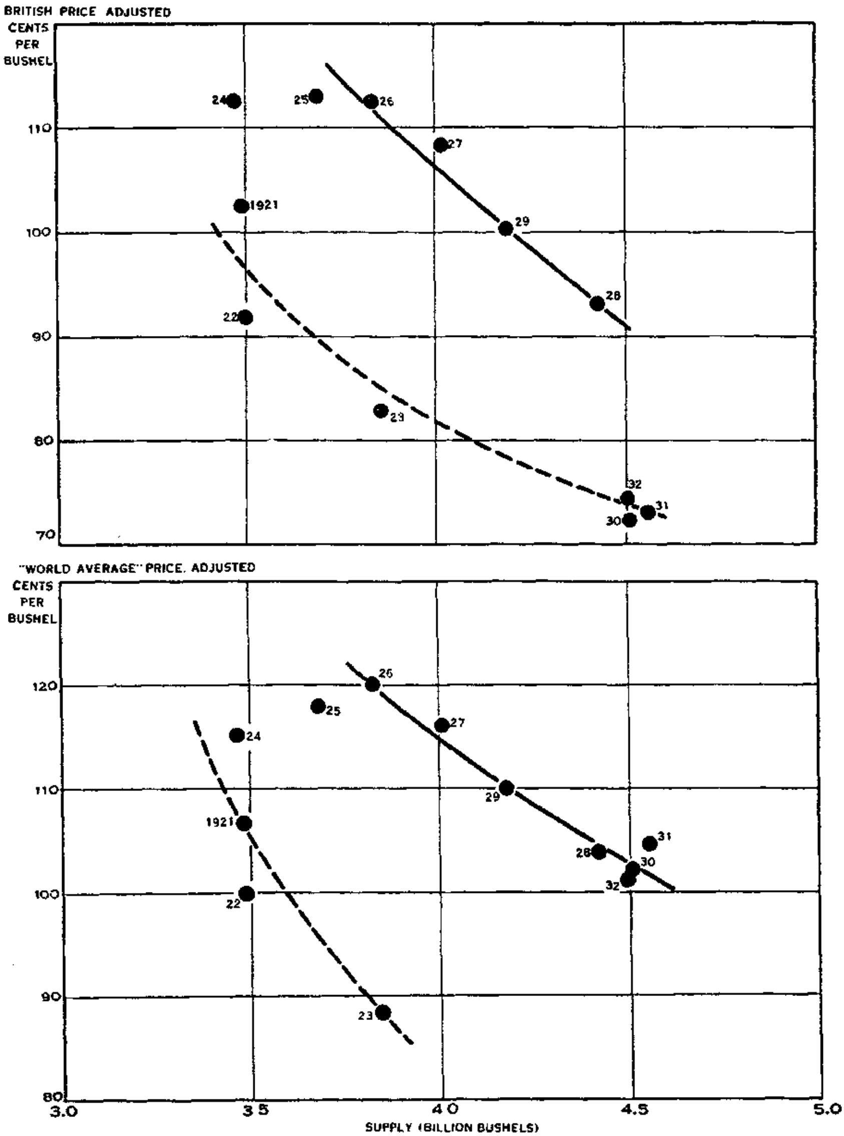
Indexes which are closely associated with the conditions of demand for domestic commodities are likely to be inadequate for use in analyzing shifts of demand for international commodities. A case in point is wheat. Chart XI shows certain price and supply relationships for that commodity. The upper section shows the relation between the price of wheat in Great Britain, adjusted for the British price level; and the estimated supply of wheat for the world excluding Russia and China. The section is somewhat similar to the upper section of the pork chart in that there tends to be two distinct groupings of the price-quantity observations. The data for the crop years 1921-22 to 1923-24 and for 1930-31 to 1932-33 group themselves fairly closely about what might be termed a "low demand curve," while the data for the other years are fairly close to the "high demand curve."

In view of the fact that the data on the wheat chart refer to calendar years rather than to July to June crop years, it may appear that these data would lend themselves to the same type of analysis as that employed in the pork price study. As a matter of fact, a procedure similar to that used in the pork price study does yield a high correlation. To quote: "the careful price analyst will not, upon obtaining such a correlation, rush forth with the explanation that the price of wheat at Liverpool is determined by the sup-

*E. J. Working, "Studies in Demand During Periods of Rapid Economic Changes," — paper presented to the Econometric Society, December 28, 1933. See *Econometrica,* Vol. 2, 1934, pp. 140-151.

WHEAT: SUPPLY AND PRICE, WORLD (EXCLUDING RUSSIA AND CHINA)

CHART XI



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ply of wheat in the world (outside Russia and China) and industrial production in the United States. A partial explanation of the high correlation is, of course, to be found in the fact that the fluctuations in business activity of the more important wheat-consuming countries of the world have been quite closely correlated in the past 15 years."

This explanation is not satisfactory since, as Working points out, there are a number of important wheat consuming countries in which wheat prices have not followed a curve parallel to that of British wheat prices in the past four years. In some of these countries "money prices" are now as high and real prices are higher than in 1926-27.

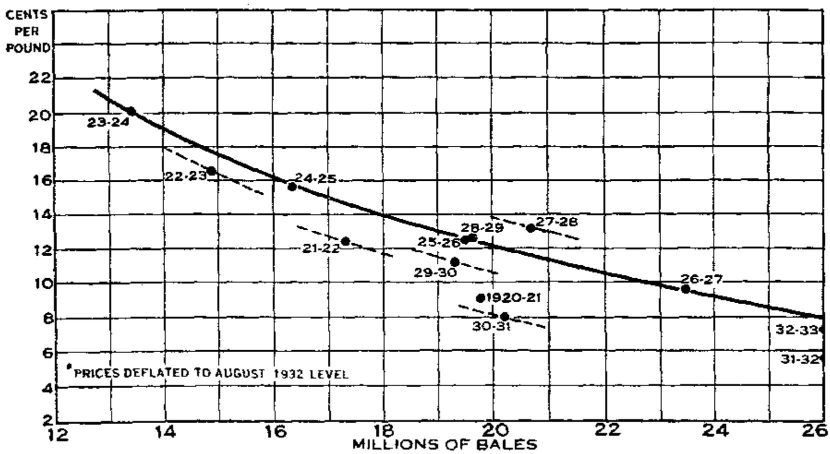
To bring the above situation out more clearly Working computed a "world" average "real" price of wheat. This was done by first adjusting for the price level the money price of wheat in seven important wheat consuming countries and converting these adjusted or real prices to terms of gold cents per bushel at the par of exchange. A weighted average of these seven series was then computed — the weighting being in proportion to the wheat consumption of the regions for which the original price series were deemed to be representative. The resulting prices as plotted against the wheat supply of the world (excluding Russia and China) are shown in the lower half of Chart XI.

From this study it appears that the price of an international commodity such as wheat is essentially a gold price. The question of whether one should use "money prices" or "real prices" in measuring demand has been a matter of general concern among agricultural price analysts for some years. This question was discussed in Chapter II. Another question that has recently arisen is: "Should the prices used be currency prices or gold prices?" Working answers this question as follows:

> "In order to make the question more concrete, let us consider this chart (Chart XII) which shows the deflated price of cotton (the price adjusted for price level changes) in relation to the supply of American cotton. Assume for sake of simplicity that, in so far as past analysis indicates, demand conditions for the 1933-34 season are expected to be similar to those of 1926-27. Under such conditions and with a supply of about 24.8 million bales, what price is to be expected? The chart would indicate about 9 cents per pound as the deflated price, but these prices on the chart all refer to years when a dollar was worth approximately

RELATION BETWEEN PRICES AND SUPPLIES OF AMERICAN COTTON

CHART XII



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as much as one-twentieth ounce of gold. With it now worth about one thirty-fourth as much as an ounce of gold, will the money price be increased accordingly?

"The answer is not the same for all commodities. In the case of such commodities as beef and pork merely taking account of prospective changes in the "all commodity" price level should give the approximate result. On the other hand, in the case of a commodity such as wheat one should estimate what the price level would have been if the United States had remained on the gold standard, adjust for that change from the August 1932 level and then multiply the result by about 1.69 or whatever is deemed to be the probable ratio of the price of gold this year to the old price of $20.67 per ounce. In the case of cotton one would come nearer the truth to adopt the procedure outlined for wheat."

The study of gasoline demand in the preceding chapter and the various studies of demand for agricultural products presented in this chapter* indicate that there is a great deal to learn about the

*Attention is also called to the now classical study of demand for potatoes made by Holbrook Working. See Holbrook Working, "The Statistical Determination of Demand Curves," *Quarterly Journal of Economics*, August 1925, Vol. XXXIX, pp. 503-543.

behavior of prices and demand other than that "demand varies inversely with price" as is often naively stated in text books.

All the studies presented so far refer to demand for consumer goods. When attempts are made to determine the demand for capital goods new problems are met. Some of these are difficult to solve, but like so many difficult things they seem to offer a gold mine of fundamental information. In the following chapter a theory of demand for capital goods is presented and in Chapter VI a statistical verification of this theory is offered.

CHAPTER V

Demand for Capital Goods

1. Definitions. So far as the author knows, no one has yet published a demand study for capital goods.* Various attempts have been made to obtain such laws but all attempts have ended in failure. One of the difficulties has been lack of adequate data, but there are other fundamental reasons. For example, another important reason why such studies have been difficult to make is that a certain fairly long period of time has to elapse before an incentive to demand is consummated in actual demand. The length of this period of time varies with the particular kind of capital goods demanded. There is, of course, a time interval for consumer goods also, but it is not nearly so long and by using yearly data the difficulty can be surmounted, as will be made apparent at the end of this chapter. In fact, there are some consumer goods for which the time interval is very short, a few minutes, an hour, a day, or a week. For capital goods the time interval is quite long; for example, two or three years on the average.

Some economists have attempted to differentiate between various kinds of economic goods on the basis of the time required to consume the product, i.e., the life of the product. Such differentiation into durable and non-durable goods is inadequate for demand studies. Another differentiation sometimes made is on the basis of expected income, i.e., capital goods are goods that are to be used to produce income. This is a much better definition for studies of demand, but, of course, it should be recognized that no matter what definition is used there will be some goods that will fall in more than one class and be difficult to classify. Thus, according to the latter differentiation an automobile used for pleasure would be a consumer good, whereas the same automobile used as a taxicab would be a capital good. Furthermore, some capital goods will be durable and others will be non-durable.

*R. H. Whitman presented a paper on the demand for pig iron before the Econometric Society in Syracuse in June, 1932, but this has not been published. The paper uses the method of trend analysis. After elimination of trends the method is essentially that of determining integral weights $K(x_i, t)$, t fixed, for prices three months in advance of demand by the method of multiple correlation. The correlations are unfortunately far from convincing, since the signs of terms are theoretically correct only three times out of five.

The price of a capital good (using the income definition) would, of course, depend upon the expected income from the good and the cost of production of the good. In some instances the expected income is the most important factor (there are usually physical factors of at least as much importance as price), whereas in other instances the price plays a role of some importance. In the work that follows p will be used to represent either the price of the good or the expected income. As far as the analysis is concerned, it is unnecessary to differentiate between the two and the careful reader will be able to differentiate and also to interpret the results in terms of either or both according to the interpretation which is applicable to the particular problem in hand. Furthermore, even though p be used only in linear relations, it does not follow that the theory applies only to linear demand laws, for if p is expected income, this might in the case of residential building be defined to be $(R - T)/C$ where R is rent, T is taxes and C is cost (a price).

2. Demand Incentives and Consummations. In Chapter III it was assumed that demand depended upon past prices and other quantities and present prices and other quantities. On this assumption an equation of demand was obtained in the form

$$y(t) = \varphi(t) + K(t,t)F(p) + \int_{-\infty}^{t} K(x,t)p(x)dx. \tag{2.1}$$

It was then assumed that for many products the integral term could be replaced by a constant factor, so that the equation of demand became

$$y(t) = \varphi(t) + K(t,t)F(p). \tag{2.2}$$

No theoretical justification for this assumption was given, but in Chapters III and IV it was shown that this equation is sufficiently general to apply to the demand for such consumer goods as gasoline, pork, wheat and cotton.

As a further problem it is now proposed to investigate the function $K(x,t)$. This study of the questions of why demand depends upon past prices and how individuals build up wants or desires throws considerable light on the nature of the function $K(x,t)$ and makes it possible to give theoretical justification to the assumption pointed out above.

To investigate the nature of $K(x, t)$ suppose that, at the time $t_i < t$, a group consisting of G_i individuals comes into contact with an incentive to purchase, $p(t_i)$. Suppose further that this group purchases M_t units and decides to purchase a total of N_i additional units of the goods as soon as money or credit becomes available to them. Some of the group, perhaps, decide not to purchase any units while others may decide to purchase more than one unit. Some, perhaps, purchase at the time t_i and decide to purchase other units in the future. The number of units $N(t_i)$ that would be purchased per unit time in the future from such decisions (without inhibitions which can be discussed separately) may be assumed to be proportional to the incentive $p(t_i)$; that is,

$$N(t_i) = ap(t_i) + b \ ,$$

where a and b are constants (or functions of time in the sense explained in Chapter II).

There will be many factors affecting the length of time required for individuals to act on an incentive. Some of these factors will be of the same relative importance, but a few, such as credit, will have important special influences. For the many small factors it can be assumed that the time frequency distribution of those acting on the incentive will obey the normal law.* The peak of the distribution will occur M units of time after the stimulus or incentive; i.e., M units of time after t_i, so that the number reacting at time $t > t_i$ to the stimulus at t_i may be taken to be approximately

$$\psi(t_i,t) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(t-t_i-M)^2/2\sigma^2} [ap(t_i) + b] \ .$$

The parameters σ and M depend on many economic and psychological factors and in the sense already explained may be assumed to be constants, at least for some periods of time.

The total number reacting at any particular time to all stimuli prior to t, i.e., to stimuli at $t_0, t_1, t_2, \cdots, t_i, t_{i+1}, \cdots, t_{n+1} = t$, is

$$\psi(t) = \sum_{i=0}^{n} \psi(t_i,t) = \sum_{i=0}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(t-t_i-M)^2/2\sigma^2} [ap(t_i) + b] \ ,$$

where $i = 0$ refers to the first time at which a stimulus or incentive was given. More exactly,

*Whittaker and Robinson, *Calculus of Observations*, pp. 168-175.

$$\psi(t) = \int_{-\infty}^{t} \frac{ap(x) + b}{\sqrt{2\pi}\,\sigma} e^{-(t-x-M)^2/2\sigma^2} dx \ ,$$

where x has been used to designate time $t_0, t_1, \cdots, t_{n+1}$.*

A simple transformation $(x-t+M)/\sqrt{2}\,\sigma = S$ leads to a form which, in some respects, is more suggestive of lags.

The quantity M_t consists of units purchased at t due to important factors operating at that particular time. For example, some buyers coming into contact with an article for the first time might decide to purchase immediately. An individual in a group G_i who had made his decision to purchase four units at $1,000 each might purchase five units at $850 each, the latter number being better suited to his needs. There would also be individuals who had not quite decided to purchase, but whose desires had been growing ever since some previous time τ. The amount purchased by this group would depend upon past prices as well as upon present price, depending less and less on past prices as the time is more and more remote. Thus, M_t might be given by a formula of the type

$$M_t = F[p(t), t] + \int_{\tau}^{t} h(t, x)\, e^{-w_1(t-x)}\, p(x) dx$$

where M_t is used to denote a function of p and t, $h(t) > 0$ and $w_1 > 0$. Here t is used to indicate that there will be present factors other than price, $p(t)$. A function of the type $\varphi(t)$ is included in F.

The demand $y(t)$ now takes the form

$$(2.3) \qquad y(t) = F[p(t), t] + \int_{\tau}^{t} h(t, x) e^{-w_1(t-x)} p(x) dx$$

$$+ \int_{-\infty}^{t} \frac{ap(x) + b}{\sqrt{2\pi}\,\sigma} e^{-(t-x-M)/2\sigma^2} dx.$$

As already pointed out, M and σ depend on many economic factors, the effects of any one of which are assumed to be small. For impor-

*Irving Fisher seems to have been the first to use the idea of a distributed lag. See Irving Fisher, "The Business Cycle Largely a Dance of the Dollar," *Journal of the American Statistical Association*, December, 1923, p. 5. For further references see Irving Fisher, *Theory of Interest*, New York, 1930, pp. 419-420.

tant special factors in the past, as, for example, credit in the case of residential building, corrections can be made by adding (or subtracting) other integral terms (corresponding to the Gram-Charlier method of representing frequencies*) or by using weighting functions other than the normal probability function (corresponding to the Pearsonian method of representing frequency distributions†). In the study of residential building presented in the next chapter the first method will be explained and used. A short discussion of the other alternative is given here.

For definiteness suppose that the factors affecting the periods of time required for the desires to purchase to materialize into actual purchases are such that decisions to purchase are translated into actual purchases according to a Pearsonian Type III frequency function. For this type of function the amount $v(t, t_i)$ purchased by the group G at the time $t > t_i$ is given by the equation

$$v(t, t_i) = v_0 e^{-\gamma(t-t_i-\mu)} \left[\frac{t-t_i}{\mu}\right]^{\gamma\mu}, \tag{2.4}$$

where v_0, the mode of purchases, is proportional to N, $v_0 = \lambda N$; where N is the total number of proposed purchases and γ and μ are parameters not involving the time t.

The curve (2.4), (t_i, γ and μ constants) has a zero at t_i, reaches a maximum μ units of time after t_i and then approaches the t-axis asymptotically. Thus, (2.4) requires an assumption that many of the group will be able to make their purchases within a relatively short time (for some products one week, for others a month and for some perhaps a year) after t_i; others will require more and more time and some will never be able to make their proposed purchases. The parameters γ and μ quite evidently depend upon monetary and credit conditions, upon the prices of competing or substitute goods and upon various other quantities. For example, convincing advertising of the product presented to the group at the time t_i may cause certain of the group to do without other products sooner than they would if the advertising were not so alluring.

As pointed out above, v_0 is proportional to N; that is,

$$v_0 = \lambda N, \text{ where } \lambda = (\gamma\,\mu)^{\gamma\,\mu+1}\;\mu\, e^{\gamma\,\mu}; \Gamma(\gamma\,\mu + 1) \text{ and } \Gamma(\gamma\,\mu + 1)$$

*See Arne Fisher, *Mathematical Theory of Probabilities*, New York, 1930, pp. 202 et seq.

†See H. L. Reitz, *Mathematical Statistics*, Chicago 1927, pp. 54 et seq.

is the gamma function for $(\gamma \mu + 1)$.* The number of proposed purchases N, will certainly be a function of the price $p(t_i)$. As a first approximation suppose that $N = a(t)p(t) + b(t)$. Then by the process of summation (integration) already described, the demand y can be taken to be

$$(2.5) \qquad y(t) = F(p(t), t) + \int_{\tau}^{t} h(t, x) e^{-w_1(t, x)} p(x) dx$$
$$+ \int_{\tau_1}^{t} \lambda(a(x)p(x) + b(x)) e^{-\gamma(t-x-\mu)} \left[(t-x)/\mu \right]^{\gamma \mu} dx .$$

Formula (2.5) is simply a special case of formula (2.1) of Chapter II obtained in a slightly different way. Quite obviously different functions $K(x, t)$ can be obtained by assuming different distributions for the consummation of purchases at the time t. Thus, the distribution

$$v = v_0 \left[1 + \frac{(t - t_i - \mu_1)}{\mu_1} \right]^{m_1} \left[1 + \frac{(t - t_i - \mu_2)}{\mu_2} \right]^{m_2};$$
$$m_1/\mu_1 = m_2/\mu_2 ,$$

and the same linear law of demand decision would lead to

$$K(x, t) = \lambda_1 a(x) \left[1 + (t - x - \mu_1)/\mu_1 \right]^{m_1}$$
$$\left[1 - (t - x - \mu_2)/\mu_2 \right]^{m_2}$$

$$\text{where } \lambda_1 = \frac{(m_1)^{m_1}(m_2)^{m_2}}{(m_1 + m_2)^{m_1 + m_2}} \frac{\Gamma(m_1 + m_2 + 2)}{\Gamma(m_1 + 1)\Gamma(m_2 + 1)} \frac{1}{\mu_1 + \mu_2}.$$

Different kernels, $K(x, t)$, can also be obtained by assuming laws of demand decision other than the linear one which has been assumed here, that is, by replacing p by a function of p. There are, therefore, a variety of forms $K(x, t)$ that are theoretically possible.

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*See, for example, D. C. Jones, *A First Course in Statistics*, London 1921, p. 221.

3. Approximations to Past Effects—Time Lags. It is now proposed to investigate conditions under which the integral terms of demand equations such as (2.3) and (2.5) can be neglected, or approximated by other expressions which for some purposes are simpler. Consider first the expression

$$\psi(t) = \int_{-\infty}^{t} \frac{ap(x)+b}{\sqrt{2\pi}\,\sigma} e^{-(t-x-M)^2/2\sigma^2} dx \ .$$

Such an expression for ψ may be mathematically correct, but it is, nevertheless, impracticable from the points of view of economic reality and of statistical analysis. Obviously, observations on p do not extend in time to $t = -\infty$ and, furthermore, if the observations did extend that far in the past all the individuals who had such early incentives would be dead. In other words it is necessary to apply certain boundary conditions of the problem.

For this purpose, ψ can be broken down into an integral from $-\infty$ to $t-t_0$, and one from $t-t_0$ to t, where t_0 is a constant sufficiently large to make the deviation of the first integral from its mean value over the infinite period of $-\infty$ to $t-t_0$ negligible; that is,

$$\psi(t) = A_0 + A_1 p_0 + \int_{t-t_0}^{t} \frac{ap(x)}{\sqrt{2\pi}\,\sigma_1} e^{(t-x-M)^2/2\sigma_1^2} dx \ , \tag{3.1}$$

where A_0 and A_1 are constants (or functions of time in the sense used in this book) and p_0 is the normal (average) value of p on the infinite range $-\infty$ to $t - t_0$.

This is the type of formula required to represent ψ if it is assumed that once an incentive is offered it will be acted upon some time in the future, the particular time at which the action occurs being determined by a great many economic and physical forces, any one of which has only a small effect.

It is especially interesting to derive the formula (3.1) since the method of proof gives important information regarding the nature of time lags. To transform from the infinite range on which ψ is defined to a finite one, write

$$\psi = \int_{-\infty}^{t} \frac{ap(x)+b}{\sqrt{2\pi}\,\sigma} e^{-(t-x-M)^2/2\sigma^2} dx$$

$$= \int_{-\infty}^{t} \frac{b}{\sqrt{2\pi}\,\sigma} e^{-(t-x-M)^2/2\sigma^2} dx$$

$$+ \int_{-\infty}^{t-t_0} \frac{ap(x)}{\sqrt{2\pi}\,\sigma} e^{-(t-x-M)^2/2\sigma^2} dx$$

$$+ \int_{t-t_0}^{t} \frac{ap(x)}{\sqrt{2\pi}\,\sigma} e^{-(t-x-M)^2/2\sigma^2} dx \; .$$

By putting $(x + M - t)/\sqrt{2}\,\sigma = z$ in the first and second integral, ψ may be written in the form

$$\psi = \int_{-\infty}^{M/\sqrt{2}\,\sigma} \left(\frac{b}{\sqrt{\pi}}\right) e^{-z^2}\, dz$$

$$+ \int_{-\infty}^{a} \left(\frac{a}{\sqrt{\pi}}\right) p(\sqrt{2}\,\sigma z + t - M) e^{-z^2}\, dz + F[p]$$

where

$$F[p] = + \int_{t-t_0}^{t} \frac{ap(x)}{\sqrt{2\pi}\,\sigma} e^{-(t-x-M)^2/2\sigma^2} dx,$$

and

$$a = (M - t_0)/\sqrt{2}\,\sigma \; .$$

Now, the first integral in ψ is a constant which may be called A_0. In the second integral let p_0 be the average value of I in the interval $-\infty \leq z \leq (M - t_0)/\sqrt{2}\,\sigma$ and let $p_\Delta\, (\sqrt{2}\,\sigma z + t - M)$ be a function such that

$$p(\sqrt{2}\,\sigma z + t - M) = p_0 + p_\Delta(\sqrt{2}\,\sigma z + t - M).$$

Then by an application of the law of the mean for integrals it follows that

$$\psi = A_0 + A_1 p_0 + \int_{-\infty}^{a} \frac{a}{\sqrt{\pi}} p_\Delta(\sqrt{2}\,\sigma z + t - M)\; e^{-z^2}\, dz + F[p]$$

$$= A_0 + A_1 p_0 + p_\Delta \; (t - M - \lambda) \frac{\bar{a}}{\sqrt{\pi}} \int_{-\infty}^{\alpha} e^{-z^2} dz + F[p] \; ,$$

where

$$A_1 = \int_{-\infty}^{\alpha} \frac{a}{\sqrt{\pi}} e^{-z^2} dz$$

and λ is a positive quantity such that $t - M - \lambda$ represents a time previous to the present time and $\bar{a}$ is an average value of a. Thus, $p_\Delta (t - M - \lambda)$ represents an average value of past fluctuations in incentive from an assumed normal value I_0.*

In all cases it is possible to find a t_0 such that

$$p_\Delta (t - M - \lambda) \frac{a}{\sqrt{\pi}} \int_{-\infty}^{\alpha} e^{-z^2} dz$$

is negligible, since the integral can be made as small as desired by proper choice of t_0. It is, therefore, always possible to write

$$\psi = A + \int_{t - t_0}^{t} \frac{ap(x)}{\sqrt{2\pi}\,\sigma} e^{-(t - x - M)^2 / 2\sigma^2} dx$$

where A is a constant or function of time defined by the equation $A = A_0 + A_1 p_0$.

If t_0 is a year or less, the effect of the integral term is completely lost when yearly data are used, since the integral gives a weighted average of p for the year. Thus, for the consumer goods, gasoline, wheat, pork and cotton, there is no need to use the integral term. For these goods a formula of the type

$$y = \varphi(t) + F[p(t), t]$$

is sufficiently general.

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*J. A. Hobson, *Theory of Functions of a Real Variable*, Vol. 1, page 617. The theorem referred to here does not apply to an integral with an infinite range, but the transformation $1/z = y$ reduces the infinite range to a finite one, the function $p(z)$ is bounded for all values of z and $(e^{-1/y})/y^2$ is also bounded on the transformed finite range. The law of the mean referred to here can thus be applied to the transformed integral and then a transformation can be made back to z.

When t_0 is greater than a year and yearly data are used, the law of the mean can be applied to the integral term. In this case the demand is given by an equation of the form

$$y(t) = \varphi(t) + F[p(t), t] + F_1[p(t-\vartheta), t],$$

or by further averaging by a formula of the type

$$y(t) = \varphi(t) + \psi[p(t-\vartheta), t].$$

This is an equation involving a fixed average time lag. In statistical work of a preliminary nature results are more likely to be obtained by the use of a fixed average lag than by the use of an integral lag. After such a preliminary study a formula involving distributed lag can be obtained. Methodology for accomplishing this is given in Appendix V.

4. Price Forecasting and Speculation. An analysis similar to the preceding can be used to determine the form a demand law should take when speculation is taken into account. Speculative demand depends upon future prices, (price used here in the general sense explained in Section 1) so that demand for goods that lend themselves to speculation depends upon past prices, present price and expected future prices; that is, the amount demanded at the time t is

$$y(t) = f[p(t_1), \cdots, p(t_n)p(t_{n+1}), \cdots, p(t_{n+\sigma}), t, p_1 \cdots, p_m],$$

where t_n stands for time t when purchases are made and $p(t_{n+1})$, $\cdots, p(t_{n+\sigma})$ are expected future prices. The analysis already developed shows that, for hypotheses similar to those made for obtaining equation (3.1) of Chapter II, this expression can be replaced by an integral,

$$y(t) = \varphi(t) + \int_{t-t_0}^{T} K(x, t)p(x)dx \ ,$$

where $T = t_{n+\sigma}$. More generally, since special weight may be attached to the present price $p(t)$,

$$y(t) = \varphi(t) + \int_{t-t_0}^{T} K(x, t)p(x)dx + a_1(t)p(t).$$

This expression can be written as

$$(4.1)\quad y(t) = \varphi(t) + a_1(t)p(t) + \int_{t-t_0}^{t} K(x,t)p(x)dx + \int_t^T \psi(x,t)p(x)dx,$$

where for purposes of clarity $K(x, t)$ for $x > t$ has been named $\psi(x, t)$. The first three terms of (4.1) can be taken to be the same as the expression (3.1) of Chapter II. Thus

$$S = \int_t^T \psi(x,t)p(x)dx$$

represents demand due to speculation.

Long range economic forecasting has never been consistently successful. In fact, in view of a recent study by Alfred Cowles, III* of the frequency of successes of stock forecasting houses, it is somewhat doubtful that any great measure of success, perhaps only success that might be expected by chance, can, at present, be obtained for even short time forecasting of such things as security price changes. There are, of course, other phenomena for which there is some hope of short time forecasting.

Many individuals seem to be engaged in the science (or guessing game) of attempting to predict what will happen to prices the next minute, the next hour, the next day, or the next week. An important speculative problem is, therefore, the one for which $T = t + 1$. For this problem the extrapolation curve for $p(x)$ may be most conveniently taken as a straight line

$$p(x) = p(t) + (x-t)dp/dt \ , \ x > t,$$

where dp/dt is the derivative of price with respect to time; that is, dp/dt is the slope of the price-time curve at the time t. The assumption made here is that estimates of future quantities are based on present evidence. A substitution of the above value of $p(x)$ in S yields

$$S = \int_t^T \psi(x,t)p(x)dx = \int_t^{t+1} \psi(x,t)\ [p(t) + (x-t)dp/dt]\,dx.$$

*Alfred Cowles, III, "Can Stock Market Forecasters Forecast?", *Econometrica,* Vol. I, 1933, pp. 309-324.

An application of the first law of the mean for integrals* followed by an integration gives

$$S = \psi(t+\vartheta, t) \int_t^{t+1} [p(t) + (x-t)dp/dt]\,dx$$
$$= \psi(t+\vartheta, t)\ [p(t) + \tfrac{1}{2}\,dp/dt]\ , \qquad 0 < \vartheta < 1.$$

It follows that equation (7) can be replaced by the simpler form

$$(4.2) \quad y(t) = \varphi(t) + A(t)p\,(t) + H(t)dp/dt + \int_{t-t_0}^{t} K(x,t)p(x)dx\ ,$$

where $A(t) = a_1(t) + \psi(t+\vartheta, t)$ and $H(t) = \psi(t+0, t)/2$.

It will be recalled that the integral equation (4.2), or rather the equation for which the lower limit $t - t_0$ is fixed, is invertible, so that it is possible to talk about forecasting either prices or consumption without stopping to reformulate the problem.

In general, when prices are rising and there is reason to believe that they will continue to rise, speculative demand is positive. When prices are falling and there is reason to believe that they will continue to fall, speculative demand is negative; that is, there is a tendency to dump. To satisfy the above conditions $H(t)$ must be a positive function.

Many times prices rise for a period, drop back slightly, rise a bit and again drop back to the preceding recession level. In such instances there is uncertainty regarding trends, that is, there is uncertainty whether prices will rise or fall. In such instances some speculative traders sell to avoid losses or sell short expecting to buy back at lower prices, some traders buy in the expectation of gain and some simply stay out of the market. This situation is characterized by the fact that $dp/dt = 0$; that is, demand is neither increased nor decreased by speculation.

The introduction of a derivative of price into the demand equation marks one of the important contributions of mathematics to economics in the past decade. This is due to G. C. Evans, who in 1925 proposed a demand equation $y(t) = a\ p(t) + b + h\ dp/dt$,

*E. W. Hobson, *Theory of Functions of a Real Variable*, Cambridge, 1927, Vol. I, p. 617. Continuity of $\psi(x, t)$ in x for all t is assumed.

where a, b and h are constants, in order to approximate the phenomenon that "when prices are going up the demand (for lumber) is insatiable, but when prices are going down it is nil until the price movement stops."* Although it is possible that the rise in price of lumber mentioned by Evans was brought about by an expected increase in price, it is probable that the price of lumber increases because of a rising demand brought about by other factors (see the study on residential building in the next chapter). Nevertheless, there is probably some speculative influence at work on lumber and there are certainly such forces at work on other products.

One difficulty about forecasting security prices is that there is always the possibility of introducing a new element that has not previously been taken into account. In general, security prices are determined by so many causes that series of security prices appear to be random series, but the series are, of course, subject to special factors, such as abandonment of gold, favorable Government pronouncements, etc.

There are some consumer goods for which it is possible to forecast demand and prices two or three months in the future. As will be seen in the subsequent chapter, it is now possible to forecast residential construction for a year or so in advance of the present time with the probability of a reasonable degree of accuracy.

5. *Demand for Competing Products.* It will be recalled that in Section 2 the prices $p_1, \cdots, p_m$ of m goods competing with the goods whose price is p were assumed to be constant with respect to time. Since the theory just developed is a dynamic one, that is, one in which forces are allowed to modify situations as time progresses and hence one in which the time element plays an important role but not necessarily *per se*, it was unnecessary to carry these parameters along in the equations. It should be remembered now, however, that, in accordance with the hypotheses made in Section 2, each of the quantities φ, A, H and K depends upon the prices of competing goods. For example, suppose that in a study of the relation of the demand for pig iron to the price of pig iron the co-

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*G. C. Evans, "The Dynamics of Monoply," *Amer. Math. Monthly*, Vol. XXXI, February, 1924, p. 77. See also Evans, *Mathematical Introduction to Economics*, New York, 1930. For the first use of an integral equation of demand, see C. F. Roos, "A Mathematical Theory of Competition," *Amer. Jour. of Math.*, Vol. 47, 1925, p. 173. See also, C. F. Roos, "Theoretical Studies of Demand," *Econometrica*, Vol. 2, 1934, pp. 73-90.

efficient $A(t)$ is determined statistically as a number a. This number a on further analysis will be found to be made up of prices of competing goods such as copper, aluminum, lumber and so forth. Thus a might be defined by a formula such as $a = a\ p_a + c\ p_c + l\ p_l$, where a, c and l are constants and p_a, p_c and p_l are the prices of aluminum, copper and lumber respectively.* More generally a, c and l may be assumed to vary with the time t so that a will be a function of time. Thus as far as the whole of the preceding analysis is concerned, the quantities φ, A, H and K may be considered to be functions of present prices of competing or substitute goods and the time t either implicitly, as already explained, or explicitly, since for seasonal goods φ and possibly other quantities may contain periodic functions of the time as, for example, cosines and sines.

The statements made above in regard to changes in φ, A, H and K with respect to time do not mean, however, that the quantities are so continuously changing that statistical laws of demand cannot be determined. On the contrary, as indicated previously, due to compensation of prices of competing goods, these quantities may remain fairly constant for periods of time that may be much in excess of five years for some commodities. In the example considered above, it might be possible for the price of aluminum to increase, the prices of copper and lumber to decrease and yet a might remain constantly equal to some given value, but such a situation would probably not prevail for long, especially if there were noticeable price movements. The question of what must be regarded as "long" may, of course, be answered differently for each product.

If the prices $p_j(t_i)$, $j = 1, 2, \cdots, m$, $i = 0, \cdots, n$, $t_0 = 0$, $t_n = t$, are not assumed to be all equal to $p_j(t)$ respectively, the theory of demand becomes much more complicated, but, nevertheless, it can be readily formulated in terms of systems of integral equations or functional equations. Thus, the simplest case of the non-speculative problem leads to a system of $m + 1$ integral equations,

$$y_k = \sum_{j=1}^{m+1} \left[\varphi_{kj}(t)p_j(t) + \int_0^t K_{kj}(x, t)p_j(x)dx \right], \quad k = 1, \cdots, m + 1.$$

There is in general no difficulty in inverting this system of Volterra integral equations to determine the $p_j(t)$ in terms of the

*For a statistical study of the effects of prices of competing goods, see Mordecai Ezekiel, "Statistical Examination of Factors Related to Lamb Prices," *Journal of Political Economy*, Vol. 35, April, 1927, p. 254.

y_k.* In fact, entirely similar equations result with the y_k replacing the p_k and resolvent kernels replacing the K_{kj}. When derivatives are introduced and a more general functional relation is assumed, a system of $m + 1$ functional-differential equations are obtained.†

6. The Elasticity of Demand. For a great while economists have been classifying demand as elastic or inelastic depending upon whether the value of the good demanded increased or decreased with a fall in price and vice versa. In 1838 Augustin Cournot proposed to separate articles into two categories depending upon whether $(\Delta y/\Delta p)p/y$ is less than or greater than one. Alfred Marshall defined the quantity $(dy/dp)/(p/y)$ to be the *coefficient of elasticity of demand.* He and H. L. Moore popularized the concept and Moore applied the concept extensively to agricultural data and used it to obtain various laws of demand.‡

Whenever the coefficient of elasticity of demand is *numerically* greater than one, the demand is said to be elastic. Whenever it is numerically less than one, the demand is inelastic and if it is numerically equal to one, the demand is neither elastic nor inelastic.

For equation (4.2) the elasticity of demand is simply $A(t)p(t)/y$. For a functional demand equation of this type a functional coefficient of elasticity of demand might be more useful.

Let δy be the variation of y corresponding to a variation, δp, in p in the sense of the calculus of variations. Then, the *functional coefficient of elasticity of demand* may be defined to be $(\delta y/\delta p)p/y$. Thus for (4.2) with $H = 0$, the variation in y would be given by $\delta y = \int_0^t K(x, t)\, \delta p dx + A(t)\delta p$. In particular, if each price of the goods from the time 0 to t were increased by a constant amount δp, the functional coefficient of elasticity of demand would be

$$\left[\int_t^0 K(x, t) dx + A(t) \right] p(t)/y(t) \ .$$

If H were not zero, there would be added a term $Hd(\delta p)/dt$ to δy.

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*Vito Volterra, *loc. cit.*, pp. 71-74.

†C. F. Roos, "A Dynamical Theory of Economics," *Journal of Political Economy,* Vol. XXXV, 1927, pp. 648-650.

‡ See, for example, Alfred Marshall, *Principles of Economics,* London, 1920, H. L. Moore, *Synthetic Economics,* New York, 1930, and Augustin Cournot, loc. cit., pp. 52-54.

It is important to notice that no attempt has been made to determine why a demand decision curve should take the form postulated, or why the number making a demand decision should depend upon the price, or why N or M_t should take any of the forms postulated. It would, of course, be possible to pursue these questions as far as one liked by analyzing human behavior, utility and so forth, but these studies properly belong to the sciences of psychology and sociology rather than to economics.

The size of a market can be increased by advertising. This could be taken into account by postulating that the number introduced to a product at a certain time depends upon the advertising expenditures. It would then be possible to determine how demand varies with advertising.

Again, if the product is one for which there will be a repeat demand, the number of prospective purchases will certainly depend upon the life of the product. Thus, one could study the effects of obsolescence, risk of damage, seasonal changes and so forth. Also, the parameters of the functions $K(x, t)$, $\varphi(t)$, etc., depend upon such things as monetary conditions, the psychology of the buying public and so forth. However, all these factors deserve consideration in their own right and cannot be adequately treated here. It may be hoped that it will be possible to push the analysis of demand further and further back to fundamental conceptions of behavior, but it must not be forgotten that empirical formulae will have to be introduced at some stage.

One might say that the theory of demand has progressed to a point where it is questionable that further theoretical work should be done until many statistical studies have been made to verify or disprove the hypotheses and conclusions so far reached. The statistical investigation is, in itself, a tremendous task. A new type of mathematical statistician is undoubtedly required to make the studies. Nevertheless, as has been indicated in Chapters III and IV and will be demonstrated in the next chapter, there are reasons to be optimistic regarding the possibilities of the discovery of statistical laws of demand both for consumer goods and for capital goods.

CHAPTER VI

FACTORS INFLUENCING RESIDENTIAL BUILDING*

1. Building Cycles. Analysis of the post-war prosperity shows that it was building which led the way out of the short post-war depression in 1921, and that it was building which started to sag early in 1928, one year before the culmination of the new era prosperity in mid-1929. The same relationship was true in previous major cycles, although the dominant type of construction was not invariably construction of buildings — it was railroad building in the major cycle of the 80's. Variation in activity in the Construction Industry is a primary cause of the major periods of boom and depression which have characterized business in the United States over the past century.

The cycle of the residential building industry is the longest and most extreme of any. The building peak may be 1,500 per cent above the trough. Typical major swings last from 10 to 20 years with a mean of about fifteen years. It has been maintained by some that these typical swings are brought about by the financial custom of extending five and ten year mortgages. This is hardly a correct statement, as will be seen by the following study, but it must be admitted that most building is done at the highest prices and the mortgages placed then usually run for five or ten years. There is, of course, always a necessity for making careful differentiation between "cause" and "accompaniment," and in a study of such a problem as residential building great care must be taken to make this distinction.

A smoothing out of the cycle in building activity would undoubtedly accomplish a great deal in smoothing out trade cycles. It is, therefore, of paramount importance to understand the causes underlying building activity.

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*This study represents a joint effort of Victor von Szeliski, Roy Wenzlick and Charles F. Roos. It was undertaken at the suggestion of Dr. Alexander Sachs when he was Director of the Division of Research and Planning of the National Recovery Administration. Max Sasuly and Victor Perlo of the Division made noteworthy contributions to the statistical technique. The study was presented to the Econometric Society at Boston, Massachusetts, December 30, 1933, as a joint paper by Victor von Szeliski and C. F. Roos. See *Econometrica*, April, 1934.

Conservative estimates indicate that there were three to five million workers directly or indirectly dependent upon the construction industry in the last decade. These are now for the most part out of work. The burden of unemployment caused by the virtual disappearance of activity in capital goods industries is too great to be shifted to consumers' goods industries, at least in a short period of time.

The purpose of this chapter is to elucidate the "causes" of variations in the volume of residential building and to measure their relative importance, thereby providing a reasoned and rational basis for national and sectional planning to smooth out the business cycle. The study leads to the conclusion that there are three main factors affecting residential building:

(a) Growth in the number of families, affecting demand for new accommodations.

(b) The net rental income on buildings as a per cent of the replacement cost. A ten per cent net rental return usually leads to a high level of construction activity; a five per cent return, to a very low level. In other words, the volume of building operations is largely determined by expected profits on the building.

(c) The supply of funds for investment, that is, the general credit situation, as measured by foreclosures. This is by far the most important factor affecting cycles in residential building.

The reader may inquire why the study is restricted to St. Louis. One compelling reason is that adequate measures of rent, volume of construction, occupancy and foreclosures are not available for any other city. Roy Wenzlick, after several years of assiduous research, has collected reliable, complete statistics for St. Louis. Without his research this study would have been impossible.

Another reason for confining attention to one city is that mechanisms are more readily revealed by microscopic analysis than by macroscopic; it is possible to analyze out cause and effect without having to depend too heavily on the hypothesis "other things being equal"; the measures are more nearly direct measures of a single quantity, instead of averages and aggregates of heterogeneous quantities of uncertain weight. The law of averages conceals as often as it reveals, and while it may sharpen the delineation of a statistical law or relationship, it may effectively bury the mechanism of which the law is but the measurement of the end result.

TABLE V

FACTORS IN RESIDENTIAL BUILDING IN ST. LOUIS, 1890-1933

Year	Family Accommodations Built	Number of Families	Increment in Families	Crude Replacement Rate	New Buildings Needed	Rent 4-Family Flat	Occupancy (Per Cent)	Rent × Occupancy	Taxes	Net Rental (8)-(9)	Cost Land + Building	Net Rental Yield	Bond Yield	Rental-Bond Yield Ratio	Foreclosure Rate
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
1890	2587	91,756					94.3								166
1891	3111						93.7								173
1892	3180						92.0								207
1893	2833						92.3								385
1894	2478						91.9								509
1895	2861						91.8								512
1896	1891						91.7								592
1897	2104					$ 964	92.0	$ 887	$102	$785	$ 8,400	0.0934			643
1898	1396					836	92.3	772	110	662	9,000	0.0736			610
1899	1390	120,519				626	92.8	581	107	474	9,200	0.0515			489
1900	1287	123,719	[illegible]200	0.004	3695	568	93.2	613	106	507	9,800	0.0517	0.0415	1.247	490
1901	1850	126,893	3174	0.004	3682	790	93.3	737	103	634	9,800	0.0647	0.0407	1.590	363
1902	2208	130,042	3149	0.004	3669	918	95.0	872	107	765	9,700	0.0788	0.0406	1.942	385
1903	2179	133,176	3134	0.004	3667	1049	96.3	1011	118	893	9,800	0.0912	0.0424	2.151	219
1904	3424	136,306	3130	0.004	3675	1159	98.3	1140	119	1021	10,000	0.1021	0.0423	2.412	168
1905	5128	139,441	3135	0.004	3693	1163	96.4	1121	120	1001	11,000	0.0910	0.0406	2.242	199
1906	6413	142,592	3151	0.004	3721	1085	96.0	1042	121	921	11,200	0.0822	0.0418	1.967	170
1907	5511	145,769	3177	0.004	3760	1086	94.2	1023	127	896	11,700	0.0766	0.0451	1.700	187
1908	6619	148,981	3212	0.004	3808	1079	93.8	1012	130	882	11,500	0.0767	0.0455	1.686	255
1909	6256	152,239	3258	0.004	3867	1006	92.4	930	130	800	11,500	0.0695	0.0433	1.606	276
1910	4897	155,555	3315	0.004	3987	1008	91.0	917	131	786	11,400	0.0690	0.0444	1.555	327
1911	4509	158,981	3426	0.004	4062	1062	90.6	875	132	743	11,400	0.0652	0.0443	1.472	418
1912	3636	162,538	3557	0.004	4207	931	90.5	843	138	705	11,500	0.0613	0.0446	1.376	441
1913	3378	166,184	3646	0.004	4311	1018	90.8	924	138	786	11,600	0.0678	0.0464	1.462	461
1914	3490	169,877	3693	0.004	4378	1044	91.2	952	144	808	11,500	0.0703	0.0458	1.535	591
1915	3455	173,574	3697	0.004	4391	1009	91.4	922	144	778	11,500	0.0676	0.0466	1.450	686
1916	2435	177,233	3659	0.004	4368	1008	91.1	918	144	774	12,100	0.0640	0.0453	1.412	675
1917	1077	180,811	3578	0.004	4301	1043	91.6	955	142	813	13,400	0.0607	0.0480	1.265	623
1918	196	184,267	3456	0.004	4202	1092	93.0	1016	156	860	14,500	0.0593	0.0523	1.134	497
1919	590	187,557	3290	0.004	4040	1117	94.0	1050	164	886	16,200	0.0547	0.0525	1.042	388
1920	585	190,640	3083	0.004	3842	1447	96.5	1396	180	1216	20,000	0.0608	0.0588	1.034	211
1921	1500	193,587	2948	0.0041	3751	1751	98.7	1728	207	1521	17,600	0.0864	0.0579	1.493	163
1922	3607	196,483	2896	0.0043	3748	1944	99.8	1940	210	1730	17,400	0.0994	0.0494	2.012	171
1923	5384	199,303	2820	0.0045	3729	2188	99.9	2186	207	1979	18,700	0.1058	0.0498	2.125	139
1924	5284	202,023	2720	0.0048	3692	2375	99.9	2380	216	2164	19,600	0.1104	0.0485	2.276	173
1925	8712	204,618	2595	0.0050	3635	2246	98.4	2204	238	1966	19,900	0.0988	0.0472	2.094	146
1926	7504	207,063	2445	0.0054	3556	2340	96.7	2255	242	2013	18,800	0.1072	0.0460	2.330	252
1927	5609	209,334	2271	0.0056	3455	2210	94.4	2070	256	1814	18,000	0.1008	0.0447	2.255	423
1928	7160	211,406	2072	0.0059	3329	2006	93.3	1872	270	1602	17,700	0.0904	0.0449	2.014	480
1929	4139	213,254	1848	0.0062	3177	2010	91.2	1830	300	1530	17,100	0.0898	0.0470	1.911	641
1930	1590	214,854	1600	0.0065	2997	1910	90.3	1710	300	1410	15,700	0.0898	0.0452	1.987	865
1931	1474	216,160	1805	0.0068	2769	1760	89.9	1565	300	1265	14,200	0.0891	0.0469	1.900	1040
1932	550	217,170	1010	0.0070	2535	1440	88.7	1260	270	990	13,100	0.0756	0.0584	1.295	1520
1933	390E	217,930	760	0.0073	2346	1150	91.0	1040	260	780	14,000	0.0557	0.0562	0.991	1950E

1940

2. *The Basic Data.* Attention is invited to Table V. which shows the number of family accommodations provided for in all the building permits issued in St. Louis from 1890 to 1933. A single family residence is counted as one unit and a twenty-four family apartment is counted as twenty-four units. Churches, industrial and commercial buildings are not counted unless they contain residential quarters.*

A detailed description of the data and sources will be found in Appendix III. The data were carefully gathered and were assembled scientifically and are believed to be quite reliable.

Column 2 shows the number of families, interpolated from Census figures, and Column 3 the first differences. Columns 4 and 5 represent values from other data as explained later on.

The sixth column gives gross rentals on a typical four family flat described below in relation to Column 11. The figures are averages of actual rentals on this type of building for 1924-1932, and estimates for 1897-1923, based on average asking rentals as advertised in newspapers. The per cent of dwellings occupied is shown in the seventh column. Figures for 1930-33 are based on actual surveys, and those for prior years are carefully estimated from a variety of indices. These indices invariably led to the same per cent of occupancy and checked with the actual survey figures. These compilations were also supplied by Mr. Wenzlick. Column 8 is the product of Columns 6 and 7.

Taxes (Column 9) are actual taxes for 1924-33 and estimated taxes for prior years obtained by linking on an index of taxes, which is the average real estate tax in St. Louis per family. This index agreed very well with actual figures for recent years. It was submitted to real estate students of long experience, who declared their satisfaction as to its substantial accuracy for the entire period.

Column 11 shows replacement cost for a typical St. Louis residential building for the period 1900-1933. This particular building was selected because it was probably duplicated in greater number with very slight variations than any other general type.† The specifications were changed several times during the period to give recognition to changes in materials available and differences in

*See *Real Estate Analyst*, October, 1932 and July, 1932.

†The variations in the cost of this building have been compared with the variations in the cost of the four other residential buildings which are reported regularly in the *Real Estate Analyst*. This comparison has demonstrated clearly that this building is typical (so far as cost fluctuations go) of residential buildings generally during this period.

TABLE VI

VARIATION IN CONSTRUCTION COST OF A FOUR-FAMILY FLAT

The chart on the page opposite shows the variations since 1907 in the cost of construction in St. Louis of the typical four-family flat which has been used in this study. On the chart the cost is separated into material and labor. The table below itemizes the material, labor and overhead costs in greater detail. Each column in the table is numbered and a brief description of the items included in each is given in the paragraphs below. Each paragraph is numbered to correspond with the column it describes.

MATERIAL

1. Cost of face brick, salmon brick, backing tile, flue lining and building stone.
2. Cost of all materials going into mortar, concrete, cement and plaster.
3. Cost of all lumber, flooring, millwork, roofing and paint.
4. Cost of all materials for plumbing, heating, electrical work, sheet metal work, iron work, hardware, tiling and accessories.
5. TOTAL MATERIAL COST.

LABOR

6. Cost of setting all stone, laying brick and pouring concrete.
7. Cost of labor on lathing and plastering.
8. Cost of carpentry, roofing, flooring, painting and builder's general supervision.
9. Cost of installing plumbing material and fixtures, wiring, heating plant and sheet metal work.
10. Cost of excavation, grading and landscaping.
11. TOTAL LABOR COST.

OVERHEAD

12. Cost of all city permits, city inspections and utility connection costs.
13. Cost of financing, interest during construction, insurance and sales commission on the building only.
14. Estimated profit made by the builder.
15. TOTAL OVERHEAD COST.
16. TOTAL COST OF CONSTRUCTION.

YEAR	MATERIAL					LABOR						OVERHEAD				TOTAL
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1932	$1563	$756	$2059	$1379	$5727	$1099	$273	$1193	$577	$185	$3327	$144	$1156	$750	$2050	$11104
1931	1698	709	2180	1480	6067	1361	322	1411	725	211	4030	140	1257	750	2147	12244
1930	1791	777	2492	1637	6697	1590	383	1617	733	280	4603	140	1335	900	2375	13675
1929	1811	877	2860	1723	7271	1610	490	1949	731	308	5088	140	1474	1000	2614	14973
1928	1859	896	2767	1890	7412	1732	560	1948	738	298	5276	140	1470	1250	2860	15548
1927	1893	921	2683	1899	7396	1777	608	1936	760	298	5379	140	1449	1300	2889	15664
1926	1977	937	2998	1972	7884	1791	613	1936	780	298	5418	140	1500	1500	3140	16442
1925	2157	988	3244	2025	8414	1870	698	1924	836	298	5646	140	1573	1750	3463	17523
1924	2161	1071	3256	2079	8567	1742	698	1910	767	298	5415	140	1552	1500	3192	17173
1923	2046	1039	3494	2055	8634	1345	585	1643	698	271	4542	140	1487	1500	3127	16303
1922	1954	983	3298	1774	8009	1308	585	1508	607	242	4250	140	1370	1250	2760	15019
1921	1980	1143	3338	2028	8486	1265	570	1581	615	242	4273	140	1398	1000	2538	15297
1920	2488	1263	4983	2596	11230	1230	534	1556	577	242	4139	140	1585	750	2475	17844
1919	2099	1072	3411	1909	8491	1306	520	1094	533	185	3638	140	1320	500	1960	14089
1918	1667	898	2835	1900	7300	1164	492	1012	487	185	3340	140	1260	500	1900	12540
1917	1478	724	2381	1817	6400	1152	474	924	486	168	3204	140	1190	500	1830	11434
1916	1337	546	2104	1457	5444	1030	469	844	462	169	2974	140	1075	500	1715	10133
1915	1231	571	1997	1201	5000	980	469	844	448	169	2910	140	995	500	1635	9545
1914	1176	571	1990	1161	4898	994	469	838	422	169	2892	140	995	600	1735	9525
1913	1350	554	1996	1089	4989	980	468	831	422	169	2870	140	1000	600	1740	9599
1912	1340	540	1946	1102	4928	980	468	830	411	169	2858	140	995	600	1735	9521
1911	1340	540	1943	1090	4913	980	468	810	411	169	2838	140	995	500	1635	9386
1910	1317	520	1954	1136	4927	980	467	790	411	169	2817	140	995	500	1635	9379
1909	1297	529	2050	1113	4989	910	467	784	411	169	2741	140	995	600	1735	9465
1908	1297	524	2019	1127	4967	910	466	783	411	169	2739	140	995	700	1835	9541
1907	1316	559	2076	1166	5117	910	465	783	411	169	2738	140	995	700	1835	9690

building practices. For example, during the past year conduit and BX replaced the knob and tube wiring formerly allowed. The kitchen drain boards in 1907 were wooden, the toilets had high wooden tanks, the bathtubs were on legs, the bath floors were not tile and the other floors were not hardwood. As most of the buildings of this type were built "open shop", labor costs were computed on what was actually paid, rather than on some scale which, at least in periods of depression, is only nominal. Overhead as here used includes cost of city permits, inspections, financing, interest during construction and estimated profit made by the builder. Details of Cost are given in Table VI.

Foreclosures were obtained directly from the recorder's books. They included all real estate property, apartments, houses, commercial buildings, etc. The number of foreclosures per year was expressed as a rate per 100,000 families.

It must be kept in mind, in connection with the foreclosure rate in St. Louis, that the laws of Missouri make it possible for the holder of a mortgage to take over the property twenty-one days after default, at an expense generally under one hundred dollars. This makes the St. Louis index far more sensitive to current credit conditions than would be the case in a state where a long period must elapse before the property in default can be taken over and where the expense of foreclosure is a sizable item.

3. *General Relationships between the Variables.* The number of new family accommodations (dwellings)* is shown in Chart XIII. It will be noticed that the data cover two complete cycles. The first cycle covers an interval of 18 years (1900-1918) and the second an interval of 15 years at least (1918-1933). The maxima are about seventeen times the minima, and upwards, an amplitude shown by very few other industries.

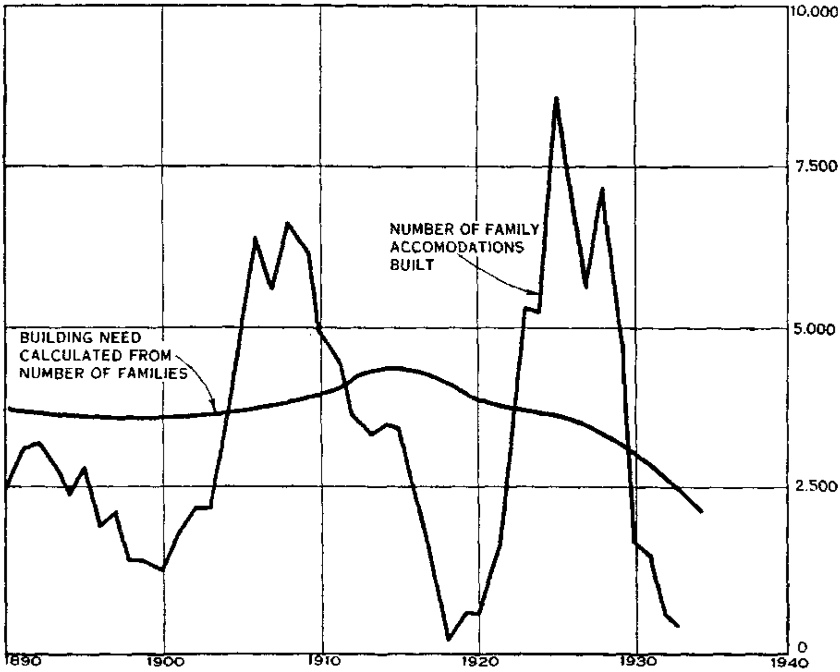
Chart XIV shows gross rents, that is, rents times occupancy, and net rents after taxes are deducted, replacement cost, and, for comparison, the volume of building.

Chart XV shows the foreclosure rates and the volume of building.

*As is fairly well known, monthly data for volume of new building constitutes what appears to be a random series; that is, one that might be obtained by drawing cards or by some other game of chance. Many of the random errors are connected ones, however, so that if a longer period of time is used as for example, a year, much of the randomness disappears. See Appendix II.

NEW RESIDENTIAL BUILDING IN ST. LOUIS

CHART XIII



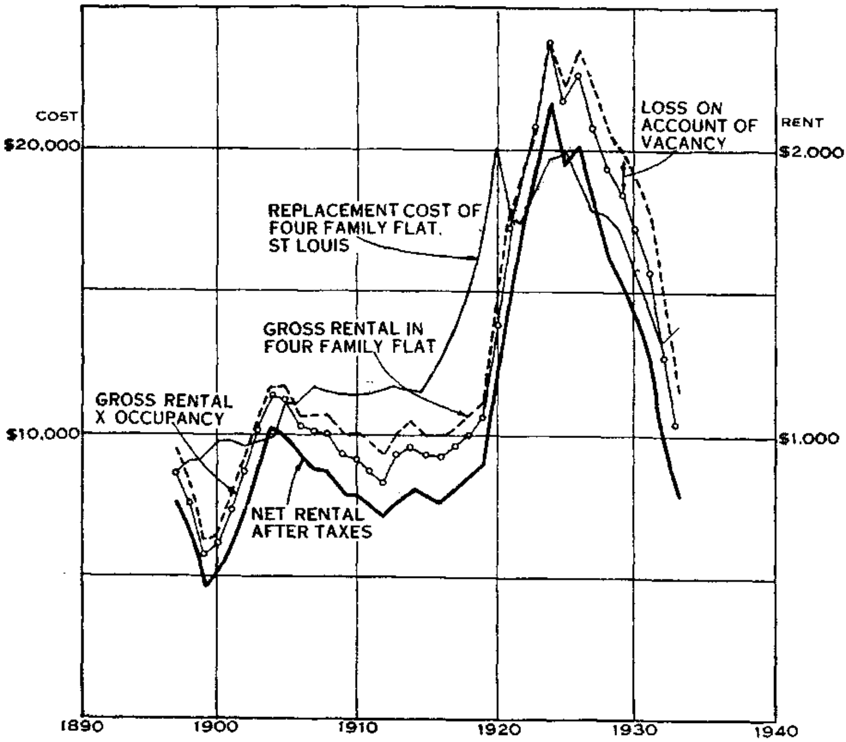
Residential building in St. Louis proper has gone through two major cycles since 1900. Physical building needs, as calculated from number of families, were fairly stable until 1920. Since then the rate of population growth has dropped sharply, with a consequent diminution of building needs.

It seems clear from even a superficial study of these charts that new building is influenced by cost of construction, rents, taxes, percentage of occupancy, and foreclosure rates.

Chart XIV shows that the replacement cost of the building described was about $10,000 at the turn of the century. From 1904 to 1907 it increased sharply to about $11,500 and then stabilized near that figure through 1915. Starting with that year, replacement cost began a new and very rapid advance, accelerating sharply to a pronounced peak in 1920. Thereafter, the cost dropped sharply during 1921 and 1922 and again rose, duplicating the 1920 peak in 1925. Then commenced a long downward sweep which culminated only when pre-war levels were reached in 1933.

RENTALS AND COST

CHART XIV



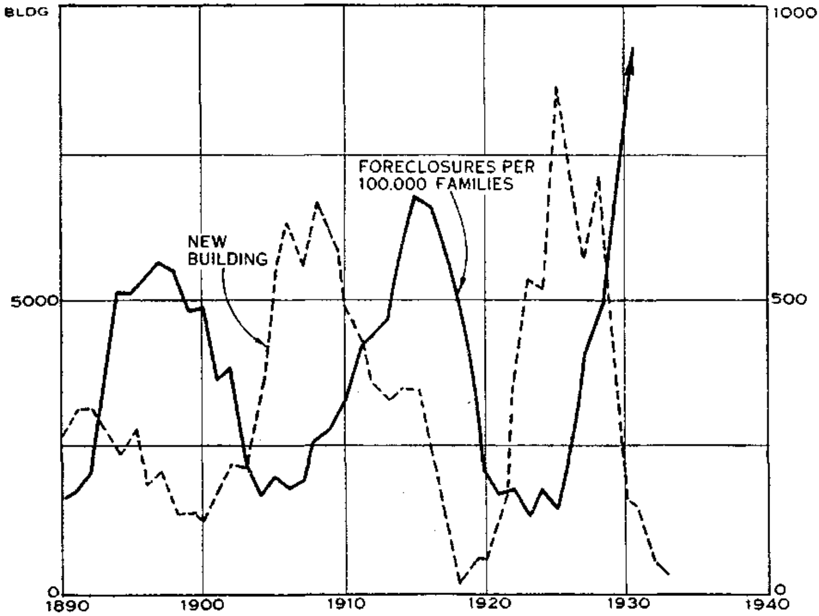
Gross rentals alone do not measure the income on property. Vacancies and taxes must be allowed for. The downward movement of rents, which started in 1926, has been aggravated by increasing vacancies and mounting taxes. Net rent after taxes is back to pre-war levels.

Replacement cost includes cost of lot, and overhead. Neither rent alone nor cost alone is significant, but their ratio is. Costs are significantly high or low only in relation to net rentals. Although rents did rise in 1918-1919-1920, costs rose so much more rapidly that demand for dwellings was choked off. But the equally high costs of 1924-25 did not affect building adversely because in the meantime rents had risen.

Cost and building volume must also be studied in conjunction with gross rental, rental and occupancy, and rents less taxes. According to Chart XIV rentals showed a sharp rise from 1900 through 1904, then a gentle sagging and stabilization until 1919. In the following year rents advanced sharply, again in 1921, and continued to rise, though less rapidly, in 1922, 1923 and 1924. In

FORECLOSURES AND NEW RESIDENTIAL BUILDING

CHART XV



The foreclosure rate, which measures the pressure of foreclosed properties on the market, and indirectly the supply of credit, is the chief determinant of the building cycle. The inverse movement of foreclosures and building is quite clear. Fluctuations in foreclosures are followed about two years later by inverse fluctuations in building.

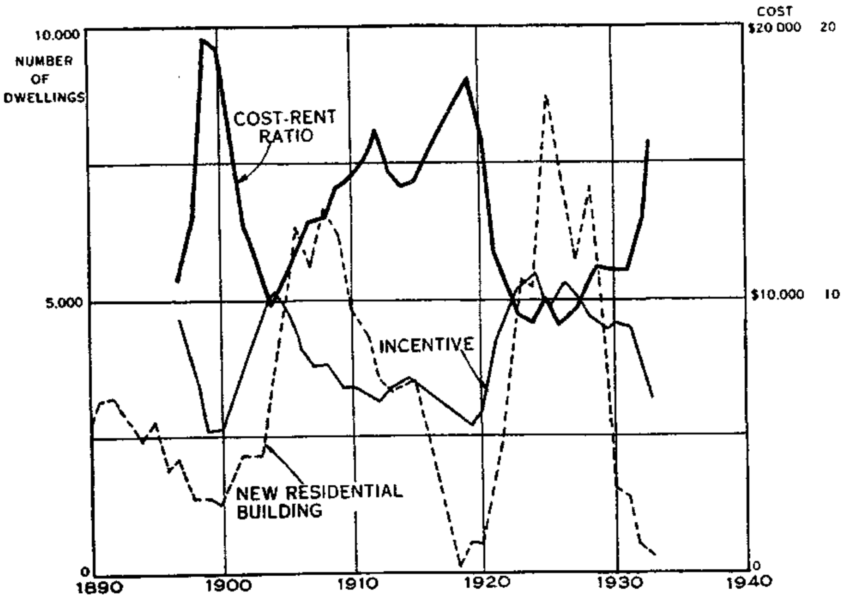
the year 1924 rents reached their peak, and thereafter they sagged slowly until 1930. With the onset of the depression the decline accelerated until in 1933 rents had dropped back to pre-war levels. Taxes show a gradual advance from 1900 through 1917, then a very persistent advance through 1929, then stabilization for two years, and finally a moderate reduction.

If these charts are studied and compared, certain general conclusions can be drawn which bear out the commonly accepted theories of the building cycle. The first boom, that of 1905-1909, was preceded by a considerable advance in rentals culminating in 1904. It was apparently brought to a stop by the persistent sagging of rentals through 1908-1910, combined with rising taxes and cost stabilized at the then high level of 1907. The sharp drop in building from 1915 to 1918 was clearly due to the sharp rise in replacement cost which was not compensated for in any way by a rise in rents (accentuated in 1918 by war-time prohibitions on building). Rents remained relatively stable. They did not start to rise until 1920, and in that year they were more than offset by a continued rise in replacement cost. It was not until rents had risen to still higher levels in 1921 and 1922 and cost had dropped in the same two years, bringing rent and cost once more into a favorable relationship, that building really got under way. It was the closing of the gap between cost and rent which generated the great building boom of the twenties. This is brought out more clearly in Chart XVI. Rents continued to rise until 1924, and attained for a short time a level which drew building up to a new all-time peak in 1925. Then the tide commenced to ebb; rents turned down while taxes continued their upward course, although these factors were offset to some extent by a moderate drop in replacement cost. The really ominous factor here was the increase in foreclosures which will be discussed below. By 1933 both rents and replacement cost approximated pre-war levels (a level which had proved only moderately stimulating in the pre-war years), but taxes continued at an extremely unfavorable level.

Chart XV furnishes a further explanation of the building cycle which reinforces the preceding analysis. The most important element is the rate of foreclosure. This has at least a three-fold influence. First, foreclosed houses are additions to the supply, so that prospective buyers may purchase a foreclosed house instead of building. Second, foreclosures affect the general value of buildings in the community in which they take place, and impair the equities and mortgages. Third, foreclosures are a measure of the state of the investment market, of the willingness of bankers to loan or to refrain from pressing debtors, of the very availability of credit itself from either banks or long term investors. This line has a distinct forecasting quality. Note that the low foreclosures of 1904-1907 correspond to the high building of 1906-1909; that the rising foreclosures from 1907-1916 correspond to the declining building of 1909-1918; that the high foreclosures of the years 1915-1917 were followed two years later by the low building of 1918-1920; that the very low foreclosures of 1921-1925 were followed by years of generally high building in 1923-1926; that the precipitous rise of foreclosures since 1925 and 1926, culminating in an all-time peak in 1933, is associated with the precipitous decline in building starting

COST OF CONSTRUCTION, INCENTIVE AND NEW BUILDING

CHART XVI



This chart shows the ratio of cost to net rental, and its reciprocal, which is a measure of incentive. Most building takes place after costs have declined to about 10 or 12 times net rentals, and the rental return on the investment (incentive) is between 8 or 10 per cent. Not much building takes place when costs are 18 to 20 times net rentals, and the incentive is only 5 or 6 per cent.
Building follows incentive and cost ratio by about 2 years on the average.
Although incentive remained high in 1929, 1930 and 1931, building dropped off because of the other and more important factor of credit supply, which contracted on an unprecedented scale.

in 1929, the end of which is not yet in sight. To summarize: high rents, low costs, low taxes and conditions measured by low foreclosures are favorable to building; and low rents, high taxes, high costs and conditions measured by high foreclosures are unfavorable to building.

While these graphical pictures certainly convey useful information, they do not synthesize into one result the action of each of the four factors, foreclosures, replacement costs, rentals, and taxes, and they do not enable one to say how important each factor is. A more intricate method of analysis is necessary, and although it is somewhat difficult to carry out, there is no escape if an adequate account of the situation is to be given.

3. *Building Need.* A quantity which is called "building need" for lack of a better term can be derived from the decennial Census data on the number of families, and the interpolated annual data. It is based on the assumption that there is, in the long run, approximately one dwelling per family. This quantity represents the long-term influence of population, as opposed to the cyclic forces of rentals, credit, and so on. The number of new family accommodations needed per year if each family is to have a dwelling is obviously the number of dwellings abandoned in a year plus the increase in the number of families.

It is possible to determine empirically the crude replacement rate (ratio of dwellings torn down or otherwise destroyed during a year to the number of dwellings at the beginning of the year) by summing the actual new buildings and setting this sum equal to the replacement rate multiplied by the sum of the number of families plus the sum of the change in the number of families. If b is the number of family accommodations needed per year and B is the number of family units built per year, D is the number of dwellings available for occupancy, r is the crude replacement rate, and m is the number of families, then over a long period of time, 1900-1933, the sum of $b =$ sum of $B = r \times$ (sum of m from 1900 to 1933) $+$ (m at 1933 minus m at 1900). This gives the crude replacement rate as .00439 approximately. But this is only an average value for the period. The replacement rate cannot be taken as a constant to determine the building need. Its value changes during the evolution of a population.*

Satisfactory values of the replacement rate cannot be computed without a knowledge of the survival curve for residential buildings and this curve is, unfortunately, unknown. The problem is rendered still more difficult by the probability that the life curve for dwellings changes with business conditions, and building technique, materials and equipment. The Gordian knot was cut, as far as this study is concerned, by putting the replacement rate equal to 0.004 during 1900-1920, and thereafter increasing it towards an assumed equilibrium value of 0.01 to be attained about 1960. (See Column 4 of Table V). The replacement rate was assumed constant through

*See A. J. Lotka, "Structure of a Growing Population," *Human Biology*, Vol. 3(4) 1931, pp. 459-493.

1920 because the rate of increase of the number of dwellings available for occupancy was fairly constant until that year. The rapid increase thereafter is by analogy with the rapid increase in the crude death rate in a population growing according to the logistic curve once the "knee" of the curve is passed, as brought out by Lotka.*

It is believed that this rough-and-ready determination of the replacement rate can be accepted as good enough, because the need for new building is determined mostly by the increase in the number of families. Building need is shown in Chart XIII.

5. *Building Incentive.* A capital goods industry (a family building unit produces the service of shelter) is stimulated by a profit incentive. A real estate investor will build when the yield on the investment is high enough. A renter will build when he can do so more cheaply than by continuing to rent. This desire to possess will, of course, not be gratified immediately since various inhibiting factors will appear. Thus, a renter may today desire to build his own home, but, because of the necessity for carrying out a lease contract, or because of numerous small causes, such as necessity for settling current accounts, "completing" the education of his children, and so forth, he may be unable to act immediately. Then, of course, there is the major inhibiting factor of the availability of funds for financing the construction activity.†

Construction experts say that the building incentive should on the average precede the actual building by two to four years,‡ and the financing factor by one to three years.

To find a measure of incentive the first step is to calculate the average gross rental per dwelling by multiplying rent by per cent occupancy. Taxes are then subtracted from the result. The remainder is net rental. Computations show that while gross rentals are still somewhat above pre-war levels, net rentals are below. The 1933 net rental was lower than that of any year but one since 1901. This is partly due to the high vacancies then existing, but mostly to high real estate taxes. Neither assessments nor rates were adjusted downwards in proportion to replacement cost.

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*See also Chapter VII.

†For a very few, the ease of financing does not act as an inhibiting factor. It must be remembered, however, that this study deals with mass reactions and not with any individual's reaction.

‡Four construction economists were interviewed on this matter. One suggested 2 years, one 2½ years, one 3 years, and one 3½ years.

Rental is income on some sum. The sum in question is, of course, the cost of the property on which the rental is received as return, so that rents, occupancy, taxes and current replacement costs may be combined in the following formula:

$$\frac{\text{Rent} \times \text{Occupancy} - \text{Taxes}}{\text{Cost}}$$

This is the rate of return on a real estate investment. It is return either to a landlord or to an owner living in the house who gets actual rental satisfaction.*

Furthermore, much property is mortgaged (about 40 per cent of all property) and there is, therefore, a further risk associated with the income, the chance of not getting the income because of foreclosure. How foreclosures, f, ought to enter into the determination of new building is not entirely clear, but close analysis reveals a likely assumption and statistical results verify the assumption.† At first glance, f appears to be of small importance; its average value is 0.005 and it was as low as 0.00139. This value would make the expectation of income equal to $0.99861 \frac{Rp - T}{C}$, where R is annual rent, P is percentage occupancy, T is taxes and C is replacement cost, a refinement hardly worth bothering about. However, this view is not correct, for

(1) The prospective builder is interested in avoiding foreclosure for more than one year; the probability of foreclosure for him is, therefore, not f, but nf, where n is an as yet undefined number of years.

(2) The money at risk is not a year's net rent, but the value of the property, or the capitalized value of rentals for many years in advance. Taking this at 15 times one year's net rent, the expectation of loss, neglecting interest, is $15\, nf(Rp - T)$, and the net incentive is $\frac{Rp - T}{C}(1 - 15nf)$. Assuming $n = 10$, $(1 - 15nf) = 0.25$, which is a factor not to be lightly disregarded.

*Not strictly. The occupancy factor does not apply to an owner who will himself live in the house he builds. The money yield in his case would be (rent — taxes) ÷ (replacement cost). However, some people argue that an important reason for owning a home is that a home owner does not have to pay for vacancies, so the hypothesis is not invalidated here.

† Economic theory rarely indicates a particular mathematical function. See footnote p. 26.

(3) Even (1) and (2) do not adequately describe the actual situation. They would if the choice were between building or not having a house or investment at all. But this is not so. The prospective builder can alternatively build, buy a house already on the market, put his money in government bonds or other securities, or invest it in a business. As a consequence, the coefficient of f cannot be determined *a priori* and indeed a statistical analysis indicates that f does not even enter linearly. Nevertheless, as a first approximation it is possible and permissible to take the expected income return E as

$$E = I(1 - Af) \ ,$$

where $I = (Rp - T)/C$ and A is a constant. To repeat, R = gross rental, p = occupancy, T = taxes, C = cost. E may be assumed to be the quantity which on the average affects the making of a decision to build.

Should not E be divided by the interest rate on long term bonds, on the hypothesis that it is the relative attractiveness of investment in building *versus* investment in bonds? This question was given careful consideration and the proposal rejected for the following reasons:

(1) The choice for the home owners in the low to middle income group is in general not home *versus* bonds, but home *versus* savings account, or continuing to pay rent *versus* building a house, or buying one already built.

(2) The decision for those in the upper income groups hinges chiefly on intangible values and the ability to finance.

(3) Many of those interested in real estate only from the investment angle confine their operations to real estate alone and do not consider bonds as a substitute.

(4) Building contractors who obtain financing for houses are not interested in bonds as a possible substitute since their profit comes from construction operations. Nearly three-quarters of all residential houses are built by speculative contractors.

(5) Since foreclosures are intimately related to general credit conditions and therefore to the bond market, the effect of bond yields, if any, is taken care of implicitly in the foreclosure functions.

(6) The really important financial factor is the availability of funds. This is not measured by interest rates.

6. *A Measure of Credit Inhibition.* It takes a little time after conditions actually become favorable before numbers of people realize that building is worth while. They build in proportion to the expected income unless financing is difficult at a later date. Besides this, there is building for other causes, by people who "just want a home anyway," and who can finance themselves if necessary. Therefore, it should be possible to write.

$$\begin{array}{l}\text{New} \\ \text{Building}\end{array} = \left\{\begin{array}{l}\text{Con-} \\ \text{stant}\end{array}\right\} \times \left\{\begin{array}{l}\text{Expected} \\ \text{Income}\end{array}\right\} + \left\{\begin{array}{l}\text{Building} \\ \text{due to} \\ \text{other causes}\end{array}\right\} - \left\{\begin{array}{l}\text{Building dis-} \\ \text{couraged by} \\ \text{financing diffi-} \\ \text{culties}\end{array}\right\} \tag{6.1}$$

This equation states simply that part of the new building is due to expected income and part to other causes, as, for example, building by people of means for self-gratification. As will be seen by the statistical analysis, these other causes appear to be remarkably constant and variations due to them can be safely neglected. Speculative builders, who are responsible for nearly three-quarters of all residential building, of course, build because they expect to sell at a profit, that is, at more than cost. Whether or not they expect to be able to sell at more than cost depends upon the expected rental income from the property, that is, a prospective buyer continues to rent until he sees a rental income advantage in owning a home.*

Now consider the inhibition of building by the withholding of credit or investment funds. As a first approximation this may be taken to be proportional to the foreclosure rate, increasing as the foreclosure rate increases and consequently decreasing as it de-

*This is not strictly true since there are intangible factors other than rental income that enter into home ownership, but, nevertheless, rental income is a very important factor and the equation has been written to take care of the "other causes."

creases.* The proportionality should not be expected to hold, however, at both extremes of foreclosures. That is, there will be some foreclosures even in prosperous times when there is no difficulty in financing and there will be financing (probably by owner's cash), no matter what the foreclosure rate is. As a first approximation, it might be expected that new building can be represented by a formula of the following type:

$$\left\{\begin{matrix}\text{Family}\\ \text{units}\\ \text{built}\end{matrix}\right\} = \text{Constant}\left\{\frac{\text{Rent}\times\left\{\begin{matrix}\text{fraction}\\ \text{of units}\\ \text{occupied}\end{matrix}\right\}-\text{Taxes}}{\text{Replacement Cost}}\right\}$$

$$\times\left\{1-\left\{\begin{matrix}\text{con-}\\ \text{stant}\end{matrix}\right\}\times\left\{\begin{matrix}\text{fore-}\\ \text{closures}\\ \text{per year}\end{matrix}\right\}\right\}$$

$$+\ \text{Constant}\dagger - (\text{constant}) \times (\text{foreclosures per year}) .$$

The constants are empirical coefficients depending upon the size and other characteristics of the city in question. They are to be determined by statistical methods.

In terms of mathematical symbols

$$B = A_0 \frac{Rp - T}{C}(1 - A_1 f) - A_2 f + A_3$$

where

R = rent in dollars
T = taxes in dollars
C = replacement cost in dollars
p = fraction of units occupied
f = number of foreclosures per unit time (year), per 100,000 families.

and A_0, A_1, A_2, A_3 are empirical constants.

*The interest rate has sometimes been used as a measure of the ease of financing. But this is not justified, for interest rates are closely regulated by laws and customs. The supply and demand for money cannot, therefore, reflect itself in price variations and must perforce work itself out as the offering or withholding of loanable funds. This is particularly true of St. Louis, where the mortgage rate has been six per cent for decades.

†Essentially building due to other causes.

The above equation can be simplified into

$$B = A_0 \frac{Rp - T}{C} - A_0 A_1 f \frac{Rp - T}{C} - A_2 f + A_3$$

The dimensions of the various quantities are as follows:

$B = MT^{-1}$	$R = T^{-1}$
T (taxes) $= T^{-1}$ (time)	$A_0 = M$
$C = 1$	$f = T^{-1}$
$p = 1$	$A_1 = T$
$(Rp - T)/C = T^{-1}$	$A_2 = M$
	$A_3 = MT^{-1}$

It seems preferable to recast the equation so that the A's will be independent of the size of the city. To accomplish this, write

$$(6.2) \quad B = b \left[A_0 \frac{Rp - T}{C} (1 - A_1 f) - A_2 f + A_3 \right]$$

where b is the quantity derived from the number of families in St. Louis.

The new A's can be obtained from the data by the method of least squares, or some other method of curve fitting. The fit can most conveniently be made to $B/b = \beta$.*

If simplicity only were wanted, a linear equation in the variables could be used, but a linear correlation between as many variables as B, b, R, T, C, P, f, (with trend eliminated) often leads to such inadmissible results as a positive coefficient for C, and the like. In many instances linear formulas are sufficient, but economic statisticians must break away from the general belief that the first two terms (constant and one other) of a Taylor's series (power series, $a_0 + a_1 x + a_2 x^2 + \ldots$) are always sufficient to represent a function. A method that is theoretically sounder is that of deducing the functional form from the conditions of the problem. This point is brought out more clearly by the problem here under consideration.

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*For a discussion of the question of the theoretical validity of time series correlations, see Appendix I.

The ratio, β, is a scalar quantity. The A's in the rewritten equation have the following dimensions.

$$A_0 = T = A_1 = A_2;$$
$$A_3 = 1$$

It will generally be found that equations written so that the coefficients to be empirically determined do not involve M are safer to use and "more rational" than those, the coefficients of which do not involve M.

Formula (6.2) gives a very satisfactory explanation of the interactions of the various factors if $[(Rp - T)/C][1 - Af]$ is led three years ahead of building, and A_2f is led two years. It is unsatisfactory, however, because for high foreclosure rates new building is negative. This is because of the A_2f term, and therefore f cannot enter linearly. A function $F(f)$ is needed which satisfies these conditions:

F very small for foreclosure rate of 10 per month (highest prosperity)
F increases slowly with f for f above 70 per month (70 represents the peak rate of previous depressions).
F approaches 1 for large values of f.

An expression of the type

$$F = 1 - h/f$$

will serve very well for the range of values which f takes, although it cannot hold for very small values of f.

The same remarks apply to the term $(1 - A_1f)$. It is best replaced by a function $W(f)$, of the type

$$W = (1 - g) + g(10)^{-Af} .$$

This approaches $(1 - g)$ as a lower limit for f large. The value $(1 - g) = 0.6$ has much to commend it, because 60 per cent of residential property is owned outright. It may be safely said, however, that as f increases, the investor, and even those who do not need to finance their homes by borrowing, become discouraged and put their money into other things; that is, high foreclosure rates dampen the incentive still further and affect the actions of those who are free of debt. Thus it is more nearly correct to put the lower power limit at .10 instead of .60* A relationship involving f that takes these requirements into account is

$W = 1 - .9(1 - 10^{-Af})$, where A is a constant and f stands for the number of foreclosures per year per 100,000 families.

*The lower limit .10 was decided upon after consultation with building experts. It was not decided upon by a least square determination or other manipulation involving curve fitting. The "best" values are not necessarily those determined by a least square fit to the observations, particularly in time series analysis. Other facts, even of a non-quantitative character, may properly play a part in determining an empirical equation.

When f is very large, 10^{-Af} is almost zero, and W is approximately 0.10. Furthermore, when $f =$ zero, $10^{-Af} = 1$ and $W = 1$. The expression for W can be written more simply as

$$W = .10 + .9(10)^{-Af} \ ,$$

Finally, in order to allow for a possible non-linear relationship between I and the other variables, an exponent α may be added. Thus the equation finally assumes the form

$$B = b(A_0 I^{\alpha} W - A_2 F + A_3) \ . \tag{6.3}$$

7. Method of Solution by Iteration. Values for the empirical constants may be obtained by a process of iteration.*

Initially, α may be taken equal to 1, and $W(f)$ and $F(f)$ taken as linear functions of f, as in equation (6.1). The lag between IW and B may be taken as three years, and between F and B as two years. Values for A_0, A_1, A_2, and A_3 may be obtained by the method of least squares.

Next, the residuals

$$\left[\frac{B}{b} - A_0 \frac{Rp - T}{C} (1 - A_1 f) - A_3 \right] + A_2 \tag{7.1}$$

may be calculated, and plotted against f. Analysis of the residuals leads to the formula

$$F = 1 - \frac{129.6}{f} \ .$$

New values may then be calculated for A_0, A_1, A_2 and A_3 by the method of least squares, the new F replacing the old linear relationship in the equation

$$B/b = \beta = A_0 I \ (1 - A_1 f) - A_2 F + A_3 \ , \tag{7.2}$$

Then (7.2) may be rewritten in the form

$$[\beta + A_2 F - A_3]/A_0 I = \text{function of } f. \tag{7.3}$$

*The method of successive iteration has been used extensively in the Bureau of Agricultural Economics of the U. S. Department of Agriculture; see Mordecai Ezekiel, *Methods of Correlation Analysis*, 1930. The validity of the process of iteration for solving normal equations was proved by Gauss and Seidel in 1847. See Whittaker and Robinson, *Calculus of Observations*, p. 255. The use of iteration methods for obtaining constants entering non-linearly, and for obtaining general graphical functions by the analysis of residuals, requires careful examination.

It is found that the quantity $W_1 = .10 + .9(10)^{-Af}$ with $A =$ 0.0006515 can be used to represent the function of f on the right-hand side of this equation.

The iteration process may be carried further by adding exponents to the I and W_1 factors, namely $\alpha = 0.86$ for I and $\gamma = 0.7$ for W_1. To determine these exponents the equation may be transformed to logarithmic form and α and γ determined by the method of least squares.

At this point the iteration process could be interrupted to re-examine the lag relationship. The quantity $A_0 I^{\alpha} W_1^{\gamma}$ plotted as a time series on transparent paper may be carefully compared with $(\beta + A_2 F - A_3)$. The curves are most nearly congruent when the lead of $I^{\alpha} W_1^{\gamma}$ over β is two and a half years instead of three years as originally assumed. The two year lead of $F(f)$ over β is confirmed by a similar test.

I_{t-3} may now be replaced by $I = \frac{1}{2}(I_{t-3} + I_{t-2})$ and W_1 (f_{t-3}) by $W_1 = W_1(\frac{1}{2} f_{t-3} + \frac{1}{2} f_{t-2})$.

A redetermination of the various constants gives the result

$$\beta = 16.63\, I^{.86}\, W_1^{.7} - 1.022F + 0.20 \;.$$

At this point it is desirable to replace the complicated expression $W_1^{.7}$ by a simpler one. For this purpose the residuals

$$(\beta + 1.022F - 0.20)/16.63 I^{0.86}$$

may be plotted against the quantity $f_1 = \frac{1}{2}(f_{t-2} + f_{t-3})$. By changing the coefficient A of f in W_1 to .00045, a new function W is found that differs only slightly from $W_1^{0.7}$.

Values of the functions

$$W = 0.1 + 0.9 \times 10^{-0.00045 f_1} \text{ and } F = 1 - 129.6/f$$

are shown in Table VII.

The formula

$$B = b[16.63\; I^{0.86}\, W - 1.022F + 0.20] \qquad (7.4)$$

gives a satisfactory representation of actual residential building as shown in Chart XVII and, therefore, the iteration process need not be carried further.*

Table VIII shows calculated building, actual building and the residuals, actual and calculated. The average residual is 585, or 0.163 of the average number of dwellings constructed. The standard deviation of the residuals, σ, is 730. The standard error of estimate cannot be computed because the observations are not independent. Until some way is found to determine the number of independent observations for time series, any attempt to relate correlation studies of time series to the theory of probability are futile since they must be based on arbitrary assumptions regarding the number of independent observations.† In view of this unsettled state of theory, it seemed undesirable to attempt to estimate the probability of getting any particular residual.

It is possible that the residuals are due to inadequate complexity of the law assumed and in part to inaccuracy of data, but, as already pointed out, great care was taken to obtain accurate data. It is extremely interesting to note, however, that certain of the residuals can be accounted for by factors which were necessarily disregarded in the analysis. Building was larger than calculated in 1914-1915. It appears that in these years certain St. Louis industries, such as the leather industries, were sharply stimulated by war orders. Factories were built on the outskirts of the city, and workmen's houses were built nearby. There was a sharp increase in the b (building need) in particular areas of the city, which is not measured by the b for the city as a whole. Building became profitable in the affected areas, because the volume of space rentable in these areas at the going price increased. In other words, the demand curve changed suddenly. In 1918 industrial activities not related to war purposes were discouraged by the Government. In 1928 a feverish expansion of credit took place as a result of the "easy money" policy. It manifested itself in a stock market boom which furnished windfall income to a larger sector of the American population than ever before. It stimulated purchases of consumers' goods, and, as seems to be indicated here, it also stimulated residential building. Actual building was about 35 per cent higher than calculated. The large negative residuals in 1931, 1932 and 1933 may

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*As good a fit as obtained here is seldom found in the domain of economics. The verification of hypothesis found here compares favorably with verifications of many physical hypotheses.

†Appendix III is devoted to a study of random errors and random alternating errors. This is a related problem.

TABLE VII

Foreclosure Functions

Foreclosures per 100,000 families	Inhibitive effect of foreclosures on capital supply	Damping effect of foreclosures on incentive
f	*F*	*W*
100	–0.296	.911
110	–0.178	.903
120	–0.080	.895
130	0.003	.887
140	0.074	.879
150	0.136	.870
160	0.190	.862
170	0.238	.854
180	0.280	.847
190	0.318	.839
200	0.352	.832
225	0.424	.813
250	0.488	.794
275	0.259	.777
300	0.568	.760
325	0.601	.743
350	0.630	.726
375	0.654	.710
400	0.676	.695
425	0.695	.680
450	0.712	.664
475	0.727	.650
500	0.741	.636
550	0.764	.609
600	0.784	.583
650	0.801	.559
700	0.815	.536
750	0.827	.514
800	0.838	.493
850	0.847	.473
900	0.856	.454
950	0.864	.437
1000	0.870	.419
1100	0.882	.388
1200	0.892	.358
1300	0.900	.334
1400	0.908	.311
1500	0.914	.290
1600	0.919	.271
1700	0.924	.256
1800	0.928	.240
1900	0.932	.226
2000	0.935	.213

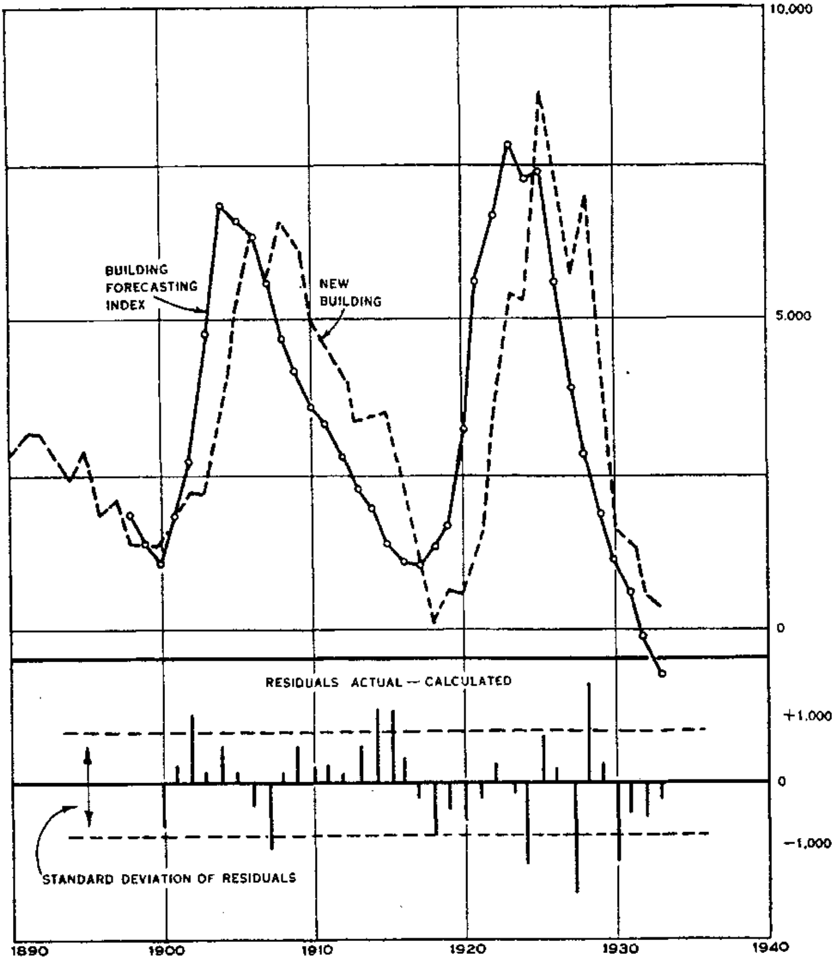
TABLE VIII

Year	$\dot{B}$ Actual	$\dot{B}$ Calc.	B–B' Residuals	f_{t-2}	F^* Residuals	F_{t-2}	$\bar{f}_{t-2}$	W^* Residuals	$\overline{W}$	I^* Residuals	$\bar{I}$	$\dot{\bar{I}}^{.86}$ Incentive	$\bar{I}^{-1}$ Cost Ratio
	1	2	3	4	5	6	7	8	9	10	11	12	13
1900	1287	1951	—664	610	.962	.785	626	.480	.572	.099	.118	.0835	11.98
1901	1850	1431	419	489	.625	.737	550	.686	.609	.103	.092	.0626	15.97
1902	2208	1041	1167	490	.454	.739	490	.867	.642	.105	.078	.0516	19.38
1903	2179	1961	218	363	.582	.641	426	.721	.678	.093	.087	.0582	17.18
1904	3424	2770	654	385	.493	.665	374	.811	.710	.119	.104	.0718	13.93
1905	5128	4881	247	219	.318	.385	302	.790	.760	.125	.120	.0850	11.76
1906	6413	6857	—444	168	.342	.223	193	.781	.838	.126	.134	.0966	10.35
1907	5511	6605	—1094	199	.602	.316	183	.710	.842	.113	.134	.0966	10.35
1908	6619	6381	238	170	.170	.231	184	.870	.842	.126	.122	.0866	11.55
1909	6256	5737	519	187	.178	.309	178	.920	.848	.122	.113	.0794	12.59
1910	4897	4676	221	255	.442	.496	221	.844	.813	.114	.110	.0766	13.05
1911	4509	4179	325	276	.451	.531	266	.827	.782	.111	.105	.0731	13.68
1912	3636	3571	65	327	.587	.602	302	.767	.757	.101	.100	.0692	14.45
1913	3378	2853	525	418	.572	.691	372	.787	.712	.108	.098	.0674	14.84
1914	3490	2295	1195	441	.443	.710	430	.851	.675	.117	.093	.0635	15.75
1915	3455	2230	1215	461	.451	.721	451	.840	.662	.119	.094	.0645	15.50
1916	2435	1961	474	591	.673	.779	526	.690	.622	.111	.100	.0690	14.49
1917	1077	1384	—307	686	.881	.810	638	.523	.566	.092	.100	.0690	14.49
1918	196	1089	—893	675	1.016	.807	680	.416	.546	.073	.096	.0658	15.20
1919	590	1053	—457	623	.902	.790	649	.488	.561	.080	.092	.0623	16.05
1920	585	1314	—729	497	.926	.740	555	.480	.607	.069	.088	.0593	16.86
1921	1500	1724	—224	388	.726	.667	442	.627	.668	.079	.084	.0562	17.79
1922	3607	3293	314	211	.309	.390	300	.817	.760	.091	.085	.0576	17.36
1923	5384	5529	—145	163	.234	.200	187	.820	.841	.103	.106	.0736	13.59
1924	5284	6689	—1405	171	.611	.239	167	.680	.856	.103	.130	.0929	10.76
1925	8712	7876	836	139	.160	.065	155	.961	.866	.156	.141	.1026	9.75
1926	7504	7327	177	173	.201	.250	156	.883	.863	.150	.147	.1079	9.27
1927	5609	7465	—1856	145	.627	.103	159	.646	.862	.106	.142	.1036	9.65
1928	7160	5528	1632	252	.011	.489	198	1.041	.833	.176	.141	.1023	9.78
1929	4139	3931	208	423	.635	.695	338	.757	.732	.148	.143	.1046	9.56
1930	1590	2857	—1267	480	1.145	.730	452	.472	.660	.097	.136	.0980	10.20
1931	1474	1897	—423	641	.950	.798	560	.531	.604	.114	.129	.0919	10.88
1932	550	1052	—502	865	1.045	.850	753	.419	.514	.103	.126	.0901	11.10
1933	300	527	—227	1040	.975	.878	952	.395	.437	.114	.126	.0905	11.05
1934		—132		1520	.80	.920	1280		.340		.118	.0835	11.98
1935		—707		1950	.86	.938	1735		.252		.097	.0665	15.04

$$F^* = \frac{-\beta - A_1 \bar{I}^{0.86}\,\overline{W} - A_3}{-A_2} \quad I^* = \frac{\beta + A_2 F - A_3}{A_1 \overline{W}} \quad W^* = \frac{\beta + A_2 F - A_3}{A_1 \bar{I}^{0.86}} \quad \bar{f} = \tfrac{1}{2}(f_{t-2} + f_{t-3}) \quad \bar{I} = \tfrac{1}{2}(I_{t-2} + I_{t-3}) \quad \overline{W} = W(\bar{f})$$

NEW BUILDING, ACTUAL AND FORECAST

CHART XVII



The composite building forecasting index combines the effects of building need (Chart XIII), incentive (Chart XVI), and foreclosures (Chart XVII). It precedes actual building by about two years. It represents the combined effect of number of families, rentals, occupancy, taxes, cost of construction, and credit supply. The standard deviation of the residuals is 790.

The formula gives negative values for building in 1934 and 1935, which are, of course, absurd. This is due to foreclosures in 1931 and 1933 going far beyond the range of previous observations.

be due to the failure of foreclosures to measure the full extent of the drying up of the supply of capital funds. The credit deflation which took place in those years was without precedent in American history. The sources of short-term credit, as well as of long-term credit, dried up.

8. *Forecasting New Building.* The composite index of new building precedes actual building. It is, therefore, a forecasting index. It must be realized, however, that the index was worked out for the more or less *laissez faire* conditions that prevailed in the past and that it will fail to work if important new factors are introduced. It is particularly valuable to tell what can be expected if nothing is done and to point out things that can be done.

On the basis of

$I(31) = .0902$	$f = 1280$
$I(32) = 0.768$	$W = .340$
$I(33) = 0.835$	$b(34) = 2200$
$f(31) = 1040$	$F(32) = .915$
$f(32) = 1520$	

there will be —132 units built in 1934, compared with about 300 in 1933. The interpretation of a negative value would be that houses destroyed by fire would not be replaced, or that buildings would be wrecked to escape taxation, and not replaced. Needless to say, the writers do not wish to be put in the position of making any such prediction. There is a simple explanation: the f's for 1931 and 1932 are far outside the range of previous observations. The formula can, of course, be revised so that negative values will not occur, but the worthwhileness of a revision is questionable. Mathematically the problem is simply one of choosing proper boundary conditions. The estimate is sufficient to indicate how bad conditions really are.

The use of two and two and a half year lag relationships puts one in the curious position of asserting that, while what happened in 1932 was very important in 1934, what happened in 1933 cannot have any effect at all, in fact, cannot be relevant. Yet it is quite certain that it is relevant. The lag used is merely an average one, as will be explained in a later section. The Government is injecting new forces into the situation which may materially shorten it or lengthen it. The shutting off of building by credit conditions has been so extreme that the loosening of those conditions might release the forces of incentive much sooner. The national income is being increased by national expenditures. Building projects such as slum improvement, which are independent of normal incentives and wholly independent of local credit conditions, are now being initiated by the Public Works Administration.

As already pointed out, all economic formulae must be used with discretion in forecasting work. The formulae are valid only so long as important new factors are not injected. Nevertheless, they can be used to measure the effects of the new factors and determine their importance. They thus furnish the important service of providing a norm for comparative purposes.

7. *Interpretation of the Partial Relationships.* The formula for volume of construction is

$$(9.1) \qquad B = [16.63\, I^{0.86}\, W - 1.022\, F + 0.207]\, b \ ,$$

where $b = rm + \Delta m$ and r at present $= 0.0073$ approximately, and $m = 217{,}000$. It is assumed that the equilibrium value of r is 0.01. If the steady state value of m is, say, 240,000, the required number b of dwellings would be 2,400, against an annual average of 3,586. That is, with a stable number of families 10 per cent greater than now, and 44 per cent greater than the average during 1900-1933, new residential building will be 33 per cent less. This is an exceptionally clear illustration of the point brought out so well by J. Maurice Clark*, that activity in capital goods industry varies with the *derivative* of consumer demand.

Unless the standard of living improves consistently over a period of years, there is clearly not so much *physical need* of new building in the future as in the past. Put in another way, the outlook for population growth being for continued slowing up, as cogently argued by Louis I Dublin and A. J. Lotka in their studies on that subject, residential building will depend more on raising the standard of living and increasing the wants of individuals, so that greater depreciation of existing structures will occur. An alternative is to invite immigration.†

The effect of credit supply as measured by foreclosures is exhibited in the accompanying chart. Foreclosures are represented as

*J. M. Clark, *Economics of Overhead Costs*, Chicago, 1923.

†In Porto Rico and other countries where poor housing standards exist increased population means dividing up existing poverty. In the United States the building of new houses gives an increased velocity of circulation to money media and a higher standard of living for all.

abscissae and new building as ordinates with the other variables held constant, that is, the variables I and W were replaced by the observed values, and transferred to the left hand side of the equation

$$(9.1) \qquad \beta = 16.63\, I^{0.86}\, W - 1.022\, F + 0.207 \ ,$$

which becomes

$$-(\beta - 16.63\, I^{0.86}\, W - 0.207)/1.022 = F + e,$$

where e is random errors. The comparison shows that the fit of the F function is fair.

The influence of a change in the foreclosure rate is apparently negligible for $f > 500$. Investors are no more inclined to advance money when $f = 500$ than when $f = 2{,}000$, as now. But if the foreclosure rate is brought much below 500 the inhibitions on capital flow disappear, and if $f < 200$ capital flows freely indeed. As shown, the inhibition function F becomes positive for $f < 130$, which means, presumably, that when investment funds are very abundant — available for the asking — people build houses for no incentive in the financial sense at all.

New building adjusted for population and incentive changes may be symbolized by $\beta(IW)$. It still remains subject to credit supply. Adjusted building is given by

$$(9.2) \qquad \beta(IW) = -1 + 129.6/f_{t-2} \ .$$

Its derivative is

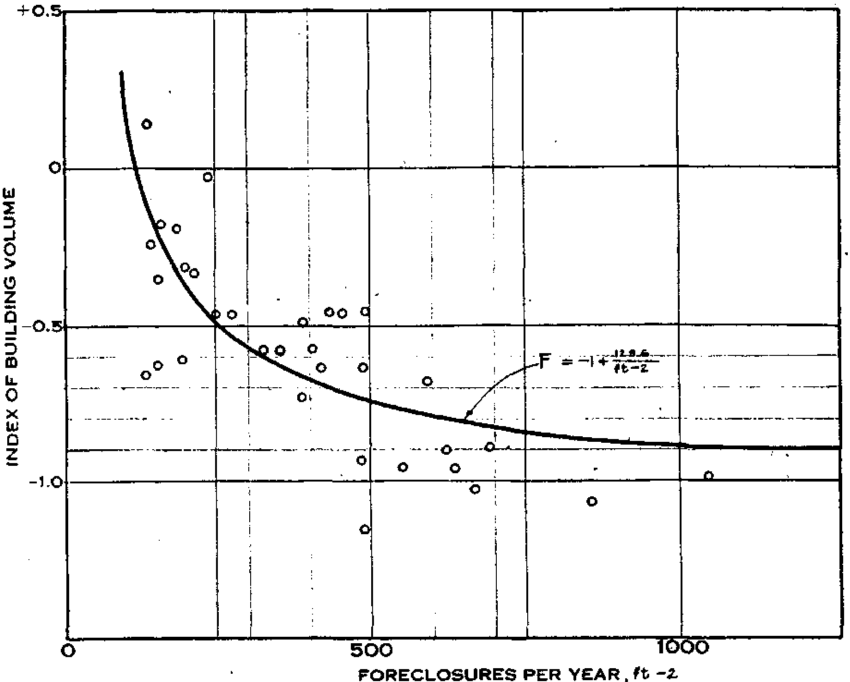
$$(9.3) \qquad \frac{\partial}{\partial f}\beta(IW) = -\, 129.6/f^2_{t-2} \ .$$

The differentiation here has been made on the assumption that f and C are independent. This is probably not strictly true, but, nevertheless, the results obtained by making this assumption (the relation between f and C is not known) probably give something like a true picture.

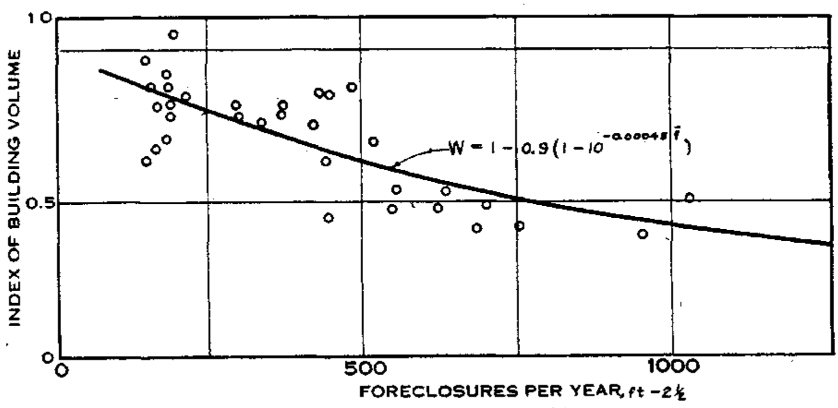
This shows how much of a change in building can be brought about by improving the foreclosure situation. It brings out the same thing shown in the chart; namely, that must get quite small before building can be much affected. In other words, a great deal must be done about the factors measured by foreclosures before building can get under way in a reasonably short time.

DEPENDENCE OF BUILDING ON CREDIT CONDITIONS

CHART XVIII



THE INDIRECT OBSERVATIONS $\frac{-(B - A_3 - A_1 I^{-0.64}\,\overline{W})}{A_2}$ PLOTTED AGAINST FORECLOSURES

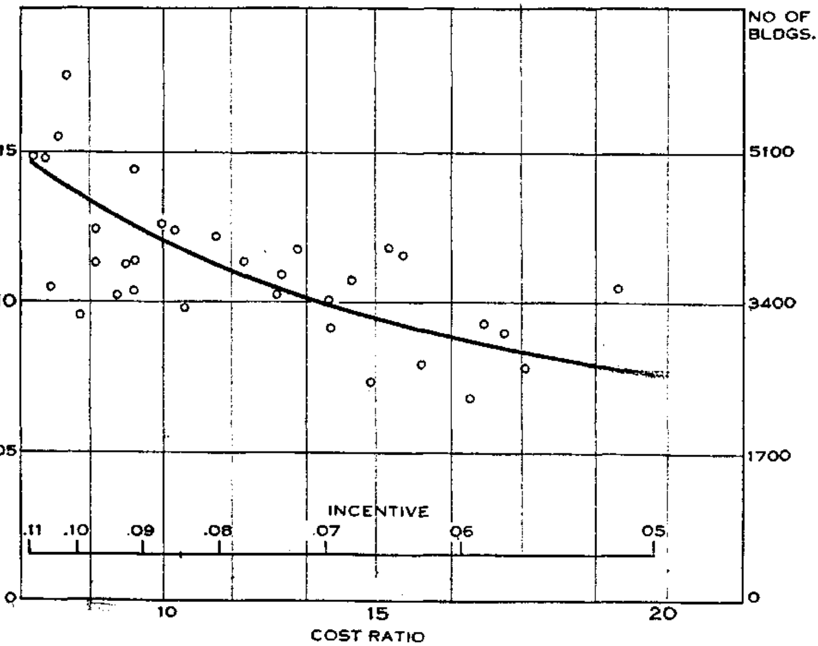


THE INDIRECT OBSERVATIONS $(B - A_3 + A_2 F)/A_1 I^{-0.06}$ PLOTTED AGAINST FORECLOSURES.

These charts are designed to show partial relationships between building and each factor. The upper one shows building with population effects, and previous incentive and foreclosure effect on incentive, held constant, plotted against foreclosures of two years previous. The lower shows building with population and previous foreclosure effect on credit supply held constant, plotted against average foreclosures of two and three years previous.

PARTIAL RELATIONSHIPS BETWEEN VARIABLES NEW BUILDING, AND COST RATIO

Chart XIX



Building volume adjusted for constant previous foreclosures, compared with cost/rent ratio. High cost/rent ratio definitely decreases volume of building.

The damping effect of foreclosures on incentive is shown on Chart XIX. Here the relationship is practically linear. A considerable effect may be secured by reducing high foreclosures, almost as much as can be effected by a similar reduction at the low end of the foreclosure scale. If building, adjusted for population, incentive and credit supply, but still subject to the damping effect of high foreclosures on incentive (as distinct from the effect of high foreclosures on credit supply), be denoted by β (IF), then

$$\beta(IF) = 1 - 0.9(1 - 10^{-.00045f}) \qquad (9.4)$$

$$= .1 + 0.9e^{-0.01037f} \ ,$$

$$\partial \beta / \partial f = -0.01037 \, e^{-0.01037f} \ . \qquad (9.5)$$

The effect of incentive changes on new building may be obtained by adjusting building for variations in foreclosures, both in their incentive damping action and in their credit supply capacity. The adjusted building denoted by $\beta(WF)$ is given by

$$\beta(WF) = I^{0.86} = \left(\frac{Rp - T}{C}\right)^{0.86}$$

$$\partial\beta/\partial I = 0.86\ I^{-0.14}$$

The size of the exponent, 0.86, shows that a doubling of the incentive will not quite cause a doubling of the corrected volume of building, but its influence is very real. The partial derivative of β with respect to I shows that more building results from raising a low incentive to a moderate incentive, than from raising a moderate incentive to a high one.

The effect of gross rents, taxes, occupancy and cost can also be deduced:

$$\beta(W\ F\ p\ T\ C) = R^{0.86}$$
$$\beta(W\ F\ R\ T\ C) = p^{0.86}$$
$$\beta(W\ F\ R\ p\ C) = T^{0.86}$$
$$\beta(W\ F\ R\ p\ T) = C^{-0.86}$$

The last is especially interesting as showing the partial relationship of building to cost. Halving the cost will not quite double the corrected volume of building. Building is inelastic in relation to cost because quite a volume of building takes place without regard to cost and its reciprocal, incentive. Cost is only one of a number of items affecting volume of residential building. It is not cost that is important, but rather the ratio net rentals, i.e., rents times occupancy minus taxes, to cost that is important. If rents advance, costs can also advance without diminishing volume. However, the building industry must make certain that rents are advancing before raising costs. Although much residential building has been done at high cost it does not follow that increasing costs will increase volume of building. Building seems to get under way when the ratio of net rent to cost is about eight per cent per annum, and, as previously pointed out, most building takes place on a basis of cost $= 10 \times$ net rentals.

10. Functional Influence of Incentive. Suppose that demand depends only upon cost two and a half years in the past. Then, obviously, if a monopolist intended to remain in business for only two and a half years, the expression obtained for new building as a function of rent, taxes, replacement cost, occupancy and foreclosures (measuring the damping of incentive) led two and a half years and foreclosures (measuring credit) led two years is sufficient to give a picture of a relationship existing between the variables and to illustrate how long leads in the industry affect the demand. It can be shown very readily, however, that a demand (or supply) equation representing an average incidence of past effects in time (a time lead) is not sufficiently precise to make it possible to predict future courses of the variable in which there is no lead (new building) with any great degree of accuracy, especially if one or two of the leading variables are greatly changed. Furthermore, it is not possible to solve a fundamental problem as that of determining cost* so that income to the industry (building industry) will be a maximum over some period of time. Only a short mathematical analysis is needed to show that this problem is impossible when the demand is given in terms of average lead.† Impossibility of such a problem can also be shown by practical considerations. Thus, if the demand depends only upon cost two and a half years in the past, a monopolist (a similar mathematical difficulty arises for competitors), if he intended to remain in business for only two and a half years, could charge as much as he pleased for that period, for his costs during the period would not affect the demand. The real trouble, of course, is that the demand equation, employing an average time lag as it does, is too rough an approximation. Averages are all right in themselves provided it is not desired to analyze what is behind the average. In the case under consideration here, it is just this analysis which is desired. The simple formula (7.4) does not adequately represent the situation.

The problem of arriving at a theoretical formula for the demand for capital goods is in itself somewhat difficult. Its theoretical aspects have, however, already been discussed in the preceding chapter and hence only a short presentation is necessary here to indicate how the theory should be applied.‡

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*Costs can be lowered by lowering the cost of raw materials, by lowering freight rates, which are part of the cost of raw materials, by lowering nominal wage rates, or by inventing new methods for pre-fabricating houses.

†See Appendix VI.

‡See also C. F. Roos, "Theoretical Studies of Demand," *Econometrica*, Vol. 2, 1934.

For brevity let

$$E(t) = \left(\frac{Rp - T}{C}\right)^{.86} W \ ,$$

where W is the function of f previously defined.

Suppose that G_i individuals come into contact with an incentive $E(t_i)$ at the time t_i. Some of these individuals decide to act on the incentive to the extent of building $N(t_i)$ units in the future. The number of units that will result from such decisions (without inhibitions which can be discussed separately) may be assumed to be proportional to $E(t_i)$; that is,

$$N(t_i) = aE(t_i) + b,$$

where a and b are constants.

There are many factors affecting the length of time required for individuals to act on an incentive. Most of these factors are of the same relative importance, but a few, such as credit, have important special influences. For the many small factors, it can be assumed that the time frequency distribution of those acting on the incentive will obey the normal law.* In the case of new building the peak of the distribution may be assumed to occur about M years after the stimulus or incentive, i.e., M years after t_i, so that the number ψ reacting per unit time at time t may be taken to be approximately

$$\psi(t_i, t) = \frac{1}{\sqrt{2\pi}\,\sigma} \, e^{-(t-t_i-M)^2/2\sigma^2} \, (aE(t_i) + b) \ .$$

The number built (without corrections for special major inhibitions) per unit at any particular time t will be the sum of the number of units built as a result of reactions to each of the stimuli given at $t_0, t_1, t_2, \cdots, t_{i+1}, \cdots, t_n$; that is,

$$\psi(t) = \sum_{i=0}^{n} \psi(t_i, t)$$

$$= \sum_{i=0}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} \, e^{-(t-t_i-M)^2/2\sigma^2} \, (a_1E(t_i) + b_1) \ ,$$

where $i = 0$ refers to the first time at which a stimulus or incentive was given. More exactly,

*For a discussion of the normal law of errors, see Whittaker and Robinson, *Calculus of Observations*, pp. 168-175.

$$\psi = \int_{-\infty}^{t} \frac{(a_1 E(x) + b_1)}{\sqrt{2\pi}\,\sigma_1} e^{-(t-x-M_1)^2/2\sigma_1^2} dx ,$$

where x has been used to designate time $t_0, t_1, \cdots, t_{i+n}$.

Evidently, observations on E and ψ do not extend in time to $t = -\infty$. It therefore becomes necessary to develop a formula that has finite limits of integration.

For this purpose, ψ can be broken down into an integral from $-\infty$ to $t - t_0$, and one from $t - t_0$ to t* where t_0 is a constant sufficiently large to make negligible the deviation of the first integral from the mean value over the infinite period $-\infty$ to $t-t_0$,† that is,

$$\psi(t) = A_{01} + A_{02}E_0 + \int_{t-t_0}^{t} \frac{a_1 E(x)}{\sqrt{2\pi}\,\sigma_1} e^{-(t-x-M_1)^2/2\sigma_1^2} dx ,$$

where A_{01} and A_{02} are constants and E_0 is the normal (average) value of E (bounded) on the infinite range $-\infty$ to $t - t_0$. This is the type of formula required to represent new building, $B(t)$, if it is assumed that once an incentive is offered it will be acted upon some time in the future, the particular time at which the action occurs being determined by a great many economic forces, any one of which has only a small effect. Corrections for economic forces of major importance, such as credit, have yet to be made.

11. Inhibiting Influence of Lack of Credit. Now, as previously mentioned, an important factor in preventing incentives $E(x)$ from producing new building according to the above "normal" law for ψ is the inhibiting effect of "lack of credit," which, as already pointed out, may be assumed to be measured by the reciprocal of the foreclosure rate.‡ Thus if this inhibiting effect be assumed to result in subtracting a certain amount of "new building" from that which would normally be built because of the incentive E, and the factor "building need" be introduced, an analysis similar to that just given would lead to a formula for new building, $B(t)$, such as the following:

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*Some far-sighted individuals may react to the incentive before it is actually present. That is, t_0 may be less than t_i.

†See Appendix IV.

‡See qualifying explanations in Sections 3 and 6.

$$(11.1) \qquad B(t) = \left[A_0 + \frac{A_1}{N_1} \int_{t-t_0}^{t} E(x) e^{-(t-x-M_1)^2/2\sigma_1^2} dx + \frac{A_2}{N_2} \int_{t-t_0}^{t} F(x) e^{-(t-x-M_2)^2/2\sigma_2^2} dx \right] b \; ,$$

where A_0, A_1, A_2, σ_1, σ_2, M_1 and M_2 are constants and

$$N_1 = \int_{t-t_0}^{t} e^{-(t-x-M_1)^2/2\sigma_1^2} dx$$

$$N_2 = \int_{t-t_0}^{t} e^{-(t-x-M_2)^2/2\sigma_2^2} dx \; ,$$

provided the quantity

$$H = E_1(t - M_1 - \lambda_1) \frac{A_1}{N_1\sqrt{\pi}} \int_{-\infty}^{(M_1-t_0)/\sqrt{2}\sigma_1} e^{-z^2} dz + F_2(t - M_2 - \lambda_2) \frac{A_2}{N_2\sqrt{\pi}} \int_{-\infty}^{(M_2-t_0)/\sqrt{2}\sigma^2} e^{-z^2} dz \; ,$$

which is a part of A_0, is small enough to be neglected.* By means of a special statistical technique developed for the purpose, it is possible to obtain values of the constants in (11.1). In fact, it can be shown† that the statistical values of these constants are

$A_0 = .443$; $A_1 = 126.002$; $A_2 = -11.909$; $M_1 = 2.25$;
$M_2 = 1.75$; $\sigma_1 = 1.00$; and $\sigma_2 = 1.25$.

12. Errors. No probable errors or standard errors for the parameters have been calculated since no generally accepted methods for calculating them are available. It can be said, however, that the parameters have the theoretically correct signs and the theoretical formula gives a plausible description of the economic situation. At the present stage of development of demand relationships

*N_1 and N_2 are necessary to prevent the exponential from having an absolute weighting effect beside the relative effect desired. In checking coefficients, the authors find that those given were obtained by erroneously using $8N_1$ and $8N_2$. This merely changes the weights slightly. Changes would be so small that recalculation was believed unnecessary.

†For the statistical technique used, see Appendix V.

this should be sufficient, but studies on other cities ought to be made.

It is interesting to return to the problem of determining whether $t_0 = 8$ is large enough to make

$$H = E_\Delta(t - M_1 - \lambda_1)\frac{A_1}{N_1\sqrt{\pi}}\int_{-\infty}^{(M_1-t_0)/\sqrt{2}\sigma_1} e^{-z^2}dz$$

$$+ F_\Delta(t - M_2 - \lambda_2)\frac{A_2}{N_2\sqrt{\pi}}\int_{-\infty}^{(M_2-t_0)/\sqrt{2}\,\sigma^2} e^{-z^2}dz$$

negligible. For this purpose it is merely necessary to substitute statistically determined values of the parameters and refer to a table of the probability integral.

A substitution of the above values of A_1, A_2, M_1, M_2, σ_1 and σ_2 in H *gives*

$$H = 7.97\,E_\Delta\left(\frac{2}{\sqrt{\pi}}\int_{-\infty}^{-4.07} e^{-z^2}dz\right)$$

$$+ .623\,F_\Delta\left(\frac{2}{\sqrt{\pi}}\int_{-\infty}^{-3.54} e^{-z^2}dz\right).$$

A reference to a table of the probability integral gives

$$H = 7.97\,E_\Delta \times .00004 - .623\,F_\Delta \times .00023$$

$$= .0003\,E_\Delta - .0001\,F_\Delta$$

The maximum value of E is 2.90 and the minimum .97, so that the maximum value of E_Δ is about .9. The maximum value of F is .92 and the minimum value .02, so that F_Δ has a maximum value in the neighborhood of .45. It follows, therefore, that

$$Max.\ H = .0003 + .0001 = .0004;$$

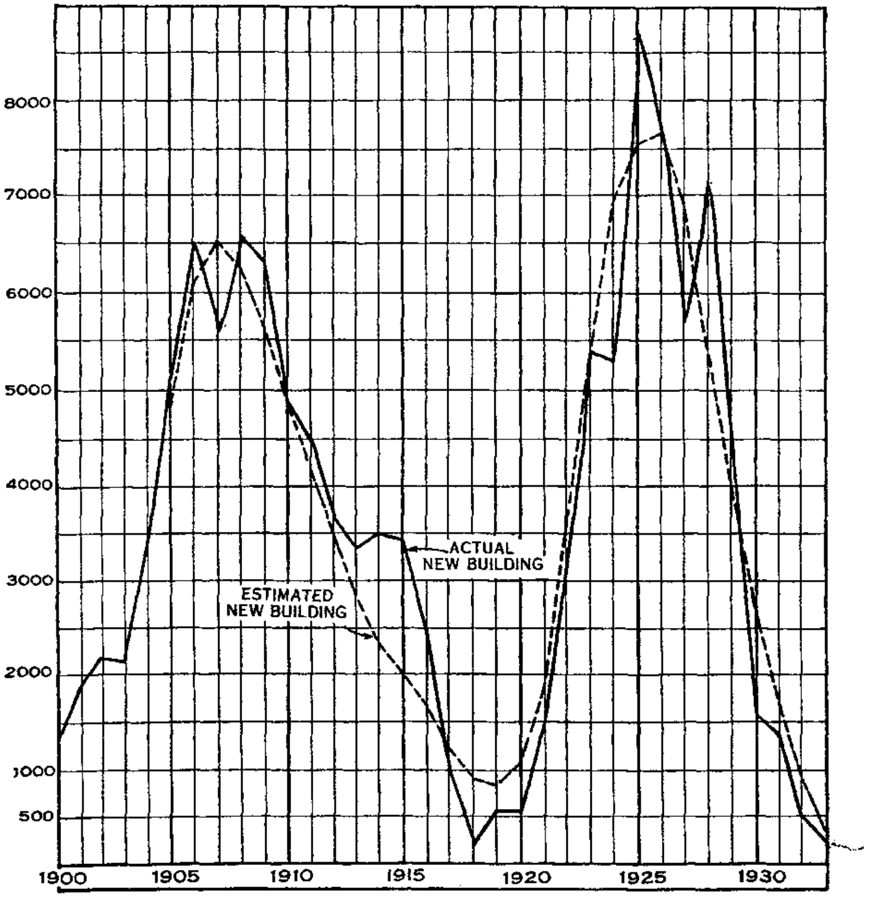
that is, there is a maximum error of one point in the fourth decimal place of the ratio B/b.

It will be observed from Chart XX that the new formulation gives a much better fit than even the remarkable fit obtained by the use of a fixed lead.

In view of the fact that a rational law has been set up and the rational law as set up agrees so exceptionally well with the obser-

ESTIMATED AND ACTUAL NEW BUILDING
IN ST. LOUIS 1900-1933

CHART XX



When a distributed lag is used, theoretical building as calculated from an integral formula and actual building agree very closely.

Shifting the estimating curve forward by two months would give a better fit. This would involve changing M_i and σ_i slightly.

vations, some confidence can be placed in the formula. It must be recognized, however, that introduction of new important factors may alter the picture. Those new factors mentioned in the previous formulation, i.e., those tending to shorten the lead, are, however, very largely taken into account by the new formulation.

TABLE IX

Year	$I(t)$ Net Rental Yield	$V = I^{.86}$	$\overline{W}$	$E = V\overline{W}$	$D_E =$ Weighted Integral of E	F	$D_F =$ Weighted Integral of F	Family Accommodations Built (Actual)	Family Accommodations Built (Estimated)
1898	.0736	.1060	.572	.0606		.785		1396	
1899	.0515	.0780	.609	.0475		.737		1390	
1900	.0517	.0782	.642	.0502		.739		1287	
1901	.0647	.0949	.678	.0643		.641		1850	
1902	.0788	.1124	.710	.0798		.665		2208	
1903	.0912	.1275	.760	.0969		.385		2179	
1904	.1021	.1405	.838	.1177		.223		3424	
1905	.0910	.1272	.842	.1071	.0890	.316	.447365	5128	4818
1906	.0822	.1166	.842	.0982	.1017	.231	.347818	6413	6071
1907	.0766	.1097	.848	.0930	.1039	.309	.294761	5511	6512
1908	.0767	.1099	.813	.0893	.0983	.496	.293447	6619	6260
1909	.0695	.1009	.782	.0789	.0922	.531	.370493	6256	5611
1910	.0690	.1003	.757	.0759	.0864	.602	.468490	4897	4870
1911	.0652	.0956	.712	.0681	.0802	.691	.556192	4509	4178
1912	.0613	.0906	.675	.0612	.0741	.710	.637805	3636	3491
1913	.0678	.0988	.662	.0654	.0680	.721	.701513	3378	2815
1914	.0703	.1019	.622	.0634	.0634	.779	.740459	3490	2322
1915	.0676	.0984	.566	.0557	.0619	.810	.776729	3455	2027
1916	.0640	.0940	.546	.0513	.0603	.807	.815199	2435	1696
1917	.0607	.0898	.561	.0504	.0562	.790	.841597	1077	1248
1918	.0593	.0881	.607	.0535	.0519	.740	.847016	196	901
1919	.0547	.0821	.668	.0548	.0501	.667	.826234	590	856
1920	.0608	.0899	.760	.0683	.0508	.390	.778498	585	1087
1921	.0864	.1217	.841	.1023	.0546	.200	.644211	1500	1915
1922	.0994	.1373	.856	.1175	.0667	.239	.450737	3607	3539
1923	.1058	.1448	.866	.1254	.0868	.065	.315945	5384	5345
1924	.1104	.1503	.863	.1297	.1056	.250	.204318	5284	6908
1925	.0988	.1366	.862	.1177	.1170	.103	.188504	8712	7539
1926	.1072	.1465	.833	.1220	.1205	.489	.169915	7504	7654
1927	.1008	.1396	.732	.1022	.1188	.695	.272362	5609	6900
1928	.0904	.1265	.660	.0835	.1137	.730	.468657	7160	5562
1929	.0898	.1258	.604	.0760	.1029	.798	.644968	4139	4064
1930	.0898	.1258	.514	.0647	.0886	.850	.759856	1590	2722
1931	.0891	.1250	.437	.0546	.0763	.878	.833578	1474	1719
1932	.0756	.1085	.340	.0369	.0661	.920	.885722	550	994
1933	.0557	.0835	.252	.0210	.0547	.938	.929371	300*	367

*Estimated building for 1934 was 300 units. Actual building was 297. In the troughs the formula gives a value about 20% too high, hence 300 instead of 367. Estimate for 1934 is 260 units.

13. Cost Maximizing Income to the Building Industry. With a law of demand of the form (11.1) it is possible to consider the problem of choosing cost C so that income over a period of time $0 \leq t \leq t_1$, where t_1 is one year, two years, three years, etc., as may be desired, is a maximum. In other words, it is possible to choose $C(t)$ so that

$$\pi = \int_0^{t_1} B(t)C(t)dt,$$

where $B(t)$ is related to $C(t)$ by means of the formula just given, is a maximum, when particular values are assumed for R, T, P and f in the period under consideration.

Let $\overline{B}$ and $\overline{C}$ be the values of B and C that satisfy (11.1) and maximize π. Write

$$B = \overline{B} + \eta\,\delta B \text{ and } C = \overline{C} + \eta\,\delta C\ ,$$

where δB and δC are arbitrary quantities.

Then

$$\delta\pi = \int_0^{t_1} (\overline{B}(t)\delta C + \overline{C}\,\delta B)\, dt$$

and

$$\delta B = .86 \int_{t-t_0}^{t} H(x,t)\,\overline{C}^{-1.86}\,\delta C\, dx\ ,$$

where

$$H(x,t) = (A_2/N_2)\,(Rp - T)^{.86}\, W e^{-(t-x-2.25)^2/2}$$

The variations here are taken in the usual sense of the calculus of variations.*

Now by making use of the fact that $C(t)$ is a known function on the range up to $x = 0$; i.e., up to $x = t - t_0 = 0$, since past costs are unalterable historical ones,

$$\delta B = -.86 \int_0^{t} H(x,t) C^{-1.86}\,\delta C\, dx$$

*For a general theory of economic equilibrium for an economy in which capital goods are important, see C. F. Roos, "A Dynamical Theory of Economics," *Journal of Political Economy*, October, 1927.

where, here and in all following equations, the bars have been dropped for simplicity of notation. By a substitution of δB in $\delta \pi$,

$$\delta \pi = \int_0^{t_1} \left[B(t)\,\delta C - .86C \int_0^{t} H(x,t) C^{-1.86}\,\delta C\,dx \right] dt\ .$$

An application of Dirichlet's formula for interchanging the order of integration for an interated integral* gives

$$\delta \pi = \int_0^{t_1} B(t)\,\delta C\,dt - .86 \int_0^{t_1} C^{-1.86}\,\delta C\,dx \int_x^{t_1} H(x,t) C\,dt\ ,$$

and by interchanging the parameters of integration x and t in the last integral,

$$\delta \pi = \int_0^{t_1} \left[B(t) - .86C^{-1.86} \int_t^{t_1} H(t,x) C(x)\,dx \right] \delta C\,dt\ .$$

Now, if δC is to be entirely arbitrary on the interval $0 \leq t \leq t_1$, it follows that

$$B(t) - .86C^{-1.86} \int_t^{t_1} H(t,x) C(x)\,dx = 0. \tag{12.1}$$

This equation should be solved with (11.1) to obtain $B(t)$ and $C(t)$. Unfortunately these equations are integral equations of non-linear type, and no direct method of solving them is known, but an approximation can be made to get linear integral equations.

In formula (11.1) replace $y_1 = C^{-1.86}$ by a linear expression $y = a - bC$ where $a - bC$ is the best fitting least squares line to the curve $y_1 = C^{-.86}$. Then (11.1), which is

$$B(t) = A_0 + (A_1/N_1) \int_{t-8}^{t} H(x,t) C^{-.86}\,dx - A_2\varphi$$

where

$$H(x,t) = (Rp - T)^{.86}\, W(f) e^{-(t-x-2.25)^2/2}$$

and

$$\varphi = (1/N_2) \int_{t-8}^{t} F(f) e^{-(t-x-1.75)^2/2 \times (2.25)^2} dx,$$

*See Whittaker and Watson, *Modern Analysis*, Cambridge, 1915, pp. 75-77.

can be written in the form

$$B(t) = A_0 + A_1 \int_{t-8}^{t} H(x,t)(a-bC)\,dx - A_2\varphi \,, \tag{12.2}$$

so that the condition (12.1) becomes

$$B(t) = A_1 b \int_{t}^{t_1} H(t,x)C(x)dx. \tag{12.3}$$

A substitution of $B(t)$ as given by (12.3) in (12.2) yields

$$+A_1 b\left[\int_{t}^{t_1} H(t,x)C(x)dx + \int_{t-8}^{t} H(x,t)C(x)dx\right]$$
$$-\left[A_0 - A_2\varphi + A_1 a \int_{t-8}^{t} C(x)H(x,t)dx\right] = 0.$$

Suppose now that $t_1 = 3$ years. As is well known, the integrals can be approximated by finite sums. For this let $\Delta x = 1$. Then

$$+A_1 b\left[\sum_{i=j}^{3} H(j,i)C(i) + \sum_{i=j-8}^{j} H(i,j)C(i)\right] - V(j) = 0$$

where

$$V(j) = A_0 + \sum_{i=j-8}^{j} G(i,j)\,,$$

$$G(i,j) = (A_2/N_2)\, F(f(i))e^{-(j-i-1.75)^2/2\times(2.25)^2} + A_1 a H(i,j)$$

and $j = 1, 2, 3$. The terms of these equations can be rearranged to give

$$A_1 b[H(1,1)C(1) + 2H(1,2)C(2) + H(1,3)C(3)]$$
$$+ \sum_{i=-7}^{0} H(i,1)C(i) - V(1) = 0$$

$$A_1 b[H(1,2)C(1) + 2H(2,2)C(2) + H(2,3)C(3)]$$
$$+ \sum_{i=-6}^{0} H(i,2)C(i) - V(2) = 0.$$

$$A_1 b[H(1,3)C(1) + H(2,3)C(2) + 2H(3,3)C(3)]$$
$$+ \sum_{i=-5}^{0} H(i,3)C(i) - V(3) = 0$$

These equations obviously in general have unique solutions for $C(1)$, $C(2)$ and $C(3)$ in terms of known quantities $C(-7), \cdots,$ $C(0)$ and quantities $H(i, j)$, $i = 1, 2, 3,$; $j = 1, 2, 3$, that can be estimated from estimated values of rents, R; taxes, T; occupancy, p; and foreclosures, f. If the determinant happens to be zero, one of the statistical constants can be changed slightly, but this brings up the question of the significance of a solution when the determinant is nearly zero. It is possible that the solution will be given in terms of random errors and if this is true the solution of course is not significant.*

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*For a related problem see Ragnar Frisch, "Pitfalls in the Statistical Construction of Demand and Supply Curves," *Frankfurter Gesellschaft für Konjunkturforschung, Veroffentlichungen, hft* 5, 1933.

Since I wrote the above, Professor Frisch has called my attention to his manuscript, "The Object of Confluence Analysis. The Danger of Including Too Many Variates in a Regression Analysis," to be published in *Nordisk Statistisk Tidskrift*. In this paper Professor Frisch treats the problem of solutions in terms of random errors.

CHAPTER VII

Growth and Decline of Industry

1. The Automobile-Building Boom of the Twenties. Growth of industry plays an extremely important role in activating national savings and putting them in plant investment. In the United States the growth of the automobile industry undoubtedly had much to do with the prosperity during the period 1921-1929. It is, of course, not implied that labor used directly in producing automobiles brought about prosperity. In the picture must be included the labor used in the oil industry, in highway construction, in demolition of old residences to be replaced by filling stations, garages, etc., and in the building of roadside inns, tourist camps, etc. There was also the collateral activity generated in the production of machinery to equip industrial plants and utility plants.

During the period of expansion brought about by the mass production of the automobile, construction activities, services in connection with filling stations, oil drilling, etc., required vast amounts of unskilled and partly skilled labor, which commanded high wages. If a technological improvement was introduced in any consumers' goods industry, this meant that someone was available for the activities mentioned above. Those making the technological changes profited because of lower cost. Labor was only temporarily displaced since jobs in the expanding industries were plentiful. High profits in these expanding industries tempted capital, so that everyone was happy. Between 1921 and 1929 the United States spent five to six billion dollars a year on industrial and utility plant expansion alone. An analysis of the industrial activity of the country discloses the fact that during this period the production of capital goods employed approximately a third of all the industrial workers.

It seems to be unnecessary to add that the expanding economy of the United States in the period 1921-1929 offered a situation quite unlike the static equilibrium that has been so often pictured by economists. It is important, therefore, to consider the economic structure of an expanding economy, i.e., a dynamic economy. Before considering this general problem, it is desirable to consider the problem of population growth, because the problems are related.

*2. Population and Sales Growth.** The Pearl-Reed population curve (the logistic) is based upon the following assumptions:

(a) physical conditions set some upper limit U to the population growth;

(b) growth in the population y is proportional to the existing population; and

(c) growth in the population is proportional to the potential expansion of population $U - y$.

The above assumptions can be summarized in the differential equation

$$dy/dt = ay(U - y) \quad , \tag{2.1}$$

where a is a constant.

The equation (2.1) is the fundamental differential equation of the autocatalytic theory. The derivative dy/dt is zero at $y = 0$ and at $y = U$ and between these points it is everywhere positive and traces a parabola which is symmetrical with respect to $U/2$.

It is evident that the assumptions (b) and (c) are not the only likely ones that can be made for the human population. All that can be safely asserted is that $dy/dt = 0$ at $y = 0$ and at $y = U$ and $dy/dt > 0$ between $y = 0$ and $y = U$. Thus, the equation

$$dy/dt = a(y + by^2 + cy^3)(U - y)^d \tag{2.2}$$

is just as reasonable as (2.1).

The equation (2.1) integrates into the form

$$y = U/[1 - e^{-k(t - t_0)}] \tag{2.3}$$

where $k = aU$ and t_0 is a constant of integration.

The logistic curve (2.3) fits the United States population fairly well, but for the population of Germany two logistics are required; that is,

$$y = U_1/[1 + e^{-k(t - t_0)}] \tag{2.4}$$

$$+ U_2/[1 + e^{-m(t - t_1)}] + C.$$

*The treatment given here is based on an unpublished manuscript by Victor von Szeliski.

The first logistic term represents the evolution of the German population in the agricultural handicraft stage of development, whereas the second represents the population made possible by the industrial revolution.*

The right-hand side of (2.1) can be split into two factors, one of which proportional to y is the physiological rate of growth which for the United States is about 3.134% per annum. The other factor represents inhibitors of growth, the physical environment chiefly.

Pearl's treatment suggests that as technology progresses, the upper limit of population growth nevertheless remains constant at $U_1 + U_2 + C$. It implies that if seven very accurate population counts had been available prior to 1800 (say) the new cycle of development that made the industrial revolution possible could have been forecast prior to the invention of machinery. The upper limit U_1 certainly exists due to a set of physical inhibitors prior to the invention. However, the technological addition U_2 cannot be considered as constant throughout the course of evolution. In fact, it is probably more nearly correct to assume U to be a continuous function of the time, $U(t)$, and dependent on the existing technology and physical facilities of the moment. The differential equation then generalizes into

$$dy/dt = ay[U(t) - y] \quad , \qquad (2.5)$$

where $U(t)$ is a continuous function and indeterminate from *a priori* considerations. Actually $U(t)$ depends upon many economic and technological factors variable in time. The quantity a may also depend on various economic factors.

Needless to say, when either a or U is not constant a curve different from the Pearl-Reed logistic is obtained. Thus, it is not surprising that some attempts to fit logistics to demand for new products have been unfruitful.

Suppose that automobiles are introduced into a stable population and that at the time t a certain automobile population, v, exists. For simplicity suppose that the upper limit will be one car per family and let m be the number of families. If replacements are ignored, the number of prospective purchasers is $m - v$.

It is reasonable to assume that if an automobile company has a car to sell, the probability that it will be able to sell the car is proportional to the number of prospects; i.e., *probability* =

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*Raymond Pearl, *Studies in Human Biology*, Baltimore, 1924.

$a(m - v)$. The quantity a will, of course, vary as the product becomes known to the population and wants are built up. If the succeeding strata of prospects have progressively less and less money, or are progressively more and more immune to sales talk, the probability will not be a linear function of $m - v$.

The quantity a will to a first approximation depend linearly on v. Also the stimulus offered to the prospects is proportional to the number of automobiles in use. At least during the early stages of exploitation of a market there is a tendency for sales outlets to expand according to an exponential law of growth. In other words, the number of people who set up as automobile dealers probably bears a more or less constant relationship to the number who purchased during the preceding interval of time. Furthermore, advertising by magazine, radio, etc., but more likely direct contact with the product, increases the number of persons who want the product. Facilities for service, showrooms, etc., have to be built up out of profits, or out of loans which depend upon expected dollar value of profits.

It is thus seen that under quite restricted assumptions the rate of change of number of automobiles in use may be taken to be

$$dv/dt = av(m - v) ,$$

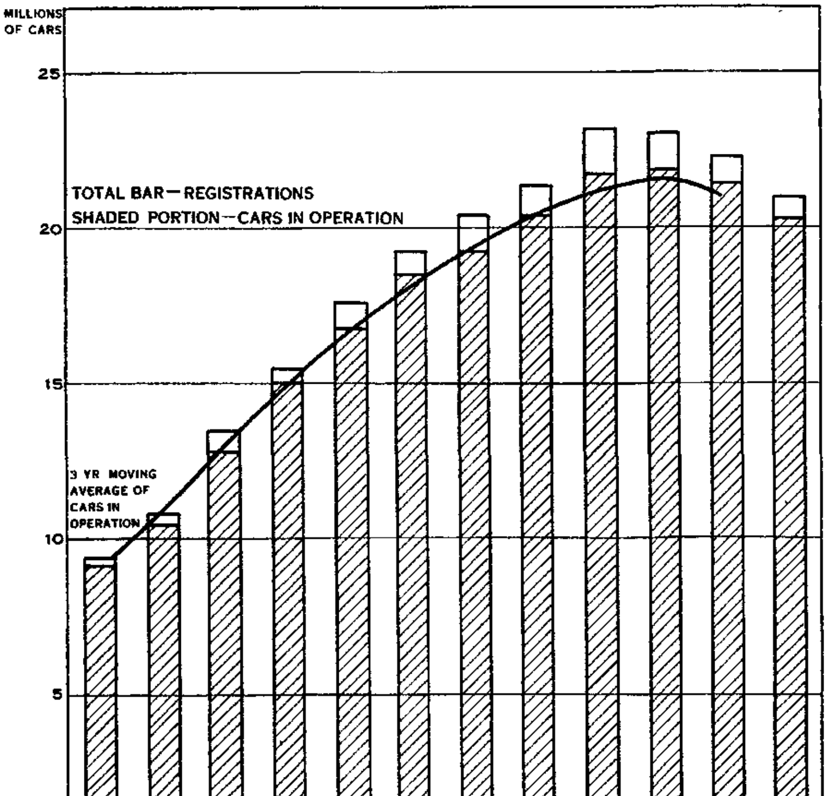
where a and M are functions of time t. Thus, a curve of growth of automobile demand should somewhat resemble a logistic curve.

3. Total Cars in Operation in the United States. In the preceding consideration of the growth in demand for automobiles, replacements were ignored. For the theory to be applicable, therefore, cars in operation must be regarded as the equivalent of v. The assumption made here is that, on the average, once a purchaser has had an automobile he will replace it or directly or indirectly influence someone else to take his place as an owner.

The best known measurement of the usage of passenger automobiles in the United States is total passenger car registrations, which is illustrated in Chart XXI. Registrations exceed cars actually in use so that corrections for this element of error should be made. The white portion at the top of each bar in this chart represents this correction and the shaded portion of the bar represents

PASSENGER CAR REGISTRATIONS

CHART XXI



Automobile registrations have shown increases from 1921-1929. The curve of growth resembles a logistic. Apparently cars in use began to stabilize (draw near to the upper asymptote) near 1929. However, stabilization is at a certain price. It is always possible for a new asymptote much higher to be set by important price concessions, new uses, etc.

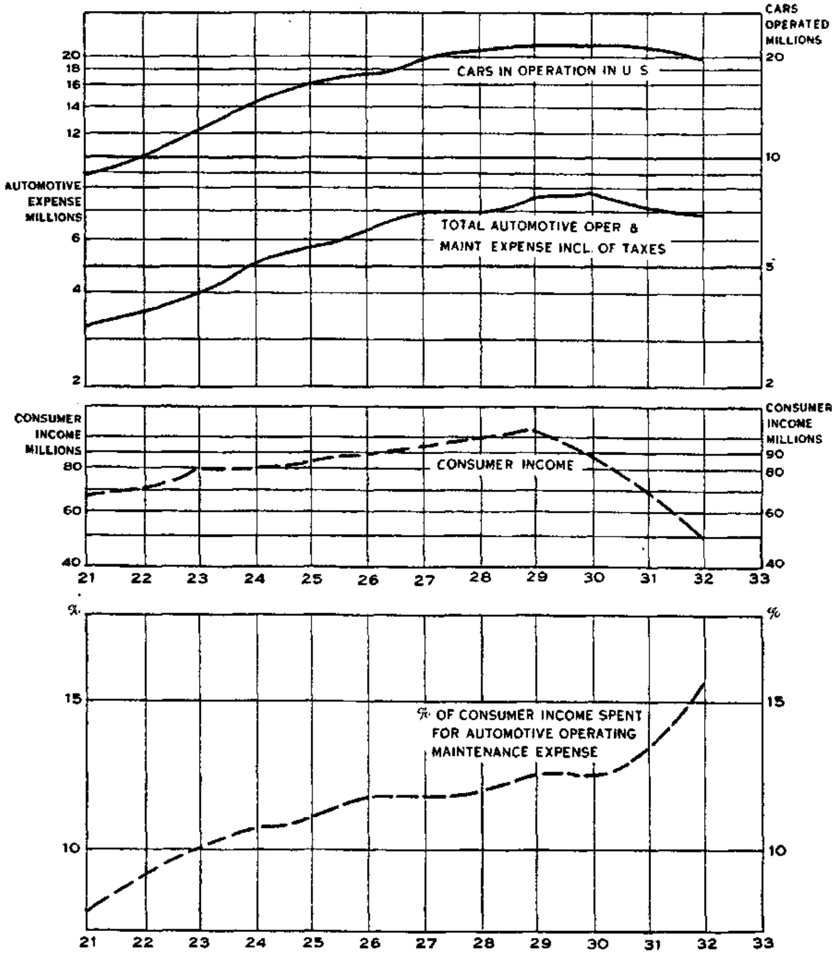
the best estimate of the cars registered which were actually in service at the end of each year from 1921-1932.*

*See Stephen Du Brul, *Analysis of the Automobile Market,* General Motors Corporation. Sections 3 and 4 of this chapter are based on Du Brul's study.

Since writing this, my attention has been called to an unpublished manuscript by Victor von Szeliski developing other aspects of the problem.

PERCENTAGE OF CONSUMER INCOME SPENT ON AUTOMOBILE OPERATION

CHART XXII



Number of cars in use and owners' expenditures for their maintenance are relatively stable, in fact, much more stable than consumer income. Automobile transportation is about as important as food and clothing in the American family's budget.

The number of cars in operation rose steadily but with decreasing rate from 1921 to 1930 when a peak was reached. Since 1930, cars in operation have been declining slightly. Earlier series show that the number of cars in operation started out slowly during the years of development of the commodity, then increased rapidly to 1925, when the increase began to taper off. An examination of the chart might indicate that, by 1929, cars in operation apparently approximated their point of stabilization. If the 1929 value approximately represents the stabilization point, rate of growth in the future will be slow and will be dependent more or less on the growth of population unless new uses are found or drastic price reductions can be made. However, severe decline in purchasing power since 1929 makes it extremely hazardous to assert that total cars in use in 1929 approximated the maximum that could be used by the population.

A comparison of cars in operation with consumer incomes shows clearly that automobile transportation is about as important as food and clothing in the American family's budget and, in many cases, even more stable than shelter. The basic demand for automotive transportation is therefore very strong — see Chart XXII.

4. New Car Domestic Sales. As pointed out, it appears that the demand for automobiles stabilized in 1929. The automobile industry is, therefore, almost entirely on a replacement basis. As Du Brul* has shown, retirements are by no means at a constant rate. On the contrary, the rate of retirement appears to vary considerably from year to year. Chart XXIII-A shows estimated car retirements, and Chart XXIII-B shows the ratio of car retirement to the straight line trend of estimated car retirement. The superimposed lines indicate the average car life corresponding to variations in the rate of retirement. In the boom year of 1929, the retirement rate was equivalent to an average car life of less than six years, whereas in 1931 the retirement rate was equivalent to an average car life of over seven years. The rate of retirement for 1932 was estimated to be equivalent to an eight year car life. Thus, the American public is able to maintain its high rate of usage in the face of a sharp drop in income by prolonging the lives of the cars that it possesses. The rate of retirement is largely a function of economic conditions.

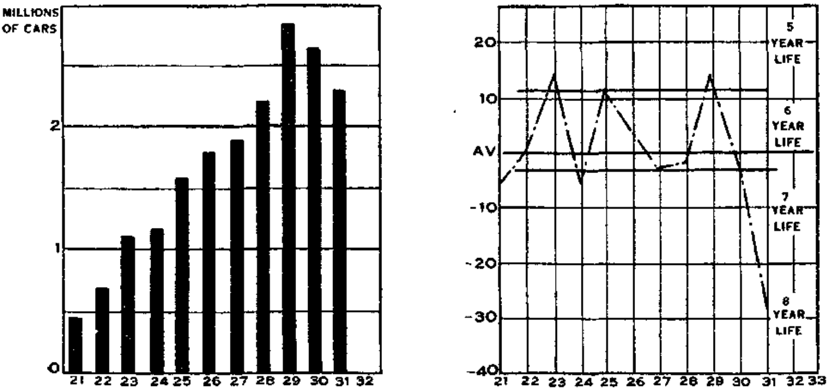
New car replacement demand is largely influenced by economic factors rather than mechanical ones. Chart XXIV contrasts total

*Stephen Du Brul, *loc. cit.*

ESTIMATED CAR RETIREMENTS

DEVIATION OF RETIREMENT FROM TREND

CHART XXIII



A B

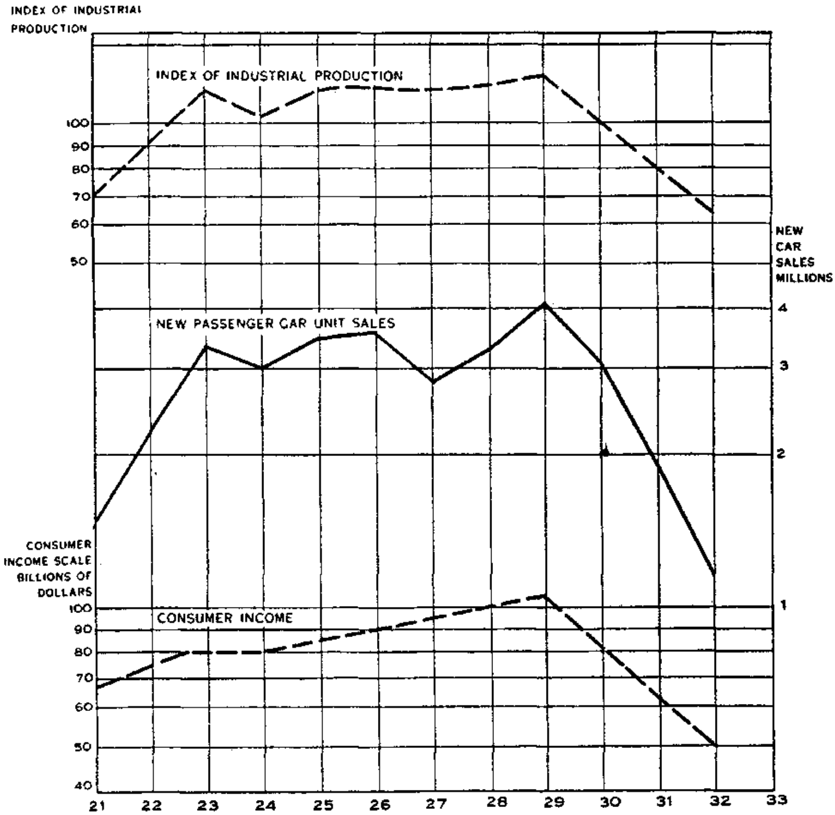
By taking the ratio of actual retirements A to the trend (average) as indicated in B it is possible to compare change in the rate of retirement more closely. For classification a series of lines have been superimposed on B to indicate the average car life corresponding to variations in the rate of retirement.

new passenger car sales from 1921 to date with consumer income during this period and an index of industrial activity. Thus, new car domestic sales depend largely upon consumer income. The percentage of total consumer income spent on new cars and the per cent of consumers' total automotive expenditures devoted to new car purchases show a substantial drop since 1929. This further indicates that new car demand is extremely sensitive to changes not only in current income but also in prospective income. This is to be expected since the purchase of a new car represents a capital expenditure to the consumer. When consumers feel confident of the future they capitalize that future in current expenditures (assisted by time payments in many cases). When, however, their future income is uncertain, they content themselves with their existing equipment.

It is possible to conclude, therefore, that the public will buy new cars when it has the money and will retire old cars more rapidly. Thus, even with stabilization in the total usage of cars, a rapid expansion in new car demand is quite possible. On the other hand,

NEW CAR SALES, CONSUMER INCOME AND INDUSTRIAL PRODUCTION

CHART XXIV



Since 1923 new car sales have followed closely consumer income. New car sales can be used as a rough index of consumer income.

a rapid contraction in new car demand can occur without any material contraction in usage.

5. *Competition of Growing and Declining Industries.* Examples of growing industries have been considered at some length. Some industries after reaching maturity may undergo a more or less steady decline irrespective of cyclical fluctuations in business. A few years ago the horse constituted the principal farm power. His

displacement by tractors and other farm machinery has been going on at a very rapid rate since the World War. Some statisticians estimate that the passing of the horse in the United States has made it necessary to retire thirty millions of acres of land that had formerly been used to grow horse feed.

Of recent years the coal industry has faced serious competitition with the oil industry. Replacement of coal power plants of steamships by oil plants has proceeded at a rapid rate. Many industrial plants and homes have changed from coal to oil or natural gas. Part of the displacement has undoubtedly been due to the inability of the coal industry to lower its price sufficiently on account of high fixed transportation costs, but much of it has been due to natural advantages of oil and gas over coal.

Lumber is an outstanding example of a declining industry. For building purposes lumber has faced serious competition with steel, cement and fireclay products. Wooden boxes have been replaced to considerable extent by paper boxes. Wooden bridges have yielded to steel structures and so forth.

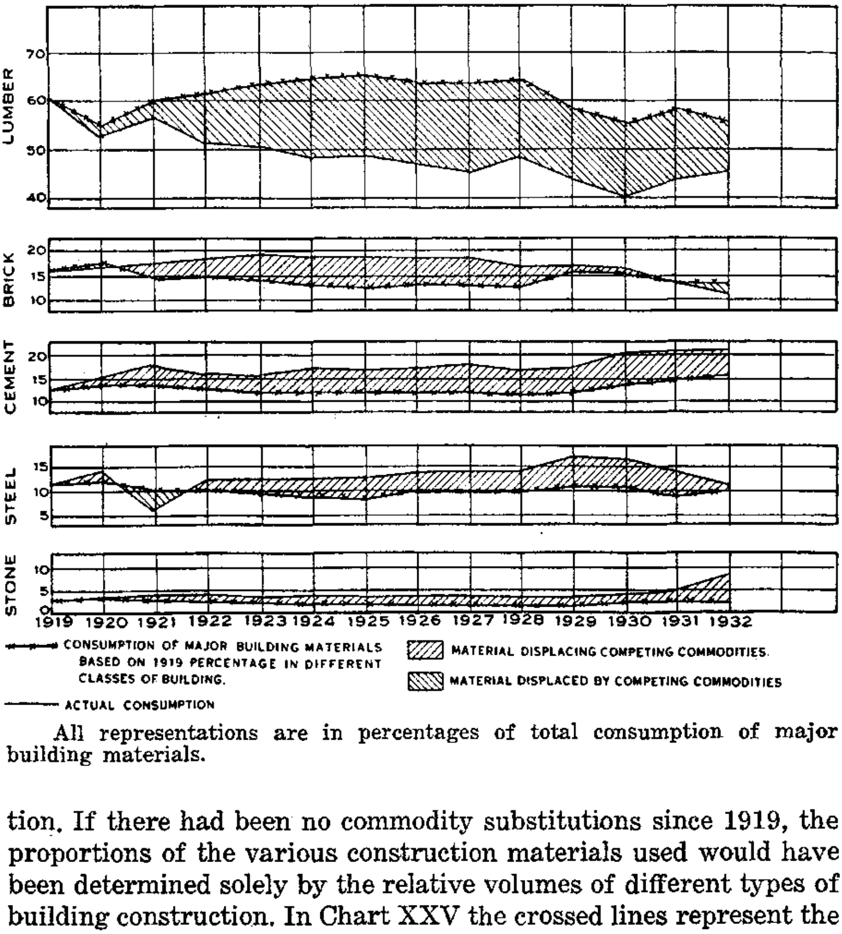
Displacement of lumber by steel, cement, stone and fire-clay products (brick, terra-cotta, tile), and other shifts between amounts of these commodities consumed, may be separated into three classifications: (1) shifts due to changes in the relative volumes of different types of building construction, (2) temporary shifts due to price competition, and (3) permanent commodity substitutions due to changes in consumer taste.* The first factor measures long term, but not quite permanent, shifts in potentialities for consumption.

An estimate of the extent of commodity substitution can be obtained by eliminating from the shifts in actual consumption the effect of shifts in types of building construction. Building can be divided into three classes: (1) residential; (2) industrial, which is here taken to include factory and commercial; and (3) public, which includes educational, institutional, Government, religious, memorial, social and recreational. The classification "public works and utilities" has been omitted since these buildings require little lumber, but in the study presented below corrections for the omissions have been made. Table X gives the percentage consumption of major building materials in the United States and Table XI gives the percentage distribution of classes of building construc-

*This study of commodity substitutions of steel, cement and clay products for lumber is based on an unpublished paper by Victor Perlo, "Displacement of Lumber in Building Construction."

SHIFTS IN RELATIVE CONSUMPTION OF MAJOR BUILDING MATERIALS

CHART XXV



All representations are in percentages of total consumption of major building materials.

tion. If there had been no commodity substitutions since 1919, the proportions of the various construction materials used would have been determined solely by the relative volumes of different types of building construction. In Chart XXV the crossed lines represent the consumption of major building materials based on 1919 percentage in different classes of building. The plain lines represent the actual consumption, and the space between represents the commodity displacement.

By 1924, twenty-five per cent of the lumber which on the basis of use in 1919 would have been used in 1924 had been displaced by competing commodities. During 1924-32 there was no significant displacement of lumber, and since 1927 there has been a trend towards lumber, which became very sharp in 1932. In the 1927-32 period the price of lumber was kept continually under its normal competitive price relative to competing commodities and there was an especially sharp decline in price in 1931, which increased 1932 relative consumption of lumber.

TABLE X
PERCENTAGE CONSUMPTION OF MAJOR BUILDING MATERIALS

YEAR	LUMBER	STEEL	CEMENT	BRICK	STONE	TOTAL
1919	59.7	10.8	11.8	15.7	2.0	100.0
1920	52.6	13.6	14.9	16.4	2.5	100.0
1921	56.4	5.9	17.2	17.3	3.2	100.0
1922	50.8	11.8	15.6	18.4	3.4	100.0
1923	50.6	11.9	15.3	19.2	3.0	100.0
1924	48.7	12.2	16.8	18.9	3.4	100.0
1925	49.1	12.2	16.3	19.1	3.3	100.0
1926	47.2	13.5	16.8	18.9	3.6	100.0
1927	45.4	13.8	18.0	19.0	3.8	100.0
1928	48.3	14.2	16.8	17.2	3.5	100.0
1929	44.1	17.2	17.3	17.4	4.0	100.0
1930	40.4	17.1	20.6	17.1	4.8	100.0
1931	44.0	14.6	21.2	14.3	5.9	100.0
1932	46.0	11.8	21.4	12.1	8.7	100.0

Brick includes common, face and vitrified brick, terra cotta, hollow building tile and fire brick. Stone includes building stone, rubble and riprap.

Stanley B. Hunt, who has studied the price competition of rayon, silk, cotton and wool fibers,* says:

> "For whereas the trend of the total poundage consumption of the four fibers as a whole has increased at a rate of about 3 per cent per annum for this period (1919-31), the individual fibers have shown decidedly different trends in that time. And while the average prices of all the fibers have declined, due principally to the general downward trend of all prices since the war, the individual fibers have shown varying declining price trends during the period.
>
> Specifically, the exhibits (Charts XXVI, XXVII and XXVIII) show that the more the price of a given fiber is

*Stanley B. Hunt, "Rayon Consumption and Price in Relation to the Other Fibers," *Textile World*, Sept. 26, 1931, p. 90-92.

reduced, the larger will be its consumption, all in relation to the other fibers, of course. This means that by certain price relationships the consumption of one fiber may be increased at the expense of another or other fibers; it does *not* mean that lower general textile prices in themselves will stimulate a demand for textiles."

TABLE XI

PERCENTAGE DISTRIBUTION OF CLASSES OF BUILDING CONSTRUCTION

YEAR	RESIDENTIAL	INDUSTRIAL	PUBLIC	TOTAL
1919	47.8	34.8	17.4	100.0
1920	37.8	35.9	26.3	100.0
1921	45.8	27.5	26.7	100.0
1922	48.0	29.8	22.2	100.0
1923	52.3	28.8	18.9	100.0
1924	54.4	25.3	20.3	100.0
1925	55.7	24.2	20.1	100.0
1926	53.0	27.6	19.4	100.0
1927	52.3	26.6	21.1	100.0
1928	54.0	27.0	19.0	100.0
1929	44.6	34.5	20.9	100.0
1930	38.2	30.9	30.9	100.0
1931	42.0	22.6	35.4	100.0
1932	36.8	21.8	41.4	100.0
1933*	38.9	36.8	24.3	100.0

*Based on data for first eleven months. Data from F. W. Dodge reports.

Note: Industrial includes Commercial and Factory Buildings.
Public includes Educational, Hospitals and Institutions, Public Buildings, Religious and Memorial, Social and Recreation, Military and Naval Buildings.

6. Wages in Expanding and Declining Industries. Wages paid for a standard type of work often differ from industry to industry. Risk or hazard of occupation, conditions of work, size of town in which the industry is located and various minor factors account for these differences in wage rates for the "same" work. There is, of course, also the question of the productivity of the labor, i.e., a comptometer operator may get paid fifteen dollars per week in one industry and another operator twenty in another industry simply because the productivity of the second operator is greater than that of the first.

In classical theories of wages the factors of production are paid according to their productivity. In actual practice this may,

PRICES OF RAYON, SILK, COTTON AND WOOL

Chart XXVI

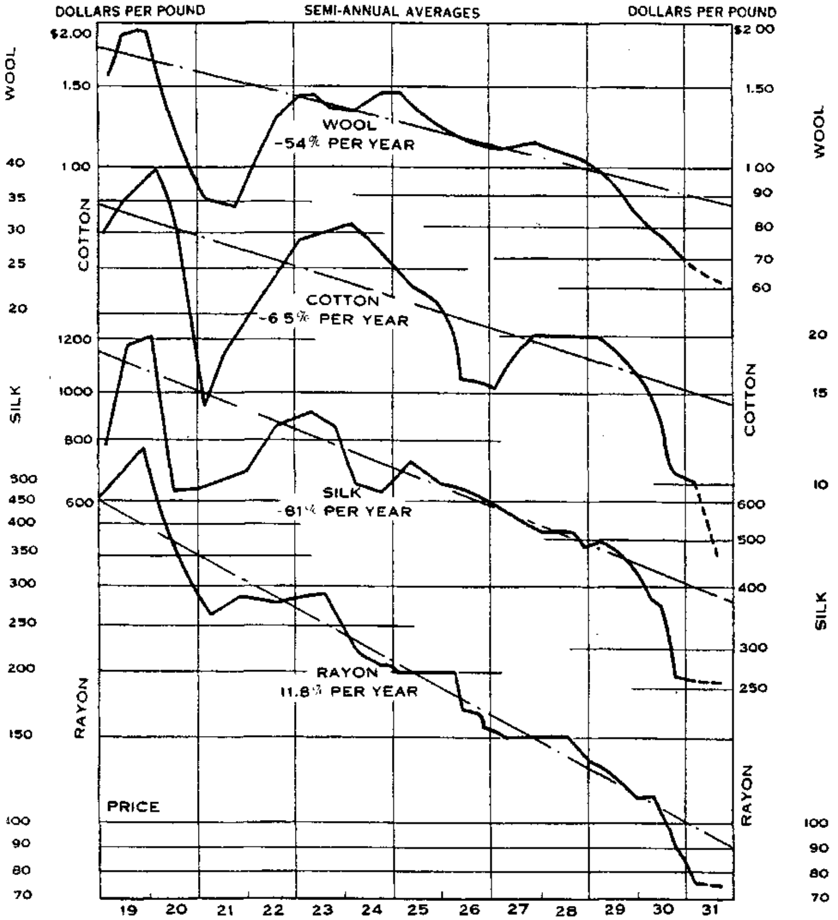


Chart shows price trends of four major fibers during the 13-year period, 1919 to 1931. Dotted lines represent estimates for concluding part of the current year. The solid lines show specific fluctuations for the 13-year period and the broken lines indicate average degree of change.

on the average, be true in a static or slowly changing economy, but it is certainly not true in a dynamic economy. In a dynamic economy the relative bargaining strength of labor and capital is of tremendous importance. Thus, in times of great depression when labor is plentiful and its productivity high it is in general paid whatever capital wishes to grant it. This may often be less than subsistence.

CONSUMPTION OF TEXTILE FIBERS

Chart XXVII

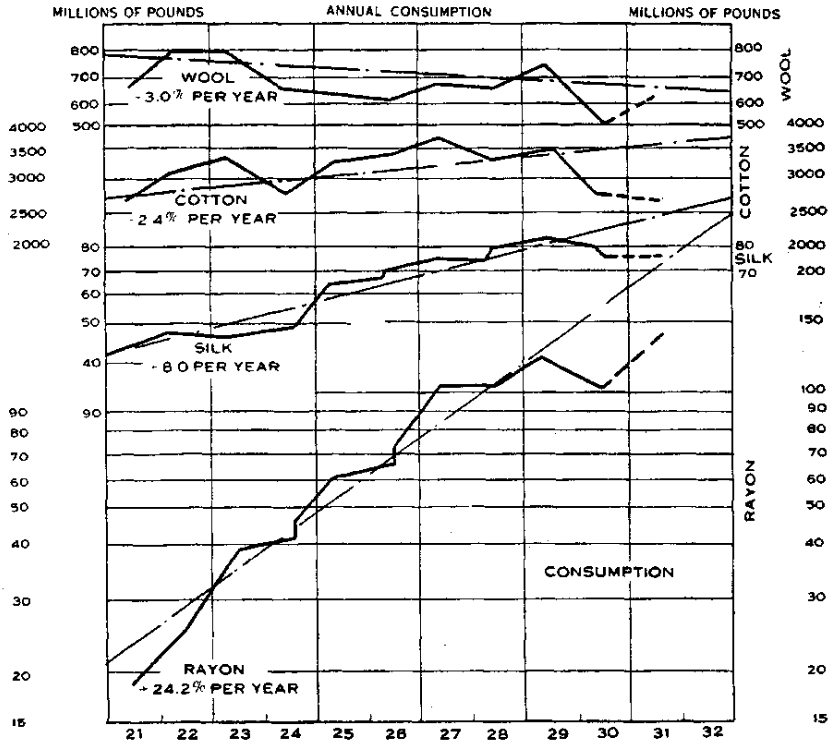
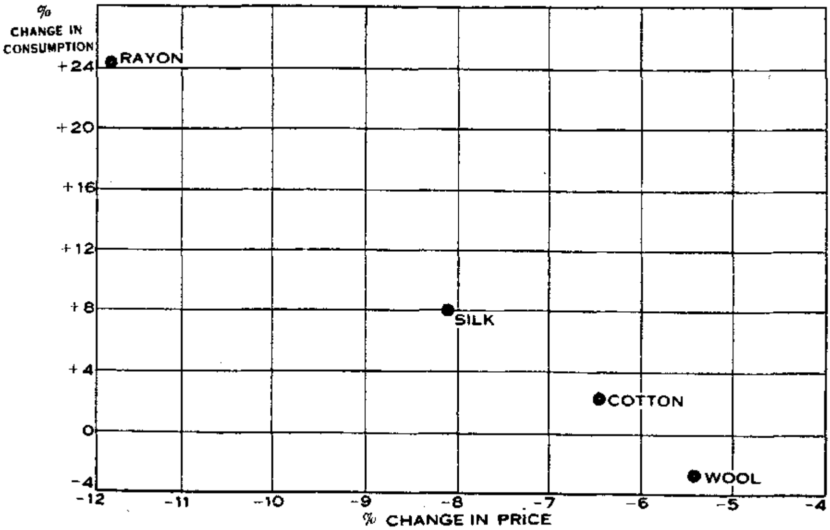


Chart shows relative consumption figures and trends of four major fibers in the decade, 1921 to 1930. Solid line traces movement of volume and dotted line represents estimates for periods for which figures are not yet available.

While an industry is growing it almost invariably pays its labor well. During periods of expansion of an industry much new labor is usually demanded by employers and such labor is almost invariably well paid, especially that part which is skilled. On the other hand, mature or declining industries usually pay low wages. Here there are almost invariably more workers than jobs and the bargaining advantage is with the employer. Workers are slow to change their occupations. Once a miner almost always a miner. Under *laissez faire* conditions it very often takes a generation to correct an oversupply of labor in a slowly declining industry, and during the period of decline wages become progressively lower except during periods of credit expansion which produce profits almost throughout the economy. Machinery installations in declining

COMPETITIVE SHIFTS BETWEEN TEXTILE FIBERS DUE TO PRICE CHANGES

CHART XXVIII



Percentage change in consumption among the competitive groups, rayon, silk, cotton and wool, seems to be proportional to percentage change in price.

or stationary industries inevitably lead to technological unemployment. Thus, for an adequate discussion of a growing economy a discussion of technological unemployment is essential. It seems advisable, however, to reserve this discussion for a later chapter.

Examples of high wage industries today are rayon, automobile, aviation and electrical. Low wage industries include cotton textiles, coal mining, shoes, and, of course, farming.

Unionization effectively increases the bargaining strength of labor. Thus, unionized labor is able in general to command higher wages than unorganized labor. Similarly, consolidations and mergers may effectively increase the bargaining strength of employers.

In small towns wages are almost invariably lower than in cities. In general, this seems to be due to the fact that numbers of employers in small towns are few and owners must compete only with the farms for the available labor supply. Building labor prices for essentially the same building are low, so that construction costs of residential building are also low. Also land values are low. Consequently, an important item of living expenses is lower in the small town, so that a "low" wage in the small town may be effectively as high as a "high" wage in the city. Also in a small town an employer may not be able to choose labor for its productivity, that is, he may constantly have to train workers.

Essentially, labor behaves like a commodity, at least so far as price is concerned. Whenever labor is scarce and there is a demand for it, it sells for a high price (at a profit to the workman), and whenever it is more plentiful than its demand it sells at a low price. The medical profession will be well paid as long as there are not enough doctors to perform the medical services demanded by the population;* engineers will command high salaries so long as new highways, new plants, new machinery are demanded in sufficient quantity to command almost all the engineering talent available, and, on the other hand, farm labor will command very low wages unless these are raised by legislative fiat.

7. *Profits.* A growing industry invariably has high profits. In fact the profit showing is what makes it possible for the industry to grow. Profits taken by the industry are in general invested in new plant and equipment, and savings from other parts of the economy are attracted into the industry. Of course, dividends can be paid in cash or in stock. If cash dividends are paid, more than the amount paid in dividends is borrowed by the industry by stock or bond issues. In a capitalistic economy the function of profit is to encourage new investment in the fields where new wants are developing.

When an industry matures, profit for the industry as a whole should theoretically be zero unless an increased velocity of circulation of money media going into goods changes the industry's markets so that it gets a new lease on life. Once the demand for a product stabilizes so that price decreases bring very little additional demand, social reasons for profits in the industry as a whole diminish or perhaps disappear entirely. Whenever productive capacity becomes sufficient to supply all demands of the economy, obviously any construction of plant in the industry other than what is necessary to replace worn out or obsolete units is social waste. Technological improvements usually provide enough profit incentive for new units to enter and force the retirement of high cost units.

*The medical profession is perhaps a rather poor example since ethical rules governing the profession almost make the profession a monoply. In other words, prices of medical services would undoubtedly decline if doctors really competed on a price basis. According to their ethics they set a fixed price (not always in the case of specialists) and compete on the basis of quality of services rendered and personality of the physician.

CHAPTER VIII

JOINT DEMAND AND LOSS LEADERS*

1. *Introduction.* Studies of automotive demand for gasoline and of factors influencing residential building indicate that growth factors are extremely important for determining the demand for a product. A new product has only a very limited market until its usefulness becomes generally known and the public must be constantly kept informed of an old product.

A new product can be made known by direct advertising of the product on its own merits, as is done in the case of a new concern making a new product, or the new product can be sold by an existing concern which uses its established reputation to help present the product. In the latter case past advertising expense for other products and good-will which make this quick introduction possible are not in general allocated to the new product. Furthermore, in its early days, such a new product need not be loaded with a proportionate share of factory and sales overhead. In this sense the new goods may be sold at a "loss" during its introductory period so that its demand will increase rapidly and make mass production possible.

In some cases sales of the new goods may greatly stimulate sales of the established line of goods. Thus, a power plant might find it practical to sell electric appliances to its customers at low cost in order to stimulate sales of power. Again, it is conceivable that a gasoline concern could afford to give away roads (indirectly through a gasoline tax) in order that its sale of gasoline might be increased and its profits sufficiently enlarged. Then, there is the important problem of chain store "loss leader merchandising."

The loss leader problem has been before the American public for over a score of years, indeed since the United States Supreme Court's decisions in the Spring of 1911 on the Standard Oil and

*This study was undertaken at the direction of Alexander Sachs when he was Director of the Division of Research and Planning of the National Recovery Administration. It represents a joint effort of C. F. Roos and H. H. Pixley. It was presented as a joint paper by Roos and Pixley at the Philadelphia meeting of the Econometric Society December 28, 1933. See *Econometrica,* April, 1934.

Tobacco cases. The Democratic platform of 1912 proposed as a remedy the "regulation of competition." The Federal Trade Commission enacted in the first Wilson Administration was in the words of Mr. Brandeis "designed to aid in preventing monopoly by preventing unfair or destructive competition, and the worst form of illegitimate competition has been found to be cut-throat competition", by which the monopolistic seller "would cut the price in the districts where a competitor establishes himself and thus destroy him, meanwhile reimbursing itself for the cut in the region by charging high prices elsewhere."

In the hearings on the original Stevens Bill before the House Committee on Interstate Commerce, January, 1915, Justice Brandeis (then Mr. Brandeis) characterized the loss leader method of competition as a species of cut-throat competition aiming at monopoly. He described graphically the destruction wrought upon the independent merchant by the "leader" or, as he preferred to call it, the "misleader" type of selling merchandise; such "unrestricted competition with its abuses and excesses, leads to monopoly, because these abuses and excesses prevent competition from functioning properly as a regulator of business" and acting "as an incentive to the securing of better quality or lower cost." Since Mr. Brandeis' analysis in 1915, the loss leader competition has gained in volume and intensity with the growth of chain stores.

The Federal Trade Commission has made a report* on loss leader merchandising to determine the extent of selling goods, other than private brands, at a loss in chain store retailing. Economic principles received little attention in this report.

The problem considered here is that of determining under what conditions one good can be sold at a loss, i.e., used as a "loss leader," in order that a sufficient amount of a second good can be sold to make a total profit greater than could be made by selling each good at a profit. The purpose of the study is to determine the effects of the use of loss leaders and companion goods on a merchant (retail, wholesale or manufacturing) as reflected in his profit and loss statement and on a consumer as reflected in the distribution of goods and in the prices paid for the goods. The problem is, of course, enormously complicated and a perfectly general solution is not yet

*Some other phases of the loss leader questions have been studied by the Federal Trade Commission in its reports on chain stores (Chain Store Leaders and Loss Leaders, Senate Document No. 51, 72nd Congress, 1st Session, submitted January 15, 1932.)

possible. However, under restrictions which will be given later, certain important general theorems or laws can be discovered.

As has been pointed out in the chain store study mentioned above, and more generally in the opening paragraphs of the present study, the term loss leader is used with various meanings. Here, by leader will be meant any good which can be used effectively by means of price concessions and advertising to increase the number of prospective customers. Thus, loss leader means a leader sold below (1) net cost (invoice price minus all discounts), or (2) gross cost (net cost plus operating expense), or (3) the usual selling price as determined either by the usual mark-up or by competition. Besides these there are other meanings which, for the purpose of the present study, have less significance. For the principal part of this study the first definition of loss leader will be used. The problems associated with the other definitions will be discussed at the end.

For simplicity the discussion will be restricted to the non-speculative case in which the merchant buys only what he can sell in the immediate future and is not interested in building up his inventory beyond the point necessary to service sales. With regard to the companion goods, which may be denoted by G_1 and G_2, the basic assumption made is that the number of customers who can be brought into contact with the good G_2, and therefore the amount of G_2 which can be sold, depends upon the quantity of G_1 sold, while the amount of G_1 that can be sold depends only slightly or not at all upon the quantity of G_2 sold. Hence, the demand for G_1 depends chiefly on the price of G_1 (consumers' purchasing power, etc., taken into account as explained in Chapters II and V and in the study of automotive demand for gasoline) while the demand for G_2 depends not only on the price of G_2 but also on the amount of G_1 sold. It is thus recognized that there are factors other than price and quantity which are important. The purpose of the study is, however, to determine what can be expected if these other factors are held constant (whether or not they can be held constant is immaterial) and price and quantity alone are varied. Price and quantity are factors that are under control by the merchant or manufacturer, whereas purchasing power, etc., are not.

It can be shown that under certain ordinary conditions the use of a loss leader results in greater profits to the merchant and a larger distribution of both goods to the consuming public. Under other conditions that seem equally likely the merchant cannot increase his profits by the use of a loss leader. A simple criterion applicable within the above-mentioned limits can be given for differentiating between these two possible cases. While the quantities of both goods, G_1 and G_2, distributed to the consuming public are increased in the case in which G_1 is used as a loss leader, the price per unit of G_2 is decreased if and only if the demand for G_2 and its cost to the merchant satisfy certain conditions which are given in Section 3. The conditions which must be satisfied are related to the well-known law of mass production, that is, the merchant, by increased purchases on his part, must be able to effect savings which when passed on to the customer increase demand sufficiently. In the special case considered in detail here it develops that if the demand can in this way be increased one hundred per cent or more over the demand that would develop if the consumer did not benefit from the merchant's discount, the price per unit of G_2 will be less under joint demand (i.e., goods sold in combination) than when G_2 is sold on its own merits.

In obtaining the above-mentioned results it is assumed that the merchant endeavors to fix his price so as to maximize his total profit per unit time arising from the sales of the two goods. The unit of time is not specified. Such questions as whether or not it is good business to sell a whole community lawn mowers in three months and then make few sales of lawn mowers over the next two or three years cannot be answered fully in terms of economic theory. Sociological questions are fundamental here. A satisfactory answer would have to take into account such factors as employment, alternative business opportunities, etc. Nevertheless, the theory that follows holds regardless of how the question of what is the proper time interval is settled; it is purposely developed from postulates which do not state how long this interval of time should be.

It should hardly be necessary to remark here that the method which is essential to examine questions of maximum and minimum is that of mathematical economics. The assumptions which define the economic situation are clearly expressed in terms of mathematical quantities and equations, and then, by the use of well-known mathematical principles only, mathematical conclusions are derived from the original quantities and equations.

2. Equations of Joint Demand. Let b_1 represent the demand for the good G_1 with respect to the merchant (not the economy) which corresponds to a zero selling price, i.e., the number of units of G_1

that would be taken if G_1 were being given away.* Let P_1 represent the selling price of G_1 and let y_1 represent the demand corresponding to the selling price P_1.

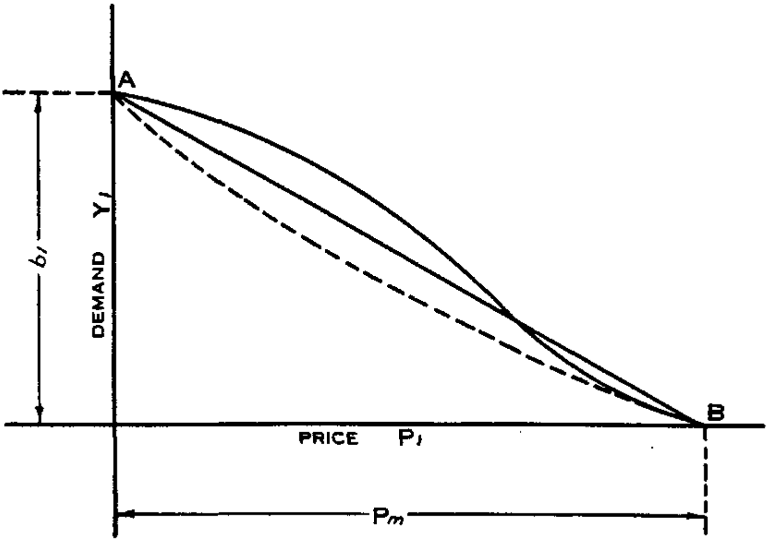


FIGURE 1

Let P_m represent the price at which demand ceases or becomes so small as to be negligible or as to force a reorganization of the manufacturing and merchandising operations involving the good G_1. If demand y_1 is graphed against price P_1 a curve joining the point $A(P_1 = 0,\ y_1 = b_1)$ to the point $B(P_1 = P_m,\ y_1 = 0)$ is obtained. Three possible forms for such a curve are shown in Figure 1. For the problem under consideration it is assumed that the demand curve is a straight line or else that it can be well approximated by a straight line. In either case the demand equation for the good G_1 is of the form

$$(2.1) \qquad y_1 = a_1 P_1 + b_1 \ ,$$

where y_1, P_1, b_1 have the meanings given above and a_1 is the slope of the line. Hence a_1 is negative, b_1 positive, and, of course, quantity

*Economists, because of poor knowledge of mathematics and incomplete economic analysis, often fall into the error of saying that demand is infinite when price is zero. For a criticism of this assumption see C. F. Roos, "Theoretical Studies of Demand," *Econometrica*, Vol. 2, 1934.

y_1 and price P_1 are always positive. The quantities a_1 and b_1 depend upon many economic, physical and psychological factors. However, since it is not proposed to study effects of changes in these factors, there is no need to complicate the study by introducing these factors in the notation. Thus, if there is a shift in the reaction of demand to price, a_1, b_1 or both change. So far as the validity of the analysis is concerned, a_1 and b_1 can both be taken to be functions of time; that is, quantities changing with changes in factors not mentioned specifically in the equation (2.1).

The number of customers who can be brought into contact with the goods G_2 as potential purchasers consists of a group of number N which is attracted to G_2 by way of G_1 and a second group which comes intending to buy G_2 or happens upon G_2 in some other way. The first group will be a certain proportion of the number of buyers of G_1 which may be assumed to be proportional to the number of sales y_1 of G_1; that is, $N = hy_1$. The second group may be assumed to be constant so far as the factors being studied are concerned.

The number of units of G_2 that would be purchased by the group N if the selling price P_2 were zero would be proportional to N, that is, equal to kN, each person taking as much as he could transport away in the period under consideration. This, of course, does not mean an infinite amount. Thus, the demand for G_2 with $P_2 = 0$ would be hky_1 where h and k are constants. When the price of G_2 is an amount P_2 greater than zero, the demand hky_1 will be modified by an amount $a_{21}P_2 + b_{21}$ (a decreasing function in which $a_{21} < 0$, $b_{21} > 0$), so that the demand due to the N customers attracted by G_1 is $y_{21} = cy_1 + a_{21}P_2 + b_{21}$ where $c = hk$.

The assumption that the part of the demand for G_2, which comes from the group of prospective customers which happens upon G_2 without having come to buy G_1, follows a law similar to the demand law (1) for G_1 requires that $y_{22} = a_{22}P_2 + b_{22}$ where $a_{22} < 0$, $b_{22} > 0$. The total demand for G_2 is the sum of y_{21} and y_{22}, that is,

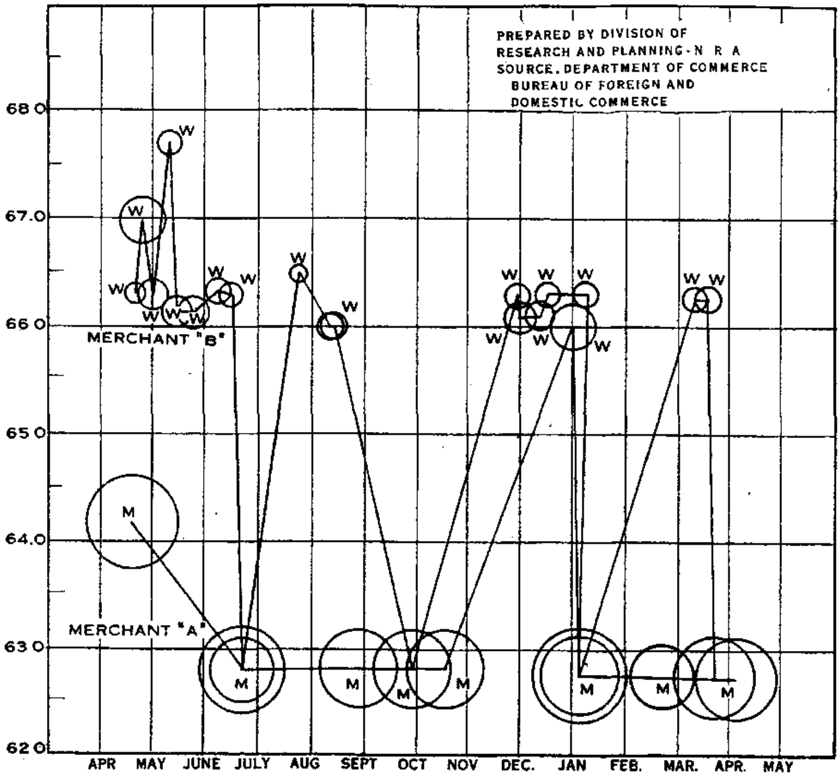
$$(2.2) \qquad y_2 = y_{21} + y_{22} = cy_1 + a_2P_2 + b_2 ,$$

where $a_2 = a_{21} + a_{22} < 0$, $b_2 = b_{21} + b_{22} > 0$, and $c > 0$.

Now let p_1 be the cost to the merchant of a unit of G_1 and let p_2 be the cost to him of a unit of G_2 and let the respective selling prices be $P_1 = p_1 + s_1$ and $P_2 = p_2 + s_2$, so that s_1 and s_2 represent the respective profits or losses per unit of goods sold.

EFFECT OF QUANTITY PURCHASED AND SOURCE OF SUPPLY ON PRICE

CHART XXIX



It is extremely difficult to define factory cost exactly, especially when rates of production vary. The direct cost of goods to a wholesaler or retailer can, of course, be determined by invoices. However, the invoice cost has long depended upon quantity purchased. The practice of quantity discounts was inaugurated to recognize that selling costs for large quantities are less than for small quantities.

This chart shows purchases of two competing merchants. Merchant B usually purchased from a wholesaler in lots of 3 to 6, paying around 66 cents for a certain commodity. Merchant A purchased from the manufacturer in lots of 24 to 48, paying about 63 cents. Several times Merchant B made purchases of 24 to 36 and paid the same low price paid by A. Note the lower prices secured by purchasing in quantity direct from the manufacturer.

M = Purchases fromManufacturer
W = Purchases from Wholesaler
The Circle Indicates the Quantity Purchased
The Position of the Circle Shows the Price Paid per Unit and the Time of Purchase

The cost prices p_1 and p_2 to the merchant usually depend upon the quantities y_1 and y_2 which he buys, it being assumed that he buys only as much as he sells over a reasonable period of time. Let p_{01} and p_{02} be the prices that the merchant would have to pay for "small" amounts of G_1 and G_2 . For larger amounts he ordinarily receives a discount. Thus, the merchant's (manufacturer's) cost prices depend upon the amounts which he buys (manufactures), and the relationships can be taken in the form

$$p_1 = (1 + d_1 y_1) p_{01} \qquad p_2 = (1 + d_2 y_2) p_{02}$$

where $-d_1 y_1$ and $-d_2 y_2$ (both positive, since d_1 and d_2 are both negative) are the discount rates for the merchant under consideration. Of course, different retail (wholesale) merchants or manufacturers would be assigned different values by the wholesale merchant (manufacturer or seller of raw materials) for the quantities p_{01}, p_{02}, d_1, d_2 depending on the size and credit standing of the retail (wholesale or manufacturing) concern involved. The prices p_{01} and p_{02} are the actual prices paid for small quantities. Hence, $-d_1 y_1$ and $-d_2 y_2$ represent actual discounts given for quantity purchasing and not the customary paper discount deducted from a published price. Chart XXIX gives some idea of how important quantity discounts can be in determining cost and by inference price.

In the preceding discussion the words retail (wholesale, manufacturers) and wholesale (manufacture, seller of raw materials) have been carried through. This will be discontinued and only the word retail used. The careful reader will be able to make the generalizations as he reads. Thus, although the following discussion may appear to apply only to retailing, it is in reality much more general.

If the equation $p_1 = (1 + d_1 y_1) p_{01} = p_{01} + d_1 p_{01} y_1$ is graphed considering p_1 and y_1 as the variables, a straight line as shown in Figure 2 is obtained. From this it might be inferred that by buying a sufficient quantity y_{10} the merchant could get the goods at a zero price. This is, of course, not true since the wholesale merchant would never allow this. There is a price p_{1m} below which he will not allow the price p_1 to go. This restriction is, however, automatically taken care of by the previous assumption that the retail merchant buys only what he can sell and he cannot increase this amount indefinitely, even by selling at a zero price.

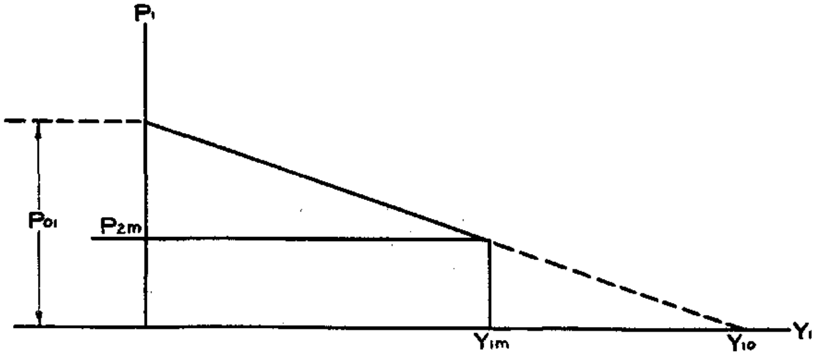


FIGURE 2

From equation (2.1), when selling price $P_1 = 0$, it follows that sales $y_1 = b_1$; hence b_1 is the largest quantity that the merchant will consider buying. By choosing d_1 properly the cost price $p_1 = (1 + d_1 y_1) p_{01}$ will always be the one for which the wholesale merchant would sell when y_1 is restricted to be between 0 and b_1. The assumption which restricts the quantities y_1 and y_2 in this way is essentially an assumption that only the non-speculative case in which the merchant is not interested in building up his inventory is being considered.

3. Maximum Profits. The selling prices P_1 and P_2 are equal to cost p_1 and p_2 plus mark up (mark down) s_1 and s_2, so that by the help of the preceding analysis

$$P_1 = p_1 + s_1 = (1 + d_1 y_1) p_{01} + s_1$$
$$P_2 = p_2 + s_2 = (1 + d_2 y_2) p_{02} + s_2 \; .$$

By substituting these values for selling prices P_1 and P_2 in equations (2.1) and (2.2) and solving them for sales y_1 and y_2, the following expressions are obtained

$$y_1 = \frac{a_1}{1 - a_1 d_1 p_{01}} (p_{01} + s_1) + \frac{b_1}{1 - a_1 d_1 p_{01}}$$

$$y_2 = \frac{c}{1 - a_2 d_2 p_{02}} \left[\frac{a_1}{1 - a_1 d_1 p_{01}} (p_{01} + s_1) + \frac{b_1}{1 - a_1 d_1 p_{01}} \right]$$
$$+ \frac{a_2}{1 - a_2 d_2 p_{02}} (p_{02} + s_2) + \frac{b_2}{1 - a_2 d_2 p_{02}}$$

In comparing changes in selling prices P_1 and P_2, relative changes, rather than unit changes are significant. For this reason the prices themselves will not be used; the price ratios obtained by dividing price P_1 by the base cost price p_{01} and price P_2 by the base cost price p_{02} are the significant quantities. Or, what accomplishes the same objective is the choosing of the units of price so that $p_{01} = p_{02} = 1$. Expressed in these units the above expressions for y_1 and y_2 become

$$(3.1) \qquad y_1 = e_1(1 + s_1) + f_1$$
$$y_2 = g[e_1(1 + s_1) + f_1] + e_2[1 + s_2] + f_2$$

where, by definition,

$$e_1 = a_1/(1 - a_1d_1) \ , \qquad f_1 = b_1/(1 - a_1d_1) \ ,$$
$$e_2 = a_2/(1 - a_2d_2) \ , \quad f_2 = b_2/(1 - a_2d_2) \ , \quad g = c/(1 - a_2d_2) \ .$$

It is clear that the conditions $1 - a_1d_1 > 0$ and $1 - a_2d_2 > 0$ must be satisfied, since otherwise e_1 and e_2 would be positive and y_1 and y_2 could be increased indefinitely by increasing the selling prices $p_1 + s_1$ and $p_2 + s_2$.

It may be assumed that the merchant desires to choose his mark up (mark down) s_1 and s_2 so that the total profits

$$E = s_1y_1 + s_2y_2$$

are as large as possible. Using the expressions (3.1) for y_1 and y_2 in E and collecting terms it follows that the total profits are given by

$$E = e_1s_1^2 + ge_1s_1s_2 + e_2s_2^2 + (e_1 + f_1)s_1$$
$$+ (ge_1 + gf_1 + e_2 + f_2)s_2 \ .$$

The total profits, E, depend upon the two variables s_1 and s_2, and for a maximum the two first partial derivatives $\partial E/\partial s_1$ and $\partial E/\partial s_2$ must be equal to zero.* Applications of the rules for differentiation to find a maximum give the equations

$$(3.2) \qquad 2e_1s_1 + ge_1s_2 = -y_{1c} \ ,$$
$$ge_1s_1 + 2e_2s_2 = -y_{2c} \ ,$$

where the notation

$$y_{1c} = e_1 + f_1 \ , y_{2c} = gy_{1c} + y_{2o} \text{ where } y_{2o} = e_2 + f_2 \ ,$$

has been introduced.

*See, for example, W. F. Osgood, *Differential and Integral Calculus*, New York, 1914, or any elementary book on calculus.

A comparison of these last three equations with equations (3.1) shows that y_{1c} and y_{2c} are the demands for G_1 and G_2 when $s_1 = s_2 = 0$, that is, they are the demands when the selling prices are equal to the cost prices p_1 and p_2, while y_{2o} is the demand for G_2 when the selling price P_2 is equal to the cost price p_2 and the demand y_1 is equal to zero.

These equations can be solved easily by well-known mathematical methods and the values of s_1 and s_2 which satisfy them can be found to be

$$
(3.3) \qquad \begin{aligned} s_1 &= (ge_1y_{2c} - 2e_2y_{1c}) / (e_1e_2 - g^2e_1^2) \ , \\ s_2 &= (2y_{2o} - gy_{1o}) / (g^2e_1 - e_2) \\ &= (y_{2c} + y_{2o}) / (g^2e_1 - e_2) \ . \end{aligned}
$$

In addition to the above first order conditions which must hold for a maximum but may also hold in other cases, second order conditions must be investigated to insure a maximum. The second order conditions are

$$
\partial^2E/\partial s_1^2 < 0 \ , \quad \partial^2E/\partial s_2^2 < 0 \ ,
$$
$$
(\partial^2E/\partial s_1\,\partial s_2)^2 - (\partial^2E/\,\partial s_1^2)\cdot(\partial^2E/\,\partial s_2^2) < 0 \ .
$$

In terms of the coefficients these three inequalities are

$$
(3.4) \qquad 2e_1 < 0 \ , \quad 2e_2 < 0 \ , \quad (ge_1)^2 - 4e_1e_2 < 0 \ .
$$

If g, e_1, e_2 are replaced by their values in terms of c, a_1, a_2, d_1, d_2, the third inequality of (3.4) becomes

$$
c^2a_1 - 4a_2(1 - a_1d_1)\ (1 - a_2d_2) < 0 \ .
$$

If the ratio a_1/a_2 is represented by R so that $a_1 = Ra_2$, a_1 can be eliminated from the inequality and it can be reduced to the form

$$
(3.5) \qquad R < 4(1 - a_2d_2)/[c^2 + 4a_2d_1(1 - a_2d_2)] \ .
$$

Hence this inequality must be satisfied by the ratio $R = a_1/a_2$ where a_1 and a_2 are the coefficients used in the demand equations (2.1) and (2.2). These coefficients represent the rate of change of the demand y_1 with respect to price P_1 and the rate of change of demand y_2 with respect to price P_2. Therefore, they can be used as

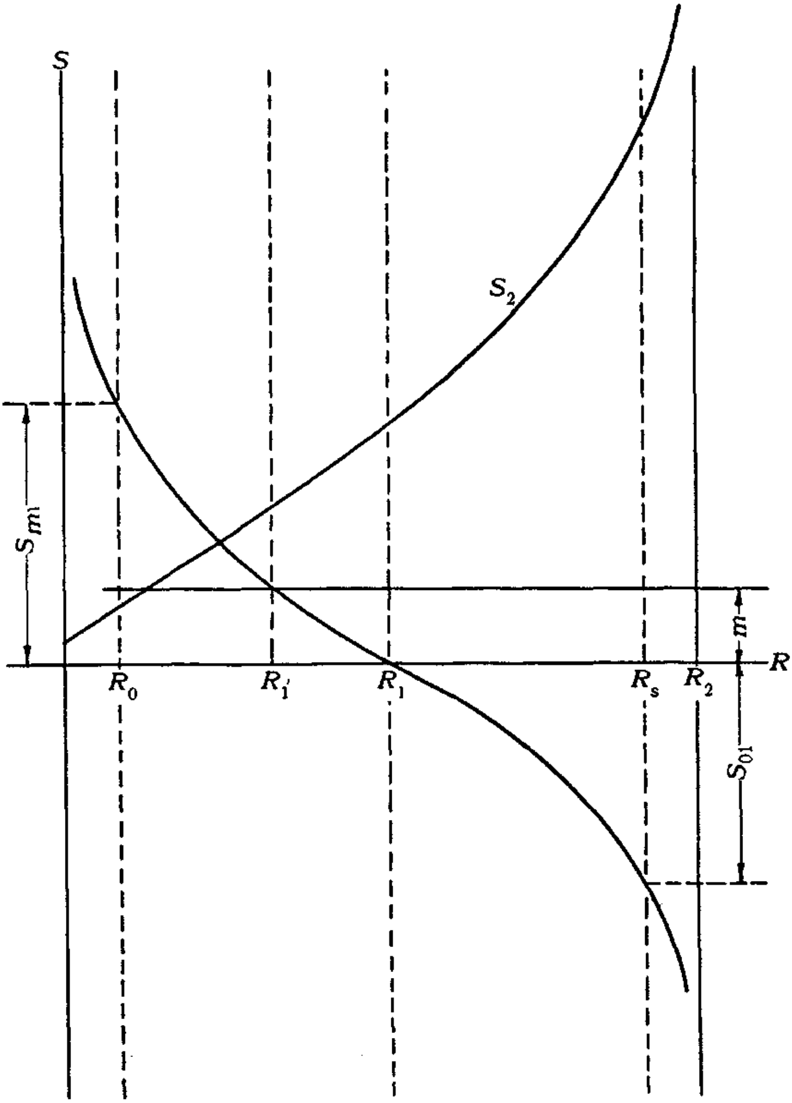


FIGURE 3

measures of the sensitiveness* of demand of the two commodities G_1 and G_2. Then, R is the ratio of the sensitiveness of demand of the two commodities and the inequality (3.5) means that the values (3.3) do not give a maximum in the case where the sensitivity of demand for G_1 is too large in comparison with that of G_2. It can be shown that when (3.5) is not satisfied there is no satisfactory maximum profit, since to maximize total profit E the merchant would set prices so that quantities sold (y_1 or y_2) would be zero or else set prices P_1 or P_2 equal to zero. None of these solutions would be considered a practical solution by a merchant. However, it can be shown that (3.5) is satisfied in many, if not most, actual cases, so that the inequality does not seriously restrict the application of the analysis. In fact, if (3.5) is not satisfied, a merchant should not use the goods G_1 and G_2 in joint demand relationship.

After restricting the problem to commodities G_1 and G_2, whose ratio of sensitiveness satisfies the inequality (3.5), it requires a detailed study of the formulas for s_1 and s_2 to reveal when an advantage can be gained by selling G_1 at a loss, and, what is just as important, when there is no such advantage. If in the formulas (3.3) the quantities g, e_1, e_2 are replaced by their values in terms of c, a_1, a_2, d_1, d_2, and if a_1 is then eliminated by the equation $a_1 = Ra_2$, the following forms are obtained:

$$s_1 = \frac{(1-a_2d_2)\ (1-Ra_2d_1)\ [(cy_{2c}+2a_2d_1y_{1c})\ R-2y_{1c}]}{-a_2R[\{c^2+4a_2d_1(1-a_2d_2)\}\ R-4(1-a_2d_2)]}$$

$$s_2 = \frac{(1-a_2d_2)^2\ (y_{2c}+y_{20})\ (1-Ra_2d_1)}{-a_2[-\{c^2+4a_2d_1(1-a_2d_2)\}\ R+4(1-a_2d_2)]}$$

The values of s_1 and s_2, when graphed for values of R from $R=0$ to $R=R_2$, where $R_2 = 4(1-a_2d_2)/[c^2+4a_2d_1(1-a_2d_2)]$ (see inequality (3.5), yield curves of the character shown in Figure 3. The value R_1 is given by $R_1 = 2y_{1c}/[cy_{2c}+2a_2d_1\ y_{1c}]$ which can be shown to lie between 0 and R_2 as indicated in the figure.

It appears from a study of the graph that it is possible to attain a maximum profit E by selling G_1 at a loss only if the ratio of the sensitiveness of demand satisfies the inequalities $R_1 < R < R_2$. Actually the possible range is somewhat smaller than this because, if it is assumed that the merchant will not sell G_1 at a zero or negative price, s_1 has a definite lower limit s_{01} (negative) which is num-

*Here sensitivity is used to refer to price sensitivity.

erically less than the cost price p_1. Corresponding to this value s_{01} is a value R_s as shown on the graph. Hence the actual limits of R for which a profitable use of G_1 as a loss leader can be made are given by the inequality

$$(3.6) \qquad R_1 < R < R_s \ .$$

There is also a lower limit R_0 to the values that R can take. This is due to the fact that if P_1, which is defined by cost plus s_1, is made large enough, the demand y_1 becomes zero. R_0 corresponds to the largest possible value s_{m1} of s_1. This value R_0 is less than R_1 as is shown in Figure 3. If R lies between R_0 and R_1, a maximum profit would result only if both G_1 and G_2 were sold at a profit.

4. Quantities Sold and Prices Paid by Consumers. If P_1 and P_2 represent the selling prices as used above and $\overline{P}_1$, $\overline{P}_2$ represent the selling prices that would result if $c = 0$ (i.e., if the demand for G_2 were independent of the demand for G_1), the following relations can be proved:

$$\overline{P}_1 - P_1 = ge_1s_2/2a_1 > 0 \qquad \text{so that } \overline{P}_1 > P_1 \quad ,$$

$$\overline{P}_2 - P_2 = [g(2e_2y_{1c} + ge_1y_{2c})]/[2a_2(g^2e_1 - 4e_2)] \ (2a_2d_2 - 1) \ .$$

Hence the price per unit of G_1 is less under joint demand than under separate demand, whether or not G_1 can be used profitably as a loss leader. It follows at once, from the nature of the demand equation, that the amount of G_1 sold increases under joint demand.

As for the sign of $\overline{P}_2 - P_2$, the fraction shown as the first factor is positive. This follows from the fact that all of the quantities involved in the fraction are positive except a_2, e_1, e_2. The factor $g^2e_1 - 4e_2$ in the denominator has been proved positive by the inequalities which follow equations (3.3). Hence $\overline{P}_2 - P_2$ has the same sign as the factor $2a_2d_2 - 1$.

The rest of the discussion is divided into two cases.

Case 1. $2a_2d_2 - 1 < 0$ or $a_2d_2 < 1/2$.

If y_{2d} represents the demand that would result if d_2 were zero; i.e., if the merchant received no discount for quantity purchasing, it follows from equations (3.1) that $y_2 = y_{2d}/(1 - a_2d_2)$ whence $a_2d_2 = (y_2 - y_{2d})/y_2$. The condition that a_2d_2 is less than $1/2$ can now be written in the form $y_2 < 2\,y_{2d}$. Therefore, this case covers the situation in which the passing on of lowered costs to the consumer results in an increased demand which is less than twice the demand without lowered costs.

Case 2. $2a_2d_2 - 1 > 0$.

By an analysis similar to the above it follows that here the demand with lowered costs is more than twice the demand without lowered costs.

In the first case

$$\overline{P}_2 - P_2 < 0 \text{ or } \overline{P}_2 < P_2 \ ;$$

i.e., there is an increase in price per unit for G_2. In the second case

$$\overline{P}_2 - P_2 > 0 \text{ or } \overline{P}_2 > P_2 \ ;$$

i.e., there is a decrease in price per unit for G_2.

An inquiry into the economic significance of the two cases developed in the last section reveals the striking fact that in the second case the situation is essentially the well-known mass production phenomenon where increased demand and lowered costs react on each other and increase demand sufficiently to make it profitable for the entrepreneur to lower prices. It is, of course, not at all uncommon for a reduction in cost with concomitant reduction in price to result in the doubling of demand. In the first case the reaction of increased demand and lowered costs does not result in a demand large enough to make lower prices profitable.

As is known, the above mass production phenomenon is not applicable to all goods G_2, nor when applicable to a good G_2 is it necessarily applicable to all phases of development of the manufacturing and merchandising of the good. Hence the possibility of the second case depends upon the nature of the good G_2. It also depends to a large extent upon the size of the merchant.

If the merchant is a small one with a small discount he is probably forced into the first case and the consumer pays more per unit for G_2. However, if the merchant is large so that his prices have an appreciable effect on the demand in the industry as a whole in his economy, he has a good chance of operating under the second case with consequent profit to the consumer. Even if he is small he may force his competitors into using G_1 as a loss leader, and, if G_2 is used by many merchants as the companion good, these might easily induce the same results as would a single large merchant.

When a good G_1 is generally used as a loss leader there is always the possibility that the "normal" selling price of the good becomes permanently lowered as in the case of certain goods such as sugar in grocery retailing and certain brands of toilet articles in drug retailing. This will lower its effectiveness as a leader at a given price but will not necessarily remove it from the class of effective leaders.

It is possible that the lowering of the normal selling price is often the result of "unsound" use of a good as a loss leader. For instance, if the demand equation for sugar for the industry as a whole were used, in the light of the preceding analysis there is doubt that this good could qualify as a loss leader for the whole industry resulting in greater consumption and profits. Sugar is one of those products for which an individual grocer's demand is highly elastic, whereas the total demand is inelastic.*

It remains to prove that the demand for G_2 always increases under joint demand. Let y_2 represent the demand for G_2 as used throughout this paper, and let $\bar{y}_2$ be the demand that would result if the demand for G_2 did not depend upon the demand for G_1, i.e., if $c = 0$. It can be shown that

$$\bar{y}_2 - y_2 = -g\frac{-2e_2y_{1c} - ge_1y_{2o}}{2(g^2e_1 - 4e_2)} < 0 \ .$$

Hence in virtue of the inequalities already established, $\bar{y}_2 < y_2$ in any case, which means that the demand for G_2 increases. This makes it easier to interpret the case in which the price of G_2 increases.

Since the demand y_2 decreases in proportion to the increase in the price P_2, it follows that any increase in the price will result in a decrease in y_2 unless it is counterbalanced by an increase in y_1. Since increases resulting from y_1 are limited, it follows that P_2 cannot be increased indefinitely if y_2 is also to increase. Since it has been shown that y_2 does increase, the conclusion is reached that any increase in P_2 is limited.

5. Drawing Trade from Competitors by the Use of Loss Leaders. The remarks made about sugar introduce another considera-

*Henry Schultz, *Statistical Laws of Demand and Supply with Special Application to Sugar,* Chicago, 1928, Chapters 1-3.

tion. The analysis of this study deals only with an individual merchant or a cooperating group of merchants. Such a merchant or group can probably use almost any staple product as a satisfactory loss leader temporarily by drawing trade away from competitors. If this is the principal way in which the increased demand for G_1 arises, there is probably a net loss to the industry. A loss always occurs in such an instance if other merchants take up the practice. However, if G_1 is a good for which the total demand is highly sensitive to price, the increase in demand which results when a merchant lowers the price may well be a real increase and not merely a shifting of sales from one merchant to another. One way in which this more desirable situation could be guaranteed would be to force a merchant to use sensitivities of demand belonging to the industry rather than those which belong to his own firm in applying the criterion (3.6). A rather serious objection, however, to such a provision is that the sensitivities of demand for whole industries have not as yet been studied sufficiently to give results useful for this purpose.

So far, no account has been taken of the institutional factor which requires that gross earnings minus invoice cost must be greater than or equal to some percentage of total invoice costs, that is,

$$(5.1) \qquad s_1 y_1 + s_2 y_2 \geqq r(p_1 y_1 + p_2 y_2) \ ,$$

where r is the customary gross profit percentage, 10%, 15%, 20%, etc., as the practice may be.

If this inequality does not hold, the commodities cannot be used in a joint demand relationship, i.e., each commodity must be sold on its own merits, at least as far as advertising the commodity is concerned. If one of the products moves too slowly on its own merits, it can be dropped.

Sometimes stores resort to such restrictions as "only one to a customer," or "for retail trade only," in order to boost the coefficient c to a sufficiently high level to make the inequality (5.1) hold. If such devices are used persistently they force the market price for G_1 down and decrease its value as a loss leader. This situation is similar to the previously discussed case of lowered market price. In view of the impracticability of the control mentioned there, an obvious way to prevent such loss leader practice is to do away in some manner with the devices mentioned above for increasing c.

6. Other Loss Leader Problems. The modified problem in which the second and third definitions of loss leader are used has some interesting aspects. If the second definition is used, so that loss leader means a leader sold below gross cost, the resulting situation can be described by modifying Figure 3 slightly. For this replace the horizontal R-axis by the horizontal $\bar{R}$-axis above it, the distance m between them representing the percentage mark-up on net cost which would be necessary to cover fixed expense only, the percentage being based on the net cost. Since p_1 and p_2 are price indices rather than prices, the overhead m is given in terms of percentage rather than cents or dollars. Then s_1's and s_2's which lie below the $\bar{R}$-axis represent loss leaders. The interval in which a profitable use of G_1 as a loss leader can be made is given in this case by the inequality

$$\bar{R}_1 < R < R_s. \tag{6.1}$$

For the third problem in which a loss leader is a leader whose price is below the usual selling price, let m represent the usual mark-up or gross profit given in terms of a percentage as before. This gives a modified position of the $\bar{R}$-axis higher than that just found above if gross profit for the department involved is greater than fixed expense. The position of $\bar{R}_1$ is shifted to the left again in such a case and the inequality (6.1) holds if the modified $\bar{R}_1$ is used.

Public utility distribution of a product such as natural gas to household consumers and industrial consumers offers another slightly different loss leader problem. The following notation is useful:

$y_2 =$ quantity of gas sold to household consumers;

$p_1 =$ field cost of a unit of gas plus pipe line cost per unit to consuming city;

$p_2 =$ the extra city pipe line and other installation cost per unit to household consumer;

$p_{01} =$ the price (including pipe line cost to the consuming city) the distributing company would pay for an amount y_{20} consumed by household consumers only at a price $(1 + r)p_{01}$ where r represents "normal" mark-up;

$d_1 =$ percentage reduction in cost due to increased amount of gas consumed by industrial concerns;

d_2 = decrease in distributing costs to small consumers (household consumers) for increase in demand; and

p_{02} = distributing cost for small amount to household consumers.

Then, obviously,

$$p_1 = (1 + d_1 y_1) p_{01} \text{ and } p_2 = (1 + d_2 y_2) p_{02} \ .$$

The demands from the two classes of consumers will, of course, be different; that is, household consumer demand will be less elastic than industrial consumer demand. The two demands may be taken to be given by

$$y_1 = a_1 P_1 + b_1 \ ,$$

$$y_2 = a_2 P_2 + b_2 \ ,$$

where P_1 and P_2 are prices to large and small consumers respectively. Obviously

$$P_1 = (1 + d_1 y_1) p_{01} + s_1 \ ,$$

$$P_2 = p_1 + p_2 + s_2 = (1 + d_1 y_1) p_{01} + (1 + d_2 y_2) p_{02} + s_2 \ ,$$

where s_1 and s_2 are the mark-ups (mark-downs) to industrial and household consumers. Then

$$y_1 = a_1[(1 + d_1 y_1) p_{01} + s_1] + b_1 \ ,$$

$$y_2 = a_2[(1 + d_1 y_1) p_{01} + (1 + d_2 y_2) p_{02} + s_2] + b_2 \ .$$

Solving these equations for y_1 and y_2 gives

$$y_1 = \frac{a_1}{1 - a_1 d_1 p_{01}} (p_{01} + s_1) + \frac{b_1}{1 - a_1 d_1 p_{01}} \ ,$$

$$y_2 = \frac{a_2 p_{01}}{1 - a_2 d_2 p_{02}} + \frac{a_2 d_1 p_{01}}{1 - a_2 d_2 p_{02}} \left[\frac{a_1}{1 - a_1 d_1 p_{01}} (p_{01} + s_1) + \frac{b_1}{1 - a_1 d_1 p_{01}} \right]$$

$$+ \frac{a_2}{1 - a_2 d_2 p_{02}} \left[p_{02} + s_2 \right] + \frac{b_2}{1 - a_2 d_2 p_{02}} \ .$$

As pointed out previously in comparing changes in selling prices P_1 and P_2, relative changes rather than unit changes are significant. For this reason the prices themselves should not be used; the price ratios obtained by dividing price P_1 by the base cost price

p_{01} and price P_2 by the base cost price p_{02} are the significant quantities. Or, as stated before, what accomplishes the same objective is the choosing of the units of price so that $p_{01} = p_{02} = 1$. Expressed in these units the preceding expressions for y_1 and y_2 become

$$(6.2) \qquad y_1 = e_1(1 + s_1) + f_1 \ ,$$
$$y_2 = g[e_1(1 + s_1) + f_1] + e_2[1 + s_2] + f_2 \ ,$$

where

$$e_1 = a_1/(1 - a_1d_1) \ , \ f_1 = b_1/(1 - a_1d_1) \ ,$$
$$e_2 = a_2/(1 - a_2d_2), \quad f_2 = b_2/(1 - a_2d_2) + a_2/(1 - a_2d_2) \ ,$$
$$g = a_2d_1/(1 - a_2d_2) \ .$$

This problem is, therefore, essentially the same as the general one already discussed; see equation (3.1).

7. *General Observations on Loss Leader Merchandising.* It is clear that it is entirely sound and, under certain restrictions, socially desirable economics to permit the practice of using loss leaders, provided the ratio of the sensitivities of demand (for the economy) of the loss leader and its companion commodity is neither too small nor too large. (See inequality (3.6).) The net effect of this is to increase the consumption of the loss leader and of the companion commodity and at the same time give merchants greater profits. This is accompanied, of course, by a decrease in the price of G_1 while the price of G_2 may be decreased or not. If increased, it is probably by a moderate amount only. Hence it is not at all necessary to require that all goods be sold above cost, nor is it desirable. Such a situation leads only to decreased consumption. In other words, the whole notion of selling every good on a cost plus basis is faulty; it is based on inadequate knowledge of joint demand.

CHAPTER IX

Production, Cost and Profit

1. Cost Theory. Cost and cost accounting have received the attention of economists and accountants for ages and yet even some of the best books on accounting theory and economic theory contain statements which reveal that their authors have little knowledge of the mathematical theories of cost which they are trying to explain. Worst of all, even the authors of the commonly accepted (or debated) classical theories of cost have apparently not fully understood the theories which they have produced. The common triple proposition that (1) free competition brings the cost of production down to a minimum, (2) under this régime the rate of remuneration of each service is equal to its marginal productivity, and (3) the whole quantity of the manufactured product is distributed among the productive services, is invariably loosely stated and more often than not, misunderstood. In reworking the statical theory of marginal productivity which is responsible for this triple proposition, Henry Schultz announces that (1) the first proposition above is essentially a condition of solution and not a theorem or consequence of the marginal productivity theory, (2) propositions two and three are true only if all the coefficients of production (quantities of the services of the factors of production which enter into the manufacture of a unit of the commodity under consideration) are *compensatory* (i.e., they form a group such that if the amount per unit produced of any one of them is decreased, this may be compensated by an increase in the others) and if the mathematical formula exhibiting this relation defines the quantity produced as a homogeneous function of the first degree in these coefficients of production.*

The classical theory of cost (marginal productivity theory) is essentially a static one and as such is open to considerable question. As previously mentioned and repeated here for emphasis, it is doubtful that society has ever reached a stage of stagnation which could give rise to static economic equilibrium and hence it is almost

*Henry Schultz, "Marginal Productivity and the General Pricing Process," *Journal of Political Economy*, Vol. 37 (1929), pp. 505-545.

certain that any theory based on an assumption of static equilibrium is inadequate. Nevertheless, parts of such theory may carry over into dynamic economics.

Some writers have postulated certain types of cost functions, as, for example, the quadratic and cubic function of G. C. Evans,* but for the most part discussion has centered around linear functions. Little consideration has been given to the problem of building up realistic cost functions out of such items as wages, power costs, raw material costs, overhead costs, etc. Again, not a great deal of attention has been given to the question of the relative flexibility of the various factors. It is one thing to say that the factors of production will be shifted so that cost is a minimum, and it is quite another thing to examine relative amounts of time required for such shifts, and possible effects on the economy, especially on such things as wages, income of various groups, and velocity of circulation of money and credit.

It seems that no one has discussed the question of how profit curves, for the individual producer and for the industry, as functions of (1) price of product and (2) units of output, depend upon the elasticity of demand and how elimination of relatively inefficient workers to obtain maximum profits can lead to decreased "purchasing power," which in turn can lead to even greater elimination of workers by a shifting of the profit peak so that a point of widespread industrial unemployment can result.

2. Coefficients of Production. In the production of each unit of almost any commodity or service G there are expenses due to raw materials and power, services of real estate, services of persons, services of capital and services of machinery. These ingredients of G may be defined to be the *factors of production of G*, and the quantities of the services of the factors of production which enter into the production of one unit of G can be defined to be the *coefficients of production of G.*

If the cost function of an actual producer of a commodity or service is analyzed, it is usually found that some of the coefficients of production, which may be denoted by f_a are constants (at least for a short period of time), others vary with the rate of production, acceleration of production and the price of the commodity or service produced, and still others exhibit a special kind of variation and

*G. C. Evans, *Mathematical Introduction to Economics,* 1930, pp. 2 and 3 and 50-60.

are compensatory. The amount of leather required to manufacture one shoe of a given kind is the same whether one shoe or a hundred shoes are produced per hour. Certain items of overhead expense, on the other hand, vary with the output per unit of time and therefore come under the second classification. Power for manufacturing shoes may be water, steam, electric, gas, or human, or all five, and the particular combination selected by a manufacturer depends upon the prices of the various services. This last group is by definition *compensatory.*

Before proceeding further it seems desirable to consider a simple example which illustrates the nature of the relation between the compensatory coefficients. Suppose that a farmer desiring to produce one thousand bushels of potatoes finds that he can do so by using 6 acres of land, 300 pounds of fertilizer, 10 pounds of calcium arsenate for spraying purposes, 70 bushels of seed, and 60 hours of labor consisting of a man and two horses. Then $f_1 = 6p_1/1000$, $f_2 = 300p_2/1000$, $f_3 = 10p_3/1000$, $f_4 = 70p_4/1000$ and $f_5 = 60p_5/1000$, where p_1, p_2, p_3, p_4, and p_5 are the prices of the factors of production. The price p_1 is, of course, not the full price of the land: it is only that part of the price which can be allocated to the production for the year. The point represented by the set $(.006p_1, .300p_2, .010p_3, .070p_4, .060p_5)$ is a point defining the compensatory relation or formula

$$(2.1) \qquad \varphi(f_1, f_2, \ldots, f_5) = 0$$

as it has invariably been taken. If the farmer can also produce 1000 pounds of potatoes by using 5 acres of land, 500 pounds of fertilizer, 9 pounds of calcium arsenate, 65 pounds of seed and 65 hours of labor, the point $(.005p_1, .500p_2, .009p_3, .065p_4, .065p_5)$ is another point helping to define φ. The totality of such points $(f_1, f_2, \ldots, f_5)$ defines the compensatory relation $\varphi(f_1, f_2, \ldots, f_5) = 0$. This is alright provided one makes sure that he applies the relation only to problems for which the seed is the same, the production rate is 1000 bushels per year and the land is similar and similarly situated climatically to the land for which the relation has been obtained, etc. A casual search of many economic books will reveal that their authors almost invariably assume that if all the factors of production are doubled the product will be doubled. In other words, these authors, without question or hesitation, willingly apply the equation (2.1) to all rates of production. Such an eminent economist as Vilfredo Pareto, however, denies the validity of this approach and labels the marginal productivity theory of Walras and Wicksteed invalid because of this tacit assumption. In fact, Pareto wrote:

> "Some authors assume that if all the factors of production are doubled the product will also double. This may be true approximately, in a certain case, but not rigorously and in general. Some expenses vary with the importance of the business (enterprise). It is certain that if we could assume another business under conditions exactly resembling those of the first, we might double all the factors and the product. But this assumption is not, in general, admissible. If, for example, one were engaged in the transportation business in Paris, it would be necessary to assume another business and another Paris. But, as this other Paris does not exist, we must consider two businesses in the same Paris, and then, we cannot assume that, when the quantities of the factors of production are doubled, the product will be doubled.
>
> "We must observe that if one cannot assume that all the coefficients of production are constant, neither can we assume that they are variable. From a certain quantity of iron ore, for example, it is impossible to extract more metallic iron than is contained in this mineral. A certain state of technical knowledge being given, the quantity of metallic iron which may be obtained from a ton of a certain ore is a fixed quantity. In other words, the quantity of ore to be employed is proportional to the quantity of iron which we wish to produce."*

In attempting to straighten out some of the difficulties of the theories of Wicksteed and Walras, Henry Shultz uses an expression of the form $\varphi(f_1, \ldots, f_m) = 0$. He employs Euler's theorem on homogeneous functions as a principal means of obtaining the theorems of the marginal productivity theory.

It would seem more natural and better to write

$$(2.2) \qquad \psi(F_1F_2, \ldots, F_m, u) = 0 \ ,$$

where $F_1, \ldots, F_m$ are the quantities of the factors of production $1, \ldots, m$ required to produce $u(t)$ units of C per unit of time. Referring again to the problem of producing potatoes one could say that the totality of points such as $(6p_1, 300p_2, 10p_3, 70p_4, 60p_5, 1000)$

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*Vilfredo Pareto, *Cours d'économie politique*, Vol. II (1897), pp. 82-83. The translation is that of Henry Schultz, *loc. cit.*, p. 517.

The criticism of Pareto is, of course, of limited usefulness now. It has been inserted here to show how chaotic some early thinking about cost was.

would determine ψ. In this problem the factors of production for all values of $u(t)$ from, say one to one thousand or more, could be determined empirically.

If F , . . . , F_m are held constant in (2.2), the expression usually determines F_1 as a function of the rate of production $u(t)$, such as $F_1 = F_1(u)$. Then $F_1(u)/u$ should be the f_1 of the early part of this section. The ratio $F_1(u)/u$ is a function of u alone. A much more general and realistic theory than the classical one can be obtained by inserting p and t as arguments of ψ in (2.2) or by saying that the technical coefficients are defined by these arguments and u and they are not given by (2.1). Conceivably, however, some of the compensatory coefficients might depend upon price p and other factors which may be represented parametrically by time t, so that it will be best to replace (2.2) by

$$(2.3) \qquad \psi(F_1, F_2, \ldots, F_m, u, p, t,) = 0 \quad ,$$

where by definition $F_\alpha = uf_\alpha$, $\quad \alpha = 1, 2, \cdots, m$.

This form of compensatory relation takes cognizance of the fact that some of the factors of production may be changed with the price of the commodity or service. For a fixed rate of production u and a given price p, the equation (2.3) defines a compensatory relation among the F's.

3. Dependence of Cost of Production on Rate of Production. Suppose that choices of compensatory factors of production have been made. The cost Q of producing and marketing u units per unit time is

$$(3.1) \qquad Q = (E_m T_m) W_m + (E_h T_h) W_h$$
$$+ (E_w T_w) W_w + (E_d T_d) W_d + \overline{C} \ ,$$

where T_m is the time of all machine employees required to produce u units; T_h is the time of all machine helpers; T_w is the time of factory white-collar employees, as clerks, stenographers, managers, buyers, etc., in the factory; T_d is the time of employees engaged in distributing or marketing the product, as salesmen, advertising managers, clerks, etc.; E_m = number of machine employees; E_h = number of employees helping with the machines but not actually operating them, as engineers, oilers, hand-workers, porters, etc.; E_w = number of white-collar employees in the factory; E_d = number of employees engaged in distributing or marketing the product; W_m = average daily wage paid to machine employees; W_w = average daily wage paid to white collar employees; W_h = average daily wage paid to machine helpers; and W_d = average daily wage paid to distributing employees; $\overline{C}$ includes the cost of power and raw material for producing u units, capital charges, such as rent, bond and mortgage interest, depreciation, risk, and an arbitrary interest on other capital invested. Thus (3.1) represents cost as a business man would define it. His definition differs materially from the economists' definition, but chiefly in the item $\overline{C}$.

If E = total number of employees in service for the industry, then obviously

$$E = E_m + E_h + E_w + E_d \ . \tag{3.2}$$

The quantities E_m, E_h, E_w, and E_d evidently depend upon the rate of production u.

The quantities used to define Q will, of course, differ in different industries. In some industries particular factors mentioned will be unimportant and factors not mentioned will be important. It is obviously impossible to produce a perfectly general theory in the limited space of a single chapter so that selection is essential. The treatment given here is merely meant to be suggestive.

Many machines normally operate at full capacity and can produce more only by having their operating time increased. Thus, for small increases in production the number of machine employees may remain the same but their hours of work will be increased, that is, the number of machine employees will increase only if new machines have to be introduced or if more than one shift becomes necessary. On the other hand, for some machines production can be speeded up by adding additional machine employees.

Whether the first or second condition prevails, the relationship between machine employees and production rate may be assumed to be linear; that is, $E_m = B_m u + C_m$ where B_m and C_m are constants for relatively short periods of time, such as periods during which technological improvements do not occur and C_m is small or zero.

The number of helpers — those engaged in production but not actually operating machines — will, in some cases at least, increase more rapidly than the first power of the rate of production, i.e.,

$$E_h = A_h u^2 + B_h u + C_h \quad . \tag{3.3}$$

As in the case of E_m, E_h is taken as a quadratic because of (1) necessity for hiring less productive labor, (2) overcrowding, (3) lax management, etc. Here, however, A_h is important only for large u.

The number of factory white-collar workers will not be greatly influenced by the rate of production. In other words, E_w might be assumed to be independent of u but it is probably better to write

$$E_w = B_w u + C_w \quad , \tag{3.4}$$

where from *a priori* considerations B_w may be expected to be small, Of course, if new plants are opened as u increases, E_w will also increase. This condition may be assumed to be also taken care of by the above formula.

It usually becomes more and more difficult to sell goods as the amount of such goods increases, so that the distributing force must usually be augmented more rapidly than is described by a linear relation. Of course, it is sometimes possible to hire more efficient distributors, but there is a limit to this. In most cases a quadratic term in u will describe the situation. Here the quadratic term is of much more importance than for manufacturing costs.

When the product is sold on a commission basis, the number of salesmen may be assumed to depend upon the value of the product; that is, upon pu. It is also inversely proportional to purchasing power. In other words, a formula describing the relation between the number of distributing employees and the rate of production and prices is likely of the form

$$E_d = A_d u^2 + B_d u + C_d + H_d pu \quad , \tag{3.5}$$

where H_d depends upon purchasing power.

Now C involves the cost of power, which may be taken as $B_c u$, the cost of raw or semi-finished material, which, of course, depends upon the number of units produced, so that it is $B_r u + C_r$, and the cost of items which for a given plant or unit are relatively independent of volume of production. The components of these latter are service charges for capital, insurance on plant and equipment, taxes, and various other items of overhead. $\overline{C}$ may also depend upon pu, due to various taxes, insurance on inventories, etc. In other words, $\bar{C}$ may be taken to be of the form

$$\bar{C} = B_c u + H_o pu + C_o + B_r u + C_r \ .$$

A substitution for E_m, E_h, E_w, E_d and C of the values just given for these in the formula (3.1) gives a cost (manufacturing and selling) formula in the form

$$Q = Au^2 + Bu + C + Hpu \ , \tag{3.6}$$

where the coefficients A, B, C and H are defined by the equations

$$\begin{aligned} A &= A_h T_h W_h + A_d T_d W_d \ , \\ B &= B_m T_m W_m + B_h T_h W_h + B_w T_w W_w + B_d T_d W_d \\ &\qquad + B_c + B_r \ , \\ C &= C_m T_m W_m + C_h T_h W_h + C_w T_w W_w + C_d T_d W_d \\ &\qquad + C_o + C_r \ , \\ H &= H_d T_d W_d + H_c \ . \end{aligned}$$

As already pointed out, the quadratic cost function (3.6) includes manufacturing and sales cost. If sales costs are not included, the coefficients A and H are small and hence a linear function can be used. However, every industry has to sell its products, so that it is more realistic to include selling cost in a formula to represent cost, and thus it is in general better to use a quadratic term. Nevertheless, it should be pointed out that industrial engineers usually treat only manufacturing costs and regard all costs as linear functions of output. For example, consider the following quotations:

"It is apparent that for the purpose of controlling expenses, the ideal method would be to keep all expenses within the sales dollar allowance established as sales volume falls and income decreases . . . . Now there are certain expenses that tend to remain constant as the volume of sales or production decreases, and there are other expenses that may readily be decreased as activity decreases, but which cannot be reduced in direct proportion to activity . . . It has been demonstrated in actual practice that it is possible to control these expenses (direct material and direct labor) in direct proportion to plant activity. . . . By a careful comparison of actual expenses with the budgeted allowances, and by charting the amount of actual expense for the various departments over a period of about two years, it became possible to establish maximum and minimum limits of variation from the straight-line relationship for the variable indirect expenses.*

4. The Element of Risk. With every industry there is associated an uncertainty or risk. Destruction of plant and equipment by fire or storm is a common menace. Technological improvements and new discoveries are risks that must be considered. As an extreme case of the latter, the almost complete passing of the horse-drawn vehicle may be cited. There are, furthermore, always the risks of competition, as, for example, the risk for the cotton, silk and wool industries, arising from the competition of the synthetic product rayon, credit losses arising from loose credit policies and from unexpected failures of debtors, and so forth.

Some risks can be provided against by insurance. Such insurance must, of course, be included in the cost function in some way, such as that already pointed out. The greater and more important business risks are, however, so intimately associated with the general management of business that an insurance company attempting to underwrite them would virtually make itself responsible for the business. As an extreme illustration it may be recalled that in 1923-25 the price of crude rubber was about $1.00 per pound. In the early part of 1930 the price touched the "phenomenally low mark" of fifteen cents per pound, and, at this price, many tire manufacturers placed large contracts. In July and August of the same year the price had dropped to ten cents per pound! Strikes must be included among risks arising from labor.

Let $r(t)$ be the money value of the total risks of management and obsolesence at the time t, i.e., let $r(t)$ be the money value of all uninsurable risks. For each firm there will be associated a risk function which will vary from firm to firm, and will, among other things, depend upon the size of the firm to which it belongs and upon the national income, or more precisely upon that part of the national income that is accessible to the firm. If the many individual risks for a particular concern are

$$r_1(t), r_2(t), \cdots, r_n(t) \ ,$$

*H. R. Mallory, "The Break-Even Chart," *Mechanical Engineering*, August, 1933, pp. 494-495. Chart 5 approximates the parabolic cost function treated here. See also, Walter Rautenstrauch, "Economic Characteristics of the Manufacturing Industries," *Mechanical Engineering*, November, 1932.

then the total risk

$$r(t) = (r_1 + \cdots + r_n)/n \ .$$

Since a remote risk is less important than an immediate one (conditions may change to eliminate the risk and, also, return of capital occurs), to obtain the present (time 0) value of a future (time t) risk it is justifiable to multiply $r(t)$ by a discount factor $E(0, t)$ where the quantity $E(0, t)$ may be defined by the formula

$$E(0, t) = e^{-\int_0^t \delta(x)dx} \ .$$

Here $\delta(x)$ is the force of interest, that is, the relative rate of increase of present value, so that $\delta = [d\,log(v^t)]/dt$, $v = (1+i)^{-1}$, where i is the "prevailing" rate of interest. The totality of these present values of future risks, R, will be the total risk at the time t_i, that is,

$$R = \int_0^{t_1} r(t)E(0, t)dt \ .$$

The period of time 0 to t_1 may be an accounting period or a year, depending upon the industry under consideration. In most practical cases the upper limit of integration for R should be greater than t_1 , but this need not give concern here. It may be argued that entrepreneurs should plan their operations long in advance, but it must be admitted that in general they do not do this. They can be absolved of some criticism because general economic conditions cannot be estimated long in advance of the present time and as long as general conditions cannot be forecast, it is useless to do long time individual forecasting. This is a logical attitude to take provided one holds that nothing can be done to establish automatic controls that could eliminate many of the long-time risks. There is little doubt, however, that fluctuations in building volume are very important factors in initiating both small and large cyclical fluctuations in business as a whole. If something can be done to stabilize building activity many long time risks will be eliminated.

Risk may, of course, be assumed to be included in C.

5. *Average and Unit Costs.* If only one unit of a commodity or service is produced per unit time, the unit cost is the overhead expense plus the *prime unit cost* which may be defined as the money cost of the raw materials used in making the unit, plus the wages

of the labor spent on it, which is paid by the hour or piece, plus the money cost of power, plus the money value of the extra wear and tear of the plant. When two units are produced the overhead expense will in general be the same as for one unit and the prime unit cost will differ little from the prime cost of the first unit, so that the total average cost will be less. Thus, if the overhead cost is C dollars and the prime cost of producing the first unit is B dollars, the average unit costs when one and two units are produced are respectively $B + C$ and $B + C/2$.

This reduction of average unit cost or law of diminishing cost will continue to operate as additional units are produced but not indefinitely, that is, in general, as already indicated, the average unit cost will not be given by the simple formula $U_c = B + C/u$ where u stands for the number of units produced per unit time. After a certain production (sales) rate is attained the average unit cost may begin to rise again due to sales resistance, bidding for labor and material, lax management and so forth. A more realistic form for U_c would, therefore, be

$$U_c = Au + B + C/u \text{ where in typical cases } C > 0 \ . \tag{5.1}$$

For simplicity, A, B and C might be assumed to be constants, but this assumption is not necessary. Whenever A, B and C are functions of the time as defined earlier, which can be adequately approximated by step functions, the conclusions will remain valid.

The curve represented by the equation (5.1) is a hyperbola whose asymptotes are $U = 0$ and $U = Au + B$. Here A, B and C are assumed constant. Only the branch of the hyperbola in the first quadrant seems to have economic significance. This branch attains a minimium at $u = u_m = \sqrt{C/A}$ as may be determined by the usual rules. By definition the quantity dU_c/du is the *marginal unit cost*. Roughly speaking, since $du_c/du = A$ approximately when u is large, A is the increase in marginal unit cost resulting from the production of an additional unit when a large number of units $u(t)$ are already being produced per unit of time. Whenever $u < u_m$, the law of diminishing cost holds since for these values of u the average unit cost U_c is a decreasing function of u. Similarly, whenever $u > u_m$ the law of increasing cost holds.

If $q(u)$ is the cost of producing u units per unit of time, then by definition $U_c = q(u)/u$ and hence

$$q(u) = Au^2 + Bu + C \ , \tag{5.2}$$

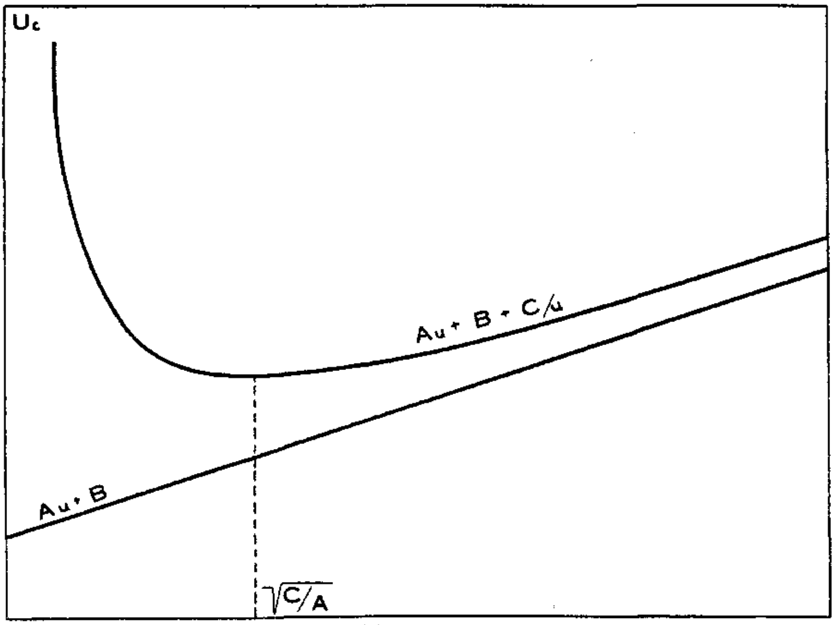

FIGURE 4

where A, B and C will usually be positive functions of the time, satisfying hypotheses previously mentioned. For a constant price p the form (5.2) is the same as (3.6) provided the B in (5.2) is taken to be the B in (3.6) plus Hp. Three dimensions would be necessary to represent the cost function (3.6) for a variable price p, and four dimensions for a variable price p and variable consumer income.

The function $q(u)$ has a minimum at the negative value $u = -B/2A$, and, as a result, assumption of the cost function (5.2) rules out the possibility of producing a larger number of units than u and destroying some of them to obtain a total cost smaller than $q(u)$.

If $q(u)$ is plotted and a tangent is drawn from the origin, the value of u which defines the point of tangency T is u_m, as may be easily proved by setting $dq/du = U_c$ and solving for u. The derivative dq/du defines a second kind of unit cost which economists call *marginal unit cost*. For the cost function (5.2), $dq/du = Au + B$ and hence B is the marginal unit cost when $u = 0$.

Since $q(u) = U_c u$, marginal unit cost and average unit cost satisfy the relation

$$dq/du = U_c + u\, d\, U_c/du$$

for all cost functions $q(u)$ which are differentiable. Whenever unit cost is decreasing, the derivative dU_c/du is negative and hence marginal unit cost is less than average unit cost by the amount $u\, du_c/du$. Furthermore, U_c is a minimum when it is equal to the marginal unit cost; that is, $dq/du = U_c$ since U_c is a minimum when $dU_c/du = 0$.

It is not true that the production rate u_m which minimizes U_c yields the highest profit to a producer. A slightly more elaborate mathematical analysis is necessary.

6. Income, Profit, and Loss. Consider the case of a producer so small that the amount which he produces has a negligible effect on the price $p(t)$. This is equivalent to assuming *free competition* — that is, that the demand for the commodity may be neglected in the calculations. Suppose that the producer sells his entire output so that his profit is $\pi = p(t)u(t) - q(u) = [p(t) - U_c(t)]u(t)$. The condition $d\pi/du = 0$ yields the equation $p(t) = dq/du$; that is, maximum profit results when the rate of production is such that the price is equal to the marginal cost of production. A warning should perhaps be issued that this latter condition, which has been so greatly revered by many economists, holds only when $p(t)$ is independent of $u(t)$ and hence does not apply to dominant producing units.

If y units are sold per unit time at the time t, the gross income will be py and hence the net profit per unit time will be

$$\pi = py - (Au^2 + Bu + C + Hpu) \ .$$

It is almost universally true that producers prefer early profits to remote or deferred ones. Waiting is an element of cost as truly as effort is, and it should be taken into account. This does not mean, however, that a producer is unwilling to forego present profits in order to obtain greater ones in the future, but it does mean that the expectation of future profits will have to be greater than actual present ones.

Suppose that the above equation defines the net profit at present time C. Then profits expected at a time t after the present must be multiplied by a suitable discount factor. Either a continuous or discontinuous compound discount factor can be used, but for purposes of mathematical analysis continuous factors are more convenient.

Now, the total net discounted profit for a period of time 0 to t_1 may be closely approximated by a sum or integral

$$\pi = \int_0^{t_1} [p(t)y(t) - Q(u, p, t)] \, E(0, t)dt \ ,$$

where $Q(u, p, t) = Au^2 + Bu + C + Hpu$ with A, B, C, H constants, or functions of time.

It will be recalled that the assumption made in regard to expected price is that each entrepreneur deduces his expected price by multiplying present price by the psychological factor γ_j. Once he has produced, or, in the case of a retailer or wholesaler, once he has purchased for sale (or in some instances where advance orders are taken before production or before purchase) he attempts to sell his product for the price $\gamma_j p_0$. In all likelihood he is able to sell at least part of the goods at the expected price, but after a short time his operations in the market tell him whether he estimated correctly or not. He may find that conditions in his market are such that he cannot dispose of his production at the estimated price, which is supposedly sufficiently high to cover overhead, etc. In this instance he may sell for less, or more likely he will offer some additional goods (thirteen for a dozen for a limited period, etc.), thus effectively reducing his price.

Let $p(t)$ represent the average market price (weighted for sales) corrected for free deals and other special discounts. Those entrepreneurs who were able to choose $\gamma_j(t)$ so that for their estimated production rate $\gamma_j(t)p_0 \leq p(t)$, in general, make profit as much as, or more than, estimated without resorting to free deals, etc., whereas entrepreneurs (operating in the same market) for whom $\gamma_j(t)p_0 > p(t)$ in general are forced actually to reduce their prices or effectively reduce them through free deals, etc.* The reduction by means of free deals, etc., can be used only within limits and when the limits are reached list prices are generally reduced. Considerable evidence can be marshalled to show that there is a natural rigidity in the price structure of the manufacturers, and for that matter of the retailers and wholesalers also, which yields only after considerable pressure.

*It is not meant that the low cost producer will not offer free deals. He is in a position to do this and often does it for the express purpose of enlarging his market.

Whenever p_0 (list price) is relatively constant, free deals, etc., may make γ_j quite small.* As γ_j shrinks, the entrepreneur usually examines his costs more closely in an endeavor to reduce them, and inevitably he makes adjustments without regard to the effects that these adjustments will have on his demand.†

Let $U(t)$ be the total demand in the markets of the entrepreneur. Then, as already shown,

$$U = ap/I^{\beta} + C(t) \ ,$$

where I stands for consumer income, or purchasing power.‡ The function $C(t)$ may be assumed to be dependent upon the prices and quantities of competing goods and upon physical and psychological quantities.

For the demand equation commonly used in elasticity of demand analysis,

$$U = aI^{\beta}/p^{\alpha} + C(t) \ .$$

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*Here the definition of γ_j is changed slightly, but the broader definition here does not alter the preceding analysis.

†If the absorption of displaced labor does not occur, however, each worker dismissed usually means a decreased velocity of circulation of monetary (currency or bank credit) media and consequently a falling off of income or purchasing power available to carry the goods off the market.

Employment of all workers seems to be necessary for greatest production. The worker who receives $10 per week spends this almost as soon as received for goods and services so that the velocity of circulation of media of exchange for him is greater than fifty times per year. The worker who receives $50 per week probably spends $25 or $30 of it as quickly as the $10 per week worker and the rest more slowly. Interest and dividends usually go to individuals whose purchasing power is largely a "power" in that they are free to exercise it as they please. They need not exercise it all for considerable time, i.e., they may hoard money or they may allow it (bank credits) to remain idle in banks. The latter does little, if any, harm unless the banks are not lending. See Chapter XI.

A price of $1.00 per unit may be high or low depending upon the purchasing power in the market. Thus, if the purchasing power is $5,000,000, the price of $1.00 is not as high as the "same price" when the purchasing power is only $1,000,000. It is, of course, not correct to say that a price of $1.00 for economy income of $5,000,000 is five times a price of $1.00 when the economy income is $1,000,000. For some luxury products the second price may be ten or twenty times as large as the first one, whereas for certain staple commodities the second may be only five or ten per cent greater than the first. This indicates why an exponent β is used. In the case of automobile demand, I is extremely important.

‡Whenever there is speculative demand present (and there usually is for manufactured goods) a term in dp/dt, derivative of price with respect to time, should be added, at least when short-time effects are being studied. In the interest of simplicity speculative effects may be neglected here.

In either case, in general the price, as determined by demand, increases with the income.*

7. *Relation of Maximum Profit to Employment.* Consider the case of an individual firm, designated for purposes of identification by the subscript j, and let its net actual income exclusive of capital charges be

$$\pi = \int_0^{t_1} [py_j - A_j u_j^2 - B_j u_j - C_j - H_j p u_j] E(0, t_1) dt ,$$

where C_j represents the overhead exclusive of capital charges and p is the average market price corrected for free deals, etc.

Now let u_j be the average rate of production per unit time; i.e., u_j is the average value of y_j over short periods of time (periods over which producers do not worry greatly about enlarged or depleted inventories).

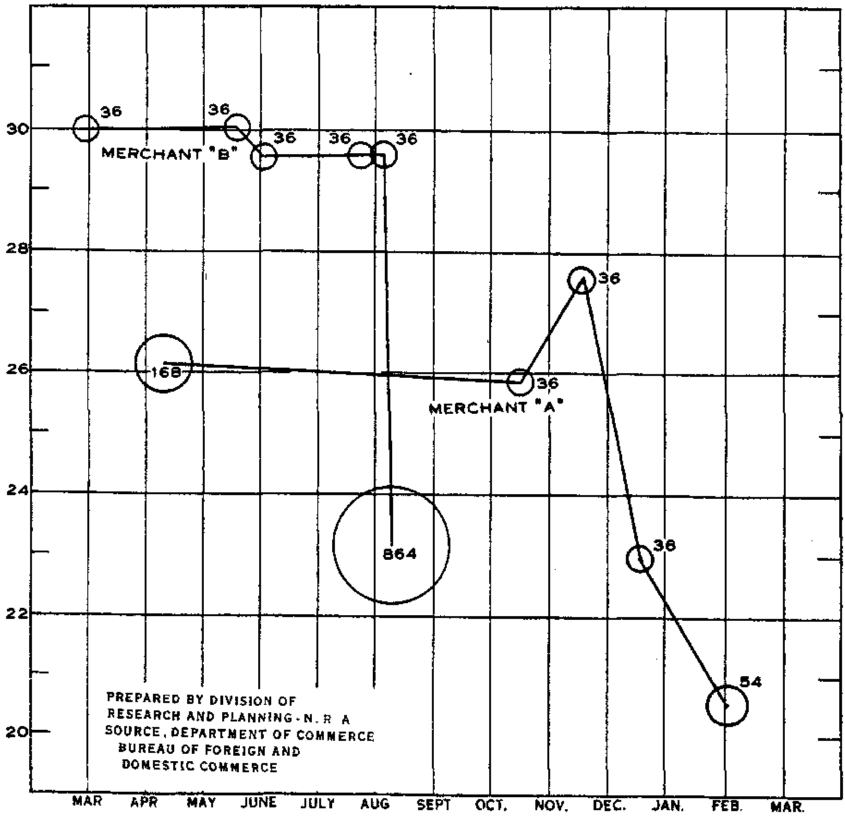
If all the producers are small, the price p may be assumed to be determined by free competition, that is, by the demand equation. In such event, for a producer j there will be a value of expected demand y_j that will maximize his profit, which will be denoted by π_j, when the expected price is $p(t)$, and any average production rate different from this will produce a smaller profit. Of course, a producer may find it advantageous to operate at two or three or even ten times the average y_j for short periods to take advantage of certain mass production factors, and then shut down for other periods.

A question that naturally arises is whether or not the average maximizing rate of production exceeds the capacity of the plant over the period for which the average is taken. If it does, and if the profit resulting is sufficiently large, there will be incentive to enlarge the plant or business. At the other extreme there will be cases for which the value of u_j that maximizes π_j is zero or negative, and, of course, there is always the possibility that the value of u_j and p that will maximize π_j will make π_j negative or so small that there is no incentive to operate. Just how large an incentive must be offered for operation must be determined in the light of the philosophy of entrepreneurs in the economy and in relation to factors that have to be paid out of profits. In general, 6.6 per cent

*The phrase "in general" is used because it is conceivable that there are products for which there is less demand in prosperous times than in depression times, for example, low rental property.

EFFECT OF DISTRIBUTOR PRESSURE AND QUANTITY DISCOUNTS ON MANUFACTURER'S PRICES

CHART XXX



Explanation: The size of each circle indicates as also does the adjacent number the quantity purchased. The position of the circle shows the price paid per unit and the time of purchase.

Merchant B was purchasing 36 units of the product at 30 cents each. Competing Merchant A purchased 168 units of the product at 26 cents and undersold B. Merchant B then purchased 864 units at 23 cents and undersold A. Merchant A then insisted that the manufacturer sell him 36 units at a smaller unit cost than he paid for 168 units. His request was granted.

profit on the capital employed is an inadequate profit since out of this must come Federal taxes, appropriations for betterments, provision of part of the surplus for pensions, etc., as well as preferred and common stock dividends.

Economic necessity has driven the individual producer to look for the most flexible medium of cost control. The major portions of costs subject to control have, of course, been wages of men and raw materials. In attempts to vary cost with volume, industries have in general placed themselves in the position of contributing to human misery by dropping men from the pay roll when sales volume fell. Recently some attempts have been made to control salaries in relation to output.*

Economists have generally argued that production rate is determined by free competition, that is, that the producer assumes the price fixed and chooses his production to maximize his profits. Another kind of competition was discussed at some length in Chapter VIII. It was pointed out there that a retail merchant has often sought to lower his price and increase his volume by getting larger discounts for quantity purchases. That quantity discounts play an important part in retail competition is indicated by Charts XXIX and XXX. In fact, Chart XXX shows that quantity buying competition can force down a manufacturer's price. If quantity merchandising is applied to a good for which the total demand for the economy is elastic, resultant greatly increased production can lead to more employment and profits than would be obtained by ordinary free competition.

During a depression when competition for purchasing power is keen, price cutting competition is very important. Instead of free competition (uniformity of prices for standardized products) each competitor tries to undersell all others. Often contracts are made at less than prime costs. In these cases workmen agree to wage cuts in order to keep at work. Localized suppliers of raw materials may also cut their prices in order to lower the cost of a producer who has to quote a low price or cease to continue in business. The assumption of free competition is, therefore, open to serious question. Another somewhat more realistic theory of competition is just as easily developed as the theory of free competition.

To repeat, the net actual income to the jth firm exclusive of capital charges can be represented by an integral

$$\pi_j = \int_0^{t_1} [pu_j - A_j u_j^2 - B_j u_j - C_j - H_j p u_j] E(0, t)\, dt \ ,$$

provided average sales are equal to average production.

---

*H. R. Mallory, *loc. cit.*

Then, for a demand equation of the first form suggested above,

$$\sum_{j=1}^{n} u_j = ap/I^{\beta} + C \ , \tag{7.1}$$

where n is the number of competing firms.

Now, at any time t the jth firm will secure a certain fraction k_{0j} of the total business,

$$U = \sum_{j=1}^{n} u_j \ ,$$

so that $u_j(t) = k_{0j}U(t)$ where $k_{0j} \leq 1$.

This relationship and (7.1) give the average market price p as

$$p = (k_j u_j - C)I^{\beta}/a \ , \tag{7.2}$$

where $k_j = 1/k_{0j}$ and hence $k_j > 1$. The actual profit obtained by the jth producer in the period $0 \leq t \leq t_1$ is then given by

$$\pi_j = \int_0^{t_1} [(1 - H_j)u_j(k_j u_j - C)I^{\beta}/a - A_j u_j^2 - B_j u_j - C_j]E(0, t)dt \ ,$$

or,

$$\pi_j = \int_0^{t_1} [-G_j u_j^2 + L_j u_j - C_j]E(0, t)\,dt \ ,$$

where

$$G_j = +A_j - (1 - H_j)k_j I^{\beta}/a$$

and

$$L_j = -B_j - (1 - H_j)CI^{\beta}/a \ .$$

In view of the inequalities $A_j > 0, H_j < 1, k_j > 0, I > 0, a < 0$, the quantity G_j is positive. The rate of production u_j that maximizes π_j is given by

$$u_j = L_j/2G_j \ ,$$

and since $G_j > 0$, it follows that $L_j > 0$, for otherwise $u_j < 0$. The inequality $u_j < 0$ could not, in general, hold, for this would mean that there would be no positive rate of production maximizing π_j.

It is possible that some producers estimate *values of production which maximize their profits per unit time,* but it is certainly not probable that the majority of producers estimate production rates anywhere near the maximizing values. There are too many variables, price, costs, consumer income, etc., involved for producers to approximate the maximizing values. Nevertheless, it is important to discuss costs on the basis of maximum obtainable profit.

A detailed study of the nature of the function

$$F_j = -G_j u_j^2 + L_j u_j - C_j$$

throws important light on the economic mechanism of a capitalistic economy. For fixed values of G_j, L_j and C_j the graph of rate of production against the profit function F_j is as represented in the accompanying figure.

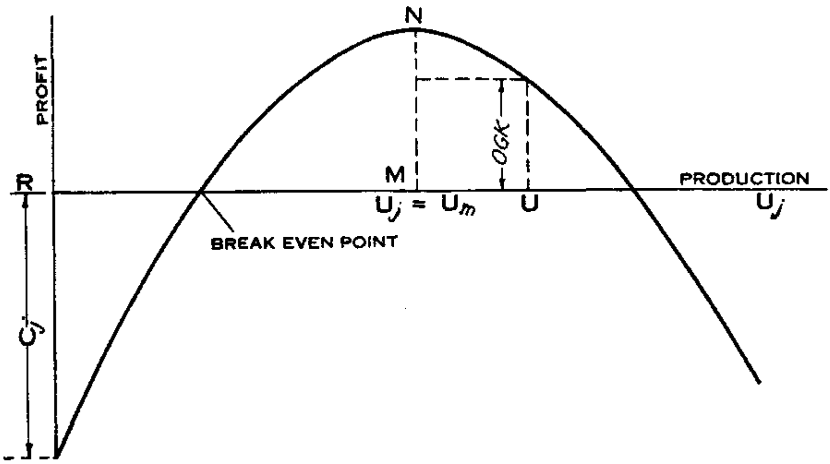


FIGURE 5

The value of $u = u_m$ is the one which maximizes F_j. The line *MR* represents fixed capital charges, so that, if the firm is operating at a rate that produces a maximum profit, *MN* is the quantity available for surplus to take care of risks and dividends.

It is important to know whether the profit curve is flat or steep near the maximum, since if the curve is flat the accounting department will be unable to locate the true maximum and production may be carried far beyond the maximizing value u_m. To determine the nature of the curve at u_m the formula for the radius of curvature is quite useful, i.e.,

$$r = [1 + (dF_j^2/du_j)]^{3/2}/[d^2F_j/du_j^2]$$
$$= + [1 + (L_j - 2G_ju_j)^2]^{3/2}/2G_j \ .$$

For u_j equal to the maximizing value u_m given by $dF_j/du_j = 0$, it follows that

$$r_m = + 1/2G_j = + a/2 \ [aA_j - (1 - H_j) k_j I^{\beta}] \ .$$

This formula shows that r_m increases with the absolute value of a, approaching an upper asymptote $+1/2A_j$. Also $r_m = 0$ when $a = 0$. It will be recalled that a is a quantity which is numerically large when the effect of change in price or demand is large.

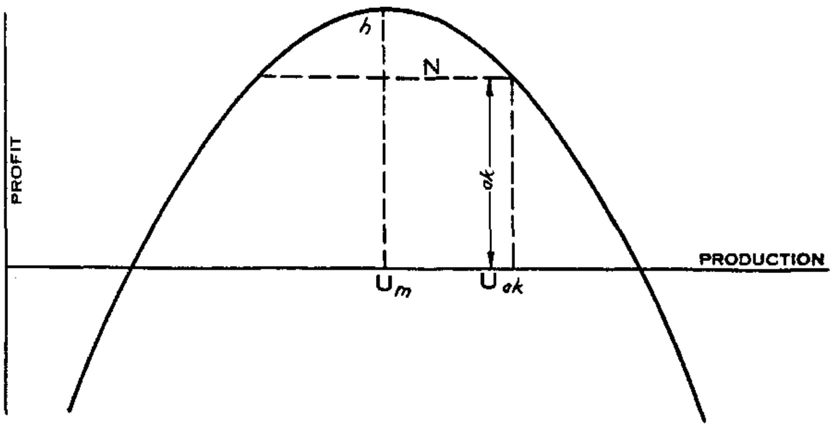


FIGURE 6

An alternate method of arriving at the same result is as follows. Let h represent maximum profit, F, minus αK where α is some per cent (.06% in some cases) of invested capital K. Let N represent u_{ak} — u_m where u_{ak} is the greatest rate of production yielding a profit of αK per annum. Then the ratio N/h is a measure of the flatness of the profit curve.

Now, if the subscript j is dropped to simplify notation,

$$F = -Gu^2 + Lu - C = -G(u - L/2G)^2 + L^2/4G - C,$$

so that maximum $F = L^2/4G - C$ since $u = L/2G$. Hence,

$$h = L^2/4G - C - (-GN^2 + L^2/4G - C)$$
$$= +GN^2 \text{ and, therefore,}$$
$$N/h = N/GN^2 = 1/GN \ ,$$
$$\text{or } N/h = \frac{1}{\sqrt{Gh}} \text{ since } N = \sqrt{h/G} \ .$$

Finally, by substituting for G its definitive quantity,

$$N/h = \frac{1}{\sqrt{h}} \frac{1}{\sqrt{A - \frac{(1-H)kI}{a}}}$$
$$= \frac{\sqrt{-a}}{\sqrt{h}\sqrt{-aA + (1-H)kI}} \ .$$

It follows, therefore, that for a small (non-sensitive demand) and h fixed, the ratio N/h is small; i.e., the profit function has a sharp peak and, also, for a large (sensitive demand) N/h is large. In other words, N/h increases with increasing a, approaching $\frac{1}{\sqrt{hA}}$ asymptotically, and N/h is zero when $a = 0$.

It can be stated, therefore, that *the profit curve for a product having sensitive demand is flatter than one for a product having non-sensitive demand.**

In the case of a monopolist who has non-sensitive demand, such as a public utility, it may, therefore, be desirable to have public regulation since the profit curve has a steep peak and the accounting department of the utility has little difficuly in locating the production rate yielding maximum profit. On the other hand, there need be no fear of an automobile monopoly since the demand is so sensitive to price changes that the profit curve should be flat. In the case of the automobile, however, consumer income is probably a more important factor than price.

A short analysis shows that

$$du_m/dI = \frac{-\beta(1-H_j)}{a}\left[BK + AC\, I\right]^{\beta-1} / \left[A - (1-H)KI^{\beta}/a\right]^2 \ ,$$

*This statement, of course, implies that the quantities A_j, H_j, k_j and I do not depend upon the sensitivity of demand.

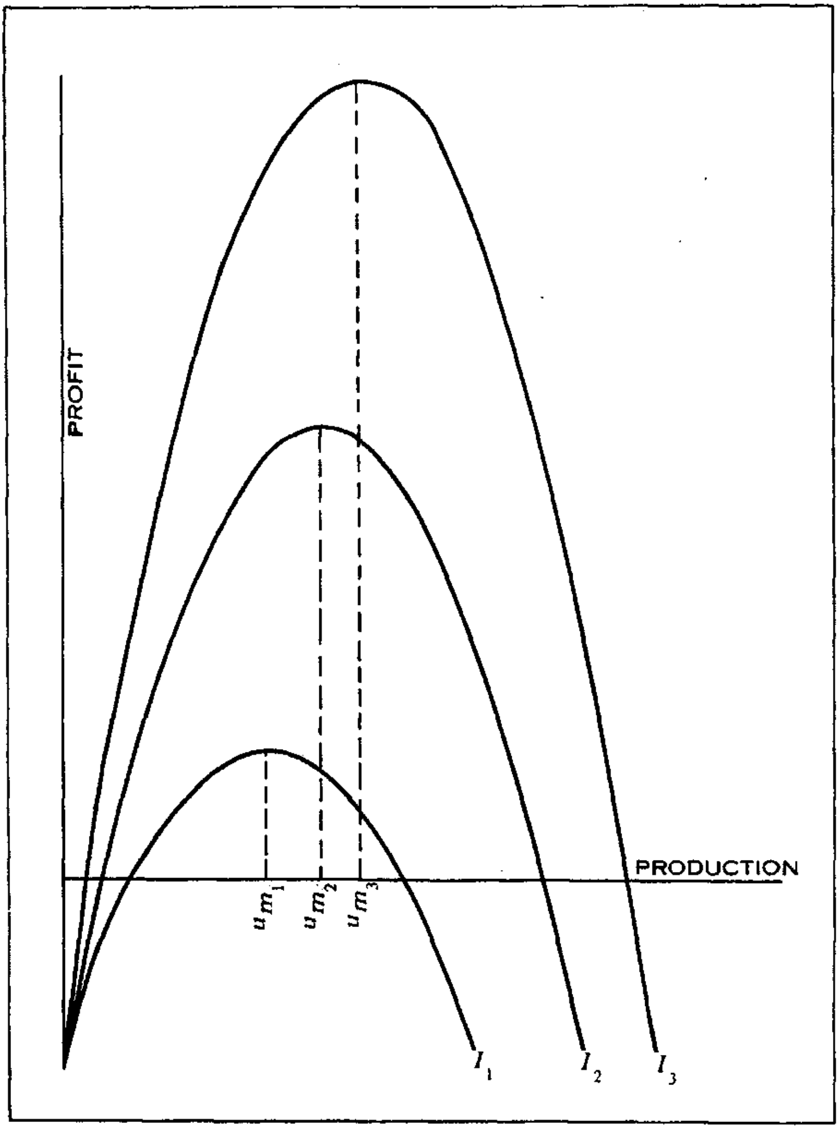


FIGURE 7

where for convenience in notation the subscripts have been dropped. It follows, therefore, that du_m/dI is positive in view of the signs of the factors defining it. This means that the production rate yielding maximum profit increases as consumer income increases. An examination of the radius of curvature shows that r_m decreases as I increases. Furthermore, if

$$I_3 > I_2 > I_1, \text{ then } F(I_3) > F(I_2) > F(I_1).$$

In other words, as consumer income increases the rate of production yielding maximum profit increases and the profit curve has decreasing curvature at the peak, i.e., the curve is more pointed. Figure 6 indicates how the profit curve changes with I.

The situation that represents highest entrepreneur profits in prosperous times is essentially an unstable one. Suppose that an entrepreneur happens to be operating at u_{m3} on the curve for I_3 and a decrease in consumer income to I_2 occurs. The entrepreneur soon finds out that he is operating beyond the point of maximum profit and lays off some of his "extra" help, usually sales force and general luxury help, and by this device succeeds in lowering the production and sales to the value u_{m2}. Since the peak of the I_2 curve is flatter than that of the I_3 curve it may be difficult for him to find the true maximum.* Thus, the entrepreneur may operate for a time at a rate of production greater than u_{m2}. During this time consumer income may drop to I_1 and in this event a more drastic change in production is required.

The preceding essentially describes a cooperative competition, since it has been assumed that the demand was known and also the proportion of it going to each producer. Thus, the theory applies to cartels and monopolies having several units. For such units price is usually held nearly constant, and is changed only if a bad guess is made in regard to consumption. Of course, in estimating future consumption, future price p must be estimated along with future consumer income, etc. These estimates depend upon present values of the variables and their trends.

For competition among units not pooling their information, the price p used by producers in making their estimates of cost and volume will usually be proportional to present price, that is, $p(t) = \gamma_j p_0$, where γ_j is a function of time which may differ among

*The assumption that the entrepreneur is able to guess the average market price is implicit here. Modifications for the cases in which he is not able to guess the average price will be obvious to the careful reader.

producers in the same industry and p_0 is present price.* If general optimism prevails $\gamma_j(t)$ will, of course, be greater than unity, whereas if pessimism prevails, γ_j will be less than unity. The function γ_j is thus somewhat of a psychological quantity.

Expected profits, as distinct from actual profits (assuming consumer income, etc., known) just discussed, over a period of time extending from present time $t = 0$ to some future time $t = t_1$, will then be given by

$$\Pi_j = \int_0^{t_1} [\gamma_j p_0 u_j - A_j u_j^2 - B_j u_j - C_j - H_j \gamma_j p_0 u_j] E(0, t)\, dt$$

$$= \int_0^{t_1} F_j(\gamma_j, u_j) dt.$$

As written, u_j is, of course, expected rate of production. The producer or entrepreneur would like to choose u_j so that Π_j is as large as possible; that is, he would like to choose his rate of production so that his expected profits over a period of time are as large as possible. It is immaterial so far as this hypothesis is concerned whether or not he is successful in his estimate of the maximizing rate of production. For a maximum Π_j well known conditions of the calculus of variations† require that $\frac{dF_j}{du_j} = 0$, that is, that

$$\gamma_j p_0 - (2Au_j + B_j + H_j \gamma_j p_0) = 0 \ ,$$

from which, if $A_j \neq 0$,‡

$$u_j = [(1 - H_j)\gamma_j p_0 - B_j]/2A_j \ . \tag{7.3}$$

The condition (7.3), when analyzed further, indicates an important role that psychology can play in determining operations of entrepreneurs. For example, if

$$\gamma_j < B_j/[(1 - H_j) p_0] \ , \tag{7.4}$$

u_j will be negative. *Thus, in the problem as formulated here, operations of entrepreneurs are largely dependent upon whether they choose optimistic or pessimistic values of* γ_j. Therefore employment

*Obviously, this proportionality may be used even though trends are employed in making estimates.

†See, for example, W. F. Osgood, *Advanced Calculus.*

‡If $A_j = 0$ no maximum exists.

depends to considerable extent upon entrepreneurial guesses of expected price, and, of course, as already pointed out, of expected consumer income.

It is, of course, possible to have losses occur during short intervals of time provided there are prospects for profits during later periods.

The size of $M N$, Figure 5, is an important factor in determining the labor policies of a firm. It may be expected that whenever $M N$ is greater than an amount which will leave, say 6 per cent, available for dividends, the business firm increases production with less caution than when only two or three per cent is available, or when nothing is available. Thus, if $M N > .06k$ where k represents capital invested, production may be carried to a value u considerably greater than u_m.

The function γ_j is not, however, the only factor that can be changed in order to insure positive values of u_j sufficiently large to give adequate profit incentives, leading to greater production and consequent greater employment.

CHAPTER X

Adjustments of Cost

1. Flexibility of Cost. As already intimated, labor costs are flexible costs of industry, and among the first to be adjusted when losses occur or appear imminent. Capital costs, bond interests, etc., and transportation are the most rigid of all costs. If raw materials are plentiful, a curtailment of some of their use may bring about drastic reductions in their costs. This is especially true for farm products for which the number of sellers (competing sellers) is very much greater than the number of buyers.

It is instructive to examine the profit function more closely to get some measure of flexibility of cost.

Substitution of the value of u_j that maximizes π_j, (7.3) of the preceding chapter, in π_j gives

$$\pi_j = \int_0^{t_1} \{[(1-H_j)^2\gamma_j^2p_0^2 - 2(1-H_j)B_j\gamma_jp_0 + B_j^2 - 4A_jC_j]/4A_jC_j\}\, E(0,t)\,dt.$$

Now, as already mentioned, the integrand may be negative over short periods of time, but these periods must be very short, for otherwise entrepreneurs will adjust their cost functions. Thus, in general, the integrand must be positive, that is, since A_j and C_j are positive, it follows that

$$(1-H_j)^2\gamma_j^2p_0^2 - 2(1-H_j)B_j\gamma_jp_0 + B_j^2 - 4A_jC_j > 0.$$

If, in particular, $H_j = 0$ and $\gamma_j = 1$, this simplifies into

$$p_0^2 - 2p_0B_j + B_j^2 - 4A_jC_j > 0 \ ,$$

where, as already pointed out, p_0 is present market price.

Evidently for a fixed price p_0 it is possible to enlarge profits by decreasing A_j and C_j, that is, either A_j, the marginal unit cost of producing and distributing when a large number of units are already being produced and distributed,* or C_j, the overhead cost, can be adjusted.

*If u_c represents unit cost, then
$u_c = q(u, p, t)/u = Au + B + C/u$, so that $du_c/du = A - C/u^2$,
and for u large C/u^2 is negligible; du_c/du is by definition marginal unit cost.

For a fixed minimum wage A_j can be adjusted only by reducing wages and salaries which are above the minimum, or by selecting employees for their productivity, and thus getting along with fewer employees, or by introducing labor-saving machinery. By recalling the origin of A_j, it will be noted that the wages and salaries principally affected are those of distributors.

An obvious way to reverse the inequality is to lower C_j. This can be accomplished in several ways:

1. C_j can be lowered by reducing the salaries of officers and executives and employees who do not contribute directly to production. In particular, in the United States white-collar employees can be reduced in salary or wages by firing and rehiring provided that the minimum wage section of the National Recovery Administration code is not violated. Furthermore, some white-collar workers and executives who might be said to furnish scenery, that is, those whose jobs are not indispensable, can be dismissed.

2. Capital charges can be reduced by refinancing, which lowers the interest charges on bonded and mortgage indebtedness, or by default of such indebtedness, but, of course, it is possible to do extensive refinancing on a long-time basis only if currency is not depreciating rapidly.* When bondholders foreclose, or ask for receivership, stock equities are wiped out or practically wiped out, and it merely becomes necessary to pay some kind of dividend to the bondholders.

In the case of concerns that have no bonded or mortgage indebtedness, the capital charge in C_j is one arbitrarily set by the accounting department. For such firms the capital charge can be temporarily lowered to zero, or even to a point below zero, that is, the concern may consume part of its own capital in an attempt to save the rest. During depressions many large concerns make extensive allowances for depreciation and appear to suffer losses, whereas in reality they are building up their liquid assets at the expense of dividends and replacements (see Chart XXVIII). This, of course, cannot be taken as a general criticism. It is only a natural consequence of a general situation.

It is also possible to increase profits by decreasing B_j, for

$$dQ_j/du_j = 2A_ju_j + B_j + H_j\gamma_jp_0$$

*In the U. S. the drastic Securities Act probably prevents this refinancing. It almost certainly was a hindrance to recovery in 1933.

so that B_j represents the marginal cost of *producing* the first unit, the production cost exclusive of overhead. It is, therefore, inconceivable that production would start unless $B_j < \gamma_j p_0$ or soon expected to be $< \gamma_j p_0$. Hence, in general, $B_j \gamma_j p_0 > B_j^2$, that is, if there is no expected profit, a profit may be possible by reducing the marginal cost of producing the first unit. This can be accomplished only by paying less for (1) raw materials (which may or may not be possible) or (2) by paying less for wages, especially productive wages. When a minimum wage has been set by law, it is possible only to reduce high productive wages and salaries, in other words, to make efforts to bring all wages and salaries to the minimum. In the case of B_j, however, a new factor enters, that is, if wages are too high it may be profitable to substitute machinery for men. Again it may be profitable to change from water power to electric power or vice versa. This introduces the principle of substitution of factors of production which will be considered more fully at the end of this chapter.

In some instances, decrease of H_j (removal of taxes, reductions of commissions and sales expenses, etc.) might suffice to reverse the inequality (7.3) of the preceding chapter.

2. Technological Unemployment. It is a truism to say that in any growing economy every technological improvement requires changes in the distribution of labor. Farmers become machine helpers, farmers and machine helpers become salesmen and so on. Sometimes demand greatly increases and additional machine workers, as well as salesmen or other sales or distributing help, are required. Some industries reach full maturity, i.e., reach a point of maximum demand, and then improvements in technology may produce unemployment of workers in the industry.

So long as new investments (residential building, public works, new enterprises, etc.) are being made in the economy so that savings are pressed into service and thus returned to the worker, reductions in cost by laying off "extra" labor to reduce costs have only beneficial effects on the economy as a whole. By this displacement of marginal help, labor is released for other productive enterprise. In such instances a higher standard of living is inevitable. The machine merely releases human labor for more productive enterprise for which a machine has not yet been designed. However, due to the fact that social progress, i.e., opportunities for investments in new enterprises, does not coincide with technological progress, this process of shifting labor is not always painless. Every age has had a group which has vociferously demanded limitation of the machine. Today is certainly no exception.

It appears to be universally true that industrial depressions lead to widespread attempts of industrialists and others to reduce costs. As profits shrink the entrepreneur inevitably is led to examine his costs carefully and to attempt to reduce them. Since labor costs are most flexible, they are the ones which first come in for adjustment. Thus, in an attempt to reduce costs, unnecessary and marginal labor may be laid off or other labor may be displaced by machinery. It is this latter that leads to the question of technological unemployment. Recently a great deal has been written about this subject. Much of this, like similar material written centuries ago, is sophomoric. A large amount of it has been written by so-called planners who know little or nothing about industry or, for that matter, about economic theory. As an exception, a recent paper by D. I. Vinogradoff may be mentioned.*

Suppose that a technological improvement is introduced into an industry. In order not to complicate the problem, suppose that a unit of time is chosen so that the output per unit time is equal to the consumption per unit time.

The following table summarizes notation that will be useful:

	Before Technological Improvement	After Technological Improvement
Demand = total output	U	U_1
Output per machine worker	v	v_1
Unit Cost	U_c	U_{c1}
Unit price	p	p_1
Number of employees at machines	E_m	E_{m1}
Number of employees assisting at machines	E_h	E_{h1}
Number of factory white collar workers	E_w	E_{w1}
Number of employees distributing and servicing	E_d	E_{d1}
Total number of employees	E	E_1

*D. I. Vinogradoff, "Effects of a Technological Improvement on Employment," *Econometrica*, Vol. 1, No. 4, October 1933, pp. 410-417.

See also, Arthur Dahlberg, *Machines, Jobs and Capitalism* and Philip Wernette, *Prices and Production*.

Obviously, $E = E_m + E_h + E_w + E_d$,

and $E_1 = E_{m1} + E_{h1} + E_{w1} + E_{d1}$.

From equations (3.2), (3.3), (3.4) and (3.5) of Chapter IX it follows that

$$(2.1) \qquad E = (A_h + A_d)U^2 + (B_m + B_h + B_w + B_d)U + (C_h + C_w + C_d) + H_d pU,$$

if C_m is taken equal to 0 as was suggested likely. Now, $B_m = \frac{1}{v}$ since the number of workmen at machines making the product is the next highest integer to U/v, i.e., $E_m = B_m U = U/v$.

In simplified form (2.1) becomes

$$E = \frac{U}{v} + \bar{B}U + HpU + AU^2 + C,$$

where

$$\bar{B} = B_h + B_w + B_d,\ H = H_d,\ A = A_h + A_d,$$

and

$$C = C_h + C_w + C_d;$$

that is, the total number of employees consists of a part U/v depending upon v, the output per machine employee, and upon another part independent of v. Let the part independent of the output per worker be designated by e so that

$$E = U/v + e \ .$$

Suppose now that due to a technological improvement, the output per workman rises from v to v_1, making it possible to sell the product at a lower price p_1. The demand in this case will be $U_1 > U$. The problem is to find E_1 and other quantities.

Evidently,

$$E_1 = U_1/v_1 + \bar{B}U_1 + Hp_1U_1 + AU_1^2 + C$$
$$= U_1/v_1 + e_1 \ .$$

The condition that total unemployment be unchanged requires that

$$U/v + e = U_1/v_1 + e_1;$$

that is, that

$$U/v - U_1/v_1 + \overline{B}(U - U_1) + A(U^2 - U_1^2) + HpU - Hp_1U_1 = 0.$$

For many products the term $A(U^2 - U_1^2)$ is small and can be neglected. In such instances it follows readily that

$$U_1/U = \frac{1/v + \overline{B} + Hp}{1/v_1 + \overline{B} + Hp_1}\ . \tag{2.2}$$

In particular, if the demand U is given by an equation of the type

$$U = apI^{-\beta} + b\ ,$$

where I stands for consumer income, the condition (2.2) becomes

$$aHI^{-\beta}p_1^2 + (aI^{-\beta}[1/v_1 + \overline{B}] + Hb)p_1 + b(\overline{B} + 1/v_1) - (apI^{-\beta} + b)\ (1/v + \overline{B} + Hp) = 0.$$

This quadratic equation in p_1 has solutions

$$p_1 = \left\{ -(aI^{-\beta}[1/v_1 + \overline{B}] + Hb) \pm \{(aI^{-\beta}[1/v_1 + \overline{B}] + Hb)^2 - 4aHI^{-\beta}(apI^{-\beta} + b)(1/v + \overline{B} + Hp)\}^{1/2} \right\} /2aHI^{-\beta}$$

If $1/v_1 + \overline{B} > H$ (see (2.1)), the first parenthesis is a positive quantity and hence the negative root is not permissible since it would yield a négative price p_1. Here

$$p_1 = \left\{ -(aI^{-\beta}[1/v_1 + \overline{B}] + Hb) + \{(aI^{-\beta}[1/v_1 + \overline{B}] + Hb)^2 - 4aHI^{-\beta}(apI^{-\beta} + b)(1/v + \overline{B} + Hp))\}^{1/2} \right\} /2aHI^{-\beta}$$

The case for which $H = 0$, i.e., no commissions, is particularly interesting because of the simplicity of the solution. In fact, for this case

$$p_1 = p[(1/v + \bar{B})/(1/v_1 + \bar{B})] + \frac{b}{a} I^{\beta} [(1/v - 1/v_1)/(1/v_1 + \bar{B})] \ , \tag{2.3}$$

if the number of employees is the same after mechanization as before.

A similar analysis shows that the number of employees after mechanization will be greater than the number before, provided

$$p_1 < p[(1/v + \bar{B})/(1/v_1 + \bar{B})] + \frac{b}{a} I^{\beta} [(1/v - 1/v_1)/(1/v_1 + \bar{B})] \ , \tag{2.4}$$

and technological unemployment will occur whenever

$$p_1 > p[(1/v + \bar{B})/(1/v_1 + \bar{B})] + \frac{b}{a} I^{\beta} [(1/v - 1/v_1)/(1/v_1 + \bar{B})] \ . \tag{2.5}$$

The above solutions are based on the assumption that the technological change does not affect consumer income, that is, that I remains unchanged. If I is changed in the process of technological improvement, the factor $(1/v + \bar{B})/(1/v_1 + B)$ becomes

$$(I_1/I)^{-\beta} (1/v + \bar{B})/(1/v_1 + B)$$

and the I in the second term becomes I_1. Otherwise the conditions (2.3), (2.4) and (2.5) are unchanged.

Now, in many industries $\bar{B}$ is much more important than $1/v$. Suppose, however, for definiteness, that before mechanization $\bar{B} = 1/v$; i.e., that there are as many white collar workers and distributors as machine employees, and suppose further that through mechanization, v, the output per machine worker, is increased *10* per cent. Then $(1/v + \bar{B})/(1/v_1 + \bar{B}) = 1.04$. Suppose also that $\beta = 1$. If technological unemployment is not to occur, then

$$p_1 \leq 1.04p(I_1/I) + .18 \frac{b}{a} I_1 .$$

If $I_1 = I/2$; that is, if the consumer income is cut in half while the technological change is being made (not necessarily because of the change), the above expression becomes

$$p_1 \leq .52p - .18(b/|a|)I_1 ,$$

where $|a|$ denotes numerical value of a, which means that *the new price must be less than half the price before the technological change if there is to be no "technological unemployment."* But this situation seldom prevails. Hence, there is no wonder that every depression brings forth those who attribute the cause thereof to technological unemployment.

Now, if $U = apI^{-\beta} + b$, as assumed in the preceding work, then $du/dp = aI^{-\beta}$, so that the conditions (2.3) can be written in the form

$$p_1 \gtreqless p\,[(1/v + \bar{B})/(1/v_1 + \bar{B})]$$
$$+ b(dp/du)\,[(1/v - 1/v_1)/(1/v_1 + \bar{B})].$$

This form may be more suggestive for some purposes.

3. The Principle of Substitution. The first two kinds of coefficients of production (Section 2 Chapter IX) are ready made, but not so with the compensatory coefficients. An entrepreneur may often have a great deal of choice left him when it comes to selecting the compensatory coefficients. As already pointed out, he may substitute machine labor for hand labor, water power for electric power and so on. Thus, if a minimum wage rate is raised, the compensatory relation may be changed so that introduction of machinery gives a lower cost. An examination of the factors of production in relation to minimum cost is essential.

If the coefficients of production are represented by $f_a(u,p,t)$ where t is a parameter in the sense explained above, and if the corresponding prices are represented by $p_a(t)$, $a = 1, 2, \cdots, m$, where m is the total number of services and commodities required to produce one unit of G per unit of time, the cost of producing $u(t)$ units per unit of time will be $\sum_{a=1}^{m} f_a(u, p, t)\, p_a(t)\, u(t)$.

This quantity is the total cost function which hereafter will be called $Q(u, p, t)$. In particular, it may be the cost function Q already derived in Chapter IX.

At the time t_1 a producer may be assumed to be using amounts of each of several factors of production, say $s \leq m$, so that the initial conditions are

$$(3.1) \qquad f_i(t_1) = f_{i1}, \qquad i = 1, 2, \cdots, s \ .$$

Some of the f_{i1} may be zero, as, for example, if the producer is not producing at the time t_1 or if he is not using a particular factor of production at that time.

As a typical problem it might be supposed that for a given rate of production $u(t)$, a given rate of sales $y(t)$ and a given sales price $p(t)$ a producer starting from the situation (3.1) desires to choose his compensatory services so that his total cost of production over the period of time (t_1, t_2) is a minimum, i.e., so that

$$Q = \int_{t_1}^{t_2} \sum_{\alpha=1}^{m} f_\alpha(u, p, t)\ p_\alpha(t)\ u(t)\ dt$$

is a minimum for $f_\alpha = F_\alpha / u$ satisfying (2.3) of Chapter IX,

$$\varphi(F_1, F_2, \cdots, F_m, u, p, t) = 0.$$

Obviously since Q is a linear functional of f_α there is a set of f_α which satisfies (2.3) and minimizes Q only if (2.3) is not linear in in the f_α and satisfies certain conditions which will be discussed later. This is a problem in the calculus of variations.* To solve the problem it can be assumed that there is a solution $\overline{F}_1(t), \cdots, \overline{F}_s(t)$ and the conditions which this solution must satisfy can then be found.

By defining a quantity $\lambda(t)$ by the equation

$$\lambda(t) = (\partial \varphi / \partial \overline{F}_s) / p_s(t) \ ,$$

the conditions for solution of the problem can be written in the symmetrical form

*See, for example, G. A. Bliss, "The Problem of Lagrange in the Calculus of Variations," *American Journal of Mathematics*, Vol. 52, 1930, pp. 673-744. However since f_α does not depend upon derivatives of u and p the problem is much simplified.

(3.2) $\partial \varphi/\partial \overline{F}_i = \lambda(t)p_i(t)$, $i = 1, 2, \cdots, s$,

where the additional equation given by $i = s$ may be taken to be the equation of definition of $\lambda(t)$.*

The equations (3.2) are also the conditions for minimum cost for a given output $u(t)$ and price $p(t)$ since in the above analysis $p(t)$ *and* $u(t)$ have been assumed constant. The partial derivative $\partial \varphi/\partial \overline{F}_i$ may be defined as the *marginal degree of productivity of the* ith factor of production. By means of this definition, which is, however, different from that usually given, the conditions (3.2) can be stated as the following theorem:

Minimum cost for a given output and a given price results when the marginal degree of productivity of each service is proportional to its price, provided that there is only one compensatory relation.

If the ith equation of (3.2) is multiplied by $\overline{F}_i$ and summed for $i = 1,2, \cdots, s$, the following equation results:

$$\sum_{i=1}^{s} \overline{F}_i \,\partial \psi/\partial \overline{F}_i = \lambda(t) \sum_{i=1}^{s} p_i(t)\overline{F}_i(t).$$

If ψ happened to be a homogeneous function of the first degree in $\overline{F}_1, \cdots, \overline{F}_s$, Euler's theorem would require that the left hand member equal $\psi = 0$, which is impossible since obviously if $\lambda(t) = 0$, there is no problem.

If $\psi(F, \cdots, F_s, u, p, t) = 0$ can be solved for u so that

$$u = \varphi(F_1, \cdots, F_s, p, t) \ ,$$

the same equations (3.5) with φ replacing ψ would hold since u has been kept constant. In other words it is possible to write also

(3.3) $\partial \varphi/\partial \overline{F}_i = \lambda(t)p_i(t)$, $i = 1,2, \cdots, s$.

Whenever $u = \varphi$ is a homogeneous equation of the first degree in $F_1, \cdots, F_s$, Euler's theorem can be used as above to obtain

$$\lambda(t) = u(t)/\sum_{i=1}^{s} p_i(t)\overline{F}_i(t) = u(t)/Q_f(u, p, t) = 1/p_c(t) \ ,$$

where by definition $Q_f(u, p, t)$ denotes that part of the total cost which is due to the compensatory coefficients $\overline{F}_i$, and $p_c(t)$ is that

*See Appendix VII.

part of the cost of producing one unit per unit of time due to the compensatory coefficients.

When this value of $\lambda(t)$ is substituted in (3.3) the following equations result:

$$(3.4) \qquad (\partial u/\partial \bar{F}) = [p_i(t)/p_c(t)] \ , \quad i = 1,2, \cdots , s \ .$$

This equation may be stated as the theorem:

For a minimum cost over a period of time (t_1, t_2) for a homogeneous compensatory relation $u = \varphi\ (\bar{F}_1, \cdots, F_s, p, t)$ of the first degree in $F_1, \cdots, F_s$, the marginal degree of productivity of each service should at every time t *of this interval be equal to the ratio of the price per unit of the service to the price per unit of the commodity.*

If one multiplies (3.4) by the increment of service $\Delta\bar{F}_i$ and rewrites the result in the form

$$((\partial u/\partial \bar{F}_i)\Delta \bar{F}_i)p_c(t) = \Delta \bar{F}_i p_i(t) \ , \qquad i = 1, \cdots, s \ ,$$

the following additional theorem can be stated:

The amount of each service used should be such that the increment in total compensatory cost Q_f due to an increment of this service is equal to the cost of this increment of service.

By an application of Euler's theorem it is possible to obtain from (3.4)

$$\sum_{i=1}^{s} \bar{F}_i\, \partial u/\partial \bar{F}_i = u(t) = \sum_{i=1}^{s} p_i(t)/p_c(t) \ ,$$

and hence the following theorem results:

The earning $\sum_{i=1}^{s} p_i(t)$ of the several compensatory services are equal to the total cost $up_c(t)$ due to them.

Finally, it should be remarked that all the theorems on this page are true only if $u = \varphi(\bar{F}_1, \cdots, \bar{F}_s, p, t)$ is a homogeneous equation of the first degree in $\bar{F}_1, \cdots, \bar{F}_s$.

For the case in which there are several products, say, for example, three, X, Y and Z, manufactured from the same commodities and services, Pareto has defined the coefficients of production to be the partial derivatives $\partial F_\alpha / \partial u_i$, $i = 1,2,3$; $\alpha = 1,2, \cdots, m$, where F_α is the quantity of the commodity or service required to manufacture u_1 units of X, u_2 units of Y and u_3 units of Z. For the special case of a single manufactured product one would, following Pareto, replace f_α by $dF_\alpha / du = f_\alpha$. If the f_α are compensatory, then according to Pareto

$$\psi(f_1, f_2, \cdots, f_m) = 0.$$

Various generalizations could be made (by considering more than one compensatory relation or by using Pareto's definition) but no new knowledge is to be gained by performing such mathematical exercises.

CHAPTER XI

Production Incentives

1. *America's Dilemma.* The farmer demands more for his products; the processor of raw materials is willing to pay more for them provided he can collect from the consumer. The laborer desires higher wages and shorter hours; the producer is willing to grant this request provided he can get more for his products. Thus, the producer is perfectly willing to pay more for raw materials and to pay higher wages provided he can collect what he pays out from the consumer, who happens to be essentially little other than the farmer and laborer. When the producer strives for a profit the consumer yells "profiteer" and "capitalist" and yet if there is no profit over an extended period of time, there is no production for the consumer to consume.

When a situation is reached in which the farmer receives almost nothing for his raw materials, the producer makes no profit, and the consumer has little to consume, it is almost inevitable that a cry for inflation should arise, any kind of inflation — devaluation, paper money, credit money, silver and so forth. If there are not enough dollars to go around, why not have more dollars? Thus the Government, representing producers, farmers and laborers (consumers all), may be injected into the scheme as the party to provide income and profits to every one of its constituent parts. It often is asked at one and the same time to provide profits to industry, high prices to farmers, high wages and low consumer prices, and with all this to balance its budget. Is such a Government faced with an impossible task? It is, unless it is able in some way to provide these profits out of increased velocity of circulation of money media going into goods, consumer goods and capital goods. This may sound unorthodox to those economists who are used to talking in terms of fixed monetary systems in which velocity of circulation of money media is supposed to be fixed by habit and any speeding up of this velocity is regarded as inflation. This attitude is apparently brought about by a more or less superficial examination of the problem — too much monetary theory and not enough economic theory, or vice versa.

Velocity of circulation of money and of goods will be discussed more fully in a later chapter but a story at this point will serve to illustrate both velocities, and some preliminary conception is essential for a complete understanding of this chapter.* This story, adapted from one told by F. Y. Edgeworth† follows: Shortly after the legalization of beer, two enterprising Americans — who may be called Pat and Mike—decided to sell beer at ten cents a glass at a certain festival. After paying license fees and buying a barrel of beer, Mike had no money but Pat had a nickel. On the way to the festival Pat became thirsty and bought a glass of beer at ten cents paying Mike the five cents as his share. Soon afterward Mike became thirsty and bought a glass of beer from Pat. Then Pat bought a glass and soon afterward Mike purchased. So they continued until the barrel was empty. A nickel having a high velocity of circulation purchased a barrel of beer for consumption!

An economic theory now receiving some attention states that the spending habits of investors, farmers, white-collar workers and laborers are different and that a depression is brought about by maladjustment of purchasing power. Thus, it is supposed that a $12 a week laborer spends all his income in a week, that is, the velocity of circulation of money for him is at least 52 times per year. A $2,000 a year worker usually spends three-quarters of his income as quickly as the laborer, but the rest he spends more slowly, that is, he places the balance in the bank, in life insurance, or in some other form of saving, and it has to be spent through the credit mechanism. Those in the higher income groups spend even more slowly — purchasing power to them may be only a power, which they may or may not exercise.

If credit is being granted by the banks, etc., this slowing down in velocity of circulation is overcome, and, in fact, the velocity can be artificially speeded up by a bank inflation. Ordinarily and certainly in times of depression, there is lost motion. It takes some time for savings to work back into wages through the banking structure. In other words, *there is a time lag.* This lag is of tremendous importance.

There is no harm in having a person save, provided someone borrows his savings and spends them. Difficulty arises when there are insufficient borrowers to meet the supply of savers. Whenever

*This chapter ought to be reread after reading Chapter XIII.

†F. Y. Edgeworth, *Economic Journal,* 1919.

the chain of circulation is broken, new links are inserted by bank credit expansion or old links are removed by failure to make new loans as old ones are paid, a situation of potential social danger appears.

If banks do not lend savings in a time of depression, thereby breaking the chain, an alternative is to force those with large incomes to spend, i.e., to find their own investment opportunities and thus complete the chain. Some who have thought superficially might suggest redistribution of wealth so that it will be placed in the hands of those who do spend. These would raise wages, shorten hours, provide farm bonuses out of taxation, lavishly hand out millions for relief, etc. Another method proposed is that of an income tax with exemptions for payrolls and raw materials. A more direct method would tax deposit accounts and money.*

A study of the nature of production incentives is essential here, since in a capitalistic economy profit incentives (not necessarily money profits) bring out new investments; that is, profit incentives effectively press savings into service.

2. Difference in Working of Profit Incentive in Capitalistic and Socialistic Economy. It must be recognized that every successful economy requires a motivating force. In a capitalistic economy the principal force is the desire for money profits. In a socialistic economy like that of Soviet Russia individual plants are required to show profits. The chief distinction, of course, is that in the capitalistic economy the entrepreneur takes the profits for himself and decides whether or not he will make new investments out of his profits or out of borrowed money on the expectation of profit, whereas in the socialistic economy the manager (corresponding to the entrepreneur) strives for profits in order that he may not incur the displeasure of the state, and the state decides what new investments shall be made. Either a capitalistic economy or a socialistic one will work, but it is extremely difficult to mix the two.

In a capitalistic economy in which profit incentives are materially reduced, entrepreneurs already in business operate in attempts to get some return on capital invested in the past but, in general, they do not make replacements. Even the entrepreneur who cannot get any return above cost continues to operate in order to preserve his organization — that is, he operates hoping that he will eventually make profits.

*This is the method proposed by Arthur Dahlberg.

Few private investments can be expected in an economy *tending* to socialism of the Russian type and unless the state aggressively pushes the socialistic program in the transition, a period in which there is a lower standard of living than would exist in a capitalistic economy or a socialistic economy is inevitable. This, of course , does not mean that social control cannot be applied to a capitalistic economy. It does mean, however, that this control cannot produce anything but harm if it reduces the profit margin too much. Here it is difficult to say just what is too much. The answer must be sought in the social philosophy of entrepreneurs in the economy. If these become active only when the profit incentive is something like five per cent per annum of the invested capital, then, obviously, any control which would seek to reduce profits to three or four per cent per annum would not call forth adequate initiative. Here, it would be necessary to introduce the element of fear or some other added drive.

There are at least three kinds of economies, e.g.,

(1) Economies in which a profit regarded as a reasonable one by an entrepreneur, and one upon which he acts, is encouraged.
(2) Economies in which the fear of the displeasure of the state or of loss of investments already made is added to a small profit incentive so that entrepreneurs are forced to become active.
(3) Socialistic economies in which entrepreneurs are managers and the state makes the new investments.

Proposition (2) is neither capitalistic nor socialistic and it is extremely doubtful that it will work. Since under it new investments are not made by private individuals, it follows that to maintain the existing standard of living the relative investment of the state in business must increase until the transition to (3) is complete. During the transition it must be expected that investments will be made chiefly in Government securities, unless monetary uncertainty forces flight of capital into durable goods least likely to be taken over by the Government or foreign currencies.

Whether one holds briefs for either capitalism or socialism, some essential advantages of capitalism are that it makes for individual freedom and greater efficiency since one is more apt to take care of his own property than the property of the state. Again, distribution of goods by bargaining (capitalistic economy) is more nearly self-policing than the distribution of goods by rationing (communistic economy). In other words, capitalism properly regulated to eliminate wasteful investments and to preserve bargaining strengths of its elements might lead to a greater standard of living than socialism properly regulated, but any conclusion must be tentative, since there are no examples of either properly regulated capitalistic or socialistic states.

Under either socialism or capitalism it seems certain that new investments should be made out of national savings or profits except that some short-time paper money can occasionally be justified, for, otherwise, a credit inflation that might prove unstable is inevitable. On the other hand, the rate of new investment should be equal to the rate of saving.

3. Monetary Requirements. In a capitalistic economy there must be reasonable certainty regarding the future value of whatever media of exchange are used as money, at least there must be this certainty as far as the investor is concerned. For example, if an investor (lender) estimates profit of five per cent on a year's investment for a stable gold dollar and then finds on recalculation that the gold dollar *for him* will decrease in value 25 per cent during the year, he must reach the conclusion that the profit incentive is not large enough. Existing equities or stocks of consumers' goods may be preferable. Thus, at least over the short period of time, as distinguished from the long period used in most gold correlations, appreciable new investments go hand in hand with stable money.

Of course, with a depreciating dollar an entrepreneur (manufacturer, wholesaler or retailer) can figure on prices of his products rising and can even estimate a higher profit than customary. In other words, a mild inflation can stimulate the production of consumer goods. It must be recognized, however, that all prices do not rise by the same relative amounts. There is always the question of the relation of present and expected supply to present and expected demand. Furthermore, many prices are more or less fixed. Behavior of prices is discussed more fully in a later chapter.

There is always the question of whether or not new investments can be postponed for other things which over short periods of time may be more desirable in the sense that they will lead to increased velocity of circulation of goods and greater incentives to invest in the not too distant future. Thus, if money depreciates slightly and if greater amounts of consumers' goods are purchased for consumption, resulting profits may after a time lead to new capital goods investments.

If, however, lack of confidence in currency and Government securities becomes so widespread that there is a flight into real property and goods, a point may be reached at which increasing prices of such property outrun wages and it becomes profitable to do a certain amount of new building. It has never yet been possible to stop such a widespread lack of confidence in currency and Government securities once it has started. New building or plant expansion in such instances is financed out of owners' capital since at this stage there are no investors, that is, investors have been practically wiped out or they have become owners. No intelligently informed person wants this situation to happen unless he has some particular selfish profit motive in mind.

A sale takes place only when buyer and seller agree on a price, and when a sale takes place goods move into channels where they are needed, leading to greater production. It is this movement of goods that creates prosperity.

There may be times when a nation finds it desirable to lower its foreign price in order to exchange goods (not simply sell goods) with its neighbors. This reduction in foreign price has in the past been accomplished in two principal ways,

(1) by providing bounties and

(2) by reduction in gold content of monetary media.

It so happens that in every economy there are certain *fixed* prices, transportation, etc., and certain other prices which change slowly. Both of these sets of prices may play important roles. In the United States the domestic market determines the price of automobiles, whereas the world market determines the price of wheat, cotton and certain other international products. Thus, a reduction in gold content of the dollar means primarily that United States' prices of goods sold in the international markets are affected. These in turn, of course, affect other prices.

It is silly to say that reducing the gold content of the dollar twenty-five per cent means that bondholders lose twenty-five per cent of their investments, since, as has been indicated and will be further demonstrated in a later chapter, all prices do not rise by twenty-five per cent. Of course, if the bondholders intend to buy gold or the international commodities whose prices are gold prices, then their dollar has depreciated twenty-five per cent. If they intend to buy transportation it has not changed. It is only in the sense just explained that a single reduction in gold content of the dollar affects the bondholder.

When the United States suspended gold payments, the downward trend in exports of automobiles began to smooth out and in a short time turned upward. Considerable progress has been made since that time, and contrary to common belief the gain has not been made at the expense of other countries. While the United States has been exporting more automobiles, the trend of automobile production in Great Britain and other manufacturing countries has also been upward. What is the explanation?

Abandonment of gold and subsequent devaluation have been accompanied by increased imports, especially imports of raw materials. An examination of markets indicates that America's chief automobile customers are getting more for their raw materials (prices of international commodities are essentially gold prices) and are buying more automobiles and, of course, some other products at prices which are cheaper to them. On the surface the net effect of this seems to be that the United States is giving away automobiles, etc., but this is not the whole picture. Increased income from sales of automobiles finds its way almost immediately into wages (fast circulating income) and into subsequent demand for more of the raw materials. Thus, in this case the velocity of circulation of goods appears to be speeded up and the United States as well as other countries benefits. The increased velocity of circulation of international goods seems to filter through to nations other than the United States and its immediate customers so that world recovery is accelerated.*

4. The Nature of Interest. In the preceding chapters the term "interest" has been used occasionally without any attempt to define what is meant by interest other than to offer hints that it is composed of risk premiums per year plus income per year on a safe investment. A short discussion of interest seems to be desirable at this point.

Present money is in general preferred to money due at some future time, present food and clothing to future food and clothing, present beauty to future beauty and so on. There is, however, a limit to the amount of present goods which is preferred to future

*Multilateral trade works much more smoothly than bilateral trade. It may be hoped that nations will eventually recognize this and stop their foolishly inefficient bilateral trade pacts.

goods of the same kind. An average family of five might spend $50 for present food because they expected to consume it in the next month, but in general this family would not wish to spend $1,000 for present food. Thus, there is a point at which the desire for an additional unit of present food is zero. The rate of interest might, therefore, be defined as the per cent of premium paid on money at one date in terms of money to be in hand one year later. In this definition the risk of not getting the money one year hence is inherent in the premium paid. Wheat or other goods might be substituted for money, but practically it is only money which is generally traded between the present and future.*

Throughout history the thought that there should be no rate of interest has persisted. Marxian socialists say that the capitalist (possessor of wealth as consisting of titles to material objects) exploits the laborer by paying him for only a part of what he produces and withholding a portion of the product of labor as interest on capital. They, therefore, describe the capitalist as one who unjustly reaps what the laborer has sown.

The Mormons who settled Utah originally possessed little capital or wealth and did not pay interest for the use of capital of others. By stinting for years, denying themselves food, clothing and leisure they created their own capital and automatically became capitalists. In citing this example, Professor Irving Fisher says:

> "It will be seen then, that capitalists are not, as such, robbers of labor, but are labor brokers who buy work at one time and sell its products at another. Their profit or gain on the transaction, if risk be disregarded, is interest, a compensation for waiting during the time elapsing between the payment to labor and the income received by the capitalist from the sale of the product of labor."†

In a dynamic society there are many factors which determine the society rate of interest — risk, population growth, velocity of circulation of money and bank credit, technological improvements leading to reduced costs, etc.

In any event capitalists must return their interest or profits to labor in order to enjoy them. They may do this by paying wages to laborers who build new capital equipment, by purchasing con-

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*Obviously grain and cotton futures are exceptions. Here the principal part of the difference in price is risk rather than interest.

†Irving Fisher, *The Theory of Interest as Determined by Impatience to Spend Income and Opportunity to Invest It*, 1930, p. 52.

sumers' goods (wages plus interest and/or profits), or by purchasing services. Part of the income, of course, goes to the Government, which in turn purchases services and consumers' goods and constructs capital goods. *So long as this circle is completed labor gets all the product of its efforts.* Of course, wages paid directly for production of a product may be only ten per cent of the final consumption price of the product, because parts of all the services mentioned — wages paid in capital construction, services of Government, services of individuals, etc., — are included in the final consumption price. *When the circle is broken, dislocations leading to a general marking down of capital values, unemployment, etc., occur.*

Interest has apparently been the subject of so much attack chiefly because *interest goes to people of means who may or may not spend it,* i.e., *they have the power to spend it or hold it.* When metallic money was the chief money, these interest receivers could be singled out as hoarders. In this sense they did withhold part of labor's share. Even in modern times interest holders often turn to gold, silver and other precious hoards when profit incentive on investment becomes small. Thus, interest takers are not blameless by any means, and yet so long as hoarding (either by individuals or by banks) is not outlawed who can blame these interest takers for seeking the largest profit incentives?

5 *Return on Capital Investment.* As pointed out in a previous chapter, residential houses are built chiefly because of net rental income incentives. An extremely high correlation has been obtained between volume of residential building in St. Louis and net income —rent times occupancy minus taxes divided by replacement cost. At present this net income or incentive is lower than it has been in fifty years and there is practically no residential building.

When a factory is considered, the question of income or profits is again of tremendous importance. It seems desirable, therefore, to give some consideration to the question of return on capital investment.

The total "expected" yearly income (risk not deducted) from the capital k is the value of expected profits Π at $t = 0$; that is, $\Pi\ E(0, t_1)$, where $E(0, t_1)$ is the discount factor described in the previous chapter. Here the price used is the *expected price*; that is, it is assumed that there is a certain element of risk in Π itself due to the uncertain estimate of price p. In many instances there is also a production risk. It is assumed here that expected price and expected production are estimated carefully so that the price and production risks are reduced as much as possible.

Now, if i is the rate of interest on an investment which for practical purposes is devoid of risk (for practical purposes i may be taken as the rate of yield of short-term Government securities), then in order that the business appear profitable the inequality

$$(\Pi - R)E(0, t_1) > ik$$

must be satisfied.

The above inequality is perhaps more suggestive in the form

$$\Pi/k > iE(t_1, 0) + R/k \quad ; \tag{5.1}$$

that is, the rate of net income must *appear* to be greater than the rate of risk by an amount equal to the value at t_2 of the rate of interest on short- term Government securities.

As long as (5.1) is expected to be satisfied for some units of an industry, producers will expand their plants and some new producers will be persuaded to enter the field of production. As new producers enter with cost functions usually more favorable than those of certain established producers, they are able to lower the price (or what is equivalent to increase the production) to the point where some of these latter are unable to maintain the inequality. Sometimes a reorganization (See Chapter X) is able to effect the necessary economies, but not always. If reorganization and its attendant adjustments do not sufficiently reduce costs, the concern usually operates for a while without providing for risk, suffers some loss because of this element, and is finally forced to default bonds or write down its assets. This may happen several times before the firm actually ceases production. In fact, the necessity for meeting interest payments forces production. Again, firms sometimes dissipate their capital in the belief that better times are "just around the corner."

Many times new concerns underestimate the risk involved, and even when the inequality does not hold they mistakenly believe that it does, and of course, as pointed out previously, there is always the problem of predicting the course of prices. As a result, even in times of general prosperity some industries taken as a whole over a number of years may fail to earn profits equivalent to the prevailing interest rates. In other words, *expectation of profit and attainment of it may be entirely distinct.*

Short-sightedness of producers in estimating capital return is well illustrated by price-fixing histories. In fact, it is so well illustrated that a short discussion of how price-fixing influences investments is desirable here even though the general subject of price fixing will be discussed in the chapter on prices.

Suppose that the total demand is given by either one of the equations

$$U = ap/I^{\beta} + C(t) \ ,$$

$$U = aI^{\beta}/p + C(t) \ .$$

Suppose that the price is fixed by agreement or by Government order at $p = p_1$, and suppose further that the production at the time the price is fixed is U_1. Several situations can result.

1. If p_1 is chosen so that it and U_1 satisfy the demand equation, no immediate harm seems to be done, but after a short time things begin to happen. For a fixed price the adjustments in rate of production and costs described in the preceding chapter take place. Some firms adjust B, some A, and some C, and profitable businesses expand. Technological research is greatly stimulated, since by lowering costs, greater profits can be realized and each entrepreneur has a definite price on which to base his calculations; i.e., he does not have to calculate on a possible decline in price due to his own disturbance of production or that of his competitors. If he understood price fixing he would take this into account, but he does not understand it. Thus, an expansion in production takes place, but an expansion in demand is only possible by increasing consumer income.

It is, of course, possible that expansion of plant may require more workmen than are laid off in the process of cost adjustment and the income of the economy may be sufficiently raised (due to increased velocity of circulation or expansion of credit) to make it possible for more goods to be sold at the fixed price than would be sold otherwise, but no definite answer can be given to this question until further quantitative studies of demand, cost and income are available.

2. If p_1 is chosen to be greater than the value of p which with U_1 and I satisfy the demand equation, sub-marginal plants already in existence make profits. If the demand is highly elastic so that

it drops appreciably as price is increased to p_1, profits of the sub-marginal firms are made at the expense of production curtailment of efficient firms, unless these firms are more efficient in distribution as well as in producing. If the latter is true the demise of the sub-marginal firms may be hurried and the way prepared for new investments on a large scale. During the transition period, i.e., during the period in which the sub-marginal firms are disappearing, there may be great decrease in quantity of goods consumed. Thus, the standard of living may be greatly lowered. If the demand is relatively inelastic, higher fixed prices mean greater profits throughout the industry. In this case new capital will be artificially attracted to the industry by the high profits, and very great expansion in capacity (over-investment) may occur.

6. Taxation and Bounties. In a capitalistic society the tax system offers a very effective tool for economic planning. By means of it production incentives can be created or destroyed. By careful choice of taxes it is possible to ruin an industry and substitute a new one in its place.

On the side of ruinous taxes consider those on timber and coal lands and agricultural lands. In many states an owner of timber lands is taxed according to the value of the standing timber. While his timber is young he can pay the tax since the tax is small, but when his timber begins to mature the tax forces him to cut regardless of market price. The market price for timber and timber products thus falls below the free competitive price equal to marginal cost, since the tax forces production beyond free production. A similar situation prevails in the cases of coal and agricultural lands. An owner of cotton land might prefer to leave the land fallow for pasture purposes or for hunting lands, etc., but he is unable to pay high taxes out of other income and in order to protect his capital investment he is forced to place his land in production. Here again the price of the product falls below marginal cost, if capital return is figured in marginal cost. Taxes on productive land are imposed in proportion to ability to pay, only if the land is put into production.

To those who may be tempted to answer these contentions by saying that a tax on land lowers the value of the land and hence the marginal cost, I merely state that adjustments in land values take place over long periods of time. This is especially true in the United States where land mortgages play important roles.

On the other hand, a high tax on old buildings would expedite their demise and would create demand for new buildings by shortening the supply of existing buildings. In the past the United States has had a growing population which was able to create new demands for building. If this population is now stabilized greater prosperity can be obtained by forcing the obsolescence of old goods through some means of taxation.

Taxes on gasoline, if not excessive and if the tax money is used to create additional physical needs for consuming gasoline, i.e., is used to build needed highways, apparently can be used to create greater demand for the product taxed.

There are, of course, many examples of governmental bounties to encourage production. The American Merchant Marine was built chiefly because the United States Government offered mail contracts. The payments made under these contracts were essentially bounties. Airmail contracts undoubtedly stimulated the building of American commercial fleets.

Government funds spent for demolishing obsolete and unsanitary buildings that are now rented would provide bounties to private initiative since the removal of such structures from the supply of buildings would tighten the rent structure and thus increase the profit incentive to build.

7. *Depreciation.* Depreciation of existing capital goods is an important factor offering powerful incentives to replace them. Attempts to preserve existing obsolete and depreciated capital goods have largely been responsible for laws protecting vested rights. Corporation bonds bear a fixed rate of interest regardless of the state of the capital goods originally offered as security so long as the quick assets of the corporation are sufficient to pay the interest. In the United States a benevolent Government created an Interstate Commerce Commission which has apparently adopted the view that it was set up to protect the interests of all railroad bondholders even though the equipment behind the bonds is obsolete, or otherwise depreciated. Furthermore, in the United States it has been customary to add the cost of tearing down obsolete buildings to the "value" of the land when floating a mortgage on a new structure to be built on the site of the old one.

A good deal of confusion exists regarding the problem of charging off depreciation. In fact, accountants are not agreed on the "best" method of allowing for depreciation. Various formulas such as the straight line formula, the sinking fund or compound interest formula, the unit cost plus formula, etc. are in more or less common use by cost accountants. Such existing theories of depreciation assume a constant rate of production. The following theory allows the rate of production to vary and in addition gives insight into the nature of value or "price" of capital goods equities. It includes all the above-mentioned theories of depreciation as special cases.

Suppose that a producer of a product or service has a cost of production which can be taken to be $Q(u, p, t)$ where $Q(u, p, t)$ is defined as in the chapter on factors of production and cost. This quantity may be taken to be the cost of production after depreciation is taken into account. Let φ be the cost of production before depreciation is taken into account.

The gross profit or rent which will be obtained from a machine in unit time will, therefore, be

$$R(t) = p(t)u(t) - \varphi(u, u', p, t) \ .$$

Here $p(t)$ and $u(t)$ are expected price and expected rate of production.

Here the acceleration of production $u' = du/dt$ has been introduced into the cost function. It may be omitted if desired without materially affecting the treament of the problem.

Now, the value of a machine to its operator at a time t_1 is the sum of the anticipated rentals which it will yield from the time t_1 to a time ω at which it is to be salvaged, each multiplied by a discount factor to allow for interest plus the salvage value also discounted. In the most general case the interest varies with the time. For the general case it is, therefore, possible to write the value of a machine to its operator at a time t_1 as

$$V = \int_{t_1}^{\omega} \left[pu - \varphi(u, u', p, t) \right] e^{-\int_{t_1}^{t} \delta(\nu) d\nu} dt \tag{7.1}$$
$$+ Se^{-\int_{t_1}^{\omega} \delta(\nu) d\nu} ,$$

where $\delta(\nu)$ is the *force of interest*, which is defined as the rate of increase of an invested sum s divided by s, and S is the salvage value of the machine at the time ω.* This salvage value S is the cost price K of the machine at the time t_1 minus the depreciation in the market value of the machine after it has been operated for the period of time t_1 to ω. The depreciation in the market value of the machine is in general a functional of the rate of production, of the price of the article produced and of the time derivatives of these quantities. It follows, therefore, that

$$S = K - \int_{t_1}^{t_2} D(u, u', p, t) E(t_2, t) dt \ , \tag{7.2}$$

where

$$E(t_2, t) = e^{-\int_{t_2}^{t} \delta(\nu) d\nu}$$

and where $D(u, u', p, t)$ is the rate of depreciation. If the value of S defined by (7.2) is substituted in equation (7.1) and the second term of the right-hand member is transposed, the following expression is obtained:

$$V - KE(t_1, t_2) = \int_{t_1}^{t_2} [pu - \varphi(u, u', p, t) - D(u, u', p, t)] \ E(t_1, t) dt \ . \tag{7.3}$$

The quantity represented by the second term of the left-hand member is the value at t_1 of a sum K necessary to replace the machine at the time t_2. In order to simplify notation in the work which follows, write this last expression in the form

$$I = \int_{t_1}^{t_2} [pu - Q(u, u', p, t)] \ E(t_1, t) \ dt, \tag{7.4}$$

where

$$Q(u, u', p, t) = \varphi(u, u', p, t) + D(u, u', p, t).$$

In general the expected price $p(t)$ will be some factor $\gamma(t)$ times the present price p_0. As already pointed out, the quantity

*The treatment given here is based on the paper by C. F. Roos, "The Problem of Depreciation in the Calculus of Variations," *Bulletin of the American Mathematical Society*, March-April, 1928. See also H. Hotelling, "A General Mathematical Theory of Depreciation," *Journal of the American Statistical Association*, September, 1925.

$\gamma(t)$ is essentially a psychological factor. It changes with business conditions. It is greater than or equal to one when prices are rising and less than one when prices are falling. The quantity was discussed more fully in the preceding chapter.

A likely assumption to make regarding the operation of the machine is that the operator (owner) will endeavor to choose his rate of production $u(t)$ so that (7.4) will be a maximum. This expression is the difference between the value of the machine at the time t_1 and the discounted cost price. This assumption is not equivalent to the assumption that the operator will endeavor to maximize V, for t_2 is a variable end-value, but it has the advantage of simplicity. For this $u(t)$ would be determined by the Euler condition of the calculus of variations, and the end-value t_2 by a transversality condition.* Suppose that the value of $u(t)$ and t_2 maximizing (7.4) have been obtained. It remains to obtain an expression for depreciation, which has been defined as the rate of decrease of value. To do this suppose that a machine is operated at a rate $u(t)$ which maximizes I, so that in the expression (7.4) the functions $u(t)$ and $p(t)$ and the end-values t_1 and t_2 are those defining the maximizing arc. Then the expression (7.4) will define the maximum value of the machine at the time t_1, and the value of the machine at any other time $t(t_1 \leqq t \leqq t_2)$ can be obtained by replacing t_1 in (7.4) by t.

It is most interesting and instructive to see what further hypotheses must be made to obtain the various depreciation theories now commonly used. If the salvage value S is assumed to be a point function of the time $t_2 = n$ at which the machine is to be salvaged instead of a functional of the rate of production and price as in this account, then on changing the parameter of integration from t to τ equation (7.1) becomes, for $t_2 = n$ and $t_1 = t$,

$$V(t) = \int_t^n [pu - \varphi(u, u', p, \tau)] E(t, \tau) d\tau + S(n) E(t, n) \quad . \tag{7.5}$$

If $\Delta(t) = -dV(t)/dt$ denotes the depreciation of the machine to its operator, by differentiating (7.3), for $t_1 = t$, $t_2 = n$, with respect to t, it is possible to write

$$\Delta(t) = -dI(t)/dt - \delta(t) K E(t, n) \quad . \tag{7.6}$$

*For a solution see Roos, "The Problem of Depreciation in the Calculus of Variations," *loc. cit.*, pp. 223-224.

If the cost of production function $\varphi(u, u', p, \tau)$ is a point function $\varphi(\tau)$ instead of the more general function φ [illegible], this formula reduces to that given by Hotelling.*

Now, by (7.5), $V(n) = S(n)$, and hence the total depreciation of the machine for the period $\tau = t$ to $\tau = n$ is quite evidently $V(t) - S(n)$. If only simplicity is desired it may be assumed that the depreciation per unit time is equal to the average depreciation $[V(t) - S(n)]/n$. The well known straight line formula for depreciation ~~will~~ then be obtained.

Again, if p, u, and φ of (7.5) are constant for a certain number of years of the machine's life and then changes abruptly at $t_2 = n$ in such a way that it is evident that the end of the useful life of the machine has come, and if the force of interest $\delta(t)$ is a constant equal to δ, the integration in (7.5) can be performed. Then,

$$(7.7) \qquad V(t) = [pu - \varphi]\{[1 - e^{-\delta(n-t)}]/\delta\} + Se^{-\delta(n-t)} \ .$$

Now, when $\delta(t) = \delta$, a constant, the discount factor $e^{-\delta}$ is equal to $(1 + i)^{-1}$, where i is the rate of interest in the ordinary sense. It follows, therefore, that

$$(7.8) \quad V(t) = [pu - \varphi]\{[1 - (1+i)^{-(n-t)}]/\delta\} + S(1+i)^{-(n-t)} \ ,$$

and, at $t = 0$, this becomes

$$(7.9) \qquad V(0) = [pu - \varphi]\{[1 - (1+i)^{-n}]/\delta\} + S(1+i)^{-n} \ .$$

If $[pu - \varphi]/\delta$ is eliminated from the equations (7.8) and (7.9), the expression

$$\begin{vmatrix} V(t) - S(1+i)^{t-n} & 1 - (1+i)^{t-n} \\ V(0) - S(1+i)^{-n} & 1 - (1+i)^{-n} \end{vmatrix} = 0$$

is obtained. By adding $-S$ times the second column to the first column, then subtracting the second row from the first row and expanding, this reduces to

$$V(0) - V(t) = [V(0) - S(n)]\ [(1+i)^t - 1]/[(1+i)^n - 1] \ .$$

When the customary notation

$$s_{\overline{t|}} = \frac{(1+i)^t - 1}{i}$$

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*Harold Hotelling, *loc cit.*

is introduced, this formula becomes

$$V(0) - V(t) = [V(0) - S(n)]s_{\overline{t|}}/s_{\overline{n|}} , \tag{7.10}$$

which is the well known formula for the accumulation of depreciation allowances at the end of the t^{th} year under the sinking-find method.*

If now the formula for $V(t+1)$ obtained from (7.10) by replacing t by $t+1$ is subtracted from (7.10), it follows that

$$V(t) - V(t+1) = [V(0) - S(n)] - [(1+i)^t/s_{\overline{n|}}] . \tag{7.11}$$

This formula states that the *depreciation for any year is equal to the depreciation charge for the first year at compound interest at the rate of i per annum.* This is, therefore, the so-called "compound interest method" of providing for depreciation.

If the indicated differentiation in (7.6) is performed and the resulting equation solved for p, the equation

$$p = [Q + \delta Ke^{-\delta(n-t)} - dV(t)/dt]/u$$

is obtained. This is equivalent to the formula by which J. S Taylor defines *unit cost plus.*†

The popular "sinking fund" or "equal annual payment" formula for depreciation (7.10) and the "compound interest formula" (7.11) have been derived from the general theory by assuming that the rate of production u, the price p and the cost of production φ are constant throughout the useful life of the machine. The unit cost plus formula can be obtained directly from the formula for depreciation as given when the price p and the rate of production u are known functions of the time.

Risk of obsolescence and certain other risks are taken into account in the discount factor $\delta(\nu)$.

8. Further Remarks on Incentives. Obviously a great deal could be written about production incentives. There are many kinds of incentives and with each there is associated a philosophical or

*Rietz, Crathorne and Rietz, *Mathematics of Finance*, New York, 1932, pp. 112-121.

†J. S. Taylor, "A Statistical Theory of Depreciation Based on Unit Cost," *Journal of the American Statistical Association*, December, 1923.

ethical question: "Is this a proper incentive to offer?" To attempt to answer such questions would require going far beyond the scope of this book. In fact, such questions have been decided differently at different times by the public and the courts. The public has often condoned or accepted certain incentives as long as they seemed "useful" and when they ceased to be useful it has turned against them.

CHAPTER XII

Behavior of Free and Restrained Prices

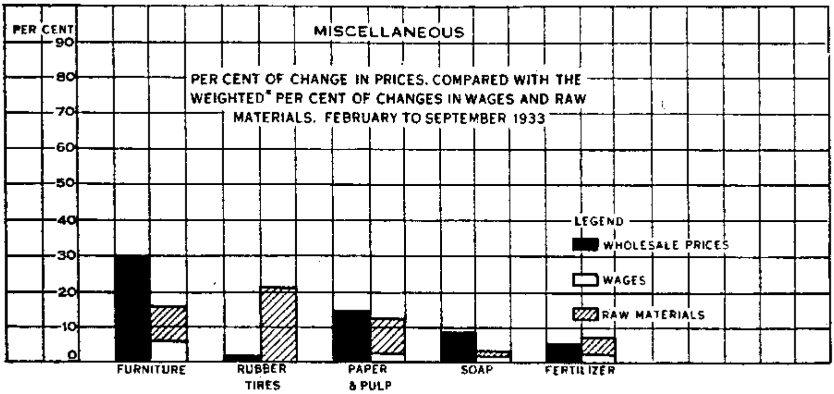
1. Introduction. For years theoretical economists have been quoting the theorem "Price equals marginal cost." Some of them have gone deeper into the problem pointing out that certain prices are fixed by monopolistic agreement or threat and that these bear little relationship to marginal cost. For example, it is no secret that Mr. Gary periodically announced steel prices for his corporation, and that his competitors then proceeded to announce the same prices. They presumably named the same prices rather than risk price-cutting competition that would ultimately drive them out of business.

Some economists have distinguished between real prices and money prices. It has already been pointed out that in the United States money prices of certain agricultural products bear little relationship to marginal cost, since production is forced by taxation beyond the free competition point. A similar situation prevails in the case of timber and timber products. Also, as has already been indicated, certain industries may become overinvested through promotional schemes of stock selling, essentially misrepresentation, and in these industries money price may be far below marginal cost unless investment "values" are drastically reduced. Development of new industries may also bring about a similar situation.

In most new industries money price remains considerably above marginal cost sometimes for a rather long period of time. In this way profits may be large during the introduction of the industry. Such profits are often reinvested in the industry and, in addition, when profits are large new capital is attracted.

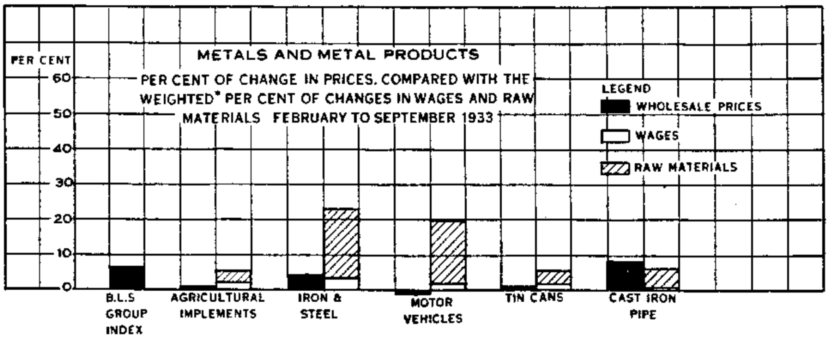
Once an industry becomes stabilized in an economy with a stable monetary unit, price is probably in general equal to marginal cost, but relatively few industries are yet stabilized and no economy has had a stable monetary unit. Such an old industry as the shoe industry has not yet become stabilized in the United States. As long as an industry has not become stabilized and as long as a monetary unit is not stable, the theorem which states that price is equal to

CHART XXXI



*Weighted according to estimated importance of wages and raw materials to value of product (Census data)

CHART XXXII



*Weighted according to estimated importance of wages and raw materials to value of product (Census data)

marginal cost is meaningless and decidedly misleading. Furthermore, it is very difficult to define marginal cost or any other "cost," since cost is composed of many elusive factors. Depreciation, overhead allowances and advertising allocation are arbitrary. When more than a single product is manufactured, cost determination is impossible except arbitrarily. In the chapter on joint demand and loss leaders it was intimated that the highest proportionate overhead would in general be assigned to the product whose demand was most inelastic if the producer were intelligently informed. Thus, demand has an important bearing on cost.

2. Movements of Wholesale Prices. In an economy in which credit is permitted there are many factors which determine prices. Cost has some influence but, as already mentioned, cost is elusive. The ratio of credit money to gold is undoubtedly important; monopolistic agreements and threats are of prime importance for some industries; systems of taxation play important roles, and, furthermore, speculation is highly important, especially over short periods of time.

An analysis of the ratio of 1929 prices to February, 1933, prices and the ratio of October, 1933, prices to February, 1933, prices of groups of commodities used in the Bureau of Labor Statistics Index of Wholesale Prices, forcefully illustrates that there are factors other than money and cost which are important, especially during periods of rapid economic changes.

Charts XXXI and XXXII compare price increases of various products with weighted increases in material and labor costs. These were compiled by utilizing price material from the United States Bureau of Labor Statistics, wage material computed from figures by that Bureau and relative weights computed from the Census of Manufacturers of the United States Census Bureau. The charts are of course subject to great error, but they are nevertheless significant in a general sort of way.

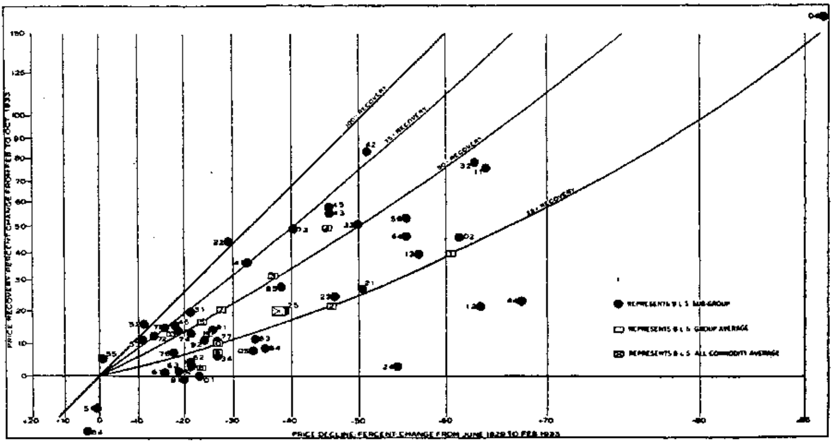
It should be noted that these charts show only changes which occurred during the period of most rapid price increases in the summer of 1933. They constitute only a rough basis of comparison since no data are available on the variation of overhead costs. They do, however, present what might be considered a liberal interpretation from the viewpoint of industry, since they assume that overhead remains approximately the same proportion of the price when the price increases. Actually with increasing prices and particularly with increasing volume, the overhead burden per unit of output ordinarily decreases. It will be noticed that with few exceptions price increase bears little, if any, relation to increases in wages and prices of raw materials. It is only in the cases of men's clothing and petroleum refining that price increases correspond to increased wages and increased prices of raw material.

The Charts XXXIII to XXXIX present on single diagrams changes between three points of time in the prices of a large number of commodities. Each dot (.) represents one commodity from the United States Bureau of Labor Statistics Wholesale Commodity Price Index. More conventional charts showing continuous curves

DISPERSION OF WHOLESALE PRICE CHANGES

B.L.S. GROUPS & SUB-GROUPS

CHART XXXIII



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through time cannot be used satisfactorily for comparing the movements of a large number of items between specific dates.

The extent of price decline for each commodity during the depression years 1929 to 1933 is represented by the distance of the dots to the right of the vertical axis; the extent of advance since February, 1933, is measured by the distance upward from the horizontal axis. The horizontal position of each dot relative to the vertical axis therefore represents the per cent of change from a base of June, 1929, to February, 1933; while its vertical position relative to the horizontal axis represents the per cent of change from a base of February, 1933, to October, 1933.

The diagonal lines mark off sectors representing the proportional part of the price decline which has been recovered.

To trace the price behavior of Douglas Fir, No. 2 turn to Building Materials Chart, Group VII, Lumber sub-group, No. 3. Douglas Fir, No. 2 Item 7. It declined 51.8 per cent. Following up from the horizontal scale on the 50 line, a dot containing the figure

SUB-GROUP AND COMMODITY NO.:	COMMODITY	HORIZONTAL LOCATION ON CHART (% Decline June '29-Feb. '33):
	ALL COMMODITIES	–37.9
1.	FARM PRODUCTS	–60.3
11.	Grains	–64.0
12.	Livestock and poultry	–63.9
13.	Other farm products	–56.8
2.	FOODS	–45.7
21.	Butter, cheese and milk	–50.3
22.	Cereal products	–29.0
23.	Fruits and vegetables	–46.2
24.	Meats	–55.0
25.	Other foods	–38.8
3.	HIDES AND LEATHER PRODUCTS	–37.1
31.	Boots and shoes	–21.5
32.	Hides and skins	–63.1
33.	Leather	–49.9
34.	Other leather products	–26.2
4.	TEXTILE PRODUCTS	–45.1
41.	Clothing	–32.0
42.	Cotton goods	–50.5
43.	Knit goods	–45.6
44.	Silk and rayon	–67.8
45.	Woolen and worsted goods	–45.6
46.	Other textile products	–17.5
5.	FUEL AND LIGHTING MATERIALS	–23.7
51.	Anthracite coal	0.6
52.	Bituminous coal	–11.4
53.	Coke	–11.2
54.	Electricity	2.9
55.	Gas	– 0.3
56.	Petroleum products	–55.2
6.	METALS AND METAL PRODUCTS	–26.4
61.	Agricultural implements	–15.4
62.	Iron and steel	–21.2
63.	Motor vehicles	–19.0
64.	Nonferrous metals	–55.9
65.	Plumbing and heating	–38.0
7.	BUILDING MATERIALS	–27.6
71.	Brick and tile	–15.7
72.	Cement	–13.6
73.	Lumber	–40.1
74.	Paint and paint materials	–21.4
75.	Plumbing and heating	–37.9
76.	Structural steel	–18.0
77.	Other building materials	–26.1
8.	CHEMICALS AND DRUGS	–23.6
81.	Chemicals	–19.8
82.	Drugs and pharmaceuticals	–21.5
83.	Fertilizer materials	–33.6
84.	Mixed fertilizers	–35.5
9.	HOUSEFURNISHING GOODS	–25.2
91.	Furnishings	–25.4
92.	Furniture	–24.3
10.	MISCELLANEOUS	–26.4
01.	Automobile tires and tubes	–23.0
02.	Cattle feed	–61.8
03.	Paper and pulp	–18.2
04.	Rubber, crude	–85.7
05.	Other miscellaneous	–33.2

3 with the figure 6 outside will be found just below the 75 per cent recovery level.*

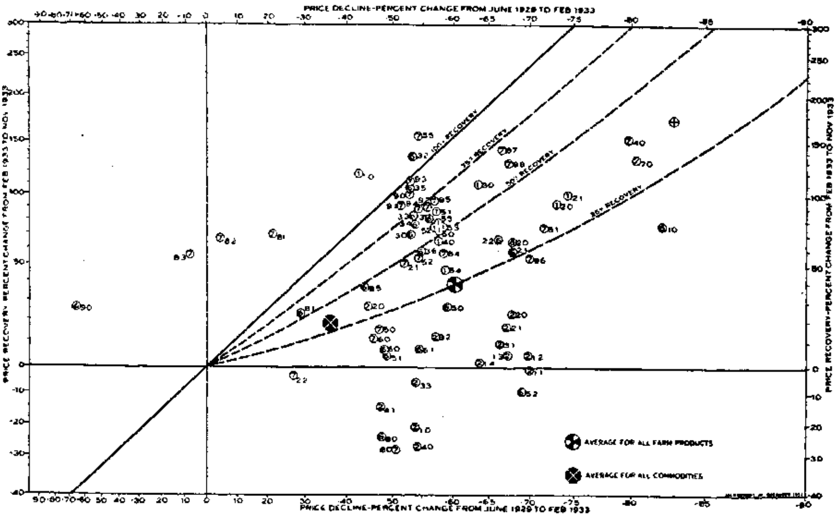
This general character of dispersion charts has long been used for depicting facts of similar variety, but the particular innovation

*The method of depicting price changes which is used in these charts was developed by H. B. Arthur of the Division of Research and Planning of the National Recovery Administration in cooperation with Larry Conant of the Bureau of Labor Statistics. H. B. Arthur and L. Conant. *Journal of American Statistical Association,* 1934.

DISPERSION OF WHOLESALE PRICE CHANGES FARM PRODUCTS

U. S. Department of Labor, Bureau of Labor Statistics, Washington

CHART XXXIV



developed here is the use of a logarithmic scale which interprets visually some of the elasticity features of prices. When a price has declined (horizontal scale) 50 per cent from its 1929 level, another decline of 50 per cent would bring it to 25 per cent of the 1929 level. This second decline would be resisted by the normal operation of supply and demand and of bankruptcies among producers of the product, probably to a greater extent that the first decline. On the chart it will be seen that the second decline is represented by as great an actual distance as the first. It will also be noted that the vertical scale representing recovery of prices is more condensed than the horizontal scale. This has been done because a price which declined 50 per cent (from $1.00 to 50c) would have to advance 100 per cent to effect a complete recovery, or a decline of 20 per cent would require a recovery of 25 per cent to reach the previous level.

The diagonal lines divide the first quadrant into five fields. Each field classifies the points within its bounds and shows what

FARM PRODUCTS

SUB-GROUP AND COMMODITY NO.:	COMMODITY	HORIZONTAL LOCATION ON CHART (% Decline June '29-Feb. '33):
1	GRAINS	
1-10	Barley	–44%
	Corn, Chicago	
1-20	Contract grades	–74%
1-21	Mixed	–75%
1-30	Oats, white, Chicago	–64%
1-40	Rye, Chicago	–58%
	Wheat	
1-50	No. 2 Chicago	–59%
1-51	No. 2 Kansas City	–58%
1-52	No. 1 Minneapolis	–57%
1-53	No. 2 Minneapolis	–58%
1-54	No. 1 Portland, Oreg.	–59%
1-55	No. 2 St. Louis	–59%
2	LIVESTOCK & POULTRY	
	Cattle, Chicago	
2-10	Calves	–55%
2-11	Cows, fair to good	–71%
2-12	Cows, good to choice	–70%
2-13	Steers, fair to good	–68%
2-14	Ditto, good to choice	–64%
	Hogs, Chicago	
2-20	Heavy	–68%
2-21	Light	–67%
	Sheep, Chicago	
2-31	Ewes	–67%
2-32	Lambs	–57%
2-33	Wethers	–55%
	Poultry, live fowls	
2-40	Chicago	–55%
2-41	New York	–49%
6 & 7	OTHER FARM PRODUCTS	
6-10	Beans, dried, N.Y.	–83%
	Cotton, middlings	
6-20	Galveston	–68%
6-21	New Orleans	–68%
6-22	New York	–68%
	Eggs, fresh	
6-30	Boston	–54%
6-31	Chicago	–57%
6-32	Cincinnati	–55%
6-33	New Orleans	–55%

SUB-GROUP AND COMMODITY NO.:	COMMODITY	HORIZONTAL LOCATION ON CHART (% Decline June '29-Feb. '33):
6-34	New York	–55%
6-35	Philadelphia	–54%
6-36	San Francisco	–56%
6 & 7	OTHER FARM PRODUCTS, Cont'd:	
	Fruits	
6-50	Apples, Chicago	–60%
6-51	Apples, New York	–50%
6-52	Apples, Portland	–70%
6-60	Lemons, Chicago	–49%
6-61	Oranges, Chicago	–57%
	Hay	
6-80	Alfalfa, Kansas City	–48%
6-81	Clover, Cincinnati	–30%
6-85	Timothy, Chicago	–46%
6-90	Hops, Portland, Oreg.	65%
	Milk	
7-20	Chicago	–45%
7-21	New York	–53%
7-22	San Francisco	–28%
7-40	Peanuts, Norfolk	–80%
	Seeds	
7-50	Alfalfa, Kansas City	–48%
7-51	Clover, Chicago	–72%
7-52	Flax, Minneapolis	–56%
7-55	Timothy, Chicago	–55%
7-60	Tobacco, leaf	–46%
7-70	Onions, Chicago	–81%
	Potatoes	
7-80	Sweet, Philadelphia	–51%
7-81	White, Boston	–22%
7-82	White, Chicago	– 5%
7-83	White, New York	6%
7-84	White, Portland	–58%
	Wool	
7-90	Fine clothing	–54%
7-91	Fine delaine	–52%
7-92	Half-blood, grease	–56%
7-93	Medium grades	–54%
7-94	Staple, fine, & medium	–55%
7-95	Half-blood, scoured	–57%
7-96	Argentine	–71%
7-97	Australian	–67%
7-98	Montevideo	–68%

B.L.S. classifications 3, 4 and 5 have been omitted to avoid unnecessary complication of the chart.

DISPERSION OF WHOLESALE PRICE CHANGES

FOODS

GROUP II

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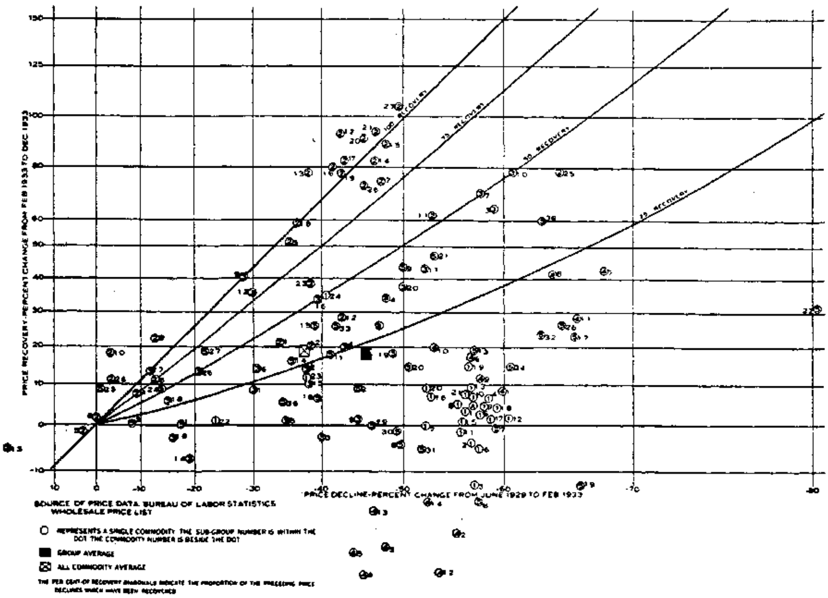
and Dep't. of Labor—Bureau of Labor Statistics

CHART XXXV



N.R.A. — 103.2

percentage of the price decline since June, 1929, has been recovered since February, 1933.

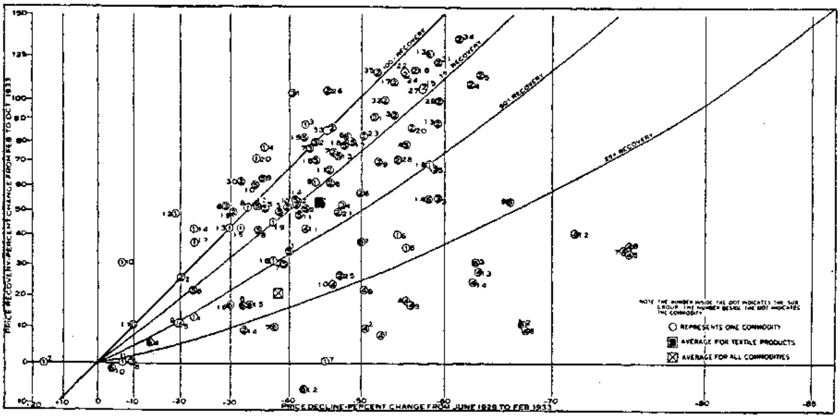
Group (1) — Farm Products: A close examination of Chart XXXIII indicates that the farm products have as a group recovered about half of their loss. Dispersion of farm prices is shown in Chart XXXIV. Their loss, of course, was one of the greatest of all losses. Since some of the principal farm products, notably wheat and to some extent cotton, have their prices determined chiefly by international forces, i.e., in relation to gold, some of the agricultural prices were undoubtedly dependent on the United States inflationary monetary program. This does not mean, however, that

II—FOODS

	Sub-Group and Commodity No. 1 COMMODITY	Horizontal Location On Chart % Decline June, '29, Feb., '33):
	1—Butter, Cheese and Milk	
	Butter, creamery	
	Boston	
	1-1 Extra	-56.0
	1-2 Firsts	-56.6
	1-3 Seconds	-56.6
	Chicago	
	1-4 Extra	-58.1
	1-5 Extra firsts	-57.5
	1-6 Firsts	-57.5
	1-7 Cincinnati, as to score	-51.6
	New Orleans	
	1-8 Fancy	-55.4
	1-9 Choice	-57.4
	New York	
	1-10 Extra	-56.9
	1-11 Firsts	-55.8
	1-12 Seconds	-60.4
	Philadelphia	
	1-13 Extra	-56.7
A	1-14 Extra firsts	-56.9
	1-15 Firsts	-56.4
	1-16 St. Louis extra	-53.1
	San Francisco	
	1-17 Extra	-58.4
	1-18 Firsts	-59.0
	Cheese, whole milk	
	1-19 Chicago	-56.7
	1-20 New York	-52.4
	1-21 San Francisco	-56.3
	Milk	
	1-22 Condensed, N. Y.	-23.7
	1-23 Evaporated, N. Y.	-37.9
	1-24 Powdered, skimmed	-40.6
	2—Cereal Products	
	Bread	
	2-1 Chicago	-33.6
	2-2 Cincinnati	-38.4
	2-3 New Orleans	-35.2
	2-4 New York	- 8.9
	2-5 San Francisco	2.6
	Cereal breakfast foods	
	2-6 Cornflakes	0.0
	2-7 Oatmeal	-47.0
	2-8 Cream of Wheat	-12.6
	Crackers	
	2-9 Soda, New York	-12.8
	2-10 Sweet	- 3.3
	Flour	
	2-11 Rye, white, Minneapolis	-52.7
	Wheat	
	2-12 Standard, patents, Buffalo	-43.0
	2-13 First clears, Buffalo	-38.1
	2-14 Short pats., Kansas City	-46.7
	2-15 Straights, Kansas City	-47.7
	2-16 Standard pats., Minneapolis	-41.7
	2-17 Second pats., Minneapolis	-42.6
	2-18 Patents, Portland Ore.	-36.3
	2-19 Short pats., St. Louis	-42.3
	2-20 Straights, St. Louis	-45.1
	2-21 Standard pats., Toledo	-46.7
C	2-22 Hominy Grits, white	-76.0
	2-23 Macaroni, Chicago	-38.4
	Meal	
C	2-24 White, mill	-76.2
	2-25 Yellow, Philadelphia	-64.3
	2-26 Pretzels	- 3.4
	Rice, New Orleans	
	2-27 Blue Rose, medium to good	-49.4
	2-28 Edith Honduras, medium to choice	-46.0
	3—Fruits and Vegetables	
	Fruits	
	Canned	
	3-1 Apples, New York	-29.4
	3-2 Apricots, cannery	-38.0
	3-3 Cherries, Chicago	-40.0
	3-4 Peaches, cannery	-42.7
	3-5 Pears, cannery	-44.0
	3-6 Pineapple, cannery	-30.5
	Fruits	
	Dried	
	3-7 Apples, evaporated	-58.0
	3-8 Apricots, evaporated	-59.1
	3-9 Currants, cleaned	-28.2
	3-10 Peaches, evaporated	-60.7
	3-11 Prunes, Calif., 50 - 60	-52.2
	3-12 Raisins, seedless	-29.7
	Fresh	
	3-13 Bananas, Honduras	+23.2
	Vegetables, canned	
	3-14 Asparagus	-18.9
	3-15 Baked Beans	-37.9
	3-16 Corn	-39.8
	3-17 Peas	-11.5
	3-18 Spinach	-16.4
	3-19 String Beans	-48.9
	3-20 Tomatoes	-50.0
	4—Meats	
	4-1 Cured beef, N. W.	-59.7
	4-2 Fresh beef, Chicago	-55.3
	4-3 Fresh beef, N. Y.	-47.4
	4-4 Fresh lamb, Chicago	-45.0
	4-5 Fresh mutton, New York	-43.8
	Pork	
	4-6 Cured bacon	-57.0
	4-7 Cured bellies, clear	-68.0
	4-8 Cured bellies, ribs	-64.0
	4-9 Cured hams, Chicago	-57.5
	4-10 Cured hams, N. Y.	-53.2
	4-11 Fresh (composite price) Chicago	-65.7
	4-12 Veal, fresh, good, Chicago	-54.1
	4-13 Poultry, dressed Chicago	-46.1
	4-14 Poultry, dressed New York	-52.0
	5—Other Foods	
	Beverages	
	5-1 Ginger Ale	-17.5
	5-2 Grape juice	-44.6
	5-3 Plain soda	- 7.7
	Cocoa	
	5-4 Beans, New York	-478
	5-5 Powdered	-34.8
	Coffee, Brazilian grades	
	5-6 Rio No. 7, N. Y.	-49.7
	5-7 Santos, No. 4, New York	-59.2
	5-8 Corpa, South Sea, dried	-57.6
	Fish	
	Canned Salmon	
	5-9 Pink, No. 1, Chi.	-50.0
B	5-10 Red, No. 1 cannery	-47.2
	Cured	
	5-11 Cod, Gloucester, Mass.	-41.3
	5-12 Herring, N. Y.	-42.8
	5-13 Mackerel, salt New York	-57.3
	5-14 Salmon, Alaska smoked	-35.5
	5-15 Glucose, New York	-39.0
	5-16 Jelly, grape	-39.3
	5-17 Lard, prime, contract	-65.5
	5-18 Molasses, N. O. fancy	-14.7
	2-19 Oleomargarine, Chicago	-65.9
	5-20 Oleo oil, extra, Chicago	-50.5
	5-21 Peanut butter, Chicago	-53.1
	5-22 Pepper, black, New York	-80.2
	5-23 Salt, American, Chicago	- 1.3
	5-24 Soup, cream of tomato	-14.3
B	5-25 Starch, corn, New York	-47.4
	Sugar	
	5-26 Granulated, New York	-21.4
	5-27 Raw 96°, New York	-21.8
	5-28 Tallow, edible, Chicago	-64.9
	5-29 Tea, Formosa, New York	-46.1
	Vegetable oil	
	5-30 Coconut, N. Y.	-49.3
	5-31 Corn, New York	-51.2
	5-32 Cottonseed, N. Y.	-63.2
	5-33 Olive, New York	-42.3
	5-34 Peanut, mill	-60.6
	5-35 Soybean, N. Y.	-63.5
	5-36 Vinegar, cider New York	-34.2

DISPERSION OF WHOLESALE PRICE CHANGES
TEXTILE PRODUCTS
GROUP IV

CHART XXXVI



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the whole agricultural situation can be solved by monetary means. It is apparently necessary for a long time solution to find new uses for farm products, to decrease production of some products, or to find foreign markets for them.

Group (2) — Food Products: Food products, Chart XXXV, which are essentially farm products plus some railroad transportation (fixed charges) plus a small amount of labor and some rent (relatively fixed), have recovered much more than half their loss, which, of course, on account of the fixed element, was not so great as that of farm products.

Group (3) — Leather Products: Leather goods, which include skins and hides from the farm group plus leathers (hides plus transportation plus some labor), shoes and leather goods (leathers plus additional labor, some rent and some power), have recovered to a point only slightly below the 1929 level.

Group (4) — Textile Products: The textile group, Chart XXXVI, has shown about the same recovery as the leather goods group. In textiles there is again the situation in which the products are composed of farm products with wages, transportation and

TEXTILE PRODUCTS

SUB-GROUP AND COMMODITY NO.:	COMMODITY	HORIZONTAL LOCATION ON CHART (% Decline June '29-Feb. '33):
1	**CLOTHING**	
1-1	Collars, soft	—22.8
1-2	Collars, stiff	+16.8
1-3	Men's cotton hdkfs.	—42.8
1-4	Women's cotton hdkfs.	—35.7
1-5	Men's linen hdkfs.	—55.7
1-6	Women's linen hdkfs.	—54.9
1-7	Men's finished hats	—45.5
1-8	Men's unfinished hats	—33.3
1-9	Overalls	—44.2
1-10	Overcoats	— 6.7
1-11	Men's dress shirts	— 6.7
1-12	Men's work shirts	—19.1
1-13	Boy's Suits	—30.0
1-14	Men's 3 piece suits	—23.0
1-15	Men's 4 piece suits	—32.0
1-16	Youth's suits	—37.3
1-17	Topcoats	—23.0
1-18	Boys' trousers	—58.9
1-19	Men's dress trousers	—38.0
1-20	Men's work trousers	—34.9
2	**COTTON GOODS**	
2-1	Broadcloth	—52.0
2-2	Damask	—20.9
2-3	Denims	—54.5
2-4	Drillings; 2.50 yds/lb.	—62.7
2-5	Drillings, 2.85 yds/lb.	—64.0
2-6	Duck, 8 oz.	—50.2
2-7	Duck, wide	—46.9
2-8	Flannel, Bleached	—46.2
2-9	Flannel, Unbleached	—52.5
2-10	Gingham	—40.0
2-11	Muslin, series 1	—46.4
2-12	Muslin, series 2	—41.1
2-13	Muslin, series 3	—59.2
2-14	Muslin, series 4	—32.1
2-15	Osnaburg	—57.8
2-16	Percale	—46.7
2-17	Print cloth, 27 inch	—54.8
2-18	Print Cloth, 38½ inch	—56.8
2-19	Sateen	—30.5
2-20	Sheeting, blchd, series 1	—56.4
2-21	Sheeting, blchd, series 2	—47.2
2-22	Sheeting, brown, series 1	—56.0
2-23	Sheeting, brown, series 2	—50.6
2-24	Sheeting, brown, series 3	—56.0
2-25	Shirting, madras	—47.1
2-26	Shirting, percale	—46.2
2-27	Ticking	—57.4
2-28	Tire Fabric, cord	—54.6
2-29	Tire Fabric, builders	—59.5
2-30	Toweling	—31.8
2-31	Yarn, northern, 10/1 cones	—59.5
2-32	Yarn, northern, 22/1 cones	—53.1
2-33	Yarn, southern	—45.9
2-34	Yarn, twisted, 20/2 weaving	—61.7
2-35	Yarn, twisted, 40/2 weaving	—52.6

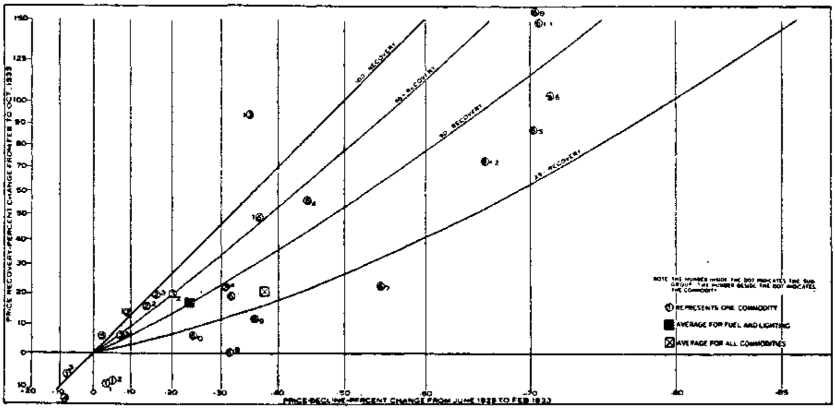
SUB-GROUP AND COMMODITY NO.:	COMMODITY	HORIZONTAL LOCATION ON CHART (% Decline June '29-Feb. '33):
3	**KNIT GOODS**	
3-1	Hosiery, men's cotton	—40.5
3-2	Hosiery, women's cotton	—44.6
3-3	Hosiery, women's rayon	—59.0
3-4	Hosiery, men's silk	—47.4
3-5	Hosiery, women's silk	—59.2
3-6	Underwear, men's cotton	—29.4
3-7	Underwear, women's cotton	—43.2
3-8	Underwear, men's woolen	— 8.9
3-9	Underwear, men's woolen	—19.6
4	**SILK AND RAYON**	
4-1	Rayon, 150 first	—52.4
4-2	Rayon, 150 second	—51.4
4-3	Rayon, 300 first	—56.3
4-4	Rayon, 300 second	—56.2
4-6	Silk, raw, Canton	—67.7
4-5	Silk, raw, steam filature	—75.6
4-7	Silk, raw, Japan	—75.2
4-8	Silk, raw, Japan, 13-15	—75.6
4-9	Yarn, spun, 62/1	—50.4
4-10	Yarn, spun, 60/2	—44.4
4-11	Yarn, imported	—42.7
4-12	Yarn, crepe, thrown	—71.9
4-13	Yarn, organize, thrown	—63.6
4-14	Yarn, tram, thrown	—63.2
5	**WOOLEN AND WORSTED GOODS**	
5-1	Broadcloth	—40.0
5-2	Crepe	—43.0
5-3	Flannel	—38.3
5-4	Suiting	—55.7
5-5	French Serge	—36.4
5-6	Sicilian cloth	—22.9
5-7	Flannel, 7 ounce	—38.8
5-8	Overcoating, Heavy	—35.1
5-9	Overcoating, Top	—35.7
5-10	Suiting, serge, 11 oz.	—35.0
5-11	Suiting, serge, 16 oz.	—41.8
5-12	Uniform serge, fine	—34.8
5-13	Uniform serge, medium	—47.2
5-14	Unfinished worsted	—41.5
5-15	Trousering	—42.5
5-16	Yard 2/32s	—44.9
5-17	Yard, 2/40s	—48.5
5-18	Yard, 2/50s	—47.7
6	**OTHER TEXTILE PRODUCTS**	
6-1	Burlap	—50.4
6-2	Hemp, manila	—67.1
6-3	Jute, raw	—63.4
6-4	Leather, heavy	—13.7
6-5	Leather, light	—20.0
6-6	Rope, cotton	—48.2
6-7	Rope, manila	—38.1
6-8	Rope, sisal	—32.1
6-9	Sisal, Mexican	—66.6
6-10	Thread, cotton	— 4.1
6-11	Thread, linen	— 9.8
6-12	Twine, binder	—42.2
6-13	Twine, cotton	—58.4
6-14	Twine, hard fiber	—58.1
6-15	Yarns, jute No. 1	—33.3
6-16	Yarns, jute No. 2	—30.0

DISPERSION OF WHOLESALE PRICE CHANGES

FUEL & LIGHTING

GROUP V

CHART XXXVII



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SUB-GROUP AND COMMODITY NO.:	COMMODITY	HORIZONTAL LOCATION ON CHART (% Decline June '29-Feb. '33):
1-	ANTHRACITE COAL	
1-1	Chestnut - - - - - - -	– 3.2
1-2	Egg - - - - - - - - -	– 5.1
1-3	Pea - - - - - - - - -	+ 8.0
2-	BITUMINOUS COAL	
2-1	Mine run - - - - - - -	– 9.0
2-2	Prepared sizes - - - -	–13.8
2-3	Screenings - - - - - -	–16.3
3-	COKE, NET PER TON	
3-1	Beehive - - - - - - -	–35.3
	Byproduct:	
3-2	Alabama - - - - - - -	–20.0
3-3	New Jersey - - - - - -	– 7.5
3-4	Ohio - - - - - - - - -	–30.7
4-	ELECTRICITY - - - - -	+ 9.0
5-	GAS - - - - - - - - - -	– 0.3

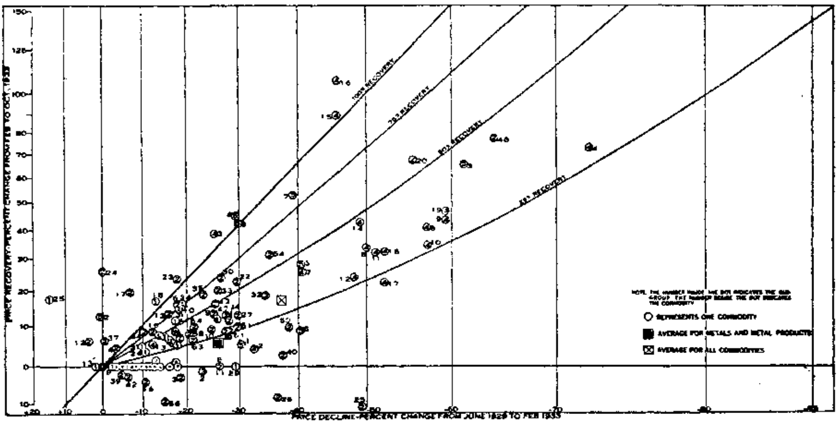
SUB-GROUP AND COMMODITY NO.:	COMMODITY	HORIZONTAL LOCATION ON CHART (% Decline June '29-Feb. '33):
6-	PETROLEUM PRODUCTS	
	FUEL OIL	
6-1	Oklahoma - - - - - - -	–37.0
6-2	Pennsylvania - - - - -	–31.7
	GASOLINE	
6-3	Natural, Oklahoma - -	–71.0
6-4	California - - - - - - -	–44.9
6-5	North Texas - - - - - -	–70.5
6-6	Oklahoma - - - - - - -	–72.2
6-7	Pennsylvania - - - - - -	–54.8
	KEROSENE	
6-8	Standard - - - - - - -	–31.2
6-9	Water White - - - - -	–36.2
	CRUDE PETROLEUM	
6-10	California - - - - - -	–24.2
6-11	Kansas-Oklahoma - - -	–70.7
6-12	Pennsylvania - - - - -	–66.1

DISPERSION OF WHOLESALE PRICE CHANGES

BUILDING MATERIALS

GROUP VII

CHART XXXVIII



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power added. Wages have been materially increased. Here recovery would be complete — in fact, many prices are above the 1929 prices — except for particular products like silk, sisal, hemp and jute, which are imported, and rayon. There is, of course, no reason why rayon should return to the 1929 price since there have been numerous improvements in the technique of production which have reduced the cost of production.

Group (5) Fuel and Lighting: In the case of fuel and lighting, Chart XXXVII, the price is depressed below the 1929 price chiefly because of depressed prices of petroleum products and anthracite coal products. Bituminous coal prices have been increased by National Recovery Administration code provisions. There is an overproduction of petroleum and possibly an encroachment of petroleum and gas upon the coal market. It might be pointed out, however, that since oil is no longer a scarce commodity, there is no logical reason why the price of crude oil should rise much above the average cost of extracting it from the ground.

Group (6) — Metals and Metal Products: Metals and metal products, Chart XXXVIII, have shown only slight recovery. This,

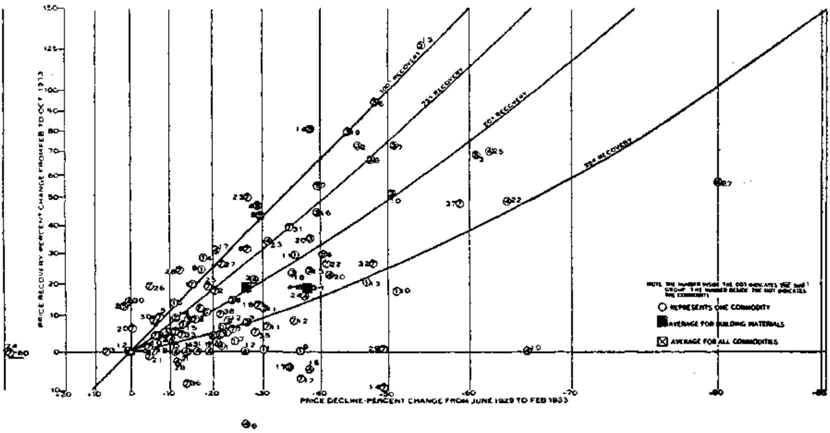
	Sub-group and Commodity No.:	Commodity	Horizontal Location on Chart (% Decline June '29-Feb. '33):
	1.	AGRICULTURAL IMPLEMENTS	
C	1-1	Grain binder	—6.0
A	1-2	Cultivator	—4.6
*	1-3	Grain drill	0.0
O	1-4	Engine, 3 h. p.	—6.2
	1-5	Hay forks	—17.4
E	1-6	Disc harrow	—8.8
E	1-7	Peg teeth harrow	—9.0
E	1-8	Spring teeth harrow	—8.4
	1-9	Harvester	—2.8
	1-10	Hoes	—19.4
C	1-11	Hay loader	—6.6
B	1-12	Hay mower	—5.2
	1-13	Corn picker	2.2
*	1-14	Corn planter	0.0
*	1-15	Tractor plow	0.0
	1-16	1 horse plow	—11.8
N	1-17	2 horse plow	—17.4
	1-18	Pumps	—14.3
S	1-19	Hand rakes	—14.1
*	1-20	Self Dump rakes	0.0
E	1-21	Side delivery rakes	—8.0
F	1-22	Cream separator	—9.7
*	1-23	Corn sheller	0.0
	1-24	Shovels	—10.8
	1-25	Spades	15.0
H	1-26	Manure spreader	—12.5
C	1-27	Grain thresher	—6.2
G	1-28	10/20 tractor	—11.4
	1-29	15/30 tractor	—29.3
G	1-30	2 horse wagon	—11.4
	1-31	Windmill	—10.5
	2.	IRON AND STEEL	
D	2-1	Angle bars	—7.6
*	2-2	Augers	0.0
	2-3	Axes	—19.1
	2-4	Bar Iron, best	—4.0
*	2-5	Bar iron, common	0.0
	2-6	Reinforcing bars	—18.0
W	2-7	Merchant, steel, bars	—17.9
	2-8	Sheet bars	—26.8
	2-9	Steel bars	—26.1
	2-10	Barrels	—25.3
	2-11	Billets	—26.2
	2-12	Boiler tubes	3.0
	2-13	Machine bolts	—16.7
	2-14	Plow bolts	—28.6
U	2-15	Stove bolts	—16.1
	2-16	Track bolts	—10.3
	2-17	Butts	—8.0
L	2-18	Sanitary cans	—14.0
R	2-19	Castings	—26.7
*	2-20	Chisels	0.0
*	2-21	Files	0.0
	2-22	Hammers	—29.8
	2-23	Hatchets	—18.3
	2-24	Knives	—0.3
	2-25	Knobs	—50.0
	2-26	Locks	—37.5
	2-27	Nails	—30.0
*	2-28	Bessemer ore	0.0
*	2-29	Non-Bessemer ore	0.0
	2-30	Basic pig iron	—27.0
	2-31	Bessemer pig iron	—18.7
	2-32	Ferromang pig iron	—35.2
	2-33	Southern foundry pig iron	—26.7
	2-34	Northern foundry pig iron	—19.1
	2-35	Malleable pig iron	—23.7
R	2-36	Spiegeleisen pig iron	—26.1
		IRON AND STEEL (Cont'd)	
	2-37	Cast iron pipe	—0.7
C	2-38	Black Steel pipe	—6.6
	2-39	Galvanized pipe	—5.2
	2-40	Jack planes	—37.9
V	2-41	Steel tank plates	—17.9
	2-42	Steel rails	—7.6
	2-43	Large rivets	—27.4
	2-44	Small rivets	—10.0
M	2-45	Rods	—16.6
T	2-46	Crosscut saws	—15.5
	2-47	Hand saws	—27.1
	2-48	Scrap steel	—64.9
	2-49	Annealed steel sheets	—28.5
	2-50	Auto body steel	—39.0
	2-51	Galvanized steel	—29.7
K	2-52	Grooved skelp	—13.6
L	2-53	Spikes	—14.3
	2-54	Strips	—35.9
X	2-55	Structural steel	—18.0
	2-56	Terneplate	—15.2
Z	2-57	Tie plates	—18.6
	2-58	Tin plate	—20.6
*	2-59	Vises	0.0
M	2-60	Annealed wire fence	—16.7
	2-61	Barbed wire fence	—29.0
P	2-62	Galvanized fence	—17.4
	2-63	Woven wire fence	—21.2
	2-64	Wood screws	—22.4
	3.	MOTOR VEHICLES	
J	3-1	Passenger cars	—14.2
	3-2	Trucks	—23.5
	4.	NONFERROUS METALS	
A	4-1	Aluminum	—4.3
	4-2	Antimony	—32.8
	4-3	Babbitt metal	—24.3
	4-4	Copper, ingot	—73.3
	4-5	Lead, pig	—57.1
	4-6	Nickel	0.0
	4-7	Lead pipe	—41.7
	4-8	Quicksilver	—50.9
	4-9	Yellow Brass rods	—60.0
	4-10	Drawn copper rods	—57.6
	4-11	Yellow brass sheets	—52.7
	4-12	Copper sheets	—49.0
	4-13	Zinc sheets	—12.5
	4-14	Silver	—49.8
	4-15	Solder	—47.2
	4-16	Tin	—46.8
	4-17	Tubes, brass	—53.1
	4-18	Brass wire	—53.7
	4-19	Copper wire	—59.9
	4-20	Zinc, pig	—56.6
	5.	PLUMBING AND HEATING	
	5-1	Heating boilers	—30.3
	5-2	Range boilers	0.6
	5-3	Water closets	—61.4
	5-4	Lavatories	—29.7
	5-5	Radiation	—40.9
	5-6	Sinks	—40.7
	5-7	Bath tubs	—40.1
	5-8	Laundry tubs	—30.7

* The asterisk indicates no change in price since June 1929.

Letters are used in some cases instead of numbers to prevent confusion.

DISPERSION OF WHOLESALE PRICE CHANGES
METALS & METAL PRODUCTS
GROUP VI

CHART XXXIX



N.R.A. — Division of Research and Planning

of course, is to be expected since prices of these products depend essentially upon demand for them in new construction, new machinery, expansions in power plant facilities, etc. Raising prices will, of course, not raise demand. Profits in consumer goods industries will have to come first. These activities will only reach the 1929 level after there are profits in consumer goods industries and after considerable new financing (bond issues, stock issues, etc.) has taken place.

Group (7) — Building Materials: Prices of building materials, Chart XXXIX, have not quite recovered to the 1929 level. It is maintained by many that it is unfortunate that they have already shown so much recovery since recovery in the price of building materials materially lowers the profit incentive to build. See Chapter VI, "Factors Influencing Residential Building." The price of lumber has recovered because of price-fixing provisions in the Lumber and Timber Products code. Demand has fallen off. Little building is at present done for purposes other than public works.

Group (8) — Chemicals and Drugs: There has been little improvement in the price of chemicals and drugs. This group as a

	SUB-GROUP AND COMMODITY NO.:	COMMODITY	HORIZONTAL LOCATION ON CHART (% Decline June '29-Feb. '33):
	1	BRICK AND TILE	
	1-1	Concrete blocks	—30.3
D	1-2	Common brick	—22.3
	1-4	Fire Clay brick	—18.6
	1-5	Front brick	—12.6
E	1-6	Paving brick	—22.4
	1-7	Sand lime brick	—25.0
	1-8	Silica brick	—18.6
	1-9	Drain tile	—36.4
	1-10	Floor tile	—51.4
	1-11	Hollow tile	—36.4
	1-12	Roofing tile	.5
	1-13	Wall tile	—47.5
	2	CEMENT	
	2-1	Buffington	—12.9
	3	LUMBER	
	3-1	Douglas fir lath	—38.1
	3-2	Pine lath	—46.7
	3-3	Red cedar	—27.7
	3-4	Chestnut	—14.9
G	3-5	Cypress	—18.8
	3-6	Douglas fir no. 1	—49.3
	3-7	Douglas fir no. 2	—51.8
	3-8	Gum	—48.7
	3-9	Hemlock	—25.3
	3-10	Maple	—50.9
	3-11	Oak	—31.1
	3-12	White pine no. 2	—23.6
	3-13	Pine flooring	—55.2
	3-14	Pine timbers	—39.1
	3-15	Ponderosa pine	—38.6
	3-16	Poplar no. 1	—40.0
	3-17	Redwood	—34.7
	3-18	Spruce	—35.5
	3-19	Cedar shingles	—45.4
	3-20	Cypress shingles	—39.1
	4	PAINT, PAINT MATERIALS	
	4-1	Enamel paint	—22.0
	4-2	Inside paint	—10.6
	4-3	Outside paint	—29.2
A	4-4	Porch paint	—20.7
G	4-5	Roof paint	—18.5
B	4-6	Varnish	—20.2
*	4-7	Barytes	0.0
	4-8	Butyl acetate	—26.6
*	4-9	Black color, bone	0.0
	4-10	Black color, carbon	—66.0
*	4-11	Black, iron oxide	0.0
	4-12	Black color, lamp	—27.3
*	4-13	Prussian blue	0.0
	4-14	Chrome green	—14.8
	4-15	Chrome yellow	—11.8
	4-16	Ethyl acetate	—38.7
	4-17	Gum	—21.4
	4-18	Red lead	—38.1
	4-19	White lead	—19.9
	4-20	Litharge	—41.1

	SUB-GROUP AND COMMODITY NO.:	COMMODITY	HORIZONTAL LOCATION ON CHART (% Decline June '29-Feb. '33):
		PAINT, PAINT MATERIALS (Cont'd)	
	4-21	Lithophone	—14.3
	4-22	Chinawood oil	—64.3
	4-23	Linseed oil	—31.4
	4-24	Putty	—38.2
	4-25	Rosin, B.	—63.2
	4-27	Shellac	—80.2
	4-28	Turpentine, N. Y.	—13.9
	4-30	Whiting	— 0.0
	4-31	Zinc oxide	—17.1
	5	SEE NOTE	
	7	OTHER BUILDING MATERIALS	
	7-1	Asphalt	—16.7
	7-2	Bars, reinforcing	—22.0
	7-3	Plaster board	8.0
	7-4	Wall board	—16.7
	7-5	Butts	— 8.0
	7-6	Doors	—28.1
	7-7	Door frames	—24.0
C	7-8	Window frames	—21.5
	7-9	Plate glass	— 5.2
	7-10	Plate glass	— 5.9
	7-11	Window glass A	—31.5
	7-12	Window glass B	—36.0
	7-13	Gravel	—11.5
	7-14	Door knobs	—50.0
	7-15	Common lime	—14.0
F	7-16	Hydrated lime	—17.9
	7-17	Locks	—37.5
	7-18	Nails	—30.0
	7-19	Black steel pipe	— 6.6
	7-20	Cast iron pipe	— 0.7
	7-21	Galvanized pipe	— 5.2
	7-22	Lead pipe	—41.7
	7-23	Sewer pipe	—27.6
	7-24	Plaster	80.
	7-25	Shingle roofing	—19.8
	7-26	Medium roofing	— 5.9
	7-27	Slate-surfaced	—22.7
	7-28	Strip shingles	—13.6
	7-29	Slate roofing	—50.0
	7-30	Building sand	— 6.3
	7-31	Window sash	—35.0
	7-32	Copper sheet	—49.0
	7-33	Zinc sheet	—12.5
*	7-34	Crushed stone	— 0.0
	7-35	Pine tar	—29.0
	7-36	Terneplate	—15.2
	7-37	Copper wire	—59.9
	7-38	Wood screws	—22.4

* The asterisk indicates no change in price since June 1929.

Letters are used in some cases instead of numbers to prevent confusion.

NOTE: 5. PLUMBING AND HEATING — See Sub-Group No. 5. METALS AND METAL PRODUCTS.

whole, however, is unimportant as far as the general price level is concerned, especially from the point of view of the index of wholesale prices of the Bureau of Labor Statistics, since it is given a weight of less than two per cent there. Furthermore, patents, advertising, etc., make up the principal parts of price when demand is high. The depressed price of fertilizers is apparently a result of the crop reduction program and low farm income.

Group (9) — House-Furnishing Goods: House furnishings, especially furniture, have shown little price recovery. These can be expected to advance in price only by having doubled-up families unscramble themselves, thus requiring new housing facilities and furnishings, or by general adding of rooms to existing buildings. At present the marriage rate is only a small per cent of normal — that is, demand for new housing and house furnishings resulting from new marriages is only a fraction of what it would normally be. The chief aggravating cause is, of course, the relatively large unemployment existing among young workers. A second cause, which is not unimportant, is the general feeling that married women should be dismissed whenever their husbands are employed. Obviously, a man who has an income of $75 a month cannot marry a woman with approximately the same income when both feel that she will be required to give up her position. The result is that a marriage is postponed.

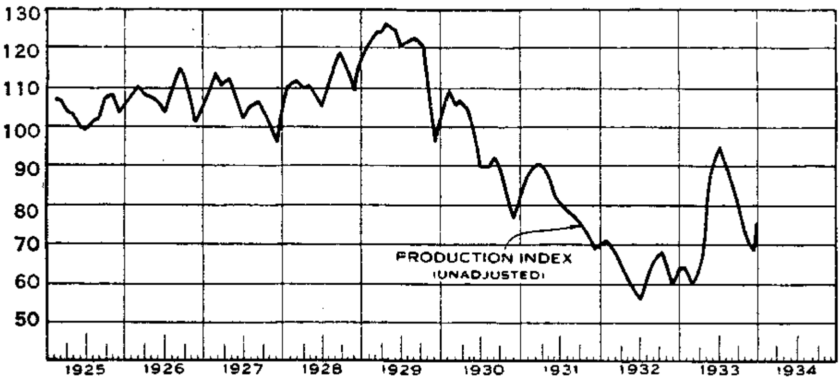
Group (10) — Miscellaneous: The miscellaneous group in the Bureau of Labor Statistics index, although it has shown little recovery and is weighted 9.23 per cent, hardly requires much analysis. The recovery of automobile tires and tubes will be effected only when there is greatly increased demand for them, as at present there is a surplus of plant equipment to supply present needs. Cattle feed is faced with competition due to a surplus of other farm products.

In general, it seems to be true that those prices which decline most recover most; that is, flexible prices are flexible on the way up as well as on the way down.

3. Inventories. It has been pointed out that entrepreneurs produce on the basis of present price and then sell their production for whatever they can get, or else they produce on the basis of orders received at a more or less fixed price. In any event there is usually a wholesale buyer or retail buyer or both between the pro-

INDICES OF INDUSTRIAL PRODUCTION

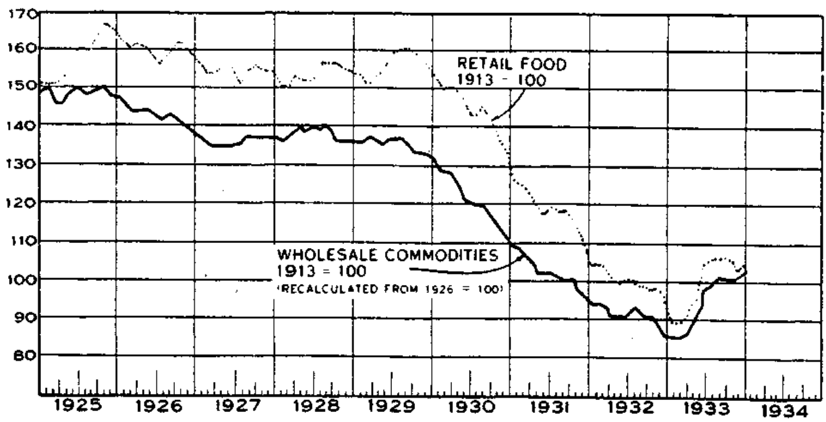
Chart XL



Source: Federal Reserve Bulletin.

INDICES OF WHOLESALE COMMODITY PRICES AND RETAIL FOOD PRICES

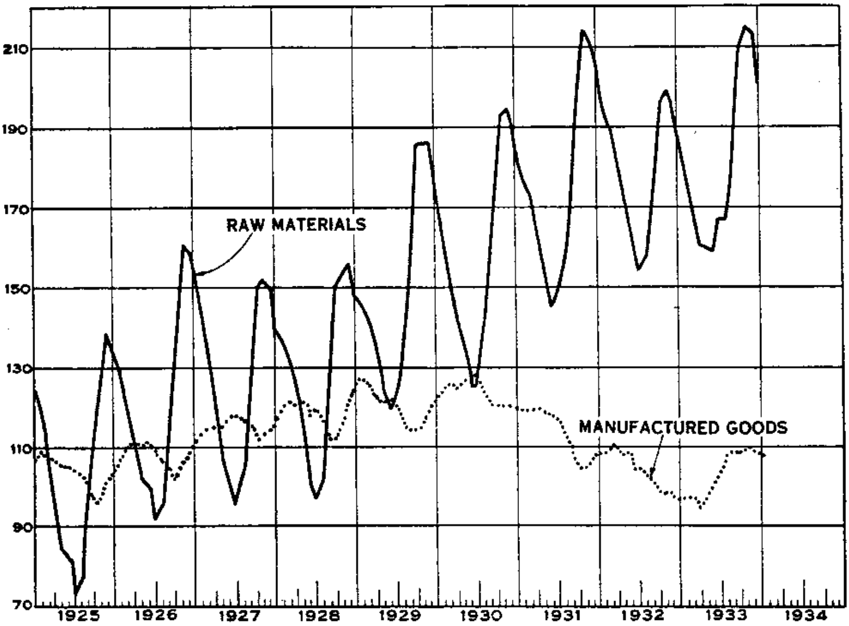
Chart XLI



Source: Bureau of Labor Statistics, Wholesale Prices and Retail Prices.

INDICES OF INVENTORIES

CHART XLII



Source: Bureau of Foreign and Domestic Commerce, Survey of Current Business.

ducer and the ultimate consumer. Factories produce on the basis of orders received from the middle groups. In times of economic stability — steady prices, steady demand, etc. — the orders placed by these middlemen usually bear definite relations to the amounts sold to the ultimate consumer.

In times of great economic shock or rising or falling price levels, this balance between factory orders and consumer demand may be greatly disturbed. Charts XL, XLI and XLII show indices of industrial production, indices of wholesale commodity prices and retail food prices and indices of inventories for the United States.

The October, 1929, crash of the securities market apparently had very little, if any, immediate effect on inventories, although seasonal decline in inventories of manufactured goods may have been slightly accentuated. However, industrial production suffered a drastic reduction, the index dropping from 137 in August to 110 in December, whereas the seasonal slump averages about 4 points. A decline in November and December, 1929 prices apparently stimulated production in the early months of 1930 and inventories again increased, but the increase was only short lived. In February inventories and production both started downward and by June, 1930, production reached its lowest point in many years. The June, 1930, crash of the stock market followed low prices and low production. Following this crash inventories were almost steadily diminished from an index value of 128 in April, 1930, to 102 in September, 1931. A slight rally occurred here and carried the index to 110 in April, 1932, from which point it declined almost steadily to 93 in the middle of May, 1933. On May 19, 1933, the National Industrial Recovery Act was introduced in Congress and almost simultaneously the index of inventories of manufactured goods started upward and continued upward through December, 1933, when the index stood at 110.

As pointed out, in the last few weeks of 1929 the production index started upward and continued to have an upward trend until the early spring of 1930, when it began to drop. The index dropped sharply with only slight interruption until it reached 87 in December, 1930. It then climbed sharply to 100 in April, 1931. Following this slight recovery the index dropped sharply again to a low of 55 in July, 1932. The index then started upward (the Federal Reserve Board began to purchase United States Government bonds in the open market in sizable amounts in May and June) and touched 66 in October, when open market purchases of bonds were practically stopped. From October, 1932, to March, 1933, the production index declined from 66 to 58. In March, 1933, the index started sharply upward and by August had reached 97. It then declined sharply, reaching 68 in December, 1933.

In general, current demand, $y(t)$, is supplied out of inventories and out of current production. Let τ be the greatest time elapsing between the manufacture of a lot of goods and the sale of the entire lot to the consumer. Thus, if 1,000,000 are manufactured today and the last of the lot passes to consumers six months from today, τ is equal to six months.

Now, in general τ varies with business conditions and is, therefore, a function of time in the sense already explained. For simplicity, however, suppose that τ is a constant. It is reasonable to assume that the frequency distribution of amounts of goods, manufactured at times between $t - \tau$ and t, making up the inventory of goods at any time t is such that the inventory v at the time t may be taken to be

$$v = \int_{t-\tau}^{t} (e^{\eta(x-t+\tau)} - e^{\mu(x-t+\tau)}) u(x) dx$$

where η and μ are negative constants. Then, if γ represents the percentage of the current production $u(t)$ passing to the consumer and τ represents the percentage of inventory passing to the consumer, it follows that

$$(3.1) \qquad y(t) = \gamma\, u(t) + \lambda \int_{t-\tau}^{t} \left[e^{\eta(x-t+\tau)} - e^{\mu(x-t+\tau)} \right] u(x) dx \,.$$

A transformation can be used to simplify this equation somewhat. In fact, if the substitution $s = t - x$ is made, it follows that $ds = -dx$; $s = o$ when $x = t$, and $s = \tau$ when $x = t - \tau$, so that

$$y(t) = \gamma\, u(t) + \lambda \int_{0}^{\tau} \left[e^{\eta(\tau-s)} - e^{\mu(\tau-s)} \right] u(t-s)\, ds \,.$$

The rate of consumption $y(t)$ is equal to the rate of production $u(t)$, whenever

$$u(t) = \lambda_1 \int_{0}^{\tau} \left[e^{\eta(\tau-s)} - e^{\mu(\tau-s)} \right] u(t-s) ds$$

where $\lambda_1 = \lambda/(1 - \lambda)$.

This is a new type of Fredholm integral equation. Furthermore, as far as the author knows, no existing mathematical theory applies to it. By analogy it may be guessed that the equation has solutions only for particular values of λ. If the equation (3.1) for $u = y$ is differentiated twice and is combined with the derivative equations it follows readily that

(3.2)

$$\frac{d^2u}{dt^2} - \left[[\eta + \mu + \lambda_1 (e^{\eta\tau} - e^{\mu\tau})] \right] \frac{du}{dt} + \eta\, \mu\, u = \lambda_1 (\eta - \mu) u(t - \tau)$$

If, as intimated, this equation has solutions only for particular values of λ_1, it follows that there can be a balance between rate of consumption and rate of production only when particular relations exist between the percentage of new goods going into stocks and the percentage of old goods being sold . In this case the equilibrium rate of production $\bar{u}(t)$ would be determinable. The nature of the equation (3.2) is such as to suggest that the solution $\bar{u}(t)$ is oscillatory. If this is true, then for almost any type of demand equation that can be admitted, the price $p(t)$ would also be oscillatory.*

A statistical example of an important kind of oscillatory production rate and price was presented in the study of factors influencing residential building. In a capital goods economy there are many prices and production rates which are fundamentally oscillatory. There are always new industries being built which spell the ultimate doom of old industries or their conversion into other allied industries.

As long as there are several of these fundamentally important production rates which are oscillatory, the price structure will be subject to strains and stresses. Under these circumstances it is extremely doubtful that equilibrium will ever be attained. At least static equilibrium in the sense of Walras, Pareto, and other static equilibrium economists, will probably never be attained.

4. Price-Fixing. There are many kinds of price-fixing and there are various techniques used for fixing prices. Sometimes prices are fixed outright as in the case of interest on indebtedness, railroad rates, license fees, chewing gum, etc. Systems of "price basing" are common. Here the plan is essentially that of establishing several base points at which prices in an industry are fixed. Prices at points other than the basing point are equal to the base point price plus freight from the base point. This plan is essentially that which appears to have been practiced in the cement industry. Under such a system industry locates itself in such a way as to profit by freight rates. The Pittsburgh Plus† Plan of the steel companies furnishes an excellent example of price basing with a single base point. Under this plan a mill in Alabama selling

*Edward Theiss has presented a paper in which he shows that the demand is oscillatory for the special case for which (3.1) takes the form $y(t) = \lambda u(t-s)$. The treatment given here antedates that of Thiess by about three years. The reason for not publishing is obvious. In fact, the author apologizes for offering an incomplete solution at this time. He does this in the hope that someone will work out the mathematical theory. Theiss, *Journal of Political Economy*, 1933.

†Federal Trade Commission, *Docket 760,* 1924.

in Alabama charged the Pittsburgh price plus freight from Pittsburgh to Alabama. Again prices of goods of a stabilized industry sold on a cost plus basis are essentially fixed.

In Germany the cartels tried various systems of price-fixing. As a rule the big cartels refrained from advancing prices to the possible limit in times of cyclical upswing and they resisted excessive cyclical price breaks with considerable success. The rigidity of cartel prices in Germany in times of business depression following the war acted to intensify the pressure on markets and to reduce the volume of sales. Prior to the war, however, business volumes in cartelized German industries did not — or to no considerable extent — exceed corresponding rates of decline in noncartelized British industries. This was apparently due to at least three factors: (1) the industrial growth of Germany occurred later than the industrial growth of Great Britian so that the pre-war period was essentially a period of expansion; (2) apart from a few cartels Germany's price and wage structure was much more flexible in the pre-war period than in the post-war period, so that the elastic part of the price structure could yield more readily to the pressure of the rigid block of cartel prices; and (3) goods not sold in the German domestic market at the high fixed prices could be dumped abroad for whatever they would bring.*

A study of all methods of price-fixing would itself require a thesis. Such a study would require examination of fair trade practices, limitation of investments either directly or through bank control, inventory control, price-fixing by means of patent rights, prohibition of sales below cost, open price associations dominated by monopolistic groups, etc. Each of the above methods can be used to fix prices. Some of the methods may give only price stabilization, but price stabilization easily becomes price fixation. Obviously methods of price-fixing and techniques by which they are made to work cannot be discussed in this work.

In order for a fixed money price to be a fixed real price, many quantities such as employment, credit, velocity of circulation of money and credit, etc., must remain fixed. A change in employment usually produces (it may be masked by an opposite force) some change in the velocity of circulation of money and very often in the ratio of credit money to fiduciary money. Similarly, changes

*The theoretical problem is similar to the utility problem of Section 6 of the chapter on joint demand and loss leaders.

in bank credits, bonds, etc., may be accompanied by changes in employment.

As pointed out already, the law of demand for either consumer or capital goods can be taken in the form

$$y(t) = AP(t)/I^{\beta} + b + \int_0^{t_0} \left[K(t, s)P(t-s)/I^{\beta}(t-s) \right] ds$$

where P represents price, I is consumer income, β is a constant defining what might be called elasticity of consumer income relative to the good whose demand is $y(t)$, t stands for time in the sense already explained and s is a parameter also representing time.

The exponent β may be expected to be different for various goods and wages. Thus a value $\beta = 2$ might apply for automobiles and a value $\beta = 1$ for bread. These values are, of course, only hypothetical. A statistical determination has never been made.

Suppose now that a price difference of 10 per cent makes no appreciable difference in demand decisions. Then since $p(t-s)$ does not change a 10 per cent increase will bring only $(A/10)P/I^{\beta}$ decrease in consumption in the next unit of time. For the next interval of time, however, the integral term will have decreased because of the price increase and the diminution in demand will be contributed by both the increase in price and its increase over the preceding interval of time. In other words, increased prices, for a fixed consumer income, are not reflected in full immediately, but after a time they increase the pressure on sales unless national income increases enough to compensate.

To see what would happen if all prices were fixed, it is necessary only to consider two products or wages since the relations may be easily generalized to n products or wages.

For two *consumer goods or wages,* the demands are given by

(4.1)

$$y_1 = a_1P_1/I^{\beta_1} + b_1 + \int_0^{t_0} [K_1(x, s)P_1(t-s)/I^{\beta_1}(t-s)]ds$$

(4.2)

$$y_2 = a_2P_2/I^{\beta_2} + b_2 + \int_0^{t_0} [K_2(x, s)P_2(t-s)/I^{\beta_2}(t-s)]ds$$

where β_1 and β_2 are different.

The income I for these two will be given by

$$I(t) = \int_t^{t+\Delta t} [P_1(t)y_1(t) + P_2(t)y_2(t)]dt$$

Let M be the average amount of fiduciary money and gold backed currency if on a gold standard, in circulation in the time t to $t + \Delta t$ over which $\int y_i(t)P_i(t)dt$

is taken, and let V be the average velocity of circulation of this money, i.e., number of times it changes hands in payment for the goods or services.

Let M' be the average amount of bank deposits subject to check in the time t to $t + \Delta t$ and other money than fiduciary money and let V' be its average velocity of circulation.

Then,

$$I = MV + M'V' = \int_t^{t+\Delta t} (P_1y_1 + P_2y_2)dt$$

or more generally if there are n transactions (consumer goods and wages but not capital goods)

$$I = MV + M'V' = \sum_{i=1}^{n} \int_t^{t+\Delta t} P_i(t)y_i(t)dt \ . \tag{4.3}$$

In reality this equation applies only to an economy in which consumers' goods are produced and existing capital assets are fixed in value. A more general "equation of exchange" applicable to a realistic capital goods economy will be considered in the next chapter.

Now suppose that all prices are increased, that is, that P_i becomes $P_i + \Delta P_i$. Then, if I remains constant,

$$\sum_{i=1}^{n} P_i(t)y_i(t)$$

must decrease. Furthermore, each y_i must decrease as is shown by (4.1) and (4.2).

It should be noted, however, that the full effect of the decrease in y_i is not immediately evident, since the immediate effects are contributed by only the first term. After a time the effects of the integral term become important and in order to keep I constant prices must be increased still further. Such adjustment can obviously take place only as long as the public will stand for decreased production and the accompanying decrease in standards of living.

Some demand seems to be stimulated by rising prices. Thus, there should be added to some of the demand equations a term in the derivative of price with respect to time, i.e., $h\ dp/dt$, where h is positive. When this element is taken into account it is easily explainable how an expected increase in price can stimulate demand. Small increases regularly administered seem to have this effect on a great many goods and wages. It should be pointed out, however, that this situation does not permit I to remain constant. Thus, if each P_i increased and a net increase occurred in the y_i (some increasing, others decreasing) there would have to be an increase in I. This increase would occur because of increases in both V and V' unless there were monetary expansions that did not introduce new factors. Changes in V and V', of course, mean changes in the spending habits of the people.

Apparently there are limits to M', V and V', so that the amount that I can be increased is strictly limited. Simply fixing all prices above present values will only result in disaster since there can be no "speculative" buying and thus all y_i must decrease unless I is expanded. Fixed prices will also fix V and V' at their present levels and even reduce V' if the fixed prices are too high. In a capital goods economy it is possible that fixed prices might result in increasing M' (bank deposits) since collateral value of inventories would be higher, but this probably would not compensate for the other important variables. In an economy in which only consumer goods are produced, it is possible to maintain higher fixed prices and the same production only by direct monetary inflation.*

As will be noted from equations (4.1) and (4.2) increased *"purchasing power" does not flow evenly to all products and wages,* so that shortly after prices are fixed, some more fixing will be required. Since it will be difficult to lower any prices, the prices of distress products and wages will have to be raised again, and this can be accomplished only by more inflation, which will again lead

*Whether or not there can be a controlled inflation need not be considered here. That there can be inflations following devaluation hardly needs to be maintained.

to unbalance. Thus one is forced to the inevitable conclusion that fixation of all prices can only lead to

1. Decreased consumption.
2. Inflation.

5. Quality Standards. One of the chief difficulties in the way of policing price-fixing is that the question of disposing of inferior goods and labor, or goods and labor not quite acceptable, introduces endless complications.

Suppose that a store purchases two hundred units of clothes, after using utmost care to select styles and colors that will appeal to its clientele. It is practically certain that at least a dozen of the suits will be carried over the usual sale period simply because they do not appeal to anyone at the fixed price of the lot. In order to dispose of these dozen units it is necessary to reduce them in price, perhaps twenty-five per cent at first. Even at this reduced price a small number will not sell, and further reductions will be necessary. It may be argued that provisions can be made for disposing of inferior goods, but who is to determine when the goods can be sold at a discount? Furthermore, what is to be done in the case of a merchant who is forced to raise cash in order to save his business? Must he be forbidden to take a twenty-five per cent loss on $50,000 worth of merchandise, i.e., a loss of $12,500, in order to save an investment of perhaps $100,000?

To insure success of any general scheme of price-fixing, not stabilization, complete (absolute) standardization of all products is necessary. It thus becomes necessary to set tolerances that the public as a group will accept.* It would be necessary to set up a rigid scheme of inspection and destroy or price lower every article that did not fall within the tolerance ranges for *all* characteristics. If off-tolerance (not necessarily inferior since off color would mean off standard) goods were permitted to be priced at lower than the fixed price there would again be a scale of prices. Some of the public would prefer to buy standard goods at the standard price, while some would prefer each of the off-standard goods for some price concession.

Some companies would undoubtedly find that they could profit most by making goods below standard and there would soon be a return to the present condition of a range of prices for goods not

*It must be remembered that producing a product that a small group desires is not the same as producing a product that can be sold commercially.

easily standardized. In other words, without rigid inspection through all phases of production and distributions (goods may be damaged even while they are being delivered, accidentally or purposely) and destruction or complete repair of all goods whose characteristics fall outside the tolerance limits, there must be scales of prices and some goods must be sold at less than the cost of production of standard goods *no matter what the standards are.*

In this connection it is important to point out that goods may be purchased in two different stores at different prices and the difference may be justified. Thus, one store may be liberal in making adjustments, may have a reputation for fair dealing, may be in a fashionable part of town so that there is an element of pride involved, etc., whereas another store may not possess these attributes. In a sense, therefore, the goods are not the same since in one case certain services are added. The above is meant to apply to branded goods made by the same manufacturer and of identical quality. The question of advertised and unadvertised goods is very much related to this one, since the advertising may be regarded as a service which builds up confidence in the product.

CHAPTER XIII

EXCHANGE IN A CAPITALISTIC ECONOMY

1. Quantity Theory of Money. In the days before banks assumed their important position in the economy and great dilution of money by credit expansion was impossible, quantity of money in circulation and its velocity of circulation were essentially the sole determinants of prices. Volumes of production in general depended upon technology, weather, etc. In these economies the price of a good was the ratio of the quantity of the good which would be exchanged for a given quantity of gold (or silver, wampum etc. in some economies) to that quantity of gold. In such economies prices and prosperity were vitally dependent upon the quantity of money in circulation since the rate of spending, which was largely fixed by habit, did not greatly vary. The Chinese 500 years before Christ knew the quantity theory of money. In fact, the story of a Chinese ruler who proposed to bring about prosperity by changing the value of his money is well known.*

Theories of money and purchasing power generally assume that the average quantity of money M in circulation during an interval of time t to $t + T$ multiplied by its velocity of circulation V in this time (number of times this money changes hands or circulates in transactions involving goods) plus the average quantity of bank credit M_c multiplied by its velocity of circulation V_c (transactions involving goods) is proportional to an average price of goods

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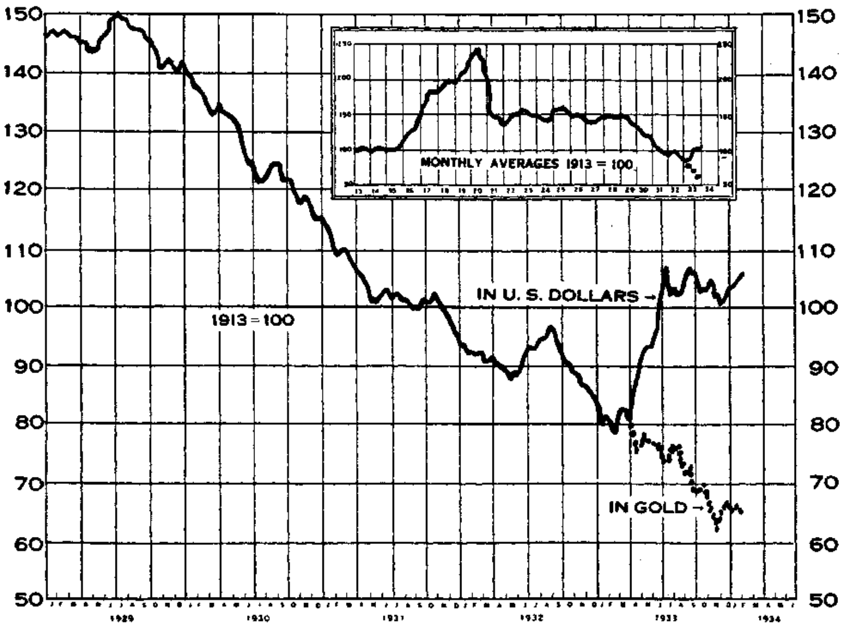
*In the thirteenth century the scholar-statesman, Ma Tuan-lin, compiled a large encyclopedia of general information based upon original Chinese sources. The particular passage given below was written twenty-five hundred years ago during the Chou Dynasty; it was translated in the Gest Chinese Research Library at McGill University:

> "In the year 524 B. C., Emperor Ching of the Chou Dynasty thought that the coins in circulation were much too light, and issued an order to mint heavier coins and banish the light ones entirely from circulation . . . .
>
> "Regarding this drastic change, Shan Mu-kung, a minister of the government, comments as follows: From time immemorial the monetary system of the country has always been so regulated as to relieve the people in time of distress (i.e., economic depression). When the inconvenience of the people is not great, more heavy coins are put into circulation for the reason that when the coins are light prices of commodities go up. In this case the heavy coins are standard value, while the light ones are auxiliaries — but they are both in circulation.

THE ANNALIST WEEKLY INDEX OF WHOLESALE COMMODITY PRICES (1913 = 100)

(Unadjusted for Seasonal Variation)

CHART XLIII



Changing the gold content of gold currencies does not immediately affect all prices. International commodities are first affected. Changes in prices of these commodities affect other prices. See also Charts XXXIII to XXXIX.

On the other hand, if the inconvenience of the people is great they cannot bear the burden of heavy money. The light money automatically becomes the standard value and the heavy auxiliary.

"Now, the Emperor has ordered the abolition of light coins completely. The immediate result of this change will be an enormous decrease in the income of the people and consequent hoarding will be inevitable. All this causes a bad effect on the national treasury and directly raises the rate of taxation. When the rate of taxation is raised, financial disaster and political discontent follow.

"Preparation should be made in advance to meet the coming of a financial crisis. If remedy is to be applied only after the crisis has arrived it may be impossible to meet the exact condition and consequently will cause great anxiety."

See Wên hsüan t'ung k'ao (Edition 1896, A. D.), Chapter 8, p. 4-15, by Ma Tuan-lin, 13th cent. A. D. Rendered by S. D. Quong.

times the quantity of goods consumed in the interval of time t to $t + T$, that is,

$$MV + M_c V_c = (\lambda/T)PQ \tag{1.1}$$

where if P is the average price of goods and Q is the quantity sold in the time t to $t + T$ the proportionality factor $\lambda = 1$. If P is used to denote a price index and Q a quantity index, the proportionality factor is not equal to unity. Actually it is only in the latter way that the equation makes sense statistically.

Equation (1.1) can be used to demonstrate the quantity theory of money provided additional assumptions are made. Thus if bank credit M_c is proportional to currency, $M_c = \gamma M$ and the velocity of circulation V_c is proportional to V that is, $V_c = \mu V$, the equation (1.1) becomes*

$$(1 - \gamma\mu) MV = (\lambda/T) PQ$$

which readily yields the proportionalities demanded by the quantity theory of money.

As a first approximation to an equation representing the exchange of goods for money the classical formula (1.1) cannot be criticized, but it is certainly no more than a first approximation. In the first place the equation does not take into account the lag between the production of a good and its consumption unless, of course, the interval of time T is sufficiently long. In the latter case the use of an average effectively buries an important element. In the second place the formulation (1.1) fails to recognize that prices of capital goods depend upon *expected* income and hence are largely psychological and that fluctuations in prices of these goods materially affect M_c, V_c and also V and M. This omission is essentially more important than the omission of the lag between production and consumption, but lag cannot be ignored.

2. Lag between Wages and Consumption. As already pointed out, Chart XLII, inventories play an important role in a capitalistic economy. There are goods purchased with raw materials and labor at prices prevailing in the past. For some goods the prices paid for labor may be those in effect six months or more before the goods pass out of inventory into final consumption. Soap is essentially in

*For refinements of this argument see J. M. Keynes, *Treatise on Money*, New York, 1930, p. 192.

this class. In other instances consumption is much more rapid. For example, according to economists of the meat packing industry the total time required to slaughter meat and dispose of it through the retail meat markets seldom exceeds two weeks.

Prices of raw materials seem to be determined largely by the supply of materials available and therefore sometimes bear little relationship to the wages required for producing them. For example, for years farmers have been complaining that they could not sell their cotton for the average cost of producing it. Whether or not the complaint is in general worthy of merit need not be answered here, but it is incontestable that prices of cotton have fluctuated widely and that the only way an assumption that the price of cotton reflects "wages" required to produce it can be justified is by a further assumption that farm wages fluctuate as violently as cotton prices. In reality, however, farm wages remain relatively stationary and farm income or profits fluctuate.*

Let p_{mi} represent average price of the ith raw material and let q_{mi} represent the quantity sold in the time t to $t + T$. This quantity may be sold several times before it actually enters a processing operation. The price p_{mi} is a weighted average so that if F is the total number of raw materials used,

$$(2.1) \qquad \sum_{i=1}^{F} p_{mi} q_{mi} \ ,$$

is the money value of all transactions involving raw materials in the time t to $t + T$.

To raw material are added transportation and sales services and then the product enters a production process and additional labor, power and management are added to produce either consumer or capital goods. A consumer good usually passes from the manufacturer to a jobber or wholesaler via additional transportation and then to the retailer who sells to the ultimate consumer (laborer, farmer, skilled mechanic, professional man, etc). Stock (inventory) shortages or stock surpluses may occur at any point in the process of manufacture and distribution. Consequently, *the price paid by the ultimate consumer for consumption goods need not be constantly proportional to the wages paid for manufacturing and*

*It is possible to beg the issue by defining farm income to be farm wages, but by the same method corporation profits could be defined to be wages to owners of capital.

distributing the goods. Let p_{gi} represent price of the ith consumption goods and let q_{gi} represent the quantity consumed (purchased) in the time t to $t + T$. In this definition all goods in process other than raw materials and all consumption goods in the usual sense are regarded as consumption goods. It might be better to make a division into producers' goods and consumers' goods, but this does not appear to be essential since additional variables would be added without increasing fundamental knowledge of flows of exchange. Then the money paid for consumption goods in the time t to $t + T$ is

$$\sum_{i=1}^{G} p_{gi}\ q_{gi} \tag{2.2}$$

where G is the total number of consumption goods sold in the time t to $t + T$.

The price of capital goods or equities depends to a large extent upon *expected income* from the goods or equities. In Chapter XI it was pointed out that this price or value could be represented by an integral or infinite sum of expected income plus a fraction of cost of the capital goods. For the present, however, denote the price of the capital goods A by p_{ci} and the quantity sold in the time t to $t + T$ by q_{ci}, so that the total money paid for capital goods is

$$\sum_{i=1}^{C} p_{ci}\ q_{ci}\ , \tag{2.3}$$

where C is the total capital goods sold in the time t to $t + T$.

Let p_{ri} represent the average price of capital services (rent, dividends, etc.) and let q_{ri} represent the quantity of such services demanded in the time t to $t + T$. Then the money paid for capital services is given by

$$\sum_{i=1}^{R} p_{ri}q_{ri}\ , \tag{2.4}$$

where R represents the number of services of capital used in the period.

If p_{si} represents the price of the ith service (wage or salary) and q_{si} the quantity of the service paid for in the time t to $t + T$, the total value of wages or salaries is equal to

$$\sum_{i=1}^{S} p_{si}q_{si}\ , \tag{2.5}$$

where S is the number of services required in the period.

The following notation will be useful:

M = average amount of currency (gold, silver, paper notes) in circulation in the interval of time t to $t + T$;

V = average number of times this currency changes hands or circulates in the time interval t to $t + T$;

M_c = average amount of bank credits subject to check in the interval of time t to $t + T$;

V_c = average velocity of circulation of M_c; and

y_c = the fraction of $M_c V_c$ used for the purchase of consumer goods and all services.

Obviously, $1 - y_c$ is the fraction of $M_c V_c$ available for (part of it may be hoarded) purchase of stocks, bonds, mortgages and capital goods.*

Let M_b represent brokers' loans, savings accounts and all money other than that already mentioned, and let V_b be its average velocity of circulation.

Then total monetary transactions are given by

$$[MV + M_c V_c + M_b V_b]T \text{ or splitting up } M_c V_c ,$$

by

$$[MV + y_c M_c V_c + (1 - y_c) M_c V_c + M_b V_b]T$$

Total monetary transactions in the time T are also given by the sum of (2.1), (2.2), (2.3), (2.4) and (2.5). It follows therefore that

$$MV + y_c M_c V_c + (1 - y_c) M_c V_c + M_b V_b = 1/T \ [\sum_{i=1}^{F} p_{mi} q_{mi}$$

$$+ \sum_{i=1}^{G} p_{gi} q_{gi} + \sum_{i=1}^{C} p_{ci} q_{ci} + \sum_{i=1}^{R} p_{ri} q_{ri} + \sum_{i=1}^{S} p_{si} q_{si}] \ .$$

This equation of exchange is, of course, an identity. Its chief value lies in the fact that it illustrates how various parts of an economy compete for purchasing power. A slightly more incisive analysis of the nature of some of the prices on the right indicates how depressions and booms may arise.

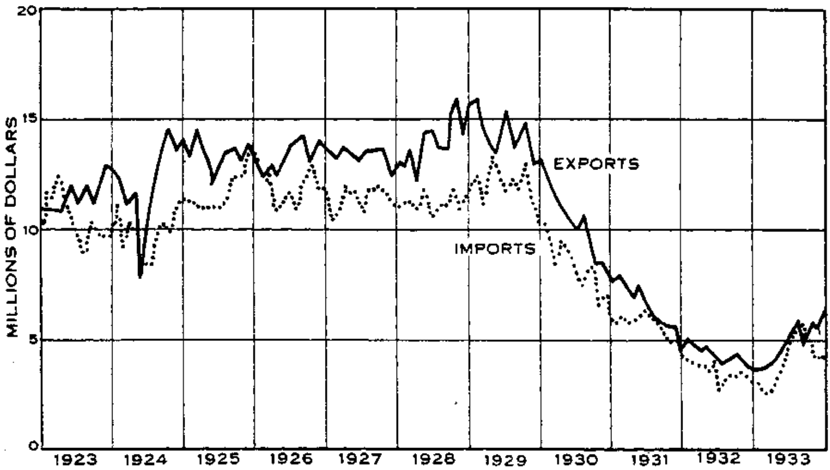
*Although related to Keynes' efficiency factor y_c is not to be confused with it.

UNITED STATES FOREIGN TRADE

Average Daily Value Adjusted for Seasonal Variation

Source: The Annalist, September 29, 1933, and February 2, 1934.

CHART XLIV



The excess of visible plus invisible exports over visible plus invisible imports can be regarded as internal inflation of purchasing power. Lure of foreign investments caused United States savings to be loaned abroad thus providing purchasing power for nations trading with the United States. Manufactured goods were thus exchanged for foreign bonds which may be regarded as paper money.

3. Prices of Capital Goods. As already pointed out, prices of capital goods depend upon expected future income and only slightly upon wages and material expended upon them. Thus, the prices, p_{ci}, are of the form: sum over some interval of time (sometimes regarded as such a long time that it can be an infinite interval of time) of $(pq - Q)$ where p represents price, q represents quantity of a good or service produced by the capital goods, and Q represents the cost of production of q, multiplied by a discount factor plus a fraction of the cost of production of the capital goods, that is,

$$p_{ci} = \sum_{j=1}^{N} \int_{t}^{t+t_1} [\gamma_{ij} p_{gj}\, q_{gj} - C\ (q_{gj})] e^{-\delta_i t}\, dt + \mu_i L_i \ ,$$

where L_i is the construction cost of the capital goods; N is the number of products or services to be produced by the capital goods;

$C(q_{gj})$ is the cost of producing the jth good or service (if a subscript s is used to denote a service) ; δ_i is a discount factor (force of interest) ; γ_{ij} is a factor used in estimating future income, and μ_i is a mortgage or bond ratio, for example, fifty or sixty per cent for real property. The quantities γ_{ij} and μ_i are largely psychological and of course, the former is more subject to fluctuation than the latter. This formulation of price applies particularly to plants. Some unassembled capital goods of course have their prices determined much like the prices of consumer goods. See Section 4.

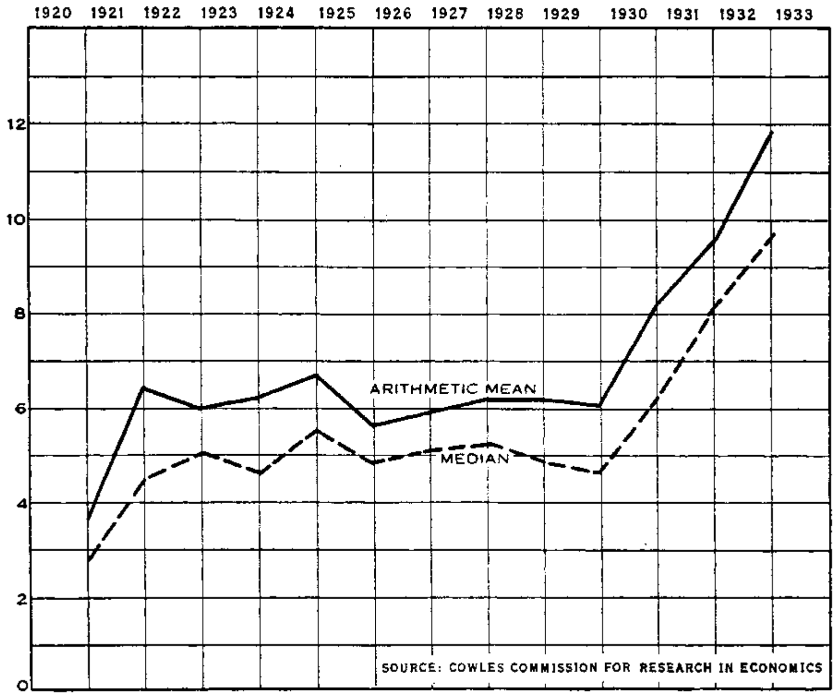
If general optimism prevails, μ_i is large and equities have high values. If pessimism prevails so that γ_{ij} becomes so small that the first term is nearly zero, brokers' loans M and even bank credit M_c will be reduced. Thus, when there are no prospects of profit, prices of bonds and money based upon these bonds are upset. Although the number of capital goods sold and their dollar value may be small, important price changes of these goods may bring serious repercussions throughout the economic system since M_c and M_b are largely based on these prices.* Thus, loss of confidence which brings about low equity values will force a downward spiral in the amount of bank credit, brokers loans and other credit money outstanding. In other words, on the left-hand side of the equation of exchange there are equity values not represented on the right-hand side which are directly affected by the prices of those which do appear on the right-hand side. Of course, disturbances also occur on the other side of the equation.

4. Prices of Raw Materials and Consumers' Goods. Consumption of raw materials going into consumers' goods is a relatively fixed quantity, i.e., over fairly long intervals of time (a year or more) consumer goods demand is relatively fixed. Excess production of raw materials goes into inventories of producers of raw materials, or speculators, or into inventories of manufacturers of consumers' goods. Manufacturers' and speculators' inventories are built up only if present price is sufficiently low and expected to advance (a short crop, production difficulties, etc.). Hence prices of raw materials depend to a large extent upon the quantity available, that is, upon the supply on hand and in sight within a year.

*Banks and investment houses do not revalue their securities daily so that short time fluctuations are relatively unimportant. A long downward or long upward sweep is the thing to be feared. See C. F. Roos, *Stabilization of Employment*, Chap. XIV.

RATIO OF CURRENT ASSETS TO CURRENT LIABILITIES FIFTY-SEVEN LARGE INDUSTRIAL CORPORATIONS

CHART XLV



Since 1929 the ratio of current assets to current liabilities for large companies has been steadily rising. This may mean 1) current liabilities have been reduced more rapidly than current assets or 2) current assets have increased more rapidly than current liabilities. At any rate a "normal" rate of circulation of current assets and current liabilities has not been maintained. Today small concerns are not considered good credit risks whereas large concerns in general do not need credit. This makes credit expansion as a means of inflation very difficult.

Salt, which is very plentiful, has a low market value. Recently the supply of petroleum became so plentiful that its price declined to ten cents per barrel.

In a surplus economy like that of the United States a slight decline in consumption of consumers' goods can bring about a drastic decline in prices of raw materials.

Decline in prices of raw materials may bring about substitute uses for the raw materials, for example, cheap cotton can be used to make bags that might otherwise be made of paper. Again high cotton prices will undoubtedly bring the rayon industry to maturity much more quickly than low cotton prices and will lead to less demand for cotton at an earlier date than would otherwise be the case. See Chart XXVIII. A decline in prices of raw materials will ultimately be reflected in reduction of prices to the consumer, but during the decline some producers are almost sure to suffer severe losses due to decline in value of their inventories.

The price $p_{gk}^{(1)}$ of a semi-finished consumer good, i.e., a good in the first stage of processing, may be expressed by a relation

$$p_{gk}^{(1)} = \gamma_1 \left[\sum_{i=1}^{M_1} p_{mi} + \sum_{j=1}^{W_1} W_j^{(1)} \right],$$

where $W_j^{(1)}$ stands for services of capital (interest) and services of labor, including direct labor to produce power. For the second stage of manufacturing

$$p_{gk}^{(2)} = \gamma_2 \left[\sum_{k=1}^{M_2} p_{gk}^{(1)} + \sum_{j=1}^{W_2} W_j^{(2)} \right]$$

and so forth for other stages of processing. Transportation and selling expense can also be included by proper modification of the definitions of W and p.

If the time of manufacture extends from $t - t_i$ to t the price of the consumer good may for simplicity be taken to be

$$p_{gi} = \int_{t-t_i}^{t} \left[\sum_{k=1} \lambda_{1k}(x, t) W_k(x) + \sum_{j=1} \lambda_{2j}(x, t) p_{mj}(x) \right] dx$$

where $\lambda_{ik}(x, t)$ is a psychological factor depending upon business conditions at each time x in the interval $t - t_i \leq x \leq t$ as well as upon business conditions at the time t. This formulation indicates how a change in the price of a raw material may be reflected through different stages of processing and distribution.

5. Interest, Rent and Dividends. The rate of interest is invariably fixed by contract. Short-term interest rates are in general (except in boom times) less than long term rates on account of the fact that the danger of losing the principal increases as the length of time of a loan increases. Short term interest rates are fairly flexible. In the United States they are considerably influenced by the discount policies of the Federal Reserve Board and by borrowings of the United States Treasury.

On the other hand, in the United States a six per cent per annum rate of interest for real estate mortgages has been in vogue for years. The rate of interest on corporation bonds has varied from four per cent per annum to seven or eight per cent per annum, the variation depending chiefly upon the expected rate of risk. Long term interest rates adjust very slowly. They become fixed by habit and are very difficult to change.

It might be expected that the force of interest δ_i in Section 3 is the long term interest rate, but it is in reality more nearly the short term rate. That is, in boom time the capitalist is tempted to use his money in financial dealings promising a high interest yield rather than to invest them in long term securities. In a depression, on the other hand, δ_i is very small to the investor. Thus δ_i and γ_i move in the same directions. In reality it may be questioned whether it is necessary to use two factors δ_i and γ_i. In view of the fact that δ_i is an interest rate and γ_i is a psychological quantity, it appears best, however, to separate them.

Dividend rates are more or less fixed by custom. A corporation paying six per cent per annum dividends usually builds up a surplus in times of prosperity and continues to pay the same rate of dividends for some time after earnings fail to justify it. Such unearned dividends are, of course, paid out of surplus built up in times of prosperity.

Variation in house rent in St. Louis is shown in the chart used in the chapter on factors influencing residential building. Rent, like interest, is fixed by both short—and—long term contracts. History indicates, however, that rent is somewhat more flexible than interest.

6. Competition for Purchasing Power. As pointed out, prices of raw materials are quite flexible whereas quantities of raw materials consumed are relatively fixed. It follows that purchasing power going to producers of raw materials is therefore quite flexible.

Purchasing power going to producers of consumers' goods is on the other hand fairly constant. It is customary to keep prices of consumers' goods fairly constant and allow profits to fluctuate with prices of raw materials. In times of rising prices inventories offer the best avenues of profit for producers of consumers' goods and similarly in times of falling prices inventories bring about losses.

Purchasing power going into capital goods is highly flexible and depends upon psychological factors. Capital goods therefore represent a highly volatile element in an economic system. Prices of capital goods depend upon confidence in the future.

Rents, interest and dividends are very stable elements in the equation of exchange. Thus, as purchasing power (left-hand side of the equation) shrinks, an increasing proportion goes to pay rent and interest and for a short time dividends also.

Wage rates or prices of labor change, but not rapidly, that is, they do not keep pace with prices. Quantity of labor employed constantly changes, however, so that purchasing power going to labor changes considerably. On the surface this fact would seem to indicate that purchases of consumers' goods would vary considerably, but it must be remembered that savings which can be drawn upon by the unemployed, gifts, charity, etc., all act to keep up purchases of consumers' goods. After prolonged unemployment of masses, i.e., when their savings are exhausted, consumption of consumers' goods necessarily falls. Here profits crumble, equities fall and repercussions occur throughout the whole system.

Examination of building statistics shows that residential building began to sag in 1926 and underwent only a slight revival in 1928, probably due to stock market profits going into building regardless of income incentive which was steadily dropping (see Chapter VI). From 1928 on, the decline in residential building has been almost precipitous. In 1929 and 1930 public works took up some of the unemployed, but abandonment of such work took the last prop out. The collapse of the stock market in the fall, due chiefly to psychological factors,* might have led to only a temporary panic had it not been for the fact that building was on a precipitous downgrade with no important growing industry to take up the slack. See Chapters VI and VII. The two effects occuring simultaneously were sufficient to produce important changes in national income and thus set up the deflationary forces already described.

Following the second stock market crash business generally set about to reduce costs. All the adjustments described in Chapter X were attempted.

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*It is not meant that the stock market had not led to an inflation of brokers' loans and bank credit which brought about profits generally, and gave the appearance of a new era.

When adjustments in cost take place rather generally the very process of decreasing production to increase profits will bring about changes in income that require smaller and smaller values of the production rate maximizing income. Thus, a shrinkage in national income can bring about an appearance of general overproduction. It must be remembered also that in this economic downtrend *expected* price decreases and the accompanying aggravating situation described briefly in inequality (7.1) of Chapter IX further reduces employment, especially in capital goods industries.

At some stage the downtrend in the psychological factor γ_j is arrested, and *expected* price increases so that production and distribution are speeded up. In this process of reversal, employment in consumer goods industries is increased. The accompanying increases in national income usually bring out new bank loans (newly created credit), which are essentially additions to national "income," and savings which have been idle or have been going into governmental loans are put into use for the construction of capital goods. As production increases further, prices of raw materials increase and inventories are increased in value. Increased values of inventories make new bank loans possible and these in turn provide more national "income", which brings about greater production and so on until a position of instability is again reached.

APPENDIX I

Correlation of Time Series

Some workers, inspired by G. Udney Yule's paper, "Why Do We Sometimes Get Nonsense Correlations Between Time Series," *Journal Royal Statistical Society*, Vol. 89, 1926, pp. 1-84, in which certain nonsense correlations are exhibited, have fallen into the natural error of maintaining that one should always "eliminate trend" and correlate the residuals. A reference to a physical problem is probably the quickest way of showing how ill-founded this opinion really is. Let $V =$ voltage, $R =$ resistance, and $I =$ current, and for definiteness suppose $R = 2$ so that $V = 2I$. Now, it is possible to vary the voltage V so that $V = 6t +$ random errors of measurement, where t is time measured from some fixed time t_0. In a physical experiment of this type, it is possible to read meters, etc., to a high degree of accuracy. Small random errors would occur in V and I due to a variety of uncontrolled causes, such as differences in temperature of the atmosphere surrounding the meters, etc. To remove the trend $6t$ from V and $3t$ from I and correlate the residuals would be to correlate random errors. A zero correlation would be found between voltage and current. This is, of course, much greater nonsense than the nonsense mentioned by Yule.

In the case of economic series, the relative size of the error is much larger than in the physical example just given. Thus, it may happen that the removal of a linear trend or a quadratic trend will leave little more than random errors. The trend in this case is the significant movement. Jordan has shown that if orthogonal functions are used to fit trends to two series and the residuals are correlated, the correlation coefficient is a function of the parameters of the trend functions, and has derived formulas describing the relationship.

Suppose that $y(t)$ and $u(t)$ represent two observed time series which it is desired to correlate. Suppose that Tchebycheff polynomial trends

$$Y = C_0 + C_1\varphi_1 + C_2\varphi_2 + \cdots + C_q\varphi_q$$

and $$U = K_0 + K_1\varphi_1 + K_2\varphi_2 + \cdots + K_q\varphi_q ,$$

where $C_0, \ldots C_q$ and $K_0, \ldots K_q$ are constants, are fitted to the series $y(t)$ and $u(t)$ respectively. Let $z = y - Y$ and $x = u - U$. Then, as Jordan has shown, the simple correlation between z and x is given by the formula

$$r = \frac{\frac{1}{n}\Sigma\, yu - C_0K_0 - C_1 K_1 \cdots C_q K_q}{\sqrt{\sigma_y^2 - C_1^2 - C_2^2 \cdots C_q^2}\ \sqrt{\sigma_u^2 - K_1^2 - K_2^2 \cdots K_q^2}}$$

where σ_y is the standard deviation of y, and n is the number of observations in the series.

If orthogonal functions different from Tchebycheff polynomials are used for example, $\sin mx$, $m = 1, 2, 3, \cdots$, a different correlation coefficient will evidently be obtained. A single term $\sin mx$ might conceivably represent a trend better than a fourth degree polynomial, and of course there is nothing sacred about a polynomial trend.

The analysis of Jordan's paper* may be carried through for general orthogonal functions as well as for the special orthogonal functions which he uses. The form of the correlation coefficient will be the same, but its value depends on the particular orthogonal functions used and on the number of terms employed in the expansions. Consequently it is not at all clear what a coefficient of correlation of deviations from trend means. In fact, it is simple to show that by using one set of orthogonal functions a correlation of almost —1.00 can be obtained between two series, whereas by using different orthogonal functions a coefficient of correlation of almost +1.00 can be obtained.

Let $w(x)$ and $u(x)$ be two functions defined on $(a\, b)$ with approximating parabolas y_n and η_n respectively, y_n and η_n being of degree n.† The correlation coefficient of the deviations $(w - y_n)$ and $(u - \eta_n)$ is defined to be

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*Charles Jordan, "Sur la Determination de la Tendance Seculaire des Grandeurs Statistiques par la Methode des Moindres Carrés". Tirage à Part de *Journal de la Societé Hongroise de Statistique*, Année 1929, No. 4.

†This is a report of a study undertaken by Emily C. Pixley and Clement Winston at the suggestion of the author.

$$r = \frac{\int_a^b (w-y_n)\ (u-\eta_n)\,dx}{\sqrt{\int_a^b (w-y_n)^2dx\ \int_a^b (u-\eta_n)^2dx}}$$

If $w(x)$ and $u(x)$ are developed as series of orthogonal functions,

$$w(x) \backsim y_n = a_0 + a_1\varphi_1 \quad \cdots \quad a_n\varphi_n$$
$$u(x) \backsim y_n = b_0 + b_1\varphi_1 \quad \cdots \quad b_n\varphi_n,$$

$[\varphi_i]$ being any complete set of orthogonal and normal functions on (a, b), (that is

$$\int_a^b \varphi_n(x)\,\varphi_m(x)\,dx \begin{array}{l} =0,\ m\neq n \\ =1,\ m=n) \end{array} \quad ,$$

the correlation coefficient may be expressed in terms of the coefficients of the expansion as follows:

$$r = \frac{\int_a^b w(x)\cdot u(x)\ dx - \sum_{i=0}^{n} a_ib_i}{\sqrt{\left[\int_a^b w^2(x)\,dx - \sum_{i=0}^{n} a_i^2\right]\left[\int_a^b u^2(x)\,dx - \sum_{i=0}^{n} b_i^2\right]}}$$

Although the coefficients a_i, b_i, vary for different sets of orthogonal functions, the form of r remains invariant. It is thus seen that in general nothing very definite can be said about the value of r obtained by this method, since it does not seem possible to obtain a relation between the different sets of coefficients.

Some examples which show considerable difference in the value of the correlation coefficients obtained by using different sets of orthogonal functions, on the same interval, were next set up.

1. Consider the interval (0,1), and the functions $w(x) = x_3$ and $u(x) = 2x-x_3$ on this interval. Using the first three terms of the expansion in Legendre polynomials it can be shown that

$$w(x) \backsim 1/4\,P_0 - 9/20\,P_1 + 1/4\,P_2$$
$$u(x) \backsim 3/4\,P_0 - 11/20\,P_1 - 1/4P_2$$

where $P_0 = 1$, $P_1 = (1-2x)$, $P_2 = (1-6x+6x^2)$.

In this case $a_0 = 1/4$, $a_1 = -9/20$, $a_2 = 1/4$

$b_0 = 3/4$, $b_1 = -11/20$, $b_2 = -1/4$.

Substituting these values in the formula for r it is found that $r = -1.00$.

On repeating the process above, representing the same functions by Fourier coefficients, a strikingly different value of the correlation coefficient is obtained. In fact, here

$$w(x) \backsim .25 + 3/2\pi^2 \cos 2\pi x + (3/2\pi^3 - 1/\pi) \sin 2\pi x$$
$$+ 3/8\pi^2 \cos 4\pi x - (3/16\pi^3 - 1/2\pi) \sin 4\pi x$$
$$u(x) \backsim .75 - 3/2\pi^2 \cos 2\pi x - (3/2\pi^3 + 1/\pi) \sin 2\pi x$$
$$- 3/8\pi^2 \cos 4\pi x - (3/16\pi^3 + 1/2\pi) \sin 4\pi x.$$

and using the values

$$a_0 = .25, \quad a_1 = 3/2\pi^2, \quad a_2 = 3/2\pi^3 - 1/\pi, \quad a_3 = 3/8\pi^2$$
$$a_4 = 3/16\pi^3 - 1/2\pi$$
$$b_0 = .75, \quad b_1 = -3/2\pi^2, \quad b_2 = -(3/2\pi^3 + 1/\pi),$$
$$b_3 = -3/8\pi^2, \quad b_4 = -(3/16\pi^3 + 1/2\pi)$$

the correlation coefficient turns out to be $r = +.976$.

2. For the interval (0,1) consider the functions $\overline{w}(x) = x^3$, $\overline{u}(x) = x^4$.

As above, it can be shown that, using Legendre polynomials, $r = 0.9921$, while using Fourier series, $r = 0.176$.

With reference to the first example, it can easily be demonstrated that for the interval (0,1), the coefficient of correlation is always -1 if w and u are expanded in a series of Legendre polynomials and $w = 2x - u$. In this case, the number of terms in the expansion of w and u is arbitrary.

If the coefficient of correlation r is to be used to indicate a linear relationship between the two functions considered, then the results presented here would tend to show that such a notion is entirely erroneous. The literature, however, seems to give methods for determining the coefficient only, with no discussion of its use or significance.

It may be of interest to study the effect of using the polynomials given by Jordan, Fisher and Sasuly to observe whether the same results are obtainable when sums instead of integrals are used. One would expect similar results.

In examining time series, it is of paramount importance to take all major factors into account. A rational, theoretical analysis based on broad acquaintance with the economic situation to be analyzed is fundamental. If all important factors are taken into account, the influence of the many neglected but minor factors (all economic quantities are related) should give a net constant effect ("systematic error", in the theory of the adjustment of observations). If the analysis is complete, there will be only random residuals left. The use of a trend in correlation analysis is therefore a confession of ignorance of fundamental factors involved. If a time-trend must be used owing to lack of observations for certain neglected factors, it can be eliminated implicitly, as in Ezekiel *Methods of Correlation Analysis.* Ragnar Frisch and Frederich V. Waugh, *Econometrica,* October 1933, show that for additive linear trends the implicit trends are identical with the linearly fitted trends. See also Ragnar Frisch, *Pitfalls in the Statistical Construction of Demand and Supply Curves.* Or, time-trends should be treated as systematic variations appearing in the residuals left after effects of the known factors have been eliminated. It is hardly necessary to refer to the high (but fallacious) correlations of data obtained by manipulating trends. All correlations of residuals from trends are therefore to be looked upon with suspicion.

APPENDIX II

Analysis of Random Errors in Time Series*

The statistical theory of time series has usually been based on the assumptions that the observations are the resultant of two main types of variation:

(a) Systematic variation: for instance, (1) secular variation (production series), (2) periodic variation, such as in the theory of the tides, and the seasonal variation in economic phenomena, (3) cyclic variation (business cycles).

(b) Random variation, chance: for instance, the residuals left after graduation of the observations by seasonal analysis and a good smoothing formula. The theory and the treatment alike have assumed that the random errors are the same kind of thing as the random errors of the classical theory of observations, developed for measurements on precision instruments.

Now, in a time series consisting of the successive readings of a precision instrument, it is assumed, and in general correctly, that the residuals, after elimination of constant and systematic error, are mutually independent of one another. Their assumed origin guarantees this independence. Thus, the residual v_1 in the observation y_1 at time t_1, and the error e_1 in that observation, are conceived to be the net result of a multitude of individually small shocks, etc., to the instrument, and defects in the sensory apparatus of the observer. Each such minute shock, twist, strain, etc., is an "elementary error," the summation of all of which is the actual error. From the hypothesis of elementary errors the well known normal law

$$\varphi = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x/\sigma)^2}$$

may be deduced. Similarly with v_2 and e_2 in the observation y_2. But there are no shocks common to e_1 and e_2, and therefore e_1 and e_2 are wholly independent of one another. Or, rather, shocks or strains which persist from one observation to the next with the same sign,

*This appendix was prepared by Victor von Szeliski.

are constant errors and systematic errors, and may be eliminated by an approximate method of graduation. There may be elementary errors and shocks common to v_1 and v_2 due to inadequate elimination of systematic error.

But in many economic time series, especially historical frequency series, the observed residuals from careful graduations do not conform to the normal law of error, and consideration of how they arise will show that they differ from the random errors and random residuals of classical theory.

True, the residuals do doubtless arise in part from causes of the same type as the random variations of theory, but in addition two successive errors may have the same "elementary error" in each, but *opposite in sign.* Thus, in building contracts a particular contract may be due to be signed on May 31 and reported in that month; but "by chance" it is recorded on June 1 and goes into that month. The May observation is of the form $B_1 - e$ and the June contract is of the form $B_2 + e$, one being increased, the other diminished, by the same amount. Or several contracts may be lumped together and assigned to one month or the other, whereas actually some were in one month and some in the other. Or where consumption of a commodity is determined by adding production and decrease in inventory, errors in the inventory figure through failure to measure it correctly or to measure it at the proper time, will enter the succeeding determination with opposite sign.

Price series may also have this type of variation; thus there is some tendency for violent changes in prices of speculative commodities and stocks on one day to be followed by reverse changes the next.

We may call errors and residuals of this type *alternating errors and alternating residuals.* A series containing them could be symbolized this way:

(1)

$$
\begin{aligned}
y_1 &= x_1 + a_0 - a_1 + e_1 \\
y_2 &= x_2 + a_1 - a_2 + e_2 \\
y_3 &= x_3 + a_2 - a_3 + e_3 \\
y_4 &= x_4 + a_3 - a_4 + e_4 \\
y_5 &= x_5 + a_4 - a_5 + e_5 \\
y_6 &= x_6 + a_5 - a_6 + e_6 \\
&\cdots\cdots\cdots \\
y_n &= x_n + a_{n-1} - a_n + e_n \ ,
\end{aligned}
$$

where $y_1, \cdots, y_6$ are the observed values; $x_1, \cdots, x_6$ are the true values; $a_0, a_1, a_2, \cdots, a_6$ are the alternating errors; and $e_1, \cdots, e_6$ are the random errors If these are summed, it follows that

$$\sum_1^n y = \Sigma x + a_0 - a_n + \Sigma e \ . \tag{2}$$

Since, in general, a_0 is independent of a_n,

$$\Sigma y = \Sigma x + \bar{a}\sqrt{2} + \bar{e}\sqrt{n} \ , \tag{3}$$

where $\bar{a}$ and $\bar{e}$ are the average absolute values of the alternating and the systematic errors to be feared. Thus if monthly observations are summed over a year, the alternating errors cancel out with the exception of the errors in the terminal months coming in from the "outside." For example, while a daily series of building contracts awarded appears to be mostly chance variation, with little observable regularity, a series of annual totals exhibits comparatively smooth cycles, with very little of the random. While alternating errors still exist at each end of the larger time interval, the size of the error is reduced relative to the size of the observation.

This analysis suggests several problems:

(1) Has a series containing alternating errors distinctive statistical characteristics, such as functions of successive differences, which would identify it as such?

(2) What is the best way of graduating the observations y?

(3) Can estimates be made of the alternating residuals, or at least of the sum of successive pairs, $a_{i-1} - a_i$, so that the series can be analyzed into:

(a) Graduated values x_i^1
(b) Alternating residuals a^1_i or $(a^1_{i-1} - a^1_i)$
(c) Random residuals e^1_i?

(4) What are the probable errors of the graduated values?

(5) What are the probable errors of constants determined by fitting curves to the observations?

(6) How should curves be fitted to such series, so as to give the best fit?

(7) Does this theory throw any light on how to calculate the standard error and reliability of coefficients of covariation between time series?

With regard to No. 7, the crux of the question appears to be: how many *independent* observations are there in a given time series? If the original series of observations $y_i \cdots$ has considerable alternating error, it may be replaced by a series

$$Y_1, Y_j, Y_{2j}, Y_{3j}, \cdots, Y_{nj}$$

where

$$Y_j = y_j + y_{j+1} + \cdots\cdots y_{2j-1}.$$

As j (the length of the unitary time interval) is increased, the alternating errors are damped out, and the successive Y's tend less and less to be repeated observations of the same quantity (if x changes slowly, y_i and y_{i+1} are clearly not independent at all, but are, to a large extent, repeated observations). Is a point finally reached where the Y's satisfy a test of independence (presumably where functions of the differences of Y take values within assigned ranges)? If so, the number of independent observations in a time series, or in a pair of correlated series, may be found, and the probable error of the coefficient of covariation determined.

APPENDIX III

Reliability of Data Relating to Residential Building

The following data on residential building apply to St. Louis proper. In the last fifteen years the chief development has taken place outside the city limits, in St. Louis County.

1) *New Building.* This is the number of family accommodations provided for as stated in permits issued by the city of St. Louis. The figures include apartment hotels, apartment houses and flats as well as single-family residences. An apartment building with 24 apartments is counted as 24. Transient hotels are excluded, and of course all buildings of a non-residential character. The figures are those of Roy Wenzlick, real estate analyst, who compiled them from the original records of the city.

2) *Number of Families.* The source is the Bureau of the Census. The definition of family, as used by the Bureau, is rather broad. It includes groups living under one roof but not related by blood, such as institutional families. For this reason, if for no other, the one-to-one relationship of family accommodations to families postulated in this study cannot be expected to hold exactly.

The Census figures were interpolated by overlapping cubics fitted in such a way that the first derivative of the interpolated function is continuous throughout.

3) *Gross Rents.* Accurate rent statistics are difficult to obtain. The most widely used rent indexes, those of the Bureau of Labor Statistics, and the National Industrial Conference Board, are based on small samples of *estimated* rentals. Rents remain one of the most important gaps in statistical information.

The figures used in this study were built up from actual rentals in recent years, and the asking rentals in the classified newspaper advertisements in other periods. For 1924 to 1932, the rental figures are averages of actual annual rentals received on fully occupied four-family flats of a certain type quite common in St. Louis. Rentals for 1897 to 1923 were estimated by linking on an index of asking rentals. The rents at which houses were offered were collected from the advertising section of the leading St. Louis newspaper, and classified according to type and number of rooms. In the earlier part of the period, up to and through 1912, observations were obtained for January of each year only, unless too few cases were found in that month, in which case additional cases were collected for February, March and so on until a sufficiently large sample had been obtained. In 1913, rents were obtained for four months, January, April, July and September; in 1914 and 1915 for January and July; in 1916 for January, February and July; in 1917 for January, April, July, October and December. They were obtained for every month in 1918, for four months in 1919, five months in 1920 and 1921, three months in 1922 and once a year (January) in 1923 to 1928 inclusive. The 1929 figure was based on rents for January and July and the 1930 figure for January, July, September, October and November. Starting in 1931, asking rentals were collected monthly.

The rents for each class and number of rooms were put on a per room basis and averaged. The per room figures were then charted on a semi-logarithmic paper. It was found that the lines ran very closely parallel with one another, a circumstance which lent color to the statistical stability of the asking rental as a measure of actual rental. An average was then picked out on the graphs by inspection. These averages ranged from a low of about $2.50 per room in 1899-1900 to a high of somewhat over $10.00 per room in 1920.

The next step was to convert these scattered observations representing isolated months into estimated averages for the year as a whole. Prior to 1913, this was accomplished simply by averaging two succeeding observations, weighting the first two-thirds and the second one-third. From April 1912 the observations were interpolated graphically by months, and then a weighting function applied. This weighting function was based on the seasonal distribution of moving in St. Louis as determined from the records of moving companies. The yearly averages determined in this way were put on an index base with 1926 = 100.

Finally, the actual rentals for the period 1924-1932 were extended backwards by use of this asking rental index. For this purpose $2,186 was taken equal to 100. This will give a correct result provided that asking rentals are subject to a constant percentage error. This can be assumed to be true in periods where the asking rental index remains relatively constant. Where, however, it is not constant, but changes rapidly, it cannot be assumed that asking rentals are always a constant fraction or multiple of actual rentals. A comparison of the index of asking rentals with the rent index of the Bureau of Labor Statistics showed wide discrepancies during the period 1919-1923 when rents were rising so rapidly. There was pretty fair agreement for other periods. The reason that the index of asking rentals does not represent true conditions during periods of rapid change of rent levels is that asking rentals are rentals on a few parcels of properties that happen to be in the market at that moment. It is a time when occupancy is high and there are few houses on the market. They do not represent the rentals actually paid on the houses. Rents change slowly because they are long-term contracts for the most part. Also, during a time of boom, landlords are carried away by the general good feeling and ask much higher prices than they expect to get.

At any rate, the index of asking rentals seems definitely at fault in the early post-war period and it was arbitrarily changed for the years 1918 to 1923, inclusive, to conform more closely to the movements of the Bureau of Labor Statistics rental index.

It is assumed that the same phenomenon occurred during the boom of 1903 and 1904, and the index of asking rentals for those years was arbitrarily lowered.

In summary, the rents given in Column 6, Table 1, are based on asking rentals for 1897-1917, Bureau of Labor Statistics rent index, adjusted, for 1918-1923, and actual rents during 1924-32-33.

The estimated rents for the earlier years must be regarded with some suspicion. It is certainly difficult to believe that the fast drop in rentals shown to have occurred from 1897 to 1899 was in accord with the fact. These were years of emergence from the deep depression of the nineties. Commodity prices, national incomes and volume of production were rising and it is exceedingly doubtful whether rentals had a sharp trend in the opposite direction.

4). *Taxes.* Taxes are averages of actual figures for the period 1924 on. They were extrapolated backwards by means of an index of the average real estate tax collections per family in St. Louis. On the assumption that in the long run the number of dwellings equals the number of families, this index gives a fairly accurate picture of the average tax burden per individual dwelling. The chief objection to the figure as it stands is that it includes taxes on non-residential property. It was not possible to eliminate these.

The purpose of putting taxes on a per family basis is that it automatically corrects for the change in the size of the average family, and the consequent change in the size of unit dwelling.

5). *Construction Costs.* Construction costs are for a typical St. Louis residential building. The particular building selected was a four-family flat. This type was probably built in greater numbers with very slight variations than any other general type during the period studied. Cost figures for this building are very much more than a simple aggregative or price index of a few building materials. They represent a very detailed and careful calculation of costs covering all items entering into building, down to the corner bead on the walls. These specifications were changed several times during the period to give recognition to changes in available materials and differences in building practice. For example, during the past year conduit and B X replaced the knob and tube wiring formerly allowed. In 1907 the kitchen drainboards were wooden, the toilets had high wooden tanks, the bath tubs were on legs, the bath floors were not tiled and the other floors were not hardwood. Labor costs were computed on what was actually paid, rather than on some scale which, at least in periods of depression, is only nominal. Further details are exhibited in the accompanying table.

Cost of lot is included in figures given in Column 11, Table 1.

6). *Change in Units of Measurement.*

One question which must be faced in regard to the foregoing data is whether the units of measurement retain their significance unchanged throughout the period. We have already alluded to the change in the size of a typical family since the turn of the century. This is to some extent taken care of by the *b*-function developed in the text, but not wholly. Some families obtained accommodations in the latter part of the period by remodeling large single-family houses built during the earlier part of the period, two or more families entering what had been when built a large single-family residence.

The rental figure represents rents on a uniform size of house throughout the period. On a per family basis, it should be relatively too low in the early years because there were more rooms per family at that time. It should be relatively high in the later years because the average family now needs fewer rooms. Fortunately the cost of land and building is also for a building of fixed size, and since rents and costs are respectively in the numerator and denominator of the incentive fraction, the change in the fundamental unit of measurement has little effect on the result. The errors, if any, compensate partially or wholly. However, taxes are not on a constant space base unit basis, but on a per family basis and therefore do not agree in regard to unit of measurement with rents and costs. Consequently, some systematic error in measurement is apparently introduced into the measurement of incentive.

There is some doubt, then, as to whether the formulas developed in the text have taken account of all relevant and important factors. It would not be surprising if the figures for actual building were found to deviate systematically from the calculated figures. That is, a trend might be expected to appear in the residuals, showing that not all variables had been taken into account.

In view of these considerations regarding possible change in the unit of measurement since 1900, there is ample justification for including some time trend factor in the equation to take care of it. There being no indications whatsoever as to what functional form this factor should take, it could be assumed to be linear. However, the writers have not introduced such a factor because the observation seemed to fit naturally without the introduction of a time-trend. The residuals do not appear to exhibit any systematic variation of a trend type. As developed in the text, fitting a trend is to be accepted only as a last resort, and after its implications have been carefully thought out. The writers are of the opinion that the compensation mentioned above has actually taken place and suggest that in many other problems in which the fitting of trends seems to be indicated, the suitable selection of functions and factors would render their use unnecessary in many instances.

APPENDIX IV

Transformation of Integral Range

To transform from the infinite range on which W is defined to a finite one, write

$$W = \int_{-\infty}^{t} \frac{b_1}{\sqrt{2\pi}\,\sigma_1} e^{-(t-x-M_1)^2/2\sigma_1^2} dx$$

$$+ \int_{-\infty}^{t} \frac{a_1 I(x)}{\sqrt{2\pi}\,\sigma_2} e^{-(t-x-M_1)^2/2\sigma_1^2} dx$$

$$= \int_{-\infty}^{t} \frac{b_1}{\sqrt{2\pi}\,\sigma_1} e^{-(t-x-M_1)^2/2\sigma_1^2} dx$$

$$+ \int_{-\infty}^{t-t_0} \frac{a_1 I(x)}{\sqrt{2\pi}\,\sigma_1} e^{-(t-x-M_1)^2/2\sigma_1^2} dx$$

$$+ \int_{t-t_0}^{t} \frac{a_1 I(x)}{\sqrt{2\pi}\,\sigma_1} e^{-(t-x-M_1)^2/2\sigma_1^2} dx \;.$$

By putting $(x + M_1 - t)/\sqrt{2}\sigma_1 = z$ in the first and second integral, W may be written in the form

$$W = \int_{-\infty}^{M_1/\sqrt{2\sigma_1}} \frac{b_1}{\sqrt{\pi}} e^{-z^2} dz$$

$$+ \int_{-\infty}^{M_1 - t_0/\sqrt{2\sigma_1}} \frac{a_1}{\sqrt{\pi}} I(\sqrt{2}\sigma_1 z + t - M_1) e^{-z^2} dz$$

$$+ \int_{t-t_0}^{t} \frac{a_1 I(x)}{\sqrt{2\pi}\,\sigma_1} e^{-(t-x-M_1)^2/2\sigma_1^2} dx \;.$$

Now, the first integral in W is a constant which may be called A_{01}. In the second integral let I_0 be the average value of I in the interval $-\infty \leq z \leq (M_1 - t_0)/\sqrt{2}\sigma_1$ and let $I_\Delta(\sqrt{2}\sigma_1 z + t - M_1)$ be a function such that $I(\sqrt{2}\sigma_1 z + t - M_1) = I_0 + I_\Delta(\sqrt{2}\sigma_1 z + t - M_1)$. Then by an application of the law of the mean for integrals,*

$$W = A_{01} + \int_{-\infty}^{(M_1 - t_0)/\sqrt{2}\,\sigma_1} \frac{a_1}{\sqrt{\pi}} I_0 e^{-z^2} dz$$

$$+ \int_{-\infty}^{(M_1 - t_0)/\sqrt{2}\,\sigma_1} \frac{a_1}{\sqrt{\pi}} I_\Delta(\sqrt{2}\sigma\, z + t - M_1) e^{-z^2} dz$$

$$+ \int_{t - t_0}^{t} \frac{a_1 I(x)}{\sqrt{2\pi}\,\sigma_1} e^{-(t - x - M_1)^2/2\sigma_1^2} dx$$

$$= A_{01} + A_{02} I_0 + I_\Delta(t - M_1 - \lambda_1) \frac{a_1}{\sqrt{\pi}} \int_{-\infty}^{(M_1 - t_0)/\sqrt{2}\,\sigma_1} e^{-z^2} dz$$

$$+ \int_{t - t_0}^{t} \frac{a_1 I(x)}{\sqrt{2\pi}\,\sigma_1} e^{-(t - x - M_1)^2/2\sigma_1^2} dx\,,$$

where

$$A_{02} = \int_{-\infty}^{(M_1 - t_0)/\sqrt{2}\,\sigma} \frac{a_1}{\sqrt{\pi}} e^{-z^2} dz$$

and λ_1 is a positive quantity such that $t - M_1 - \lambda_1$ represents a time previous to the present time; thus, $I_\Delta(t - M_1 - \lambda_1)$ represents an average value of past fluctuations in incentive from an assumed normal I_0.

---

*J. A. Hobson, *Theory of Functions of a Real Variable* Volume 1, page 617. The theorem referred to here does not apply to an integral with an infinite range, but the transformation $1/z = y$ reduces the infinite range to a finite one, the function I is bounded for all values of z and $(e^{-1/y^2})/y^2$ is also bounded on the transformed finite range. The law of the mean referred to here can thus be applied to the transformed integral and after this application a transformation be made back to z.

It remains to find a t_0 such that

$$I_{\Delta}(t - M_1 - \lambda_1)\,\frac{a_1}{\sqrt{\pi}}\int_{-\infty}^{(M\,-t_0)/\sqrt{2}\,\sigma_1} e^{-z^2}\,dz$$

is negligible, i.e., so that, for statistical purposes, the error in taking

$$W = A + \int_{t-t_0}^{t} \frac{a_1 I(x)}{\sqrt{2\pi}\,\sigma_1}\, e^{-(t-x-M_1)^2/2\sigma_1^2}\,dx\,,$$

where A is a constant, instead of in the form (7.4), Chapter VI, is negligible. Obviously, since I_{Δ} is bounded above and below and

$$E = \int_{-\infty}^{(M-t_0)/\sqrt{2}\,\sigma_1} e^{-z^2}\,dz$$

can be made as small as desired by taking t_0 sufficiently large, it follows that $I_{\Delta}\, a_1 E/\sqrt{\pi}$ can be made as small as desired.

The problem for two or more integrals differs little from the one just discussed. In fact,

$$B(t) = A_{01} + \int_{-\infty}^{t} \frac{a_1 I(x)}{\sqrt{\pi}\,\sigma_1}\, e^{-(t-x-M_1)^2/2\sigma_1^2}\,dx$$

$$+ A_{11} + \int_{-\infty}^{t} \frac{a_2 F(x)}{\sqrt{\pi}\,\sigma_2}\, e^{-(t-x-M_2)^2/2\sigma_2^2}\,dx$$

can be written in the form

$$B(t) = A_{01} + A_{02}I_0 + I_{\Delta}\,(t - M_1 - \lambda_1)\,\frac{a_1}{\sqrt{\pi}}\int_{-\infty}^{(M_1-t_0)/\sqrt{2}\,\sigma_1} e^{-z^2}\,dz$$

$$+ \int_{t-t_0}^{t} \frac{a_1 I(x)}{\sqrt{2\pi}\,\sigma_1}\, e^{-(t-x-M_1)^2/2\sigma_1^2}\,dx$$

$$+ A_{11} + A_{12}F_0 + F_{\Delta}(t - M_2 - \lambda_2)\,\frac{a_2}{\sqrt{\pi}}\int_{-\infty}^{(M_2-t_0)/\sqrt{2}\sigma_2} e^{-z^2}\,dz$$

$$+ \int_{t-t_0}^{t} \frac{a_2 F(x)}{\sqrt{2\pi}\,\sigma_2} e^{-(t-x-M_2)^2/2\sigma_2^2}\, dx$$

where $F(x) = 1 - A/f(x)$, where $f(x)$ is the foreclosure rate, $A = .108$, that is, the constant determined in Part I, and F_0 and F_Δ are defined in a manner analogous to the manner in which I_0 and I_Δ were defined. Now, what is desired is a t_0 large enough so that

$$H = I_\Delta\,(t-M_1-\lambda_1)\,\frac{a_1}{\sqrt{\pi}} \int_{-\infty}^{(M_1-t_0)/\sqrt{2}\,\sigma_1} e^{-z^2}\, dz$$

$$+ F_\Delta\,(t-M_2-\lambda_2)\,\frac{a_2}{\sqrt{\pi}} \int_{-\infty}^{(M_2-t_0)/\sqrt{2}\sigma_2} e^{-z^2}\, dz$$

will be negligible, so that it will be possible to write

$$B(t) = A_{03} + \frac{A_1}{\sqrt{\pi}\,\sigma_1} \int_{t-t_0}^{t} I(x) e^{-(t-x-M_1)^2/2\sigma_1^2}\, dx$$

$$+ \frac{A_2}{\sqrt{\pi}\,\sigma_2} \int_{t-t_0}^{t} F(x) e^{-(t-x-M_2)^2/2\sigma_2^2}\, dx$$

where A_{03}, A_1, A_2, σ_1, σ_2, M_1 and M_2 are constants. Such a t_0 can obviously be found.

APPENDIX V

Technique for Fitting Formulas Involving Integrals

Formula (11.1) in Chapter VI is itself not readily adaptable to methods of fitting, but a further approximation can be made. For this purpose replace the functions

$$\varphi(f) = \frac{1}{\sqrt{\pi}\,\sigma_i}\, e^{-(t-x-M_i)^2/2\sigma_i^2}, \quad i = 1, 2,$$

by lines as indicated in the accompanying figure. To obtain a function $E_1(t)$ with "moving lag" as illustrated, it is merely necessary to obtain the quantities

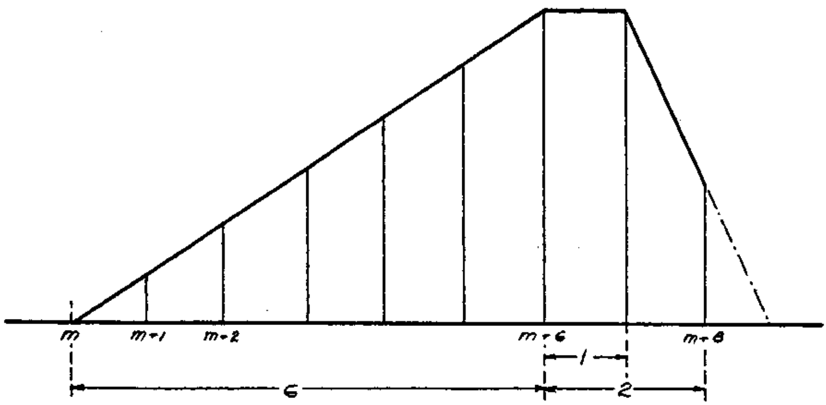

FIGURE 8

$$S_m{}^0 = E(1) + \cdots + E(m)$$

$$S_m{}^1 = mE(1) + \cdots + E(m) =$$

$$\sum_{i=1}^{m} (m - i + 1)E(i),$$

where 1 is the first year in the series, and $m = t - t_0$, since E_1 is given by the formula*

*See Max Sasuly: *Trend Analysis of Statistics,* published by Brookings Institution, 1934, Chapter X.

$$E_1(m+\mu+\nu) = \frac{2}{\mu\,\nu(\mu+\nu+2)} \left\{ \nu\,[S^1_{m-1} - S_{1\,m+\mu-1}] + \mu[S^1_{m+\mu+\nu} - S^1_{m+\mu}] \right\}$$

As a first approximation one can take $m = t - 8$, $\mu = 6$, $\nu = 2$, so that

$$E_1(t) = 1/60\ \{2[S^1_{t-9} - S^1_{t-5}] + 6[S^1_t - S^1_{t-2}]\}\ .$$

For the foreclosure function $F_1(t)$ with distributed lag, the straight line approximation can be taken to be as shown in Figure 9, and then if $m = t - 8$, $\mu = 8$,

$$F_1(f) = 1/36\,(T^1_{t-9} - T^1_t + 9T^0_t)\ ,$$

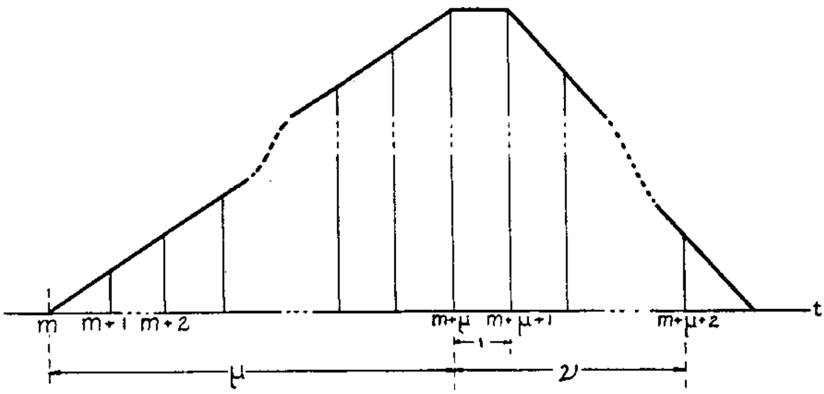

FIGURE 9

where T^0 and T^1 are defined in ways analogous to the ways in which S^0_m and S^1_m were defined, but with the quantities $F(1), \cdots, F(m)$ replacing the quantities $E(1), \cdots, E(m)$.

Now, if A_0, A_1, A_2 and A_3 are the constants determined previously, using lags (formula 7.1), and if the effect of $E(t)$ is small and if, furthermore, $B(t)$ can be represented by the sum of definite integrals which are approximately proportional to $E_1(t)$ and $F_1(t)$ and a constant term, then it should be possible to write as an approximation

$$B(t) = A_{01} + A_{11}K_{01}E_1(t) + A_{21}K_{02}F_1(t)\ ,$$

where K_{01} and K_{02} are effectively constant weight functions, which can be determined by the method of least squares.

When K_{01} and K_{02} are known, the maximum heights of the weighting functions indicated in Figure 8 and Figure 9 are known. It is then possible to replace the broken line weights functions by normal probability functions, thus determining approximate values for M_1, σ_1 and M_2, σ_2 in (11.1) ; i.e., neglecting the effect of $A_1E(t)$.

Once approximate values of σ_i, M_i, $i = 1, 2$, are obtained, the coefficients A_0, A_1, A_2 and A_3 can be determined by the method of least squares, and better values of M_i and σ_i determined by means of a Taylor expansion. When M_1, M_2, σ_1 and σ_2 were corrected once* it was found that

$$A_0 = .443; \quad A_1 = 126.002; \quad A_2 = -11.909; \quad M_1 = 2.25;$$

$$M_2 = 1.75; \quad \sigma_1 = 1.00; \text{and } \sigma_2 = 1.25 \; .$$

*Only a qualitative analysis seemed necessary for the σ_i. One knowing the mechanics of a formula like (11.1) can tell almost by inspection what quantities if recalculated might lead to better "fits". Empirical estimates involve smaller errors than those involved in approximating to the σ_i with the first term of a Taylor expansion.

APPENDIX VI

INADEQUACY OF LAG FORMULAS FOR MAXIMIZING INCOME OVER PERIODS OF TIME

The formula

$$(7.4) \qquad B(t) = [16.63\, I_{-2.5}{}^{.86}\, W_{-2.5} - 1.022\, F + 0.20]\, b \ ,$$

where $W_{-2.5}$ is the chance of not losing the income through foreclosure, etc., and b is the number of family accommodations needed each year, gives an excellent representation of new building as a function of the variables R = rent, T = taxes, C = replacement cost, P = per cent of occupancy, and f = number of foreclosures per year per 100,000 families.

In order to obtain a C that will maximize income — that is, maximize $v = B(t)C(t)$ — it is obviously incorrect to multiply (7.4) by $C(t)$ and differentiate, since C in the bracket above is $C(t-2.5)$ and not $C(t)$. It is natural to ask here if the difficulty can be bridged by means of the calculus of variations,* that is, by maximizing

$$V = \int_0^\tau C(t)B(t)dt \ ,$$

where τ represents some future time, 1, 2, 3, 4, etc., years in advance of present time $t = 0$. This is not possible. The introduction of leads of 2.5 and 2 years in the formula for $B(t)$ makes the calculus of variations problem one which in general does not have a solution. Only a short mathematical analysis is needed to demonstrate this.

For simplicity, write

$$B(t) = G_0(t-2.5)\{C(t-2.5)\}^{-.86} + G_1(t-2) \ ,$$

where

$$G_0(t-2.5) = A_0[(Rp-T)]W$$

and

$$G_1(t-2) = A_1/f(t-2) + A_2 \ ,$$

*See for example, W. F. Osgood, *Advanced Calculus*, New York, 1925.

and suppose that $\tau = 5$.

Then, it is desired to maximize (if possible)

$$V = V_5 = \int_0^5 B(t)C(t)dt = \int_0^5 C(t)[G_0(t-2.5)\{C(t-2.5)\}^{-.86} + G_1(t-2)]dt$$

$$= \int_{2.5}^{5} C(t)[G_0(t-2.5)\{C(t-2.5)\}^{-.86} + G_1(t-2)]dt$$

$$+ \int_{2.5}^{5} C(t)[G_0(t-2.5)\{C(t-2.5)\}^{-.86} + G_1(t-2)]dt \ .$$

Write*

$$(1) \qquad \begin{array}{lll} y_0(t) \text{ for } C(t) \text{ on} & -2.5 \leq t \leq 0 \\ y_1(t) \text{ for } C(t) \text{ on} & 0 \leq t \leq 2.5 \\ y_2(t) \text{ for } C(t) \text{ on} & 2.5 \leq t \leq 5 \end{array}$$

In the second integral put $t = t_1 + 2.5$. Then $C(t-2.5)$ for $2.5 \leq t \leq 5$ becomes $C(t_1)$ for $0 \leq t_1 \leq 2.5$ and $C(t)$ for $2.5 \leq t \leq 5$ becomes $C(t_1 + 2.5)$ for $0 \leq t_1 \leq 2.5$. By means of (1), the integral V_5 becomes

$$V_5 = \int_0^{2.5} y_1[G_0 y_0^{-.86} + G_1]dt + \int_0^{2.5} y_2[G_0^{(1)} y_1^{-.86} + G^{(1)}]dt \ ,$$

where $G_0^{(1)} = G_0(t + 2.5)$ and $G_1^{(1)} = G_1(t + 2.5)$; that is,

$$V_5 = \int_0^{2.5} \sum_{i=1}^{2} y_i(G_0^{(i-1)} y_{i-1}^{-.86} + G_1^{(i-1)})\, dt.$$

Here the function y_0 is known, so that the only variables are y_1 and y_2. If the integrand be denoted by F, the necessary conditions for a maximum are $\partial F/\partial y_1 = \partial F/\partial y_2 = 0$, or,

$$G_0 y_0^{-.86} + G_1 - .86\, G_0^{(1)}\, y_2 y_1^{-1.86} = 0 \ .$$

and

$$G_0 y_1^{-.86} + G_1^{(1)} = 0 \ .$$

*The treatment given here was suggested by Professor Arnold Dresden, Swarthmore College.

From these equations, y_1 and y_2 can in general be determined, but there is a joker. If the period of time had been only two and a half years,

$$V = V_{2.5} = \int_0^{2.5} y_1 [G_0 y_0^{-.86} + G_1] dt \ ,$$

where the only variable is y_1, the first necessary condition is $G_0 y_0^{-.86} + G_1 = 0$ on $(0,2.5)$, and this is, in general, not satisfied. Furthermore, if a nine-year period had been used, still different results would have been obtained. From the point of view of economics this mathematical analysis is entirely relevant.

APPENDIX VII

Problem of Lagrange in the Calculus of Variations

To find the values of f_a satisfying

$$\psi(F_1, F_2, \cdots, F_m, u, p, t) = 0 \ ,$$

where $F_a = f_a u$, so that

$$Q = \int_{t_1}^{t_2} \sum_{a=1}^{m} f_a(u, p, t) p_a(t) u(t) dt$$

is a minimum, it can be assumed that there is a solution

$$\overline{F}_1(t), \cdots, \overline{F}_s(t)$$

and the conditions which this solution must satisfy can then be found.

Let $\mu_1(t), \cdots, \mu_s(t)$ be s functions of t which possess continuous derivatives of the first order on the range $t_1 \leq t \leq t_2$ and vanish when $t = t_1$. Form the functions

$$(1) \qquad F_i(t) = \overline{F}_i(t) + \varepsilon_i \mu_i(t), \qquad i = 1, 2, \cdots, s \ ,$$

where $\varepsilon_1, \cdots, \varepsilon_s$ are parameters. This family of functions evidently contains $\overline{F}_i(t)$ for the parametric values $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_s = 0$. The parameters $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_s$ are not independent since the functions $\overline{F}_1, \overline{F}_2, \cdots, \overline{F}_s$ must satisfy *(2.3)* of Chapter IX; that is,

$$(2) \qquad \psi(\overline{F}_1 + \varepsilon_1 \mu_1, \cdots, \overline{F}_s + \varepsilon_s \mu_s, u, p, t) = 0 \ .$$

When one substitutes the family (1) in Q_f, it becomes a function of $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_s$ as follows:

$$Q(\varepsilon_1, \cdots, \varepsilon_s) = \int_{t_1}^{t_2} \sum_{i=1}^{s} (\overline{F}_i + \varepsilon_i \mu_i) p_i(t) u(t) dt$$

$$+\int_{t_1}^{t_2}\sum_{j=s+2}^{m} F_j(t)p_j(t)u(t)dt \ .$$

This function $Q(\varepsilon_1, \cdots, \varepsilon_s)$ is to be a minimum when $\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_s = 0$ for $\varepsilon_1, \cdots, \varepsilon_s$ satisfying (2). The relation (2) may be taken to define one of the parameters ε_i in terms of the others. Any one of the ε_i may be taken to be this dependent variable. For the sake of definiteness assume that $\varepsilon_1, \cdots \varepsilon_{s-1}$ are the independent variables defining ε_s by (2).

For a minimum of $Q(\varepsilon_1, \cdots, \varepsilon_s)$ the partial derivatives with respect to the independent variables must vanish and hence

$$\partial Q/\partial \varepsilon_j = \int_{t_1}^{t_2} \mu_j(t)p_j(t)u(t)dt$$

$$+ \int_{t_1}^{t_2} \mu_s(t)(\partial \varepsilon_s/\partial \varepsilon_j)p_s \ (t)u(t)dt = 0$$

for $j = 1,2, \cdots, s—1$ when $\varepsilon_1 = \varepsilon_2 = \cdots \varepsilon_s = 0$.

Now a differentiation of (2) yields

$$(\partial \psi/\partial \overline{F}_j)\mu_j(t) + (\partial \psi/\partial \overline{F}_s)\mu_s(t) \ (\partial \varepsilon_s/\partial \varepsilon_j) = 0, \ j = 1,2, \cdots, s—1,$$

so that

$$\mu_s(t)\partial \varepsilon_s/\partial \varepsilon_j = — [(\partial \psi/\partial \overline{F}_j)/ \ (\partial \psi/\partial \overline{F}_s)] \ \mu_j(t) \ .$$

When this latter value is substituted in $\partial Q/\partial \varepsilon_j$ one obtains

$$\int_{t_1}^{t_2} \{p_j(t) — p_s(t)[(\partial \psi/\partial F_j)/ (\partial \psi/\partial F_s)]\}u(t)u_j(t)dt = 0$$

for $j = 1, 2, \ldots, s—1$.

The functions $\mu_j(t)$ are arbitrary except that they possess continuous derivatives of the first order and vanish when $t = t_1$. As a result the coefficient of each $\mu_j(t)$ must vanish for every value of t in the interval (t_1,t_2), that is,

$$(3) \qquad p_j(t) — p_s(t) \ [(\partial \psi/\partial \overline{F}_j)/ \ (\partial \psi/\partial \overline{F}_s)] = 0$$

except at values of t for which $u(t)$ is zero; for, suppose that at a point $t = \tau$ of this interval, the left-hand side were, say, positive.

Since it represents a continuous function (by hypothesis) it will be positive throughout a certain neighborhood of τ. One could then choose $\mu_j(t)$ to be positive throughout this neighborhood and zero everywhere else in the interval (t_1, t_2). It would follow that the integral could not be zero and a contradiction would result.

A system of $(s-1)$ partial differential equations of the type (3) will in general have unique solutions in terms of $(s-1)$ arbitrary constants. In general, therefore, because of the history of the problem, one may assume that these $(s-1)$ equations plus $(s-1)$ independent initial conditions defining the status of the F's at $t = t_1$ uniquely determine the s coefficients of production.

By defining a quantity $\lambda(t)$ by the equation

$$\lambda(t) = (\partial\,\psi/\partial\,\bar{F}_s)p_s(t)$$

the conditions (3) can be written in the more symmetrical form

$$\partial\,\psi/\partial\,\bar{F}_i = \lambda(t)p_i(t)\ , \qquad i = 1,2,\ldots,s\ ,$$

where the additional equation given by $i = s$ may be taken to be the equation of definition of $\lambda(t)$.

INDEX