Appendix for "A Dynamic Theory of Resource Wars"—Not for Publication

Proofs from Section 2

Proof of Lemma 1

For any $\lambda \in (0,1)$, it must be the case that

$$\begin{aligned} \log u'(x) - \log u'(\lambda x) &= \log u'\left(\exp^{\log x}\right) - \log u'\left(\exp^{\log(\lambda x)}\right) \\ &= \int_{\log \lambda x}^{\log x} \left(\frac{d \log u'(\exp^z)}{dz}\right) dz \end{aligned} \tag{A-1}$$

By Assumption 1,

$$\frac{d \log u'(\exp^z)}{dz} = \frac{\exp^z u''(\exp^z)}{u'(\exp^z)} \geq -\frac{1}{\underline{\sigma}}. \tag{A-2}$$

Substitution of (A-2) into (A-1) implies that

$$\log u'(x) - \log u'(\lambda x) \geq \int_{\log \lambda x}^{\log x} \left(-\frac{1}{\underline{\sigma}}\right) dz = \frac{1}{\underline{\sigma}} \log \lambda,$$

which means that

$$u'(\lambda x) < \lambda^{-1/\underline{\sigma}} u'(x). \tag{A-3}$$

To see why this ensures that $V(w(m_t) e_t)$ is bounded from below for any $w(m_t) e_t > 0$, consider the consumption path given by $e_{t+k+1} = \lambda e_{t+k}$ for all $k \geq 1$ for $\lambda \in (0,1)$ with λ chosen such that $\beta \lambda^{-1/\underline{\sigma}} < 1$, where this is possible by Assumption 1.2. Under this consumption path country A consumes $(1-\lambda)\lambda^k w(m_t) e_t$ units of resources at date k and the concavity of $u(\cdot)$ implies that

$$\begin{aligned} u\left((1-\lambda)\lambda^k w(m_t) e_t\right) &> u(w(m_t) e_t) \\ &\quad - u'\left((1-\lambda)\lambda^k w(m_t) e_t\right)\left(1-(1-\lambda)\lambda^k\right) w(m_t) e_t \\ &> u(w(m_t) e_t) - u'\left((1-\lambda)\lambda^k w(m_t) e_t\right) w(m_t) e_t \end{aligned} \tag{A-4}$$

where we have used the fact that $1-(1-\lambda)\lambda^k < 1$. From (A-3), (A-4) becomes

$$u\left((1-\lambda)\lambda^k w(m_t) e_t\right) > u(w(m_t) e_t) - \left(\lambda^{-1/\underline{\sigma}}\right)^k (1-\lambda)^{-1/\underline{\sigma}} u'(w(m_t) e_t) w(m_t) e_t. \tag{A-5}$$

Therefore,

$$V\left(w\left(m_t\right)e_t\right) \geq \sum_{k=0}^{\infty} \beta^k u\left(\left(1-\lambda\right)\lambda^k w\left(m_t\right)e_t\right) \geq$$
$$\frac{u\left(w\left(m_t\right)e_t\right)}{1-\beta} - \frac{\left(1-\lambda\right)^{-1/\underline{\sigma}} u'\left(w\left(m_t\right)e_t\right)w\left(m_t\right)e_t}{1-\beta\lambda^{-1/\underline{\sigma}}} > -\infty.$$

Therefore, $V\left(w\left(m_t\right)e_t\right)$ is bounded from below.■

Proof of Proposition 1

The first-order condition to (8) defines $m^*\left(e\right)$ as

$$l'\left(m\right) = V'\left(w\left(m\right)e\right)w'\left(m\right)e \text{ for all } e. \tag{A-6}$$

Given the solution to (4), the envelope condition implies that

$$V'\left(w\left(m_t\right)e_t\right) = \beta^k u'\left(x_{t+k}\right) \text{ for all } k \geq 0. \tag{A-7}$$

Substitution of (A-7) into (A-6) followed by implicit differentiation yields

$$\left(\frac{l''\left(m_t\right)}{\beta^k u''\left(x_{t+k}\right)w'\left(m_t\right)e_t} - \frac{u'\left(x_{t+k}\right)w''\left(m_t\right)}{u''\left(x_{t+k}\right)w'\left(m_t\right)}\right)\frac{dm_t}{de_t} = \frac{dx_{t+k}}{de_t} + \frac{u'\left(x_{t+k}\right)}{u''\left(x_{t+k}\right)e_t}. \tag{A-8}$$

Summing (5) and (6) gives $\sum_{k=0}^{\infty} x_{t+k} = w\left(m_t\right)e_t$, and differentiating this yields

$$\sum_{k=0}^{\infty}\frac{dx_{t+k}}{de_t} = w\left(m_t\right) + w'\left(m_t\right)e_t\frac{dm_t}{de_t}. \tag{A-9}$$

Taking the sum of (A-8) overall $k \geq 0$ and substituting into (A-9), we obtain

$$\frac{dm_t}{de_t} = \frac{w\left(m_t\right)\left(1+\sum_{k=0}^{\infty}\frac{u'\left(x_{t+k}\right)}{u''\left(x_{t+k}\right)x_{t+k}}\frac{x_{t+k}}{w\left(m_t\right)e_t}\right)}{\sum_{k=0}^{\infty}\left(\frac{l''\left(m_t\right)}{\beta^k u''\left(x_{t+k}\right)w'\left(m_t\right)e_t} - \frac{u'\left(x_{t+k}\right)w''\left(m_t\right)}{u''\left(x_{t+k}\right)w'\left(m_t\right)}\right) - w'\left(m_t\right)e_t}. \tag{A-10}$$

Since the denominator is negative, (A-10) is positive if and only if the numerator is negative. If

$$-u'\left(x_{t+k}\right)/\left(u''\left(x_{t+k}\right)x_{t+k}\right) > 1 \text{ for all } x_{t+k},$$

then the numerator is negative since from (**??**), $\sum_{k=0}^{\infty} \frac{x_{t+k}}{w(m_t)\, e_t} = 1$, and the opposite holds if

$$-u'(x_{t+k}) / \left(u''(x_{t+k})\, x_{t+k}\right) < 1 \text{for all } x_{t+k}.$$

■

Proofs from Section 3

Definition of Strategies at $e_t = 0$ for $u(0) = -\infty$

As noted in the text, when the endowment equals 0 and $u(0) = -\infty$, then in the unperturbed economy the payoff from war and from peace may both equal $-\infty$. We determine whether or not war occurs in this case by explicitly looking at the economy with cost of war $\upsilon > 0$ for country A as specified in Definition 1. Let

$$U^C(e) = \sum_{t=0}^{\infty} \beta^t \left(u(\widetilde{x}_t(e)) - u'(\widetilde{x}_t(e))\, \widetilde{x}_t(e)\right) \tag{A-11}$$

$$\text{for } \{\widetilde{x}_t(e), \widetilde{e}_t(e)\}_{t=0}^{\infty} \text{ which satisfies}$$
$$u'(\widetilde{x}_{t+1}(e)) = (1/\beta)\, u'(\widetilde{x}_t(e)),$$
$$\widetilde{e}_{t+1}(e) = \widetilde{e}_t(e) - \widetilde{x}_t(e), \text{ and } \widetilde{e}_0(e) = e.$$

$U^C(e)$ corresponds to equilibrium welfare of country A in a permanently peaceful competitive equilibrium starting from endowment e at date 0, where $\widetilde{x}_t(e)$ and $\widetilde{e}_t(e)$ correspond to the resource consumption and resource endowment, respectively, at date t in such an equilibrium. The constraint that $\widetilde{x}_t(e) \leq \overline{x}$ is ignored since it does not bind as a consequence of Assumption 1.3.

For cost of war $\upsilon \geq 0$, we define

$$F_\upsilon(e) \equiv U^C(e) - \left(V(w(m^*(e))\, e) - l(m^*(e)) - \upsilon\right). \tag{A-12}$$

$F_\upsilon(e)$ corresponds to the difference in country A's welfare between a permanently peaceful competitive equilibrium and war with optimal armament $m^*(e)$ starting from endowment e when the cost of war is equal to υ. In what follows, we will not separately give the expressions for the case where $\upsilon = 0$, which can be readily obtained from the expressions here by setting $\upsilon = 0$. Following the fourth requirement of the definition of MPCE, we will determine the behavior of country A at zero endowment (when $u(0) = -\infty$) from this function $F_\upsilon(e)$. In particular, given this function, our definition in the text implies:

Observation (**Equilibrium Selection**) Suppose that $f_{t-1} = 0$ and $e_t = 0$. Then $f_t = 0$ only if $\lim_{e \to 0} F_\upsilon(e) > 0$.

Note that this definition also subsumes the case for which $u(0) > -\infty$, as in this case $\lim_{e\to 0} F_\upsilon(e) = \upsilon > 0$ and thus $f_t = 0$ at $e_t = 0$. The following lemma and its corollary are useful to simplify the analysis of country A's equilibrium decisions. Because all of our results in this Appendix are true for any value of $\upsilon > 0$, we do not qualify the next lemma and other lemmas and propositions with "fix some $\upsilon > 0$".

Lemma 4 *Starting from any e_t^*, country A's payoff $U_A(e_t^*)$ must satisfy*

$$U_A(e_t^*) = \max\left\{u\left(x_t^{A*}\right) - p_t^* x_t^{A*} + \beta U_A\left(e_{t+1}^*\right), V\left(w\left(m^*\left(e_t\right)\right)e_t^*\right) - l\left(m^*\left(e_t^*\right)\right) - \upsilon\right\} \quad \text{(A-13)}$$

Proof. By definition of MPCE, $U_A(e_t^*)$ equals (14) for some equilibrium sequence $\left\{e_{t+k}^*, p_{t+k}^*, x_{t+k}^{S*}, x_{t+k}^{A*}\right\}_{k=0}^{\infty}$ which does not depend on m_t chosen by country A. Therefore without loss of generality country A can make a joint decision over choice of (f_t, m_t) to maximize its payoff (14), which would be either setting $f_t = 1$ and $m_t = m^*(e_t)$, or $f_t = 0$ and $m_t = 0$. ■

The immediate implication of this lemma is the following corollary.

Corollary 3 *In any MPCE, without loss of generality country A's strategies in state e can be restricted to choosing no armament and no attack with probability $\mu(e)$ and armament $m^*(e)$ and attack with probability $1-\mu(e)$.*

Proof of Lemma 2

We prove the existence of MPCE using the properties of F_υ. We construct equilibria for three separate cases: (i) $\lim_{e\to 0} F_\upsilon(e) \le 0$; (ii) $\lim_{e\to 0} F_\upsilon(e) > 0$; and there does not exist $e \le e_0$ such that $F_\upsilon(e) < 0$; and (iii) $\lim_{e\to 0} F_\upsilon(e) > 0$ and there exists $e \le e_0$ such that $F_\upsilon(e) < 0$. We prove each case in a separate lemma. Throughout we use the result of Corollary 3 that allows us to restrict strategies of country A to not arm and not attack with probability $\mu(e_t)$ and arm $m^*(e_t)$ and attack with probability $1-\mu(e_t)$

Lemma 5 *If $\lim_{e\to 0} F_\upsilon(e) \le 0$ then there exists an equilibrium in which war occurs in period 0 with probability 1.*

Proof. First, note that if $u(0)$ is finite then $\lim_{e\to 0} F_\upsilon(e) = \upsilon$. Therefore $\lim_{e\to 0} F_\upsilon(e) \le 0$ implies that $u(0) = -\infty$.

We construct an equilibrium (γ^*, μ^*) in which war occurs with probability 1 in period 0. Let $\left\{e_0^*, p_0^*, x_0^{S*}, x_0^{A*}\right\} = \{e_0, u'(e_0), e_0, e_0\}$ and $\left\{e_t^*, p_t^*, x_t^{S*}, x_t^{A*}\right\} = \{0, u'(0), 0, 0\}$ for all $t > 0$. Let $\gamma^* = \left\{e_t^*, p_t^*, x_t^{S*}, x_t^{A*}\right\}_{t=0}^{\infty}$. Let strategies of country A be $\mu^*(e_0) = 0$ and $\mu^*(0) = 0$.

To verify that this is an equilibrium we need to check that country A does not gain from deviating from strategy μ^*. The payoff of country A from choosing no armament and no war in period 0 is given by

$$u(e_0) - u'(e_0)e_0 + U_A(0) = -\infty,$$

where the equality follows from $u(0) = -\infty$. The payoff of country A from playing $\mu^*(e_0)$ is $V(w(m^*(e_0))) - l(m^*(e_0)) - \upsilon > -\infty$, therefore it is the best response for country A to play $\mu(e_0) = 0$. The observation in Section 7 implies that $\mu(e_t) = 0$ is the best response in the states in which $e_t = 0$.

To see that γ^* is an equilibrium, note that $\mu(e_1) = 0$ implies that $\Pr\{f_1 = 0\} = 0$. Then (3), (10), (12), and (13) imply that $\left\{p_0^*, x_0^{S*}, x_0^{A*}\right\} = \{u'(e_0), e_0, e_0\}$ and $e_1^* = 0$, completing the proof. ■

Lemma 6 *If* $\lim_{e\to 0} F_\upsilon(e) > 0$ *and there does not exist* $e \leq e_0$ *such that* $F_\upsilon(e) < 0$, *then there exists an equilibrium with permanent peace.*

Proof. In an equilibrium with permanent peace country A sets $\mu^*(e) = 1$ for all $e \leq e_0$, and equilibrium allocations $\gamma^* = \{\widetilde{e}_t(e_0), u'(\tilde{x}_t(e_0)), \tilde{x}_t(e_0), \tilde{x}_t(e_0)\}_{t=0}^{\infty}$ where $\{\widetilde{e}_t(e_0), \tilde{x}_t(e_0)\}_{t=0}^{\infty}$ are the competitive equilibrium allocations with permanent peace defined in (A-11). At every date t the payoff for country A along the equilibrium path is given by $U^C(\widetilde{e}_t(e_0))$. Since $\widetilde{e}_t(e_0) \leq e_0$ for all t,

$$
\begin{aligned}
0 &\leq F_\upsilon(\widetilde{e}_t(e_0)) \\
&= U^C(\widetilde{e}_t(e_0)) - \left(V(w(m^*(\widetilde{e}_t\,(e_0)))\widetilde{e}_t\,(e_0)) - l(m^*(\widetilde{e}_t\,(e_0))) - \upsilon\right),
\end{aligned}
$$

which implies $\mu^*(\widetilde{e}_t) = 1$ is the best response of country A. Given that country A never attacks, γ^* satisfies optimization conditions (3), (10), (12), and (13). ■

Lemma 7 *If* $\lim_{e\to 0} F_\upsilon(e) > 0$ *and there exists* $e \leq e_0$ *such that* $F_\upsilon(e) < 0$ *then an MPCE exists.*

Proof. Define $\widehat{e} > 0$ s.t. $F_\upsilon(\widehat{e}) = 0$ and $F_\upsilon(e) > 0$ for all $e \in [0, \widehat{e})$. Such $\widehat{e}$ exists because F_υ is continuous, $F_\upsilon(0) > 0$ and $F_\upsilon(e) < 0$ for some e. Let $\widehat{\widehat{e}}$ be defined implicitly by $\widetilde{e}_1(\widehat{\widehat{e}}) = \widehat{e}$. $\widehat{\widehat{e}}$ represents a value of initial endowment of resources such that in competitive equilibrium with permanent peace, remaining resource reserves in period 1 are equal to $\widehat{e}$. We construct equilibria for three different cases depending on the values of $F_\upsilon(e_0)$ and e_0 relative to $\widehat{\widehat{e}}$.

Case 1. Suppose $e_0 \leq \widehat{\widehat{e}}$ and $F_\upsilon(e_0) \geq 0$. We construct an equilibrium with permanent peace.

Define $\gamma^* = \{\widetilde{e}_t(e_0), u'(\tilde{x}_t(e_0)), \tilde{x}_t(e_0), \tilde{x}_t(e_0)\}_{t=0}^{\infty}$ and $\mu_t^*(\widetilde{e}_t(e_0)) = 1$ for all t. The proof of this case is analogous to proof of Lemma 6.

Case 2. Suppose $e_0 \leq \widehat{\widehat{e}}$ and $F_\upsilon(e_0) < 0$. We construct an equilibrium in which war occurs with probability 1 in period 0. In this case define $\gamma^* = \{\widetilde{e}_t(e_0), u'(\tilde{x}_t(e_0)), \tilde{x}_t(e_0), \tilde{x}_t(e_0)\}_{t=0}^{\infty}$ and $\mu_0^*(e_0) = 0$, $\mu_t^*(\widetilde{e}_t(e_0)) = 1$ for all $t > 0$. Given these strategies of country A, (μ^*, γ^*) is an equilibrium for the same reasons as described in the proof of Lemma 6. Since $F_\upsilon(e_0) < 0$, country A obtains higher utility under war and and thus $\mu_0^*(e_0) = 0$ is a best response in period

0. To verify that $\mu_t^*(\widetilde{e}_t(e_0)) = 1$ for all $t > 0$, note that $e_0 \leq \widehat{\widehat{e}}$ implies that $\widetilde{e}_1(e_0) \leq \widetilde{e}_1(\widehat{\widehat{e}}) = \widehat{e}$.[24] Therefore in any period $t > 0$

$$\begin{aligned} & U_A(\widetilde{e}_t(e_0)) - (V(w(m^*(\widetilde{e}_t(e_0)))\widetilde{e}_t(e_0)) - l(m^*(\widetilde{e}_t(e_0))) - \upsilon) \\ = \ & U^C(\widetilde{e}_t(e_0)) - (V(w(m^*(\widetilde{e}_t(e_0)))\widetilde{e}_t(e_0)) - l(m^*(\widetilde{e}_t(e_0))) - \upsilon) \\ = \ & F_\upsilon(\widetilde{e}_t(e_0)) \geq F_\upsilon(\widehat{e}) = 0. \end{aligned}$$

Therefore peace is a dominated strategy for country A in all $t > 0$.

Case 3. Suppose $e_0 > \widehat{\widehat{e}}$. We construct an equilibrium in which resource endowment in period 1 is equal to $\widehat{e}$ followed by permanent peace from $t \geq 2$. Probabilities of war in periods 0 and 1 depend on the initial conditions.

Let

$$\left(e_0^*, p_0^*, x_0^{S*}, x_0^{A*}\right) = \left(e_0, u'\left(e_0 - \widehat{e}\right), e_0 - \widehat{e}, e_0 - \widehat{e}\right)$$

and

$$\left(e_t^*, p_t^*, x_t^{S*}, x_t^{A*}\right) = \left(\widetilde{e}_{t-1}\left(\widehat{e}\right), u'\left(\widetilde{x}_{t-1}\left(\widehat{e}\right)\right), \widetilde{x}_{t-1}\left(\widehat{e}\right), \widetilde{x}_{t-1}\left(\widehat{e}\right)\right) \text{ for all } t \geq 1.$$

Let $\mu^*(e_1^*) = u'(e_0 - \widehat{e})/\beta u'(\tilde{x}_0(\widehat{e}))$. Note that $\mu^*(e_1^*)$ is equal to 1 for $e_0 = \widehat{\widehat{e}}$ and monotonically converges to 0 as $e_0 \to \infty$. Therefore $\mu^*(e_1^*)$ is a well-defined probability. Set $\mu^*(e_t^*) = 1$ for all $t \geq 2$. Under this construction $\left\{e_t^*, p_t^*, x_t^{S*}, x_t^{A*}\right\}_{t=0}^\infty$ satisfies conditions (3), (10), (12), and (13) (since they do not depend on the probability of war in period 0, $\mu^*(e_0)$). To check that constructed strategies are also best response for country A starting from period 1, note that by construction $e_1^* = \widehat{e}$ and $e_t^* < \widehat{e}$ for all $t \geq 2$. Since $F_\upsilon(\widehat{e}) = 0$, country A is indifferent between war and peace and is weakly better off randomizing between the two outcomes with probabilities $\mu^*(e_1^*)$ and $1 - \mu^*(e_1^*)$. Since $e_t^* < \widehat{e}$ for $t \geq 2$, $F_\upsilon(e_t^*) > 0$ for $t \geq 2$, and therefore $\mu^*(e_t^*) = 1$ is a best response analogously to Case 1.

Finally we need to construct $\mu^*(e_0)$. Note that under proposed equilibrium strategies country A is indifferent between permanent peace and attack in period 1, and therefore its payoff period 1 is $U^C(\widehat{e})$. Therefore, if country A does not attack in period 0, its payoff is given by $u\left(e_0 - \widehat{e}\right) - u'\left(e_0 - \widehat{e}\right)\left(e_0 - \widehat{e}\right) + \beta U^C\left(\widehat{e}\right)$. Then we set $\mu^*(e_0) = 1$ if

$$u\left(e_0 - \widehat{e}\right) - u'\left(e_0 - \widehat{e}\right)\left(e_0 - \widehat{e}\right) + \beta U^C\left(\widehat{e}\right) \geq V\left(w\left(m^*\left(e_0\right)\right)e_0\right) - l\left(m^*\left(e_0\right)\right) - \upsilon,$$

and set $\mu^*(e_0) = 0$ otherwise. This completes construction of the equilibrium. ■

[24] This follows, for example, because the competitive equilibrium is efficient and thus equilibrium allocations $\{\tilde{e}_t\}_{t=0}^\infty$ can be found recursively from

$$J(e_t) = \max_{e_{t+1}} u(e_t - e_{t+1}) + \beta J(e_{t+1}).$$

Concavity of J implies that e_{t+1} is increasing in e_t.

Proof of Proposition 3

First we prove a preliminary result about properties of MPCE. By Corollary 3, without loss of any generality, we can restrict attention to only two actions of country A in each period, to not arm and not attack with probability $\mu^*(e_t^*)$ and to arm $m^*(e_t^*)$ and attack with probability $1-\mu^*(e_t)$.

Lemma 8 *Let (γ, μ) be an MPCE. Suppose that $\mu_t^* = \mu^*(e_t^*) > 0$ for all t. Then*

1. *Country A must weakly prefer permanent peace to war,*

$$\sum_{k=0}^{\infty} \beta^k \left(u\left(x_{t+k}^*\right) - p_{t+k}^* x_{t+k}^*\right) \geq V\left(w\left(m^*\left(e_t^*\right)\right) e_t^*\right) - l\left(m^*\left(e_t^*\right)\right) - \upsilon \tag{A-14}$$

for all t, with strict equality if country A attacks with a positive probability (i.e. $\mu(e_t^*) < 1$).

2. *The payoff in the event of no war satisfies*

$$\sum_{k=0}^{\infty} \beta^k \left(u\left(x_{t+k}^*\right) - p_{t+k}^* x_{t+k}^*\right) = K_t e_t^{*1-1/\sigma} - \frac{1}{(1-\beta)(1-1/\sigma)} \tag{A-15}$$

where

$$K_t = \frac{1}{\sigma} \frac{1}{1-1/\sigma} \frac{\left(1 + \sum_{k=1}^{\infty} \beta^k \left(\prod_{l=1}^{k} \left(\beta \mu_{t+l}^*\right)^{\sigma}\right)^{1-1/\sigma}\right)}{\left(1 + \sum_{k=1}^{\infty} \prod_{l=1}^{k} \left(\beta \mu_{t+l}^*\right)^{\sigma}\right)^{1-1/\sigma}}. \tag{A-16}$$

Moreover, K_t is bounded from below, and K_t is bounded from above by

$$K^C = \frac{1}{\sigma} \frac{1}{1-1/\sigma} \left(1-\beta^{\sigma}\right)^{-1/\sigma}. \tag{A-17}$$

3. *(x_t^*, e_t^*) for all $e_t^* > 0$ must satisfy*

$$\frac{x_t^*}{e_t^*} \geq 1 - \beta^{\sigma}. \tag{A-18}$$

4. *Country A's payoff in the event of war satisfies*

$$V\left(w\left(m^*\left(e_t^*\right)\right) e_t^*\right) = w\left(m^*\left(e_t^*\right)\right)^{1-1/\sigma} \left(1-\beta^{\sigma}\right)^{-1/\sigma} \frac{1}{1-1/\sigma} e_t^{*1-1/\sigma} - \frac{1}{(1-\beta)(1-1/\sigma)}. \tag{A-19}$$

Proof. Since peace occurs with a positive probability at any $t+k \geq t$, the equilibrium payoff for country A should be equal to

$$U_A(e_t^*) = u(x_t^*) - p_t^* x_t^* + \beta U_A(e_{t+1}^*).$$

Iterating forward, this implies that

$$U_A(e_t^*) = \sum_{k=0}^{\infty} \beta^k \left(u\left(x_{t+k}^*\right) - p_{t+k}^* x_{t+k}^* \right)$$

for all $t+k \geq 0$. Substitution into (A-13) implies that (A-14) must hold, with strict equality if $\mu\left(e_t^*\right) < 1$. This establishes part (i).

Consider any $\{\mu_t^*\}_{t=0}^{\infty}$ with $\mu_t^* > 0$ for all t. Optimal extraction for firms requires that

$$\mu_{t+1}^* p_{t+1}^* = \frac{1}{\beta} p_t^*. \tag{A-20}$$

If instead $\mu_{t+1}^* p_{t+1}^* > \frac{1}{\beta} p_t^*$, then from condition (12) $x_t^{A*} > 0$ since $p_t^* < \infty$. From (10) $x_t^{S*} = 0$, but this implies that $x_t^{S*} \neq x_t^{A*}$ which violates (13). If instead $\mu_{t+1}^* p_{t+1}^* < \frac{1}{\beta} p_t^*$, then analogous arguments imply that $x_{t+1}^{A*} > 0$ and $x_{t+1}^{S*} = 0$ which violates (13). (A-20) together with (12) implies that

$$x_{t+1}^* = \left(\beta \mu_{t+1}^*\right)^{\sigma} x_t^*. \tag{A-21}$$

Forward substitution on (3) implies that

$$\sum_{k=0}^{\infty} x_{t+k}^* \leq e_t^*. \tag{A-22}$$

(A-22) must bind, since if this were not the case, a firm would be able to increase some x_{t+k}^* by $\epsilon > 0$ and increase its profits. Substitutions of (A-21) into (A-22), noting that the latter binds, yields

$$x_t^* \left(1 + \sum_{k=1}^{\infty} \prod_{l=1}^{k} \left(\beta \mu_{t+l}^*\right)^{\sigma} \right) = e_t^*. \tag{A-23}$$

Equation (A-23) together with the fact that $\mu_t \in (0, 1]$ for all $t > 0$ implies that

$$e_t^* > 0 \text{ and } \frac{x_t^*}{e_t^*} = \frac{1}{1 + \sum_{k=1}^{\infty} \prod_{l=1}^{k} \left(\beta \mu_{t+l}^*\right)^{\sigma}} \geq 1 - \beta^{\sigma} > 0 \text{ for all } t\,. \tag{A-24}$$

Substitution of $p^*_{t+k} = u'\left(x^*_{t+k}\right)$ into (A-15) yields

$$\sum_{k=0}^{\infty} \beta^k \left(\frac{1}{\sigma} \frac{x^{*1-1/\sigma}_{t+k}}{1-1/\sigma} \right) - \frac{1}{(1-\beta)(1-1/\sigma)} = K_t e^{*1-1/\sigma}_t - \frac{1}{(1-\beta)(1-1/\sigma)} \tag{A-25}$$

where we used (A-21) and (A-23) to get (A-16).

We are left to show that K_t is bounded from above and below. The maximization of the left hand side of (A-25) subject to the resource constraint (3) implies that $x^*_{t+1} = \beta^\sigma x^*_t$ so that the maximum of the left hand side of (A-25) is

$$\frac{1}{\sigma} \frac{1}{1-1/\sigma} (1-\beta^\sigma)^{-1/\sigma} e^{*1-1/\sigma}_t - \frac{1}{(1-\beta)(1-1/\sigma)}. \tag{A-26}$$

Since $e^{*1-1/\sigma}_t > 0$ by (A-24), this means that

$$K_t \leq K^C = \frac{1}{\sigma} \frac{1}{1-1/\sigma} (1-\beta^\sigma)^{-1/\sigma}, \tag{A-27}$$

so that K_t is bounded from above. To see that K_t is bounded from below, note that if $\sigma > 1$, (A-16) implies that

$$K_t \geq \frac{1}{\sigma} \frac{1}{1-1/\sigma} (1-\beta^\sigma)^{1-1/\sigma}$$

since

$$\begin{aligned} \frac{\left(1 + \sum_{k=1}^{\infty} \beta^k \left(\prod_{l=1}^{k} \left(\beta \mu^*_{t+l}\right)^\sigma \right)^{1-1/\sigma} \right)}{\left(1 + \sum_{k=1}^{\infty} \prod_{l=1}^{k} \left(\beta \mu^*_{t+l}\right)^\sigma \right)^{1-1/\sigma}} &\geq \frac{1}{\left(1 + \sum_{k=1}^{\infty} \prod_{l=1}^{k} \beta^\sigma \right)^{1-1/\sigma}} \\ &= (1-\beta^\sigma)^{1-1/\sigma} \end{aligned}$$

If instead $\sigma < 1$, then (A-14) implies that under any armament level $m > 0$,

$$K_t e^{*1-1/\sigma}_t - \frac{1}{(1-\beta)(1-1/\sigma)} \geq V(w(m) e^*_t) - l(m) - \upsilon. \tag{A-28}$$

The first order conditions which define (4) imply that $x_{t+1} = \beta^\sigma x_t$ which given (5) and (6) implies that

$$V(w(m) e^*_t) = w(m)^{1-1/\sigma} (1-\beta^\sigma)^{-1/\sigma} \frac{1}{1-1/\sigma} e^{*1-1/\sigma}_t - \frac{1}{(1-\beta)(1-1/\sigma)}. \tag{A-29}$$

Together with (A-28), this means that

$$\begin{aligned}K_t &\geq w(m)^{1-1/\sigma}(1-\beta^\sigma)^{-1/\sigma}\frac{1}{1-1/\sigma}-\frac{l(m)+\upsilon}{e_t^{*1-1/\sigma}}\\ &\geq w(m)^{1-1/\sigma}(1-\beta^\sigma)^{-1/\sigma}\frac{1}{1-1/\sigma}-\frac{l(m)+\upsilon}{e_0^{1-1/\sigma}}\end{aligned}$$

where we have used the fact that $e_t^* \leq e_0$. This means that K_t is bounded from below.

This establishes part (ii) of the lemma. Part (iii) follows from (A-24), and part (iv) follows by substitution of $m^*(e_t)$ in for m in (A-29). ∎

Now we are ready to prove Proposition 3. Here we prove a stronger version of Proposition 3 that shows that if at any node of the game (both on and off equilibrium path) war does not occur with probability 1, then permanent peace must follow after that node.

Proposition 13 *Let (γ^*, μ^*) be an MPCE. Suppose that $\mu^*(e_T^*) > 0$ for some $e_T^* > 0$. Then $\mu^*(e_t) = 1$ for all $t > T$. Moreover, $U_A(e_T^*) = U^C(e_T^*)$ where $U^C(e_T^*)$ is a payoff in permanent peace defined in equation (A-11) and $\{x_t^*\}_{t=T}^{\infty}$ satisfies (16).*

Proof. First, note that using the same arguments as those used in Proposition 2 we can establish that if $\mu^*(e_T^*) > 0$ for some $e_T^* > 0$ then $\mu^*(e_t) > 0$ for all $t > T$. Now substituting from Lemma 8 into equation (A-14), we obtain

$$K_t e_t^{*1-1/\sigma} \geq w(m^*(e_t^*))^{1-1/\sigma}(1-\beta^\sigma)^{-1/\sigma}\frac{1}{1-1/\sigma}e_t^{*1-1/\sigma} - l(m^*(e_t^*)) - \upsilon. \tag{A-30}$$

We now show that (A-30) cannot hold with equality which proves that there cannot be equilibrium randomization by country A between war and peace. Suppose (A-30) holds with equality at some date $t > T$. We consider two cases separately: case 1, when there is some finite date $\hat{T}$ after which country A never attacks, and case 2, when $\mu_t^* < 1$ infinitely often.

Case 1. Suppose there is some $\hat{T}$ such that $\mu_{\hat{T}}^* < 1$ and $\mu_t^* = 1$ for all $t > \hat{T}$. In this case, since country A is indifferent between war and peace at $\hat{T}$ and weakly prefers peace at $\hat{T}-1$ and $\hat{T}+1$ to war using the same armament as at $\hat{T}$, it follows that:

$$K_{\hat{T}+1}e_{\hat{T}+1}^{*1-1/\sigma} \geq w\left(m^*\left(e_{\hat{T}}^*\right)\right)^{1-1/\sigma}(1-\beta^\sigma)^{-1/\sigma}\frac{1}{1-1/\sigma}e_{\hat{T}+1}^{*1-1/\sigma} - l\left(m^*\left(e_{\hat{T}}^*\right)\right) - \upsilon \tag{A-31}$$

$$K_{\hat{T}}e_{\hat{T}}^{*1-1/\sigma} = w\left(m^*\left(e_{\hat{T}}^*\right)\right)^{1-1/\sigma}(1-\beta^\sigma)^{-1/\sigma}\frac{1}{1-1/\sigma}e_{\hat{T}}^{*1-1/\sigma} - l\left(m^*\left(e_{\hat{T}}^*\right)\right) - \upsilon \tag{A-32}$$

$$K_{\hat{T}-1}e_{\hat{T}-1}^{*1-1/\sigma} \geq w\left(m^*\left(e_{\hat{T}}^*\right)\right)^{1-1/\sigma}(1-\beta^\sigma)^{-1/\sigma}\frac{1}{1-1/\sigma}e_{\hat{T}-1}^{*1-1/\sigma} - l\left(m^*\left(e_{\hat{T}}^*\right)\right) - \upsilon \tag{A-33}$$

Since $\mu_t^* = 1$ for all $t \geq \hat{T}+1$, from (A-16), it must be the case that $K_{\hat{T}+1} = K_{\hat{T}} = K^C$ for K^C defined in (A-17), and since $\mu_{\hat{T}} \in (0,1)$, it must be that $K^C > K_{\hat{T}-1}$ since war is chosen with positive probability at $\hat{T}$. Moreover, it must be that

$$K^C - w\left(m^*\left(e^*_{\hat{T}}\right)\right)^{1-1/\sigma}(1-\beta^\sigma)^{-1/\sigma}\frac{1}{1-1/\sigma} < 0$$

in order that (A-32) hold. Equations (A-31) – (A-33) therefore imply that

$$\frac{e^{*1-1/\sigma}_{\hat{T}}}{e^{*1-1/\sigma}_{\hat{T}+1}} \geq 1 \text{ and } \frac{e^{*1-1/\sigma}_{\hat{T}}}{e^{*1-1/\sigma}_{\hat{T}-1}} \geq 1.$$

If $\sigma < 1$, then by (3) this implies that $e^*_{\hat{T}+1} = e^*_{\hat{T}}$ so that $x^*_{\hat{T}} = 0$ which violates (A-18). If instead $\sigma > 1$, then this implies that $e^*_{\hat{T}} \geq e^*_{\hat{T}-1}$ which implies $x^*_{\hat{T}-1} = 0$, which violates (A-18). This establishes that it country A cannot be indifferent between attack and not attack in period T, which implies that it must choose $f_t = 0$ with probability 1.

Case 2. Suppose $\mu^*_t < 1$ infinitely often.

Consider sequence $s^1 = \{\mu^*_t, K_t\}^\infty_{t=0}$ where K_t is defined by (A-16). By Lemma 8, there exists some compact set S such that $(\mu^*_t, K_t) \in S$ for all t. Therefore we can select a convergent subsequence s^2 within s^1 (where K_t converges to some K^*). Consider three consecutive elements of s^2, denoted by $n-1$, n, and $n+1$. Weak preference for peace at $n-1$ and $n+1$ together with indifference to peace at n using armament $m^*(e^*_n)$ implies:

$$K_{n+1}e^{*1-1/\sigma}_{n+1} \geq w(m^*(e^*_n))^{1-1/\sigma}(1-\beta^\sigma)^{-1/\sigma}\frac{1}{1-1/\sigma}e^{*1-1/\sigma}_{n+1} - l(m^*(e^*_n)) - \upsilon \quad \text{(A-34)}$$

$$K_n e^{*1-1/\sigma}_{n} = w(m^*(e^*_n))^{1-1/\sigma}(1-\beta^\sigma)^{-1/\sigma}\frac{1}{1-1/\sigma}e^{*1-1/\sigma}_{n} - l(m^*(e^*_n)) - \upsilon \quad \text{(A-35)}$$

$$K_{n-1}e^{*1-1/\sigma}_{n-1} \geq w(m^*(e^*_n))^{1-1/\sigma}(1-\beta^\sigma)^{-1/\sigma}\frac{1}{1-1/\sigma}e^{*1-1/\sigma}_{n-1} - l(m^*(e^*_n)) - \upsilon \quad \text{(A-36)}$$

Equations (A-34) and (A-35) imply that

$$\left(K_{n+1} - w(m^*(e^*_n))^{1-1/\sigma}(1-\beta^\sigma)^{-1/\sigma}\frac{1}{1-1/\sigma}\right)e^{*1-1/\sigma}_{n+1} \geq \left(K_n - w(m^*(e^*_n))^{1-1/\sigma}(1-\beta^\sigma)^{-1/\sigma}\frac{1}{1-1/\sigma}\right)e^{*1-1/\sigma}_{n} \quad \text{(A-37)}$$

and equations (A-35) and (A-36) imply that

$$\left(K_{n-1} - w(m^*(e^*_n))^{1-1/\sigma}(1-\beta^\sigma)^{-1/\sigma}\frac{1}{1-1/\sigma}\right)e^{*1-1/\sigma}_{n-1} \geq \left(K_n - w(m^*(e^*_n))^{1-1/\sigma}(1-\beta^\sigma)^{-1/\sigma}\frac{1}{1-1/\sigma}\right)e^{*1-1/\sigma}_{n} \quad \text{(A-38)}$$

Note that it cannot be that

$$\lim_{n\to\infty}\left\{K_n - w\left(m^*\left(e_n^*\right)\right)^{1-1/\sigma}\left(1-\beta^\sigma\right)^{-1/\sigma}\frac{1}{1-1/\sigma}\right\} = 0, \tag{A-39}$$

since if this were the case, then given the indifference condition, it would violate (A-35) since $\upsilon > 0$. Therefore, (A-39) cannot hold and the left hand side of (A-39) must be negative for (A-35) to be satisfied. Then (A-37), (A-38) and the fact that K_n converges to some K^* imply that

$$\lim_{n\to\infty}\left(\frac{e_n^{*1-1/\sigma}}{e_{n+1}^{*1-1/\sigma}}\right) \geq 1 \text{ and } \lim_{n\to\infty}\left(\frac{e_n^{*1-1/\sigma}}{e_{n-1}^{*1-1/\sigma}}\right) \geq 1,$$

which given (3) implies that if either $\sigma < 1$ or $\sigma > 1$, then $\lim_{n\to\infty} e_{n+1}^*/e_n^* = 1$, but this violates (A-18) which requires that $e_{t+1}^*/e_t^* \leq \beta^\sigma < 1$ for all t which implies from (3) that $e_{n+1}^*/e_n^* \leq \beta^\sigma$ for all n. This establishes that it is not possible for $\mu^*(e_t^*) < 1$ for $t \geq T$ in an equilibrium in which war continues occurring forever with positive probability, and this completes the proof of the first part of the proposition.

Finally, since country A weakly prefers peace in state e_T^*, $U_A(e_T^*) = U^C(e_T^*)$ and $\{x_t^*\}_{t=T}^\infty$ must satisfy (16). ■

Proof of Proposition 4

To prove this proposition we construct the function F_υ as defined in (A-12) and use Lemmas 5, 6, and 7 to establish the existence of equilibrium in which either war occurs with probability 1 in period 0 or there is a permanent peace depending on the assumptions in Proposition 4. Next we use Proposition 13 to rule out other equilibria. Similarly to the proofs of all preceding lemmas, we use Corollary 3 to restrict our attention to only two strategies for country A, not arm and not attack with probability $\mu(e)$ and arm $m^*(e)$ and attack with probability $1-\mu(e)$.

First we derive payoffs from the permanent peace $U^C(e)$ and war $V\left(w\left(m^*\left(e\right)\right)e\right)$. Set $\mu_t = 1$ for all t and use Lemma 8 to show that

$$U^C(e) = \frac{1}{\sigma}\frac{1}{1-1/\sigma}\left(1-\beta^\sigma\right)^{-1/\sigma} e^{1-1/\sigma} - \frac{1}{\left(1-1/\sigma\right)\left(1-\beta\right)}. \tag{A-40}$$

Then $F_\upsilon(e)$ is equal to

$$F_\upsilon(e) = \frac{1}{1-1/\sigma}\left(1-\beta^\sigma\right)^{-1/\sigma} e^{1-1/\sigma}\left(1/\sigma - \left[w\left(m^*\left(e\right)\right)\right]^{1-1/\sigma}\right) + l(m^*(e)) + \upsilon. \tag{A-41}$$

Part 1: Consider the case when $\sigma > 1$. First we show that there exists a unique $\widehat{e}$ such that $F_\upsilon(e) > 0$ for all $e < \widehat{e}$ and $F_\upsilon(e) < 0$ for all $e > \widehat{e}$. Then it follows immediately from Lemma 6 that there exists an equilibrium that has no war along the equilibrium path if $e_0 < \widehat{e}$ and we show using Lemma 7 there exists an equilibrium in which war occurs with probability 1 in period 0 if $e_0 > \widehat{e}$.

Claim 1. If $\sigma > 1$ then there exists a unique $\widehat{e}$ such that $F_v(e) > 0$ for all $e < \widehat{e}$ and $F_v(e) < 0$ for all $e > \widehat{e}$.

Note that $F_v(0) = v > 0$. Differentiating F_v in (A-41) and using the optimality condition (A-6) for $m^{*\prime}(e)$, we obtain

$$F_v'(e) = (1-\beta^\sigma)^{-1/\sigma} e^{-1/\sigma} \left(1/\sigma - [w(m^*(e))]^{1-1/\sigma}\right). \tag{A-42}$$

If $\sigma > 1$ then from Proposition 1 $m^*(e)$ is increasing in e. Therefore $F_v(e)$ has at most one peak and it can cross zero at most once. If it crosses zero, let $\widehat{e}$ be a solution to $F_v(\widehat{e}) = 0$. If $F_v(e)$ does not cross zero we set $\widehat{e} = \infty$.

Claim 2. If $F_v(e) > 0$ for all $e \le e_0$, then there exists no equilibrium in which war occurs with positive probability.

Claim 2 together with Claim 1 immediately imply that if $\sigma > 1$ and $e_0 < \widehat{e}$ then there exists no equilibrium in which war occurs with positive probability.

Suppose there exists an equilibrium in which war occurs with a positive probability at date 0. More formally, suppose there exists an equilibrium (γ^*, μ^*) such that $\mu^*(e_0) < 1$.

First suppose that $\mu^*(e_1^*) = 0$. In this case (10) and (3) imply that $x_0^{A*} = e_0$ and $e_1^* = 0$. When $\sigma > 1$, then $F_v(0) > 0$, and by Observation 7 $\mu^*(0) = 1$. Therefore $\mu^*(e_1^*) = 1$ leading to a contradiction.

Now suppose that $\mu^*(e_1^*) > 0$. In this case from Proposition 13, $U_A(e_1^*) = U^C(e_1^*)$. Then

$$\begin{aligned} U_A(e_1^*) - (V(w(m^*(e_1^*))e_1^*) - l(m^*(e_1^*)) - v) &= U^C(e_1^*) - (V(w(m^*(e_1^*))e_1^*) - l(m^*(e_1^*)) - v) \\ &= F_v(e_1^*) > F_v(\widehat{e}) = 0, \end{aligned}$$

where the strict inequality follows from the definition $\widehat{e}$. This implies that peace is strictly preferred to attack and therefore $\mu^*(e_1^*) = 1$.

If $\mu^*(e_1^*) = 1$ so that peace occurs with probability 1 in period 1, then $\left(x_0^{A*}, p_0^*\right) = (\tilde{x}_0(e_0), u'(\tilde{x}_0(e_0)))$ where $\tilde{x}_0(e_0)$ is a permanent peace allocation defined in (A-11), and $e_1^* = \tilde{e}_1(e_0)$. Since country A attacks in period 0 with positive probability, it must be true that

$$V(w(m^*(e_0))e_0) - l(m^*(e_0)) - v \ge u(x_0^{A*}) - p_0^* x_0^{A*} + \beta U_A(e_1^*). \tag{A-43}$$

Substitute $\left(x_0^{A*}, p_0^*\right) = (\tilde{x}_0(e_0), u'(\tilde{x}_0(e_0)))$ and $U_A(e_1^*) = U^C(e_1^*)$ into equation (A-43) and regroup terms to get

$$\begin{aligned} 0 &\ge u(\tilde{x}_0(e_0)) - u'(\tilde{x}_0(e_0))\tilde{x}_0(e_0) + \beta U^C(\tilde{e}_1(e_0)) - (V(w(m^*(e_0))e_0) - l(m^*(e_0)) - v) \\ &= U^C(e_0) - (V(w(m^*(e_0))e_0) - l(m^*(e_0)) - v) \\ &= F_v(e_0) > 0 \end{aligned}$$

which is a contradiction. Therefore there cannot exist an equilibrium with $\mu^*(e_0) > 0$ and Lemma 6 establishes existence of equilibrium with $\mu^*(e_0) = 1$.

Claim 3. If $\sigma > 1$ and $e_0 > \widehat{e}$, then there exists no equilibrium in which peace occurs with positive probability in period 0.

Suppose $e_0 > \widehat{e}$ and there exists an equilibrium in which country A chooses peace with positive probability in period 0, i.e., $\mu^*(e_0) > 0$. By Proposition 13,

$$\begin{aligned} 0 &\leq U_A(e_0) - (V(w(m^*(e_0))e_0) - l(m^*(e_0)) - \upsilon) \\ &= U^C(e_0) - (V(w(m^*(e_0))e_0) - l(m^*(e_0)) - \upsilon) = F_\upsilon(e_0) < 0 \end{aligned}$$

which is a contradiction. Therefore in any MPCE $\mu^*(e_0) = 0$.

Part 2: Suppose $\sigma < 1$ and let $\widehat{w} = (1/\sigma)^{1/(1-1/\sigma)}$. By construction, $\widehat{w} \in (0,1)$.

Claim 4. If $\sigma < 1$ and $\lim_{m \to \bar{m}} w(m) < \widehat{w}$, then there exists no equilibrium in which war occurs with positive probability.

We prove that in this case $F_\upsilon(e) > 0$ for all e, so that we can apply Claim 2 of the proof of this proposition directly to establishes this result.

In order to prove that $F_\upsilon(e) > 0$ for all e, we show that $F_\upsilon'(e) < 0$ for all e and that $\lim_{e\to\infty} F(e) > 0$. We can establish that $F_\upsilon'(e) < 0$ for all e from (A-42); this is true given that $w(m^*(e)) < \widehat{w}$ for all e. To establish that $\lim_{e\to\infty} F_\upsilon(e) > 0$, consider first the value of $\lim_{e\to\infty} m^*(e)$. Suppose that $\lim_{e\to\infty} m^*(e) = \underline{m} > 0$. Since $m^*(e)$ is the optimal armament, it must satisfy (A-6). The first order condition which characterizes (8) taking into account (4) and (17) implies

$$(1-\beta^\sigma)^{-1/\sigma} e^{1-1/\sigma} = \frac{l'(m^*(e))}{[w(m^*(e))]^{-1/\sigma} w'(m^*(e))}. \tag{A-44}$$

If $\lim_{e\to\infty} m^*(e) = \underline{m} > 0$, then this would violate (A-44) since the left-hand side of (A-44) would converge to 0 whereas the right-hand side of (A-44) would converge to a positive number. Therefore, $\lim_{e\to\infty} m^*(e) = 0$ which implies that

$$\lim_{e\to\infty} (V(w(m^*(e))e) - l(m^*(e)) - \upsilon) = -\frac{1}{(1-\beta)(1-1/\sigma)} - \upsilon, \tag{A-45}$$

so that $\lim_{e\to\infty} F_\upsilon(e) = \upsilon > 0$.[25] This establishes that $F_\upsilon(e) > 0$ for all e. Claim 4 then follows from Claim 2.

[25] (A-45) follows because by definition

$$V(w(m^*(e))e) - l(m^*(e)) - \upsilon \leq -\frac{1}{(1-\beta)(1-1/\sigma)} - \upsilon$$

and because optimality of $m^*(e)$ requires

$$\lim_{e\to\infty} (V(w(m^*(e))e) - l(m^*(e)) - \upsilon) \geq \lim_{e\to\infty} (V(\epsilon e) - l(\epsilon) - \upsilon) = -\frac{1}{(1-\beta)(1-1/\sigma)} - l(\epsilon) - \upsilon$$

for any $\epsilon > 0$ chosen to be arbitrarily small.

Claim 5. If $\sigma < 1$ and $\lim_{m \to \bar{m}} w(m) > \widehat{w}$, then there exists no equilibrium in which peace occurs with positive probability in period 0.

First we show that in this case $\lim_{e \to 0} F_\upsilon(e) = -\infty$. The existence of the pure-strategy equilibrium with immediate war then follows from Lemma 5 and we will use Proposition 13 to rule out existence of equilibria with a positive probability of peace in period 0.

Let us show that $\lim_{e \to 0} F_\upsilon(e) = -\infty$. Note that when $\sigma < 1$, Proposition 1 that $m^*(e)$ is decreasing in e. Suppose that $\lim_{e \to 0} m^*(e) = \overline{m}' < \overline{m}$. This would violate (A-44) since the left-hand side of (A-44) approaches ∞ as e approaches 0, whereas the right-hand side of (A-44) approaches $l'(\overline{m}') / \left[[w(\overline{m}')]^{-1/\sigma} w'(\overline{m}') \right] < \infty$, yielding a contradiction. Therefore $\lim_{e \to 0} m^*(e) = \overline{m}$ and $\lim_{e \to 0} w(m^*(e)) > \widehat{w}$. Now consider $\lim_{e \to 0} F_\upsilon(e)$ which satisfies:

$$\lim_{e \to 0} F_\upsilon(e) = \lim_{e \to 0} \left(V\left(w\left(m^*(e)\right) e\right) - l\left(m^*(e)\right) - \upsilon\right) \left(\frac{U^C(e)}{V\left(w\left(m^*(e)\right) e\right) - l\left(m^*(e)\right) - \upsilon} - 1 \right). \tag{A-46}$$

The first term on the right-hand side of (A-46) converges to $-\infty$. The limit of the second term is positive since after substituting $U^C(e)$ from (A-40) and $V(w(m^*(e))e)$ from (A-19) and applying the L'Hopital's rule (together with the optimality condition (A-6)), we obtain

$$\begin{aligned} \lim_{e \to 0} \frac{U^C(e)}{V\left(w\left(m^*(e)\right) e\right) - l\left(m^*(e)\right) - \upsilon} &= \lim_{e \to 0} \frac{dU^C(e)/de}{d\left(V\left(w\left(m^*(e)\right) e\right) - l\left(m^*(e)\right) - \upsilon\right)/de} \\ &= \frac{1/\sigma}{\lim_{m \to \overline{m}} \left[w(m)\right]^{1-1/\sigma}} > 1. \end{aligned}$$

Therefore $\lim_{e \to 0} F_\upsilon(e) = -\infty$. Since $\lim_{e \to 0} F_\upsilon(e) = -\infty$ and F_υ is continuous, there exists $\widehat{e} > 0$ such that $F_\upsilon(e) < 0$ for all $e < \widehat{e}$.

Now we are ready to prove that there exist no equilibrium in which peace occurs with a positive probability in period 0. Suppose such an equilibrium (γ^*, μ^*) exists with $\mu^*(e_0) > 0$. In this case by Proposition 13, $\mu^*(e_t^*) = 1$ for all $t > 0$ and $x_t^{A*} = \tilde{x}_t(e_0)$ for all t. From the proof of Lemma 8 it follows that $e_t^* = \tilde{e}_t(e_0) = \beta^{\sigma t} e_0$. Therefore there exists some T such that $e_T^* < \widehat{e}$. Since peace is the best response for country A in state e_T^*, its payoff $U_A(e_T^*)$ should be greater then the payoff from war, so that

$$\begin{aligned} 0 &\leq U_A(e_T^*) - \left(V\left(w\left(m^*\left(e_T^*\right)\right) e_T^*\right) - l\left(m^*\left(e_T^*\right)\right) - \upsilon\right) \\ &= U^C(e_T^*) - \left(V\left(w\left(m^*\left(e_T^*\right)\right) e_T^*\right) - l\left(m^*\left(e_T^*\right)\right) - \upsilon\right) \\ &= F_\upsilon(e_T^*) < 0, \end{aligned}$$

where the last inequality follows from the fact that $e_T^* < \widehat{e}$. This is a contradiction.■

Proof of Proposition 5

We establish this result is several steps. The following preliminary lemma is useful since it implies that the payments made by country A which equal $u'(x)\,x$ in equilibrium rise to infinity as resource consumption x declines to zero. It also implies that the utility of 0 resource consumption is $-\infty$.

Lemma 9 *Suppose that there exists some* $\overline{\sigma} < 1$ *such that* $-u'(x)/(xu''(x)) \leq \overline{\sigma}$ *for all* $x \geq 0$. *Then (i)* $\lim_{x\to 0} u'(x)\,x = \infty$ *and (ii)* $u(0) = -\infty$.

Proof. Part 1. For any $x \in (0,1)$, it must be the case that

$$\begin{aligned} \log u'(1) - \log u'(x) &= \log u'\left(\exp^{\log 1}\right) - \log u'\left(\exp^{\log x}\right) \\ &= \int_{\log x}^{\log 1} \left(\frac{d \log u'(\exp^{z})}{dz}\right) dz. \end{aligned} \tag{A-47}$$

Analogous arguments to those in the proof of Lemma 1 imply that since $xu''(x)/u'(x) \leq -1/\overline{\sigma}$, it must be the case that (A-47) implies that

$$\log u'(1) - \log u'(x) \leq -\left(\log(1) - \log(x)\right)/\overline{\sigma},$$

which means that

$$u'(x) \geq u'(1)\,x^{-1/\overline{\sigma}}. \tag{A-48}$$

Therefore,

$$u'(x)\,x \geq u'(1)\,x^{1-1/\overline{\sigma}}. \tag{A-49}$$

The right hand side of (A-49) approaches ∞ as x approaches 0 since $1 - 1/\overline{\sigma} < 0$. Therefore, given (A-49), it must be that $\lim_{x\to 0} u'(x)\,x = \infty$.

Part 2. The concavity of $u(\cdot)$ implies that for any $\alpha \in (0,1)$ and $x > 0$,

$$u(x) - u(\alpha x) \geq u'(x)\,x\,(1-\alpha). \tag{A-50}$$

Suppose that $u(0)$ is finite. Then the left hand side of (A-50) approaches 0 as x approaches 0. However, by part 1, the right hand side of (A-50) approaches ∞ as x approaches 0. This means that $u(0)$ cannot be finite so that $u(0) = -\infty$. ■

We can show that if $\lim_{m\to\bar{m}} w(m)$ is sufficiently close to 1, there does not exist an equilibrium in which there is a positive probability of peace for all t. To make this argument, note that Lemma 4 and Corollary 3 both hold in the case with an extraction limit so that country A at e_t chooses $\mu^*(e_t)$, where $\mu^*(e_t)$ corresponds to the probability of peace with zero armament and $1 - \mu^*(e_t)$ corresponds to the probability of war with armament $m^*(e_t)$. Given $e_t^* > 0$, define

$\left\{\widetilde{\widetilde{x}}_{t+k}\left(e_t^*\right)\right\}_{k=0}^{\infty}$ as follows:

$$\left\{\widetilde{\widetilde{x}}_{t+k}\left(e_t^*\right)\right\}_{k=0}^{\infty} = \arg \max_{\{x_{t+k}\}_{k=0}^{\infty}} \sum_{k=0}^{\infty} \beta^k \left(u\left(x_{t+k}\right) - u'\left(x_{t+k}\right) x_{t+k}\right) \text{ s.t. } \sum_{k=0}^{\infty} x_{t+k} = e_t^*. \quad \text{(A-51)}$$

Lemma 10 *Suppose there exists some $\overline{\sigma} < 1$ such that $-u'(x) / (xu''(x)) \leq \overline{\sigma}$ for all $x \geq 0$, and suppose that $\lim_{m \to \overline{m}} w(m)$ is sufficiently close to 1. Then the following must be true:*

1. *There does not exist an MPCE with $\mu_t^* > 0$ for all t.*
2. *$\lim_{e \to 0} F_{\upsilon}(e) \leq 0$ so that $\mu^*(0) = 0$.*
3. *Suppose there exists an MPCE in which war occurs with probability 1 before some finite date T. Then it is necessary that $x_t = \overline{x}$ if war has not yet occurred (i.e., if $f_t = 0$).*

Proof. Part 1. We prove this in three steps.

Step 1. Suppose that $\mu_t^* = \mu^*(e_t^*) > 0$ for all t. Then country A must weakly prefer permanent peace to war at all dates so that

$$\sum_{k=0}^{\infty} \beta^k \left(u\left(x_{t+k}^*\right) - p_{t+k}^* x_{t+k}^*\right) \geq V\left(w\left(m^*\left(e_t^*\right)\right) e_t^*\right) - l\left(m^*\left(e_t^*\right)\right) - \upsilon \quad \text{(A-52)}$$

for all e_t^* along the equilibrium path. This is because since peace occurs with a positive probability at any $t+k \geq t$, the equilibrium payoff for country A should be equal to

$$U_A(e_t^*) = u(x_t^*) - p_t^* x_t^* + \beta U_A(e_{t+1}^*).$$

Iterating forward, this implies that

$$U_A(e_t^*) = \sum_{k=0}^{\infty} \beta^k \left(u\left(x_{t+k}^*\right) - p_{t+k}^* x_{t+k}^*\right)$$

for all $t + k \geq 0$. Substitution into (A-13) implies that (A-52) must hold.

Step 2. If (A-52) holds at e_t^*, then the below inequality also holds at e_t^* for any $m \in (0, \overline{m}]$

$$\sum_{k=0}^{\infty} \beta^k \left(w\left(m\right)^{-1/\underline{\sigma}}\left(1 - w\left(m\right)\right) - 1\right) u'\left(\widetilde{\widetilde{x}}_{t+k}\left(e_t^*\right)\right) \widetilde{\widetilde{x}}_{t+k}\left(e_t^*\right) \geq -l\left(m\right) - \upsilon \quad \text{(A-53)}$$

for $\underline{\sigma}$ defined in Assumption 1. To see why, note that given (A-51) and the the fact that $p_{t+k}^* = u'\left(x_{t+k}^*\right)$ from (12), the left hand side of (A-52) is bounded from above as follows:

$$\sum_{k=0}^{\infty} \beta^k \left(u\left(x_{t+k}^*\right) - p_{t+k}^* x_{t+k}^*\right) \leq \sum_{k=0}^{\infty} \beta^k \left(u\left(\widetilde{\widetilde{x}}_{t+k}\left(e_t^*\right)\right) - u'\left(\widetilde{\widetilde{x}}_{t+k}\left(e_t^*\right)\right) \widetilde{\widetilde{x}}_{t+k}\left(e_t^*\right)\right). \quad \text{(A-54)}$$

The right hand side of (A-52) is bounded from below as follows for all $m \in (0, \overline{m}] > 0$:

$$V\left(w\left(m^*\left(e_t^*\right)\right)e_t^*\right) - l\left(m^*\left(e_t^*\right)\right) - \upsilon \geq V\left(w\left(m\right)e_t\right) - l\left(m\right) - \upsilon \tag{A-55}$$
$$\geq \sum_{k=0}^{\infty} \beta^k u\left(w\left(m\right)\tilde{\tilde{x}}_{t+k}\left(e_t^*\right)\right) - l\left(m\right) - \upsilon.$$

The first inequality in (A-55) follows from the fact that country A can choose to go to war with any feasible m. The second inequality in (A-55) follows from the fact that, conditional on m, $\{x_{t+k}\}_{k=0}^{\infty} = \left\{w\left(m\right)\tilde{\tilde{x}}_{t+k}\left(e_t^*\right)\right\}_{k=0}^{\infty}$ for all $k \geq 0$ is a feasible solution to (4). Moreover, the concavity of $u\left(\cdot\right)$ implies that

$$u\left(w\left(m\right)\tilde{\tilde{x}}_{t+k}\left(e_t^*\right)\right) > u\left(\tilde{\tilde{x}}_{t+k}\left(e_t^*\right)\right) - u'\left(w\left(m\right)\tilde{\tilde{x}}_{t+k}\left(e_t^*\right)\right)\tilde{\tilde{x}}_{t+k}\left(e_t^*\right)\left(1 - w\left(m\right)\right). \tag{A-56}$$

Combination of (A-52) with (A-54), (A-55), and (A-56) implies that

$$\sum_{k=0}^{\infty} \beta^k \left(\left(u'\left(w\left(m\right)\tilde{\tilde{x}}_{t+k}\left(e_t^*\right)\right)\left(1 - w\left(m\right)\right) - u'\left(\tilde{\tilde{x}}_{t+k}\left(e_t^*\right)\right)\right)\tilde{\tilde{x}}_{t+k}\left(e_t^*\right)\right) \geq -l\left(m\right) - \upsilon. \tag{A-57}$$

To see why (A-57) implies (A-53), note that analogous arguments to those of Lemma 9 imply that since $-u'\left(x\right)/xu''\left(x\right) \geq \underline{\sigma}$ for all x, it must be that given $w\left(m\right) \in (0,1)$:

$$\log u'\left(\tilde{\tilde{x}}_{t+k}\left(e_t^*\right)\right) - \log u'\left(w\left(m\right)\tilde{\tilde{x}}_{t+k}\left(e_t^*\right)\right) \geq -\left(\log\left(\tilde{\tilde{x}}_{t+k}\left(e_t^*\right)\right) - \log\left(w\left(m\right)\tilde{\tilde{x}}_{t+k}\left(e_t^*\right)\right)\right)/\underline{\sigma},$$

so that

$$u'\left(w\left(m\right)\tilde{\tilde{x}}_{t+k}\left(e_t^*\right)\right) \leq w\left(m\right)^{-1/\underline{\sigma}} u'\left(\tilde{\tilde{x}}_{t+k}\left(e_t^*\right)\right). \tag{A-58}$$

Substitution of (A-58) into (A-57) implies (A-53).

Step 3. We now prove that there does not exist an MPCE with $\mu_t^* = \mu^*\left(e_t^*\right) > 0$ for all t since (A-53) cannot hold for all e_t^*. We establish that e_t^* must converge to zero and we prove that (A-53) cannot hold as e_t^* approaches 0. Suppose that e_t^* did not converge to zero. From (3) this would imply that x_t^* converges to zero so that $x_t^* < \overline{x}$ for some t. However, if this is the case, then a firm would be able to increase some x_t^* by $\epsilon > 0$ arbitrarily small along the equilibrium path and increase its profits. Therefore, e_t^* must converge to zero. Now consider (A-53) for some $m \in (0, \overline{m})$ as e_t^* converges to zero. Since $\lim_{m \to \bar{m}} w\left(m\right)$ is sufficiently close to 1, m can be chosen such that

$$w\left(m\right)^{-1/\underline{\sigma}}\left(1 - w\left(m\right)\right) - 1 < 0 \tag{A-59}$$

since $w\left(m\right)^{-1/\underline{\sigma}}\left(1 - w\left(m\right)\right) - 1 = -1 < 0$ if $w\left(m\right) = 1$. The right hand side of (A-53) is bounded from below by some finite number as e_t^* converges to zero. Now consider the left hand side of (A-53). Since feasibility requires $\tilde{\tilde{x}}_{t+k}\left(e_t^*\right) \leq e_t^*$, and since e_t^* converges to 0, it follows that

$\widetilde{\widetilde{x}}_{t+k}\left(e_t^*\right)$ converges to 0. >From Lemma 9, this implies that $u'\left(\widetilde{\widetilde{x}}_{t+k}\left(e_t^*\right)\right)\widetilde{\widetilde{x}}_{t+k}\left(e_t^*\right)$ approaches ∞ as e_t^* approaches 0. Given (A-59), this implies that the left hand side of (A-53) approaches $-\infty$ as e_t^* approaches 0. Therefore, (A-53) cannot hold as e_t^* approaches 0.

Part 2. Consider $F_\upsilon(e)$ defined in (A-12). Since $U^C\left(e_t^*\right)$ is bounded from above by the right hand side of (A-54) it follows that analogous arguments to those of part (i) imply that $\lim_{e\to 0} F_\upsilon(e) = -\infty \leq 0$. Therefore, by the observation in Section 7, $\mu^*(0) = 0$.

Part 3. This is proved by backward induction. Let T correspond to the earliest date at which war occurs with probability 1. We prove that $x_{T-1}^* = \overline{x}$, and we follow the argument by proving that if $x_{t+1}^* = \overline{x}$, then it is necessary for $x_t^* = \overline{x}$ for all $t+1 \leq T-1$.

Since T is the earliest date with war with probability 1, this means that $\mu^*\left(e_T^*\right) = 0$ and $\mu^*\left(e_t^*\right) > 0$ for $t \leq T-1$. Since $\mu^*(0) = 1$ by part (ii), it must be that $e_{T-1}^* > 0$. Since country A weakly prefers peace to war at $T-1$, this implies that

$$u\left(x_{T-1}^*\right) - p_{T+1}^* x_{T+1}^* + \beta\left(V\left(w\left(m^*\left(e_T^*\right)\right)\right) - l\left(m^*\left(e_T\right)\right) - \upsilon\right) \geq V\left(w\left(m^*\left(e_{T-1}^*\right)\right)\right) - l\left(m^*\left(e_{T-1}\right)\right) - \upsilon, \tag{A-60}$$

where the right hand side of (A-60) exceeds $-\infty$ since $e_{T-1}^* > 0$. Consider firm behavior at $T-1$. Given (10), it follows that firms choose $x_{T-1}^* = \min\left\{e_{T-1}^*, \overline{x}\right\}$. Suppose it were the case that $e_{T-1}^* \leq \overline{x}$. Then this would imply from (3) that $e_T^* = 0$. However, given Lemma 9 and given (4), this implies that $V\left(w\left(m^*\left(e_T^*\right)\right)\right) - l\left(m^*\left(e_T\right)\right) = -\infty$, which means that the left hand side of (A-60) equals $-\infty$ which is below the right hand side of (A-60), leading to a contradiction. Therefore, $x_{T-1}^* = \overline{x}$.

Now suppose that $x_{t+1}^* = \overline{x}$ for $t+1 \leq T-1$. Since $x_t^* \leq \overline{x}$, this implies that $u'\left(x_t^*\right) > \beta u'\left(x_{t+1}^*\right)\mu^*\left(e_{t+1}^*\right)$. Since (12) implies that $u'\left(x_t\right) = p_t$, this means that $p_t > \beta p_{t+1}\mu^*\left(e_{t+1}^*\right)$ so that (10) implies that $x_t^* = \overline{x}$. Forward iteration on this argument implies that $x_t = \overline{x}$ if war has not yet occurred. ■

Lemma 10 implies that if an MPCE exists, then war occurs with probability 1 before some finite date T with $x_t = \overline{x}$ if war has not yet occurred (i.e., if $f_t = 0$). This establishes part (ii) of Proposition 5. It also establishes part (iii) since it is not possible for $x_t = \overline{x} > e_0$ for any t by (3). We are left to prove part (i) by showing that an MPCE exists.

We construct an equilibrium (γ^*, μ^*). Let

$$\left\{e_t^*, p_t^*, x_t^{S*}, x_t^{A*}\right\} = \left\{\begin{array}{c}\max\left\{e_0 - t\overline{x}, 0\right\}, u'(\min\left\{\max\left\{e_0 - t\overline{x}, 0\right\}, \overline{x}\right\}),\\ \min\left\{\max\left\{e_0 - t\overline{x}, 0\right\}, \overline{x}\right\}, \min\left\{\max\left\{e_0 - t\overline{x}, 0\right\}, \overline{x}\right\}\end{array}\right\} \tag{A-61}$$

for all $t \geq 0$. (A-61) implies that at date t, firms extract $\overline{x}$ if $\overline{x}$ is below e_t, and they otherwise extract e_t. Given this sequence, we can define the strategy of country A as follows. If $e_t = 0$, let $\mu^*(e_t) = 0$. If instead $e_t > 0$, let $\mu^*(e_t)$ correspond to the highest value of $\mu\left(e_t\right) \in \{0, 1\}$ which solves the following program given e_t:

$$\max_{\{\mu(e_{t+k})\}_{k=0}^{\infty}} \sum_{k=0}^{\infty} \beta^k \left(\begin{array}{c} \mu(e_{t+k})(u(x_{t+k}) - u'(x_{t+k})x_{t+k}) \\ +(1-\mu(e_{t+k})) \prod_{l=0}^{k-1} \mu(e_{t+l})(V(w(m^*(e_{t+k}))e_{t+k}) - l(m^*(e_{t+k})) - \upsilon) \end{array} \right) \tag{A-62}$$

s.t.

$$x_{t+k} = \min\{\max\{e_t - k\overline{x}, 0\}, \overline{x}\} \text{ for all } k, \tag{A-63}$$

$$e_{t+k} = \max\{e_t - k\overline{x}, 0\} \text{ for all } k, \tag{A-64}$$

$$\mu(e_{t+k}) = 0 \text{ if } \mu(e_{t+k-1}) = 0 \;\; k, \tag{A-65}$$

$$\mu(e_{t+k}) = 0 \text{ if } e_{t+k} = 0 \text{ for all } k\text{, and} \tag{A-66}$$

$$\mu(e_{t+k}) \in \{0,1\} \text{ for all } k. \tag{A-67}$$

This value of $\mu^*(e_t)$ exists since the objective maximized in (A-62) is well defined. To see why, note that this objective is bounded from below since $\mu(e_t) = 0$ is a potential solution which yields $V(w(m^*(e_t))e_t) - l(m^*(e_t))$ which is well defined. To see that this objective is bounded from above, note that from (A-63) and (A-64), $u(x_{t+k}) - u'(x_{t+k})x_{t+k} \leq u(e_t)$. Moreover, since $l(m^*(e_{t+k})) \geq 0$ and $e_{t+k} \leq e_t$,

$$V(w(m^*(e_{t+k}))e_{t+k}) - l(m^*(e_{t+k})) \leq V(e_{t+k}) \leq V(e_t) \leq u(e_t)/(1-\beta),$$

where the last inequality uses the fact that $u(x_{t+k}) \leq u(e_t)$ in (4) which defines $V(e_t)$.

To verify that this is an equilibrium we need to check that country A does not gain from deviating from strategy μ^*. By part (ii) of Lemma 10, $\mu(e_t) = 0$ is the best response if $e_t = 0$. Given this strategy and given (A-61), the program in (A-62)−(A-67) corresponds to the objective with a maximum equal to $U_A(e_t)$. Therefore, the value of $\mu(e_t)$ which solves (A-62) − (A-67) is optimal.

To see that γ^* is an equilibrium, we need only check (10) since (3), (12), and (13) are satisfied if $x_t^* = \min\{e_t^*, \overline{x}\}$ for all t under (A-61). To show that firm behavior given prices and future war probabilities is optimal with $x_t^* = \min\{e_t^*, \overline{x}\}$, we consider three cases.

Case 1. Suppose that $e_t^* = 0$. Then $x_t^* = \min\{e_t^*, \overline{x}\}$ is the only feasible firm strategy as a consequence of (3).

Case 2. Suppose that $e_t^* > 0$ and $\mu^*(e_{t+1}^*) = 0$ so that war occurs with probability 1 at $t+1$. Given (10), it follows that firms choose $x_t^* = \min\{e_t^*, \overline{x}\}$.

Case 3. Suppose that $e_t^* > 0$ and $\mu^*(e_{t+1}^*) > 0$ so that peace occurs with some probability at $t+1$. We can prove that in this case, $x_t^* = \overline{x}$. To this end, the following claim is useful.

Claim 1. Suppose that $e_t^* > 0$ and $\mu^*(e_{t+1}^*) > 0$. Then the following must be true of μ^*. There exists some $k' > 1$ where $e_{t+k'}^* > 0$, $\mu^*\left(e_{t+k'}^*\right) = 0$, and $\mu^*\left(e_{t+k}^*\right) > 0$ for $1 \leq k < k'$. To see why, suppose it were not the case that $\mu^*\left(e^*_{t+k'}\right)=0$ for some $k'\geq 1$. Then from (A-61), there exists some k' such that $e^*_{t+k'}=0$ for which $\mu^*\left(e^*_{t+k'}\right)=0$ by definition, yielding a contradiction. To see why $e^*_{t+k'}>0$, suppose it were instead the case that $e^*_{t+k'}=0$. From Lemma 9 and (4), this implies that $V\left(w\left(m^*\left(e^*_{t+k'}\right)\right)\right)-l\left(m^*\left(e_{t+k'}\right)\right)=-\infty$. Since $\mu^*\left(e^*_{t+k'-1}\right)>0$, it follows that country A's continuation value conditional on choosing peace at $t+k'-1$ satisfies

$$\begin{aligned} -\infty &= u\left(x^*_{t+k'-1}\right)-p^*_{t+k'-1}x^*_{t+k'-1}+\beta\left(V\left(w\left(m^*\left(e^*_{t+k'}\right)\right)\right)-l\left(m^*\left(e_{t+k'}\right)\right)-\upsilon\right) \\ &< V\left(w\left(m^*\left(e^*_{t+k'-1}\right)\right)\right)-l\left(m^*\left(e_{t+k'-1}\right)\right)-\upsilon \end{aligned}$$

so that country A could make itself strictly better off by choosing $\mu(e^*_{t+k'-1})=0$ and achieving $V\left(w\left(m^*\left(e^*_{t+k'-1}\right)\right)\right)-l\left(m^*\left(e_{t+k'-1}\right)\right)$.

Using the above claim, let us prove that that $x^*_t=\overline{x}$ in case 3, consider $k'\geq 1$ as defined in the above claim. Since $e^*_{t+k'}=e^*_t-k'\overline{x}>0$ this means that $p^*_{t+k}=u'\left(\overline{x}\right)$ for all $0\leq k<k'$, where we have used (A-61) to solve for $e^*_{t+k'}$ and p^*_{t+k}. We can use this observation to show that $x^*_{t+k}=\overline{x}$ for all $0\leq k<k'$ in the firm's problem. We prove this by solving the firm's problem by backward induction in two steps. First, we show that if $k=k'-1$, then $x^*_{t+k}=\overline{x}$. Second, we show that if $x^*_{t+k}=\overline{x}$ then $x^*_{t+k-1}=\overline{x}$ for $0\leq k<k'$. To prove the first part, suppose that $k=k'-1$. At $t+k'-1$, case 2 applies so that $x^*_{t+k'-1}=\min\left\{e^*_{t+k'-1},\overline{x}\right\}$. Because $e^*_{t+k'}=e^*_t-k'\overline{x}>0$, it follows that $e^*_{t+k'-1}=e^*_t-(k'-1)\,\overline{x}>\overline{x}$, which means that $x^*_{t+k'-1}=\overline{x}$. To prove the second part, suppose that $x^*_{t+k}=\overline{x}$ and $\mu(e^*_{t+k})>0$ for $0<k<k'$. It follows that (10) applies at $t+k-1$ with $e^*_{t+k-1}=e^*_t-(k-1)\,\overline{x}>0$. Since $\beta p^*_{t+k}\mu\left(e^*_{t+k}\right)=\beta u'\left(\overline{x}\right)\mu\left(e^*_{t+k}\right)<u'\left(x^*_{t+k-1}\right)$ for all $x^*_{t+k-1}\leq\overline{x}$ and since (12) implies that $p^*_{t+k-1}=u'\left(x^*_{t+k-1}\right)$, it follows that $\beta p^*_{t+k}\mu\left(e^*_{t+k}\right)<p^*_{t+k-1}$ so that $x^*_{t+k-1}=\overline{x}$ by (10). By backward induction, this implies that $x^*_t=\overline{x}$. ■

Proofs from Section 4

Proof of Lemma 3

Following the discussion in the text, the existence of an MPME is guaranteed by the existence of a function $U_S\left(e_t\right)$ which satisfies (24). Substitute (3) and (23) into (21), which holds as equality, to obtain

$$-c_t=G\left(e_{t+1},e_t\right)\equiv u\left(e_t-e_{t+1}\right)+\beta\left(V\left(w\left(m^*\left(e_{t+1}\right)\right)e_{t+1}\right)-l\left(m^*\left(e_{t+1}\right)\right)\right)-V\left(w\left(m^*\left(e_t\right)\right)e_t\right). \tag{A-68}$$

Substituting (20) and (A-68) into (24), we can write $U_S\left(e_t\right)$ as:

$$U_S\left(e_t\right)=\max_{f_t=\{0,1\},e_{t+1}\in[0,e_t]}\left\{\left(1-f_t\right)\left[G\left(e_{t+1},e_t\right)+\beta U_S\left(e_{t+1}\right)\right]+f_t\psi\right\} \tag{A-69}$$

To show that $U_S(e_t)$ exists and is well-defined, note that (23) and (21) imply that

$$\begin{aligned} U_A(e_t) &= \sum_{k=0}^{\infty} \beta^k \begin{pmatrix} (1 - f_{t+k})(u(e_{t+k} - e_{t+k+1}) + c_{t+k} - l(m^*(e_{t+k}))) \\ + \left(f_{t+k} \prod_{l=0}^{k-1} (1 - f_{t+l}) \right) V(w(m^*(e_{t+k})) e_{t+k}) \end{pmatrix} \\ &= V(w(m^*(e_t)) e_t) - l(m^*(e_t)), \end{aligned}$$

so that

$$\begin{aligned} U_S(e_t) &= \sum_{k=0}^{\infty} \beta^k \left(-(1 - f_{t+k}) c_{t+k} + \left(f_{t+k} \prod_{l=0}^{k-1} (1 - f_{t+l}) \right) \psi \right) \qquad \text{(A-70)} \\ &= \sum_{k=0}^{\infty} \beta^k \begin{pmatrix} (1 - f_{t+k})(u(e_{t+k} - e_{t+k+1}) - \beta l(m^*(e_{t+k+1}))) \\ + \left(f_{t+k} \prod_{l=0}^{k-1} (1 - f_{t+l}) \right) (\psi + V(w(m^*(e_{t+k})) e_{t+k})) \end{pmatrix} - V(w(m^*(e_t)) e_t) \end{aligned}$$

for a given equilibrium sequence $\{f_{t+k}, e_{t+k+1}\}_{k=0}^{\infty}$. Consider the following problem:

$$\begin{aligned} \widetilde{U}_S(e_t) &= \max_{\substack{\{f_{t+k}, e_{t+k+1}\}_{k=0}^{\infty} \\ f_{t+k} = \{0,1\}, \\ e_{t+k+1} \in [0, e_{t+k}]}} \left\{ \sum_{k=0}^{\infty} \beta^k \begin{pmatrix} (1 - f_{t+k})(u(e_{t+k} - e_{t+k+1}) - \beta l(m^*(e_{t+k+1}))) \\ + \left(f_{t+k} \prod_{l=0}^{k-1} (1 - f_{t+l}) \right) (\psi + V(w(m^*(e_{t+k})) e_{t+k})) \end{pmatrix} \right\} \\ &\quad - V(w(m^*(e_t)) e_t). \end{aligned} \qquad \text{(A-71)}$$

Since $f_t = 1$ is feasible, (A-71) is bounded from below by ψ. Moreover, since $l(m^*(e_t)) \geq 0$, $e_{t+k} - e_{t+k+1} \leq e_{t+k} \leq e_t$, and $V(w(m^*(e_{t+k})) e_{t+k}) \leq V(e_{t+k}) \leq V(e_t)$, given $e_t > 0$, $\widetilde{U}_S(e_t)$ defined in (A-71) is less than

$$\max_{\substack{\{f_{t+k},\}_{k=0}^{\infty} \\ f_{t+k} = \{0,1\}}} \left\{ \sum_{k=0}^{\infty} \beta^k \begin{pmatrix} (1 - f_{t+k}) u(e_t) \\ + \left(f_{t+k} \prod_{l=0}^{k-1} (1 - f_{t+l}) \right) (\psi + V(e_t)) \end{pmatrix} \right\} - V(w(m^*(e_t)) e_t) < \infty,$$

where the last inequality uses the facts that (i) $V(e_t)$ and $u(e_t)$ are bounded from above; (ii) in view of Assumption 1 in the text, $V(w(m^*(e_t)) e_t)$ is bounded from below for $e_t > 0$ (and thus $w(m^*(e_t)) e_t > 0$), ensuring that $\widetilde{U}_S(e_t)$ is also bounded from above for $e_t > 0$. Therefore, the solution to (A-71) exists and $\widetilde{U}_S(e_t)$ is well-defined for $e_t > 0$. This then implies that we can rewrite (A-71) recursively as

$$\widetilde{U}_S(e_t) = \max_{f_t = \{0,1\}, e_{t+1} \in [0, e_t]} \left\{ (1 - f_t) \left[G(e_{t+1}, e_t) + \beta \widetilde{U}_S(e_{t+1}) \right] + f_t \psi \right\}, \qquad \text{(A-72)}$$

as desired. It is also straightforward to see that $\widetilde{U}_S(e_t)$ in (A-71), and thus in (A-72), is uniquely defined. This follows simply from the observation that any MPME is given by (A-71) (and vice versa), and we have already established that for any $e_t > 0$, $\widetilde{U}_S(e_t)$ is bounded. ■

Proof of Proposition 6

This is proved by a variational argument which considers a specific perturbation on the solution in which starting from e_t, the choice of e_{t+1} is increased by $\epsilon \gtreqless 0$ arbitrarily small, where this increase is accommodated by a decrease in x_t by ϵ and an increase in x_{t+1} by ϵ.

Let e^*_{t+1} denote the optimal choice of e_{t+1} starting from e_t. Since $f_{t+1} = 0$, then $f_t = 0$. Using this observation, equation (A-69) becomes:

$$U_S(e_t) = u\left(e_t - e^*_{t+1}\right) + \beta\left[V\left(w\left(m^*\left(e^*_{t+1}\right)\right)e^*_{t+1}\right) - l\left(m^*\left(e^*_{t+1}\right)\right)\right] - V\left(w\left(m^*\left(e_t\right)\right)e_t\right) + \beta U_S\left(e^*_{t+1}\right). \tag{A-73}$$

Since $f_{t+1} = 0$, (A-73) also holds replacing e_t with e^*_{t+1} and e^*_{t+1} with e^*_{t+2}, where e^*_{t+2} denotes the optimal choice of e_{t+2} starting from e^*_{t+1}.

Optimality requires that the solution at e_t weakly dominates the choice of $e^*_{t+1} + \epsilon$ for $\epsilon \gtreqless 0$. Let $x^*_t = e_t - e^*_{t+1}$ and let $x^*_{t+1} = e^*_{t+1} - e^*_{t+2}$. Optimality of the choice of e^*_{t+1} implies

$$\begin{aligned} &u\left(x^*_t\right) + \beta\left[V\left(w\left(m^*\left(e^*_{t+1}\right)\right)e^*_{t+1}\right) - l\left(m^*\left(e^*_{t+1}\right)\right)\right] + \beta U_S\left(e^*_{t+1}\right) \geq \\ &u\left(x^*_t - \epsilon\right) + \beta\left[V\left(w\left(m^*\left(e^*_{t+1} + \epsilon\right)\right)\left(e^*_{t+1} + \epsilon\right)\right) - l\left(m^*\left(e^*_{t+1} + \epsilon\right)\right)\right] + \beta U_S\left(e^*_{t+1} + \epsilon\right). \end{aligned} \tag{A-74}$$

Since starting from $e^*_{t+1} + \epsilon$ country S can always choose policy e^*_{t+2} associated with e^*_{t+1} together with $f_t = 0$, this implies that

$$\begin{aligned} U_S\left(e^*_{t+1} + \epsilon\right) \geq\ & U_S\left(e^*_{t+1}\right) + u\left(x^*_{t+1} + \epsilon\right) - u\left(x^*_{t+1}\right) \\ & + V\left(w\left(m^*\left(e^*_{t+1}\right)\right)e^*_{t+1}\right) - V\left(w\left(m^*\left(e^*_{t+1} + \epsilon\right)\right)\left(e^*_{t+1} + \epsilon\right)\right). \end{aligned} \tag{A-75}$$

Combining (A-74) with (A-75) we achieve:

$$\begin{aligned} &\left[u\left(x^*_t\right) - u\left(x^*_t - \epsilon\right)\right] - \beta\left[u\left(x^*_{t+1} + \epsilon\right) - u\left(x^*_{t+1}\right)\right] \\ &+ \beta\left[l\left(m^*\left(e^*_{t+1} + \epsilon\right)\right) - l\left(m^*\left(e^*_{t+1}\right)\right)\right] \geq 0. \end{aligned} \tag{A-76}$$

Divide both sides of (A-76) by $\epsilon \gtreqless 0$ and take the limit as ϵ approaches 0. This yields:

$$u'\left(x_t\right) - \beta u'\left(x_{t+1}\right) + \beta l'\left(m^*\left(e_{t+1}\right)\right) m^{*\prime}\left(e_{t+1}\right) = 0. \tag{A-77}$$

Since $l'(\cdot) > 0$, (A-77) implies that $u'(x_{t+1}) > (<)\,(1/\beta)\,u'(x_t)$ if $m^{*\prime}(e_{t+1}) > (<)\,0$. ■

Proof of Proposition 7

Part 1. Suppose that (28) holds. We can prove by contradiction that the equilibrium cannot involve war for any e_t. Suppose there exists an MPME in which war occurs for some e_t. Consider an offer by country S in state e_t that satisfies $x_t^o = (1-\beta^\sigma) w(m^*(e_t)) e_t$ and

$$-c_t^o = u(x_t^o) + \beta \left(V\left(w\left(m^*\left(e_t - x_t^o\right)\right)\left(e_t - x_t^o\right)\right) - l\left(m^*\left(\left(e_t - x_t^o\right)\right)\right)\right) - V\left(w\left(m^*\left(e_t\right)\right)e_t\right). \quad \text{(A-78)}$$

This offer makes country A indifferent between accepting it, and rejecting it and declaring war. We show next that the payoff for country S from making this offer strictly exceeds the payoff from war ψ, which implies that there exists a strategy for country S that gives it a higher payoff than the payoff from war.

Payoff for country S from offer (x_t^o, c_t^o) is

$$\begin{aligned}
& u(x_t^o) + \beta \left(V\left(w\left(m^*\left(e_t - x_t^o\right)\right)\left(e_t - x_t^o\right)\right) - l\left(m^*\left(e_t - x_t^o\right)\right)\right) - V\left(w\left(m^*\left(e_t\right)\right)e_t\right) \quad \text{(A-79)}\\
& + \beta U_S(e_t - x_t^o) \\
\geq\ & u(x_t^o) + \beta \left(V\left(w\left(m^*\left(e_t - x_t^o\right)\right)\left(e_t - x_t^o\right)\right) - l(\bar{m})\right) - V\left(w\left(m^*\left(e_t\right)\right)e_t\right)) + \beta\psi \\
\geq\ & u(x_t^o) + \beta \left(V\left(w\left(m^*\left(e_t - x_t^o\right)\right)e_t - x_t^o\right) - l(\bar{m})\right) - V\left(w\left(m^*\left(e_t\right)\right)e_t\right)) + \beta\psi \\
\geq\ & u(x_t^o) + \beta \left(V\left(w\left(m^*\left(e_t\right)\right)e_t - x_t^o\right) - l(\bar{m})\right) - V\left(w\left(m^*\left(e_t\right)\right)e_t\right)) + \beta\psi
\end{aligned}$$

The first inequality follows from (24) and $-l(m^*((e_t - x_t^o))) \geq -l(\overline{m})$. The second inequality holds because $w(m^*(e_t - x_t^o)) \leq 1$. The third inequality holds because Proposition 1 and $\sigma < 1$ imply that $w(m^*(e_t - x_t^o)) \geq w(m^*(e_t))$.

Note that x_t^o was chosen so that it is the optimal amount of resource extraction for country A when it owns $w(m^*(e_t))e_t$ of resources (i.e. it is the optimal x_t in the maximization problem (4)). Therefore

$$u(x_t^o) + \beta V\left(w\left(m^*\left(e_t\right)\right)e_t - x_t^o\right) = V\left(w\left(m^*\left(e_t\right)\right)e_t\right). \quad \text{(A-80)}$$

Substitute (A-80) into the right-hand side of (A-79) to show that payoff from offer (x_t^o, c_t^o) for country S is bounded from below by $-\beta l(\overline{m}) + \beta\psi$, which exceeds ψ if (28) holds. Therefore war cannot occur for any e_t.

Part 2. Suppose preferences satisfy (17) for $\sigma < 1$ and $w(\overline{m}) > (1/\sigma)^{1/(1-1/\sigma)}$, then war occurs with probability 1 in the MPCE by Proposition 4. Suppose that (28) also holds. Then war is avoided in the MPME by part 1. To show that this is possible, suppose that $l(m) = m$ and $w(m) = 2m - m^2$ for $\overline{m} = 1$. Then the condition that $w(\overline{m}) > (1/\sigma)^{1/(1-1/\sigma)}$ is satisfied and any value of $\psi < -\beta/(1-\beta)$ satisfies (28).

Part 3. Suppose $\sigma < 1$ and (29) holds. Suppose that war never occurs along the equilibrium path. Using the fact that constraint (21) must hold with equality to substitute for c_t, and using

(3), (23, and (24) , the optimality of a permanently peace equilibrium implies that for all $e_t \leq e_0$:

$$U_S(e_t) = \max_{e_{t+1}} \left\{ \begin{array}{c} u(e_t - e_{t+1}) + \beta V(w(m^*(e_{t+1}))e_{t+1}) - \beta l(m^*(e_{t+1})) \\ -V(w(m^*(e_t))e_t) + \beta U_S(e_{t+1}) \end{array} \right\} \geq \psi \qquad \text{(A-81)}$$

Forward iteration on (A-81) implies that the equilibrium sequence $\{x_t^*, e_t^*\}_{t=0}^{\infty}$ must satisfy

$$\begin{aligned} U_S(e_0) &= \sum_{t=0}^{\infty} \beta^t (u(x_t^*) - l(m^*(e_t^*))) - (V(w(m^*(e_0))e_0) - l(m^*(e_0))) \qquad \text{(A-82)} \\ &\leq \sum_{t=0}^{\infty} \beta^t u(x_t^*) - \frac{\beta l(m^*(e_0^*))}{1-\beta} - V(w(m^*(e_0))e_0) \\ &\leq V(e_0) - \frac{\beta l(m^*(e_0^*))}{1-\beta} - V(w(m^*(e_0))e_0). \end{aligned}$$

The first inequality in (A-82) follows from the fact that $e_{t+1} \leq e_t$ from (3) and from Proposition 1 which establishes that $m^{*\prime}(e) < 0$ so that $l(m^*(e_{t+1})) \geq l(m^*(e_t))$ for all e_t. The second inequality in (A-82) follows from the fact that the maximization of $\sum_{t=0}^{\infty} \beta^t u(x_t)$ s.t. (3) yields $V(e_0)$. Given (29), the last inequality implies that $U_S(e_0) < \psi$ which means that the best response for country S at $t = 0$ is to make any offer that violates (21) and leads to war. Therefore, war must occur along the equilibrium path.

Part 4. Suppose $\sigma < 1$, $w(\overline{m}) < (1/\sigma)^{1/(1-1/\sigma)}$, and (29) is satisfied. By part 3, war occurs in the MPME. In the MPCE, by Proposition 4 war does not occur. To show that it is possible for $w(\overline{m}) < (1/\sigma)^{1/(1-1/\sigma)}$ and (29) to be satisfied, suppose that

$$l(m) = m \text{ and } w(m) = \eta m/(m+\delta)$$

for $\delta > 0$. Let $\overline{m} = \infty$ so that $w(\overline{m}) = \eta$. Suppose that η satisfies

$$\eta < (1/\sigma)^{1/(1-1/\sigma)},$$

which is always feasible for η sufficiently low. Suppose that

$$\frac{1-\beta}{1/\sigma - 1} < \beta \qquad \text{(A-83)}$$

which is always feasible for σ sufficiently low. Finally, suppose that ψ and e_0 satisfy

$$\psi(1-\beta) > e_0^{1-1/\sigma}(1-\beta^\sigma)^{-1/\sigma} \times \qquad \text{(A-84)}$$
$$\left(-\frac{1-\beta}{1/\sigma-1} + w(m^*(e_0))^{1-1/\sigma}\left(\frac{1-\beta}{1/\sigma-1} - \beta\frac{\delta}{m^*(e_0)+\delta}\right)\right).$$

This is possible because ψ can be chosen to be arbitrarily close to zero from below and because the right hand side of (A-84) becomes negative for sufficiently high e_0. This is because $m^*(e_0)$

declines towards 0 as e_0 rises by the arguments in claim 4 in the proof of part 2 of Proposition 4 which, means given (A-83), that the second term on the right hand side of (A-84) becomes negative for high e_0. In this situation, the first-order condition which characterizes $m^*(e)$ given (8) implies

$$1 = w\left(m^*(e)\right)^{-1/\sigma} w'\left(m^*(e)\right)\left(1-\beta^\sigma\right)^{-1/\sigma} e^{1-1/\sigma},$$

which by some algebraic manipulation yields

$$l\left(m^*(e)\right) = m^*(e) = w\left(m^*(e)\right)^{1-1/\sigma} \frac{\delta}{m^*(e)+\delta}\left(1-\beta^\sigma\right)^{-1/\sigma} e^{1-1/\sigma}.$$

which means that

$$\left(V(e_0) - V\left(w\left(m^*(e_0)\right)e_0\right)\right)(1-\beta) - \beta l\left(m^*(e_0)\right)$$

equals the right hand side of (A-84) so that (29) is satisfied.■

Proof of Proposition 8

This follows from the same variational argument as used in the proof of Proposition 6.■

Proof of Proposition 9

The same variational argument as used in the proof of Proposition 6 implies that if $x_t < \overline{x}$, then

$$u'(x_t) - \beta u'(x_{t+1}) + \beta l'\left(m^*(e_{t+1})\right) m^{*\prime}(e_{t+1}) \leq 0 \tag{A-85}$$

and if $x_{t+1} < \overline{x}$, then

$$u'(x_t) - \beta u'(x_{t+1}) + \beta l'\left(m^*(e_{t+1})\right) m^{*\prime}(e_{t+1}) \geq 0. \tag{A-86}$$

We use this observation to prove each part of the proposition.

Part 1. In this situation, Proposition 1 implies that $m^{*\prime}(e_{t+1}) > 0$. Suppose by contradiction that $x_{t+1} = \overline{x}$ but that $x_t < \overline{x}$. In this situation, (A-85), would be violated. This means that if $x_{t+1} = \overline{x}$, then $x_t = \overline{x}$, which implies that there exists a $T \geq 0$ for which $x_t = \overline{x}$ if $t \leq T$ and for which $x_t < \overline{x}$ if $t > T$, where the last observation follows from (3) which implies that $x_t < \overline{x}$ for some t. Since $x_t < \overline{x}$ if $t > T$, this means that (A-85) and (A-86) imply (A-77) which means given that $m^{*\prime}(e_{t+1}) > 0$ that $u'(x_t) < \beta u'(x_{t+1})$ if $t > T$.

Part 2. In this situation, Proposition 1 implies that $m^{*\prime}(e_{t+1}) < 0$. We can show that the constraint that $x_t < \overline{x}$ never binds, which given (A-85) and (A-86) implies (A-77). Together with the fact that $m^{*\prime}(e_{t+1}) < 0$ this means that $u'(x_t) < \beta u'(x_{t+1})$ for all t which completes the proof of the proposition. To show that $x_t < \overline{x}$ never binds, consider the relaxed problem of country S which ignores capacity constraints starting from any e_t. We can show that the solution admits $x_t < \widetilde{x}_0(e_t) < \overline{x}$ for $\widetilde{x}_0(e_t)$ defined in (A-11). To see why, note that first order conditions imply (A-77) and $u'(x_{t+k}) > \beta u'(x_{t+k+1})$ for all k. Suppose it were the case that $x_t \geq \widetilde{x}_0(e_t)$. This would mean given the definition of $\widetilde{x}_k(e_t)$ in (A-11) that $x_{t+k} > \widetilde{x}_k(e_t)$ for all $k \geq 1$, violating (3). Therefore, the solution to country S's relaxed problem implies that $x_t < \overline{x}$ so that the capacity constraint never binds, implying that $u'(x_t) < \beta u'(x_{t+1})$ for all t.■

Proofs from Section 6

Proof of Proposition 10

Part 1. Define

$$\widetilde{V}_i(e_t) = V\left(w\left(m_i^*(e_t), \{m_j^*(e_t)\}_{j=1,j\neq i}^N\right) e_t\right)$$

Given the discussion in the text, country S's program can be written as:

$$U_S(e_t) = \max_{\{x_{it}\geq 0, c_{it}\}_{i=1}^N} \left\{-\sum_{i=1}^N c_{it} + \beta U_S(e_{t+1})\right\} \text{ s.t. (31) and} \tag{A-87}$$

$$u(x_{it}) + c_{it} + \beta\left(\widetilde{V}_i(e_{t+1}) \quad - l(m_i^*(e_{t+1}))\right) = \widetilde{V}_i(e_t) \text{ for all } i \tag{A-88}$$

Now consider the solution given that $f_t = f_{t+1} = 0$. Let x_{it}^* and e_{t+1}^* denotes the implied optimal choice of e_{t+1} starting from e_t so that

$$U_S(e_t) = \sum_{i=1}^N \left(u(x_{it}^*) + \beta\left[\widetilde{V}_i(e_{t+1}^*) - l(m^*(e_{t+1}^*))\right] - \widetilde{V}_i(e_t)\right) + \beta U_S(e_{t+1}^*). \tag{A-89}$$

Since $f_{t+1} = 0$, (A-89) also holds replacing e_t with e_{t+1}^* and e_{t+1}^* with e_{t+2}^*, where e_{t+2}^* denotes the optimal choice of e_{t+2} starting from e_{t+1}^*. Optimality requires that the solution at e_t weakly dominates the choice of $e_{t+1}^* + \epsilon$ for $\epsilon \gtreqless 0$ where this is achieved by reducing x_{it}^* by ϵ. Optimality of the choice of e_{t+1}^* implies

$$\begin{aligned} u(x_{it}^*) + \beta\sum_{j=1}^N \left[\widetilde{V}_j(e_{t+1}^*) - l(m_j^*(e_{t+1}^*))\right] + \beta U_S(e_{t+1}^*) \geq \\ u(x_{it}^* - \epsilon) + \beta\sum_{j=1}^N \left[\widetilde{V}_j(e_{t+1}^* + \epsilon) - l(m_j^*(e_{t+1}^* + \epsilon))\right] + \beta U_S(e_{t+1}^* + \epsilon). \end{aligned} \tag{A-90}$$

Since starting from $e_{t+1}^* + \epsilon$ country S can always choose policy e_{t+2}^* associated with e_{t+1}^* so that x_{it+1}^* is increased by ϵ this implies that

$$\begin{aligned} U_S(e_{t+1}^* + \epsilon) \quad \geq \quad & U_S(e_{t+1}^*) + u(x_{it+1}^* + \epsilon) - u(x_{it+1}^*) \\ & + \sum_{j=1}^N \left[\widetilde{V}_j(e_{t+1}^*) - \widetilde{V}_j(e_{t+1}^* + \epsilon)\right]. \end{aligned} \tag{A-91}$$

Combining (A-90) with (A-91) we achieve:

$$\left[u\left(x_{it}^{*}\right)-u\left(x_{it}^{*}-\epsilon\right)\right]-\beta\left[u\left(x_{it+1}^{*}+\epsilon\right)-u\left(x_{it+1}^{*}\right)\right] \\ +\sum_{j=1}^{N}\beta\left[l\left(m_{j}^{*}\left(e_{t+1}^{*}+\epsilon\right)\right)-l\left(m_{j}^{*}\left(e_{t+1}^{*}\right)\right)\right]\geq 0. \tag{A-92}$$

Divide both sides of (A-92) by $\epsilon \gtreqless 0$ and take the limit as ϵ approaches 0. This yields:

$$u'\left(x_{it}\right)-\beta u'\left(x_{it+1}\right)+\sum_{j=1}^{N}\beta l'\left(m_{j}^{*}\left(e_{t+1}\right)\right)m_{j}^{*\prime}\left(e_{t+1}\right)=0. \tag{A-93}$$

Since $l'(\cdot)>0$, (A-93) implies that $u'\left(x_{it+1}\right)>(<)\left(1/\beta\right)u'\left(x_{it}\right)$ if $m_{j}^{*\prime}\left(e_{t+1}\right)>(<)\,0$ for all j. Since $m_{i}^{*\prime}\left(e_{t}\right)=m_{j}^{*\prime}\left(e_{t+1}\right)$ for all j, this implies that this depends only on the sign of $m_{i}^{*\prime}\left(e_{t}\right)$.■

Part 2. At each t, given e_t, equilibrium profile of armaments $\mathbf{m}_t^*$ is such that m_{it} is the same for all countries, which implies that $w_i\left(m_i,\mathbf{m}_{-it}\right)=\eta/N$ and that

$$\begin{aligned} w_{i_{m_{it}}}\left(m_{it},\mathbf{m}_{-it}\right) &= \eta h'\left(m_i\right)\frac{\sum_{j\neq i}h\left(m_j\right)}{\left[\sum_j h\left(m_j\right)\right]^2} \\ &= \frac{h'\left(m_{it}\right)}{h\left(m_{it}\right)}w_i\left(m_{it},\mathbf{m}_{-it}\right)\left(1-\frac{1}{\eta}w_i\left(m_{it},\mathbf{m}_{-it}\right)\right)=\eta\frac{h'\left(m_{it}\right)}{h\left(m_{it}\right)}\frac{N-1}{N^2}. \end{aligned}$$

This implies that the first-order condition which characterizes equilibrium armament $m_i^*\left(e_t\right)$ is uniquely defined by

$$V'\left(\eta e_t/N\right)\eta e_t\frac{N-1}{N^2}\frac{h'\left(m_i^*\left(e_t\right)\right)}{h\left(m_i^*\left(e_t\right)\right)}=l'\left(m_i^*\left(e_t\right)\right). \tag{A-94}$$

Given the solution to (4), the envelope condition implies that

$$V'\left(\eta e_t/N\right)=\beta^k u'\left(x_{it+k}\right)\ \text{ for all } k\geq 0. \tag{A-95}$$

Substitution of (A-95) into (A-94) followed by implicit differentiation yields

$$\frac{u'\left(x_{it+k}\right)}{u''\left(x_{it+k}\right)}\left(\frac{l''\left(m_i^*\left(e_t\right)\right)}{l'\left(m_i^*\left(e_t\right)\right)}+\left[\frac{h'\left(m_i^*\left(e_t\right)\right)}{h\left(m_i^*\left(e_t\right)\right)}-\frac{h''\left(m_i^*\left(e_t\right)\right)}{h'\left(m_i^*\left(e_t\right)\right)}\right]\right)\frac{dm_{it}}{de_t}=\frac{dx_{it+k}}{de_t}+\frac{u'\left(x_{it+k}\right)}{u''\left(x_{it+k}\right)e_t} \tag{A-96}$$

Summing up (5) and (6) one obtains

$$\sum_{k=0}^{\infty}x_{it+k}=\eta e_t/N \tag{A-97}$$

differentiation of which implies

$$\sum_{k=0}^{\infty} \frac{dx_{it+k}}{de_t} = \eta/N.$$

Taking the sum of (A-96) for all $k \geq 0$ and substitution into the above equation yields

$$\frac{dm_{it}}{de_t} = \frac{\frac{\eta}{N}\left(1+\sum_{k=0}^{\infty} \frac{u'(x_{it+k})}{u''(x_{it+k})\, x_{it+k}} \frac{x_{it+k}}{\eta e_t/N}\right)}{\left(\frac{l''(m_i^*(e_t))}{l'(m_i^*(e_t))} + \left[\frac{h'(m_i^*(e_t))}{h(m_i^*(e_t))} - \frac{h''(m_i^*(e_t))}{h'(m_i^*(e_t))}\right]\right) \sum_{k=0}^{\infty} \frac{u'(x_{it+k})}{u''(x_{it+k})}} \tag{A-98}$$

Since the denominator is negative, (A-98) is positive if and only if the numerator is negative. If $-u'(x_{it+k})/u''(x_{it+k})\, x_{it+k} > 1$ for all x_{it+k} then the numerator is negative since from (A-97), $\sum_{k=0}^{\infty} \frac{x_{t+k}}{(\eta e_t/N)} = 1$, and the opposite holds if $-u'(x_{it+k})/u''(x_{it+k})\, x_{it+k} < 1$ for all x_{it+k}. ■

Proof of Proposition 11

We proceed first by proving that $U_S(e_t)$ is uniquely defined in the symmetric MPME, and then we guess and verify a function for $U_S(e_t)$ in order to prove the properties of the equilibrium allocations described in the proposition. Given the symmetry of the equilibrium $\widetilde{V}_i(e_t)$ and $m_i^*(e_t)$ are the same across countries, so that they can be denoted by $\widetilde{V}(e_t)$ and $m^*(e_t)$, respectively, and all countries receive the same resource consumption equal to $(e_t - e_{t+1})/N$. Define

$$G(e_{t+1}, e_t) = N\left(u\left(\frac{1}{N}(e_t - e_{t+1})\right) + \beta\left(\widetilde{V}(e_{t+1}) \ - l(m^*(e_{t+1}))\right) - \widetilde{V}(e_t)\right).$$

Given (17) and (33), (A-94) implies (34). Therefore, $G(e_{t+1}, e_t)$ can be rewritten as:

$$G(e_{t+1}, e_t) = N\left(\begin{array}{c} \frac{\left(\frac{1}{N}(e_t - e_{t+1})\right)^{1-1/\sigma}}{1-1/\sigma} \\ +\beta\left(\frac{(1-\beta^\sigma)^{-1/\sigma}}{1-1/\sigma}\left(\frac{e_{t+1}}{N}\right)^{1-1/\sigma} - \left(\frac{N-1}{N}\right)(1-\beta^\sigma)^{-1/\sigma}\left(\frac{e_{t+1}}{N}\right)^{1-1/\sigma}\right) \\ -\frac{(1-\beta^\sigma)^{-1/\sigma}}{1-1/\sigma}\left(\frac{e_t}{N}\right)^{1-1/\sigma} \end{array}\right) \tag{A-99}$$

Substitution of (A-88) into (A-87) implies that country S's optimal offer satisfies

$$U_S(e_t) = \max_{e_{t+1} \in [0, e_t]} G(e_{t+1}, e_t) + \beta U_S(e_{t+1}) \tag{A-100}$$

By analogous arguments to those of Lemma 3, there is a unique $U_S(e_t)$. Let us guess and verify that $U_S(e_t)$ satisfies

$$U_S(e_t) = Q\frac{e_t^{1-1/\sigma}}{1-1/\sigma} \tag{A-101}$$

for some constant $Q > 0$. It is straightforward to see that under this assumption, and given that the second line of (A-99) is increasing and concave in e_{t+1}, the program defined by (A-100) is strictly concave and yields a unique solution characterized by first order conditions. The first order conditions and the envelope condition for the program defined in (A-100) yield:

$$\left(\frac{1}{N}\right)^{-1/\sigma}\begin{pmatrix} -\left(e_t-e_{t+1}\right)^{-1/\sigma} \\ +\beta\left(\left(1-\beta^{\sigma}\right)^{-1/\sigma}\left(1\ -\left(1-1/\sigma\right)\left(\frac{N-1}{N}\right)\right)\right)e_{t+1}^{-1/\sigma}\end{pmatrix} = -\beta Q e_{t+1}^{-1/\sigma} \quad \text{(A-102)}$$

$$\left(\frac{1}{N}\right)^{-1/\sigma}\left(\left(e_t-e_{t+1}\right)^{-1/\sigma}-\left(1-\beta^{\sigma}\right)^{-1/\sigma}e_t^{-1/\sigma}\right) = Q e_t^{-1/\sigma} \quad \text{(A-103)}$$

Define $\rho \in (0,1)$ such that the e_{t+1} which satisfies (A-102) and (A-103) also satisfies $e_{t+1} = \rho^{\sigma} e_t$. Substitution of $e_{t+1} = \rho^{\sigma} e_t$ into (A-102) and (A-103) allows us to combine both equations to cancel out for Q, so that ρ satisfies

$$\left(1-1/\sigma\right)\left(\frac{N-1}{N}\right)\left(1-\beta^{\sigma}\right)^{-1/\sigma} = \left(1-\frac{\rho}{\beta}\right)\left(1-\rho^{\sigma}\right)^{-1/\sigma}, \quad \text{(A-104)}$$

which implies that ρ is independent of e_t and Q. Given (A-103), this means that Q must satisfy

$$Q = \left(\frac{1}{N}\right)^{-1/\sigma}\left(\left(1-\rho^{\sigma}\right)^{-1/\sigma}-\left(1-\beta^{\sigma}\right)^{-1/\sigma}\right) \quad \text{(A-105)}$$

for ρ defined in (A-104). To complete the proof, we can substitute in for e_{t+1} and Q on the right hand side of (A-100) using the fact that $e_{t+1} = \rho^{\sigma} e_t$ and that Q is defined by (A-105) for ρ defined in (A-104), and this confirms that the original guess in (A-101) is correct.

To prove the first part of the proposition, note that since $e_{t+1} = \rho^{\sigma} e_t$, then this implies that $x_{it} = \left(e_t - e_{t+1}\right)/N = \left(1-\rho^{\sigma}\right)e_t/N$. Therefore,

$$u'\left(x_{it+1}\right) = \left[\left(1-\rho^{\sigma}\right)e_{t+1}/N\right]^{-1/\sigma} = (1/\rho)\left[\left(1-\rho^{\sigma}\right)e_t/N\right]^{-1/\sigma} = (1/\rho)\,u'\left(x_{it}\right).$$

The second part of the proposition follows from the fact that the left hand side of (A-104) is positive (negative) if $\sigma > (<)\,1$. Therefore, if $\sigma > (<)\,1$, then for the right hand side of (A-104) to be positive (negative) it must be the case that $\rho < (>)\,\beta$. To prove the third part of the proposition note that the derivative of the right-hand side of (A-104) with respect to ρ has the same sign as:

$$-\frac{1}{\beta}+\left(\frac{1}{\rho}-\frac{1}{\beta}\right)\frac{\rho^{\sigma}}{1-\rho^{\sigma}} \quad \text{(A-106)}$$

which must be negative. This is because if $\sigma < 1$, then $\rho > \beta$ so that (A-106) is negative and if $\sigma > 1$, then $\rho < \beta$ and (A-106) cannot be greater than

$$-\frac{1}{\beta}+\left(\frac{1}{\rho}-\frac{1}{\beta}\right)\frac{\rho}{1-\rho} = \frac{1}{1-\rho}\left(-\frac{1}{\beta}+1\right) < 0.$$

Therefore, ρ is uniquely defined. It follows that if $\sigma < 1$, the left-hand side of (A-104) declines as N rises, so that ρ rises as N rises. Alternatively, if $\sigma > 1$, the the left-hand side of (A-104) rises as N rises, so that ρ declines as N rises, which completes the argument.■

Proof of Proposition 12

Part 1. Given the discussion in the text, country S's program can be written as:

$$U_S\left(e_t\right) = \max_{x_t \geq 0, c_t} \left\{-c_t - l\left(m_S^*\left(e_t\right)\right) + \beta U_S\left(e_{t+1}\right)\right\} \text{ s.t. (3) and}$$

$$u\left(x_t\right) + c_t + \beta\left[V\left(w\left(m_A^*\left(e_{t+1}\right), m_S^*\left(e_{t+1}\right)\right)e_{t+1}\right) - l\left(m_A^*\left(e_{t+1}\right)\right)\right] = V\left(w\left(m_A^*\left(e_t\right), m_S^*\left(e_t\right)\right)e_t\right).$$

Now consider the solution given that $f_t = f_{t+1} = 0$. Let e_{t+1}^* denotes the implied optimal choice of e_{t+1} starting from e_t so that

$$\begin{aligned} U_S\left(e_t\right) &= u\left(e_t - e_{t+1}^*\right) - l\left(m_S^*\left(e_t\right)\right) + \beta\left[V\left(w\left(m_A^*\left(e_{t+1}\right), m_S^*\left(e_{t+1}\right)\right)e_{t+1}\right) - l\left(m_A^*\left(e_{t+1}\right)\right)\right] \\ &\quad - V\left(w\left(m_A^*\left(e_t\right), m_S^*\left(e_t\right)\right)e_t\right) + \beta U_S\left(e_{t+1}^*\right). \end{aligned} \quad \text{(A-107)}$$

Follow the same perturbation arguments as in the proof of Proposition 6. This yields:

$$\begin{aligned} &\left[u\left(x_t^*\right) - u\left(x_t^* - \epsilon\right)\right] - \beta\left[u\left(x_{t+1}^* + \epsilon\right) - u\left(x_{t+1}^*\right)\right] \\ &+\beta\left[l\left(m_A^*\left(e_{t+1}^* + \epsilon\right)\right) - l\left(m_A^*\left(e_{t+1}^*\right)\right) + l\left(m_S^*\left(e_{t+1}^* + \epsilon\right)\right) - l\left(m_S^*\left(e_{t+1}^*\right)\right)\right] \geq 0. \end{aligned} \quad \text{(A-108)}$$

Divide both sides of (A-108) by $\epsilon \gtrless 0$ and take the limit as ϵ approaches 0. This yields:

$$u'\left(x_t\right) - \beta u'\left(x_{t+1}\right) + \beta l'\left(m_A^*\left(e_{t+1}\right)\right)m_A^{*\prime}\left(e_{t+1}\right) + \beta l'\left(m_S^*\left(e_{t+1}\right)\right)m_S^{*\prime}\left(e_{t+1}\right) = 0. \quad \text{(A-109)}$$

Since $l'(\cdot) > 0$, (A-109) implies that $u'\left(x_{t+1}\right) > (<) \left(1/\beta\right)u'\left(x_t\right)$ if $m_A^{*\prime}\left(e_{t+1}\right) > (<) 0$ and $m_S^{*\prime}\left(e_{t+1}\right) > (<) 0$.

Part 2. Analogous arguments to those of part 2 of Proposition 10 imply that $m_A^*\left(e_t\right)$ and $m_S^*\left(e_t\right)$ increase (decrease) in e_t if $-u'(x)/\left(xu''(x)\right) > (<) 1$ for all x. ■

Monopolistic Environment without Armament

Here we briefly consider the implications of allowing country A to engage in war without the possibility for armament. In particular, suppose that

$$w(m) = \bar{w} \in (0, 1] \text{ for all } m, \quad \text{(A-110)}$$

which implies that country A never invests in armament in equilibrium.

It is then straightforward to see that wars do not occur in any period. This is because country S can always structure offers to country A so as to replicate the outcome of war while making itself better off by avoiding war which costs it ψ.

Formally, if country A attacks country S over any stock of the resource e_t, country A's payoff is $V(\bar{w}e_t)$ and its path of extraction of the resource following the war $\{\tilde{x}_{t+k}(\bar{w}e_t)\}_{k=0}^{\infty}$ is a solution to (4) when $w(m) = \bar{w}$. Note that it satisfies

$$V(\bar{w}e_t) = u\left(\tilde{x}_t(\bar{w}e_t)\right) + \beta V(\bar{w}e_t - \tilde{x}_t(\bar{w}e_t)). \tag{A-111}$$

It is feasible for country S to make offers in equilibrium that replicate the payoff of country A in the event of war. In fact, we can show a stronger statement that country S in any period can make an offer that makes both countries strictly better off than having a war. Consider an offer $\tilde{z}_t = \{\tilde{x}_t\left(\bar{w}e_t\right), \epsilon\}$ where $\epsilon \in \left(0, -\left(1-\beta\right)\psi\right)$. Since the payoff of country A in period $t+1$ is bounded by the payoff from attacking country S, $V(\bar{w}\left(e_t - \tilde{x}_t(\bar{w}e_t)\right))$, its payoff in period t from accepting offer $\tilde{z}_t$ satisfies

$$\begin{aligned} u(\tilde{x}_t(\bar{w}e_t)) + \epsilon + \beta U_A(e_t - \tilde{x}_t(\bar{w}e_t)) \quad &> \quad u(\tilde{x}_t(\bar{w}e_t)) + \beta V(\bar{w}e_t - \tilde{x}_t(\bar{w}e_t)) \\ &= \quad V(\bar{w}e_t) \end{aligned}$$

where the last line uses (A-111). This means country A is made strictly better off accepting this alternative offer.

Similarly, the payoff of country S in period $t+1$ is bounded by the payoff from being attacked ψ, since country S can always make an offer which is rejected.[26] Therefore, country S's payoff following the acceptance of the offer is

$$-\epsilon + \beta U_S(e_t - \tilde{x}_t(\bar{w}e_t)) \geq -\epsilon + \beta\psi.$$

Since $-\epsilon + \beta\psi > \psi$, country S is made strictly better off so that war cannot be an equilibrium with any endowment e_t.

Since wars are never an equilibrium, country S makes an offer z_t to extract the maximum surplus from country A subject to avoiding war. We can then show that such an offer always satisfies the Hotelling rule. Formally, country S's maximization problem is

$$U_S\left(e_t\right) = \max_{x_t \geq 0, c_t} \left\{-c_t + \beta U_S\left(e_{t+1}\right)\right\} \tag{A-113}$$

subject to (3),

$$u\left(x_t\right) + c_t + \beta U_A\left(e_{t+1}\right) \geq V\left(\bar{w}e_t\right). \tag{A-114}$$

[26]Formally, starting from any e_t, country S can offer $\{0, 0\}$, which yields a payoff $\beta U_S\left(e_t\right)$ if it does not lead to war and ψ if it leads to war. This implies that

$$U_S\left(e_t\right) \geq \min\left\{\beta U_S\left(e_t\right), \psi\right\} = \psi, \tag{A-112}$$

where we have used the fact that if it were the case that $\beta U_S\left(e_t\right) < \psi < 0$, (A-112) would imply $U_S\left(e_t\right) \geq 0$, yielding a contradiction.

With the same argument as in the text, the participation constraint is given by (A-114) and this constraint must bind; if it did not, country S could strictly improve its payoff by offering a lower value of c_t to country A. Therefore, in this case, $U_A(e_t) = V(\bar{w}e_t)$ for all e_t so that country A is indifferent between attacking and not attacking country S in every period. Therefore, the maximization problem of country S can be written as a maximization of (A-113) subject to (3), and

$$u(x_t) + c_t + \beta V(\bar{w}e_{t+1}) \geq V(\bar{w}e_t).$$

The first-order conditions to this problem establishes that x_t must satisfy Hotelling rule (16).[27]

It is optimal for country S to equalize country S's marginal rate of substitution over x to the marginal rate of transformation since this is the most efficient means of extracting payments from country A. As an illustration of this intuition, suppose that $\beta u'(x_{t+1}) > u'(x_t)$. If country S extracts ϵ units of resources less in period t and $\epsilon > 0$ more in period $t+1$, holding everything fixed, it changes payoff of country A by $(\beta u'(x_{t+1}) - u'(x_t))\,\epsilon > 0$, which relaxes constraint (A-114). This allows country S to reduce c_t and hence increase the payments it receives from country A. If instead $\beta u'(x_{t+1}) < u'(x_t)$, then analogous arguments imply that country S could improve its payoff by extracting $\epsilon > 0$ units of resources more in period t and ϵ less in period $t+1$.

We summarize the results of this section in the following proposition:

Proposition 14 *Suppose $w(\cdot)$ satisfies (A-110). Then in any MPME:*

1. *War never occurs.*
2. *The equilibrium sequence of resource extraction, x_t, satisfies (16) for all t.*

Competition Among Suppliers

In this part of the Appendix we consider an environment which includes M resource-rich countries, denoted by $s = 1, ..., M$, as well as N resource-poor countries, $i = 1, ..., N$. The law of motion of the endowment of each resource-rich country is given by

$$e_{t+1}^s = e_t^s - \sum_{i=1}^{N} x_{it}^s \qquad \text{(A-115)}$$

for each s, where $x_{it}^s \geq 0$ denotes the extraction of country s which is sold to country i. Clearly, $\sum_{s=1}^{M} x_{it}^s = x_{it}$ corresponds to the consumption of the resource by the households in country i and $\sum_{s=1}^{M} e_t^s = e_t$ to the global resource endowment. We assume that each country s holds some initial endowment e_0/M. Country s transfers c_{it}^s units of the consumption good to each country

[27] To take the first-order condition one needs to assume that $U_S(e)$ is differentiable. One can prove the same result without assuming differentiability by following the same steps as in the proof of Proposition 6.

i, which implies that the instantaneous utility to country s is $-\sum_{i=1}^{N} c_{it}^{s}$. The instantaneous utility to country i from its consumption of the resource and the consumption good is equal to

$$u(x_{it}) + c_{it},$$

where $c_{it} = \sum_{s=1}^{M} c_{it}^{s}$. All countries discount the future at the rate β.

Suppose to simplify the discussion here that if any country i goes to war, this causes a "world war" by all resource-poor countries against all resource-rich countries, where each resource-poor country is able to capture a fraction $w_i(m_{it}, \mathbf{m}_{-it})$ of the reserves of each research-rich country. As in subsection 6.1, country i's payoff from war is $V(w_i(m_{it}, \mathbf{m}_{-it}) e_t) - l(m_{it})$.

It is straightforward to observe that the structure of MPCE in this extended environment with multiple resource-poor countries is similar to Proposition 2. In particular, in the pure-strategy equilibrium, war can only take place at date $t = 0$ and the Hotelling rule applies throughout. In what follows, we focus on MPME.

In MPME, at each date t, each country s simultaneously makes a take-it-or-leave-it offer to every country i, $\{x_{it}^{so}, c_{it}^{s0}\}$, consisting of a quantity of resource to be traded in exchange of the consumption good. For simplicity, we assume that rejection of any offer from any country s by any country i automatically leads to world war. The analysis of the monopolistic environment is complicated because of the size of the state space, which now consists of the remaining endowment of each resource-rich country. In addition to this state vector, the offers of resource-rich countries depend on the vector of armaments of all resource-poor countries and the war decision of each resource-poor country also depends on the entire vector of offers of resource-rich countries. Here, to simplify the analysis we simply give a flavor of the results in the context of a two-period model, with periods $t = 0, 1$. This enables us to solve for the equilibrium using backward induction. Moreover, to further simplify the discussion, we assume that preferences and technologies satisfy (17) and (33), and we focus on "symmetric equilibria," where along the equilibrium path (when all resource-rich countries have the same remaining endowment), all countries use symmetric strategies.[28] An immediate implication of this is that, because resource-poor countries i all choose the same armament along the equilibrium path, $w_i(m_{i1}, \mathbf{m}_{-i1}) = 1/N$.

As in subsection 6.1 all resource-poor countries make their armament decisions to maximize their continuation payoff from war, which implies that the armament levels of country i at dates 1 and 0 satisfy

$$\begin{aligned} m_1^*(e_1) &= \left(\frac{N-1}{N}\right)\left(\frac{e_1}{N}\right)^{1-1/\sigma} \text{ and} \\ m_0^*(e_0) &= \left(\frac{N-1}{N}\right)\left(\frac{e_0}{N}\right)^{1-1/\sigma} (1+\beta^{\sigma})^{1/\sigma}, \end{aligned} \tag{A-116}$$

[28] More formally, we focus on pure-strategy equilibria that have the following Markovian property: an offer from s to i depend only on the payoff relevant variables, and thus not on the identities of countries s and i. This of course does not imply that choices off-the-equilibrium path, where endowments are unequal, will be symmetric.

where we have used the fact that countries arm symmetrically along the equilibrium path.[29] The first equation uses the fact that country i at date 1 competes over resource endowment e_1 and thus would acquire total resource equal to $w_i(m_{i1}, \mathbf{m}_{-i1}) e_1$ in the case of war. It shows that, as in our baseline model, a lower remaining endowment of the resource increases (decreases) the armaments of resource-poor countries if the demand elasticity for the resource is less than (greater than) one. The second equation takes into account that at date 0, country i competes over resource endowment e_0 and following war, it would smooth its consumption of the resource so that $x_1 = \beta^\sigma x_0$, which gives $x_0 = w_i(m_{i0}, \mathbf{m}_{-i0}) e_0 / (1+\beta^\sigma)$.

Given this armament strategy, we now consider the offer made by (some) country s at date 1. At date 1, each country i invests $m_1^*(e_1)$ and would consume e_1/N in the case of war. Moreover, since date 1 is the last period, without war, we also have that each country s transfers e_1^s/N to each country i. Since each country i receives $\sum_{s=1}^M e_1^s/N = e_1/N$ with peace as well as with war, no transfer of the consumption good will take place along the equilibrium path at date 1, and thus $c_{it}^s = 0$ for all i and s. Consequently, for any $\{e_1^s\}_{s=1}^M$, the equilibrium at date 1 entails each country s and each country i receiving 0 units of the consumption good and each country i investing $m_1^*(e_1)$ and consuming e_1/N of the resource endowment. Therefore, every country i's continuation value at date 1 given the aggregate endowment e_1 is

$$\frac{(e_1/N)^{1-1/\sigma}}{1-1/\sigma} - \left(\frac{N-1}{N}\right)\left(\frac{e_1}{N}\right)^{1-1/\sigma}.$$

Consider the offer by a given country s at date 0 given this continuation equilibrium. At date 0, each country i invests $m_0^*(e_0)$ in armament. Moreover, since we consider symmetric equilibria, every rival producer to country s makes some offer x' and c' to every country i. Therefore, for country i to accept the offer from country s, we need that

$$\frac{((M-1)x' + x_{i0}^s)^{1-1/\sigma}}{1-1/\sigma} + (M-1)c' + c_{i0}^s + \beta\left(\frac{(e_1/N)^{1-1/\sigma}}{1-1/\sigma} - \left(\frac{N-1}{N}\right)\left(\frac{e_1}{N}\right)^{1-1/\sigma}\right) \tag{A-117}$$

$$\geq \left(\frac{e_0}{N}\right)^{1-1/\sigma}(1+\beta^\sigma)^{1/\sigma}.$$

where the resource constraint implies that

$$e_1 = e_0 - \left(N(M-1)x' + \sum_{i=1}^N x_{i0}^s\right). \tag{A-118}$$

(A-117) ensures that the the welfare of country i from accepting the offers made by all countries s weakly exceeds the payoff from war at date 0. Clearly, in a symmetric equilibrium, $x_{i0}^s = x'$ and

[29] If a country were to choose a different level of armament, all resource-rich countries would make the same offer to this resource-poor country making it indifferent between war and no war (in the same way that all resource-poor countries are indifferent between war and no war along the equilibrium path).

$c_{i0}^s = c'$. Given that country s receives 0 units of the consumption good at date 1 independently of its offer at date 0, it solves the following problem of maximizing its period 0 consumption:

$$\max_{\{x_{i0}^s, c_{i0}^s\}_{i=1}^N} -\sum_{i=1}^N c_{i0}^s \text{ s.t. (A-117) and (A-118)},$$

where constraint (A-117) will necessarily bind, since country s could otherwise strictly increase its payoff by making a less generous offer. The first-order condition of this problem implies the following relationship between resource consumption at dates 0 and 1:

$$x_{i0}^{-1/\sigma} = \beta x_{i1}^{-1/\sigma} + \beta(1-1/\sigma)\left(\frac{N-1}{N}\right)\left(\frac{e_1}{N}\right)^{-1/\sigma}. \tag{A-119}$$

Equation (A-119) shows that our main conclusions regarding the MPME are preserved in this environment. In particular, if preferences are inelastic, i.e., $\sigma < 1$, then $x_{i0}^{-1/\sigma} > \beta x_{i1}^{-1/\sigma}$ and thus resource are extracted at a slower pace relative to the Hotelling rule. The opposite conclusion holds if preferences are elastic, i.e., $\sigma > 1$. The intuition for this result is the same as in our benchmark environment. A resource-rich country internalizes the effect of its resource extraction decision on the armament of all resource-poor countries in the next period as captured by equation (A-116). This result is summarized in the following proposition.

Proposition 15 *Consider the symmetric MPME of the two-period economy with M resource-rich and N resource-poor countries and suppose that preferences and technologies satisfy* (17) *and* (33)*. Then:*

$$\begin{aligned} \beta u'(x_{i1}) &> u'(x_{0t}) \text{ if } m_1^{*\prime}(e_1) > 0 \text{ and} \\ \beta u'(x_{i1}) &< u'(x_{0t}) \text{ if } m_1^{*\prime}(e_1) < 0. \end{aligned}$$

Alternative Preferences

A natural question is the extent to which our conclusions depend on our assumption of quasi-linear preferences for country A. In this subsection, we focus on MPME and show that the general insights in Proposition 6 continue to hold. More specifically, consider an environment in which the instantaneous utility to country A is equal to

$$u(x_t, c_t, -m_t),$$

where $u(\cdot)$ is increasing and globally concave in x_t, c_t, and $-m_t$. Let $\lim_{x\to 0} u_x(\cdot) = \infty$ and $\lim_{x\to\infty} u_x(\cdot) = 0$. For simplicity, we assume that $u(\cdot)$ is defined for all values of $c_t \gtreqless 0$.[30]

[30] The analysis of MPCE in this case is similar to the baseline environment since $u(0,0,0)$ is either finite or equal to $-\infty$. Therefore, a direct application Proposition 2 shows that in any pure-strategy equilibrium, war can only occur in the initial period.

Note that in this environment, the Hotelling rule can be written as:

$$u_x\left(x_{t+1},c_{t+1},-m_{t+1}\right)/u_c\left(x_{t+1},c_{t+1},-m_{t+1}\right)=\left(1/\beta\right)u_x\left(x_t,c_t,-m_t\right)/u_c\left(x_t,c_t,-m_t\right),$$

so that the marginal rate of substitution between the resource and the consumption good is increasing in the discount rate.

Consider the order of events and define the MPME as in Section 4. In this environment, we can define:

$$\widetilde{V}\left(e_t\right)=\max_{\{x_{t+k},e_{t+k+1}\}_{k=0}^{\infty},m_t}u\left(x_t,0,-m_t\right)+\sum_{k=1}^{\infty}\beta^k u\left(x_{t+k},0,0\right)$$

subject to (5)-(7). Here $\widetilde{V}\left(e_t\right)$ corresponds to the highest continuation value that country A can achieve in the event of war and is the analogue of $V\left(w\left(m^*\left(e_t\right)\right)e_t\right)-l\left(m^*\left(e_t\right)\right)$ in the quasi-linear case. Let $m^*\left(e_t\right)$ correspond to the value of m_t associated with $\widetilde{V}\left(e_t\right)$.

Proposition 16 *In an MPME,*

$$u_x\left(x_{t+1},c_{t+1},-m_{t+1}\right)/u_c\left(x_{t+1},c_{t+1},-m_{t+1}\right)>(<)\left(1/\beta\right)u_x\left(x_t,c_t,-m_t\right)/u_c\left(x_t,c_t,-m_t\right)$$

if

$$m^{*\prime}\left(e_{t+1}\right)+\frac{\widetilde{V}'\left(e_{t+1}\right)}{u_m\left(x_{t+1},c_{t+1},-m_{t+1}\right)}\left(1-\frac{u_c\left(x_{t+1},c_{t+1},-m_{t+1}\right)}{u_c\left(x_t,c_t,-m_t\right)}\right)>(<)\,0.$$

Proof. Analogous arguments as in the proof of Proposition 6 imply that $m_t=m^*\left(e_t\right)$, that

$$U_A\left(e_t\right)=\widetilde{V}\left(e_t\right),$$

and that country S's optimal offer must satisfy:

$$U_S\left(e_t\right)=\max_{x_t\geq 0,c_t}\left\{-c_t+\beta U_S\left(e_{t+1}\right)\right\}\text{ s.t. (3) and}$$
$$u\left(x_t,c_t,-m^*\left(e_t\right)\right)+\beta\widetilde{V}\left(e_{t+1}\right)=\widetilde{V}\left(e_t\right).$$

Let e^*_{t+1} denote the implied optimal value of e_{t+1} starting from e_t, and let e^*_{t+2} denote the implied optimal value of e_{t+2} starting from e^*_{t+1}. Let $\widetilde{c}_t\left(\epsilon\right)$ and $\widetilde{c}_{t+1}\left(\epsilon\right)$, respectively, solve:

$$u\left(e_t-e^*_{t+1}-\epsilon,\widetilde{c}_t\left(\epsilon\right),-m^*\left(e_t\right)\right)+\beta\widetilde{V}\left(e^*_{t+1}+\epsilon\right)=\widetilde{V}\left(e_t\right)\text{ and}\qquad\text{(A-120)}$$
$$u\left(e^*_{t+1}-e^*_{t+2}+\epsilon,\widetilde{c}_{t+1}\left(\epsilon\right),-m^*\left(e^*_{t+1}+\epsilon\right)\right)+\beta\widetilde{V}\left(e^*_{t+2}\right)=\widetilde{V}\left(e^*_{t+1}+\epsilon\right)\qquad\text{(A-121)}$$

for $\epsilon \gtrless 0$. Note that by implicit differentiation:

$$\begin{aligned}\widetilde{c}_t'(0) &= \frac{u_x(x_t, c_t, -m_t) - \beta \widetilde{V}'(e_{t+1})}{u_c(x_t, c_t, -m_t)} \\ \widetilde{c}_{t+1}'(0) &= \frac{-u_x(x_{t+1}, c_{t+1}, -m_{t+1}) + u_m(x_{t+1}, c_{t+1}, -m_{t+1}) m^{*\prime}(e_{t+1}) + \widetilde{V}'(e_{t+1})}{u_c(x_{t+1}, c_{t+1}, -m_{t+1})}\end{aligned}$$

Optimality requires that

$$\begin{aligned}-\widetilde{c}_t(0) + \beta U_S(e_{t+1}^*) &\geq -\widetilde{c}_t(\epsilon) + \beta U_S(e_{t+1}^* + \epsilon) \\ &\geq -\widetilde{c}_t(\epsilon) + \beta\left(-\widetilde{c}_{t+1}(\epsilon) + \widetilde{c}_{t+1}(0) + U_S(e_{t+1}^*)\right)\end{aligned}$$

which implies that

$$\widetilde{c}_t(0) - \widetilde{c}_t(\epsilon) \leq \beta\left(\widetilde{c}_{t+1}(\epsilon) - \widetilde{c}_{t+1}(0)\right). \tag{A-122}$$

Divide both sides of (A-122) by $\epsilon \gtrless 0$ and take the limit as ϵ approaches 0 so as to achieve:

$$-\widetilde{c}_t'(0) = \beta \widetilde{c}_{t+1}'(\epsilon),$$

which by substitution yields:

$$\begin{aligned}\frac{u_x(x_{t+1}, c_{t+1}, -m_{t+1})}{u_c(x_{t+1}, c_{t+1}, -m_{t+1})} &= \frac{1}{\beta}\frac{u_x(x_t, c_t, -m_t)}{u_c(x_t, c_t, -m_t)} + \frac{u_m(x_{t+1}, c_{t+1}, -m_{t+1})}{u_c(x_{t+1}, c_{t+1}, -m_{t+1})} m^{*\prime}(e_{t+1}) \\ &\quad + \widetilde{V}'(e_{t+1})\left(\frac{1}{u_c(x_{t+1}, c_{t+1}, -m_{t+1})} - \frac{1}{u_c(x_t, c_t, -m_t)}\right),\end{aligned}$$

which completes the proof since $u_c(\cdot), u_m(\cdot) > 0$. ■

Proposition 16 states that the shadow price of the resource increases faster (slower) if armament increases (decreases) in the size of the total resource endowment, which is similar to Proposition 6. Nevertheless, in relating this rate of growth to the rate of time preference, Proposition 16 differs from Proposition 6 because the rate of growth of the shadow price not only depends on $m^{*\prime}(e_{t+1})$ but also on an additional term (which was equal to zero when preferences were quasi-linear). This term emerges because even in the absence of endogenous armament, there will be distortions in the growth rate of the shadow price provided that the marginal utility of the consumption good is time varying. Intuitively, when country A's marginal utility from the consumption good is lower, it is cheaper for country S to extract payments from country A while still ensuring that country A does not declare war. Therefore, if the marginal utility of the consumption good is higher (lower) today relative to tomorrow, country S will deplete more (less) of the endowment today. Proposition 16 therefore shows that in addition to this force, the sign of $m^{*\prime}(e_{t+1})$ continues to play the same role as in the quasi-linear case.[31]

[31] It may be conjectured that in a richer environment with additional smoothing instruments such as bonds, this marginal utility of consumption will not vary significantly along the equilibrium path so that the dominating effect would come from $m^{*\prime}(e_{t+1})$.