A Preferred-Habitat Model of Term Premia, Exchange Rates, and Monetary Policy Spillovers

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Abstract

We develop a two-country model in which currency and bond markets are populated by different investor clienteles, and segmentation is partly overcome by global arbitrageurs with limited capital. Our model accounts for the empirically documented violations of Uncovered Interest Parity (UIP) and the Expectations Hypothesis, and for how UIP violations depend on bond maturity, investment horizon, and yield curve slope differentials. Large-scale purchases of long-maturity bonds lower domestic and foreign bond yields, and depreciate the currency. Conventional monetary policy is transmitted to domestic and international bond yields as well, but its international transmission is weaker than for unconventional policy.

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1 Introduction

How is monetary policy transmitted domestically and internationally? The standard international macroeconomics model with perfect capital mobility and floating exchange rates (e.g., Gali (2015)) delivers sharp answers. These follow from the Expectations Hypothesis (EH) and the Uncovered Interest Parity (UIP), which hold in the standard model up to constant risk premia. Because of EH, the yield curve in each country depends only on expectations of the domestic short rate, which is controlled by the domestic central bank. Hence, Quantitative Easing (QE) purchases of long-maturity bonds by the central bank, keeping short rates unchanged, have no effect on the yield curve. Moreover, each country's yield curve is fully insulated from other countries' monetary policy. Insulation arises because according to UIP, short-rate differentials across countries are absorbed into the exchange rate, whose expected movements compensate investors for these differentials. The insulation result is a slightly broader statement of the well-known Friedman-Obstfeld-Taylor Trilemma: with perfect capital mobility, a floating exchange rate provides monetary policy autonomy, not just in setting short rates, but also in shaping the domestic yield curve.[1]

Four broad empirical observations cast doubt on the validity of the standard model. First, a large literature starting with Bilson (1981) and Fama (1984) documents strong violations of UIP: currency carry trade (CCT) strategies that borrow in currencies with low short rates and invest in currencies with high short rates earn abnormally high expected returns. Second, a similarly large literature starting with Fama and Bliss (1987) and Campbell and Shiller (1991) documents strong violations of EH: bond carry trade (BCT) strategies that borrow in maturities with low interest rates and invest in maturities with high interest rates earn abnormally high expected returns. Third, risk premia in currencies and bonds are connected. For example, Chernov and Creal (2020) and Lloyd and Marin (2020) find that yield curve slope differentials predict the CCT's profitability, and Lustig, Stathopoulos, and Verdelhan (2019) find that the CCT's profitability declines when that trade is carried out with long-maturity rather than short-maturity bonds. Fourth, a growing body of evidence surveyed in Bhattarai and Neely (2018) suggests that central banks' QE purchases had a significant impact not only on domestic yields but also on exchange rates and foreign yields.[2]

In this paper we develop a two-country model in which currency and bond markets are populated by different investor clienteles, and segmentation is partly overcome by global arbitrageurs with limited capital. Our model accounts for the predictability patterns of currency and bond returns documented empirically. It also delivers sharply different im-

[1] Obstfeld, Shambaugh, and Taylor (2010) provide a modern articulation of the Trilemma.

[2] See also Bauer and Neely (2014), Neely (2015), Curcuru, De Pooter, and Eckerd (2018), Curcuru, Kamin, Li, and Rodriguez (2018), Fratzscher, Lo Duca, and Straub (2018) and Dedola, Georgiadis, Grab, and Mehl (2020).

plications about monetary policy transmission than the standard model. QE purchases of long-maturity bonds in our model lower domestic and foreign bond yields, and depreciate the currency, consistent with empirical evidence. Conventional monetary policy is transmitted to domestic and international bond yields as well, but its international transmission is weaker than for unconventional policy. A common theme in our analysis is that when accounting for realistic frictions in financial markets, floating exchange rates provide limited insulation, at odds with the Trilemma.

Our model, presented in Section 2, is set in continuous time and infinite horizon. In each of the two countries, home and foreign, a central bank sets the short rate exogenously. There are three types of investors: currency traders, bond investors, and global arbitrageurs. Currency traders express an exogenous demand for the foreign currency, which can depend on the real exchange rate. Examples of such traders are exporters and importers. Bond investors form clienteles, each of which expresses a demand for a bond of a specific country and maturity. The demand can depend on the bond's price. Examples of such clienteles are pension funds, holding long-maturity bonds to hedge their liabilities, or money-market funds, whose mandates require them to hold short-maturity bonds. Demands for currency and bonds are hit by shocks, which generate additional variation to short-rate shocks. Arbitrageurs can trade the currency and bonds of both countries. They maximize mean-variance utility over instantaneous changes in wealth. Their risk aversion parameter can capture in reduced form capital or Value-at-Risk constraints. Examples of global arbitrageurs are macro hedge funds and global banks.

Section 3 derives the equilibrium, as a solution to a scalar non-linear system. The exchange rate and bond prices are log affine functions of five state variables: the two short rates, a currency demand factor, and two bond demand factors. When arbitrageur risk aversion is zero, UIP and EH hold, and monetary policy transmission is as in the standard model. When instead risk aversion is non-zero, UIP and EH are violated. The violations of UIP and EH are tightly linked because arbitrageurs are the marginal investors in all markets.

Section 4 specializes the model to the case where there are no recurring demand shocks and short rates are independent across countries. In that case, we can show key mechanisms and results analytically. Consider the transmission of conventional monetary policy. Following a cut to the home short rate, arbitrageurs find it attractive to enter into the CCT, by borrowing in the home currency and investing in the foreign currency. If the demand by currency traders is price-elastic, then both the exchange rate, defined as the price of foreign currency in terms of home currency, and arbitrageurs' holdings of foreign currency rise in equilibrium. The expected return of the CCT rises as well, as arbitrageurs must be compensated for the risk of entering into that trade. The rise in the CCT's expected return attenuates the transmission of monetary policy to the exchange rate, which appreciates less than implied by UIP. This attenuation effect parallels Gabaix and Maggiori (2015), who model exchange-rate dynamics without a yield curve.

A similar attenuation effect arises in the home bond market. The short-rate cut prompts arbitrageurs to also enter into the home BCT, by borrowing in the home short rate and investing in long-maturity home bonds. If the demand by home bond investors is price-elastic, then both home bond prices and arbitrageurs' holdings of home bonds rise in equilibrium. The expected return of the home BCT rises as well, as arbitrageurs must be compensated for the risk of entering into that trade. The rise in the BCT's expected return attenuates the transmission of monetary policy to domestic bond yields, which drop less than implied by EH. This attenuation effect parallels Vayanos and Vila (2021), who model yield-curve dynamics in a closed economy.

In addition to the above attenuation effects, our model of joint exchange-rate and yield-curve dynamics delivers a propagation effect of conventional policy to foreign bond yields. Propagation occurs through global risk management by arbitrageurs. By entering into the CCT in response to the home short-rate cut, arbitrageurs become more exposed to the risk that the foreign short rate drops and the foreign currency depreciates. Foreign bonds provide a natural hedge for that risk because their price rises when the foreign short rate drops. Hence, arbitrageurs increase their demand for foreign bonds, causing foreign bond yields to drop.

Our model delivers additional new predictions on the transmission of unconventional policies such as QE and foreign exchange intervention. Following QE purchases of home bonds by the home central bank, prices of home bonds rise. Arbitrageurs accommodate the increased demand for home bonds by holding fewer such bonds. This renders them less exposed to a rise in the home short rate and more willing to hold foreign currency, which depreciates when the home short rate rises. Arbitrageurs also become more willing to hold foreign bonds, which hedge the foreign currency position against a drop in the foreign short rate. Hence, QE purchases depreciate the home currency and lower foreign bond yields. A similar argument implies that sterilized purchases of foreign currency by the home or foreign central bank lower home bond yields and raise foreign ones.

Section 5 complements the analytical results with a quantitative exercise based on the full model with all five state variables. We take the two countries to be the US and the Eurozone, and estimate the model parameters by comparing empirical and model-implied moments of exchange rates, bond yields and trading volume. Our estimates of bond demand elasticity are in the same ballpark as those in Krishnamurthy and Vissing-Jorgensen (2012) and Koijen and Yogo (2020). The estimated model matches the evidence in Fama and Bliss (1987, FB) and Campbell and Shiller (1991, CS) on violations of EH, with the caveat that the model-implied FB coefficients do not increase with maturity and the CS coefficients do not decrease. The model also matches the evidence in Bilson (1981) and Fama (1984) on violations of UIP, and the related evidence in Lustig, Stathopoulos, and Verdelhan (2019, LSV) that the CCT's profitability declines when that trade is carried out with long-maturity bonds. The intuition in the case of LSV is that because the BCT is profitable, long-maturity bonds in high short-rate countries are expected to underperform relative to the short rate. The model also matches the evidence in Chernov and Creal (2020) and Lloyd and Marin (2020) that the CCT is less profitable when the slope of the yield curve in the high short-rate country is higher than in the low short-rate country. The intuition is that a high slope indicates low bond demand, which causes the currency to appreciate and its future expected return to decline.

In our estimated model, unconventional monetary policy has significant effects on bond yields, and these are transmitted almost one-to-one across countries. QE purchases in either country equal to 10% of GDP with a half-life of seven years reduce the home and foreign intermediate-maturity yields by about 50 basis points. By contrast, conventional policy has pronounced effects on domestic but not on foreign yields. The difference arises partly because positive correlation in short rates across countries magnifies the international transmission of unconventional policy and dampens that of conventional policy. In particular, in response to QE at home, arbitrageurs reduce their positions in home bonds, and rebuild their exposure to short-rate risk by buying positively correlated foreign bonds. Our results suggest that flexible exchange rates have better insulation properties under conventional than unconventional monetary policy.

Our paper is part of a recent literature that emphasizes the role of segmented markets, financial intermediaries and limits of arbitrage for macroeconomics. In Gabaix and Maggiori (2015), households in each of two countries can only invest in a domestic bond, while intermediaries can invest in the bonds of both countries. Because intermediary positions are constrained, UIP fails to hold, and exchange rates are influenced by financial flows as in an earlier literature on portfolio balance (e.g., Kouri (1976) and Driskill and McCafferty (1980)). In Itskhoki and Mukhin (2021), the exchange rate is determined by households who can only invest in a domestic bond, risk-averse intermediaries who can overcome this segmentation, and noise traders. Shocks to noise-trader demand generate UIP deviations, and account for more than 90% of exchange rate fluctuations but for only a small fraction of output fluctuations. Other frictional models of exchange rates with financial flows and noise traders include Jeanne and Rose (2002), Evans and Lyons (2002), Hau and Rey (2006), Bacchetta and van Wincoop (2010, 2021) and Bruno and Shin (2015). These papers focus on the determination of the exchange rate and do not model the yield curve.

Vayanos and Vila (2021, VV) develop a closed-economy model of the yield curve with preferred-habitat investors who can invest in specific maturities and risk-averse arbitrageurs. Our model extends VV to two countries, adds the currency market, and assumes global arbitrageurs. Ray (2019) and Droste, Gorodnichenko, and Ray (2021) embed VV into a New Keynesian model and study the transmission of conventional and unconventional monetary policy to the domestic economy. Closest to our work is Greenwood, Hanson, Stein, and Sunderam (2020), who develop independently a model of currency and bond markets with preferred-habitat investors and global arbitrageurs. They derive results analogous to those that we present in Section 4 and explore additionally Covered Interest Parity violations and arbitrageur heterogeneity, but do not estimate their model or quantify its effects. They assume only two bond maturities, while we assume a continuum. This makes our model more suitable for quantification and for examining how different policies affect the shape of the domestic and foreign yield curves.

Our paper is also related to a recent literature that examines how convenience yields affect exchange rates and interest rates. Engel and Wu (2018) and Jiang, Krishnamurthy, and Lustig (2021a) construct convenience yields by comparing home government bonds to synthetic counterparts constructed by buying foreign government bonds and swapping the foreign into the home currency. They find that the home currency appreciates when the home convenience yield rises. Jiang, Krishnamurthy, and Lustig (2021b) show that investor preferences for safe dollar assets underlie the global financial cycle (Rey, 2013) whereby US monetary policy transmits to the rest of the world. Our model can capture investor preferences for currencies and bonds of a specific country through the demand factors, and can quantify the effects of each type of demand.

Finally, our paper is related to DSGE models of monetary policy transmission. Closest to our work is the two-country model of Alpanda and Kabaca (2020). They find that QE purchases have large international spillover effects, which exceed those of conventional monetary policy. Portfolio balance effects in their model arise from bond holdings entering directly in agents' utility functions, while we partly endogenize them through mean-variance optimization by arbitrageurs.[3]

2 Model

Time is continuous and goes from zero to infinity. There are two countries, Home (H) and Foreign (F). We define the exchange rate as the units of home currency that one unit of foreign currency can buy, and denote it by e_t at time t. An increase in e_t corresponds to a home currency depreciation.

In each country $j = H, F$, a continuum of zero-coupon government bonds can be traded. The bonds' maturities lie in the interval $(0, T)$, where T can be finite or infinite. The country-j bond with maturity τ at time t pays off one unit of country j's currency at time $t+\tau$. We denote by $P_{jt}^{(\tau)}$ the time-t price of that bond, expressed in units of country j's currency, and by $y_{jt}^{(\tau)}$ the bond's yield. The yield is the spot rate for maturity τ, and

[3] See also Andres, Lopez-Salido, and Nelson (2004) and Chen, Curdia, and Ferrero (2012), for closed-economy DSGE models of QE in which agents face transaction costs and position limits when trading bonds.

is related to the price through

$$y_{jt}^{(\tau)} = -\frac{\log\left(P_{jt}^{(\tau)}\right)}{\tau}. \tag{2.1}$$

The country-j and time-t short rate i_{jt} is the limit of the yield $y_{jt}^{(\tau)}$ when τ goes to zero. We take i_{jt} as exogenous, and describe its dynamics later in this section (eq. (2.8)). An exogenous i_{jt} can be interpreted as the result of actions that the central bank in country j takes when targeting the short nominal rate by elastically supplying liquidity.

There are three types of agents: arbitrageurs, currency traders, and bond investors. Arbitrageurs are competitive and maximize a mean-variance objective over instantaneous changes in wealth. We express their wealth in units of the home currency, thus assuming that the home currency is the riskless asset for them. We allow arbitrage to be global or segmented. When arbitrage is global, arbitrageurs can invest in the currency and bonds of both countries. When instead arbitrage is segmented, arbitrageurs can invest in the currency of the home country (the riskless asset), and in a single additional asset class: foreign currency for some arbitrageurs, home bonds for others, and foreign bonds for the remainder. Our main analysis and results concern global arbitrage. We use segmented arbitrage as a benchmark.

In the case of global arbitrage, we denote by W_t the arbitrageurs' time-t wealth, by W_{Ht} and W_{Ft} their net position in home and foreign-currency instruments, respectively, and by $X_{Ht}^{(\tau)}d\tau$ and $X_{Ft}^{(\tau)}d\tau$ their position in the home and foreign bonds with maturities in $[\tau, \tau + d\tau]$, respectively, all expressed in units of the home currency.

The arbitrageurs' budget constraint is

$$\begin{aligned} W_{t+dt} = &\left(W_{Ht} - \int_0^T X_{Ht}^{(\tau)}d\tau\right)(1 + i_{Ht}dt) + \int_0^T X_{Ht}^{(\tau)}\frac{P_{H,t+dt}^{(\tau-dt)}}{P_{Ht}^{(\tau)}}d\tau \\ &+ \left(W_{Ft} - \int_0^T X_{Ft}^{(\tau)}d\tau\right)(1 + i_{Ft}dt)\frac{e_{t+dt}}{e_t} + \int_0^T X_{Ft}^{(\tau)}\frac{P_{F,t+dt}^{(\tau-dt)}e_{t+dt}}{P_{Ft}^{(\tau)}e_t}d\tau. \end{aligned} \tag{2.2}$$

The first term in (2.2) corresponds to a position in the home short rate, the second term to a position in home bonds, the third term to a position in the foreign short rate, and the fourth term to a position in foreign bonds. In the third term, $W_{Ft} - \int_0^T X_{Ft}^{(\tau)}d\tau$ units of the home currency are converted at time t to units of the foreign currency by dividing by e_t. They earn the foreign short rate between time t and $t + dt$, and are converted back at time $t + dt$ to units of the home currency by multiplying by e_{t+dt}. In the fourth term, $X_{Ft}^{(\tau)}$ units of the home currency are converted at time t to units of the foreign currency by dividing by e_t, and then to units of the foreign bond with maturity τ by dividing by $P_{Ft}^{(\tau)}$, the price of the bond in foreign currency. They are converted back at time $t + dt$ to units of the home currency by multiplying by $P_{F,t+dt}^{(\tau-dt)}e_{t+dt}$.

Subtracting $W_t = W_{Ht} + W_{Ft}$ from both sides of (2.2) and rearranging, we find

$$dW_t = W_t i_{Ht} dt + W_{Ft}\left(\frac{de_t}{e_t} + (i_{Ft} - i_{Ht})dt\right) + \int_0^T X_{Ht}^{(\tau)}\left(\frac{dP_{Ht}^{(\tau)}}{P_{Ht}^{(\tau)}} - i_{Ht}dt\right)d\tau + \int_0^T X_{Ft}^{(\tau)}\left(\frac{d(P_{Ft}^{(\tau)}e_t)}{P_{Ft}^{(\tau)}e_t} - \frac{de_t}{e_t} - i_{Ft}dt\right)d\tau. \tag{2.3}$$

If arbitrageurs invest all their wealth in the home short rate, then the instantaneous change dW_t in their wealth is $W_t i_{Ht} dt$, the first term in (2.3). Arbitrageurs can earn additional returns by investing in foreign currency, home bonds, and foreign bonds. The additional return from foreign currency derives from the *currency carry trade* (CCT) and corresponds to the second term in (2.3). The additional returns from home and foreign bonds derive from the home and foreign *bond carry trades* (BCT) and correspond to the third and fourth terms in (2.3). We return to the CCT and BCT in the next section.

The optimization problem of a global arbitrageur is

$$\max_{W_{Ft},\{X_{jt}^{(\tau)}\}_{\tau\in(0,T),j=H,F}} \left[\mathbb{E}_t(dW_t) - \frac{a}{2}\mathbb{V}\text{ar}_t(dW_t)\right]. \tag{2.4}$$

The coefficient $a \geq 0$ characterizes the arbitrageur's risk aversion. It can capture in reduced form capital or Value-at-Risk constraints. By possibly redefining a, we assume that global arbitrageurs are in measure one.

In the case of segmented arbitrage, the budget constraint of any given arbitrageur is derived from (2.3) by setting two of the terms to zero. For an arbitrageur who can invest only in foreign currency, the third and fourth terms are zero ($X_{Ht}^{(\tau)} = X_{Ft}^{(\tau)} = 0$); for an arbitrageur who can invest only in home bonds, the second and fourth terms are zero ($W_{Ft} = X_{Ft}^{(\tau)} = 0$); and for an arbitrageur who can invest only in foreign bonds, with a zero net position in foreign-currency instruments, the second and third terms are zero ($W_{Ft} = X_{Ht}^{(\tau)} = 0$). The optimization problem is derived from (2.4) by restricting the choice variables accordingly. We denote by a_e, a_H, and a_F, respectively, the risk-aversion coefficient of an arbitrageur who can invest in foreign currency, home bonds, and foreign bonds. By possibly redefining (a_e, a_H, a_F), we assume that each type of arbitrageur is in measure one.

Currency traders generate a downward-sloping demand for foreign currency as a function of the exchange rate e_t, as in Hau and Rey (2006). These agents can be interpreted as exporters and importers, or as central banks intervening in currency markets. For example, when e_t is low, the flow demand for foreign currency arising from exporters and importers due to expenditure switching may increase, as in Gabaix and Maggiori (2015), and this may push up the stock demand for foreign currency. Similarly, when e_t is low, the central bank in the home country may want to increase its stock of foreign currency, perhaps to stabilize the currency. For tractability, we assume that the demand of currency traders, expressed in units of the home currency, is affine and decreasing in the logarithm of the exchange rate:

$$Z_{et} = -\alpha_e \log(e_t) - (\zeta_{et} + \theta_e \gamma_t), \tag{2.5}$$

where $\alpha_e \geq 0$ is a slope coefficient, ζ_{et} is a deterministic term, θ_e is a constant, and γ_t is a demand risk factor. We describe the dynamics of γ_t and motivate the deterministic term ζ_{et} later in this section.

The demand (2.5) for foreign currency is expressed in the spot market. Most of the trading volume in currency markets, however, originates in forwards and swaps (BIS, 2019). We show in Appendix A that under global arbitrage, Covered Interest Parity (CIP) holds and the demand for foreign-currency forwards and swaps can be expressed as a combination of demands for spot currency and for domestic and foreign bonds. Therefore, our model can accommodate foreign-currency forwards and swaps by redefining currency and bond demands.

Bond investors have preferences ("habitats") for specific countries and maturities. For example, pension funds in the home country prefer long-maturity home bonds because these match their pension liabilities, which are long term and denominated in home currency. At the other end of the maturity spectrum, home money-market funds are required by their mandates to hold short-maturity home bonds. For tractability, we assume that preferences take an extreme form, where investors demand only the bond closest to their preferred characteristics. That is, investors with preferences for country j and maturity τ at time t hold a position $Z_{jt}^{(\tau)}$ in the country-j bond with maturity τ and hold no other bond. We assume that maturity preferences cover the interval $(0, T)$, and investors with preferences for country j and maturities in $[\tau, \tau + d\tau]$ are in measure $d\tau$. We express the position $Z_{jt}^{(\tau)}$ in units of the home currency, and assume that it is affine and decreasing in the logarithm of the bond price:

$$Z_{jt}^{(\tau)} = -\alpha_j(\tau) \log\left(P_{jt}^{(\tau)}\right) - \beta_{jt}^{(\tau)}. \tag{2.6}$$

The slope coefficient $\alpha_j(\tau) \geq 0$ is constant over time but can depend on country j and maturity τ. The intercept coefficient $\beta_{jt}^{(\tau)}$ can depend on t, τ, and j. With a slight abuse of language, we refer to $\alpha_j(\tau)$ and $\beta_{jt}^{(\tau)}$ as demand slope and demand intercept, respectively.

The demand intercept $\beta_{jt}^{(\tau)}$ takes the form

$$\beta_{jt}^{(\tau)} = \zeta_j(\tau) + \theta_j(\tau)\beta_{jt}, \tag{2.7}$$

where $(\zeta_j(\tau), \theta_j(\tau))$ are constant over time but can depend on country j and maturity τ, and β_{jt} is independent of τ but can depend on country j and time t. We refer to β_{jt} as a demand risk factor. Vayanos and Vila (2021) provide an optimizing foundation for the demand specification (2.6)-(2.7) in a setting where investors form overlapping generations consuming at the end of their life and are infinitely risk-averse.

The 5×1 vector $q_t \equiv (i_{Ht}, i_{Ft}, \gamma_t, \beta_{Ht}, \beta_{Ft})^\top$ follows the process

$$dq_t = \Gamma(\bar{q} - q_t)dt + \Sigma dB_t, \tag{2.8}$$

where $\bar{q}$ is a constant 5×1 vector, (Γ, Σ) are constant 5×5 matrices, B_t is a 5×1 vector $(B_{iHt}, B_{iFt}, B_{\gamma t}, B_{\beta Ht}, B_{\beta Ft})^\top$ of independent Brownian motions, and $\top$ denotes transpose. Equation (2.8) nests the case where the factors $(i_{Ht}, i_{Ft}, \gamma_t, \beta_{Ht}, \beta_{Ft})$ are mutually independent, and the case where they are correlated. Independence arises when the matrices (Γ, Σ) are diagonal. When instead Σ is non-diagonal, shocks to the factors are correlated, and when Γ is non-diagonal, the drift (instantaneous expected change) of each factor depends on all other factors. We assume that the eigenvalues of Γ have positive real parts so that q_t is stationary. Equation (2.8) implies that the long-run mean of a stationary q_t is $\bar{q}$. We set the long-run means of the demand factors to zero ($\bar{q}_3 = \bar{q}_4 = \bar{q}_5 = 0$). This is without loss of generality since we can redefine ζ_{et} and $\{\zeta_j(\tau)\}_{j=H,F}$ to include a non-zero long-run mean. We set the supply of foreign currency and home and foreign bonds to zero by redefining investor demand to be net of supply.

Key to the tractability of our model is that all demand functions are expressed in terms of the same numeraire, which is the riskless asset for arbitrageurs. The numeraire can be the currency of one of the two countries, and we take it to be the home currency. One limiting feature of this assumption is that the home currency must be the riskless asset for all arbitrageurs, even foreign ones. Our assumption also precludes that exchange-rate movements, holding foreign bond yields constant, affect foreign bond demand in home currency terms.

Our model can be given both a nominal and a real interpretation. Our presentation so far focuses on the nominal interpretation: the exchange rate is the price of one currency relative to the other, bonds pay in currency units, preferences of arbitrageurs concern their nominal wealth, the demand of currency traders is a function of the nominal exchange rate, and preferences of bond investors concern their nominal consumption. A difficulty with the nominal interpretation is that the demand of currency traders such as exporters and importers is better viewed as a function of the real, rather than the nominal, exchange rate. To make the nominal interpretation compatible with a real currency demand, we can replace the nominal exchange rate e_t in (3.1) by the real exchange rate. This amounts to keeping e_t inside the logarithm and adding $\alpha_e(\log(p_{Ft}) - \log(p_{Ht}))$ to ζ_{et}, where p_{jt} is the price level in country $j = H, F$. Hence, under the nominal interpretation, we can take ζ_{et} to be $\alpha_e(\log(p_{Ft}) - \log(p_{Ht}))$. Our formal analysis in subsequent sections can accommodate this interpretation as long as we ignore inflation risk, i.e., assume that

$\log(p_{Ft}) - \log(p_{Ht})$ is deterministic.

An alternative interpretation of our model is real: the exchange rate e_t is the real exchange rate defined as the price of goods in one country relative to the other, bonds pay in units of goods with a real price $P_{jt}^{(\tau)}$, preferences of arbitrageurs concern their real wealth, the demand of currency traders depends on the real exchange rate, and preferences of bond investors concern their real consumption. Under the real interpretation, we can take ζ_{et} to be a constant, ζ_e.

In what follows, we present the nominal interpretation of the model in the special case where the inflation rate is constant in each country: $\zeta_{et} = \zeta_e + \alpha_e(\pi_F - \pi_H)t$, where π_j is the constant inflation rate in country j and ζ_e is a constant.

3 Equilibrium

In this section we characterize the equilibrium with global arbitrage. We start by conjecturing a functional form for the equilibrium exchange rate and bond yields. Using that functional form, we next simplify the arbitrageurs' objective and derive their first-order conditions. We finally combine the first-order conditions with market clearing, and confirm that equilibrium prices are as conjectured.

We conjecture that the equilibrium exchange rate and bond yields are log-affine functions of q_t. That is, there exist six scalars $(\{A_{ije}\}_{j=H,F}, A_{\gamma e}, \{A_{\beta je}\}_{j=H,F}, C_e)$ and twelve functions $(\{A_{ijj'}(\tau)\}_{j,j'=H,F}, \{A_{\gamma j}(\tau)\}_{j=H,F}, \{A_{\beta jj'}(\tau)\}_{j,j'=H,F}, \{C_j(\tau)\}_{j=H,F})$ that depend only on τ, such that

$$\log e_t = -\left[A_e^\top q_t + C_e + (\pi_F - \pi_H)t\right], \tag{3.1}$$

$$\log P_{jt}^{(\tau)} = -\left[A_j(\tau)^\top q_t + C_j(\tau)\right], \tag{3.2}$$

where

$$A_e \equiv (A_{iHe}, -A_{iFe}, A_{\gamma e}, A_{\beta He}, -A_{\beta Fe})^\top, \tag{3.3}$$

$$A_j(\tau) \equiv (A_{iHj}(\tau), A_{iFj}(\tau), A_{\gamma j}(\tau), A_{\beta Hj}(\tau), A_{\beta Fj}(\tau))^\top. \tag{3.4}$$

Applying Ito's Lemma to (3.1) and using (2.8), we find

$$\frac{de_t}{e_t} = \mu_{et}dt - A_e^\top \Sigma dB_t, \tag{3.5}$$

where

$$\mu_{et} \equiv A_e^\top \Gamma(q_t - \bar{q}) - (\pi_F - \pi_H) + \frac{1}{2}A_e^\top \Sigma\Sigma^\top A_e. \tag{3.6}$$

Applying Ito's Lemma to (3.2) for $j = H$ and using (2.8), we find

$$\frac{dP_{Ht}^{(\tau)}}{P_{Ht}^{(\tau)}} = \mu_{Ht}^{(\tau)}dt - A_H(\tau)^\top \Sigma dB_t, \tag{3.7}$$

where

$$\mu_{Ht}^{(\tau)} \equiv A'_H(\tau)^\top q_t + C'_H(\tau) + A_H(\tau)^\top \Gamma(q_t - \bar{q}) + \frac{1}{2} A_H(\tau)^\top \Sigma\Sigma^\top A_H(\tau). \tag{3.8}$$

Likewise, (3.2) for $j = F$ and (3.1) together imply

$$\frac{d(P_{Ft}^{(\tau)} e_t)}{P_{Ft}^{(\tau)} e_t} - \frac{de_t}{e_t} = \mu_{Ft}^{(\tau)}dt - A_F(\tau)^\top \Sigma dB_t, \tag{3.9}$$

where

$$\mu_{Ft}^{(\tau)} \equiv A'_F(\tau)^\top q_t + C'_F(\tau) + A_F(\tau)^\top \Gamma(q_t - \bar{q}) + \frac{1}{2} A_F(\tau)^\top \Sigma\Sigma^\top \left(A_F(\tau) + 2A_e\right). \tag{3.10}$$

To derive the arbitrageurs' first-order conditions, we substitute (3.5), (3.7) and (3.9) into the budget constraint (2.3) and write the optimization problem (2.4) as

$$\max_{W_{Ft}, \{X_{jt}^{(\tau)}\}_{\tau \in (0,T), j=H,F}} \Bigg[W_{Ft}\left(\mu_{et} + i_{Ft} - i_{Ht}\right) + \sum_{j=H,F} \int_0^T X_{jt}^{(\tau)} \left(\mu_{jt}^{(\tau)} - i_{jt}\right) d\tau$$
$$- \frac{a}{2} \left(W_{Ft} A_e + \sum_{j=H,F} \int_0^T X_{jt}^{(\tau)} A_j(\tau) d\tau \right)^\top \Sigma\Sigma^\top \left(W_{Ft} A_e + \sum_{j=H,F} \int_0^T X_{jt}^{(\tau)} A_j(\tau) d\tau \right) \Bigg]. \tag{3.11}$$

The first-order condition with respect to W_{Ft} is

$$\mu_{et} + i_{Ft} - i_{Ht} = A_e^\top \lambda_t, \tag{3.12}$$

and the first-order condition with respect to $X_{jt}^{(\tau)}$ is

$$\mu_{jt}^{(\tau)} - i_{jt} = A_j(\tau)^\top \lambda_t, \tag{3.13}$$

where $j = H, F$, $\lambda_t \equiv (\lambda_{iHt}, \lambda_{iFt}, \lambda_{\beta Ht}, \lambda_{\beta Ft}, \lambda_{\gamma t})^\top$ and

$$\lambda_t \equiv a\Sigma\Sigma^\top \left(W_{Ft} A_e + \sum_{j=H,F} \int_0^T X_{jt}^{(\tau)} A_j(\tau) d\tau \right). \tag{3.14}$$

The first-order condition (3.12) describes the arbitrageurs' risk-return trade-off when investing in the *currency carry trade* (CCT). We term CCT the trade of borrowing short-term in the home country, exchanging the borrowed amount in the foreign currency, investing it short-term in the foreign country, and exchanging it back in the home currency.[4] The CCT's return is $\frac{de_t}{e_t} + (i_{Ft} - i_{Ht})dt$, equal to the return on foreign currency plus that on the foreign-home short-rate differential.

If arbitrageurs invest an extra unit of home currency in the CCT, then their annualized expected return increases by $\mu_{et} + i_{Ft} - i_{Ht}$. This is the left-hand side of (4.2), and we refer to it as the CCT's expected return (omitting that it is annualized). The right-hand side of (4.2) is the increase in the arbitrageurs' portfolio risk, times their risk-aversion coefficient a. The increase in portfolio risk is equal to the covariance between the return on the CCT and the return on the arbitrageurs' portfolio. The covariance is the product of the vectors that describe the sensitivity of each return to the risk factors, times the factors' covariance matrix $\Sigma\Sigma^\top$. The sensitivity of the CCT's return to the factors is A_e, and the sensitivity of the portfolio return is $W_{Ft}A_e + \sum_{j=H,F} \int_0^T X_{jt}^{(\tau)} A_j(\tau) d\tau$.

The first-order condition (3.13) describes the arbitrageurs' risk-return trade-off when investing in the *bond carry trade* (BCT) in country j. We term BCT in country j the trade of borrowing short-term in that country and investing the borrowed amount in that country's bonds.[5] The return on the BCT in the home country and for maturity τ is $\frac{dP_{Ht}^{(\tau)}}{P_{Ht}^{(\tau)}} - i_{Ht}dt$, equal to the return on the home bond with maturity τ minus that on the home short rate. The return on the BCT in the foreign country, expressed in home-currency terms, is $\frac{d(P_{Ft}^{(\tau)}e_t)}{P_{Ft}^{(\tau)}e_t} - \frac{de_t}{e_t} - i_{Ft}dt$. This is equal to the return on the foreign bond with maturity τ, expressed in home-currency terms, minus that on foreign currency, minus that on the foreign short rate.

If arbitrageurs invest an extra unit of home currency in the BCT for country j and maturity τ, then their annualized expected return increases by $\mu_{jt}^{(\tau)} - i_{jt}$. This is the left-hand side of (4.3), and we refer to it as the BCT's expected return. The right-hand side of (4.3) is the increase in the arbitrageurs' portfolio risk, times their risk-aversion coefficient a. The increase in portfolio risk is equal to the covariance between the return on the BCT in country j and for maturity τ, and the return on the arbitrageurs' portfolio.

The first-order conditions (3.12) and (3.13) can be interpreted in the language of no-arbitrage models. No-arbitrage in continuous time requires that there exist prices specific to each risk factor and common across assets, such that the expected return of any zero-cost portfolio is equal to the sum across factors of the portfolio's sensitivity to each factor times the factor's price. The factor prices are the elements of the vector λ_t.

We next combine the first-order conditions with market clearing. Clearing in the

[4]For simplicity, we deviate from market terminology, according to which the CCT borrows in the currency with the low interest rate.

[5]For simplicity, we deviate from market terminology, according to which the BCT borrows at maturities with a low interest rate.

currency market requires

$$W_{Ft} + Z_{et} = 0. \tag{3.15}$$

Clearing in the market for country j bonds with maturity τ requires

$$X_{jt}^{(\tau)} + Z_{jt}^{(\tau)} = 0. \tag{3.16}$$

Using (3.15) and (3.16), we can write λ_t as

$$\lambda_t = a\Sigma\Sigma^\top \left(-Z_{et}A_e - \sum_{j=H,F} \int_0^T Z_{jt}^{(\tau)} A_j(\tau) d\tau \right). \tag{3.17}$$

Substituting the demands Z_{et} and $\{Z_{jt}^{(\tau)}\}_{j=H,F}$ from (2.6) and (2.5), respectively, we can write (3.17) as

$$\begin{aligned}
\lambda_t &= a\Sigma\Sigma^\top \Bigg(\left(\alpha_e \log(e_t) + \zeta_e + \alpha_e(\pi_F - \pi_H)t + \theta_e \gamma_t\right) A_e \\
&\quad + \sum_{j=H,F} \int_0^T \left(\alpha_j(\tau) \log\left(P_{jt}^{(\tau)}\right) + \zeta_j(\tau) + \theta_j(\tau)\beta_{jt} \right) A_j(\tau) d\tau \Bigg) \\
&= a\Sigma\Sigma^\top \Bigg(\left(\zeta_e + \theta_e\gamma_t - \alpha_e \left(A_e^\top q_t + C_e\right)\right) A_e \\
&\quad + \sum_{j=H,F} \int_0^T \left(\zeta_j(\tau) + \theta_j(\tau)\beta_{jt} - \alpha_j(\tau) \left(A_j(\tau)^\top q_t + C_j(\tau)\right)\right) A_j(\tau) d\tau \Bigg),
\end{aligned} \tag{3.18}$$

where the second step follows from (3.1) and (3.2). Substituting λ_t from (3.18) and μ_{et} from (3.6) into (3.12), we find an equation that are affine in q_t. We find two additional affine equations by substituting λ_t from (3.18), $\mu_{Ht}^{(\tau)}$ from (3.8) and $\mu_{Ft}^{(\tau)}$ from (3.10) into (3.12) for $j = H, F$. Identifying linear and constant terms yields a system of scalar equations and ODEs, which can be solved for the equilibrium exchange rate and bond yields. To state these equations, we denote by $(\mathcal{E}_{iH}, \mathcal{E}_{iF}, \mathcal{E}_\gamma, \mathcal{E}_{\beta H}, \mathcal{E}_{\beta F})$ the five 5×1 vectors that correspond to the five consecutive columns of the 5×5 identity matrix.

Proposition 3.1. *When arbitrage is global, the exchange rate e_t is given by (3.1) and bond prices $P_{jt}^{(\tau)}$ in country $j = H, F$ are given by (3.2), with (A_e, C_e) solving*

$$MA_e - \mathcal{E}_{iH} + \mathcal{E}_{iF} = 0, \tag{3.19}$$

$$-A_e^\top \Gamma \bar{q} - (\pi_F - \pi_H) + \frac{1}{2} A_e^\top \Sigma\Sigma^\top A_e = A_e^\top \lambda_C, \tag{3.20}$$

and $(A_j(\tau), C_j(\tau))$ *solving*

$$A_j'(\tau) + MA_j(\tau) - \mathcal{E}_{ij} = 0, \tag{3.21}$$

$$C_j'(\tau) - A_j(\tau)^\top \Gamma \bar{q} + \frac{1}{2} A_j(\tau)^\top \Sigma\Sigma^\top \left(A_j(\tau) + 2A_e 1_{\{j=F\}} \right) = A_j(\tau)^\top \lambda_C, \tag{3.22}$$

with the initial conditions $A_j(0) = C_j(0) = 0$*, and*

$$M \equiv \Gamma^\top - a \Bigg((\theta_e \mathcal{E}_\gamma - \alpha_e A_e) A_e^\top + \sum_{j=H,F} \int_0^T (\theta_j(\tau)\mathcal{E}_{\beta j} - \alpha_j(\tau) A_j(\tau)) A_j(\tau)^\top d\tau \Bigg) \Sigma\Sigma^\top, \tag{3.23}$$

$$\lambda_C \equiv a\Sigma\Sigma^\top \left((\zeta_e - \alpha_e C_e) A_e + \sum_{j=H,F} \int_0^T (\zeta_j(\tau) - \alpha_j(\tau) C_j(\tau)) A_j(\tau) d\tau \right). \tag{3.24}$$

Equation (3.21) is a linear ODE system in the 5×1 vector $A_j(\tau)$. We solve it taking the 5×5 matrix M as given. Taking M as given, we also solve the linear scalar system (3.19) in A_e. We then substitute $(A_e, \{A_j(\tau)\}_{j=H,F})$ in (3.23) and derive M as solution to a system of nonlinear scalar equations. In the general case, the system consists of 25 equations, and we solve it numerically as described in Appendix C. In the case where there is no demand risk, the system's dimensionality drops substantially and we can derive analytical results, as shown in Section 4.

Our model yields Uncovered Interest Parity (UIP) and the Expectations Hypothesis (EH) as a special case. Both properties hold when arbitrageurs are risk-neutral ($a = 0$). Setting $a = 0$ in (3.12) yields $\mu_{et} = i_{Ht} - i_{Ft}$, as under UIP. Setting $a = 0$ in (3.13) yields $\mu_{jt}^{(\tau)} \equiv i_{jt}$, as under EH. Since for $a = 0$, (3.23) implies $M = \Gamma^\top$ and (3.24) implies $\lambda_C = 0$, (3.19) implies $A_e = (\Gamma^{-1})^\top (\mathcal{E}_{iH} - \mathcal{E}_{iF})$ and (3.20) implies

$$\bar{i}_H - \pi_H = \bar{i}_F - \pi_F + \frac{1}{2} (\mathcal{E}_{iH} - \mathcal{E}_{iF})^\top \Gamma^{-1} \Sigma\Sigma^\top \left(\Gamma^{-1}\right)^\top (\mathcal{E}_{iH} - \mathcal{E}_{iF}), \tag{3.25}$$

where $\bar{i}_j$ denotes the unconditional mean of the nominal short rate in country j. According to (3.25), the unconditional mean $\bar{i}_j - \pi_j$ of the real interest rate in country j is equal across home and foreign, up to a convexity adjustment (the final term in (3.25)). Equality of average real rates is a restriction on model parameters. It must hold for $a = 0$ because of the stationarity of the real exchange rate, which is implicit in the conjectured form (3.1). Indeed, if average real rates differed across countries and arbitrageurs were risk neutral, then the real exchange rate would appreciate on average or depreciate on average forever, violating stationarity.

When arbitrageurs are risk averse, a stationary equilibrium can exist even when average real rates differ across countries and even in the limit when risk aversion goes to zero. This is because any difference in average real rates is absorbed in equilibrium by an adjustment in currency risk premia. The currency of the country with a higher average real rate is permanently stronger and earns a positive premium. Arbitrageurs have a high demand for that currency, through their position in the CCT, and earn the positive premium. Currency traders have offsetting low demand if their demand is price-elastic ($\alpha_e > 0$). In the limit when arbitrageurs' risk aversion a goes to zero, risk premia remain non-zero and arbitrageurs' position in the CCT becomes arbitrarily large. Corollary 3.1 summarizes these results.

Corollary 3.1. *Suppose that arbitrage is global.*

- *When arbitrageurs are risk-neutral ($a = 0$), UIP and EH hold: the expected return on foreign currency is $\mu_{et}^{UIP} \equiv i_{Ht} - i_{Ft}$, and the expected return on country-j bonds is $\mu_{jt}^{(\tau)EH} \equiv i_{jt}$. Stationarity of the real exchange rate, as per the conjecture (3.1), requires (3.25).*
- *When arbitrageurs' risk aversion goes to zero ($a \to 0$), the expected return on foreign currency goes to μ_{et}^{UIP} and the expected return on country-j bonds goes to $\mu_{jt}^{(\tau)EH}$ only when (3.25) holds. Stationarity of the real exchange rate does not require (3.25) if $\alpha_e > 0$.*

4 No Demand Risk

In this section we study the case where the demand for foreign currency and home and foreign bonds does not vary stochastically: the demand factors $(\gamma_t, \beta_{Ht}, \beta_{Ft})$ are equal to their mean of zero in steady state. For simplicity we also assume that the home and foreign short rates (i_{Ht}, i_{Ft}) are independent and that one-off shocks to the demand factors do not affect the short rates or other demand factors. Our assumptions amount to taking the matrices (Γ, Σ) in (2.8) to be diagonal and to setting $\Sigma_{3,3} = \Sigma_{4,4} = \Sigma_{5,5} = 0$. Setting $(\Gamma_{1,1}, \Gamma_{2,2}, \bar{q}_1, \bar{q}_2, \Sigma_{1,1}, \Sigma_{2,2}) \equiv (\kappa_{iH}, \kappa_{iF}, \bar{i}_H, \bar{i}_F, \sigma_{iH}, \sigma_{iF})$, we can write the dynamics of the country-j short rate as

$$di_{jt} = \kappa_{ij}(\bar{i}_j - i_{jt})dt + \sigma_{ij} dB_{ijt}. \tag{4.1}$$

The simplifying assumptions in this section allow us to study the equilibrium analytically. We begin by analyzing segmented arbitrage and then return to global arbitrage.

4.1 Segmented Arbitrage

When arbitrage is segmented, the first-order condition of the arbitrageurs in each market reflects their own risk aversion and portfolio composition. The first-order condition (3.12) in the currency market becomes

$$\mu_{et} + i_{Ft} - i_{Ht} = A_e^\top \lambda_{et}, \tag{4.2}$$

where $A_e \equiv (A_{iHe}, -A_{iFe})$ and $\lambda_{et} \equiv a_e W_{Ft}(\sigma_{iH}^2 A_{iHe}, -\sigma_{iF}^2 A_{iFe})$. Relative to (3.12), the definition of A_e is modified to include the CCT's sensitivity to the short rates only because these are the sole risk factors. The definition of λ_t is also modified and includes the subscript e since when arbitrage is segmented, factor prices differ across bond and currency markets. Factor prices λ_{et} in the currency market reflect the risk aversion a_e of currency arbitrageurs and their position W_{Ft} in foreign currency. They do not depend on positions in home or foreign bonds since currency arbitrageurs do not hold such positions.

The first-order condition (3.13) in the country-j bond market becomes

$$\mu_{jt}^{(\tau)} - i_{jt} = A_j(\tau)\lambda_{jt}, \tag{4.3}$$

where $A_j(\tau) \equiv A_{ijj}(\tau)$ and $\lambda_{jt} \equiv a_j \sigma_{ij}^2 \int_0^T X_{jt}^{(\tau)} A_{ijj}(\tau) d\tau$. Relative to (3.13), the definition of A_e is modified to include the BCT's sensitivity to the country-j short rate only because with segmented arbitrage this is the only risk factor for country-j bonds. The definition of λ_t is also modified and includes the subscript j. Factor prices λ_{jt} in the country-j bond market reflect the risk aversion a_j of country-j bond arbitrageurs and their positions $X_{jt}(\tau)$ in these bonds.

Solving for equilibrium in the currency market reduces to a nonlinear scalar equation (Proposition B.1 in Appendix B). The same is true for equilibrium in the country-j bond market (Proposition B.2 in Appendix B). Using our characterization of equilibrium we next derive analytically equilibrium properties.

4.1.1 Short-Rate Shocks and Carry Trades

Proposition 4.1. *Suppose that arbitrage is segmented. Following a drop in the home short rate or a rise in the foreign short rate, the foreign currency appreciates ($A_{iHe} > 0$, $A_{iFe} > 0$). When additionally currency arbitrageurs are risk-averse ($a_e > 0$) and the demand of currency traders is price-elastic ($\alpha_e > 0$),*

- *The foreign currency does not appreciate all the way to the level implied by UIP: $A_{iHe} < A_{iHe}^{UIP}$, $A_{iFe} < A_{iFe}^{UIP}$.*
- *The expected return of the CCT rises: $\frac{\partial(\mu_{et}+i_{Ft}-i_{Ht})}{\partial i_{Ht}} < 0$ and $\frac{\partial(\mu_{et}+i_{Ft}-i_{Ht})}{\partial i_{Ft}} > 0$.*

When the home short rate drops or the foreign short rate rises, the foreign currency appreciates. These movements are in the direction implied by UIP. The foreign currency does not appreciate all the way to the value implied by UIP, however. To explain the mechanism, we begin by assuming that the exchange rate remains the same as before the shock. The drop in i_{Ht} or rise in i_{Ft} render the CCT more profitable, raising its expected return $\mu_{et} + i_{Ft} - i_{Ht}$ and inciting arbitrageurs to increase their holdings W_{Ft} of the foreign currency. This puts upward pressure on the exchange rate. When the demand by currency traders is price-elastic, their holdings Z_{et} decrease as the foreign currency appreciates and those of currency arbitrageurs W_{Ft} increase in equilibrium. Risk-averse arbitrageurs, however, do not trade all the way to the point where e_t reaches its UIP value. Instead, in a spirit similar to Gabaix and Maggiori (2015), the CCT's expected return $\mu_{et} + i_{Ft} - i_{Ht}$ remains higher than before the shock to compensate arbitrageurs for the risk generated by their larger foreign-currency position. The exchange rate adjusts all the way to its UIP value when currency arbitrageurs are risk-neutral or when the demand by currency traders is price-inelastic. In the latter case this is because arbitrageurs' activity causes prices to rise up to the point where there is no change in W_{Ft}.

Proposition 4.1 implies that the difference between the foreign and the home short rate predicts positively the CCT's future return. This is consistent with the evidence in Bilson (1981) and Fama (1984), who document that following an increase in the foreign-minus-home short-rate differential, the expected return on the foreign currency typically increases. Moreover, even in samples where it decreases, it does so less than implied by UIP. Hence, the CCT becomes more profitable.

Proposition 4.2. *Suppose that arbitrage is segmented. Following a drop in the short rate in country j, bond yields drop in that country ($A_{ijj}(\tau) > 0$) and do not change in the other country ($A_{ij'j}(\tau) = 0$ for $j' \neq j$). When additionally bond arbitrageurs in country j are risk-averse ($a_j > 0$) and the demand of bond investors in that country is price-elastic ($\alpha_j(\tau) > 0$):*

- *Bond yields do not drop all the way to the value implied by the EH: $A_{ijj}(\tau) < A^{EH}_{ijj}(\tau)$.*
- *The expected return of the BCT rises: $\frac{\partial\left(\mu^{(\tau)}_{jt} - i_{jt}\right)}{\partial i_{jt}} < 0$.*

When the short rate in country j drops, bond prices in that country rise (and bond yields drop) because of a standard discounting effect. Prices do not rise all the way to the value implied by the EH, however. To explain the mechanism, we begin by assuming that bond prices remain the same as before the shock. The drop in the short rate renders the BCT in country j more profitable, raising its expected return $\mu^{(\tau)}_{jt} - i_{jt}$ and inciting arbitrageurs in country j to increase their bond holdings $X^{(\tau)}_{jt}$. This puts upward pressure on bond prices $P^{(\tau)}_{jt}$. When the demand by bond investors in country j is price-elastic, their holdings $Z^{(\tau)}_{jt}$ decrease as bond prices rise and those of bond arbitrageurs $X^{(\tau)}_{jt}$ increase in equilibrium. Risk-averse arbitrageurs, however, do not trade all the way to the point where bond prices reach their EH value. Instead, as Vayanos and Vila (2021) show for a closed economy, the BCT's expected return $\mu^{(\tau)}_{jt} - i_{jt}$ remains higher than before the shock to compensate arbitrageurs for the risk generated by their larger bond position. Bond prices adjust all the way to their EH value when bond arbitrageurs in country j are risk-neutral or when the demand by bond investors in country j is price-inelastic.

Proposition 4.2 implies that the slope of the term structure in country j predicts positively the BCT's future return in that country. Indeed, slope and future return vary over time only because of the country j short rate i_{jt}, and are both high when i_{jt} is low. A positive relationship between the slope of the term structure and the BCT's future return is documented in Fama and Bliss (1987), but is inconsistent with the EH according to which the BCT's expected return should be zero. Campbell and Shiller (1991) document a related violation of the EH: the slope of the term structure in country j predicts negatively changes in future long rates in that country.

4.1.2 Demand Shocks

We next determine how the exchange rate and bond yields respond to changes in the demand for foreign currency and bonds. Since we assume no demand risk in this section, we take the demand changes to be unanticipated and one-off. Demand changes by currency traders correspond to shocks to the demand factor γ_t. Demand changes by bond investors in country j correspond to shocks to the demand factor β_{jt}. Following the shocks, the demand factors revert deterministically to their mean of zero. The effects of unanticipated and one-off shocks are the limit of those under anticipated and recurring shocks (Section 5) when the shocks' variance goes to zero.

Without loss of generality, we take θ_e to be positive, which means that an increase in γ_e corresponds to a drop in demand for foreign currency. We take $\theta_j(\tau)$ to be positive for all τ, which means that an increase in β_{jt} corresponds to a drop in demand for the bonds of country j.

Proposition 4.3. *Suppose that arbitrage is segmented, $\theta_e > 0$ and $\theta_j(\tau) > 0$ for all τ.*

- *An unanticipated one-off drop in currency traders' demand for foreign currency (increase in γ_e) causes the foreign currency to depreciate if currency traders are risk-averse ($a_e > 0$). It has no effect on bond yields.*
- *An unanticipated one-off drop in investor demand for the bonds of country j (increase in β_{jt}) raises bond yields in country j if bond arbitrageurs in that country are risk-averse ($a_j > 0$). It has no effect on bond yields in the other country and on the exchange rate.*

When arbitrage is segmented, changes to the demand for an asset class—foreign currency, home bonds, foreign bonds—affect that asset class only. When, for example, the demand for bonds in country j drops, these bonds become cheaper and their yields increase, while foreign currency and bonds in the other country are unaffected.

4.1.3 Monetary Policy Transmission

We next summarize the implications of the segmented-arbitrage model for the domestic and international transmission of monetary policy. Consider first conventional monetary easing at home, modelled as an unanticipated cut to the home short rate i_{Ht} by the central bank. The rate cut propagates along the home term structure, although less than implied by EH (Proposition 4.2). Moreover, the home currency depreciates, although less than implied by UIP (Proposition 4.1). Propagation is imperfect (compared to EH and UIP) because bond and foreign-currency arbitrageurs must be compensated for the change in their portfolio holdings. The cut to the home short rate does not affect the foreign term structure (Proposition 4.2), and hence has no effect on foreign monetary conditions. In that sense, the model with segmented arbitrage features *full insulation*.

Consider next quantitative easing (QE) at home, consisting of purchases of home bonds by the central bank and modelled as an unanticipated drop in the demand factor β_{Ht}. QE lowers home bond yields (Proposition 4.3). It does not effect the foreign term structure, and hence has no effect on foreign monetary conditions. Once again, the model with segmented arbitrage features *full insulation*. An additional implication of the insulation present in the segmented-arbitrage model is that QE transmits differently to the domestic economy than conventional monetary easing. Indeed, while the exchange rate drops in response to conventional easing, it remains unaffected with QE (Proposition 4.3).

Even though the segmented-arbitrage model delivers deviations from UIP and the EH, its implications for policy spillovers are similar to the standard model. This similarity masks differences in the underlying mechanisms. In the segmented-arbitrage model, the insulation of the term structure arises entirely from the assumption that the home and foreign bond markets are segmented. In Trilemma terms, insulation is due to restrictions on capital flows. As we show in the next section, when arbitrageurs are active in all markets, insulation breaks down.

4.2 Global Arbitrage

The first-order conditions are (3.12) and (3.13), with $A_e \equiv (A_{iHe}, -A_{iFe})$, $A_j(\tau) \equiv (A_{iHj}(\tau), A_{iFj}(\tau))$ and $\lambda_t \equiv (\lambda_{iHt}, \lambda_{iFt})$ with

$$\lambda_{ijt} \equiv a\sigma_{ij}^2 \left(W_{Ft} A_{ije} (-1)^{1_{\{j=F\}}} + \sum_{j'=H,F} \int_0^T X_{j't}^{(\tau)} A_{ijj'}(\tau) d\tau \right). \tag{4.4}$$

The definitions of A_e and $A_j(\tau)$ include the CCT's and BCT's sensitivity to only the short rates because these are the only risk factors. Solving for equilibrium reduces to a system of three nonlinear scalar equations (Proposition B.3 in Appendix B).

4.2.1 Short-Rate Shocks and Carry Trades

Proposition 4.4. *Suppose that arbitrage is global.*

- *The effects of short-rate shocks on the exchange rate and on the CCT's expected* *return have the same properties as in Proposition 4.1, except that the price-elasticity condition can hold for currency traders or bond investors ($\alpha_e > 0$ or $\alpha_j(\tau) > 0$).*

- *The effects of shocks to the country-j short rate i_{jt} on bond yields in country j and on the BCT's expected return have the same properties as in Proposition 4.2, except that the price-elasticity condition can hold for currency traders or bond investors ($\alpha_e > 0$ or $\alpha_j(\tau) > 0$).*
- *When arbitrageurs are risk-averse ($a > 0$) and the demand by currency traders is price-elastic ($\alpha_e > 0$), a drop in i_{jt} causes bond yields in country $j' \neq j$ to drop ($A_{j'j}(\tau) > 0$) and the BCT's expected return to drop ($\frac{\partial\left(\mu^{(\tau)}_{j't} - i_{j't}\right)}{\partial i_{jt}} > 0$).*
- *The effect of i_{jt} on bond yields is smaller in country j' than in country j ($A_{jj}(\tau) > A_{j'j}(\tau)$).*

Bond yields respond to shocks differently under global and segmented arbitrage. Under segmented arbitrage, a shock to the short rate i_{jt} in country j affects bond yields in that country only. By contrast, under global arbitrage, and provided that $a\alpha_e > 0$, the shock affects bond yields in both countries, even though the short rate $i_{j't}$ in country $j' \neq j$ does not change. When i_{jt} drops, bond yields in both countries drop.

Short-rate shocks are transmitted across countries because global arbitrageurs engage in the CCT and use the bond market to hedge. Recall that under both segmented and global arbitrage, a drop in the home short rate i_{Ht} raises the profitability of the CCT, making it more attractive to arbitrageurs. When the demand by currency traders is price-elastic, the arbitrageurs' equilibrium investment in the CCT increases. Because arbitrageurs hold more foreign-currency instruments (higher W_{Ft}), they become more exposed to the risk that the foreign short rate i_{Ft} drops and the foreign currency depreciates. Global arbitrageurs hedge that risk by buying foreign bonds because their price rises when i_{Ft} drops. The arbitrageurs' activity pushes the prices of foreign bonds up and their yields down.

An additional consequence of hedging by global arbitrageurs is greater under-reaction of the exchange rate and bond yields to short rates. When i_{Ht} drops, arbitrageurs invest more in the CCT and in the home BCT. Each of these trades exposes them to a rise in i_{Ht}. Hence, global arbitrageurs are less eager than segmented arbitrageurs to buy foreign currency and home bonds following a drop in i_{Ht}, and the expected return of the CCT and the home BCT increase more than under segmented arbitrage. In particular, when the demand by currency traders is inelastic and that by bond investors is elastic, a drop in i_{Ht} raises the CCT's expected return under global arbitrage but leaves it unaffected under segmented arbitrage. Likewise, when the demand by home bond investors is price-inelastic and that by currency traders is elastic, a drop in i_{Ht} raises the home BCT's expected return under global arbitrage but leaves it unaffected under segmented arbitrage.

We next turn to variants of the CCT studied in the empirical literature. One variant is a hybrid CCT in which the trading horizon is short but the trading instruments are long-term. Borrowing in the home country and investing in the foreign country is done with the respective τ-year bonds, and the positions are held for a short horizon dt. The return of the hybrid CCT in home-currency units is

$$\frac{d(P_{Ft}^{(\tau)}e_t)}{P_{Ft}^{(\tau)}e_t} - \frac{dP_{Ht}^{(\tau)}}{P_{Ht}^{(\tau)}} = \left(\frac{de_t}{e_t} + (i_{Ft} - i_{Ht})dt\right) + \left(\frac{d(P_{Ft}^{(\tau)}e_t)}{P_{Ft}^{(\tau)}e_t} - \frac{de_t}{e_t} - i_{Ft}dt\right) - \left(\frac{dP_{Ht}^{(\tau)}}{P_{Ht}^{(\tau)}} - i_{Ht}dt\right). \tag{4.5}$$

Hence, the hybrid CCT can be viewed as a combination of (i) the basic CCT, (ii) a long position in the foreign BCT, and (iii) a short position in the home BCT.

A second variant is a long-horizon CCT, in which borrowing in the home country and investing in the foreign country is done with the respective τ-year bonds, and the positions are held until the bonds' maturity. The return of the long-horizon CCT in home-currency units and log terms is

$$\log\left(\frac{e_{t+\tau}}{P_{Ft}^{(\tau)}e_t}\right) - \log\left(\frac{1}{P_{Ht}^{(\tau)}}\right) = \int_t^{t+\tau}\left(\log\left(\frac{e_{s+ds}}{e_s}\right) + i_{Fs}ds - i_{Hs}ds\right) + \left(\tau y_{Ft}^{(\tau)} - \int_t^{t+\tau} i_{Fs}ds\right) - \left(\tau y_{Ht}^{(\tau)} - \int_t^{t+\tau} i_{Hs}ds\right), \tag{4.6}$$

where the equality follows from (2.1). Hence, the long-horizon CCT can be viewed as the combination of (i) a sequence of basic CCTs, (ii) a long position in a long-horizon foreign BCT, and (iii) a short position in a long-horizon home BCT. The long-horizon BCT in country j involves buying bonds in country j and financing that position by borrowing short-term and rolling over. Proposition 4.5 characterizes the annualized expected returns of the hybrid CCT and the long-horizon CCT.

Proposition 4.5. *Suppose that arbitrage is global, arbitrageurs are risk-averse ($a > 0$), and the demand by currency traders or by bond investors is price-elastic ($\alpha_e > 0$ or $\alpha_j(\tau) > 0$).*

- *The hybrid CCT's and the long-horizon CCT's expected returns rise following a drop in the home short rate i_{Ht} or a rise in the foreign short rate i_{Ft}, provided that the maturity τ of the bonds involved in these trades lies in an interval $(0, \tau^*)$. The threshold τ^* is infinite when countries are symmetric.*
- *The sensitivity of the hybrid CCT's expected return to (i_{Ht}, i_{Ft}) is smaller than for the basic CCT. The sensitivity of the long-horizon CCT's expected return to (i_{Ht}, i_{Ft}) is smaller than for the corresponding sequence of basic CCTs.*
- *When maturity τ goes to infinity, regardless of whether $(a, \alpha_e, \alpha(\tau))$ are non-zero:*

– *The expected returns of the hybrid CCT and long-horizon CCT go to zero.*
– *The difference in real yields across countries goes to zero.*

Short-rate shocks move the expected returns of the hybrid CCT and the long-horizon CCT in the same direction as for the basic CCT, except possibly when the maturity τ of the bonds involved in these trades is long. The effects of short-rate shocks on the hybrid CCT and the long-horizon CCT are smaller than for the corresponding basic CCTs because the shocks' effects through the BCTs work in the opposite direction. Consider, for example, a drop in the home short rate. Proposition 4.4 implies that the expected return of the basic CCT increases, but so does the expected return of the home BCT, which enters as a short position in the hybrid CCT and the long-horizon CCT.

When the maturity τ of the bonds involved in the hybrid CCT and the long-horizon CCT goes to infinity, the effects of short-rate shocks through the BCTs offset fully those through the basic CCT. As a consequence, short-rate shocks have no effect on the hybrid CCT's and the long-horizon CCT's expected returns. The expected returns of both trades go to zero. These results are consistent with Lustig, Stathopoulos, and Verdelhan (2019), who find that that short rates lose their predictive power for the return of the hybrid CCT, while they predict strongly the return of the basic CCT. They are also consistent with Chinn and Meredith (2004), who find that UIP cannot be rejected over long horizons.

The annualized expected returns of the hybrid CCT and the long-horizon CCT go to zero when maturity τ goes to infinity because with a stationary real exchange rate, risk remains bounded when τ increases. Consider a trader who enters into the long-horizon CCT at time t. Per unit of home currency invested, the trader pays the fixed amount $\left(1 + y_{Ht}^{(\tau)}\right)^{\tau}$ in home currency and receives the fixed amount $\left(1 + y_{Ft}^{(\tau)}\right)^{\tau}$ in foreign currency at time $t + \tau$. The trader faces only exchange-rate risk. Since with a stationary real exchange rate that risk remains bounded when τ goes to infinity, the difference in real interest rates across countries must go to zero: if it did not, then the long-horizon CCT (or its reverse) would offer the trader a near-riskless profit for large τ. For the same reason, a stationary real exchange rate implies that the expected return of the long-horizon CCT goes to zero. The expected return of the hybrid CCT goes to zero as well because that trade is identical to the long-horizon CCT except that it is unwound at time $t + dt$.

4.2.2 Demand Shocks

Under global arbitrage, shocks to the demand for an asset class (foreign currency, home bonds, foreign bonds) affect all other assets. This is in contrast to segmented arbitrage, where only the asset class for which demand changes is affected (Proposition 4.3).

Proposition 4.6. *Suppose that arbitrage is global, arbitrageurs are risk-averse ($a > 0$), the functions $(\alpha_H(\tau), \alpha_F(\tau))$ are non-increasing, and $\theta_e > 0$. A drop in currency traders' demand for foreign currency (increase in γ_t):*

- *Causes the foreign currency to depreciate.*
- *Raises bond yields in the home country.*
- *Lowers bond yields in the foreign country.*

A drop in currency traders' demand for foreign currency causes it to depreciate, as in Proposition 4.3. Additionally, hedging by global arbitrageurs causes home bond prices to drop and foreign bond prices to rise. Indeed, arbitrageurs accommodate the drop in demand for foreign currency by holding more of it. Hence, they become more exposed to a rise in the home short rate i_{Ht} and to a decline in the foreign short rate i_{Ft}. This makes them less willing to hold home bonds, which drop in price when i_{Ht} rises, and more willing to hold foreign bonds, which rise in price when i_{Ft} drops.

Proposition 4.7. *Suppose that arbitrage is global, arbitrageurs are risk-averse ($a > 0$), the functions $(\alpha_H(\tau), \alpha_F(\tau))$ are non-increasing, and the function $\theta_j(\tau)$ is positive. A drop in investor demand for the bonds of country j (increase in β_{jt}):*

- *Raises bond yields in country j.*
- *Raises bond yields in country $j' \neq j$ when the demand by currency traders is price-elastic ($\alpha_e > 0$).*
- *Causes the foreign currency to depreciate if $j = H$, and to appreciate if $j = F$.*

A drop in investor demand for home bonds depresses their prices, as in Proposition 4.3. Additionally, hedging by global arbitrageurs causes prices for foreign bonds to drop and the foreign currency to depreciate. Indeed, arbitrageurs accommodate the drop in demand for home bonds by holding more such bonds. Hence, they become more exposed to a rise in the home short rate i_{Ht}. This makes them less willing to hold foreign currency, which depreciates when i_{Ht} rises. If the demand by currency traders is price-elastic, then arbitrageurs hold less foreign currency in equilibrium. Hence, they become less exposed to a drop in the foreign short rate i_{Ft} and less willing to hold foreign bonds, which rise in price when i_{Ft} drops. A drop in demand for foreign bonds has symmetric effects.

4.2.3 Monetary Policy Transmission

We next summarize the implications of the global-arbitrage model for the domestic and international transmission of monetary policy. Consider first conventional monetary easing at home, modelled as an unanticipated cut to the home short rate i_{Ht}. The rate cut propagates imperfectly along the home term structure and causes the home currency to depreciate (Proposition 4.4). These effects are as in the case of segmented arbitrage. Unlike in that case, however, yields on foreign bonds decrease, even though the foreign short rate remains unchanged. Hence, foreign monetary conditions are affected by domestic monetary conditions, and so the global arbitrage model features *imperfect insulation.*

Insulation is imperfect in the sense that foreign monetary policy does not fully control the foreign yield curve and thus the monetary impulse to the foreign economy.

Consider next QE purchases of home bonds, modelled as an unanticipated drop in the demand factor β_{Ht}. This policy decreases home bond yields (Proposition 4.7), as in the case of segmented arbitrage. Unlike in that case, however, yields on foreign bonds decrease and the home currency depreciates. Hence, foreign monetary conditions are affected by domestic monetary conditions. Once again, the model with global arbitrage features *imperfect insulation*. For both conventional and unconventional policies, monetary conditions co-move positively: easing at home eases abroad and vice versa. Moreover, both types of policies affect the exchange rate: conventional easing causes the home currency to depreciate, and QE does the same.

The imperfect insulation result with global arbitrage is at odds with the Trilemma. Indeed, according to the Trilemma, a country that wants to maintain monetary autonomy must either let its currency float or impose capital controls. In our model, the exchange rate is floating, and thus the Trilemma would imply that countries can maintain monetary autonomy, in the narrow sense that they can set the monetary impulse to their economy independently from monetary conditions in the rest of the world. Under global arbitrage, however, each country's term structure is influenced by monetary conditions in the other country. Hence, a floating exchange rate does not yield monetary autonomy, at odds with the Trilemma. Full insulation is possible under segmented arbitrage because of the lack of capital mobility implied by segmentation and not because of the floating exchange rate.

Insulation is imperfect under global arbitrage because arbitrageurs rebalance their entire portfolio of bonds and currencies in response to shocks, to optimize their exposures to the risk factors. Through that rebalancing, factor prices are equalized across bonds and currencies. For example, if arbitrageurs become more exposed to foreign short-rate risk through their currency position, they adjust their home and foreign bond position to attenuate that risk exposure. If arbitrageurs are risk-neutral, they do not do global rebalancing, and the prices of the risk factors are always zero.

5 Demand Risk

In this section we return to the full model analyzed in Section 3 with global arbitrage and stochastic demand for bonds and foreign currency. We estimate the model parameters by comparing empirical to model-implied moments, and use the estimated model to assess quantitatively the domestic and international transmission of monetary policy.

5.1 Estimation

We reduce the model parameters in the estimation to a set that is manageable yet sufficiently rich. We assume the functions $\{\alpha_j(\tau)\}_{j=H,F}$ that describe how the demand slope of preferred-habitat investors depends on bond maturity τ, and the functions $\{\theta_j(\tau)\}_{j=H,F}$ that describe how shocks to the demand factors affect the demand intercept for maturity τ take the exponential form

$$\alpha_j(\tau) = \alpha_{j0} \exp(-\alpha_{j1}\tau), \tag{5.1}$$

$$\theta_j(\tau) = \theta_{j0}\tau \exp(-\theta_{j1}\tau), \tag{5.2}$$

for positive scalars $(\alpha_{j0}, \alpha_{j1}, \theta_{j0}, \theta_{j1})$. The function $\theta_j(\tau)$ is positive and hump-shaped with a peak at $\frac{1}{\theta_{j1}}$. Thus, shifts to the demand factor β_{jt} shift the demand for bonds of all maturities in the same direction, with the effect being more pronounced at maturity $\frac{1}{\theta_{j1}}$. The function $\alpha_j(\tau)\tau$, which describes the demand slope when demand is expressed as function of yield rather than price, has the same functional form as $\theta_j(\tau)$, with a peak at $\frac{1}{\alpha_{j1}}$. We take the demand parameters $(\alpha_{j0}, \alpha_{j1}, \theta_{j0}, \theta_{j1})$ to be the same across home and foreign, and drop the subscript j. In the case of θ_{j0} this is a normalization, as we explain below. We set the maximum maturity T to infinity.

We take the mean-reversion matrix Γ to be diagonal except for the non-zero terms $\Gamma_{3,1}$ and $\Gamma_{3,2}$. Thus, risk factors do not respond to each other's movements except for the currency demand factor γ_t that responds to short-rate movements in both countries. We take the covariance matrix Σ to be diagonal except for the non-zero term $\Sigma_{1,2}$ (setting $\Sigma_{2,1}$ to zero is without loss of generality as the data only identify $\Sigma\Sigma^\top$). Thus, innovations to risk factors are independent except for those to short rates. As with the demand parameters, we take the mean reversion and volatility of the demand factors to be the same across home and foreign, setting $\Gamma_{4,4} = \Gamma_{5,5}$ and $\Sigma_{4,4} = \Sigma_{5,5}$. The restrictions on (Γ, Σ) simplify the estimation of the model and the interpretation of the results, while allowing us to capture two key features of the data: the correlation between short rates across countries, and the gradual response of exchange rates to short-rate shocks. With these restrictions, and with analogous notation to that in Section 4, we can write Γ and Σ as

$$\Gamma = \begin{bmatrix} \kappa_{iH} & 0 & 0 & 0 & 0 \\ 0 & \kappa_{iF} & 0 & 0 & 0 \\ \kappa_{\gamma,iH} & \kappa_{\gamma,iF} & \kappa_\gamma & 0 & 0 \\ 0 & 0 & 0 & \kappa_\beta & 0 \\ 0 & 0 & 0 & 0 & \kappa_\beta \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sigma_{iH} & 0 & 0 & 0 & 0 \\ \sigma_{iH,iF} & \sigma_{iF} & 0 & 0 & 0 \\ 0 & 0 & \sigma_\gamma & 0 & 0 \\ 0 & 0 & 0 & \sigma_\beta & 0 \\ 0 & 0 & 0 & 0 & \sigma_\beta \end{bmatrix}. \tag{5.3}$$

We do not require estimates of the long-run mean $\bar{q}$ of the vector q_t of risk factors, the constant terms $(\zeta_e, \{\zeta_j(\tau)\}_{j=H,F})$ in the demand intercepts, and the inflation rates (π_H, π_F). This is because these parameters concern long-run averages, while the results in this section depend only on the part of our model that describes responses to shocks.

Our assumptions leave us eighteen parameters to estimate: two currency demand parameters (α_e, θ_e), four bond demand parameters $(\alpha_0, \alpha_1, \theta_0, \theta_1)$, six elements of Γ, five elements of Σ, and the arbitrageurs' risk-aversion coefficient a. Our estimation procedure does not identify the two demand intercept parameters (θ_e, θ_0) and the arbitrageur risk-aversion coefficient a. The parameters (θ_e, θ_0) are not identified because they affect our target moments only through their products $(\theta_e\sigma_\gamma, \theta_0\sigma_\beta)$ with the volatility parameters of the corresponding demand factors. We focus on these products and drop $(\theta_e, \theta_0, \sigma_\gamma, \sigma_\beta)$ as separate parameters. The parameter a is not identified because it affects our target moments only through its products $(a\alpha_e, a\theta_e\sigma_\gamma, a\alpha_0, a\theta_0\sigma_\beta)$ with the demand parameters. Intuitively, exchange rates and bond yields can be volatile if demand shocks are modest and arbitrageurs highly risk-averse, or if shocks are large and arbitrageur risk aversion is low. Identifying a separately is important for assessing the effects of QE, so we bring additional information to calibrate it.

We estimate the fifteen (=18-3) parameters via Generalized Method of Moments by targeting moments of exchange rates, bond yields, and trading volume. We take the home country to be the United States and the foreign country to be the Eurozone. We use the Deutschemark as the foreign currency prior to introduction of the Euro in 01/1999, and the German yield curve as the foreign yield curve. We focus on the US and the Eurozone because they are roughly comparable in size and because a long time-series of zero-coupon bond yields for a large set of maturities is available. Our sample of exchange rates and bond yields is monthly. It starts in 06/1986, which is when zero-coupon bond yields are consistently available for maturities up to 20 years, and ends in 04/2021. We source US yields from the Federal Reserve and German yields from the Bundesbank.[6] As in previous sections, the units of time t and maturity τ are years. We source annual volume data by maturity covering the period 2002-2020 for the US from the FR 2004 dataset.[7]

A first set of target moments concern one-year yields. We include them to obtain information on the dynamics of short rates (parameters $\kappa_{iH}, \kappa_{iF}, \sigma_{iH}, \sigma_{iF}, \sigma_{iH,iF}$). These moments are: the standard deviation of one-year yields $y_{jt}^{(1)}$ and of their annual change $\Delta y_{jt}^{(1)} \equiv y_{j,t+1}^{(1)} - y_{jt}^{(1)}$, and the standard deviation of the one-year yield differential $y_{Ht}^{(1)} - y_{Ft}^{(1)}$ between home and foreign.

A second set of moments concern the exchange rate. We include them to obtain information on the slope of currency demand and on the dynamics of the currency demand

[6]https://www.federalreserve.gov/data/nominal-yield-curve.htm (Gurkaynak, Sack, and Wright (2007)) and https://www.bundesbank.de/en/statistics/money-and-capital-markets/interest-rates-and-yields/term-structure-of-interest-rates.

[7]https://www.newyorkfed.org/markets/counterparties/primary-dealers-statistics.

factor (parameters $\alpha_e, \kappa_\gamma, \kappa_{\gamma,iH}, \kappa_{\gamma,iF}, \theta_e\sigma_\gamma$). These moments are: the standard deviation of the annual (log) exchange rate change $\Delta \log e_t \equiv \log e_{t+1} - \log e_t$; the correlation between $\Delta \log e_t$ and the one-year yield differential $y^{(1)}_{Ht} - y^{(1)}_{Ft}$; the correlation between $\Delta \log e_t$ and the annual change $\Delta y^{(1)}_{jt}$ in the home and the foreign one-year yield; and the correlation between the five-year change in the exchange rate $\Delta^{(5)} \log e_t \equiv \log e_{t+5} - \log e_t$ and the five-year yield differential $y^{(5)}_{Ht} - y^{(5)}_{Ft}$.

A third set of moments concern yields across all maturities up to twenty years. We include them to obtain information on the slope of bond demand and on the dynamics of the bond demand factors (parameters $\alpha_0, \kappa_\beta, \theta_0\sigma_\beta$). These moments are: the standard deviation of yields $y^{(\tau)}_{jt}$ and of their annual change $\Delta y^{(\tau)}_{jt} \equiv y^{(\tau)}_{j,t+1} - y^{(\tau)}_{jt}$; the correlation between the annual changes $\Delta y^{(1)}_{jt}$ in one-year yields and $\Delta y^{(\tau)}_{jt}$ in all other yields; and the standard deviation of yield differentials $y^{(\tau)}_{Ht} - y^{(\tau)}_{Ft}$ for all maturities.

A final set of moments concern trading volume. We include them to obtain information on the shape of bond demand (parameters α_1, θ_1). These moments are the trading volume of US government bonds with maturities between zero and three years, and with maturities between eleven and thirty years, as a fraction of total US government bond trading volume (denoted by $\tilde{V}_H(0 \le \tau \le 3)$ and $\tilde{V}_H(11 \le \tau \le 30)$, respectively).

The total number of target moments is $N = 12 + 7(\mathcal{N}_T - 1)$, where $\mathcal{N}_T$ is the number of bond maturities (we subtract one to not double-count the one-year maturity). With maturities going from one to twenty years in annual increments, $\mathcal{N}_T$ equals twenty and the number of target moments is 145 (=12 + 7 × 19). We refer to the twelve moments that do not depend on maturity as scalar.

We estimate the model by choosing the vector $\boldsymbol{\rho}$ of 17 parameters that minimizes

$$L(\boldsymbol{\rho}) = \sum_{n=1}^{N} w_n(\hat{m}_n - m_n(\boldsymbol{\rho}))^2, \tag{5.4}$$

the weighted sum of squared differences between the empirical target moments $\{\hat{m}_n\}_{n=1,..,N}$ and their model-implied counterparts $\{m_n(\boldsymbol{\rho})\}_{n=1,..,N}$, which are functions of $\boldsymbol{\rho}$. We set the weights w_n to one for scalar moments and to $\frac{1}{\mathcal{N}_T}$ for moments that depend on maturity, so that each type of moment receives the same weight (for moments corresponding to the one-year maturity, we use $1 + \frac{1}{\mathcal{N}_T}$). The methodology to calculate the model-implied moments is described in Appendix C.

We finally calibrate a. Since a is the coefficient of arbitrageurs' absolute risk aversion, it is equal to the coefficient γ of their relative risk aversion divided by their wealth W. We set $\gamma = 2$, in line with common estimates. An estimate for W can be derived by identifying arbitrageurs with hedge funds. The assets of hedge funds in the fixed-income, macro and balanced categories in 2020 were about 5% of US GDP in that year.[8] Taking

[8] https://www.barclayhedge.com/solutions/assets-under-management/hedge-fund-assets-under-management/

US GDP as the numeraire, we can thus set $W = 5\%$. We use that value as a lower bound for W since arbitrageurs can include additional agents such as global banks and multinational corporations, and use 20% as an upper bound. The implied bounds for a are $2/5\% = 40$ and $2/20\% = 10$.

5.2 Model Fit

Table 1 compares the empirical and model-implied scalar moments. Figure 1 does the same for the moments that depend on maturity. The empirical moments are the red circles in Figure 1, and the model-implied moments are the blue solid lines. Standard deviations are reported in percentages throughout this section (e.g., $\sigma\left(y_{Ht}^{(1)}\right) = 2.622\%$ and $\sigma(\Delta \log e_t) = 10.186\%$ in Table 1). The model fits well the large set of target moments, both across maturities and across countries.

Moment	Data	Model	Moment	Data	Model
$\sigma\left(y_{Ht}^{(1)}\right)$	2.622	2.614	$\rho\left(\Delta \log e_t, (y_{Ht}^{(1)} - y_{Ft}^{(1)})\right)$	-0.105	-0.096
$\sigma\left(\Delta y_{Ht}^{(1)}\right)$	1.273	1.254	$\rho\left(\Delta \log e_t, \Delta y_{Ht}^{(1)}\right)$	-0.095	-0.214
$\sigma\left(y_{Ft}^{(1)}\right)$	2.822	2.853	$\rho\left(\Delta \log e_t, \Delta y_{Ft}^{(1)}\right)$	0.048	0.071
$\sigma\left(\Delta y_{Ft}^{(1)}\right)$	1.09	1.174	$\rho\left(\Delta^{(5)} \log e_t, (y_{Ht}^{(5)} - y_{Ft}^{(5)})\right)$	0.12	0.06
$\sigma\left((y_{Ht}^{(1)} - y_{Ft}^{(1)})\right)$	1.816	1.717	$\tilde{V}_H(0 \leq \tau \leq 3)$	0.361	0.378
$\sigma(\Delta \log e_t)$	10.186	10.183	$\tilde{V}_H(11 \leq \tau \leq 30)$	0.08	0.116

Table 1: Scalar Moments in the Data and the Model

Table 2 reports the estimated model parameters. Innovations to the home short rate have somewhat higher standard deviation ($\sigma_{iH} = 1.43$) and lower persistence ($\kappa_{iH} = 0.126$) than to the foreign short rate ($\sqrt{\sigma_{iH,iF}^2 + \sigma_{iF}^2} = 1.294$ and $\kappa_{iF} = 0.0896$), and are positively correlated (correlation $\frac{\sigma_{iH,iF}}{\sqrt{\sigma_{iH,iF}^2+\sigma_{iF}^2}} = 0.814$). Innovations to the demand for foreign assets by currency traders are somewhat less persistent ($\kappa_\gamma = 0.134$) than to short rates, and respond negatively to the home short rate ($\kappa_{\gamma,iH} = -0.267$) and positively to the foreign short rate ($\kappa_{\gamma,iF} = 0.252$). Thus, a drop in the home short rate or a rise in the foreign short rate causes the demand for foreign assets to rise, holding the exchange rate constant. Innovations to bond demand are more persistent than to short rates and currency demand ($\kappa_\beta = 0.0501$).

The slope α_e of currency demand can be derived by dividing $a\alpha_e$ by a. It ranges from 7.34 ($=\frac{73.4}{10}$) for the lower bound of a to 1.83 ($=\frac{73.4}{40}$) for the upper bound. Thus, a 1% drop in the exchange rate raises the demand for foreign assets by an amount ranging from 7.34% ($= 7.34 \times 1\%$) to 1.83% of US GDP. The slope parameter α_0 of bond demand can likewise be derived by dividing $a\alpha_0$ by a, and ranges from 0.474 to 0.119. To interpret this coefficient, consider a uniform 0.1% rise in home or foreign bond yields. This translates

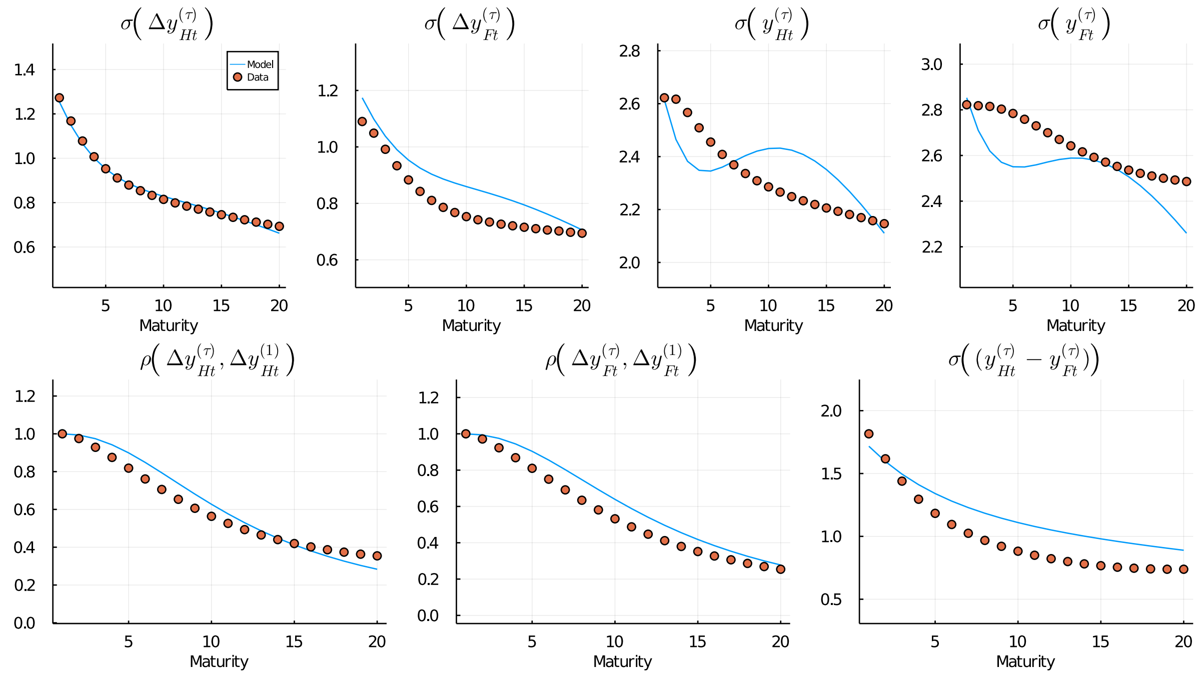

Figure 1: Maturity-Dependent Moments in the Data and the Model

Parameter	Value	Parameter	Value	Parameter	Value
κ_{iH}	0.126	κ_γ	0.134	$a\sigma_\beta\theta_0$	90.6
κ_{iF}	0.0896	$\kappa_{\gamma,iH}$	-0.267	$a\alpha_e$	73.4
σ_{iH}	1.43	$\kappa_{\gamma,iF}$	0.252	$a\alpha_0$	4.74
σ_{iF}	0.751	$a\sigma_\gamma\theta_e$	763.0	α_1	0.144
$\sigma_{iH,iF}$	1.05	κ_β	0.0501	θ_1	0.374

Table 2: Estimated Model Parameters

to a price drop of the corresponding τ-year bond by $\tau \times 0.1\%$, which for a ten-year bond is 1%, the same as for the exchange-rate exercise. The aggregate bond demand across maturities rises by an amount ranging from 2.295% ($= \int_0^\infty \alpha_0 \exp(-\alpha_1\tau)\tau d\tau \times 0.1\% = \frac{\alpha_0}{\alpha_1^2} \times 0.1\% = \frac{0.474}{0.144^2} \times 0.1\%$) of US GDP to 0.574%. By comparison, Krishnamurthy and Vissing-Jorgensen (2012) estimate that a 0.1% drop in the spread between AAA-rated US corporate bonds and US government bonds raises government bond demand by 5.9% of US GDP. Their estimate is about four times as large as the midpoint of ours. This discrepancy may arise because an increase in government bond yields in our estimation can be accompanied by an increase in corporate bond yields (which mitigates the increase in the spread). Another estimate comes from Koijen and Yogo (2020), who find that a 1% price drop in the price of long-maturity bonds (maturity of one year or longer) raises their demand by foreign investors by 1.9%. This estimate is not directly comparable to ours as we do not distinguish whether preferred-habitat investors for a country's bonds are home or foreign.

We next examine the implications of our estimated model for the predictability of

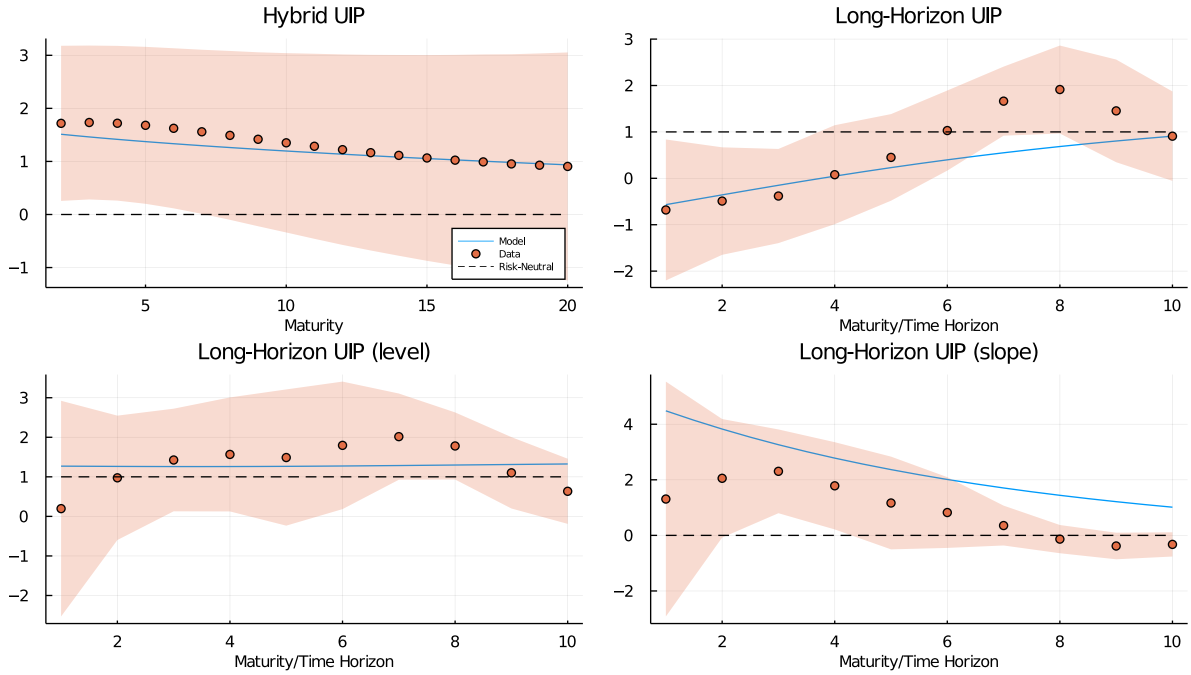

Figure 2: Coefficients of Regressions on Uncovered Interest Parity

currency and bond returns. We do so by running common regressions in the asset pricing literature and comparing the empirical coefficients computed within our sample to the coefficients implied by our model. The empirical coefficients are the red circles in Figures 2 and 3, and the model-implied coefficients are the blue solid lines. The black dashed lines show the UIP benchmark in Figure 2 and the EH benchmark in Figure 3. Shaded areas are 95% confidence intervals. The calculations of the model-implied coefficients are in Appendix C.

Figure 2 reports coefficients for various types of UIP regressions. The top left panel concerns the hybrid UIP regression of Lustig, Stathopoulos, and Verdelhan (2019, LSV), in which the return over horizon $\Delta\tau$ of the hybrid CCT constructed using bonds with maturity τ is regressed on the foreign-minus-home $\Delta\tau$-year yield differential. This regression nests as a special case, for $\tau = \Delta\tau$, the standard UIP regression of Bilson (1981) and Fama (1984). Under the UIP, the LSV coefficient should be zero. The empirical coefficients are positive and statistically significant for short maturities, consistent with Bilson (1981) and Fama (1984). They decline with maturity and become statistically insignificant for long maturities. This is consistent with LSV, although LSV's coefficients, computed over multiple currency pairs rather than over only Dollar/Euro as in our estimation, are closer to zero. The model-implied coefficients are close to the empirical coefficients.

The top right panel in Figure 2 concerns the long-horizon UIP regression of Chinn and Meredith (2004, CM), in which the rate of foreign currency depreciation over horizon $\Delta\tau$ is regressed on the foreign-minus-home $\Delta\tau$-year yield differential. Under the UIP, the CM coefficient should be one. The empirical coefficients are not statistically different from zero at short maturities, although confidence intervals are large because we use only one currency pair. As horizon increases, the empirical coefficients go to one, consistent with CM. The model-implied coefficients approach one as maturity increases, somewhat more slowly than the empirical ones.

The bottom two panels concern regressions run in Chernov and Creal (2020) and Lloyd and Marin (2020), in which the rate of foreign currency depreciation over horizon $\Delta\tau$ is regressed on the foreign-minus-home $\Delta\tau$-year yield differential (level—same regressor as in CM), and the foreign-minus-home slope differential (slope). Under UIP, the level coefficient should be one and the slope coefficient should be zero. As with the CM regression, the coefficients using only one currency pair are imprecisely estimated, but the point estimates are consistent with the literature. In particular, the slope coefficient is positive, meaning that for a given yield differential, the foreign-minus-home CCT is less profitable when the foreign-minus-home slope differential is larger. Our model implies a positive slope coefficient, as in the data. Indeed, suppose that the demand for foreign bonds by preferred-habitat investors is temporarily low. This pushes up foreign bond yields, raising the foreign-minus-home slope differential. It also causes the foreign currency to appreciate temporarily (Proposition 4.7), and its future expected return to decline. As in the data, the predictability of slope in our model is primarily over short and medium maturities.

Overall, our model matches well the UIP regression evidence. This is helped by our using the one- and five-year coefficients in the long-horizon UIP regression as target moments. (Our target moments include the numerator and denominator in each of the two coefficients.) Yet, these are only two data points in one panel in Figure 2, and our model matches closely the evidence in all four panels, including in the bottom right panel which is generated by a different mechanism. Our model also matches reasonably well the EH regression evidence in Figure 3, which is fully untargeted.

The top left and top right panels in Figure 3 concern the Fama and Bliss (1987, FB) regression in the home and foreign country, respectively. FB regress the excess return over horizon $\Delta\tau$ of the bond with maturity τ on the difference between the forward rate between maturities $\tau - \Delta\tau$ and τ, and the $\Delta\tau$-year spot rate (yield). The bottom left and bottom right panels concern the Campbell and Shiller (1991, CS) regression in the home and foreign country, respectively. CS regress the change over horizon $\Delta\tau$ in the yield of a bond with initial maturity τ on a scaled difference between the spot rates for maturities τ and $\Delta\tau$. Under the EH, the FB coefficient should be zero and the CS coefficient should be one. The empirical coefficients are consistent with the findings of FB and CS: the FB coefficient is positive and increasing with maturity, and the CS coefficient is negative and decreasing with maturity.

The model-implied FB and CS coefficients have the same sign as their empirical counterparts but not the same monotonicity. The FB coefficient is positive and around one for all maturities. It is close to its empirical counterpart for maturities ranging from five

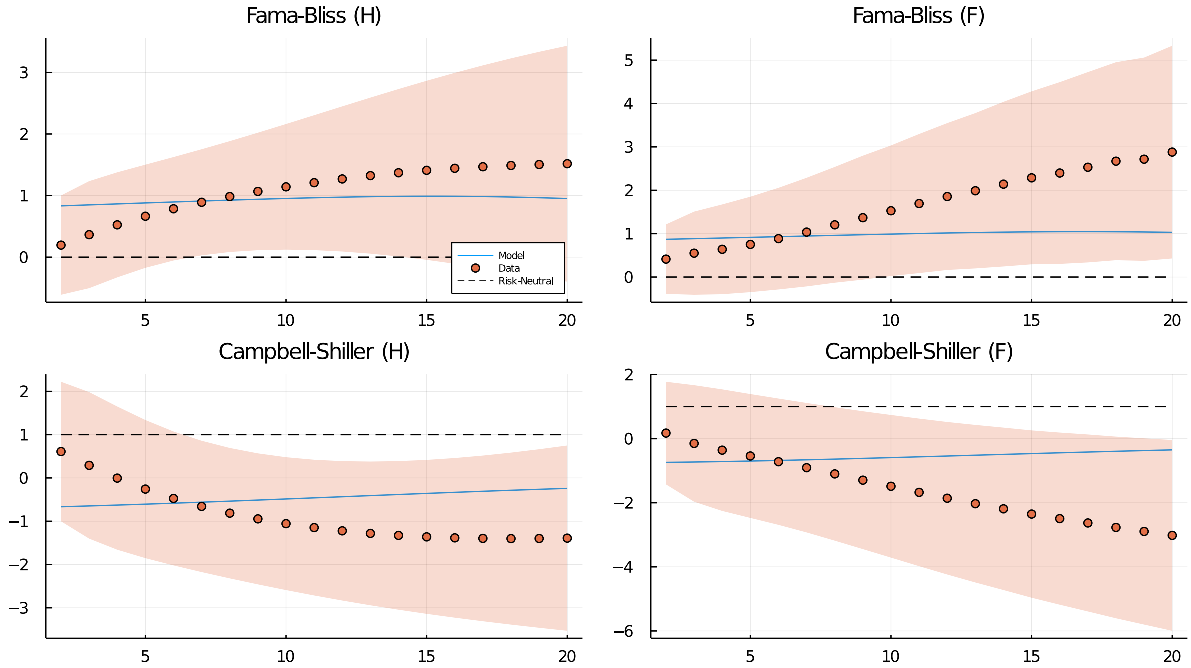

Figure 3: Coefficients of Regressions on Expectations Hypothesis

to ten years, but becomes smaller for longer maturities. The CS coefficient is negative and around minus one for short maturities, and increases slightly with maturity. It is close to its empirical counterpart for maturities ranging from five to ten years, but becomes smaller in absolute value for longer maturities. Thus, our model matches closely the empirical deviations from EH for maturities ranging from five to ten years, but not for longer maturities as these deviations become larger in the data. Yet, even for the longer maturities, our model generates sizeable deviations from EH.

5.3 Monetary Policy

We next use our estimated model to study the domestic and international transmission of monetary policy. We start with conventional policy, and consider a cut to the short rate by the central bank in country j. We assume that the cut is unanticipated, occurs at time zero, and is unwound over time at a rate κ_{ij}^{MP} possibly different from κ_{ij}. We model the short-rate movement induced by the cut as a separate, additive component $\Delta i_{jt} = \Delta i_{j0} e^{-\kappa_{ij}^{MP} t}$ of the short-rate process. We set $\Delta i_{j0} = -0.25$, implying a cut of 25 basis points (bps), and $\kappa_{ij}^{MP} = 0.75$, implying a half-life of the cut of about a year.

The top left and top right panels of Figure 4 show, respectively, how a cut to the home short rate affects the home and foreign structures at time zero and how it affects the exchange rate over time. The bottom left and right panels show the same for a cut to the foreign short rate. The home term structure is shown in blue and the foreign term structure in red. Exchange-rate movements are measured as percentage price changes.

The cut affects the term structure in the country where it originates, but has essentially

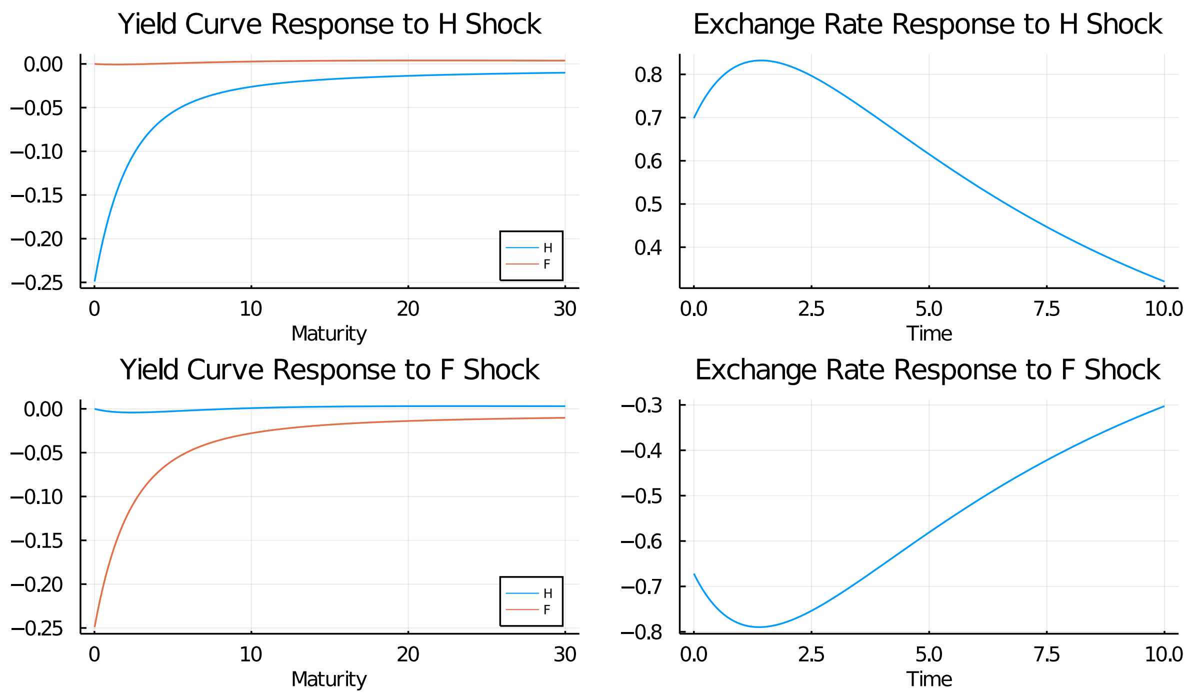

Figure 4: Conventional Monetary Policy – Short Rate Cut

no effect on the other country's term structure. The international transmission is weak partly because short rates are positively correlated across countries. Indeed, in response to a cut to the home short rate, arbitrageurs build long positions in home bonds and foreign currency (Proposition 4.4) but hedge only mildly the currency position with foreign bonds. This is because a drop in the foreign short rate, which generates losses in the currency position, is positively correlated with a drop in the home short rate, which generates gains in home bonds and foreign currency. If short rates are independent, with the same volatilities as in our estimation (i.e., $(\sigma_{iH}, \sigma_{iF}) = (1.43, 1.294)$), then the effect of the cut to the home short rate on foreign yields ranges from 7.5% to 15.5% that on home yields for intermediate maturities between five and ten years.

The response of the exchange rate is hump-shaped. Following the cut to the home short rate, the exchange rate jumps up (i.e., the foreign currency appreciates) by 0.7%, then rises further by 0.15% over the next two years, and then declines gradually to its pre-cut value. The exchange-rate response is hump-shaped because demand for foreign assets rises gradually following the cut to the home short rate ($\kappa_{\gamma,iH} < 0$). The demand effect is dominant: in its absence, the exchange rate would jump up by roughly 0.15%, only one-half of the effect under UIP ($\frac{0.25\%}{\kappa_{ij}^{MP}} = 0.33\%$).

We next turn to unconventional policy, and consider QE purchases by the central bank in country j. We assume that the purchases are unanticipated, occur at time zero, and are unwound over time at a rate $\kappa_{\beta j}^{QE}$ possibly different from $\kappa_{\beta j}$. We model the net addition to the central bank's position in the bond with maturity τ as a separate, additive component $\theta_j^{QE}(\tau)\Delta\beta_{jt}$ of the demand-intercept process, where $\theta_j^{QE}(\tau)$ has the

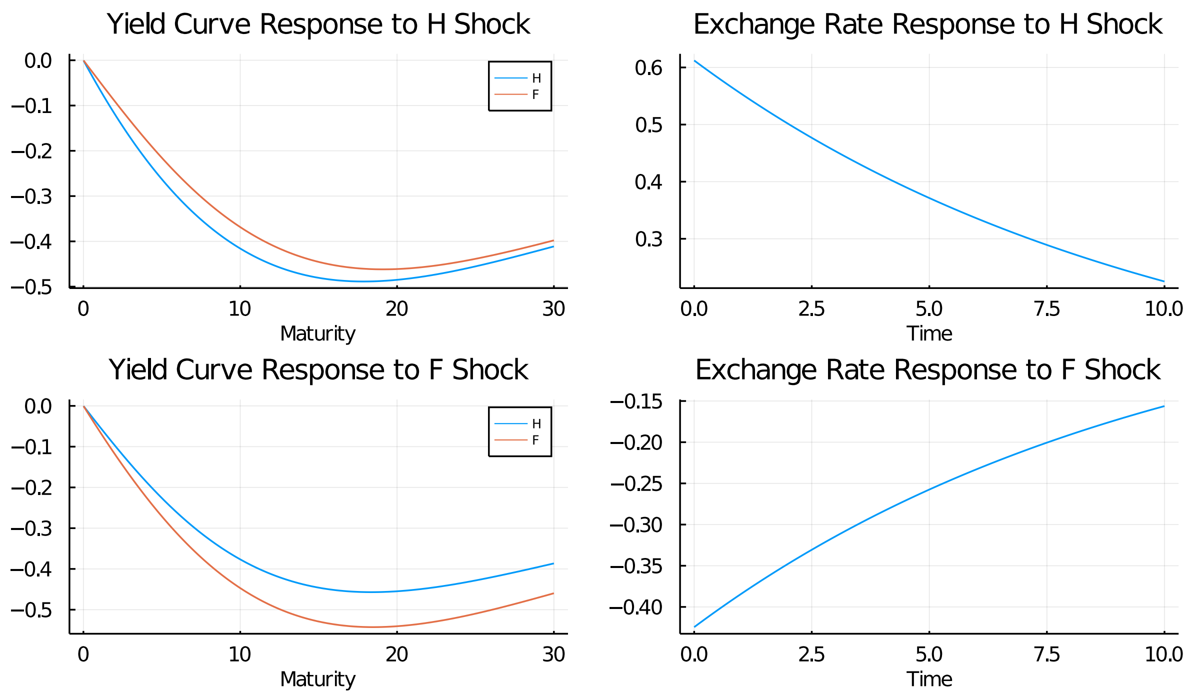

Figure 5: Unconventional Monetary Policy – Bond Purchases

exponential form (5.2) and $\Delta\beta_{jt} = \Delta\beta_{j0}e^{-\kappa_{\beta j}^{QE}t}$. We allow the parameters $(\theta_0^{QE}, \theta_1^{QE})$ to differ from (θ_0, θ_1), and normalize $\Delta\beta_{j0}$ to one. We set $\kappa_{\beta j}^{QE} = 0.1$, implying a half-life of QE purchases of about seven years, θ_1^{QE} to 0.2, implying that purchases are maximized at the five-year maturity $(\frac{1}{\theta_1^{QE}} = 5)$, and $\theta_0^{QE} = 0.004$, implying that purchases are 10% of US GDP $(\frac{\theta_0^{QE}}{(\theta_1^{QE})^2} = 0.1)$.

The top left and top right panels of Figure 5 show, respectively, how QE purchases in the home country affect the home and foreign term structures at time zero and how they affect the exchange rate over time. The bottom left and right panels show the same for QE purchases in the foreign country. The coloring and units are the same as in Figure 4. Figure 5 sets arbitrageur risk aversion a to 40. When $a = 10$, the effects are one-quarter of those in the figure.

QE purchases have sizeable effects on the term structure in the country where they originate: they reduce intermediate yields by between 40-50bps. The mid-point of this estimate and of the one for $a = 10$ is about 30bps, slightly lower than in the literature: according to Wiliams (2014), a consensus estimate is that QE purchases of 10% of GDP reduce the ten-year yield by 35-65bps.

QE purchases affect yields in the non-originating country almost as much as in the originating country: the non-originating country's yield curve responds by nearly 90% relative to the originating country's yield curve. Hence, conventional and unconventional policy differ sharply in their transmission to international bond yields: spillovers are non-existent for the former, and sizeable for the latter. In terms of the exchange rate response, the effect of QE is comparable to that of conventional policy. QE in the home country causes the foreign currency to appreciate by about 0.6%, while QE in the foreign country causes the foreign currency to depreciate by about 0.45%. The effect is largest on impact and reverts towards zero as QE purchases are unwound.

The international transmission of unconventional policy to bond yields is strong partly because short rates are positively correlated across countries and partly because the demand by currency traders is sufficiently price-elastic. The intuition for the effect of correlation is as follows. In response to QE in the home country, arbitrageurs reduce their positions in home bonds. They rebuild their exposure to short-rate risk by buying positively correlated foreign bonds, thus exerting downward pressure to those bonds' yields. If we counterfactually assume that short rates are independent (with the same volatilities as in our estimation), then ten-year foreign yields respond by nearly 80% relative to home yields in response to a home QE shock. If we maintain correlated short rates and assume that currency elasticities are much smaller than implied by the data (reducing $a\alpha_e$ three-fold from 74.3 to 25.0), we find that a home QE shock moves ten-year foreign yields by roughly 70% relative to home yields. It is only when we assume both that short rates are independent and currency traders are much less price-elastic than implied by the data that we find significantly smaller QE spillover effects: a home QE shock moves ten-year foreign yields by only 30% relative to home yields. Hence, our result that the spillover effects to international bond yields are stronger for unconventional than for conventional policy is quantitatively robust.[9]

6 Concluding Remarks

We model exchange rates and bond yields as determined by the interaction of different investor clienteles. Global arbitrageurs partially integrate domestic and foreign asset markets, but imperfect risk-bearing capacity leads to deviations from the predictions of standard models. Beyond making sense of the predictability patterns of currency and bond returns found in the data, our model has important implications for the transmission of monetary policy. Both conventional and unconventional monetary policy cause global arbitrageurs to rebalance their portfolio of currencies and bonds, to optimally manage their risk exposure. The joint determination of currency and bond risk premia implies that policy shocks in one country have spillovers in the domestic bond market, the currency market, and the foreign bond market. Our estimated model shows that the international spillovers of large-scale asset purchases are particularly strong: domestic QE purchases of long-maturity bonds push down foreign bond yields by nearly the same amount as domestic bond yields, in addition to depreciating the currency.

[9] Our result is similarly robust when we vary the bond elasticities.

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Appendix – For Online Publication

A Forwards and Swaps

We show that the equilibrium with a currency forward market is equivalent to one without it but with different currency and bond demands. The equivalence result extends to swaps because they are portfolios of forwards.

We model the demand for currency forwards as follows. Currency traders with preferences for forward contracts with maturities in $[\tau, \tau + d\tau]$ are in measure $d\tau$, and their demand, expressed in units of the home currency, is

$$Z_{et}^{(\tau)} = -(\zeta_e(\tau) + \theta_e(\tau)\gamma_t), \tag{A.5}$$

where $(\zeta_e(\tau), \theta_e(\tau))$ are functions of τ.

Since global arbitrageurs can trade costlessly home and foreign bonds and foreign currency in the spot market, Covered Interest Parity (CIP) holds. Moreover, the demand $Z_{et}^{(\tau)}$ for the foreign-currency forward contract with maturity τ is equivalent to the combination of (i) a demand $Z_{et}^{(\tau)}$ for foreign currency in the spot market, (ii) a demand $Z_{et}^{(\tau)}$ for the foreign bond with maturity τ, and (iii) a demand $-Z_{et}^{(\tau)}$ for the home bond with maturity τ. Hence, the equilibrium with the forward market is equivalent to one without it but with the demands (i)-(iii) added to the currency demand (2.5) and the bond demand (2.6). The demand for foreign currency becomes

$$Z_{et} + \int_0^T Z_{et}^{(\tau)} d\tau = -\alpha_e \log(e_t) - (\zeta_{et} + \theta_e \gamma_t) - \int_0^T (\zeta_e(\tau) + \theta_e(\tau)\gamma_t) d\tau$$

instead of Z_{et}. The demand for country j bonds with maturity τ becomes

$$Z_{jt}^{(\tau)} + (-1)^{1_{\{j=H\}}} Z_{et}^{(\tau)} = -\alpha_j(\tau) \log\left(P_{jt}^{(\tau)}\right) - (\zeta_j(\tau) + \theta_j(\tau)\beta_{jt}) - (-1)^{1_{\{j=H\}}} (\zeta_e(\tau) + \theta_e(\tau)\gamma_t)$$

instead of $Z_{jt}^{(\tau)}$. Forwards induce a negative correlation between currency and home bond demands, and a positive correlation between currency and foreign bond demands.

B Proofs

Proof of Proposition 3.1: Using the definitions of $(\mathcal{E}_{iH}, \mathcal{E}_{iF}, \mathcal{E}_{\gamma}, \mathcal{E}_{\beta H}, \mathcal{E}_{\beta F})$, we can write (3.18) as

$$
\begin{aligned}
& a\Sigma\Sigma^{\top}\left(A_e\left(\theta_e\mathcal{E}_{\gamma}-\alpha_e A_e\right)^{\top}+\sum_{j=H,F}\int_0^T A_j(\tau)\left(\theta_j(\tau)\mathcal{E}_{\beta j}-\alpha_j(\tau)A_j(\tau)\right)^{\top}d\tau\right)q_t \\
& + a\Sigma\Sigma^{\top}\left(\left(\zeta_e-\alpha_e C_e\right)A_e+\sum_{j=H,F}\int_0^T\left(\zeta_j(\tau)-\alpha_j(\tau)C_j(\tau)\right)A_j(\tau)\right) \\
& = -(M-\Gamma^{\top})^{\top}q_t+\lambda_C,
\end{aligned} \tag{B.1}
$$

where the second step follows from the definitions of (M, λ_C) in the statement of the proposition. We next substitute $(\mu_{et}, \{\mu_{jt}^{(\tau)}\}_{j=H,F}, \lambda_t)$ from (3.6), (3.8), (3.10) and (B.1) into the arbitrageurs' first-order condition. Substituting into (3.12) and identifying terms in q_t and constant terms, we find (3.19) and (3.20), respectively. Substituting into (3.13) and identifying terms in q_t and constant terms, we find (3.21) and (3.22), respectively. ∎

Proof of Corollary 3.1: The results for $a = 0$ follow from the arguments before the corollary' statement. When a goes to zero, (3.19), (3.21) and (3.23) imply that M goes to $\Gamma^{\top}$ and $(A_e, \{A_j(\tau)\}_{j=H,F})$ have the finite limits

$$
\begin{aligned}
& \lim_{a\to 0} A_e = \left(\Gamma^{-1}\right)^{\top}\left(\mathcal{E}_{iH}-\mathcal{E}_{iF}\right), \\
& \lim_{a\to 0} A_j(\tau) = \left(\Gamma^{-1}\right)^{\top}\left(I-e^{-\Gamma^{\top}\tau}\right)\mathcal{E}_{ij}.
\end{aligned}
$$

When (3.25) holds, (3.20), (3.24) (3.24) are met with λ_C having a zero limit and $(C_e, \{C_j(\tau)\}_{j=H,F})$ having finite limits. Equation (3.18) then implies that λ_t goes to zero, which means from (3.12) and (3.13) that UIP and EH hold in the limit. When instead (3.25) does not hold, (3.20) implies that $A_e^{\top}\lambda_C$ has a non-zero limit. When, in addition, $\alpha_e > 0$, (3.20), (3.24) (3.24) are met with λ_C having a non-zero limit, $\{C_j(\tau)\}_{j=H,F}$ having finite limits, and C_e going to plus or minus infinity at the rate $\frac{1}{a}$. Equation (3.18) then implies that λ_t does not go to zero, which means from (3.12) and (3.13) that UIP and EH do not hold. ∎

Propositions B.1 and B.2 characterize the equilibrium in the currency and bond market, respectively, under segmented arbitrage and the parameter restrictions assumed in Section 4.

Proposition B.1. *Suppose that arbitrage is segmented, the matrices (Γ, Σ) are diagonal, and $\Sigma_{3,3} = \Sigma_{4,4} = \Sigma_{5,5} = 0$. The exchange rate e_t is given by (3.1), with (A_{iHe}, A_{iFe}) positive and equal to the unique solution of*

$$
\kappa_{ij}A_{ije} - 1 = -a_e\alpha_e A_{ije}\left(\sigma_{iH}^2 A_{iHe}^2 + \sigma_{iF}^2 A_{iFe}^2\right), \tag{B.2}
$$

and C_e solving

$$
\begin{aligned}
&-\kappa_{iH}\bar{i}_H A_{iHe} + \kappa_{iF}\bar{i}_F A_{iFe} - (\pi_F - \pi_H) + \frac{1}{2}\sigma_{iH}^2 A_{iHe}^2 + \frac{1}{2}\sigma_{iF}^2 A_{iFe}^2 \\
&= a_e(\zeta_e - \alpha_e C_e)\left(\sigma_{iH}^2 A_{iHe}^2 + \sigma_{iF}^2 A_{iFe}^2\right). \qquad \text{(B.3)}
\end{aligned}
$$

Proof of Proposition B.1: The first-order condition (4.2) follows from (3.12) by keeping only the term $W_{Ft}A_e$ in the parenthesis in (3.14), taking Σ to be diagonal with $\Sigma_{3,3} = \Sigma_{4,4} = \Sigma_{5,5} = 0$, and replacing a by a_e. Proceeding as in the derivation of (3.18) and using $\gamma_t = \beta_{Ht} = \beta_{Ft} = 0$, we find $\lambda_{et} = (\lambda_{eHt}, \lambda_{eFt})$ with

$$\lambda_{ejt} = a_e\sigma_{ij}^2\left[\zeta_e - \alpha_e\left(A_{iHe}i_{Ht} - A_{iFe}i_{Ft} + C_e\right)\right]A_{ije}(-1)^{1_{\{j=F\}}} \qquad \text{(B.4)}$$

for $j = H, F$. Since (Γ, Σ) are diagonal with $\Sigma_{3,3} = \Sigma_{4,4} = \Sigma_{5,5} = 0$ and $\gamma_t = \beta_{Ht} = \beta_{Ft} = 0$, we can write (3.6) as

$$\mu_{et} = -A_{iHe}\kappa_{iH}(\bar{i}_H - i_{Ht}) + A_{iFe}\kappa_{iF}(\bar{i}_F - i_{Ft}) - (\pi_F - \pi_H) + \frac{1}{2}A_{iHe}^2\sigma_{iH}^2 + \frac{1}{2}A_{iFe}^2\sigma_{iF}^2. \qquad \text{(B.5)}$$

Substituting λ_{et} from (B.4) and μ_{et} from (B.5) into (4.2), we find an equation that is affine in (i_{Ht}, i_{Ft}). Equation (B.2) follows by identifying the linear terms in (i_{Ht}, i_{Ft}), and (B.3) follows by identifying the constant terms.

When $a\alpha_e = 0$, (B.2) has the unique solution $(A_{iHe}, A_{iFe}) = \left(\frac{1}{\kappa_{iH}}, \frac{1}{\kappa_{iF}}\right)$, which is positive. Consider next the case $a\alpha_e > 0$. A solution (A_{iHe}, A_{iFe}) to (B.2) must be positive, as can be seen by writing that equation as

$$\left[\kappa_{ij} + a_e\alpha_e\left(\sigma_{iH}^2 A_{iHe}^2 + \sigma_{iF}^2 A_{iFe}^2\right)\right]A_{ije} = 1. \qquad \text{(B.6)}$$

Since $(A_{iHe}, A_{iFe}, a\alpha_e)$ are positive, the right-hand side of (B.2) is negative. Therefore, the left-hand side is negative as well, which implies $A_{iHe} < \frac{1}{\kappa_{Hj}}$ and $A_{iFe} < \frac{1}{\kappa_{Fj}}$. Dividing (B.2) written for $j = H$ by (B.2) written for $j = F$, we find

$$\frac{1 - \kappa_{iH}A_{iHe}}{1 - \kappa_{iF}A_{iFe}} = \frac{A_{iHe}}{A_{iFe}} \Leftrightarrow A_{iHe} = \frac{A_{iFe}}{1 + (\kappa_{iH} - \kappa_{iF})A_{iFe}}. \qquad \text{(B.7)}$$

Equation (B.7) determines A_{iHe} as an increasing function of $A_{iFe} \in \left[0, \frac{1}{\kappa_{iF}}\right]$, equal to zero for $A_{iFe} = 0$, and equal to $\frac{1}{\kappa_{iH}}$ for $A_{iFe} = \frac{1}{\kappa_{iF}}$. Substituting A_{iHe} as a function of A_{iFe} in (B.6) written for $j = F$, we find an equation in the single unknown A_{iFe}. The left-hand side of that equation is increasing in A_{iFe}, is equal to zero for $A_{iFe} = 0$, and is equal to a value larger than one for $A_{iFe} = \frac{1}{\kappa_{iF}}$. Hence, that equation has a unique solution $A_{iFe} \in \left(0, \frac{1}{\kappa_{iF}}\right)$. Given that solution, (B.7) determines $A_{iHe} \in \left(0, \frac{1}{\kappa_{iH}}\right)$ uniquely. Given

(A_{iHe}, A_{iFe}), (B.3) determines C_e uniquely if $\alpha_e > 0$. If $\alpha_e = 0$, then the restriction

$$\bar{i}_H - \pi_H = \bar{i}_F - \pi_F + \left(\frac{1}{2} - a_e\zeta_e\right)\left(\frac{\sigma_{iH}^2}{\kappa_{iH}^2} + \frac{\sigma_{iF}^2}{\kappa_{iF}^2}\right) \tag{B.8}$$

on model parameters must be imposed and C_e is indeterminate. ∎

Proposition B.2. *Suppose that arbitrage is segmented, the matrices* (Γ, Σ) *are diagonal, and* $\Sigma_{3,3} = \Sigma_{4,4} = \Sigma_{5,5} = 0$. *Bond prices* $P_{jt}^{(\tau)}$ *in country* $j = H, F$ *are given by (3.2), with* $A_{ij'j}(\tau)$ *equal to zero for* $j' \neq j$ *and* $(A_{ijj}(\tau), C_{ij}(\tau))$ *equal to the unique solution of the system*

$$A'_{ijj}(\tau) + \kappa_{ij}A_{ijj}(\tau) - 1 = -a_j\sigma_{ij}^2 A_{ijj}(\tau)\int_0^T \alpha_j(\tau)A_{ijj}(\tau)^2 d\tau, \tag{B.9}$$

$$C'_j(\tau) - \kappa_{ij}\bar{i}_j A_{ijj}(\tau) + \frac{1}{2}\sigma_{ij}^2 A_{ijj}(\tau)\left(A_{ijj}(\tau) - 2A_{iFe}1_{\{j=F\}}\right)$$
$$= a_j\sigma_{ij}^2 A_{ijj}(\tau)\int_0^T \left[\zeta_j(\tau) - \alpha_j(\tau)C_j(\tau)\right]A_{ijj}(\tau)d\tau, \tag{B.10}$$

with the initial conditions $A_{ijj}(0) = C_j(0) = 0$.

Proof of Proposition B.2: The first-order condition (4.3) follows from (3.13) by keeping only the term $\int_0^T X_{jt}^{(\tau)} A_j(\tau)d\tau$ in the parenthesis in (3.14), taking Σ to be diagonal with $\Sigma_{3,3} = \Sigma_{4,4} = \Sigma_{5,5} = 0$, replacing a by a_j, and conjecturing that in equilibrium $A_{ij'j}(\tau) = 0$ for $j' \neq j$. Proceeding as in the derivation of (3.18) and using $\gamma_t = \beta_{Ht} = \beta_{Ft} = 0$ and $A_{ij'j}(\tau) = 0$ for $j' \neq j$, we find

$$\lambda_{jt} = a_j\sigma_{ij}^2\left(\int_0^T \left[\zeta_j(\tau) - \alpha_j(\tau)\left(A_{ijj}(\tau)i_{jt} + C_j(\tau)\right)\right]A_{ijj}(\tau)d\tau\right). \tag{B.11}$$

Since (Γ, Σ) are diagonal with $\Sigma_{3,3} = \Sigma_{4,4} = \Sigma_{5,5} = 0$, $\gamma_t = \beta_{Ht} = \beta_{Ft} = 0$, and $A_{ij'j}(\tau) = 0$ for $j' \neq j$, we can write (3.8) and (3.10) as

$$\mu_{jt}^{(\tau)} = A'_{ijj}(\tau)i_{jt} + C'_j(\tau) - A_{ijj}(\tau)\kappa_{ij}(\bar{i}_j - i_{jt}) + A_{ijj}(\tau)\left(A_{ijj}(\tau) - 2A_{iFe}1_{\{j=F\}}\right)\sigma_{ij}^2. \tag{B.12}$$

Substituting λ_{jt} from (B.11) and μ_{jt} from (B.12) into (4.3), we find an equation that is affine in i_{jt}. Equation (B.9) follows by identifying the linear terms in i_{jt}, and (B.10) follows by identifying the constant terms. The initial conditions $A_{ijj}(0) = C_j(0) = 0$ follow because the price of a bond with zero maturity is its face value, which is one. Since the affine equation holds when (B.9) and (B.10) hold, our conjecture $A_{ij'j}(\tau) = 0$ for $j' \neq j$ is validated.

Solving (B.9) with the initial condition $A_{ijj}(0) = 0$, we find

$$A_{ijj}(\tau) = \frac{1 - e^{-\kappa_{ij}^* \tau}}{\kappa_{ij}^*}, \tag{B.13}$$

with

$$\kappa_{ij}^* \equiv \kappa_{ij} + a_j \sigma_{ij}^2 \int_0^T \alpha_j(\tau) A_{ijj}(\tau)^2 d\tau. \tag{B.14}$$

Substituting $A_{ijj}(\tau)$ from (B.13) into (B.14), we find the equation

$$\kappa_{ij}^* - \kappa_{ij} + a_j \sigma_{ij}^2 \int_0^T \alpha_j(\tau) \left(\frac{1 - e^{-\kappa_{ij}^* \tau}}{\kappa_{ij}^*} \right)^2 d\tau = 0 \tag{B.15}$$

in the single unknown κ_{ij}^*. The left-hand side of (B.15) is increasing in κ_{ij}^*, is negative for $\kappa_{ij}^* = \kappa_{ij}$, and goes to infinity when κ_{ij}^* goes to infinity. Hence, (B.15) has a unique solution $\kappa_{ij}^* > \kappa_{ij}$. Given κ_{ij}^*, (B.13) determines $A_{ijj}(\tau)$ uniquely.

Solving (B.10) with the initial condition $C_j(\tau) = 0$, we find

$$C_j(\tau) = \kappa_{ij}^* \bar{i}_j^* \int_0^\tau A_{ijj}(\tau) d\tau - \frac{1}{2} \sigma_{ij}^2 \int_0^\tau A_{ijj}(\tau)^2 d\tau, \tag{B.16}$$

with

$$\kappa_{ij}^* \bar{i}_j^* \equiv \kappa_{ij} \bar{i}_j + a_j \sigma_{ij}^2 \int_0^T \left[\zeta_j(\tau) - \alpha_j(\tau) C_j(\tau) \right] A_{ijj}(\tau) d\tau + \sigma_{ij}^2 A_{iFe} 1_{\{j=F\}}. \tag{B.17}$$

Substituting $C_j(\tau)$ from (B.16) into (B.17), we find

$$\bar{i}_j^* = \frac{\kappa_{ij} \bar{i}_j + a_j \sigma_{ij}^2 \int_0^T \zeta_j(\tau) A_{ijj}(\tau) d\tau + \sigma_{ij}^2 A_{iFe} 1_{\{j=F\}} + \frac{1}{2} a_j \sigma_{ij}^4 \int_0^T \alpha_j(\tau) \left(\int_0^\tau A_{ijj}(\tau')^2 d\tau' \right) A_{ijj}(\tau) d\tau}{\kappa_{ij}^* \left[1 + a_j \sigma_{ij}^2 \int_0^T \alpha_j(\tau) \left(\int_0^\tau A_{ijj}(\tau') d\tau' \right) A_{ijj}(\tau) d\tau \right]}. \tag{B.18}$$

Given $\bar{i}_j^*$, (B.16) determines $C_j(\tau)$ uniquely. ∎

Proof of Proposition 4.1: The property $A_{ije} > 0$ is shown in the proof of Proposition B.1. The UIP value of A_{ije} is $A_{ije}^{UIP} \equiv \frac{1}{\kappa_{ij}}$, as can be seen from (B.2) by setting $a_e = 0$. When $a_e > 0$ and $\alpha_e > 0$, the proof of Proposition B.1 shows $A_{ije} < \frac{1}{\kappa_{ij}}$. Differentiating (B.5) with respect to i_{Ht} and i_{Ft}, we find

$$\frac{\partial(\mu_{et} + i_{Ft} - i_{Ht})}{\partial i_{Ht}} = \kappa_{iH} A_{iHe} - 1 < 0,$$
$$\frac{\partial(\mu_{et} + i_{Ft} - i_{Ht})}{\partial i_{Ft}} = -\kappa_{iF} A_{iFe} + 1 > 0,$$

respectively. ∎

Proof of Proposition 4.2: The properties $A_{ijj}(\tau) > 0$ and $A_{ij'j} = 0$ for $j' \neq j$ are shown in the proof of Proposition B.2. The EH value of $A_{ijj}(\tau)$ is $A^{EH}_{ijj}(\tau) \equiv \frac{1-e^{-\kappa_{ij}\tau}}{\kappa_{ij}}$, as can be seen from (B.13) and (B.14) by setting $a_j = 0$. When $a_j > 0$ and $\alpha_j(\tau) > 0$, (B.14) implies $\kappa^*_{ij} > \kappa_{ij}$ and (B.13) implies $A_{ijj}(\tau) < A^{EH}_{ijj}(\tau)$. Differentiating (B.12) with respect to i_{jt}, we find

$$\frac{\partial\left(\mu^{(\tau)}_{jt} - i_{jt}\right)}{\partial i_{jt}} = A'_{ijj}(\tau) + \kappa_{ij}A_{ijj}(\tau) - 1 = (\kappa_{ij} - \kappa^*_{ij})A_{ijj}(\tau) < 0,$$

where the second step follows from (B.13). ∎

Proof of Proposition 4.3: Consider an one-off increase in γ_t at time zero, and denote by κ_γ the rate at which γ_t reverts to its mean of zero. Equation (B.4) is modified to

$$\lambda_{ejt} = a_e\sigma^2_{ij}\left[\zeta_e + \theta_e\gamma_t - \alpha_e\left(A_{iHe}i_{Ht} - A_{iFe}i_{Ft} + A_{\gamma e}e_t + C_e\right)\right]A_{ijt}(-1)^{1_{\{j=F\}}} \tag{B.19}$$

and (B.5) is modified to

$$\mu_{et} = -A_{iHe}\kappa_{iH}(\bar{i}_H - i_{Ht}) + A_{iFe}\kappa_{iF}(\bar{i}_F - i_{Ft}) + A_{\gamma e}\kappa_\gamma\gamma_t - (\pi_F - \pi_H) + \frac{1}{2}A^2_{iHe}\sigma^2_{iH} + \frac{1}{2}A^2_{iFe}\sigma^2_{iF}. \tag{B.20}$$

Substituting λ_{ejt} from (B.19) and μ_{et} from (B.20) into (4.2), we find an equation that is affine in $(i_{Ht}, i_{Ft}, \gamma_t)$. Identifying the linear terms in γ_t yields

$$\begin{aligned} &\kappa_\gamma A_{\gamma e} = a_e(\theta_e - \alpha_e A_{\gamma e})\left(A^2_{iHe}\sigma^2_{iH} + A^2_{iFe}\sigma^2_{iF}\right) \\ &\Rightarrow A_{\gamma e} = \frac{a_e\theta_e\left(A^2_{iHe}\sigma^2_{iH} + A^2_{iFe}\sigma^2_{iF}\right)}{\kappa_\gamma + a_e\alpha_e\left(A^2_{iHe}\sigma^2_{iH} + A^2_{iFe}\sigma^2_{iF}\right)}. \end{aligned} \tag{B.21}$$

When $\alpha_e > 0$, (B.21) implies $A_{\gamma e} > 0$ because $\theta_e > 0$. Hence, an increase in γ_t causes the foreign currency to depreciate. Since bonds in each country are traded by a separate set of agents than those trading foreign currency, their prices do not depend on γ_t.

Consider next an one-off increase in β_{jt} at time zero, and denote by $\kappa_{\beta j}$ the rate at which β_{jt} reverts to its mean of zero. Equation (B.11) is modified to

$$\lambda_{jt} = a_j\sigma^2_{ij}\left(\int_0^T\left[\zeta_j(\tau) - \alpha_j(\tau)\left(A_{ijj}(\tau)i_{jt} + A_{\beta jj}(\tau)\beta_{jt} + C_j(\tau)\right)\right]A_{ijj}(\tau)d\tau\right), \tag{B.22}$$

and (B.12) is modified to

$$\mu_{jt}^{(\tau)} = A'_{ijj}(\tau)i_{jt} + A'_{\beta jj}(\tau)\beta_{jt} + C'_j(\tau) - A_{ijj}(\tau)\kappa_{ij}(\bar{i}_j - i_{jt}) + A_{\beta jj}(\tau)\kappa_{\beta j}\beta_{jt}$$
$$+ \frac{1}{2}A_{ijj}(\tau)\left(A_{ijj}(\tau) - 2A_{iFe}1_{\{j=F\}}\right)\sigma_{ij}^2. \tag{B.23}$$

Substituting λ_{jt} from (B.22) and μ_{jt} from (B.23) into (4.3), we find an equation that is affine in (i_{jt}, β_{jt}). Identifying the linear terms in β_{jt} yields

$$A'_{\beta jj}(\tau) + \kappa_{\beta j}A_{\beta jj}(\tau) = a_j\sigma_{ij}^2 A_{ijj}(\tau)\int_0^T \left[\theta_j(\tau) - \alpha_j(\tau)A_{\beta jj}(\tau)\right]A_{ijj}(\tau)d\tau. \tag{B.24}$$

Solving (B.24) with the initial condition $A_{\beta jj}(\tau) = 0$, we find

$$A_{\beta jj}(\tau) = \bar{\lambda}_{ij\beta j}\int_0^\tau A_{ijj}(\tau')e^{-\kappa_{\beta j}(\tau-\tau')}d\tau', \tag{B.25}$$

where

$$\bar{\lambda}_{ij\beta j} \equiv a_j\sigma_{ij}^2\int_0^T \left[\theta_j(\tau) - \alpha_j(\tau)A_{\beta jj}(\tau)\right]A_{ijj}(\tau)d\tau. \tag{B.26}$$

Substituting $A_{\beta jj}(\tau)$ from (B.25) into (B.26) and solving for $\bar{\lambda}_{ij\beta j}$, we find

$$\bar{\lambda}_{ij\beta j} = \frac{a_j\sigma_{ij}^2\int_0^T \theta_j(\tau)A_{ijj}(\tau)d\tau}{1 + a_j\sigma_{ij}^2\int_0^T \alpha_j(\tau)\left(\int_0^\tau A_{ijj}(\tau')e^{-\kappa_{\beta j}(\tau-\tau')}d\tau'\right)A_{ijj}(\tau)d\tau}. \tag{B.27}$$

When $a_j > 0$, (B.27) implies $\lambda_{ij\beta j} > 0$ and (B.25) implies $A_{\beta jj}(\tau) > 0$ because $(\theta_j(\tau), A_{ijj}(\tau))$ are positive. Hence, an increase in β_{jt} raises bond yields in country j. Since the foreign currency and bonds in country j' are traded by different agents than those trading bonds in country j, their prices do not depend on β_{jt}. ∎

Proposition B.3. *Suppose that arbitrage is global, the matrices* (Γ, Σ) *are diagonal, and* $\Sigma_{3,3} = \Sigma_{4,4} = \Sigma_{5,5} = 0$. *The exchange rate* e_t *is given by (3.1) and bond prices* $P_{jt}^{(\tau)}$ *in country* $j = H, F$ *are given by (3.2), with* $(\{A_{ije}\}_{j=H,F}, C_e)$ *solving*

$$\kappa_{ij}A_{ije} - 1 = a\sigma_{ij}^2\bar{\lambda}_{ijj}A_{ije} - a\sigma_{ij'}^2\bar{\lambda}_{ij'j}A_{ij'e}, \tag{B.28}$$

$$-\kappa_{iH}\bar{i}_H A_{iHe} + \kappa_{iF}\bar{i}_F A_{iFe} - (\pi_F - \pi_H) + \frac{1}{2}\sigma_{iH}^2 A_{iHe}^2 + \frac{1}{2}\sigma_{iF}^2 A_{iFe}^2 \tag{B.29}$$
$$= a\sigma_{iH}^2\bar{\lambda}_{iHC}A_{iHe} - a\sigma_{iF}^2\bar{\lambda}_{iFC}A_{iFe},$$

and $(A_{ijj}(\tau), A_{ijj'}(\tau), C_j(\tau))$ *solving*

$$A'_{ijj}(\tau) + \kappa_{ij}A_{ijj}(\tau) - 1 = a\sigma_{ij}^2\bar{\lambda}_{ijj}A_{ijj}(\tau) + a\sigma_{ij'}^2\bar{\lambda}_{ij'j}A_{ij'j}(\tau), \tag{B.30}$$

$$A'_{ij'j}(\tau) + \kappa_{ij'}A_{ij'j}(\tau) = a\sigma_{ij}^2\bar{\lambda}_{ijj'}A_{ijj}(\tau) + a\sigma_{ij'}^2\bar{\lambda}_{ij'j'}A_{ij'j}(\tau), \tag{B.31}$$

$$\begin{aligned} &C'_j(\tau) - \kappa_{ij}\bar{i}_jA_{ijj}(\tau) - \kappa_{ij'}\bar{i}_{j'}A_{ij'j}(\tau) + \frac{1}{2}\sigma_{ij}^2A_{ijj}(\tau)\left(A_{ijj}(\tau) - 2A_{iFe}1_{\{j=F\}}\right) \\ &+ \frac{1}{2}\sigma_{ij'}^2A_{ij'j}(\tau)\left(A_{ij'j}(\tau) + 2A_{iHe}1_{\{j=F\}}\right) = a\sigma_{ij}^2\bar{\lambda}_{ijC}A_{ijj}(\tau) + a\sigma_{ij'}^2\bar{\lambda}_{ij'C}A_{ij'j}(\tau), \end{aligned} \tag{B.32}$$

with the initial conditions $A_{ijj}(0) = A_{ijj'}(0) = C_j(0) = 0$*, where* $j' \neq j$ *and*

$$\bar{\lambda}_{ijj} \equiv -\alpha_eA_{ije}^2 - \sum_{k=H,F}\int_0^T \alpha_k(\tau)A_{ijk}(\tau)^2d\tau, \tag{B.33}$$

$$\bar{\lambda}_{ijj'} \equiv \alpha_eA_{ije}A_{ij'e} - \sum_{k=H,F}\int_0^T \alpha_k(\tau)A_{ijk}(\tau)A_{ij'k}(\tau)d\tau, \tag{B.34}$$

$$\bar{\lambda}_{ijC} \equiv (\zeta_e - \alpha_eC_e)A_{ije}(-1)^{1_{\{j=F\}}} + \sum_{k=H,F}\int_0^T (\zeta_k(\tau) - \alpha_k(\tau)C_k(\tau))A_{ijk}(\tau)d\tau. \tag{B.35}$$

Proof of Proposition B.3: The first-order conditions in Section 4.2 follow from (3.12) and (3.13) by taking Σ to be diagonal with $\Sigma_{3,3} = \Sigma_{4,4} = \Sigma_{5,5} = 0$. Proceeding as in the derivation of (3.18) and using $\gamma_t = \beta_{Ht} = \beta_{Ft} = 0$, we find

$$\begin{aligned} \lambda_{ijt} = a\sigma_{ij}^2 &\Bigg([\zeta_e - \alpha_e(A_{iHe}i_{Ht} - A_{iFe}i_{Ft} + C_e)]A_{ije}(-1)^{1_{\{j=F\}}} \\ &+ \sum_{j'=H,F}\int_0^T [\zeta_{j'}(\tau) - \alpha_{j'}(\tau)(A_{iHj'}(\tau)i_{Ht} + A_{iFj'}(\tau)i_{Ft} + C_{j'}(\tau))]A_{ijj'}(\tau)d\tau \Bigg) \\ \equiv{}& a\sigma_{ij}^2\left(\bar{\lambda}_{ijj}i_{jt} + \bar{\lambda}_{ijj'}i_{j't} + \bar{\lambda}_{ijC}\right). \end{aligned} \tag{B.36}$$

Since (Γ, Σ) are diagonal with $\Sigma_{3,3} = \Sigma_{4,4} = \Sigma_{5,5} = 0$ and $\beta_{Ht} = \beta_{Ft} = \gamma_t = 0$, we can write (3.8) and (3.10) as

$$\begin{aligned} \mu_{jt}^{(\tau)} \equiv{}& A'_{iHj}(\tau)i_{Ht} + A'_{iFj}(\tau)i_{Ft} + C'_j(\tau) - A_{iHj}(\tau)\kappa_{iH}(\bar{i}_H - i_{Ht}) - A_{iFj}(\tau)\kappa_{iF}(\bar{i}_F - i_{Ft}) \\ &+ \frac{1}{2}A_{iHj}(\tau)\left(A_{iHj}(\tau) + 2A_{iHe}1_{\{j=F\}}\right)\sigma_{iH}^2 + \frac{1}{2}A_{iFj}(\tau)\left(A_{iFj}(\tau) - 1_{\{j=F\}}2A_{iFe}\right)\sigma_{iF}^2. \end{aligned} \tag{B.37}$$

Substituting λ_t from (B.36) and μ_{et} from (B.5) into (3.12) (for the definitions of (A_e, λ_t) in Section 4.2), we find an equation that is affine in (i_{Ht}, i_{Ft}). Equation (B.28) follows by identifying the linear terms in (i_{Ht}, i_{Ft}), and (B.29) follows by identifying the constant terms. Substituting λ_t from (B.36) and $\mu_{jt}^{(\tau)}$ from (B.37) into (3.13) (for the definitions of $(A_j(\tau), \lambda_t)$ in Section 4.2), we find an equation that is affine in (i_{Ht}, i_{Ft}). Equations (B.30) and (B.31) follow by identifying the linear terms in (i_{Ht}, i_{Ft}), and (B.32) follows by identifying the constant terms. Solving the system of (B.28)-(B.35) reduces to solving a system of three nonlinear scalar equations. Indeed, taking $\bar{\lambda}_{iHH}$, $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH}$ and $\bar{\lambda}_{iFF}$ as given, we can solve the linear scalar system (B.28) in (A_{iHe}, A_{iFe}), the system (B.30) and (B.31) of two linear ODEs in $(A_{iHH}(\tau), A_{iFH}(\tau))$ (setting $(j, j') = (H, F)$), and the same system (B.30) and (B.31) of two linear ODEs in $(A_{iHF}(\tau), A_{iFF}(\tau))$ (setting $(j, j') = (F, H)$). We can then substitute back into the definitions of $\bar{\lambda}_{iHH}$, $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH}$ and $\bar{\lambda}_{iFF}$ to derive the system of three nonlinear scalar equations. Given a solution of that system, (B.32) determines $(C_H(\tau), C_F(\tau))$ uniquely, and (B.29) determines C_e uniquely if $\alpha_e > 0$. If $\alpha_e = 0$, then a parameter restriction analogous to (B.8) must be imposed and C_e is indeterminate. The results in Section 4.2 hold for any solution $\bar{\lambda}_{iHH}$, $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH}$ and $\bar{\lambda}_{iFF}$. ∎

Proof of Proposition 4.4: We start by proving a series of lemmas.

Lemma B.1. *The matrix*

$$M_i \equiv \begin{pmatrix} \kappa_{iH} - a\sigma_{iH}^2 \bar{\lambda}_{iHH} & -a\sigma_{iF}^2 \bar{\lambda}_{iFH} \\ -a\sigma_{iH}^2 \bar{\lambda}_{iHF} & \kappa_{iF} - a\sigma_{iF}^2 \bar{\lambda}_{iFF} \end{pmatrix} \tag{B.38}$$

has two positive eigenvalues.

Proof: The characteristic polynomial of M_i is

$$\Pi(\lambda) \equiv \left(\kappa_{iH} - a\sigma_{iH}^2 \bar{\lambda}_{iHH} - \lambda\right)\left(\kappa_{iF} - a\sigma_{iF}^2 \bar{\lambda}_{iFF} - \lambda\right) - a^2\sigma_{iH}^2\sigma_{iF}^2 \bar{\lambda}_{iHF}\bar{\lambda}_{iFH}. \tag{B.39}$$

For $\lambda = 0$, $\Pi(\lambda)$ takes the value

$$\begin{aligned}
\Pi(0) &= \left(\kappa_{iH} - a\sigma_{iH}^2 \bar{\lambda}_{iHH}\right)\left(\kappa_{iF} - a\sigma_{iF}^2 \bar{\lambda}_{iFF}\right) - a\sigma_{iH}^2\sigma_{iF}^2 \bar{\lambda}_{iHF}\bar{\lambda}_{iFH} \\
&> a^2\sigma_{iH}^2\sigma_{iH}^2 \left(\bar{\lambda}_{iHH}\bar{\lambda}_{iFF} - \bar{\lambda}_{iHF}\bar{\lambda}_{iFH}\right) \\
&= a^2\sigma_{iH}^2\sigma_{iH}^2 \Bigg[\left(\alpha_e A_{iHe}^2 + \int_0^T \alpha_H(\tau) A_{iHH}(\tau)^2 d\tau + \int_0^T \alpha_F(\tau) A_{iHF}(\tau)^2 d\tau\right) \\
&\quad \times \left(\alpha_e A_{iFe}^2 + \int_0^T \alpha_H(\tau) A_{iFH}(\tau)^2 d\tau + \int_0^T \alpha_F(\tau) A_{iFF}(\tau)^2 d\tau\right) \\
&\quad - \left(\alpha_e A_{iHe} A_{iFe} - \int_0^T \alpha_H(\tau) A_{iHH}(\tau) A_{iFH}(\tau) d\tau + \int_0^T \alpha_F(\tau) A_{iHF}(\tau) A_{iFF}(\tau) d\tau\right)^2\Bigg].
\end{aligned} \tag{B.40}$$

The second step in (B.40) follows because $(\kappa_{iH}, \kappa_{iF})$ are positive and because (B.33) implies that $(\bar{\lambda}_{iHH}, \bar{\lambda}_{iFF})$ are non-positive. The third step in (B.40) follows from (B.33) and (B.34). The Cauchy-Schwarz inequality associated to the scalar product

$$X \cdot Y \equiv \alpha_e xy + \int_0^T \alpha_H(\tau) X_H(\tau) Y_H(\tau) d\tau + \int_0^T \alpha_F(\tau) X_F(\tau) Y_F(\tau) d\tau$$

where $X \equiv (x, X_H(\tau), X_F(\tau))$, $Y \equiv (y, Y_H(\tau), Y_F(\tau))$, (x, y) are scalars, and $(X_H(\tau), X_F(\tau), Y_H(\tau), Y_F(\tau))$ are functions of τ, implies that (B.40) is non-negative. Hence, $\Pi(0) > 0$.

For $\lambda = \kappa_{iH} - a\sigma_{iH}^2 \bar{\lambda}_{iHH}$ and $\lambda = \kappa_{iF} - a\sigma_{iF}^2 \bar{\lambda}_{iFF}$, $\Pi(\lambda)$ takes the value $-a^2\sigma_{iH}^2\sigma_{iF}^2\bar{\lambda}_{iHF}\bar{\lambda}_{iFH}$, which is non-positive because (B.34) implies $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH}$. Since $(\kappa_{iH}, \kappa_{iF})$ are positive and $(\bar{\lambda}_{iHH}, \bar{\lambda}_{iFF})$ are non-positive, $\kappa_{iH} - a\sigma_{iH}^2 \bar{\lambda}_{iHH}$ and $\lambda = \kappa_{iF} - a\sigma_{iF}^2 \bar{\lambda}_{iFF}$ are positive. Since $\Pi(\lambda)$ is a quadratic function of λ, is positive for $\lambda = 0$, is non-positive for two positive values of λ, and goes to infinity when λ goes to infinity, it has two positive roots. ∎

The matrix M_i plays an important role in the determination of (A_{iHe}, A_{iFe}) and $(A_{iHH}(\tau), A_{iFF}(\tau), A_{iHF}(\tau), A_{iFH}(\tau))$. Equation (B.28) gives rise to the linear system

$$M_i \begin{pmatrix} A_{iHe} \\ A_{iFe} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{B.41}$$

Since M_i has two positive eigenvalues, it is invertible, and hence (B.41) can be solved for (A_{iHe}, A_{iFe}). Equations (B.30) and (B.31) give rise to the linear system

$$\begin{pmatrix} A_{iHH}(\tau) \\ A_{iFH}(\tau) \end{pmatrix}' + M_i \begin{pmatrix} A_{iHH}(\tau) \\ A_{iFH}(\tau) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{B.42}$$

for $(j, j') = (H, F)$, and to

$$\begin{pmatrix} A_{iHF}(\tau) \\ A_{iFF}(\tau) \end{pmatrix}' + M_i \begin{pmatrix} A_{iHF}(\tau) \\ A_{iFF}(\tau) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{B.43}$$

for $(j, j') = (F, H)$. Since M_i has two positive eigenvalues, the solutions $(A_{iHH}(\tau), A_{iFH}(\tau))$ to (B.42) and $(A_{iHF}(\tau), A_{iFF}(\tau))$ to (B.43) go to finite limits when τ goes to infinity.

Lemma B.2. *The normalized factor prices $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH}$ are non-negative.*

Proof: Suppose, proceeding by contradiction, that $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH}$ are negative. The solution to (B.41) is

$$A_{iHe} = \frac{\kappa_{iF} - a\sigma_{iF}^2(\bar{\lambda}_{iFF} + \bar{\lambda}_{iFH})}{\left(\kappa_{iH} - a\sigma_{iH}^2\bar{\lambda}_{iHH}\right)\left(\kappa_{iF} - a\sigma_{iF}^2\bar{\lambda}_{iFF}\right) - a^2\sigma_{iH}^2\sigma_{iF}^2\bar{\lambda}_{iHF}\bar{\lambda}_{iFH}}, \tag{B.44}$$

$$A_{iFe} = \frac{\kappa_{iH} - a\sigma_{iH}^2(\bar{\lambda}_{iHH} + \bar{\lambda}_{iHF})}{\left(\kappa_{iH} - a\sigma_{iH}^2\bar{\lambda}_{iHH}\right)\left(\kappa_{iF} - a\sigma_{iF}^2\bar{\lambda}_{iFF}\right) - a^2\sigma_{iH}^2\sigma_{iF}^2\bar{\lambda}_{iHF}\bar{\lambda}_{iFH}}. \tag{B.45}$$

The denominator in (B.44) and (B.45) is $\Pi(0) > 0$. The numerators in (B.44) and (B.45) are positive because $(\kappa_{iH}, \kappa_{iF})$ are positive and $(a\bar{\lambda}_{iHH}, a\bar{\lambda}_{iFF}, a\bar{\lambda}_{iHF}, a\bar{\lambda}_{iFH})$ are non-positive. Hence, (A_{iHe}, A_{iFe}) are positive.

When $a = 0$, (B.31) with the initial conditions $A_{iHF}(0) = A_{iFH}(0) = 0$ implies $A_{iHF}(\tau) = A_{iFH}(\tau) = 0$ for all $\tau > 0$. Since, in addition, $A_{iHe} > 0$ and $A_{iFe} > 0$, (B.34) implies $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH} \geq 0$, a contradiction.

When $a > 0$, (B.30) and (B.31) with the initial conditions $A_{iHH}(0) = A_{iFF}(0) = A_{iHF}(0) = A_{iFH}(0) = 0$ imply $A'_{iHH}(0) = A'_{iFF}(0) = 1$ and $A'_{iHF}(0) = A'_{iFH}(0) = 0$. Moreover, differentiating (B.31), we find $A''_{iFH}(0) = a\sigma_{iH}^2\bar{\lambda}_{iHF}A'_{iHH}(0) < 0$ and $A''_{iHF}(0) = a\sigma_{iF}^2\bar{\lambda}_{iFH}A'_{iFF}(0) < 0$. Hence, $A_{iHH}(\tau) > 0$, $A_{iFF}(\tau) > 0$, $A_{iHF}(\tau) < 0$ and $A_{iFH}(\tau) < 0$ for τ close to zero. We define τ_0 by

$$\tau_0 \equiv \sup_{\tau}\{A_{iHH}(\tau') > 0,\ A_{iFF}(\tau') > 0,\ A_{iHF}(\tau') < 0 \text{ and } A_{iFH}(\tau') < 0 \text{ for all } \tau' \in (0, \tau)\}.$$

If τ_0 is finite, then (i) $A_{iHH}(\tau_0) = 0$, $A'_{iHH}(\tau_0) \leq 0$, $A_{iFF}(\tau_0) \geq 0$, $A_{iHF}(\tau_0) \leq 0$ and $A_{iFH}(\tau_0) \leq 0$, or (ii) $A_{iHH}(\tau_0) > 0$, $A_{iFF}(\tau_0) = 0$, $A'_{iFF}(\tau_0) \leq 0$, $A_{iHF}(\tau_0) \leq 0$ and $A_{iFH}(\tau_0) \leq 0$, or (iii) $A_{iHH}(\tau_0) > 0$, $A_{iFF}(\tau_0) > 0$, $A_{iHF}(\tau_0) = 0$, $A'_{iHF}(\tau_0) \geq 0$ and $A_{iFH}(\tau_0) \leq 0$, or (iv) $A_{iHH}(\tau_0) > 0$, $A_{iFF}(\tau_0) > 0$, $A_{iHF}(\tau_0) < 0$, $A_{iFH}(\tau_0) = 0$ and $A'_{iFH}(\tau_0) \geq 0$. Case (i) yields a contradiction because (B.30) for $(j, j') = (H, F)$, $A_{iHH}(\tau_0) = 0$, $A_{iFH}(\tau_0) \leq 0$ and $\bar{\lambda}_{iFH} < 0$ imply $A'_{iHH}(\tau_0) \geq 1$. Case (ii) yields a contradiction by using the same argument as in Case (i) and switching H and F. Case (iii) yields a contradiction because (B.31) for $(j, j') = (H, F)$, $A_{iHH}(\tau_0) > 0$, $A_{iFH}(\tau_0) = 0$ and $\bar{\lambda}_{iHF} < 0$ imply $A'_{iFH}(\tau_0) < 0$. Case (iv) yields a contradiction by using the same argument as in Case (iii) and switching H and F. Therefore, τ_0 is infinite, which means $A_{iHH}(\tau) > 0$, $A_{iFF}(\tau) > 0$, $A_{iHF}(\tau) < 0$ and $A_{iFH}(\tau) < 0$ for all $\tau > 0$. Since, in addition, $A_{iHe} > 0$ and $A_{iFe} > 0$, (B.34) implies $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH} \geq 0$, a contradiction. Therefore, $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH}$ are non-negative. ∎

Lemma B.3. *The functions $A_{iHH}(\tau)$ and $A_{iFF}(\tau)$ are positive for all $\tau > 0$.*

- *When $a > 0$ and $\alpha_e > 0$, the functions $A_{iHF}(\tau)$ and $A_{iFH}(\tau)$ are positive for all $\tau > 0$.*
- *When $a = 0$ or $\alpha_e = 0$, the functions $A_{iHF}(\tau)$ and $A_{iFH}(\tau)$ are zero.*

Proof: Consider first the case $a > 0$ and $\alpha_e > 0$. If $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH} = 0$, then (B.31) with the initial conditions $A_{iHF}(0) = A_{iFH}(0) = 0$ implies $A_{iHF}(\tau) = A_{iFH}(\tau) = 0$ for all $\tau > 0$. Since, in addition, (B.44) and (B.45) imply $A_{iHe} > 0$ and $A_{iFe} > 0$, (B.34) implies $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH} > 0$, a contradiction. Hence, Lemma B.2 implies $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH} > 0$.

Equations (B.30) and (B.31) with the initial conditions $A_{iHH}(0) = A_{iFF}(0) = A_{iHF}(0) = A_{iFH}(0) = 0$ imply $A'_{iHH}(0) = A'_{iFF}(0) = 1$ and $A'_{iHF}(0) = A'_{iFH}(0) = 0$. Moreover, differentiating (B.31), we find $A''_{iFH}(0) = a\sigma_{iH}^2\bar{\lambda}_{iHF}A'_{iHH}(0) > 0$ and $A''_{iHF}(0) =$

$a\sigma_{iF}^2\bar{\lambda}_{iFH}A'_{iFF}(0) > 0$. Hence, $A_{iHH}(\tau) > 0$, $A_{iFF}(\tau) > 0$, $A_{iHF}(\tau) > 0$ and $A_{iFH}(\tau) > 0$ for τ close to zero. We define τ_0 by

$$\tau_0 \equiv \sup_{\tau}\{A_{iHH}(\tau') > 0,\ A_{iFF}(\tau') > 0,\ A_{iHF}(\tau') > 0 \text{ and } A_{iFH}(\tau') > 0 \text{ for all } \tau' \in (0,\tau)\}.$$

If τ_0 is finite, then (i) $A_{iHH}(\tau_0) = 0$, $A'_{iHH}(\tau_0) \leq 0$, $A_{iFF}(\tau_0) \geq 0$, $A_{iHF}(\tau_0) \geq 0$ and $A_{iFH}(\tau_0) \geq 0$, or (ii) $A_{iHH}(\tau_0) > 0$, $A_{iFF}(\tau_0) = 0$, $A'_{iFF}(\tau_0) \leq 0$, $A_{iHF}(\tau_0) \geq 0$ and $A_{iFH}(\tau_0) \geq 0$, or (iii) $A_{iHH}(\tau_0) > 0$, $A_{iFF}(\tau_0) > 0$, $A_{iHF}(\tau_0) = 0$, $A'_{iHF}(\tau_0) \leq 0$ and $A_{iFH}(\tau_0) \geq 0$, or (iv) $A_{iHH}(\tau_0) > 0$, $A_{iFF}(\tau_0) > 0$, $A_{iHF}(\tau_0) > 0$, $A_{iFH}(\tau_0) = 0$ and $A'_{iFH}(\tau_0) \leq 0$. Case (i) yields a contradiction because (B.30) for $(j,j') = (H,F)$, $A_{iHH}(\tau_0) = 0$, $A_{iFH}(\tau_0) \geq 0$ and $\bar{\lambda}_{iFH} > 0$ imply $A'_{iHH}(\tau_0) \geq 1$. Case (ii) yields a contradiction by using the same argument as in Case (i) and switching H and F. Case (iii) yields a contradiction because (B.31) for $(j,j') = (H,F)$, $A_{iHH}(\tau_0) > 0$, $A_{iFH}(\tau_0) = 0$ and $\bar{\lambda}_{iHF} > 0$ imply $A'_{iFH}(\tau_0) > 0$. Case (iv) yields a contradiction by using the same argument as in Case (iii) and switching H and F. Therefore, τ_0 is infinite, which means $A_{iHH}(\tau) > 0$, $A_{iFF}(\tau) > 0$, $A_{iHF}(\tau) > 0$ and $A_{iFH}(\tau) > 0$ for all $\tau > 0$.

Consider next the case $a = 0$. The properties of $(A_{iHH}(\tau), A_{iFF}(\tau), A_{iHF}(\tau), A_{iFH}(\tau))$ follow because (B.30) with the initial conditions $A_{iHH}(0) = A_{iFF}(0) = 0$ implies $A_{iHH}(\tau) = A^{EH}_{iHH}(\tau) \equiv \frac{1-e^{-\kappa_{iH}\tau}}{\kappa_{iH}} > 0$ and $A_{iFF}(\tau) = A^{EH}_{iFF}(\tau) \equiv \frac{1-e^{-\kappa_{iF}\tau}}{\kappa_{iF}} > 0$, and (B.31) with the initial conditions $A_{iHF}(0) = A_{iFH}(0) = 0$ implies $A_{iHF}(\tau) = A_{iFH}(\tau) = 0$.

Consider finally the case $a > 0$ and $\alpha_e = 0$. Suppose, proceeding by contradiction, that $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH}$ are positive. The argument in the case $a > 0$ and $\alpha_e > 0$ implies that $(A_{iHH}(\tau), A_{iFF}(\tau), A_{iHF}(\tau), A_{iFH}(\tau))$ are positive for all $\tau > 0$. Since $\alpha_e = 0$, (B.34) implies $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH} \leq 0$, a contradiction. Hence, Lemma B.2 implies $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH} = 0$. Since $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH} = 0$, (B.31) with the initial conditions $A_{iHF}(0) = A_{iFH}(0) = 0$ implies $A_{iHF}(\tau) = A_{iFH}(\tau) = 0$. Since $A_{iHF}(\tau) = A_{iFH}(\tau) = 0$, (B.30) with the initial conditions $A_{iHH}(0) = A_{iFF}(0) = 0$ implies that $(A_{iHH}(\tau), A_{iFF}(\tau))$ are positive for all $\tau > 0$. ∎

Lemma B.4. *The functions $A_{iHH}(\tau)$ and $A_{iFF}(\tau)$ are increasing. When $a > 0$ and $\alpha_e > 0$, the functions $A_{iHF}(\tau)$ and $A_{iFH}(\tau)$ are also increasing.*

Proof: Consider first the case $a > 0$ and $\alpha_e > 0$. Equations $A'_{iHH}(0) = A'_{iFF}(0) = 1$, $A'_{iHF}(0) = A'_{iFH}(0) = 0$, $A''_{iFH}(0) = a\sigma_{iH}^2\bar{\lambda}_{iHF}A'_{iHH}(0) > 0$ and $A''_{iHF}(0) = a\sigma_{iF}^2\bar{\lambda}_{iFH}A'_{iFF}(0) > 0$ imply $A'_{iHH}(\tau) > 0$, $A'_{iFF}(\tau) > 0$, $A'_{iHF}(\tau) > 0$ and $A'_{iFH}(\tau) > 0$ for τ close to zero. We define τ_0 by

$$\tau_0 \equiv \sup_{\tau}\{A'_{iHH}(\tau') > 0,\ A'_{iFF}(\tau') > 0,\ A'_{iHF}(\tau') > 0 \text{ and } A'_{iFH}(\tau') > 0 \text{ for all } \tau' \in (0,\tau)\}.$$

If τ_0 is finite, then (i) $A'_{iHH}(\tau_0) = 0$, $A''_{iHH}(\tau_0) \leq 0$, $A'_{iFF}(\tau_0) \geq 0$, $A'_{iHF}(\tau_0) \geq 0$ and $A'_{iFH}(\tau_0) \geq 0$, or (ii) $A'_{iHH}(\tau_0) > 0$, $A'_{iFF}(\tau_0) = 0$, $A''_{iFF}(\tau_0) \leq 0$, $A'_{iHF}(\tau_0) \geq 0$ and

$A_{iFH}(\tau_0)' \geq 0$, or (iii) $A'_{iHH}(\tau_0) > 0$, $A'_{iFF}(\tau_0) > 0$, $A'_{iHF}(\tau_0) = 0$, $A''_{iHF}(\tau_0) \leq 0$ and $A'_{iFH}(\tau_0) \geq 0$, or (iv) $A'_{iHH}(\tau_0) > 0$, $A'_{iFF}(\tau_0) > 0$, $A'_{iHF}(\tau_0) > 0$, $A'_{iFH}(\tau_0) = 0$ and $A''_{iFH}(\tau_0) \leq 0$. To analyze Cases (i)-(iv), we use

$$A''_{ijj}(\tau) + \kappa_{ij} A'_{ijj}(\tau) = a\sigma_{ij}^2 \bar{\lambda}_{ijj} A'_{ijj}(\tau) + a\sigma_{ij'}^2 \bar{\lambda}_{ij'j} A'_{ij'j}(\tau), \tag{B.46}$$

$$A''_{ij'j}(\tau) + \kappa_{ij'} A'_{ij'j}(\tau) = a\sigma_{ij}^2 \bar{\lambda}_{ijj'} A'_{ijj}(\tau) + a\sigma_{ij'}^2 \bar{\lambda}_{ij'j'} A'_{ij'j}(\tau), \tag{B.47}$$

which follow from differentiating (B.30) and (B.31), respectively.

Case (i) yields a contradiction. Indeed, if $A''_{iHH}(\tau_0) = 0$, then (B.46) for $(j, j') = (H, F)$, $A'_{iHH}(\tau_0) = 0$ and $\bar{\lambda}_{iFH} > 0$ imply $A'_{iFH}(\tau_0) = 0$. The unique solution to the linear system of ODEs (B.46) and (B.47) for $(j, j') = (H, F)$ with the initial condition $(A'_{iHH}(\tau_0), A'_{iFH}(\tau_0)) = (0, 0)$ is the function that equals (0,0) for all τ. This yields a contradiction because $(A'_{iHH}(0), A'_{iFH}(0)) = (1, 0)$. Hence, $A''_{iHH}(\tau_0) < 0$, which combined with (B.46) for $(j, j') = (H, F)$, $A'_{iHH}(\tau_0) = 0$ and $\bar{\lambda}_{iFH} > 0$ implies $A'_{iFH}(\tau_0) < 0$, again a contradiction. Case (ii) yields a contradiction by using the same argument as in Case (i) and switching H and F. Case (iii) yields a contradiction because (B.47) for $(j, j') = (H, F)$, $A'_{iHH}(\tau_0) > 0$, $A'_{iFH}(\tau_0) = 0$ and $\bar{\lambda}_{iHF} > 0$ imply $A''_{iFH}(\tau_0) > 0$. Case (iv) yields a contradiction by using the same argument as in Case (iii) and switching H and F. Therefore, τ_0 is infinite, which means that $(A_{iHH}(\tau), A_{iFF}(\tau), A_{iHF}(\tau), A_{iFH}(\tau))$ are increasing.

In the case $a = 0$ or $\alpha_e = 0$, Lemma B.3 implies $A_{iHF}(\tau) = A_{iFH}(\tau) = 0$. Since $A_{iHF}(\tau) = A_{iFH}(\tau) = 0$, (B.30) with the initial conditions $A_{iHH}(0) = A_{iFF}(0) = 0$ implies that $A_{iHH}(\tau)$ and $A_{iFF}(\tau)$ are increasing. ∎

Lemma B.5. *The scalars A_{iHe} and A_{iFe} are positive.*

Proof: Consider first the case $a > 0$ and $\alpha_e > 0$. Since $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH} > 0$ and $A_{iHH}(\tau) > 0$, $A_{iFF}(\tau) > 0$, $A_{iHF}(\tau) > 0$ and $A_{iFH}(\tau) > 0$ for all $\tau > 0$ (Lemma B.3), (B.34) implies $A_{iHe} A_{iFe} > 0$. Hence, (A_{iHe}, A_{iFe}) are either both positive or both negative. Suppose, proceeding by contradiction, that they are both negative. Equations (B.44) and (B.45) imply

$$\kappa_{iH} - a\sigma_{iH}^2 \bar{\lambda}_{iHH} < a\sigma_{iH}^2 \bar{\lambda}_{iHF}, \tag{B.48}$$

$$\kappa_{iF} - a\sigma_{iF}^2 \bar{\lambda}_{iFF} < a\sigma_{iF}^2 \bar{\lambda}_{iFH}. \tag{B.49}$$

Since the left-hand side in each of (B.48) and (B.49) is positive, (B.48) and (B.49) imply

$$\Pi(0) = \left(\kappa_{iH} - a\sigma_{iH}^2 \bar{\lambda}_{iHH}\right)\left(\kappa_{iF} - a\sigma_{iF}^2 \bar{\lambda}_{iFF}\right) - a\sigma_{iH}^2 \sigma_{iF}^2 \bar{\lambda}_{iHF} \bar{\lambda}_{iFH} < 0,$$

a contradiction. Hence, (A_{iHe}, A_{iFe}) are positive.

Consider next the case $a = 0$. Equation (B.28) implies $A_{iHe} = A_{iHe}^{UIP} \equiv \frac{1}{\kappa_{Hj}} > 0$ and $A_{iFe} = A_{iFe}^{UIP} \equiv \frac{1}{\kappa_{Fj}} > 0$. Consider finally the case $\alpha_e = 0$ and $a > 0$. Since $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH} = 0$ and $(\bar{\lambda}_{iHH}, \bar{\lambda}_{iFF})$ are non-positive, (B.44) and (B.45) imply that (A_{iHe}, A_{iFe}) are positive. ∎

Lemma B.6. *The functions $A_{iHH}(\tau) - A_{iHF}(\tau)$ and $A_{iFF}(\tau) - A_{iFH}(\tau)$ are positive for all $\tau > 0$.*

Proof: In the case $a = 0$ or $\alpha_e = 0$, the lemma follows from Lemma B.3. To prove the lemma in the case $a > 0$ and $\alpha_e > 0$, we proceed in two steps. In Step 1, we show that $A_{iHH}(\tau) - A_{iHF}(\tau)$ and $A_{iFF}(\tau) - A_{iFH}(\tau)$ are positive in the limit when τ goes to infinity. In Step 2, we show that $A_{iHH}(\tau) - A_{iHF}(\tau)$ and $A_{iFF}(\tau) - A_{iFH}(\tau)$ are either increasing in τ, or increasing and then decreasing. The lemma follows by combining these properties with $A_{iHH}(0) - A_{iHF}(0) = A_{iFF}(0) - A_{iFH}(0) = 0$.

Step 1: Limit at infinity. Since the matrix M has two positive eigenvalues, the functions $(A_{iHH}(\tau), A_{iFF}(\tau), A_{iHF}(\tau), A_{iFH}(\tau))$ go to finite limits when τ goes to infinity. These limits solve the system of equations

$$\kappa_{ij} A_{ijj}(\infty) - 1 = a\sigma_{ij}^2 \bar{\lambda}_{ijj} A_{ijj}(\infty) + a\sigma_{ij'}^2 \bar{\lambda}_{ij'j} A_{ij'j}(\infty), \tag{B.50}$$

$$\kappa_{ij'} A_{ij'j}(\infty) = a\sigma_{ij}^2 \bar{\lambda}_{ijj'} A_{ijj}(\infty) + a\sigma_{ij'}^2 \bar{\lambda}_{ij'j'} A_{ij'j}(\infty), \tag{B.51}$$

which are derived from (B.30) and (B.31) by setting the derivatives to zero. Subtracting (B.51) for $(j, j') = (F, H)$ from (B.50) for $(j, j') = (H, F)$, we find

$$\begin{aligned} &\kappa_{iH}(A_{iHH}(\infty) - A_{iHF}(\infty)) - 1 \\ &= a\sigma_{iH}^2 \bar{\lambda}_{iHH}(A_{iHH}(\infty) - A_{iHF}(\infty)) + a\sigma_{iF}^2 \bar{\lambda}_{iFH}(A_{iFH}(\infty) - A_{iFF}(\infty)). \end{aligned} \tag{B.52}$$

Subtracting (B.51) for $(j, j') = (H, F)$ from (B.50) for $(j, j') = (F, H)$, we similarly find

$$\begin{aligned} &\kappa_{iF}(A_{iFF}(\infty) - A_{iFH}(\infty)) - 1 \\ &= a\sigma_{iH}^2 \bar{\lambda}_{iHF}(A_{iHF}(\infty) - A_{iHH}(\infty)) + a\sigma_{iF}^2 \bar{\lambda}_{iFF}(A_{iFF}(\infty) - A_{iFH}(\infty)). \end{aligned} \tag{B.53}$$

The solution to the system of (B.52) and (B.53) is

$$A_{iHH}(\infty) - A_{iHF}(\infty) = \frac{\kappa_{iF} - a\sigma_{iF}^2(\bar{\lambda}_{iFF} + \bar{\lambda}_{iFH})}{\left(\kappa_{iH} - a\sigma_{iH}^2 \bar{\lambda}_{iHH}\right)\left(\kappa_{iF} - a\sigma_{iF}^2 \bar{\lambda}_{iFF}\right) - a^2 \sigma_{iH}^2 \sigma_{iF}^2 \bar{\lambda}_{iHF} \bar{\lambda}_{iFH}} = A_{iHe}, \tag{B.54}$$

$$A_{iFF}(\infty) - A_{iFH}(\infty) = \frac{\kappa_{iH} - a\sigma_{iH}^2(\bar{\lambda}_{iHH} + \bar{\lambda}_{iHF})}{\left(\kappa_{iH} - a\sigma_{iH}^2 \bar{\lambda}_{iHH}\right)\left(\kappa_{iF} - a\sigma_{iF}^2 \bar{\lambda}_{iFF}\right) - a^2 \sigma_{iH}^2 \sigma_{iF}^2 \bar{\lambda}_{iHF} \bar{\lambda}_{iFH}} = A_{iFe}, \tag{B.55}$$

where the second equality in (B.54) and (B.55) follows from (B.44) and (B.45), respectively. Since (A_{iHe}, A_{iFe}) are positive (Lemma B.5), so are $(A_{iHH}(\infty)-A_{iHF}(\infty), A_{iFF}(\infty)-A_{iFH}(\infty))$.

Step 2: Monotonicity. Equations (B.30) and (B.31) with the initial conditions $A_{iHH}(0) = A_{iFF}(0) = A_{iHF}(0) = A_{iFH}(0) = 0$ imply $A'_{iHH}(0) = A'_{iFF}(0) = 1 > 0$ and $A'_{iHF}(0) = A'_{iFH}(0) = 0$. Hence, $A'_{iHH}(\tau) - A'_{iHF}(\tau) > 0$ and $A'_{iFF}(\tau) - A'_{iFH}(\tau) > 0$ for τ close to zero. We define τ_0 by

$$\tau_0 \equiv \sup_{\tau}\{A'_{iHH}(\tau') - A'_{iHF}(\tau') > 0 \text{ and } A'_{iFF}(\tau') - A'_{iFH}(\tau') > 0 \text{ for all } \tau' \in (0,\tau)\}.$$

If τ_0 is infinity, then $A_{iHH}(\tau)-A_{iHF}(\tau)$ and $A_{iFF}(\tau)-A_{iFH}(\tau)$ are increasing in τ. Suppose instead that τ_0 is finite. Then, either (i) $A'_{iHH}(\tau_0)-A'_{iHF}(\tau_0) = 0$, $A''_{iHH}(\tau_0)-A''_{iHF}(\tau_0) \leq 0$ and $A'_{iFF}(\tau_0) - A'_{iFH}(\tau_0) \geq 0$, or (ii) $A'_{iHH}(\tau_0) - A'_{iHF}(\tau_0) > 0$, $A'_{iFF}(\tau_0) - A'_{iFH}(\tau_0) = 0$ and $A''_{iFF}(\tau_0) - A''_{iFH}(\tau_0) \leq 0$. To analyze Cases (i) and (ii), we use

$$\begin{aligned} &A'_{iHH}(\tau) - A'_{iHF}(\tau) + \kappa_{iH}(A_{iHH}(\tau) - A_{iHF}(\tau)) - 1 \\ &= a\sigma^2_{iH}\bar{\lambda}_{iHH}(A_{iHH}(\tau) - A_{iHF}(\tau)) + a\sigma^2_{iF}\bar{\lambda}_{iFH}(A_{iFH}(\tau) - A_{iFF}(\tau)), \end{aligned} \tag{B.56}$$

which follows by subtracting (B.31) for $(j,j') = (F,H)$ from (B.50) for $(j,j') = (H,F)$, and

$$\begin{aligned} &A'_{iFF}(\tau) - A'_{iFH}(\tau) + \kappa_{iF}(A_{iFF}(\tau) - A_{iFH}(\tau)) - 1 \\ &= a\sigma^2_{iH}\bar{\lambda}_{iHF}(A_{iHF}(\tau) - A_{iHH}(\tau)) + a\sigma^2_{iF}\bar{\lambda}_{iFF}(A_{iFF}(\tau) - A_{iFH}(\tau)), \end{aligned} \tag{B.57}$$

which follows by subtracting (B.51) for $(j,j') = (H,F)$ from (B.50) for $(j,j') = (F,H)$. Differentiating (B.56) and (B.57), we find

$$\begin{aligned} &A''_{iHH}(\tau) - A''_{iHF}(\tau) + \kappa_{iH}(A'_{iHH}(\tau) - A'_{iHF}(\tau)) \\ &= a\sigma^2_{iH}\bar{\lambda}_{iHH}(A'_{iHH}(\tau) - A'_{iHF}(\tau)) + a\sigma^2_{iF}\bar{\lambda}_{iFH}(A'_{iFH}(\tau) - A'_{iFF}(\tau)) \end{aligned} \tag{B.58}$$

and

$$\begin{aligned} &A''_{iFF}(\tau) - A''_{iFH}(\tau) + \kappa_{iF}(A'_{iFF}(\tau) - A'_{iFH}(\tau)) \\ &= a\sigma^2_{iH}\bar{\lambda}_{iHF}(A'_{iHF}(\tau) - A'_{iHH}(\tau)) + a\sigma^2_{iF}\bar{\lambda}_{iFF}(A'_{iFF}(\tau) - A'_{iFH}(\tau)), \end{aligned} \tag{B.59}$$

respectively. Equations (B.58) and (B.59) are a linear system of ODEs in the functions $(A'_{iHH}(\tau) - A'_{iHF}(\tau), A'_{iFF}(\tau) - A'_{iFH}(\tau))$.

Consider first Case (i). If $A''_{iHH}(\tau_0)-A''_{iHF}(\tau_0) = 0$, then (B.58), $A'_{iHH}(\tau_0)-A'_{iHF}(\tau_0) = 0$ and $\bar{\lambda}_{iFH} > 0$ imply $A'_{iFF}(\tau_0) - A'_{iFH}(\tau_0) = 0$. The unique solution to the linear system of ODEs (B.58) and (B.59) with the initial condition $(A'_{iHH}(\tau_0) - A'_{iHF}(\tau_0), A'_{iFF}(\tau_0) -$

$A'_{iFH}(\tau_0)) = (0,0)$ is the function that equals (0,0) for all τ. This yields a contradiction because $(A'_{iHH}(0)-A'_{iHF}(0), A'_{iFF}(0)-A'_{iFH}(0)) = (1,1)$. Hence, $A''_{iHH}(\tau_0)-A''_{iHF}(\tau_0) < 0$, which combined with (B.58), $A'_{iHH}(\tau_0) - A'_{iHF}(\tau_0) = 0$ and $\bar{\lambda}_{iFH} > 0$ implies $A'_{iFF}(\tau_0) - A'_{iFH}(\tau_0) > 0$. Since $A'_{iHH}(\tau_0) - A'_{iHF}(\tau_0) = 0$ and $A''_{iHH}(\tau_0) - A''_{iHF}(\tau_0) < 0$, $A'_{iHH}(\tau) - A'_{iHF}(\tau) < 0$ for τ larger than and close to τ_0. We define τ'_0 by

$$\tau'_0 \equiv \sup_{\tau}\{A'_{iHH}(\tau') - A'_{iHF}(\tau') < 0 \text{ and } A'_{iFF}(\tau') - A'_{iFH}(\tau') > 0 \text{ for all } \tau' \in (\tau_0, \tau)\}.$$

If τ'_0 is finite, then either (ia) $A'_{iHH}(\tau_0) - A'_{iHF}(\tau_0) = 0$, $A''_{iHH}(\tau_0) - A''_{iHF}(\tau_0) \geq 0$ and $A'_{iFF}(\tau_0) - A'_{iFH}(\tau_0) \geq 0$, or (ib) $A'_{iHH}(\tau_0) - A'_{iHF}(\tau_0) < 0$, $A'_{iFF}(\tau_0) - A'_{iFH}(\tau_0) = 0$ and $A''_{iFF}(\tau_0) - A''_{iFH}(\tau_0) \leq 0$. In Case (ia), the same argument as for τ_0 implies $A''_{iHH}(\tau'_0) - A''_{iHF}(\tau'_0) > 0$, which combined with (B.58), $A'_{iHH}(\tau_0) - A'_{iHF}(\tau_0) = 0$ and $\bar{\lambda}_{iFH} > 0$ implies $A'_{iFF}(\tau'_0) - A'_{iFH}(\tau'_0) < 0$, a contradiction. In Case (ib), the same argument as for τ_0 implies $A''_{iFF}(\tau'_0) - A''_{iFH}(\tau'_0) < 0$, which combined with (B.59), $A'_{iFF}(\tau_0) - A'_{iFH}(\tau_0) = 0$ and $\bar{\lambda}_{iHF} > 0$ implies $A'_{iHH}(\tau'_0) - A'_{iHF}(\tau'_0) > 0$, a contradiction. Therefore, τ'_0 is infinite, which means that $A_{iFF}(\tau) - A_{iFH}(\tau)$ is increasing, and $A_{iHH}(\tau) - A_{iHF}(\tau)$ is increasing in $(0, \tau_0)$ and decreasing in (τ_0, ∞).

Consider next Case (ii). A symmetric argument by switching H and F implies that $A_{iHH}(\tau) - A_{iHF}(\tau)$ is increasing, and $A_{iFF}(\tau) - A_{iFH}(\tau)$ is increasing in $(0, \tau_0)$ and decreasing in (τ_0, ∞). ∎

Using Lemmas B.1-B.6, we next prove the proposition. Since (A_{iHe}, A_{iFe}) are positive (Lemma B.5), (3.1) implies $\frac{\partial e_t}{\partial i_{Ht}} < 0$ and $\frac{\partial e_t}{\partial i_{Ft}} > 0$. When $a > 0$ and $\alpha_e > 0$, (B.33) implies that $(\bar{\lambda}_{iHH}, \bar{\lambda}_{iFF})$ are negative, and the proof of Lemma B.3 implies that $(\bar{\lambda}_{iHF}, \bar{\lambda}_{iFH})$ are positive. Hence,

$$a\sigma_{iH}^2 \bar{\lambda}_{iHH} A_{iHe} - a\sigma_{iF}^2 \bar{\lambda}_{iFH} A_{iFe} < 0, \tag{B.60}$$

$$a\sigma_{iF}^2 \bar{\lambda}_{iFF} A_{iFe} - a\sigma_{iH}^2 \bar{\lambda}_{iHF} A_{iHe} < 0. \tag{B.61}$$

Equations (B.60) and (B.61) also hold when $a > 0$, $\alpha_e = 0$ and $(\alpha_H(\tau), \alpha_F(\tau))$ are positive. This is because (B.33) again implies that $(\bar{\lambda}_{iHH}, \bar{\lambda}_{iFF})$ are negative, and the proof of Lemma B.3 implies $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH} = 0$. Combining (B.60) and (B.61) with (B.28), we find $A_{iHe} < \frac{1}{\kappa_{iH}} \equiv A_{iHe}^{UIP}$ and $A_{iFe} < \frac{1}{\kappa_{iF}} \equiv A_{iFe}^{UIP}$. Combining (B.60) and (B.61) with (3.12) (for the definitions of (A_e, λ_t) in Section 4.2) and (B.36), we find $\frac{\partial(\mu_{et}+i_{Ft}-i_{Ht})}{\partial i_{Ht}} < 0$ and $\frac{\partial(\mu_{et}+i_{Ft}-i_{Ht})}{\partial i_{Ft}} > 0$. This establishes the first bullet point of the proposition.

Since $(A_{iHH}(\tau), A_{iFF}(\tau))$ are positive for all $\tau > 0$ (Lemma B.3), (2.1) and (3.2) imply that $(\frac{\partial y_{Ht}^{(\tau)}}{\partial i_{Ht}}, \frac{\partial y_{Ft}^{(\tau)}}{\partial i_{Ft}})$ are positive. When $a > 0$ and $\alpha_e > 0$, Lemma B.3 implies that $(A_{iHF}(\tau), A_{iFH}(\tau))$ are positive for all $\tau > 0$, and Lemma B.4 implies that $(A_{iHF}(\tau), A_{iFH}(\tau))$

are increasing. Equation (B.31) for $(j, j') = (H, F)$ implies

$$a\sigma_{iH}^2 \bar{\lambda}_{iHF} A_{iHH}(\tau) + a\sigma_{iF}^2 \bar{\lambda}_{iFF} A_{iFH}(\tau) > 0. \tag{B.62}$$

Multiplying both sides of (B.62) by $\frac{\bar{\lambda}_{iHH}}{\bar{\lambda}_{iHF}} < 0$, we find

$$\begin{aligned} &a\sigma_{iH}^2 \bar{\lambda}_{iHH} A_{iHH}(\tau) + a\sigma_{iF}^2 \frac{\bar{\lambda}_{iHH}\bar{\lambda}_{iFF}}{\bar{\lambda}_{iHF}} A_{iFH}(\tau) < 0 \\ &\Rightarrow a\sigma_{iH}^2 \bar{\lambda}_{iHH} A_{iHH}(\tau) + a\sigma_{iF}^2 \bar{\lambda}_{iFH} A_{iFH}(\tau) < 0, \end{aligned} \tag{B.63}$$

where the second step follows from $A_{iFH}(\tau) > 0$ and from the inequality $\bar{\lambda}_{iHH}\bar{\lambda}_{iFF} - \bar{\lambda}_{iHF}\bar{\lambda}_{iFH} < 0$ established in the proof of Lemma B.1. We likewise find

$$a\sigma_{iF}^2 \bar{\lambda}_{iFH} A_{iFF}(\tau) + a\sigma_{iH}^2 \bar{\lambda}_{iHH} A_{iHF}(\tau) > 0, \tag{B.64}$$

$$\Rightarrow a\sigma_{iF}^2 \bar{\lambda}_{iFF} A_{iFF}(\tau) + a\sigma_{iH}^2 \bar{\lambda}_{iHF} A_{iHF}(\tau) < 0, \tag{B.65}$$

by switching H and F. Equations (B.63) and (B.65) hold also when $a > 0$, $\alpha_e = 0$ and $(\alpha_H(\tau), \alpha_F(\tau))$ are positive. Indeed, the proof of Lemma B.3 implies $\bar{\lambda}_{iHF} = \bar{\lambda}_{iFH} = 0$, and since $(A_{iHH}(\tau), A_{iFF}(\tau))$ are positive, (B.33) implies that $(\bar{\lambda}_{iHH}, \bar{\lambda}_{iFF})$ are negative. Combining (B.63) and (B.65) with (B.30), we find $A_{iHH}(\tau) < \frac{1-e^{-\kappa_{iH}\tau}}{\kappa_{iH}} \equiv A_{iHH}^{EH}(\tau)$ and $A_{iFF}(\tau) < \frac{1-e^{-\kappa_{iF}\tau}}{\kappa_{iF}} \equiv A_{iFF}^{EH}(\tau)$. Combining (B.63) and (B.65) with (3.13) (for the definitions of $(A_j(\tau), \lambda_t)$ in Section 4.2) and (B.36), we find $\frac{\partial\left(\mu_{Ht}^{(\tau)} - i_{Ht}\right)}{\partial i_{Ht}} < 0$ and $\frac{\partial\left(\mu_{Ft}^{(\tau)} - i_{Ft}\right)}{\partial i_{Ft}} < 0$. This establishes the second bullet point of the proposition.

When $a > 0$ and $\alpha_e > 0$, $(A_{iHF}(\tau), A_{iFH}(\tau))$ are positive for all $\tau > 0$, and hence (2.1) and (3.2) imply that $(\frac{\partial y_{Ht}^{(\tau)}}{\partial i_{Ft}}, \frac{\partial y_{Ft}^{(\tau)}}{\partial i_{Ht}})$ are positive. Moreover, combining (B.62) and (B.64) with (3.13) and (B.36), we find $\frac{\partial\left(\mu_{Ht}^{(\tau)} - i_{Ht}\right)}{\partial i_{Ft}} > 0$ and $\frac{\partial\left(\mu_{Ft}^{(\tau)} - i_{Ft}\right)}{\partial i_{Ht}} > 0$. This establishes the third bullet point of the proposition. The fourth bullet point follows from Lemma B.6, (2.1) and (3.2). ∎

Proof of Proposition 4.5: Combining (3.12) and (3.13) (for the definitions of $(A_e, A_j(\tau), \lambda_t)$ in Section 4.2) with (4.5), we can write the expected return of the hybrid CCT as

$$\mu_{hCCTt}^{(\tau)} \equiv \lambda_{iHt}(A_{iHe} + A_{iFH}(\tau) - A_{iHH}(\tau)) - \lambda_{iFt}(A_{iFe} + A_{iHF}(\tau) - A_{iFF}(\tau)). \tag{B.66}$$

Using (B.36), we find

$$\frac{\partial \mu^{(\tau)}_{hCCTt}}{\partial i_{Ht}} = a\sigma^2_{iH}\bar{\lambda}_{iHH}(A_{iHe} + A_{iHF}(\tau) - A_{iHH}(\tau)) - a\sigma^2_{iF}\bar{\lambda}_{iFH}(A_{iFe} + A_{iFH}(\tau) - A_{iFF}(\tau)), \tag{B.67}$$

$$\frac{\partial \mu^{(\tau)}_{hCCTt}}{\partial i_{Ft}} = a\sigma^2_{iH}\bar{\lambda}_{iHF}(A_{iHe} + A_{iHF}(\tau) - A_{iHH}(\tau)) - a\sigma^2_{iF}\bar{\lambda}_{iFF}(A_{iFe} + A_{iFH}(\tau) - A_{iFF}(\tau)). \tag{B.68}$$

When $a > 0$, and $\alpha_e > 0$ or $\alpha_j(\tau) > 0$, $(\bar{\lambda}_{iHH}, \bar{\lambda}_{iFF})$ are negative. Since, in addition, $(\bar{\lambda}_{iHF}, \bar{\lambda}_{iFH})$ are non-negative, (A_{iHe}, A_{iFe}) are positive and $A_{iHH}(0) - A_{iHF}(0) = A_{iFF}(0) - A_{iFH}(0) = 0$, (B.67) and (B.68) imply that there exists a threshold $\tau^* > 0$ such that $\frac{\partial \mu^{(\tau)}_{hCCTt}}{\partial i_{Ht}} < 0$ and $\frac{\partial \mu^{(\tau)}_{hCCTt}}{\partial i_{Ft}} > 0$ for all $\tau \in (0, \tau^*)$. Since at least one of $(A_{iHH}(\tau) - A_{iHF}(\tau), A_{iFF}(\tau) - A_{iFH}(\tau))$ is increasing (proof of Lemma B.4), they are both increasing when countries are symmetric. Since, in addition, $(A_{iHH}(\infty) - A_{iHF}(\infty), A_{iFF}(\infty) - A_{iFH}(\infty)) = (A_{iHe}, A_{iFe})$ (proof of Lemma B.6), (B.67) and (B.68) imply that when countries are symmetric, $\frac{\partial \mu^{(\tau)}_{hCCTt}}{\partial i_{Ht}} < 0$ and $\frac{\partial \mu^{(\tau)}_{hCCTt}}{\partial i_{Ft}} > 0$ for all $\tau > 0$, which means $\tau^* = \infty$.

Combining

$$\mu_{CCTt} \equiv \mu_{et} + i_{Ft} - i_{Ht} = \lambda_{iHt}A_{iHe} - \lambda_{iFt}A_{iFe},$$

which gives the expected return of the basic CCT and follows from (3.12) (for the definitions of (A_e, λ_t) in Section 4.2), with (B.36), (B.67) and (B.68), we find

$$\frac{\partial \left(\mu^{(\tau)}_{hCCTt} - \mu_{CCTt}\right)}{\partial i_{Ht}} = \bar{\lambda}_{iHH}(A_{iHF}(\tau) - A_{iHH}(\tau)) - \bar{\lambda}_{iHF}(A_{iFH}(\tau) - A_{iFF}(\tau)) > 0, \tag{B.69}$$

$$\frac{\partial \left(\mu^{(\tau)}_{hCCTt} - \mu_{CCTt}\right)}{\partial i_{Ft}} = \bar{\lambda}_{iFH}(A_{iHF}(\tau) - A_{iHH}(\tau)) - \bar{\lambda}_{iFF}(A_{iFH}(\tau) - A_{iFF}(\tau)) < 0, \tag{B.70}$$

where the inequalities follow because $(\bar{\lambda}_{iHH}, \bar{\lambda}_{iFF})$ are negative, $(\bar{\lambda}_{iHF}, \bar{\lambda}_{iFH})$ are non-negative, and $(A_{iHH}(\tau) - A_{iHF}(\tau), A_{iFF}(\tau) - A_{iFH}(\tau))$ are positive for all $\tau > 0$ (Lemma B.6). Hence, the sensitivity of the hybrid CCT's expected return to (i_{Ht}, i_{Ft}) is smaller (less negative in the case of i_{Ht} and less positive in the case of i_{Ft}) than for the basic CCT. Since $(A_{iHH}(\infty) - A_{iHF}(\infty), A_{iFF}(\infty) - A_{iFH}(\infty)) = (A_{iHe}, A_{iFe})$, (B.66) implies that $\mu^{(\tau)}_{hCCTt}$ goes to zero when τ goes to infinity.

Using (3.1), (3.2), (4.6) and $\gamma_t = \beta_{Ht} = \beta_{Ft} = 0$, we can write the return of the long-horizon CCT as

$$A_{iHe}i_{Ht} - A_{iFe}i_{Ft} + C_e + (\pi_F - \pi_H)t - (A_{iHe}i_{H,t+\tau} - A_{iFe}i_{F,t+\tau} + C_e + (\pi_F - \pi_H)(t+\tau))$$
$$+ A_{iFF}(\tau)i_{Ft} + A_{iHF}(\tau)i_{Ht} + C_F(\tau) - (A_{iHH}(\tau)i_{Ht} + A_{iFH}(\tau)i_{Ft} + C_H(\tau)).$$

Hence, (4.1) implies that the annualized expected return of the long-horizon CCT is

$$\mu^{(\tau)}_{\ell CCTt} \equiv \frac{1}{\tau}\left[A_{iHe}(1-e^{-\kappa_{iH}\tau})(i_{Ht}-\bar{i}_H) - A_{iFe}(1-e^{-\kappa_{iF}\tau})(i_{Ft}-\bar{i}_F) - (\pi_F-\pi_H)\tau \right.$$
$$\left. + A_{iFF}(\tau)i_{Ft} + A_{iHF}(\tau)i_{Ht} + C_F(\tau) - (A_{iHH}(\tau)i_{Ht} + A_{iFH}(\tau)i_{Ft} + C_H(\tau))\right], \tag{B.71}$$

and its sensitivity to (i_{Ht}, i_{Ft}) is

$$\frac{\partial \mu^{(\tau)}_{\ell CCTt}}{\partial i_{Ht}} = \frac{1}{\tau}\left[A_{iHe}(1-e^{-\kappa_{iH}\tau}) + A_{iHF}(\tau) - A_{iHH}(\tau)\right], \tag{B.72}$$
$$\frac{\partial \mu^{(\tau)}_{\ell CCTt}}{\partial i_{Ft}} = \frac{1}{\tau}\left[-A_{iFe}(1-e^{-\kappa_{iF}\tau}) + A_{iFF}(\tau) - A_{iFe}(\tau)\right]. \tag{B.73}$$

When $a > 0$, and $\alpha_e > 0$ or $\alpha_j(\tau) > 0$, $A_{iHe} < \frac{1}{\kappa_{iH}}$ and $A_{iFe} < \frac{1}{\kappa_{iF}}$. Since, in addition, $A'_{iHH}(0) = A'_{iFF}(0) = 1$ and $A'_{iHF}(0) = A'_{iFH}(0) = 0$, the derivative of (B.72) with respect to τ at $\tau = 0$ is negative, and the derivative of (B.73) with respect to τ at $\tau = 0$ is positive. Hence, there exists a threshold $\tau^* > 0$ such that $\frac{\partial \mu^{(\tau)}_{\ell CCTt}}{\partial i_{Ht}} < 0$ and $\frac{\partial \mu^{(\tau)}_{\ell CCTt}}{\partial i_{Ft}} > 0$ for all $\tau \in (0, \tau^*)$. When countries are symmetric, we set $\kappa_r \equiv \kappa_{iH} = \kappa_{iF}$, $\sigma_r \equiv \sigma_{iH} = \sigma_{iF}$, $A_{ie} \equiv A_{iHe} = A_{iFe}$, $\Delta A(\tau) \equiv A_{iHH}(\tau) - A_{iHF}(\tau) = A_{iFF}(\tau) - A_{iFH}(\tau)$, $\Delta\bar{\lambda} \equiv \bar{\lambda}_{iHH} - \bar{\lambda}_{iHF} = \bar{\lambda}_{iFF} - \bar{\lambda}_{iFH} < 0$. Taking the difference between (B.30) and (B.31) yields

$$\Delta A'(\tau) + \kappa_r \Delta A(\tau) - 1 = a\sigma_r^2 \Delta\bar{\lambda}\Delta A(\tau),$$

which integrates to

$$\Delta A(\tau) = A_{ie}\left(1 - e^{-(\kappa_r - a\sigma_r^2\Delta\bar{\lambda})\tau}\right)$$

since $\Delta A(0) = 0$ and $\Delta A(\infty) = A_{ie}$. Substituting into (B.72) and (B.73), we find

$$\frac{\partial \mu^{(\tau)}_{\ell CCTt}}{\partial i_{Ht}} = -\frac{\partial \mu^{(\tau)}_{\ell CCTt}}{\partial i_{Ft}} = \frac{1}{\tau}A_{ie}(e^{-(\kappa_r - a\sigma_r^2\Delta\bar{\lambda})\tau} - e^{-\kappa_r\tau}) < 0. \tag{B.74}$$

Hence, $\tau^* = \infty$.

The annualized expected return of the sequence of basic CCTs is

$$\mu_{CCTt}^{(\tau)} \equiv \frac{1}{\tau} E_t \int_t^{t+\tau} \left(\lambda_{iHt'} A_{iHe} - \lambda_{iFt'} A_{iFe}\right) dt'.$$

Using (4.1) and (B.36), we find

$$\begin{aligned}
\frac{\partial \mu_{CCTt}^{(\tau)}}{\partial i_{Ht}} &= \frac{1-e^{-\kappa_{iH}\tau}}{\kappa_{iH}\tau}\left(a\sigma_{iH}^2 \bar{\lambda}_{iHH} A_{iHe} - a\sigma_{iF}^2 \bar{\lambda}_{iFH} A_{iFe}\right) \\
&= \frac{1-e^{-\kappa_{iH}\tau}}{\kappa_{iH}\tau}(\kappa_{iH} A_{iHe} - 1),
\end{aligned} \tag{B.75}$$

where the second step follows from (B.28). We likewise find

$$\frac{\partial \mu_{CCTt}^{(\tau)}}{\partial i_{Ft}} = -\frac{1-e^{-\kappa_{iF}\tau}}{\kappa_{iF}\tau}(\kappa_{iF} A_{iFe} - 1). \tag{B.76}$$

Combining (B.72) and (B.75), we find

$$\frac{\partial \left(\mu_{\ell CCTt}^{(\tau)} - \mu_{CCTt}^{(\tau)}\right)}{\partial i_{Ht}} = \frac{1}{\tau}\left[\frac{1-e^{-\kappa_{iH}\tau}}{\kappa_{iH}} + A_{iHF}(\tau) - A_{iHH}(\tau)\right] > 0,$$

where the inequality sign follows from (B.56) by noting that the left-hand side of (B.56) is negative. Combining (B.73) and (B.76), we likewise find

$$\frac{\partial \left(\mu_{\ell CCTt}^{(\tau)} - \mu_{CCTt}^{(\tau)}\right)}{\partial i_{Ft}} = \frac{1}{\tau}\left[-\frac{1-e^{-\kappa_{iF}\tau}}{\kappa_{iF}} + A_{iFF}(\tau) - A_{iFH}(\tau)\right] < 0.$$

Hence, the sensitivity of the long-horizon CCT's expected return to (i_{Ht}, i_{Ft}) is smaller (less negative in the case of i_{Ht} and less positive in the case of i_{Ft}) than for the corresponding sequence of basic CCTs. Since $(A_{iHH}(\tau), A_{iFF}(\tau), A_{iHF}(\tau), A_{iFH}(\tau))$ go to finite limits when τ goes to infinity, (B.71) implies that $\mu_{\ell CCTt}^{(\tau)}$ goes to

$$\begin{aligned}
&\lim_{\tau\to\infty}\frac{C_F(\tau)}{\tau} - \lim_{\tau\to\infty}\frac{C_H(\tau)}{\tau} - (\pi_F - \pi_H) \\
&= \kappa_{iF}\bar{i}_F\left(A_{iFF}(\infty) - A_{iFH}(\infty)\right) - \kappa_{iH}\bar{i}_H\left(A_{iHH}(\infty) - A_{iHF}(\infty)\right) \\
&\quad - \frac{1}{2}\sigma_{iF}^2\left[A_{iFF}(\infty)\left(A_{iFF}(\infty) - 2A_{iFe}\right) - A_{iFH}(\infty)^2\right] \\
&\quad + \frac{1}{2}\sigma_{iH}^2\left[A_{iHH}(\infty)^2 - A_{iHF}(\infty)\left(A_{iHF}(\infty) + 2A_{iHe}\right)\right] \\
&\quad + a\sigma_{iF}^2\bar{\lambda}_{iFC}\left(A_{iFF}(\infty) - A_{iFH}(\infty)\right) - a\sigma_{iH}^2\bar{\lambda}_{iHC}\left(A_{iHH}(\infty) - A_{iHF}(\infty)\right) - (\pi_F - \pi_H) \\
&= \kappa_{iF}\bar{i}_F A_{iFe} - \kappa_{iH}\bar{i}_H A_{iHe} + \frac{1}{2}\sigma_{iF}^2 A_{iFe}^2 + \frac{1}{2}\sigma_{iH}^2 A_{iHe}^2 \\
&\quad + a\sigma_{iF}^2\bar{\lambda}_{iFC} A_{iFe} - a\sigma_{iH}^2\bar{\lambda}_{iHC} A_{iHe} - (\pi_F - \pi_H) = 0,
\end{aligned}$$

where the second step follows from (B.32) by noting $\lim_{\tau\to\infty}\frac{C_j(\tau)}{\tau} = \lim_{\tau\to\infty} C_j'(\tau)$, the third step follows from $(A_{iHH}(\infty) - A_{iHF}(\infty), A_{iFF}(\infty) - A_{iFH}(\infty)) = (A_{iHe}, A_{iFe})$, and the fourth step follows from (B.29). Since

$$\lim_{\tau\to\infty}\frac{C_F(\tau)}{\tau} - \lim_{\tau\to\infty}\frac{C_H(\tau)}{\tau} - (\pi_F - \pi_H) = y_F^{(\infty)} - y_H^{(\infty)} - (\pi_F - \pi_H),$$

the difference in real yields across countries becomes zero in the limit τ goes to infinity. ∎

We next prove a lemma that we use in subsequent proofs.

Lemma B.7. *When $a > 0$ and $\alpha_e > 0$, the functions $\left(\frac{A_{iFH}(\tau)}{A_{iHH}(\tau)}, \frac{A_{iHF}(\tau)}{A_{iFF}(\tau)}\right)$ are increasing.*

Proof: The functions $(A_{iHH}(\tau), A_{iFH}(\tau))$ solve the system (B.42) of linear ODEs with constant coefficients. The solution is an affine function of $(e^{-\nu_1\tau}, e^{-\nu_2\tau})$, where (ν_1, ν_2) are the eigenvalues of the matrix M. Because of the initial conditions $A_{iHH}(0) = A_{iFH}(0) = 0$, we can write the solution as a linear function of $\left(\frac{1-e^{-\nu_1\tau}}{\nu_1}, \frac{1-e^{-\nu_2\tau}}{\nu_2}\right)$. Because $(A_{iHH}'(0), A_{iFH}'(0)) = (1, 0)$, the coefficients of the linear terms sum to one for $A_{iHH}(\tau)$ and to zero for $A_{iFH}(\tau)$. Hence, we can write the solution as

$$A_{iHH}(\tau) = \frac{1-e^{-\nu_1\tau}}{\nu_1} + \phi_{HH}\left(\frac{1-e^{-\nu_2\tau}}{\nu_2} - \frac{1-e^{-\nu_1\tau}}{\nu_1}\right), \tag{B.77}$$

$$A_{iFH}(\tau) = \phi_{FH}\left(\frac{1-e^{-\nu_2\tau}}{\nu_2} - \frac{1-e^{-\nu_1\tau}}{\nu_1}\right), \tag{B.78}$$

for scalars (ϕ_{HH}, ϕ_{FH}). The eigenvalues (ν_1, ν_2) are positive (Lemma B.1), and without loss of generality we can set $\nu_1 > \nu_2$. Since $A_{iFH}(\tau)$ is positive when $a > 0$ and $\alpha_e > 0$ (Lemma B.3), $\phi_{FH} > 0$. Since

$$\frac{A_{iHH}(\tau)}{A_{iFH}(\tau)} = \frac{\frac{1-e^{-\nu_1\tau}}{\nu_1}}{\phi_{FH}\left(\frac{1-e^{-\nu_2\tau}}{\nu_2} - \frac{1-e^{-\nu_1\tau}}{\nu_1}\right)} + \frac{\phi_{HH}}{\phi_{FH}} = \frac{1}{\phi_{HF}\left(\frac{\nu_1}{\nu_2}\frac{1-e^{-\nu_2\tau}}{1-e^{-\nu_1\tau}} - 1\right)} + \frac{\phi_{HH}}{\phi_{FH}},$$

and the function $(\nu_1, \nu_2, \tau) \longrightarrow \frac{1-e^{-\nu_2\tau}}{1-e^{-\nu_1\tau}}$ increases in τ because its derivative has the same sign as $\frac{e^{\nu_1\tau}-1}{\nu_1} - \frac{e^{\nu_2\tau}-1}{\nu_2}$, the function $\frac{A_{iHH}(\tau)}{A_{FH}(\tau)}$ is decreasing. Hence, the inverse function $\frac{A_{iFH}(\tau)}{A_{HH}(\tau)}$ is increasing. A similar argument using (B.43) establishes that $\frac{A_{iHF}(\tau)}{A_{FF}(\tau)}$ is increasing. ∎

Proof of Proposition 4.6: Consider a one-off increase in γ_t at time zero, and denote by κ_γ the rate at which γ_t reverts to its mean of zero. Equation (B.36) is modified to

$$\lambda_{ijt} = a\sigma_{ij}^2 \Bigg(\left[\zeta_e + \theta_e\gamma_t - \alpha_e\left(A_{iHe}i_{Ht} - A_{iFe}i_{Ft} + A_{\gamma e}\gamma_t + C_e\right)\right] A_{ije}(-1)^{1_{\{j=F\}}}$$
$$+ \sum_{j'=H,F} \int_0^T \left[\zeta_{j'}(\tau) - \alpha_{j'}(\tau)\left(A_{iHj'}(\tau)i_{Ht} + A_{iFj'}(\tau)i_{Ft} + A_{\gamma j'}(\tau)\gamma_t + C_{j'}(\tau)\right)\right] A_{ijj'}(\tau)d\tau \Bigg)$$
$$\equiv a\sigma_{ij}^2\left(\bar{\lambda}_{ijj}i_{jt} + \bar{\lambda}_{ijj'}i_{j't} + \bar{\lambda}_{ij\gamma}\gamma_t + \bar{\lambda}_{ijC}\right) \tag{B.79}$$

(B.5) is modified to (B.20), and (3.8) and (3.10) are modified to

$$\begin{aligned}\mu_{jt}^{(\tau)} \equiv & A'_{iHj}(\tau)i_{Ht} + A'_{iFj}(\tau)i_{Ft} + A'_{\gamma j}(\tau)\gamma_t + C'_j(\tau) \\ & - A_{iHj}(\tau)\kappa_{iH}(\bar{i}_H - i_{Ht}) - A_{iFj}(\tau)\kappa_{iF}(\bar{i}_F - i_{Ft}) + A_{\gamma j}(\tau)\kappa_\gamma\gamma_t \\ & + \frac{1}{2}A_{iHj}(\tau)\left(A_{iHj}(\tau) + 2A_{iHe}1_{\{j=F\}}\right)\sigma_{iH}^2 + \frac{1}{2}A_{iFj}(\tau)\left(A_{iFj}(\tau) - 1_{\{j=F\}}2A_{iFe}\right)\sigma_{iF}^2.\end{aligned} \tag{B.80}$$

Substituting λ_t from (B.79) and μ_{et} from (B.20) into (3.12) (for the definitions of (A_e, λ_t) in Section 4.2), we find an equation that is affine in $(i_{Ht}, i_{Ft}, \gamma_t)$. Identifying the linear terms in γ_t yields

$$\kappa_\gamma A_{\gamma e} = a\sigma_{iH}^2\bar{\lambda}_{iH\gamma}A_{iHe} - a\sigma_{iF}^2\bar{\lambda}_{iF\gamma}A_{iFe}. \tag{B.81}$$

Substituting λ_t from (B.79) and $\mu_{jt}^{(\tau)}$ from (B.80) into (3.13) (for the definitions of $(A_j(\tau), \lambda_t)$ in Section 4.2), we find an equation that is affine in $(i_{Ht}, i_{Ft}, \gamma_t)$. Identifying the linear terms in γ_t yields

$$A'_{\gamma j}(\tau) + \kappa_\gamma A_{\gamma j}(\tau) = a\sigma_{iH}^2\bar{\lambda}_{iH\gamma}A_{iHj}(\tau) + a\sigma_{iF}^2\bar{\lambda}_{iF\gamma}A_{iFj}(\tau). \tag{B.82}$$

Solving (B.82) with the initial condition $A_{\gamma j}(0) = 0$, we find

$$A_{\gamma j}(\tau) = a\sigma_{iH}^2\bar{\lambda}_{iH\gamma}\int_0^\tau A_{iHj}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau' + a\sigma_{iF}^2\bar{\lambda}_{iF\gamma}\int_0^\tau A_{iFj}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau', \tag{B.83}$$

We next substitute $A_{\gamma e}$ from (B.81) and $\{A_{\gamma j}(\tau)\}_{j=H,F}$ from (B.83) into

$$\bar{\lambda}_{ij\gamma} \equiv (\theta_e - \alpha_e A_{\gamma e})A_{ije}(-1)^{1_{\{j=F\}}} - \int_0^T \alpha_H(\tau)A_{\gamma H}(\tau)A_{ijH}(\tau)d\tau - \int_0^T \alpha_F(\tau)A_{\gamma F}(\tau)A_{ijF}(\tau)d\tau, \tag{B.84}$$

which follows from the definition of $\bar{\lambda}_{ij\gamma}$ in (B.79). We find

$$(1 + a\sigma_{iH}^2 z_{\gamma HH})\bar{\lambda}_{iH\gamma} + a\sigma_{iF}^2 z_{\gamma FH}\bar{\lambda}_{iF\gamma} = \theta_e A_{iHe}, \tag{B.85}$$

$$a\sigma_{iH}^2 z_{\gamma HF}\bar{\lambda}_{iH\gamma} + (1 + a\sigma_{iF}^2 z_{\gamma FF})\bar{\lambda}_{iF\gamma} = -\theta_e A_{iFe}, \tag{B.86}$$

where

$$\begin{aligned}
z_{\gamma HH} &= \frac{\alpha_e}{\kappa_\gamma} A_{iHe}^2 + \int_0^T \alpha_H(\tau) A_{iHH}(\tau) \left[\int_0^\tau A_{iHH}(\tau') e^{-\kappa_\gamma(\tau-\tau')} d\tau'\right] d\tau \\
&\quad + \int_0^T \alpha_F(\tau) A_{iHF}(\tau) \left[\int_0^\tau A_{iHF}(\tau') e^{-\kappa_\gamma(\tau-\tau')} d\tau'\right] d\tau, \\
z_{\gamma FF} &= \frac{\alpha_e}{\kappa_\gamma} A_{iFe}^2 + \int_0^T \alpha_H(\tau) A_{iFH}(\tau) \left[\int_0^\tau A_{iFH}(\tau') e^{-\kappa_\gamma(\tau-\tau')} d\tau'\right] d\tau \\
&\quad + \int_0^T \alpha_F(\tau) A_{iFF}(\tau) \left[\int_0^\tau A_{iFF}(\tau') e^{-\kappa_\gamma(\tau-\tau')} d\tau'\right] d\tau,
\end{aligned}$$

$$\begin{aligned}
z_{\gamma HF} &= -\frac{\alpha_e}{\kappa_\gamma} A_{iHe} A_{iFe} + \int_0^T \alpha_H(\tau) A_{iFH}(\tau) \left[\int_0^\tau A_{iHH}(\tau') e^{-\kappa_\gamma(\tau-\tau')} d\tau'\right] d\tau \\
&\quad + \int_0^T \alpha_F(\tau) A_{iFF}(\tau) \left[\int_0^\tau A_{iHF}(\tau') e^{-\kappa_\gamma(\tau-\tau')} d\tau'\right] d\tau, \\
z_{\gamma FH} &= -\frac{\alpha_e}{\kappa_\gamma} A_{iHe} A_{iFe} + \int_0^T \alpha_H(\tau) A_{iHH}(\tau) \left[\int_0^\tau A_{iFH}(\tau') e^{-\kappa_\gamma(\tau-\tau')} d\tau'\right] d\tau \\
&\quad + \int_0^T \alpha_F(\tau) A_{iHF}(\tau) \left[\int_0^\tau A_{iFF}(\tau') e^{-\kappa_\gamma(\tau-\tau')} d\tau'\right] d\tau.
\end{aligned}$$

Equations (B.85) and (B.86) form a linear system of two equations in the two unknowns $(\bar{\lambda}_{iH\gamma}, \bar{\lambda}_{iF\gamma})$. Its solution is

$$\bar{\lambda}_{iH\gamma} = \frac{\theta_e}{\Delta_{z\gamma}} \left[(1 + a\sigma_{iF}^2 z_{\gamma FF}) A_{iHe} + a\sigma_{iF}^2 z_{\gamma FH} A_{iFe}\right] \tag{B.87}$$

$$\bar{\lambda}_{iF\gamma} = -\frac{\theta_e}{\Delta_{z\gamma}} \left[(1 + a\sigma_{iH}^2 z_{\gamma HH}) A_{iFe} + a\sigma_{iH}^2 z_{\gamma HF} A_{iHe}\right], \tag{B.88}$$

where

$$\Delta_{z\gamma} \equiv (1 + a\sigma_{iH}^2 z_{\gamma HH})(1 + a\sigma_{iF}^2 z_{\gamma FF}) - a^2\sigma_{iH}^2\sigma_{iF}^2 z_{\gamma HF} z_{\gamma FH}.$$

To complete the proof, we proceed in three steps. In Step 1, we show that $\Delta_{z\gamma}$ is positive. In Step 2, we show that $A_{\gamma e}$ is positive. This proves the first statement in the proposition. In Step 3, we show that $A_{\gamma H}(\tau)$ is positive and $A_{\gamma F}(\tau)$ is negative. This proves the second and third statements in the proposition.

Step 1: $\Delta_{z\gamma}$ is positive. Since $(z_{\gamma HH}, z_{\gamma FF})$ are non-negative, $\Delta_{z\gamma} > 0$ under the sufficient condition

$$z_{\gamma HH} z_{\gamma FF} \geq z_{\gamma HF} z_{\gamma FH}. \tag{B.89}$$

The function

$$\begin{aligned}
F(\mu) &\equiv z_{\gamma HH} + \mu(z_{\gamma HF} + z_{\gamma FH}) + \mu^2 z_{\gamma FF} \\
&= \frac{\alpha_e}{\kappa_\gamma} \left(A_{iHe} - \mu A_{iFe}\right)^2 \\
&\quad + \int_0^T \alpha_H(\tau) \left[A_{iHH}(\tau) + \mu A_{iFH}(\tau)\right] \left[\int_0^T \left[A_{iHH}(\tau) + \mu A_{iFH}(\tau)\right] e^{-\kappa_\gamma(\tau-\tau')} d\tau'\right] d\tau \\
&\quad + \int_0^T \alpha_F(\tau) \left[A_{iHF}(\tau) + \mu A_{iFF}(\tau)\right] \left[\int_0^T \left[A_{iHF}(\tau) + \mu A_{iFF}(\tau)\right] e^{-\kappa_\gamma(\tau-\tau')} d\tau'\right] d\tau
\end{aligned}$$

is non-negative for all μ if

$$F_0 \equiv \int_0^T \alpha(\tau) A(\tau) \left[\int_0^\tau A(\tau') e^{-\kappa_\gamma(\tau-\tau')} d\tau'\right] d\tau$$

is non-negative for a non-negative and non-increasing $\alpha(\tau)$. Since

$$F_0 = \int_0^T \phi(\tau)\Phi(\tau) \left[\int_0^\tau \Phi(\tau') d\tau'\right] d\tau,$$

where

$$\begin{aligned}
\phi(\tau) &\equiv \alpha(\tau) e^{-2\kappa_\gamma \tau}, \\
\Phi(\tau) &\equiv A(\tau) e^{\kappa_\gamma \tau},
\end{aligned}$$

integration by parts implies

$$F_0 = \frac{1}{2}\phi(T) \left[\int_0^T \Phi(\tau) d\tau\right]^2 - \frac{1}{2} \int_0^T \phi'(\tau) \left[\int_0^\tau \Phi(\tau') d\tau'\right]^2 d\tau. \tag{B.90}$$

The first term in the right-hand side of (B.90) is non-negative because $\alpha(\tau)$ is non-negative, and the first term is non-positive because $\alpha(\tau)$ is non-increasing. Therefore, F_0 is non-negative. Since $F(\mu)$ is quadratic in μ, its non-negativity for all μ implies

$$\begin{aligned}
&4 z_{\gamma HH} z_{\gamma FF} \geq (z_{\gamma HF} + z_{\gamma FH})^2 \\
&\Rightarrow z_{\gamma HH} z_{\gamma FF} \geq \frac{1}{4}(z_{\gamma HF} + z_{\gamma FH})^2 = z_{\gamma HF} z_{\gamma FH} + \frac{1}{4}(z_{\gamma HF} - z_{\gamma FH})^2 \geq z_{\gamma HF} z_{\gamma FH}.
\end{aligned}$$

Therefore, (B.89) holds.

Step 2: $A_{\gamma e}(\tau)$ is positive. Substituting $(\bar{\lambda}_{iH\gamma}, \bar{\lambda}_{iF\gamma})$ from (B.87) and (B.88) into

(B.81), and using the definitions of $(z_{\gamma HH}, z_{\gamma FF}, z_{\gamma HF}, z_{\gamma FH})$ and that $(\theta_e, \Delta_{z\gamma})$ are positive, we find $A_{\gamma e} > 0$ if

$$Z_{\gamma H} A_{iHe} + Z_{\gamma F} A_{iFe} > 0, \tag{B.91}$$

where

$$\begin{aligned}
Z_{\gamma H} &\equiv \sigma_{iH}^2(1 + a\sigma_{iF}^2 z_{\gamma FF})A_{iHe} + a\sigma_{iH}^2\sigma_{iF}^2 z_{\gamma FH} A_{iFe} \\
&= \sigma_{iH}^2 A_{iHe} \\
&\quad + a\sigma_{iH}^2\sigma_{iF}^2 \int_0^T \alpha_H(\tau)[A_{iHe}A_{iFH}(\tau) + A_{iFe}A_{iHH}(\tau)] \left[\int_0^\tau A_{iFH}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right] d\tau \\
&\quad + a\sigma_{iH}^2\sigma_{iF}^2 \int_0^T \alpha_F(\tau)[A_{iHe}A_{iFF}(\tau) + A_{iFe}A_{iHF}(\tau)] \left[\int_0^\tau A_{iFF}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right] d\tau, \\
Z_{\gamma F} &\equiv \sigma_{iF}^2(1 + a\sigma_{iH}^2 z_{\gamma HH})A_{iFe} + a\sigma_{iH}^2\sigma_{iF}^2 z_{\gamma HF} A_{iHe} \\
&= \sigma_{iF}^2 A_{iFe} \\
&\quad + a\sigma_{iH}^2\sigma_{iF}^2 \int_0^T \alpha_H(\tau)[A_{iHe}A_{iFH}(\tau) + A_{iFe}A_{iHH}(\tau)] \left[\int_0^\tau A_{iHH}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right] d\tau \\
&\quad + a\sigma_{iH}^2\sigma_{iF}^2 \int_0^T \alpha_F(\tau)[A_{iHe}A_{iFF}(\tau) + A_{iFe}A_{iHF}(\tau)] \left[\int_0^\tau A_{iHF}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right] d\tau.
\end{aligned}$$

Since $(A_{iHe}, A_{iFe}, Z_{\gamma H}, Z_{\gamma F})$ are positive, (B.91) holds.

Step 3: $A_{\gamma H}(\tau)$ is positive and $A_{\gamma F}(\tau)$ is negative. We prove that $A_{\gamma H}(\tau)$ is positive. The proof that $A_{\gamma F}(\tau)$ is negative is symmetric. Substituting $(\bar{\lambda}_{iH\gamma}, \bar{\lambda}_{iF\gamma})$ from (B.87) and (B.88) into (B.83) for $j = H$, and using the definitions of $(z_{\gamma HH}, z_{\gamma FF}, z_{\gamma HF}, z_{\gamma FH})$ and that $(\theta_e, \Delta_{z\gamma})$ are positive, we find $A_{\gamma H}(\tau) > 0$ if

$$Z_{\gamma H} \int_0^\tau A_{iHH}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau' - Z_{\gamma F} \int_0^\tau A_{iFH}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau' > 0. \tag{B.92}$$

Since $(A_{iHH}(\tau), Z_{\gamma H}, Z_{\gamma F})$ are positive, $A_{iFH}(\tau)$ is non-negative and $\frac{A_{iFH}(\tau)}{A_{iHH}(\tau)}$ is non-decreasing, (B.92) holds under the sufficient condition

$$Z_{\gamma H} A_{iHH}(\infty) - Z_{\gamma F} A_{iFH}(\infty) > 0. \tag{B.93}$$

Using the definitions of $(Z_{\gamma H}, Z_{\gamma F})$, we can write (B.93) as

$$\begin{aligned}
&\sigma_{iH}^2 A_{iHe} A_{iHH}(\infty) - \sigma_{iF}^2 A_{iFe} A_{iFH}(\infty) \\
&+ a\sigma_{iH}^2 \sigma_{iF}^2 \int_0^T \alpha_H(\tau)[A_{iHe}A_{iFH}(\tau) + A_{iFe}A_{iHH}(\tau)] \\
&\times \left[\int_0^\tau [A_{iFH}(\tau')A_{iHH}(\infty) - A_{iHH}(\tau')A_{iFH}(\infty)]\, e^{-\kappa_\gamma(\tau-\tau')} d\tau'\right] d\tau \\
&+ a\sigma_{iH}^2 \sigma_{iF}^2 \int_0^T \alpha_F(\tau)[A_{iHe}A_{iFF}(\tau) + A_{iFe}A_{iHF}(\tau)] \\
&\times \left[\int_0^\tau [A_{iFF}(\tau')A_{iHH}(\infty) - A_{iHF}(\tau')A_{iFH}(\infty)]\, e^{-\kappa_\gamma(\tau-\tau')} d\tau'\right] d\tau > 0.
\end{aligned} \tag{B.94}$$

Equation (B.31) for $(j, j') = (H, F)$ implies

$$A_{iFH}(\tau) = \frac{a\sigma_{iH}^2 \bar{\lambda}_{iHF} A_{iHH}(\tau)}{\kappa_{iF} - a\sigma_{iF}^2 \bar{\lambda}_{iFF}} - \frac{A'_{iFH}(\tau)}{\kappa_{iF} - a\sigma_{iF}^2 \bar{\lambda}_{iFF}}, \tag{B.95}$$

which for $\tau = \infty$ becomes

$$A_{iFH}(\infty) = \frac{a\sigma_{iH}^2 \bar{\lambda}_{iHF} A_{iHH}(\infty)}{\kappa_{iF} - a\sigma_{iF}^2 \bar{\lambda}_{iFF}}. \tag{B.96}$$

Equation (B.30) for $(j, j') = (F, H)$ implies

$$A_{iFF}(\tau) = \frac{a\sigma_{iH}^2 \bar{\lambda}_{iHF} A_{iHF}(\tau)}{\kappa_{iF} - a\sigma_{iF}^2 \bar{\lambda}_{iFF}} + \frac{1 - A'_{iFF}(\tau)}{\kappa_{iF} - a\sigma_{iF}^2 \bar{\lambda}_{iFF}}. \tag{B.97}$$

Using (B.95)-(B.97) to simplify the terms in the first, second and fourth lines of (B.94), and dividing throughout by $\frac{a\sigma_{iH}^2 \sigma_{iF}^2 A_{iHH}(\infty)}{\kappa_{iF} - a\sigma_{iF}^2 \bar{\lambda}_{iFF}} > 0$, we find that (B.94) is equivalent to

$$\begin{aligned}
&\left(\frac{\kappa_{iF}}{a\sigma_{iF}^2} - \bar{\lambda}_{iFF}\right) A_{iHe} - \bar{\lambda}_{iHF} A_{iFe} \\
&- \int_0^T \alpha_H(\tau)[A_{iHe}A_{iFH}(\tau) + A_{iFe}A_{iHH}(\tau)] \left[\int_0^\tau A'_{iFH}(\tau') e^{-\kappa_\gamma(\tau-\tau')} d\tau'\right] d\tau \\
&+ \int_0^T \alpha_F(\tau)[A_{iHe}A_{iFF}(\tau) + A_{iFe}A_{iHF}(\tau)] \left[\int_0^\tau (1 - A'_{iFF}(\tau')) e^{-\kappa_\gamma(\tau-\tau')} d\tau'\right] d\tau > 0.
\end{aligned} \tag{B.98}$$

Equations (B.33) and (B.34) imply

$$\begin{aligned}
&-\bar{\lambda}_{iFF}A_{iHe}-\bar{\lambda}_{iHF}A_{iFe}\\
&=\int_0^T \alpha_H(\tau)A_{iFH}(\tau)[A_{iHe}A_{iFH}(\tau)+A_{iFe}A_{iHH}(\tau)]d\tau\\
&\quad+\int_0^T \alpha_F(\tau)A_{iFF}(\tau)[A_{iHe}A_{iFF}(\tau)+A_{iFe}A_{iHF}(\tau)]d\tau. \qquad \text{(B.99)}
\end{aligned}$$

We next substitute (B.99) into (B.98). Noting that $1-A'_{iFF}(\tau)>0$, which follows from (B.30) for $(j,j')=(F,H)$ and (B.65), and that $(A_{iHH}(\tau), A_{iFF}(\tau), A_{iHe}, A_{iFe})$ are positive and $(A_{iHF}(\tau), A_{iFH}(\tau))$ are non-negative, we find that (B.98) holds under the sufficient condition

$$\int_0^T \alpha_H(\tau)[A_{iHe}A_{iFH}(\tau)+A_{iFe}A_{iHH}(\tau)]\left[A_{FH}(\tau)-\int_0^\tau A'_{iFH}(\tau')e^{-\kappa_\gamma(\tau-\tau')}d\tau'\right]d\tau\geq 0,$$

which, in turn, holds because

$$A_{iFH}(\tau)-\int_0^\tau A'_{iFH}(\tau)e^{-\kappa_\gamma(\tau-\tau')}d\tau'\geq A_{iFH}(\tau)-\int_0^\tau A'_{iFH}(\tau')d\tau'=A_{iFH}(0)=0.$$

∎

Proof of Proposition 4.7: We prove the proposition in the case $j=H$. The proof for the case $j=F$ is symmetric. Consider a one-off increase in β_{Ht} at time zero, and denote by $\kappa_{\beta H}$ the rate at which β_{Ht} reverts to its mean of zero. The counterparts of (B.81) and (B.83) are

$$\kappa_{\beta H}A_{\beta He}=a\sigma_{iH}^2\bar{\lambda}_{iH\beta}A_{iHe}-a\sigma_{iF}^2\bar{\lambda}_{iF\beta}A_{iFe}, \qquad \text{(B.100)}$$

$$A_{\beta Hj}(\tau)=a\sigma_{iH}^2\bar{\lambda}_{iH\beta}\int_0^\tau A_{iHj}(\tau')e^{-\kappa_{\beta H}(\tau-\tau')}d\tau'+a\sigma_{iF}^2\bar{\lambda}_{iF\beta}\int_0^\tau A_{iFj}(\tau')e^{-\kappa_{\beta H}(\tau-\tau')}d\tau, \qquad \text{(B.101)}$$

where

$$\begin{aligned}
\bar{\lambda}_{ij\beta}\equiv&-\alpha_e A_{\gamma e}A_{ije}(-1)^{1_{\{j=F\}}}\\
&+\int_0^T\left[\theta_H(\tau)-\alpha_H(\tau)A_{\beta HH}(\tau)\right]A_{ijH}(\tau)d\tau-\int_0^T\alpha_F(\tau)A_{\beta HF}(\tau)A_{ijF}(\tau)d\tau \qquad \text{(B.102)}
\end{aligned}$$

is the counterpart of (B.84). The counterparts of (B.85) and (B.86) are

$$(1 + a\sigma_{iH}^2 z_{\beta HH})\bar{\lambda}_{iH\beta} + a\sigma_{iF}^2 z_{\beta FH}\bar{\lambda}_{iF\beta} = \int_0^T \theta_H(\tau) A_{iHH}(\tau) d\tau, \tag{B.103}$$

$$a\sigma_{iH}^2 z_{\beta HF}\bar{\lambda}_{iH\beta} + (1 + a\sigma_{iF}^2 z_{\beta FF})\bar{\lambda}_{iF\beta} = \int_0^T \theta_H(\tau) A_{iFH}(\tau) d\tau, \tag{B.104}$$

respectively, where

$$\begin{aligned}
z_{\beta HH} &= \frac{\alpha_e}{\kappa_{\beta H}} A_{iHe}^2 + \int_0^T \alpha_H(\tau) A_{iHH}(\tau) \left[\int_0^\tau A_{iHH}(\tau') e^{-\kappa_{\beta H}(\tau-\tau')} d\tau'\right] d\tau \\
&\quad + \int_0^T \alpha_F(\tau) A_{iHF}(\tau) \left[\int_0^\tau A_{iHF}(\tau') e^{-\kappa_{\beta H}(\tau-\tau')} d\tau'\right] d\tau, \\
z_{\beta FF} &= \frac{\alpha_e}{\kappa_{\beta H}} A_{iFe}^2 + \int_0^T \alpha_H(\tau) A_{iFH}(\tau) \left[\int_0^\tau A_{iFH}(\tau') e^{-\kappa_{\beta H}(\tau-\tau')} d\tau'\right] d\tau \\
&\quad + \int_0^T \alpha_F(\tau) A_{iFF}(\tau) \left[\int_0^\tau A_{iFF}(\tau') e^{-\kappa_{\beta H}(\tau-\tau')} d\tau'\right] d\tau,
\end{aligned}$$

$$\begin{aligned}
z_{\beta HF} &= -\frac{\alpha_e}{\kappa_{\beta H}} A_{iHe} A_{iFe} + \int_0^T \alpha_H(\tau) A_{iFH}(\tau) \left[\int_0^\tau A_{iHH}(\tau') e^{-\kappa_{\beta H}(\tau-\tau')} d\tau'\right] d\tau \\
&\quad + \int_0^T \alpha_F(\tau) A_{iFF}(\tau) \left[\int_0^\tau A_{iHF}(\tau') e^{-\kappa_{\beta H}(\tau-\tau')} d\tau'\right] d\tau, \\
z_{\beta FH} &= -\frac{\alpha_e}{\kappa_{\beta H}} A_{iHe} A_{iFe} + \int_0^T \alpha_H(\tau) A_{iHH}(\tau) \left[\int_0^\tau A_{iFH}(\tau') e^{-\kappa_{\beta H}(\tau-\tau')} d\tau'\right] d\tau \\
&\quad + \int_0^T \alpha_F(\tau) A_{iHF}(\tau) \left[\int_0^\tau A_{iFF}(\tau') e^{-\kappa_{\beta H}(\tau-\tau')} d\tau'\right] d\tau.
\end{aligned}$$

The solution to the linear system of (B.85) and (B.86) is

$$\bar{\lambda}_{iH\beta} = \frac{1}{\Delta_{z\beta}} \left[(1 + a\sigma_{iF}^2 z_{\beta FF}) \int_0^T \theta_H(\tau) A_{iHH}(\tau) d\tau - a\sigma_{iF}^2 z_{\beta FH} \int_0^T \theta_H(\tau) A_{iFH}(\tau) d\tau\right], \tag{B.105}$$

$$\bar{\lambda}_{iF\beta} = \frac{1}{\Delta_{z\beta}} \left[(1 + a\sigma_{iH}^2 z_{\beta HH}) \int_0^T \theta_H(\tau) A_{iFH}(\tau) d\tau - a\sigma_{iH}^2 z_{\beta HF} \int_0^T \theta_H(\tau) A_{iHH}(\tau) d\tau\right], \tag{B.106}$$

where

$$\Delta_{z\beta} \equiv (1 + a\sigma_{iH}^2 z_{\beta HH})(1 + a\sigma_{iF}^2 z_{\beta FF}) - a^2 \sigma_{iH}^2 \sigma_{iF}^2 z_{\beta HF} z_{\beta FH}.$$

The same argument as in the proof of Proposition 4.6 implies $\Delta_{z\beta} > 0$.

To complete the proof, we proceed in three steps. In Step 1, we show that $(z_{\beta HF}, z_{\beta FH})$

are non-positive, and are zero when $\alpha_e = 0$. In Step 2, we show that $A_{\beta HH}(\tau)$ is positive, and that $A_{\beta HF}(\tau)$ is positive when $\alpha_e > 0$ and zero when $\alpha_e = 0$. This proves the first and second statements in the proposition. In Step 3, we show that $A_{\beta He}$ is positive. This proves the third statement in the proposition.

Step 1: $(z_{\beta HF}, z_{\beta FH})$ **are non-positive, and are zero when** $\alpha_e = 0$**.** Since Lemma B.3 implies that $A_{iFH}(\tau)$ is non-negative and $A_{iFF}(\tau)$ is positive, and Lemma B.4 implies that $A_{iHH}(\tau)$ is increasing and $A_{iHF}(\tau)$ is non-decreasing,

$$\begin{aligned}
z_{\beta HF} \leq & -\frac{\alpha_e}{\kappa_{\beta H}} A_{iHe}A_{iFe} + \int_0^T \alpha_H(\tau)A_{iFH}(\tau)\left[\int_0^\tau A_{iHH}(\tau)e^{-\kappa_{\beta H}(\tau-\tau')}d\tau'\right]d\tau \\
& + \int_0^T \alpha_F(\tau)A_{iFF}(\tau)\left[\int_0^\tau A_{iHF}(\tau)e^{-\kappa_{\beta H}(\tau-\tau')}d\tau'\right]d\tau \\
\leq & -\frac{\alpha_e}{\kappa_{\beta H}} A_{iHe}A_{iFe} + \int_0^T \alpha_H(\tau)A_{iFH}(\tau)\frac{A_{iHH}(\tau)}{\kappa_{\beta H}}d\tau + \int_0^T \alpha_F(\tau)A_{iFF}(\tau)\frac{A_{iHF}(\tau)}{\kappa_{\beta H}} \\
= & -\frac{\bar{\lambda}_{iHF}}{\kappa_{\beta H}} \leq 0,
\end{aligned}$$

where the second step follows because $(A_{iHH}(\tau), A_{iFF}(\tau))$ are positive and $(A_{iHF}(\tau), A_{iFH}(\tau))$ are non-negative, the third step follows from (B.34), and the fourth step follows from Lemma B.2. The inequality $z_{\beta FH} \leq 0$ follows similarly.

When $\alpha_e = 0$, Lemma B.3 implies $A_{iHF}(\tau) = A_{iFH}(\tau) = 0$. Therefore, $z_{\beta HF} = z_{\beta FH} = 0$.

Step 2: $A_{\beta HH}(\tau)$ **is positive, and** $A_{\beta HF}(\tau)$ **is positive when** $\alpha_e > 0$ **and zero when** $\alpha_e = 0$**.** Since $(\Delta_{z\beta}, \theta_H(\tau), A_{iHH}(\tau))$ are positive, $(A_{iFH}(\tau), z_{\beta FF})$ are non-negative, and $z_{\beta FH} \leq 0$, (B.105) implies $\bar{\lambda}_{iH\beta} > 0$. When $\alpha_e > 0$, $A_{iFH}(\tau) > 0$. Since, in addition, $z_{\beta HH} \geq 0$ and $z_{\beta FH} \leq 0$, (B.106) implies $\bar{\lambda}_{iF\beta} > 0$. When $\alpha_e = 0$, (B.106) and $A_{iFH}(\tau) = z_{\beta HF} = 0$ imply $\bar{\lambda}_{iF\beta} = 0$.

Since $(\bar{\lambda}_{iH\beta}, A_{iHH}(\tau))$ are positive and $(\bar{\lambda}_{iF\beta}, A_{iFH}(\tau))$ are non-negative, (B.101) implies $A_{\beta HH}(\tau) > 0$. When $\alpha_e > 0$, $A_{iHF}(\tau) > 0$. Since, in addition, $(\bar{\lambda}_{iH\beta}, \bar{\lambda}_{iF\beta}, A_{iFF}(\tau))$ are positive, (B.101) implies $A_{\beta HF}(\tau) > 0$. When $\alpha_e = 0$, (B.101) and $A_{iHF}(\tau) = \bar{\lambda}_{iF\beta} = 0$ imply $A_{\beta HF}(\tau) = 0$.

Step 3: $A_{\beta He}$ **is positive.** Substituting $(\bar{\lambda}_{\beta HH}, \bar{\lambda}_{\beta HF})$ from (B.105) and (B.106) into (B.100), and using the definitions of $(z_{\beta HH}, z_{\beta HF}, z_{\beta FH}, z_{\beta FF})$, we find $A_{\beta He} > 0$ if

$$Z_{\beta H}\int_0^T \theta_H(\tau)A_{iHH}(\tau)d\tau - Z_{\beta F}\int_0^T \theta_H(\tau)A_{iFH}(\tau)d\tau > 0, \tag{B.107}$$

where

$$
\begin{aligned}
Z_{\beta H} &\equiv \sigma_{iH}^2(1 + a\sigma_{iF}^2 z_{\beta FF})A_{iHe} + a\sigma_{iH}^2\sigma_{iF}^2 z_{\beta HF}A_{iFe} \\
&= \sigma_{iH}^2 A_{iHe} \\
&\quad + a\sigma_{iH}^2\sigma_{iF}^2 \int_0^T \alpha_H(\tau)A_{iFH}(\tau)\left[\int_0^\tau [A_{iHe}A_{iFH}(\tau') + A_{iFe}A_{iHH}(\tau')]e^{-\kappa_{\beta H}(\tau-\tau')}d\tau'\right]d\tau \\
&\quad + a\sigma_{iH}^2\sigma_{iF}^2 \int_0^T \alpha_F(\tau)A_{iFF}(\tau)\left[\int_0^\tau [A_{iHe}A_{iFF}(\tau') + A_{iFe}A_{iHF}(\tau')]e^{-\kappa_{\beta H}(\tau-\tau')}d\tau'\right]d\tau, \\
Z_{\beta F} &\equiv \sigma_{iF}^2(1 + a\sigma_{iH}^2 z_{HH})A_{iFe} + a\sigma_{iH}^2\sigma_{iF}^2 z_{HF}A_{iHe} \\
&= \sigma_{iF}^2 A_{iFe} \\
&\quad + a\sigma_{iH}^2\sigma_{iF}^2 \int_0^T \alpha_H(\tau)A_{iHH}(\tau)\left[\int_0^\tau [A_{iHe}A_{iFH}(\tau') + A_{iFe}A_{iHH}(\tau')]e^{-\kappa_{\beta H}(\tau-\tau')}d\tau'\right]d\tau \\
&\quad + a\sigma_{iH}^2\sigma_{iF}^2 \int_0^T \alpha_F(\tau)A_{iHF}(\tau)\left[\int_0^\tau [A_{iHe}A_{iFF}(\tau') + A_{iFe}A_{iHF}(\tau')]e^{-\kappa_{\beta H}(\tau-\tau')}d\tau'\right]d\tau.
\end{aligned}
$$

Since $(\theta_H(\tau), A_{iHH}(\tau))$ are positive, $A_{iFH}(\tau)$ is non-negative, and $\frac{A_{iFH}(\tau)}{A_{iHH}(\tau)}$ is non-decreasing (increasing when $a > 0$ and $\alpha_e > 0$ from Lemma B.7, and zero when $a = 0$ or $\alpha_e = 0$), the ratio $\frac{\int_0^T \theta_H(\tau)A_{iFH}(\tau)d\tau}{\int_0^T \theta_H(\tau)A_{iHH}(\tau)d\tau}$ is bounded above by $\frac{A_{iFH}(\infty)}{A_{iHH}(\infty)}$. Since, in addition $(Z_{\beta H}, Z_{\beta F})$ are positive, (B.107) holds for all positive functions $\theta_H(\tau)$ under the sufficient condition

$$
Z_{\beta H}A_{iHH}(\infty) - Z_{\beta F}A_{iFH}(\infty) > 0. \tag{B.108}
$$

Using the definitions of $(Z_{\beta H}, Z_{\beta F})$, we can write (B.108) as

$$
\begin{aligned}
&\sigma_{iH}^2 A_{iHe}A_{iHH}(\infty) - \sigma_{iF}^2 A_{iFe}A_{iFH}(\infty) \\
&+ a\sigma_{iH}^2\sigma_{iF}^2 \int_0^T \alpha_H(\tau)\left[A_{iFH}(\tau)A_{iHH}(\infty) - A_{iHH}(\tau)A_{iFH}(\infty)\right] \\
&\times \left[\int_0^\tau [A_{iHe}A_{iFH}(\tau') + A_{iFe}A_{iHH}(\tau')]e^{-\kappa_{\beta H}(\tau-\tau')}d\tau'\right]d\tau \\
&+ a\sigma_{iH}^2\sigma_{iF}^2 \int_0^T \alpha_F(\tau)\left[A_{iFF}(\tau)A_{iHH}(\infty) - A_{iHF}(\tau)A_{iFH}(\infty)\right] \\
&\times \left[\int_0^\tau [A_{iHe}A_{iFF}(\tau') + A_{iFe}A_{iHF}(\tau')]e^{-\kappa_{\beta H}(\tau-\tau')}d\tau'\right]d\tau > 0.
\end{aligned} \tag{B.109}
$$

Using (B.95)-(B.97) to simplify the terms in the first, second and fourth lines of (B.109), and dividing throughout by $\frac{a\sigma_{iH}^2\sigma_{iF}^2 A_{iHH}(\infty)}{\kappa_{iF}-a\sigma_{iF}^2\bar{\lambda}_{iFF}} > 0$, we find that (B.109) is equivalent to

$$
\begin{aligned}
&\left(\frac{\kappa_{iF}}{a\sigma_{iF}^2} - \bar{\lambda}_{iFF}\right) A_{iHe} - \bar{\lambda}_{iHF} A_{iFe} \\
&- \int_0^T \alpha_H(\tau) A'_{iFH}(\tau) \left[\int_0^\tau [A_{iHe}A_{iFH}(\tau') + A_{iFe}A_{iHH}(\tau')] e^{-\kappa_{\beta H}(\tau-\tau')} d\tau'\right] d\tau \\
&+ \int_0^T \alpha_F(\tau)(1 - A'_{iFF}(\tau)) \left[\int_0^\tau [A_{iHe}A_{iFF}(\tau') + A_{iFe}A_{iHF}(\tau')] e^{-\kappa_{\beta H}(\tau-\tau')} d\tau'\right] d\tau > 0.
\end{aligned}
\tag{B.110}
$$

We next substitute (B.99) into (B.110). Noting that $1-A'_{iFF}(\tau) > 0$ and that $(A_{iHH}(\tau), A_{iFF}(\tau), A_{iHe}, A_{iFe})$ are positive and $(A_{iHF}(\tau), A_{iFH}(\tau))$ are non-negative, we find that (B.110) holds under the sufficient condition

$$
\begin{aligned}
&\int_0^T \alpha_H(\tau) \Big\{ A_{iFH}(\tau)[A_{iHe}A_{iFH}(\tau) + A_{iFe}A_{iHH}(\tau)] d\tau \\
&-A'_{iFH}(\tau) \left[\int_0^\tau [A_{iHe}A_{iFH}(\tau') + A_{iFe}A_{iHH}(\tau')] e^{-\kappa_{\beta H}(\tau-\tau')} d\tau'\right]\Big\} d\tau \geq 0,
\end{aligned}
$$

which, in turn, holds under the sufficient condition

$$
\begin{aligned}
&\int_0^T \alpha_H(\tau) \Big\{ A_{iFH}(\tau)[A_{iHe}A_{iFH}(\tau) + A_{iFe}A_{iHH}(\tau)] d\tau \\
&-A'_{iFH}(\tau) \left[\int_0^\tau [A_{iHe}A_{iFH}(\tau') + A_{iFe}A_{iHH}(\tau')] d\tau'\right]\Big\} d\tau \geq 0.
\end{aligned}
\tag{B.111}
$$

Equation (B.111) holds under the sufficient condition that the function

$$
G(\tau) \equiv \frac{A_{iFH}(\tau)}{\int_0^\tau [A_{iHe}A_{iFH}(\tau') + A_{iFe}A_{iHH}(\tau')] d\tau'}
$$

is non-increasing because the term in curly brackets in (B.111) is the negative of the numerator of $G'(\tau)$. The function $G'(\tau)$ is non-increasing under the sufficient condition that the function

$$
G_1(\tau) \equiv \frac{A'_{iFH}(\tau)}{A_{iHe}A_{iFH}(\tau) + A_{iFe}A_{iHH}(\tau)}
$$

is non-increasing. Equation (B.31) for $(j, j') = (H, F)$ implies

$$
\begin{aligned}
G_1(\tau) &= \frac{a\sigma_{iH}^2 \bar{\lambda}_{iHF} A_{iHH}(\tau) + (a\sigma_{iF}^2 \bar{\lambda}_{iFF} - \kappa_{iF}) A_{iFH}(\tau)}{A_{iHe}A_{iFH}(\tau) + A_{iFe}A_{iHH}(\tau)} \\
&= \frac{a\sigma_{iH}^2 \bar{\lambda}_{iHF} + (a\sigma_{iF}^2 \bar{\lambda}_{iFF} - \kappa_{iF}) \frac{A_{iFH}(\tau)}{A_{iHH}(\tau)}}{A_{iHe}\frac{A_{iFH}(\tau)}{A_{iHH}(\tau)} + A_{iFe}}.
\end{aligned}
$$

Since $\bar{\lambda}_{iFH} \geq 0$, $\bar{\lambda}_{iFF} \leq 0$ and $\frac{A_{iFH}(\tau)}{A_{iHH}(\tau)}$ is non-decreasing, $G_1(\tau)$ is non-increasing. ∎

C Numerical Solution and Model Estimation

C.1 Numerical Solution

We derive a system of 25 nonlinear scalar equations in the elements of the 5×5 matrix M. We adopt the exponential specification (5.1) and (5.2) for the functions $\{(\alpha_j(\tau), \theta_j(\tau))\}_{j=H,F}$, and set $T = \infty$. Using the exponential specification and $T = \infty$, we can compute the integrals involving $A_j(\tau)$ in the definition (3.23) of M as Laplace transforms. The Laplace transforms can be computed from polynomial functions of M. Computing them does not require solving the ODE system (3.21), which would entail computing eigenvalues and eigenvectors of M.

We define the Laplace transform

$$\mathcal{A}_j(s) \equiv \int_0^\infty A_j(\tau) e^{-s\tau} d\tau$$

of $A_j(\tau)$, and

$$\mathcal{X}_j(s) \equiv \int_0^\infty X_j(\tau) e^{-s\tau} d\tau$$

of $X_j(\tau) \equiv A_j(\tau) A_j(\tau)^\top$. Multiplying (3.21) by $e^{-s\tau}$, taking integrals of both sides from zero to infinity, and using the property that the Laplace transform of $A'_j(\tau)$ is s times that of $A_j(\tau)$ (this property follows from integration by parts), we find

$$(sI + M)\mathcal{A}_j(s) = \frac{1}{s}\mathcal{E}_{ij} \Rightarrow \mathcal{A}_j(s) = \frac{1}{s}(sI + M)^{-1}\mathcal{E}_{ij}, \tag{C.1}$$

where I denotes the 5×5 identity matrix. Multiplying (3.21) from the right by $A_j(\tau)^\top$, and adding to the resulting equation its transpose, we find

$$A'_j(\tau)A_j(\tau)^\top + A_j(\tau)A'_j(\tau)^\top + MA_j(\tau)A_j(\tau)^\top + A_j(\tau)A_j(\tau)^\top M^\top - \mathcal{E}_{ij}A_j(\tau)^\top - A_j(\tau)\mathcal{E}_j^\top = 0. \tag{C.2}$$

Multiplying (C.2) by $e^{-s\tau}$, taking integrals of both sides from zero to infinity, and using the definition of $X_j(\tau)$ and the property that the Laplace transform of $X'_j(\tau)$ is s times that of $X_j(\tau)$, we find

$$\left(\frac{s}{2}I + M\right)\mathcal{X}_j(s) + \mathcal{X}_j(s)\left(\frac{s}{2}I + M\right)^\top = \mathcal{E}_{ij}\mathcal{A}_j(s)^\top + \mathcal{A}_j(\tau)\mathcal{E}_{ij}^\top. \tag{C.3}$$

Equation (C.3) is a Lyapunov equation, and has a unique solution $\mathcal{X}_j(s)$ under the sufficient condition that the eigenvalues of $\frac{s}{2}I + M$ have positive real parts. Its solution can be computed by solving a system of fifteen linear scalar equations (as $\mathcal{X}_j(s)$ is a symmetric matrix).

Using the Laplace transforms $(\mathcal{A}_j(s), \mathcal{X}_j(s))$ and the exponential specifications (5.1) and (5.2), we can compute the integrals involving $A_j(\tau)$ in the definition of M as follows

$$\int_0^\infty \theta_j(\tau)\mathcal{E}_{\beta j}A_j(\tau)^\top d\tau = -\theta_{j0}\mathcal{E}_{\beta j}\mathcal{A}_j'(\theta_{j1})^\top, \tag{C.4}$$

$$\int_0^\infty \alpha_j(\tau)A_j(\tau)A_j(\tau)^\top d\tau = \alpha_{j0}\mathcal{X}_j(\alpha_{j1}). \tag{C.5}$$

Deriving (C.4) requires additionally the property that the Laplace transform of $\tau A_j(\tau)$ is minus the derivative of that of $A_j(\tau)$. The derivative $\mathcal{A}_j'(s)$ can be computed as function of $\mathcal{A}_j(s)$ by differentiating (C.1):

$$\mathcal{A}_j(s) + (sI+M)\mathcal{A}_j'(s) = -\frac{1}{s^2}\mathcal{E}_{ij} \Rightarrow \mathcal{A}_j'(s) = -(sI+M)^{-1}\left(\mathcal{A}_j(s) + \frac{1}{s^2}\mathcal{E}_{ij}\right). \tag{C.6}$$

Using (C.4) and (C.5), together with

$$A_e = M^{-1}\left(\mathcal{E}_{iH} - \mathcal{E}_{iF}\right), \tag{C.7}$$

which follows from (3.19), we can write (3.23) as

$$\begin{aligned} M \equiv \Gamma^\top - a\Bigg[&\left(\theta_e\mathcal{E}_\gamma - \alpha_e M^{-1}\left(\mathcal{E}_{iH} - \mathcal{E}_{iF}\right)\right)\left(\mathcal{E}_{iH} - \mathcal{E}_{iF}\right)^\top \left(M^{-1}\right)^\top \\ &- \sum_{j=H,F}\left(\theta_{j0}\mathcal{E}_{\beta j}\mathcal{A}_j'(\theta_{j1})^\top + \alpha_{j0}\mathcal{X}_j(\alpha_{j1})\right)\Bigg]\Sigma\Sigma^\top. \end{aligned} \tag{C.8}$$

The right-hand side of (C.8) is a function of M, derived from (C.1), (C.3) and (C.6). Therefore, (C.8) forms a system of 25 nonlinear scalar equations in the 25 elements of M. Given M, we derive $A_j(\tau)$ by solving the ODE system (3.21), and we obtain A_e from (C.7). Given $A_j(\tau)$ and A_e, we solve for $C_j(\tau)$ and C_e from (3.20), (3.22) and (3.24).

We solve the system of 25 nonlinear scalar equations using a continuation algorithm.

- Step 0 of the algorithm solves the system for zero risk aversion $a^{(0)} = 0$. The solution is $M = \Gamma^\top$.
- Step $i+1$ of the algorithm solves the system for risk aversion $a^{(i+1)} = a^{(i)} + s^{(i+1)}$, where $a^{(i)}$ is risk aversion for step i and $s^{(i+1)}$ is a small step size. The solution $M^{(i)}$ in step i is used as initial condition for solving the system in step $i+1$. This ensures that the solution in step $i+1$ is found quickly and is close to the solution in step i.
- The algorithm ends when $a^{(i+1)} = a$.

If there are multiple solutions for M, the continuation algorithm picks the solution that converges to the unique solution $M = \Gamma^\top$ when risk aversion goes to zero.

C.2 Model Estimation

For each vector $\boldsymbol{\rho}$ of parameters, we solve the model numerically and compute the weighted sum $L(\boldsymbol{\rho})$ of squared differences between the empirical moments and their model-implied counterparts. To compute the model-implied moments of exchange rates and bond yields, we first compute the unconditional covariance and autocovariance of the state vector q_t. Integrating (2.8), we find

$$q_t = \bar{q} + \int_{-\infty}^{t} e^{-\Gamma(t-t')}\Sigma dB_{t'}. \tag{C.9}$$

Equation (C.9) implies that the unconditional covariance of q_t is

$$\mathbb{C}\text{ov}(q_t, q_t^\top) = \left[\int_{-\infty}^{t} e^{-\Gamma(t-t')}\Sigma\Sigma^\top e^{-\Gamma^\top(t-t')}dt'\right] \equiv \hat{\Sigma}. \tag{C.10}$$

Differentiating (C.10) with respect to t and noting that the derivative is zero, we find

$$\Gamma\hat{\Sigma} + \hat{\Sigma}\Gamma^\top = \Sigma\Sigma^\top, \tag{C.11}$$

which is a Lyapunov equation and has a unique solution $\hat{\Sigma}$ because the eigenvalues of Γ have positive real parts. The unconditional autocovariance of q_t is

$$\begin{aligned}
\mathbb{C}\text{ov}(q_t, q_s^\top) &= \int_{-\infty}^{t} e^{-\Gamma(t-t')}\Sigma\Sigma^\top e^{-\Gamma^\top(s-t')}dt' \\
&= \left[\int_{-\infty}^{t} e^{-\Gamma(t-t')}\Sigma\Sigma^\top e^{-\Gamma^\top(t-t')}dt'\right] e^{-\Gamma^\top(s-t)} \\
&= \hat{\Sigma}e^{-\Gamma^\top(s-t)},
\end{aligned} \tag{C.12}$$

for $s > t$, where the last step in (C.12) follows from (C.10).

Bond yields and log exchange rates in the model are affine functions of the state vector q_t. The covariance between two such affine functions $Xq_t + X_0$ and $Yq_s + Y_0$ for 1×5 constant vectors (X, Y), scalars (X_0, Y_0), and $s > t$ is

$$\mathbb{C}\text{ov}(Xq_t + X_0, Yq_s + Y_0) = X\mathbb{C}\text{ov}(q_t, q_s^\top)Y^\top. \tag{C.13}$$

C.3 Predictive Regressions

Bilson (1981) and Fama (1984) perform the regression

$$\frac{1}{\Delta\tau}\log\left(\frac{e_t}{e_{t+\Delta\tau}}\right) = a_{\text{UIP}} + b_{\text{UIP}}\left(y_{Ft}^{(\Delta\tau)} - y_{Ht}^{(\Delta\tau)}\right) + e_{t+\Delta\tau}.$$

The dependent variable is the rate of foreign currency depreciation over horizon $\Delta\tau$. The independent variable is the foreign-minus-home $\Delta\tau$-year yield differential. Bilson (1981) and Fama (1984) assume that the horizon $\Delta\tau$ is short (monthly). Chinn and Meredith (2004) perform the same regression for longer horizons. The coefficient b_{UIP} of this regression depends on second moments of bond yields and log exchange rates, and can be computed as described in (C.13).

Lustig, Stathopoulos, and Verdelhan (2019) perform the regression

$$\frac{1}{\Delta\tau}\log\left(\frac{P_{F,t+\Delta\tau}^{(\tau-\Delta\tau)}e_{t+\Delta\tau}}{P_{Ft}^{(\tau)}e_t}\right) - \frac{1}{\Delta\tau}\log\left(\frac{P_{H,t+\Delta\tau}^{(\tau-\Delta\tau)}}{P_{Ht}^{(\tau)}}\right) = a_{\text{LSV}} + b_{\text{LSV}}\left(y_{Ft}^{(\Delta\tau)} - y_{Ht}^{(\Delta\tau)}\right) + e_{t+\Delta\tau}.$$

The dependent variable is the return over horizon $\Delta\tau$ of the hybrid CCT constructed using bonds with maturity τ. The independent variable is the foreign-minus-home $\Delta\tau$-year yield differential. Since log bond prices are affine functions of the state vector q_t, the coefficient b_{LSV} of this regression can be computed as described in (C.13).

Chernov and Creal (2020) and Lloyd and Marin (2020) perform the regression

$$\frac{1}{\Delta\tau}\log\left(\frac{e_t}{e_{t+\Delta\tau}}\right) = a_{\text{UIP}\ell s} + b_{\text{UIP}\ell}\left(y_{Ft}^{(\Delta\tau)} - y_{Ht}^{(\Delta\tau)}\right) + b_{\text{UIPs}}\left[\left(y_{Ft}^{(\tau_2)} - y_{Ft}^{(\tau_1)}\right) - \left(y_{Ht}^{(\tau_2)} - y_{Ht}^{(\tau_1)}\right)\right] + e_{t+\Delta\tau}.$$

The dependent variable is the rate of foreign currency depreciation over horizon $\Delta\tau$. The independent variables are the foreign-minus-home $\Delta\tau$-year yield differential and the foreign-minus-home slope differential between years τ_1 and τ_2. The coefficients b_{UIPs} and $b_{\text{UIP}\ell}$ of this regression can be computed as described in (C.13).

Fama and Bliss (1987) perform the regression

$$\frac{1}{\Delta\tau}\log\left(\frac{P_{j,t+\Delta\tau}^{(\tau-\Delta\tau)}}{P_{jt}^{(\tau)}}\right) - y_{jt}^{(\Delta\tau)} = a_{\text{FB}} + b_{\text{FB}}\left(f_{jt}^{(\tau-\Delta\tau,\tau)} - y_{jt}^{(\Delta\tau)}\right) + e_{t+\Delta\tau}.$$

The dependent variable is the log return over horizon $\Delta\tau$ of the country-j bond with maturity τ in excess of the $\Delta\tau$-year spot rate (yield). The independent variable is the slope of the country-j term structure as measured by the difference between the forward rate between maturities $\tau - \Delta\tau$ and τ, and the $\Delta\tau$-year spot rate. Since log bond prices are affine functions of the state vector q_t, and the forward rate is

$$f_{jt}^{(\tau-\Delta\tau,\tau)} = -\frac{\log\left(\frac{P_{jt}^{(\tau)}}{P_{jt}^{(\tau-\Delta\tau)}}\right)}{\Delta\tau},$$

the coefficient b_{FB} of this regression can be computed as described in (C.13).

Campbell and Shiller (1991) perform the regression

$$y_{j,t+\Delta\tau}^{(\tau-\Delta\tau)} - y_{jt}^{(\tau)} = a_{\text{CS}} + b_{\text{CS}}\frac{\Delta\tau}{\tau-\Delta\tau}\left(y_{jt}^{(\tau)} - y_{jt}^{(\Delta\tau)}\right) + e_{t+\Delta\tau}.$$

The dependent variable is the change over horizon $\Delta\tau$ in the yield of a country-j bond with initial maturity τ. The independent variable is the difference between the country-j spot rates for maturities τ and $\Delta\tau$, normalized so that b_{CS} is equal to one under the EH. The coefficient b_{CS} of this regression can be computed as described in (C.13).