ALGORITHM DESIGN:
A FAIRNESS-ACCURACY FRONTIER

ANNIE LIANG[†], JAY LU[§], AND XIAOSHENG MU[‡]

February 27, 2023

Abstract. Algorithm designers increasingly optimize not only for accuracy but also fairness, defined as how similar accuracy is across demographic groups. We study the tradeoff between fairness and accuracy via a fairness-accuracy frontier, which consists of the optimal points (for a fixed set of inputs) across a broad range of preferences over fairness and accuracy. Our results identify a simple property of the inputs, *group-balance*, which qualitatively determines the shape of the frontier. We further study an information-design problem where the designer flexibly regulates the inputs (e.g., by coarsening an input or banning its use) but the algorithm is chosen by another agent. Whether it is optimal to ban an input generally depends on the designer's preferences. But when inputs are group-balanced, then excluding group identity is strictly suboptimal for all designers, and when the designer has access to group identity, then it is strictly suboptimal to exclude any informative input.

1. Introduction

Decisions such as which patients should receive treatment or which borrowers should receive loans are increasingly guided by the predictions of algorithms (Roth and Kearns, 2019). A recent literature establishes that the error rates of commercially-deployed algorithms often differ substantially across racial and gender groups (Arnold et al., 2021; Fuster et al., 2021). For example, patients assigned the same risk score by a widely-used healthcare algorithm were shown to have substantially different actual health risks depending on their race (Obermeyer et al., 2019); the false-positive rate of an algorithm used to predict criminal reoffense was shown to be twice as high for Black defendants as for White defendants (Angwin and Larson, 2016); and the accuracy of facial-recognition technologies vary substantially across demographic groups (Klare et al., 2012).

There is a long tradition in economics of studying equity-efficiency tradeoffs in settings as diverse as taxation (Saez and Stantcheva, 2016; Dworczak et al., 2021), policing (Persico,

We thank Nageeb Ali, Simon Board, Krishna Dasaratha, Will Dobbie, Sergiu Hart, Peter Hull, Navin Kartik, Yair Livne, Sendhil Mullainathan, Derek Neal, and Kyohei Okumura for helpful comments, and National Science Foundation Grant SES-1851629 for financial support. We also thank Andrei Iakovlev and Aristotle Magganas for valuable research assistance on this project.
[†]Northwestern University.
[§]UCLA.
[‡]Princeton University.

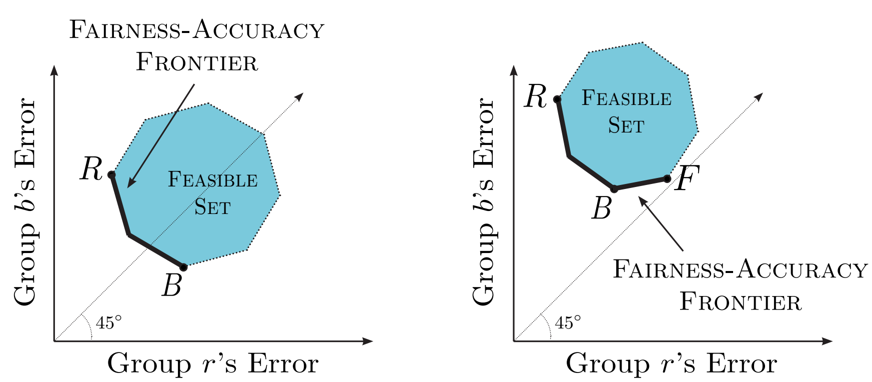
2002; Jung et al., 2020), and college admissions (Chan and Eyster, 2003; Ellison and Pathak, 2021). The context of algorithmic predictions presents a new equity-efficiency tradeoff, manifested as the tradeoff between accuracy (the overall error rate of the algorithm) and fairness (how similar the algorithm's error rate is across pre-defined groups). This tradeoff is governed in substantial part by the inputs to the algorithm and their statistical relationship to group identity—for example, whether these inputs are systematically more informative for one group than another. Algorithmic inputs can be observed, manipulated, and regulated, raising the following fundamental questions: How does the tradeoff between fairness and accuracy depend on the information available for prediction? Which informational environments create a tension between fairness and accuracy, and which ameliorate it? While the tradeoff between fairness and accuracy has been empirically computed in specific applications (Wei and Niethammer, 2020; Chohlas-Wood et al., 2021; Little et al., 2022), substantially less is known about how the available information shapes the tension between these goals in general.

To examine these questions, we define and study a *fairness-accuracy frontier*. The frontier consists of those outcomes that are optimal for various objective functions, which reflect a wide range of views on how to optimally trade off fairness and accuracy. We prove two types of results about the frontier. First, we identify simple properties of the algorithmic inputs that determine the qualitative shape of this frontier. Second, we take an information-design perspective on understanding how constraints on information can induce certain desired outcomes. Specifically, we consider an interaction between a regulator flexibly constraining the inputs and an agent setting the algorithm, and characterize what part of the fairness-accuracy frontier the designer can achieve through appropriate design of the inputs. We also examine whether it might be optimal for the designer to exclude an input altogether (e.g., excluding group identity in the context of medical predictions, or a test score in the context of college admissions).

In our model, a designer chooses an algorithm that takes observed covariates as inputs (e.g., image scans, lab tests, records of prior hospital visits) and outputs a decision (e.g., whether to recommend a medical procedure). The algorithm's consequences for any given individual are measured using a loss function, which can be interpreted as the inaccuracy or the harm of the decision. We aggregate losses within two groups, group r (red) and group b (blue). Each group's *error* is the expected loss for individuals of that group. An algorithm is understood to be more accurate if it implies lower errors for both groups, and more fair if it implies a smaller difference between the two groups' errors.

To understand the tradeoff between fairness and accuracy, we define the class of *fairness-accuracy (FA) preferences* to be all preferences over group error pairs that are consistent with the following order: one pair of group errors *FA-dominates* another if the former involves smaller errors for both groups (greater accuracy) and also a smaller difference between group errors (greater fairness).[1] This partial order is consistent with a broad range of designer preferences, including Utilitarian designers (who minimize the aggregate error in the population), Rawlsian designers (who minimize the greater of the two group errors), and Egalitarian designers (who minimize the absolute difference between group errors). Some of these preferences correspond directly to optimization problems that have been proposed for use in practice.[2] We define the *fairness-accuracy frontier* to be the set of all feasible group error pairs that are FA-undominated within the feasible set, i.e., there is no feasible error pair that improves simultaneously on accuracy and fairness.

A simple property of the algorithm's inputs turns out to be critical for determining the shape of the fairness-accuracy frontier. Say that a covariate vector is *group-balanced* if given this covariate vector, group r's optimal algorithm (i.e. the one that gives r the smallest error over all feasible algorithms) yields a lower error for group r than for group b, and if the reverse is true for group b's optimal algorithm. Otherwise, say that the covariate vector is *group-skewed*. It is difficult to anticipate in advance of an empirical analysis which of group-balance or group-skew is more typical in practice. One reason to expect group-skew is if the covariates have the same implications for both groups but are measured more accurately for one group than the other—say, if medical data is recorded more accurately for high-income patients than low-income patients.



(a) Inputs are Group-Balanced (b) Inputs are Group-Skewed

Figure 1. The Fairness-Accuracy Frontier.

[1] We do not take a stance on the normative desirability of these preferences, instead interpreting our class as encompassing the broad range of designer preferences that could be relevant in practice.

[2] For example, optimizing a Rawlsian preference is equivalent to implementing group distributionally robust optimization (Sagawa et al., 2020), and optimizing an Egalitarian preference is equivalent (on a restricted domain) to maximizing accuracy subject to equality of error rates (as considered in Hardt et al. (2016) among others).

Our first result says that depending on whether the covariate vector is group-balanced or group-skewed, the fairness-accuracy frontier takes either of two possible forms, as depicted in Figure 1. In both cases, the frontier is a part of the lower boundary of the *feasible set*, namely the error pairs that are implementable using some algorithm that takes the covariate vector as input. But in the case of group-balanced inputs, the fairness-accuracy frontier is the part of the lower boundary that begins at the point that is best for group r (labeled R) and ends at the point that is best for group b (labeled B). This is precisely the set of all feasible error pairs that cannot be simultaneously reduced in both coordinates. In the case of group-skewed inputs, the frontier again includes those points, but now additionally includes a positively-sloped part (in Figure 1, the segment from B to the fairness-maximizing point F) along which both groups' errors increase but the gap between their errors decreases. This characterization of the frontier tells us that a policy proposal that increases errors for both groups, but reduces the gap between group errors, can only be justified by fairness considerations if the covariate vector is group-skewed.

We next consider the important special case where group identity is an input to the algorithm. We show that the feasible set and frontier simplify as depicted in Figure 2: The feasible set is a rectangle, and the fairness-accuracy frontier is a single line segment along which the disadvantaged group (i.e., the group with the higher error) receives its minimal feasible error. If we consider a comparative static exercise in which a baseline covariate vector is augmented to include group identity, then a corollary of this characterization is that access to group identity must reduce the disadvantaged group's error regardless of the designer's fairness-accuracy preferences.

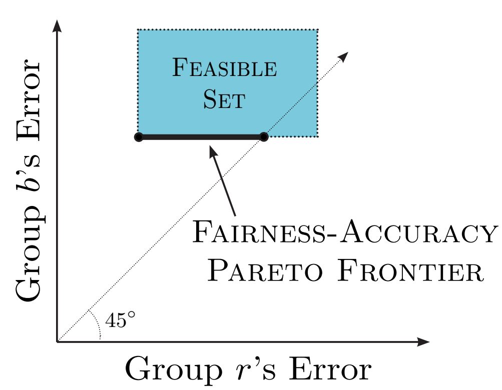


Figure 2. Depiction of the fairness-accuracy frontier in the case where X reveals G.

In the second part of the paper, we investigate what happens if the designer does not choose the algorithm, but instead regulates the inputs of the algorithm. This question is motivated by settings where a designer has fairness concerns, but the agent setting the algorithm does not. For example, a judge (agent) determining sentencing may seek to maximize the number of correct verdicts, while a policymaker (designer) may additionally prefer that the accuracy of the judge's verdicts is equitable across certain social groups. In these cases, the policymaker can impose regulation that restricts the inputs available to the algorithm, for example, by legally banning the use of a specific input.

We model this as an information design problem (Kamenica and Gentzkow, 2011) where the designer chooses a garbling of the available inputs, and an agent chooses an algorithm (based on the garbling) to maximize accuracy. Under weak conditions, it turns out to be without loss for the designer to only control the algorithm's inputs. That is, any error pair that a designer would choose to implement given full control of the algorithm can also be achieved by appropriately garbling the inputs.

We next consider whether the optimal garbling might involve excluding a covariate entirely from use in the algorithm. We demonstrate two results: First, excluding group identity as an algorithmic input is strictly welfare-reducing for all designers (with FA preferences) if and only if the permitted covariates are group-balanced. Second, when group identity is permitted as an input, then completely excluding any other covariate makes every designer strictly worse off, so long as that covariate satisfies a mild condition that we call decision-relevance. When applied to the policy question of whether to permit standardized test scores in admissions decisions, the latter result suggests that so long as group identity is a permissible input into admission decisions,[3] then excluding test scores is welfare-reducing for *all* designers with the power to garble covariates. On the other hand, if group identity is not permitted as an input into college admissions decisions (as is the case in the states of California and Michigan), then the optimal garbling of covariates for some designer preference may indeed involve completely excluding test scores, and we provide an example to this effect.

1.1. **Related Literature.** Our paper is motivated by recent problems in the literature on algorithmic bias (Section 1.1.3), but assumes a novel perspective on these questions based on approaches from two literatures in economic theory: the literature on information design (Section 1.1.1) and the literature on social preferences and inequality (Section 1.1.2). Building on the former, we model the interaction between a designer flexibly regulating inputs and an agent setting the algorithm. Building on the latter, we focus on understanding equity-efficiency tradeoffs, and consider a wide class of preferences that reflects heterogeneity in social preferences.

1.1.1. *Information Design.* One contribution of our paper is the casting of the design of algorithmic inputs as an information design problem (see Kamenica (2019) and Bergemann and Morris (2019) for recent surveys). This approach complements previous frameworks for modeling the regulation of algorithms, in which regulators communicate information via

[3]This is currently true in most states in the US, pending the decision of *Students for Fair Admissions, Inc. v. President and Fellows of Harvard College.*

cheap talk (Cowgill and Stevenson, 2020) or impose restrictions directly on the algorithm (Yang and Dobbie, 2020; Rambachan et al., 2021; Blattner et al., 2022). We view the garbling of inputs as a potentially effective policy tool, which can be implemented through a variety of technological or legal commitments,[4] and deserves further attention within the context of algorithmic fairness.

Conversely, problems regarding algorithmic fairness motivate analyses that depart from typical information design problems in a few interesting ways. First, the Sender in our framework cannot choose a completely flexible information structure, but is constrained to garblings of a primitive covariate vector. Second, motivated by heterogeneous attitudes toward fairness (Section 1.1.2), we focus on a frontier of solutions with respect to a wide class of Sender preferences. Our results in Section 4.2 describe how the frontier of solutions changes with respect to changes in the underlying information. We focus on special cases of this comparative static that are of interest given our motivation (e.g., adding or removing group identity), but a more general solution (analogous to Curello and Sinander (2022)'s recent work on comparative statics with respect to the Sender's utility function) would be an interesting avenue for future work.

Finally, at the broader intersection of information design and algorithms, Ichihashi (2023) considers optimal information acquisition for crime deterrence, and Caplin et al. (2023) draws a connection between different machine learning objectives and costly information design.

1.1.2. *Social Preferences and Inequality.* The literature on social preferences documents substantial heterogeneity in how individuals assess efficiency-equity tradeoffs (Andreoni and Miller, 2002; Fehr and Schmidt, 1999; Fisman et al., 2007; Sullivan, 2022), which is reflected in our broad class of FA-preferences. In this literature, social preferences are preferences over individual payoffs rather than preferences over group errors, but most have analogues in our setting. For example, the "social welfare approach" aggregates individual payoffs using differential weights (Charness and Rabin, 2002; Saez and Stantcheva, 2016; Dworczak et al., 2021), and is nested in our class of FA preferences (if we interpret individual payoffs as group errors). We additionally allow for a direct penalty for unequal outcomes, as in the models of "difference aversion" or "inequity aversion" (Loewenstein et al., 1989; Bolton and Ockenfels, 2000; Fehr and Schmidt, 1999).[5]

There is a separate literature studying the equity-efficiency tradeoffs of affirmative action programs. Specifically, Lundberg (1991) and Chan and Eyster (2003) model affirmative

[4]For example, organizations such as the US Census Bureau, Google, Apple, and Microsoft are committed to differential privacy initiatives (Dwork and Roth, 2014), which take various forms of adding noise to user inputs. Yang and Dobbie (2020) summarizes the existing law on algorithmic regulation and proposes new legal policies for mitigating algorithmic bias.

[5]Another part of this literature is concerned with intentions and reciprocity (Rabin, 1993; Charness and Rabin, 2002) and is outside of our model.

action as a ban on the use of group identity in admissions decisions, and show that this can lead organizations to condition on proxies in a way that reduces both efficiency and equity. (A similar point is made in Agan and Starr (2018) regarding the use of prior criminal history in hiring decisions in "ban-the-box" policies.) Ellison and Pathak (2021) empirically quantify the equity and efficiency losses of race-neutral affirmative action (based on geographic proxies for race) as compared to plans that explicitly consider race. These papers are related to our study of the impact of excluding group identity, but focus on how a designer's optimal algorithm given group identity compares to the optimal algorithm without. Our analogous comparative static is in the context of an information design problem, where the designer controls the inputs to the algorithm but does not choose the algorithm itself. We examine how the frontier of achievable outcomes changes when the designer can design a group-dependent garbling versus when the designer must choose a group-independent garbling. These analyses are not nested; see Section 4.2.1 for more detail.

1.1.3. *Algorithmic Bias.* The recent literature on algorithmic bias has emerged around the concern that algorithms have error rates that differ substantially across social and demographic groups (see Kleinberg et al. (2018) and Cowgill and Tucker (2020) for overviews). In this literature and in the accompanying policy discussion (e.g, Angwin and Larson (2016)), algorithms are often considered to be "less fair" if the harms of the algorithm are more unequally borne across groups, with this comparison formalized as the disparity in error rates across groups (Hardt et al., 2016; Kleinberg et al., 2017; Chouldechova, 2017).[6] A growing body of empirical work documents and quantifies these disparate impacts (Obermeyer et al., 2019; Arnold et al., 2021; Fuster et al., 2021).

The tradeoff between accuracy (overall error rate of the algorithm in the population) and fairness (discrepancy between error rates across social groups) is a special kind of equity-efficiency tradeoff. A common approach for resolving this tradeoff is to posit a particular objective criterion (Hardt et al., 2016; Diana et al., 2021). Other papers identify improvements with respect to both objectives simultaneously (Rose, 2021; Feigenberg and Miller, 2021). Our paper is closest to a smaller part of this literature, which engages with the tension between fairness and accuracy by quantifying fairness-accuracy tradeoffs for specific loss functions (Menon and Williamson, 2018) or for specific empirical applications (Wei and Niethammer, 2020; Chohlas-Wood et al., 2021; Little et al., 2022). We are interested in how this fairness-accuracy tradeoff is moderated by the inputs to the algorithm in general, and provide simple conditions on the inputs that qualitatively govern this tradeoff independently of other details of the loss function or informational environment.

[6]A notable exception is the concept of individual fairness proposed in Dwork et al. (2012).

2. Framework

2.1. **Setup and Notation.** There is a population of individuals, where each individual is described by a *covariate vector* X taking values in the finite set $\mathcal{X}$, a *type* Y taking values in the finite set $\mathcal{Y}$,[7] and a *group identity* G taking values r or b.[8] Throughout we think of G, X, Y as random variables with joint distribution $\mathbb{P}$, and use $p_g \equiv \mathbb{P}(G = g) > 0$ to denote the fraction of the population that belongs to group $g \in \{r, b\}$. We impose no assumptions on the joint distribution,[9] permitting for example each of the following:

Example 1 (X reveals or closely proxies for G). The group identity may be an input in the covariate vector X, or predictable from inputs in the covariate vector X. For example, Bertrand and Kamenica (2020) show that data on consumption patterns permits near perfect classification of gender and a fairly accurate prediction of other group identities such as income bracket, race, and political ideology.

Example 2 (Biased Covariates). The value of an input in X may be systematically biased depending on group identity. For example, if G is income bracket, Y is ability, and X is a test score that can be improved through better access to test prep, the distribution $\mathbb{P}$ may have the property that at every ability level, the conditional distribution of test scores is shifted higher for students in the high-income bracket (i.e., the distribution of $X \mid Y = y, G = r$ first-order stochastically dominates $X \mid Y = y, G = b$ at every $y \in \mathcal{Y}$).

Example 3 (Asymmetrically Informative Covariates). The inputs in X may be more informative about Y for one group than the other. For example, in Obermeyer et al. (2019), a patient's health care costs are more predictive of their health care needs for White patients than for Black patients, and Rothstein (2004) shows that SAT scores are more informative about future college grades for high-income students than low-income students.

A designer chooses an *algorithm* $a : \mathcal{X} \to \Delta(\mathcal{D})$ that maps covariate vectors into distributions over decisions in $\mathcal{D} = \{0, 1\}$. Let $\mathscr{A}_X$ denote the set of all algorithms. Some motivating examples of types, group identities, covariate vectors, and decisions are given below:

[7]We make the assumption of finiteness to simplify various notations in the exposition. Most of our results generalize to infinite covariate values and/or infinite types.

[8]Throughout, we assume the definition of the relevant groups to be a primitive of the setting, determined by sociopolitical precedent and outside the scope of our model.

[9]We view $\mathbb{P}$ as the population distribution on which the algorithm is both trained and tested. An interesting direction for future work would be to permit the data that the algorithm is trained on to differ in distribution from the data on which the algorithm's errors are evaluated. For example, the data on which the algorithm is trained may reflect historical biases that are no longer descriptive of the current environment. Another interesting direction would be to study optimal sampling of data on which to train the algorithm (in which case $\mathbb{P}$ is endogenous); for example, Che et al. (2019) show that biased data sampling can create a dynamic feedback loop that reinforces inequities.

Healthcare. Y is need of treatment, G is socioeconomic class, and the decision is whether the individual receives treatment. The covariate vector X includes possible attributes such as image scans, number of past hospital visits, family history of illness, and blood tests.

Credit scoring. Y is creditworthiness, G is gender, and the decision is whether the borrower's loan request is approved. The covariate vector X includes possible attributes such as purchase histories, social network data, income level, and past defaults.

Bail. Y is whether an individual is high-risk or low-risk of criminal reoffense, G is race, and the decision is whether the individual is released on bail. The covariate vector X includes possible attributes such as the individual's past criminal record, psychological evaluations, family criminal background, frequency of moves, or drug use as a child.[10]

Job hiring. Y is whether a job applicant is high or low quality, G is citizenship, and the decision is whether the applicant is hired. The covariate vector X includes possible attributes such as past work history, resume, and references.

The consequence of choosing decision d for an individual whose true type is y is evaluated using a (potentially group-dependent) loss function $\ell : \mathcal{D} \times \mathcal{Y} \times \mathcal{G} \to \mathbb{R}$.[11] We further aggregate these losses across individuals within each group:

Definition 1. For any algorithm $a \in \mathscr{A}_X$ and group $g \in \{r, b\}$, the *group g error* is

$$e_g(a) := \mathbb{E}_{D \sim a(X)} \left[\ell(D, Y, g) \mid G = g \right].$$

That is, group g's error is the average loss for members of group g. For example, if the type Y is binary and $\ell(d, y, g) = \mathbb{1}(d \neq y)$, then $e_g(a)$ is the total probability of a type I or type II error. Other loss functions may put different weights on different kinds of errors. We view the choice of the right loss function as application-specific, and demonstrate results that hold for arbitrary ℓ.

Each algorithm a implies a pair of group errors $(e_r(a), e_b(a))$. Throughout this paper, an *improvement in accuracy* means a reduction in both group errors, while an *improvement in fairness* means a reduction in the absolute difference between the group errors.[12] This

[10] These example covariates are based on the survey used by the Northpointe COMPAS risk tool. See for reference: `https://www.documentcloud.org/documents/2702103-Sample-Risk-Assessment-COMPAS-CORE.html`.

[11] For example, if G is socioeconomic background, Y is creditworthiness, the decision is whether to grant a loan, and the loss function corresponds to financial cost, then a bank manager may experience greater losses from predicting creditworthiness incorrectly for the wealthy group.

[12] This formulation is consistent with much of the literature on algorithmic fairness, but does not take into account all important fairness considerations. For example, perfect prediction of criminal offense (Y) by the algorithm for both groups does not address historical inequities that have shaped differential base rates of Y across groups. Moreover, as Kasy and Abebe (2021) point out, an algorithm that is fair in the narrow context of one decision may perpetuate or exacerbate inequalities within a larger context. We leave to future work the interesting question of how these algorithmic design decisions might impact decisions in a larger dynamic game.

approach nests many of the various fairness criteria that have been proposed in the literature (see Mehrabi et al. (2022) for a recent survey) under a particular choice of a loss function. For example, if the type Y is binary and $\ell(d, y, g) = \mathbb{1}(d \neq y)$, then $e_r(g) = e_b(g)$ corresponds to equality of misclassification rates, while if $\ell(d, y, g) = \mathbb{1}(d = 1, y = 0)$ then $e_r(g) = e_b(g)$ corresponds to equality of false positive rates (Kleinberg et al., 2017; Chouldechova, 2017). And if

$$\ell(d, y, g) = \begin{cases} \frac{P(Y=y)}{P(Y=y|G=g)} & \text{if } d = 1 \\ 0 & \text{otherwise} \end{cases}$$

then $e_r(g) = e_b(g)$ corresponds to equality of equalized odds (Hardt et al., 2016). See Appendix A for further details.

In Section 5, we discuss an extension of the fairness criterion to any $|\phi(e_r) - \phi(e_b)|$ where ϕ is continuous and strictly increasing, which includes the ratio of error rates as a special case (setting $\phi(e) = \log(e)$). We also discuss in Section 5 an extension of our framework when fairness and accuracy are evaluated using different loss functions.

2.2. **Fairness-Accuracy Preferences.** The designer has a preference ordering over group error pairs $e = (e_r, e_g) \in \mathbb{R}^2$. We consider the set of all preferences that are consistent with the following weak criterion.

Definition 2. The *fairness-accuracy (FA) dominance* relation $>_{FA}$ is the partial order on $\mathbb{R}^2$ satisfying $(e_r, e_b) >_{FA} (e'_r, e'_b)$ if $e_r \leq e'_r$, $e_b \leq e'_b$, and $|e_r - e_b| \leq |e'_r - e'_b|$, with at least one of these inequalities strict.[13]

That is, if it is possible to simultaneously increase accuracy (reducing errors for both groups) and also increase fairness (reducing the gap between these errors), then all designers must prefer this.

Definition 3. A *fairness-accuracy (FA) preference* $\succeq$ is any total order on $\mathbb{R}^2$ such that $e \succ e'$ whenever $e >_{FA} e'$.

It is straightforward to see that these orders are unchanged if $|e_r - e_b|$ is replaced with $\phi(|e_r - e_b|)$ where ϕ is a strictly increasing function.

The class of FA preferences reflects a broad range of views on how to trade off fairness and accuracy, including the following special cases that have been proposed in the literature.

[13]Kleinberg and Mullainathan (2019) define an admissions rule to be a strict improvement over another if it improves both efficiency (the average type of an admitted applicant) and equity (the fraction of admitted students who belong to the disadvantaged group), which is similar to our FA dominance relation but non-nested, as it involves two loss functions. The FA-dominance relation in Online Appendix O.1 generalizes both orders.

Example 4 (Utilitarian). The designer evaluates errors $e = (e_r, e_b)$ according to the weighted sum in the population. That is, let

$$w_u(e) = -p_r e_r - p_b e_b$$

and let $\succeq_u$ be the ordering represented by w_u, i.e. $e \succeq_u e'$ if and only if $w_u(e) \geq w_u(e')$. (Note that the minority population, which has a lower weight by definition, will be naturally discounted as a group in this evaluation.) A designer with preferences $\succeq_u$ is called *Utilitarian* (Harsanyi, 1953, 1955).

Example 5 (Rawlsian). The designer evaluates errors $e = (e_r, e_b)$ according to the greater error. That is, let

$$w_r(e) = -\max\{e_r, e_b\}$$

and let $\succeq_r$ be the corresponding ordering represented by w_r.[14] A designer with preferences $\succeq_r$ is called *Rawlsian* (Rawls, 1971).

Example 6 (Egalitarian). The designer evaluates errors $e = (e_r, e_b)$ according to their difference. That is, let

$$w_e(e) = -|e_r - e_b|$$

and let $\succeq_e$ be the lexicographic order that first evaluates errors according to w_e and then compares ties using the Utilitarian utility w_u. A designer with preferences $\succeq_e$ is called *Egalitarian* (Parfit, 2002).

Example 7 (Constrained Optimization). The designer evaluates errors $e = (e_r, e_b)$ according to

$$w_c(e) = (1-\lambda)\, w_u(e) + \lambda w_e(e)$$

for some $\lambda \in [0, 1]$ (breaking ties with $\succeq_e$ when $\lambda = 1$). The optimal choices here correspond to the solutions of the following constrained optimization problem

$$\min_{a \in \mathscr{A}_X} p_r e_r(a) + p_b e_b(a) \text{ s.t. } |e_r(a) - e_b(a)| \leq c$$

when the constraint is satisfiable.[15] The special case of $c = 0$ (as considered in Hardt et al. (2016)) returns the Egalitarian solution. This is a standard approach in the algorithmic fairness literature (Ferry et al., 2022; Menon and Williamson, 2018; Corbett-Davis et al., 2017; Agarwal et al., 2018).

Example 8 (Accuracy then Fairness). The designer evaluates errors $e = (e_r, e_b)$ by first evaluating accuracy and then fairness. That is, $e \succ e'$ if $e_r \leq e'_r$ and $e_b \leq e'_b$ with at least one

[14] This approach is also known as *group distributionally robust optimization* (Sagawa et al., 2020; Hansen et al., 2022).

[15] The constant λ corresponds to the Lagrange multiplier in the optimization problem. Note that while the preference induced by w_c is complete, the constrained optimization yields an incomplete ordering (for example, two errors that are both not feasible cannot be ranked).

strict, and if not, they are then compared using w_e. This is the approach recently proposed by Viviano and Bradic (2023).

Our consideration of this wide class of preferences is motivated in part by the experimental literature on social preferences, which documents substantial heterogeneity across individuals' views on how to trade off equity and efficiency. In particular, when given the choice between different allocations of payoffs across individuals, some experimental subjects choose Pareto-dominated allocations that are more equal (corresponding in our setting to choice of (e_r, e_b) over (e'_r, e'_b) where $e_r > e'_r$ and $e_b > e_b$ but $|e_r - e_b| < |e'_r - e'_b|$). These are minority preferences in the population (Andreoni and Miller, 2002; Charness and Rabin, 2002), but constitute 31% of subjects in an experiment in Fisman et al. (2007). We view the class of FA preferences as encompassing a broad range of designer preferences that may be relevant in practice.

2.3. **The Fairness-Accuracy Frontier.** Fixing any covariate vector X, we define the feasible set of group error pairs to be those pairs that can be implemented by some algorithm that takes X as input. The fairness-accuracy frontier is the set of all group error pairs that are FA-undominated in the feasible set.

Definition 4. The *feasible set* given covariate vector X is

$$\mathcal{E}(X) \equiv \{(e_r(a), e_b(a)) : a \in \mathscr{A}_X\}$$

Definition 5. The *fairness-accuracy (FA) frontier* given X, denoted $\mathscr{F}(X)$, is the set of all error pairs $e \in \mathcal{E}(X)$ that are FA-undominated, i.e. there does not exist another error pair $e' \in \mathcal{E}(X)$ satisfying $e' >_{FA} e$.

The FA frontier consists of all group error pairs that are optimal under some FA preference. Furthermore, it is minimal in the sense that every point in the FA frontier is uniquely optimal for some FA preference, so we cannot exclude any points without hurting some designer. We discuss these alternate characterizations in Appendix O.6.

3. The Fairness-Accuracy Frontier

In Section 3.1, we define the property of *group-balance* that will play a key role in several of our results. In Section 3.2, we characterize the frontier and its implications for the kinds of fairness-accuracy tradeoffs that emerge. In Section 3.3, we provide further results for two important special cases: when group identity is an input in the algorithm and when group identity is independent of type conditional on the covariate vector.

3.1. **Key Property: Group-Balance.** For all covariate vectors X, the feasible set $\mathcal{E}(X)$ is closed and convex (Lemma B.1). It includes the following special points.

Definition 6 (Group Optimal Points). For any covariate vector X, define

$$R_X \equiv \underset{(e_r,e_b)\in\mathcal{E}(X)}{\arg\min}\ e_r$$

to be the feasible point that minimizes group r's error, and define

$$B_X \equiv \underset{(e_r,e_b)\in\mathcal{E}(X)}{\arg\min}\ e_b$$

to be the feasible point that minimizes group b's error. In both cases, if the minimizer is not unique, break ties by choosing the point that minimizes the other group's error. We use G_X to denote the group optimal point for group g.

Group optimal points can be easily derived. For instance, to calculate R_X, set the algorithm to choose the optimal decision for group r for each realization of X (breaking ties in favor of group b).[16] R_X is then the error pair resulting from this algorithm.

Definition 7 (Fairness Optimal Point). For any covariate vector X, define

$$F_X \equiv \underset{(e_r,e_b)\in\mathcal{E}(X)}{\arg\min}\ |e_r - e_b|$$

to be the point that minimizes the absolute difference between group errors. If the minimizer is not unique, we choose the point that further minimizes either group's error.[17]

While R_X and B_X respectively denote the points that minimize group r and b's errors, the group whose error is minimized need not be the group with the lower error. For example, suppose X is a binary score where the conditional distribution $(X, Y) \mid G$ is described by:

$G = r$	$X=0$	$X=1$
$Y=0$	$3/8$	$1/8$
$Y=1$	$1/8$	$3/8$

$G = b$	$X=0$	$X=1$
$Y=0$	$1/3$	$1/6$
$Y=1$	$1/6$	$1/3$

Let the loss function ℓ be the misclassification rate; that is, $\ell(d, y, g) = \mathbb{1}(d \neq y)$. Then the b-optimal point B_X is achieved by the algorithm that maps $X = 1$ to $d = 1$ and $X = 0$ to $d = 0$, which leads to a *higher* error for group b than group r ($1/3$ compared to $1/4$). We will define such a covariate vector to be r-skewed.

Definition 8. Covariate vector X is:

- *r-skewed* if $e_r < e_b$ at R_X and $e_r \leq e_b$ at B_X
- *b-skewed* if $e_b < e_r$ at B_X and $e_b \leq e_r$ at R_X
- *group-balanced* otherwise

[16]Throughout, when we say "the optimal decision for group g at realization x," we mean any decision $d^* \in \arg\min_{d\in\mathcal{D}} \mathbb{E}[\ell(d, Y, g) \mid X = x, G = g]$.

[17]This point is the same regardless of which group is used to break the tie.

If X is g-skewed for either group g, then we say it is *group-skewed*.

In words, X is r-skewed if group r's error is smaller than group b's error not only at the r-optimal point R_X, but also at the b-optimal point B_X. Geometrically, this means that R_X and B_X fall to the same side of the 45 degree line. In contrast, the covariate vector X is group-balanced if at each group's optimal point, its error is lower than that of the other group, implying that R_X and B_X fall to opposite sides of the 45 degree line.

Loosely speaking, a covariate vector is group-balanced if it is possible to disentangle accurate predictions for one group from accurate predictions for another. This might be, for example, because the meaning of the covariate vector is group-dependent (e.g., larger realizations of X imply larger realizations of Y for group r but smaller realizations of Y for group b),[18] or because different covariates in the covariate vector are predictive for either group (e.g., $X = (X_1, X_2)$ where X_1 is uninformative about Y for group r and X_2 is uninformative about Y for group b). In contrast, we would expect a covariate vector to be group-skewed if it is systematically more informative about one group than the other (e.g., if $Y \mid X = x, G = r$ is more dispersed than $Y \mid X = x, G = b$ for every x).

3.2. **Characterization of the Frontier.** Depending on whether the covariate vector X is group-balanced or group-skewed, the fairness-accuracy frontier $\mathscr{F}(X)$ takes either of two forms. In the result below, we use *lower boundary between two points* to mean the part of the boundary of the set that lies between the two points and below the line segment connecting the two.

Theorem 1. *The fairness-accuracy frontier $\mathscr{F}(X)$ is the lower boundary of the feasible set $\mathcal{E}(X)$ between*

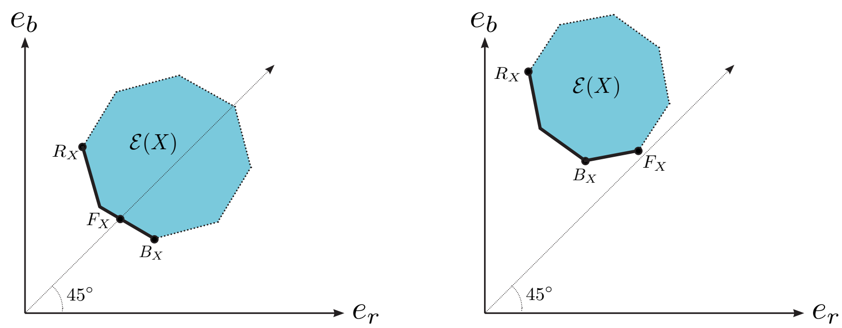
(a) *R_X and B_X if X is group-balanced*
(b) *G_X and F_X if X is g-skewed*

These two cases are depicted in Figure 3. When X is group-balanced and R_X and B_X are distinct, the two points fall on opposite sides of the 45-degree line (Panel (a)), and the fairness-accuracy frontier is that part of the lower boundary of the feasible set connecting these two points. This corresponds precisely to the set of all points (e_r, e_b) such that no other feasible point (e'_r, e'_b) is component-wise smaller, which we subsequently call the *Pareto frontier*. When X is r-skewed (Panel (b)), then both R_X and B_X fall on the same side of the 45-degree line, and the fairness-accuracy frontier consists not only of the usual Pareto

[18]For example, let subjects be borrowers, Y be creditworthiness, X be frequency of address changes, and G be an income bracket. Suppose frequent address changes (high X) signal higher creditworthiness for high-income borrowers (e.g., because these borrowers primarily move for new opportunities) but lower creditworthiness for low-income borrowers (e.g., because these borrowers primarily move due to evictions). Then the algorithm (based on this covariate) that maximizes accuracy for the high-income group will lead to a lower error for the high-income group, and vice versa.

frontier connecting R_X to B_X, but additionally a positively sloped line segment connecting the Pareto frontier to F_X.

Thus, the usual Pareto frontier and the fairness-accuracy frontier differ if and only if the covariate vector is group-skewed, implying the following corollary.



(a) X is group-balanced

(b) X is r-skewed

FIGURE 3. Example feasible set and fairness-accuracy frontier for (a) a group-balanced covariate vector X and (b) an r-skewed covariate vector X.

Corollary 1. *Suppose F_X is distinct from R_X and B_X. Then if and only if X is group-skewed, there are points $e, e' \in \mathscr{F}(X)$ satisfying $e_r \leq e'_r$ and $e_b \leq e'_b$ with at least one inequality strict.*

This corollary says that if the covariate vector is group-balanced, then no two points on the fairness-accuracy frontier can be Pareto-ranked. Thus, a policy proposal that increases errors for both groups, but reduces the gap between group errors, cannot be optimal under any fairness-accuracy preference. On the other hand, if inputs are group-skewed, then the frontier has a positively-sloped segment along which every pair of points can be Pareto-ranked. On this part of the frontier, the only way to decrease the gap in errors (given the available information) is to increase errors for both groups. In practice, moving along this part of the frontier could correspond to choosing to ignore certain available information.[19]

Suppose it were possible to acquire new covariates that turned a group-skewed covariate vector into a group-balanced covariate vector. Corollary 1 implies that such a change would not only (weakly) improve the fairness-accuracy frontier, but also change the nature of the fairness-accuracy conflict, eliminating the need to consider Pareto-dominated outcomes as

[19]The choice to exclude test scores from admissions decisions is arguably such an example—test scores are predictive of college grades for all of the relevant demographic groups (see Section A.5 of Systemwide Academic Senate (2020)), but are more predictive for applicants in some groups than others (Rothstein, 2004). In Section 4.2.2 we return to this application, interpreting the exclusion of test scores slightly differently—not as a choice made by the agent setting the algorithm, but as an informational regulation imposed by a designer whose preferences are different from those of the agent.

a means to improve fairness. It would be interesting to explore such implications in more detail in a model of endogenously chosen covariates.

3.3. **Special Cases.** In the important case where group identity is an algorithmic input, the feasible set and fairness-accuracy frontier simplify further.

Definition 9. Say that X *reveals* G if the conditional distribution $G \mid X = x$ is degenerate for every realization x of X.

Proposition 1. *Suppose X reveals G. Then the feasible set $\mathcal{E}(X)$ is a rectangle whose sides are parallel to the axes, and the fairness-accuracy frontier $\mathscr{F}(X)$ is the line segment from $R_X = B_X$ to F_X.*

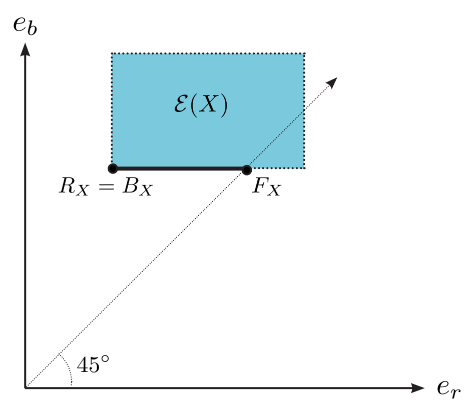


FIGURE 4. Example feasible set and fairness-accuracy frontier when X reveals G.

An example of such a feasible set and fairness-accuracy frontier are depicted in Figure 4. One endpoint, the Utilitarian-optimal point labeled $R_X = B_X$, gives both groups their minimal feasible error. The other endpoint, the Egalitarian-optimal F_X, maximizes fairness. Everywhere along the fairness-accuracy frontier $\mathscr{F}(X)$, the worse-off (higher error) group receives its minimal feasible error, so every point on the frontier is optimal for a Rawlsian designer. It is straightforward to see from this result that if we consider augmenting any covariate vector X to include G, the error for the group that was "worse-off" under X (i.e., had the higher error) must reduce regardless of which FA preference the designer holds.

Another interesting condition (nesting the previous) is one in which the covariates satisfy the following conditional independence property.

Definition 10. Say that X *creates conditional independence* if $G \perp\!\!\!\perp Y \mid X$.

A covariate vector that satisfies this property contains all of the information in group identity that is relevant for predicting Y.[20] We characterize the fairness-accuracy frontier for such covariate vectors, in the case that the loss function is group independent.

Proposition 2. *Suppose* $\ell(\cdot,\cdot,r) = \ell(\cdot,\cdot,b)$ *and* X *creates conditional independence. Then* $\mathscr{F}(X)$ *is that part of the lower boundary of the feasible set from the point* $B_X = R_X$ *to the point* F_X*.*

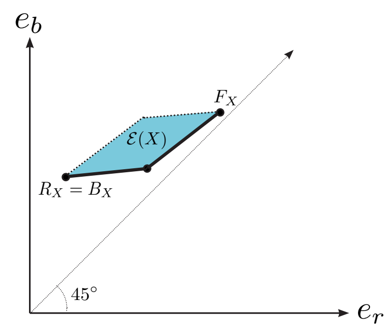


FIGURE 5. Depiction of the fairness-accuracy frontier under assumption of conditional independence of G and Y.

Figure 5 depicts an example fairness-accuracy frontier for a covariate vector satisfying Conditional Independence. The left point is the (shared) group optimal point $R_X = B_X$, and the right endpoint is the fairness optimal point F_X. From $R_X = B_X$ to F_X, the fairness-accuracy frontier consists entirely of positively sloped line segments. Thus, everywhere along the frontier, the two groups' errors move in the same direction, implying that the only way to improve fairness is to decrease accuracy uniformly across groups, and that the only difference across designers that matters is how they choose to resolve strong fairness-accuracy conflicts of this form.

4. Input Design

We have so far assumed that the designer directly chooses the best algorithm to maximize a preference that (weakly) responds to both fairness and accuracy. This is a good description of some settings; for example, a company may internalize both fairness and accuracy concerns in its hiring algorithm. But often the algorithm is set by an agent who does not care about fairness across groups, while the inputs used by the algorithm are constrained by a designer who does. For example, a judge (agent) determining sentencing may seek to maximize the

[20]This kind of conditional independence appears for example when the coefficient on group identity is zero in a regression of Y on observables, e.g. Ludwig and Mullainathan (2021) find that race (G) is not predictive of a criminal's risk (Y) conditional on arrest (X) in their data.

number of correct verdicts, while a policymaker (designer) may additionally prefer that the accuracy of the judge's verdicts is equitable across certain social groups. Or, a bank (agent) may seek to maximize profit from loan issuance, while a regulator (designer) may require that the rate at which individuals are incorrectly denied loans does not differ too sharply across groups. In these settings, the designer can often influence the algorithm indirectly by passing regulation that constrains the algorithm's inputs. For example, Chan and Eyster (2003) report that as part of an effort to influence Berkeley law school's admissions policy in 1997, UC Berkeley administrators coarsened candidates' LSAT scores into intervals and reported this coarsened variable to the law school admissions committee.

In Section 4.1, we model this interaction as an information design problem in which the designer constrains the inputs of the algorithm, while the algorithm is chosen by an accuracy-minded agent. In Section 4.2, we ask whether the designer should completely exclude an input such as group identity or a test score.

4.1. **Input Design Versus Algorithm Design.** A designer chooses a *garbling* of the covariate vector X, which is represented as a stochastic map $T : \mathcal{X} \to \Delta(\mathcal{T})$ taking realizations of X into distributions over the possible realizations of T (assumed without loss to be finite). Examples include:

Example 9 (Banning an Input). $X = (X_1, X_2, X_3)$ and $T(x_1, x_2, x_3) = (x_1, x_2)$ with probability 1.

Example 10 (Coarsening the Input). The set of realizations $\mathcal{X} = \{1, 2, 3, 4\}$ is partitioned into $\{\{1, 2\}, \{3, 4\}\}$, and $T(x)$ reports (with probability 1) the partition element to which x belongs.

Example 11 (Adding Noise). $T(x) = x + \varepsilon$ where the noise term ε takes value $+1$ or -1 with equal probability.

We view these garblings as information policies that the designer can plausibly commit to by law. Real examples of garblings are abundant: The "ban-the-box" campaign (Agan and Starr, 2018) restricted employers from using criminal history as an input into hiring decisions (similar to Example 9); the College Board coarsens a test-taker's answers into an integer-valued score between 400 and 1600 (similar to Example 10); and organizations such as the US Census Bureau, Apple, and Google add noise to users' inputs under differential privacy initiatives (similar to Example 11).[21]

The agent chooses an algorithm $a : \mathcal{T} \to \Delta(\mathcal{D})$ that takes as input the garbled variable chosen by the designer. The agent's utility function is

$$-\alpha_r \cdot e_r(a) - \alpha_b \cdot e_b(a)$$

[21]See Garfinkel et al. (2018) for an example reference.

for some constants $\alpha_r, \alpha_b \geq 0$, with the special case $\alpha_g = p_g$ returning the Utilitarian preference.[22,23] (We prove additional results in Appendix O.4 for the case in which some coefficient α_g is negative, in which case the agent prefers to *increase* error for one of the two groups, thus falling outside of our class of FA preferences.) We can rewrite this utility as

$$\begin{aligned}\alpha_r e_r(a) + \alpha_b e_b(a) &= \sum_g \alpha_g \mathbb{E}\left[\ell\left(a\left(T\right), Y, g\right) \mid G = g\right] \\ &= \sum_{t \in \mathcal{T}} p_t \sum_{y,g} \frac{\alpha_g}{p_g} \cdot \mathbb{P}\left(Y = y, G = g \mid T = t\right) \cdot \ell\left(a(t), y, g\right),\end{aligned}$$

where p_t is the probability of $T = t$. Thus the agent's problem of minimizing ex-ante error is equivalent to the following ex-post problem[24]

$$a\left(t\right) \in \arg\min_{d \in \mathcal{D}} \sum_{y,g} \frac{\alpha_g}{p_g} \cdot \mathbb{P}\left(Y = y, G = g \mid T = t\right) \cdot \ell\left(d, y, g\right). \tag{1}$$

Definition 11. The pair of group errors (e_r, e_b) is *implemented by* T if there exists an algorithm a_T satisfying (1) such that $(e_r, e_b) = (e_r(a_T), e_b(a_T))$.

Definition 12. The *input-design feasible set* given X consists of all error pairs that the designer can implement using a garbling of X:

$$\mathcal{E}^*(X) \equiv \{(e_r, e_b) : (e_r, e_b) \text{ is implemented by a garbling } T \text{ of } X\}.$$

The *input-design fairness-accuracy frontier* $\mathscr{F}^*(X)$ is the set of error pairs $e \in \mathcal{E}^*(X)$ with the property that no other $e' \in \mathcal{E}^*(X)$ satisfies $e' >_{FA} e$.

The following proposition says that under relatively weak conditions, it is without loss to have control only of the algorithm's inputs: Any error pair that a designer would choose to implement in the unconstrained problem (i.e., given control of the algorithm) can also be achieved under input design. To state the result, we define

$$e_0 = \min_{d \in \mathcal{D}} \left(\alpha_r \cdot \mathbb{E}[\ell(d, Y, r) \mid G = r] + \alpha_b \cdot \mathbb{E}[\ell(d, Y, b) \mid G = b]\right)$$

to be the best payoff that the agent can achieve given no information, and

$$H = \{(e_r, e_b) : \alpha_r e_r + \alpha_b e_b \leq e_0\}$$

[22]The agent's utility may involve weights different from the utilitarian weights if errors for the two groups are differentially costly for the agent. For example, suppose the agent is a bank manager and group b is wealthier than group r. In this case, loans for group b may be of higher value, so that incorrectly classifying creditworthy individuals in group b is more costly. This corresponds to scaling the loss ℓ for group b by $\alpha_b/p_b > 1$.

[23]We view the typical setting as one in which the regulator has fairness concerns that the agent does not share, but the reverse case (in which the agent has fairness concerns that the regulator does not share) is also interesting. See Section 5 for a brief discussion of some technical complications that arise in this case.

[24]When the agent's utility is non-linear in group errors, the ex-ante and ex-post problems are not equivalent in general.

to be the halfspace including all error pairs that improve the agent's payoff relative to no information.

Proposition 3 (When Input Design is Without Loss)**.** *The following hold:*

(a) *Suppose X is group-balanced. Then, $\mathscr{F}^*(X) = \mathscr{F}(X)$ if and only if $R_X, B_X \in H$.*
(b) *Suppose X is g-skewed. Then, $\mathscr{F}^*(X) = \mathscr{F}(X)$ if and only if $G_X, F_X \in H$.*

This result follows from the subsequent lemma, which says that the input-design feasible set is equal to the intersection of the unconstrained feasible set and H, with an analogous statement relating the fairness-accuracy frontiers. A version of this lemma has been demonstrated in Alonso and Câmara (2016) and Ichihashi (2019), although we provide an independent argument in Appendix 1 for completeness.

Lemma 1. *For every covariate vector X, the input-design feasible set is $\mathcal{E}^*(X) = \mathcal{E}(X) \cap H$ and the input-design fairness-accuracy frontier is $\mathscr{F}^*(X) = \mathscr{F}(X) \cap H$.*

Clearly the designer cannot hold the agent to a payoff lower than what the agent can guarantee with no information, so $\mathcal{E}^*(X) \subseteq \mathcal{E}(X) \cap H$. In the other direction, we need to show that every point in $\mathcal{E}(X) \cap H$ can be implemented by a garbling of X. The proof is by construction: If the designer garbles X into recommendation of the decision, then the obedience constraints reduce precisely to the condition that the agent's payoff is improved relative to no information, i.e., the error pair belongs to H. This yields the lemma, and Figure 6 provides an illustration of how Proposition 3 is implied by Lemma 1.

These results tell us that input design is always sufficient to recover part of the original fairness-accuracy frontier. Moreover, so long as certain points (R_X and B_X in the case of a group-balanced X, R_X and F_X in the case of an r-skewed X, or B_X and F_X in the case of a b-skewed X) improve the agent's payoffs relative to no information, then the designer can induce the agent to choose the designer's most preferred outcome even without explicit control of the algorithm. Conversely, when these conditions do not hold, then input design is limiting for some designers.

4.2. **Excluding a Covariate.** Constraints on algorithmic inputs sometimes take the form of a ban on use of a specific covariate. For example, protected group identities such as race, religion and gender are illegal inputs into lending and hiring decisions,[25] and the University of California university system recently excluded consideration of standardized test scores from their admissions decisions.[26]

[25]For example, the Equal Opportunity Act forbids any creditor to discriminate on the basis of "race, color, religion, national origin, sex or marital status, or age" (see `https://files.consumerfinance.gov/f/201306_cfpb_laws-and-regulations_ecoa-combined-june-2013.pdf`), and Title VII of the Civil Rights Act prohibits discrimination by employers on the basis of "race, color, religion, sex, or national origin" except in cases where the protected trait is an occupational qualification.

[26]See `https://www.nytimes.com/2021/05/15/us/SAT-scores-uc-university-of-california.html`.

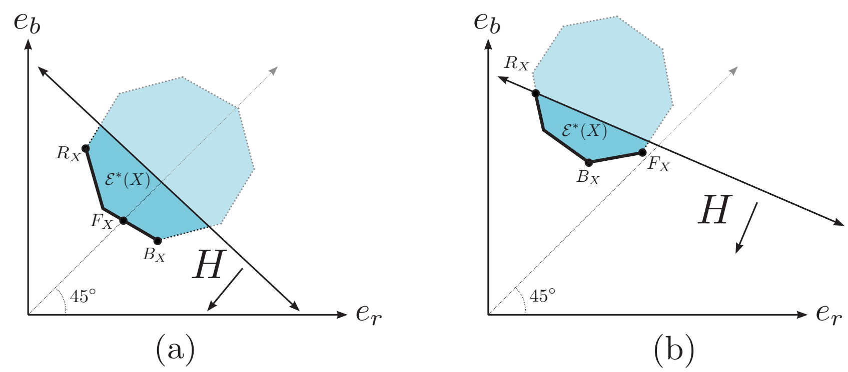


FIGURE 6. Depiction of an example input-design fairness-accuracy frontier for (a) a group-balanced covariate vector X and (b) an r-skewed covariate vector X. In Panel (a), it is sufficient to check $R_X, B_X \in H$ to determine whether the entire unconstrained fairness-accuracy frontier belongs to H. In Panel (b), it is sufficient to check whether $R_X, F_X \in H$. This condition is failed in the figure, so some designer cannot implement his favorite unconstrained outcome using input design.

Since the designer and agent have (potentially) misaligned preferences, it can be optimal for the designer to ban an input.[27] But for two important classes of inputs, we will show that excluding the input is strictly worse for all designers with FA-preferences.

Definition 13. Say that *excluding covariate vector X' over X uniformly worsens the (input design) frontier* if every point in $\mathscr{F}^*(X)$ is FA-dominated by a point in $\mathscr{F}^*(X, X')$.

To interpret this condition, recall that $\mathscr{F}^*(X)$ is the frontier of error pairs that can be implemented by some garbling of X, while $\mathscr{F}^*(X, X')$ is the frontier of error pairs that can be implemented by some garbling of (X, X'). So any point that belongs to $\mathscr{F}^*(X, X')$ but not to $\mathscr{F}^*(X)$ can only be implemented if the garbling chosen by the designer includes information about X'. When excluding X' over X uniformly worsens the frontier, then no designer's optimal garbling excludes X', and so a ban on X' is not optimal for any designer in our class.

4.2.1. *Excluding Group Identity.* First let $X' = G$, so that the comparison is between the frontier implemented by garblings of X and the frontier implemented by garblings of (X, G). The property of group balance (suitably strengthened) turns out to be critical for whether exclusion of G uniformly worsens the frontier.

Definition 14. Say that X is *strictly group-balanced* if $e_r < e_b$ at R_X and $e_b < e_r$ at B_X.

[27]Adding group identity leads to a Blackwell-improvement in information; thus, it is well understood that access to this variable must weakly improve the designer's payoffs when the designer has control of the algorithm (see Menon and Williamson (2018), Agarwal et al. (2018), Lipton et al. (2018), and Rambachan et al. (2021) among others). This is no longer generally the case when the designer cannot choose the algorithm.

Relative to group-balance, strict group-balance rules out covariate vectors X for which $R_X = B_X = F_X$.

Proposition 4. *Suppose $R_X, B_X \in H$. Then, excluding G over X uniformly worsens the frontier if and only if X is strictly group-balanced.*[28]

To show this result, we first demonstrate that the minimal (and maximal) feasible error for both groups is the same given X and given (X, G). Geometrically, this means that the feasible set given (X, G) is the smallest rectangle containing the feasible set given X. When X is group-balanced, then $\mathscr{F}^*(X)$ is characterized by Part (a) of Theorem 1 while $\mathscr{F}^*(X, G)$ is characterized by Proposition 1 (using the equivalence in Proposition 3 for both cases). As depicted in Panel (a) of Figure 7, the fairness-accuracy frontier given X does not intersect with the frontier given (X, G), so every point on the frontier given X is FA-dominated by a point on the frontier given (X, G). On the other hand, when X is group-skewed, the two frontiers necessarily overlap as depicted in Panel (b).

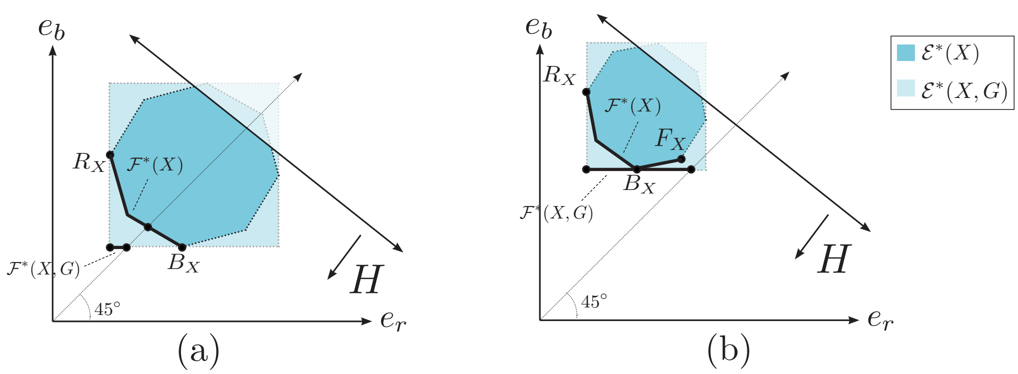


FIGURE 7. (a) X is strictly group-balanced and excluding G over X uniformly worsens the input-design frontier; (b) X is r-skewed and excluding G over X does not uniformly worsen the input-design frontier.

Proposition 4 says that so long as X is strictly group-balanced, then every designer is made strictly better off by being given access to group identity.[29] That is, every designer can find a way of combining the information in G and X—for example, by adding noise to X for individuals in one group but not the other—which induces the agent to choose an algorithm that the agent would not have chosen given any garbling of X alone. In contrast, if X is not strictly group-balanced, then there is at least one designer for whom no garbling of (X, G)

[28]The assumption $R_X, B_X \in H$ makes the above result easier to state as an if-and-only-if condition. But it follows from our proof of Proposition 4 that even when this assumption fails, strict group-balance is a sufficient condition for the frontier to uniformly worsen when excluding G.

[29]We show in Appendix O.4 that this result extends even to a case where the agent is adversarial against one of the groups (i.e., preferring to increase that group's error) so long as the agent is not "too strongly" adversarial.

strictly improves over garblings of X. For example, we see in Panel (b) of Figure 7 that the Rawlsian designer's payoffs is not improved by access to G.

Our results complement papers such as Chan and Eyster (2003), which compare choice between decision rules based on (X, G) to choice between decision rules based on X alone (in a single-agent setting), and show that the latter can be worse in terms of both equity and efficiency. Our analyses are distinct for a few reasons. First, we consider a strategic interaction between a designer and agent with misaligned preferences, so it may be that the designer prefers not to give the agent access to group identity because of how the agent will use this information, even while there is *some* use of this information that the designer would consider an improvement. Second, the property of a uniform worsening of the frontier does not in general rank the information policy of revealing X versus revealing (X, G). That is, it may be that excluding G over X uniformly worsens the frontier, but the designer's payoff is lower from revealing (X, X') than from revealing X alone.

Nevertheless, our result relates to and builds on previous findings that *disparate treatment* (use of different rules for individuals in different groups) may be necessary to preclude *disparate impact* (disparate harms across groups).[30] Specifically, Proposition 4 implies that to reduce disparate impact, it may be necessary to impose information policies that are asymmetric across groups. Interestingly, this may not involve sending G as an input, so the algorithm can be formally group-blind (thus not exhibiting disparate treatment).[31] Nevertheless, if we consider the total procedure—taking into account both information design and algorithm design—then two individuals who are otherwise identical but belong to different groups may receive different distributions of outcomes. This distinction brings up an interesting question regarding how disparate treatment should be conceptualized in settings where both information design and algorithm design are present.

4.2.2. *Excluding a Covariate When Group Identity is Known.* Next compare the frontier implemented by garblings of (X, G) with the frontier implemented by garblings of (X, G, X'), where X and X' are arbitrary covariate vectors.

Definition 15. Say that X' is *decision-relevant over X for group g* if there are realizations (x, x') and $(x, \tilde{x}')$ of (X, X') that have strictly positive probability conditional on $G = g$, where

$$\{1\} = \arg\min_{d \in \mathcal{D}} \mathbb{E}[\ell(d, Y, g) \mid X = x, X' = x', G = g]$$

[30]This tension between disparate treatment and disparate impact is noted in explicitly in works such as Chouldechova (2017) and Rambachan et al. (2021), and is implied by results in Chan and Eyster (2003).
[31]The algorithm exhibits disparate treatment if, holding all other covariates equal, it yields different outputs depending on the individual's group identity. See `https://www.justice.gov/crt/book/file/1364106/download` for definitions of disparate treatment and impact.

while

$$\{0\} = \arg\min_{d \in \mathcal{D}} \mathbb{E}[\ell(d, Y, g) \mid X = x, X' = \tilde{x}', G = g].$$

This weak condition requires only that there is some individual in group g for whom the decision that maximizes (expected) accuracy is different given X and given (X, X'). For example, if X' is a test score and X is high school GPA, then X' is decision-relevant for group g when taking the test score into consideration reverses the admission decision for at least one individual in group g relative to the decision based on GPA alone. Systemwide Academic Senate (2020) report that test scores indeed satisfy this property for relevant demographic groups.[32]

Proposition 5. *Choose arbitrary covariate vectors X and X'.*

(a) *If (X, G) is g-skewed, then excluding X' over X uniformly worsens the frontier if and only if X' is decision-relevant over X for group $g' \neq g$.*
(b) *If (X, G) is group-balanced, then excluding X' over X uniformly worsens the frontier if and only if X' is decision-relevant over X for both groups.*

When X' is decision-relevant over X for the disadvantaged group, then the minimal feasible error for that group given (X, G, X') is strictly lower than the minimal feasible error given (X, G) only. So the fairness-accuracy frontier is pushed towards the origin (either downwards or towards the left), as in Panel (a) of Figure 8. On the other hand, when X' is not decision-relevant over X for the disadvantaged group, then the new fairness-accuracy frontier must remain a line that overlaps with the previous frontier (see Panel (b) of Figure 8), so there is some FA preference for which excluding X' is at least weakly (and possibly strictly) worse. This yields part (a) of the result. Part (b) pertains to a knife-edge case: If (X, G) is group-balanced then the minimal feasible error is the same for both groups. For a uniform worsening of the frontier to occur, access to X' over X must reduce the minimal feasible error for both groups.

One application of Proposition 5 relates to the question of whether to ban test scores in admissions decisions. Our result suggests that so long as group identities are permissible inputs for college admission decisions, then excluding test scores is welfare-reducing for all designers with the ability to garble available covariates. On the other hand, if group identity is not permitted as an input into college admissions decisions, then it may be better for a sufficiently fairness-minded designer to completely exclude test scores. With regards to the pending Supreme Court case *Students for Fair Admissions, Inc. v. President and Fellows of Harvard College*, our result suggests that if affirmative action is banned nationwide, then

[32]Specifically, Section A of Systemwide Academic Senate (2020) finds that test scores are predictive of college success, predictive above other covariates (such as GPA), and and predictive for all demographic groups that they consider (with individuals disaggregated by factors such as parental education, family income, and racial/ethnic identity).

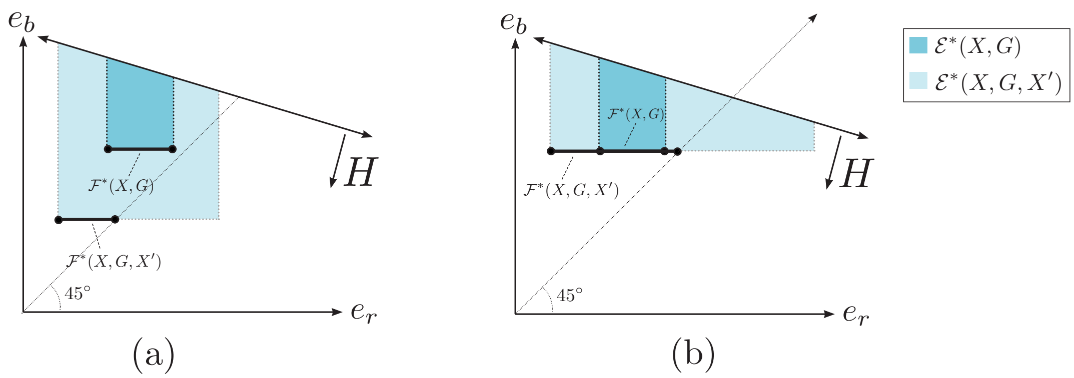


FIGURE 8. (a) Example in which X' is decision-relevant for group b, and excluding X' uniformly worsens the frontier; (b) Example in which X' is not decision-relevant for group b, and excluding X' does not uniformly worsen the frontier.

universities with certain FA preferences will have more reason to ban use of test scores in admissions decisions.

While our framework abstracts away from many important features of the college admissions process—including access to testing (Garg et al., 2021) and test-optional admissions (Dessein et al., 2022))—the link between the availability of group identity and the value of additional information, such as test scores, is one that we believe holds more generally. The crucial point is that when group identity is available, then the designer can tailor how the additional information is used for each group separately. For example, the designer could selectively report test scores only for standout students in the disadvantaged group.[33] In this sense, access to group identity has a positive spillover effect for the value of other covariates, guaranteeing that there is some (possibly group-dependent) garbling of the other information that aligns the agent and designer's incentives.

We conclude with the following simple example, which illustrates the contrast between access to an auxiliary covariate X' alone versus access to the pair (X', G).

Example 12. Suppose $\mathcal{Y} = \{0,1\}$ and Y and G are independently and uniformly distributed, i.e., $\mathbb{P}(Y = y, G = g) = 1/4$ for any $y \in \{0,1\}$ and $g \in \{r,b\}$. Let X be a null signal; that is, $X = x_0$ with probability one. Further let X' be a binary signal with the following

[33]Indeed, Systemwide Academic Senate (2020) reports that one use of test scores at UC Berkeley (prior to the university's move to test-blind admissions in 2021) was to identify otherwise ineligible applicants from relatively disadvantaged backgrounds.

conditional probabilities $\mathbb{P}(X' \mid Y, G)$: [34]

	$X'=1$	$X'=0$
$Y=1$	1	0
$Y=0$	0	1

$G=r$

	$X'=1$	$X'=0$
$Y=1$	0.6	0.4
$Y=0$	0.4	0.6

$G=b$

Thus, X' is perfectly informative about the individuals in group r, and imperfectly informative about those in group b. Suppose the loss function is $\ell(d,y,g)=\mathbb{1}(d\neq y)$, and the agent is Utilitarian ($\alpha_r=p_r=1/2$ and $\alpha_b=p_b=1/2$).

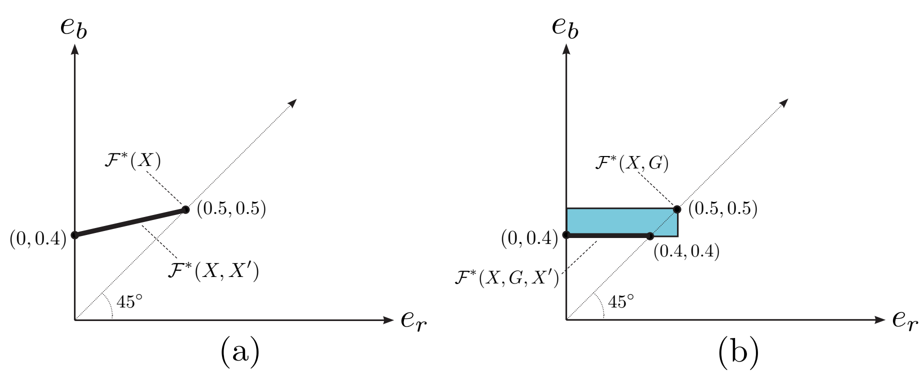


FIGURE 9. (a) A comparison of the input-design fairness-accuracy frontiers given X versus given (X,X'); (b) A comparison of the input-design fairness-accuracy frontiers given (X,G) versus given (X,G,X')

.

The input-design feasible set given X only is the singleton $\{(0.5,0.5)\}$, which delivers a payoff of 0 to the Egalitarian designer. But if the designer chooses any nontrivial garbling of (X,X'), the agent will use what he learns about X' to maximize aggregate accuracy. Since this information is inevitably more informative about group r than about group b, conditioning decisions on this information increases the gap between the two group errors, reducing the designer's payoff.[35] So it is strictly optimal for the designer to exclude all information about X'. In more detail, the fairness-accuracy frontier given (X,X') is the line segment connecting $(0,0.4)$ with $(0.5,0.5)$ (see Panel (a) of Figure 9),[36] and any nontrivial garbling of (X,X') leads to a point on this frontier that is different from $(0.5,0.5)$, yielding a strictly negative payoff for the designer.

[34]In this example, neither covariates X nor X' reveal group identity. Thus, this example falls outside of the settings considered in the previous two subsections.

[35]While we assume an Egalitarian designer here for simplicity, a similar construction is possible for any designer who places sufficient weight on fairness considerations.

[36]Indeed this is also the input-design feasible set. See Appendix B.9 for details.

In contrast, Panel (b) of Figure 9 demonstrates the comparison between the fairness-accuracy frontiers $\mathcal{F}^*(X,G)$ and $\mathcal{F}^*(X,G,X')$. Here we see that the Egalitarian designer is able to achieve the superior outcome $(0.4, 0.4)$ by choosing an appropriate garbling of (X,G,X'). Thus while making information about X' available to the agent is strictly harmful for the designer when group identity is not available, this ceases to be true once the designer can condition the garbling of X' on G.

5. Extensions

5.1. **Different loss functions for evaluating fairness and accuracy.** When defining the partial order $>_{FA}$ we use the same loss function to evaluate accuracy and fairness. In some cases, the designer may wish to evaluate accuracy using one loss function and fairness using another. (For example, the designer may wish to minimize the misclassification rate subject to equality of false positive rates.) In Appendix O.1 we develop a more general version of our framework that allows for different loss functions, and extend Theorem 1 under an assumption that the accuracy and fairness loss functions are not "directly opposed" to one another. In this result, our group-balance condition is generalized to a condition of whether the fairness-maximizing point F_X belongs to usual Pareto frontier. When this condition is satisfied, then the fairness-accuracy frontier is identical to the usual Pareto frontier; otherwise, the fairness-accuracy frontier is the union of the Pareto frontier and a positively-sloped sequence of lines, along which every pair of points has the property that one point involves higher errors for both groups but greater fairness.

5.2. **Beyond absolute difference for evaluating fairness.** Our main analysis assumes that (un)fairness is evaluated according to the absolute difference of errors between the two groups, i.e. $|e_r - e_b|$. A natural extension is to consider $|\phi(e_r) - \phi(e_b)|$ where ϕ is some continuous strictly increasing function. For instance, if ϕ is log, then this corresponds to evaluating fairness using the ratio of errors rather than their difference. Our main characterization (Theorem 1) holds for any such ϕ with the fairness optimal point F_X suitably defined.[37] We further demonstrate that the frontier becomes larger (smaller) whenever ϕ is concave (convex). Thus, for example, evaluating fairness using ratios instead of absolute difference results in a larger frontier, although the qualitative properties of this frontier are unchanged.

5.3. **Other agent preferences in the input design problem.** Section 4 considers misaligned incentives between a designer controlling inputs and an agent setting the algorithm. There, we assume that the agent cares about accuracy and prefers for both group errors

[37]To see why, first note that no interior point can be on the frontier. Otherwise, we can always find some $\epsilon_1, \epsilon_2 > 0$ such that $|\phi(e_r - \epsilon_1) - \phi(e_b - \epsilon_2)| \leq |\phi(e_r) - \phi(e_b)|$ so $(e_r - \epsilon_1, e_b - \epsilon_1) >_{FA} (e_r, e_b)$ yielding a contradiction. The rest of the proof follows as in Theorem 1.

to be lower. In Appendix O.4, we consider what happens when this misalignment is more extreme and the agent is adversarial (i.e. negatively biased) towards one of the two groups, preferring that group's error to be higher. We generalize several results from Section 4 and show that even if the agent is negatively biased, it can still be optimal for the designer to provide information about group identity (so long as the bias is not too extreme).

Two other potential generalizations would permit the agent and designer to have different loss functions, or permit the agent to care about fairness.[38] In both cases, the set of points that the agent prefers over the prior (what we defined to be H) is no longer a halfspace from the designer's perspective. Moreover, non-linearities in the agent's objective function imply that the agent's ex-ante and ex-post problems may be different, and so it is relevant whether the agent commits to the algorithm or chooses the decision after the realization of the garbling. We consider these problems beyond the scope of the present paper, and leave them as open questions for future work.

5.4. **Capacity constraints.** In our main model, we allow the designer unconstrained choice of any algorithm. In a few of the applications of interest, there may be an additional capacity constraint on the algorithm, e.g., if only a fixed number of students can be admitted in admissions decisions. One way to formulate a capacity constraint is a restriction on the ex-ante probability of assignment of decision $d = 1$ (e.g., admit). In this case, the set of error pairs satisfying the constraint can be shown to be a convex set, so the feasible set is simply the intersection between the feasible set (as we have defined) and the convex set of error pairs that satisfy this capacity constraint. Our Theorem 1 then applies for this new feasible set, although the fairness-accuracy frontier as characterized in Proposition 1 may no longer be a horizontal line.

5.5. **More than two groups or two decisions.** We have assumed that there are two groups $\mathcal{G} = \{r, b\}$. Some of our results, such as Proposition 3, can be shown to directly extend for any finite $\mathcal{G}$. However, in order to extend our other results, we would first have to specify a definition of fairness for multiple groups. One possible generalization of the FA-dominance relationship is to say that a vector of group errors $(e_g)_{g\in\mathcal{G}}$ FA-dominates another vector $(e'_g)_{g\in\mathcal{G}}$ if $e_g \leq e'_g$ for every group g, and also $|e_g - \frac{1}{|\mathcal{G}|}\sum_{g\in\mathcal{G}} e_g| \leq |e'_g - \frac{1}{|\mathcal{G}|}\sum_{g\in\mathcal{G}} e'_g|$ for every $g \in \mathcal{G}$, with at least one inequality holding strictly. That is, fairness is improved if each group's error is closer to the average group error. We expect our characterization in Theorem 1 to extend qualitatively in this case.

We have also assumed that there are two decisions $\mathcal{D} = \{0, 1\}$. All of our results in Section 3 about the unconstrained problem directly extend for any finite $\mathcal{D}$. However, Lemma 1 (the relationship between the input-design fairness-accuracy frontier and the unconstrained

[38]Our result does include the special case when the agent's loss function $\ell_a = \alpha_g \ell_d$ is just a group-specific multiple of the designer's loss function. This is mathematically equivalent to the setup in Section 4

fairness-accuracy frontier) relies on the assumption of a binary decision. We leave a characterization of the input design frontier for this more general case to future work.

Appendix A. Fairness Criteria in the Literature

We review here certain fairness criteria that have appeared in the literature, and explain how these criteria can be accommodated within our framework.

A.1. **Statistical Parity.** This criterion seeks equality in decisions, namely that the proportion of either group receiving the two decisions is the same (Dwork et al., 2012). Formally, an algorithm a satisfies statistical parity if

$$\mathbb{E}(a(X) = 1 \mid G = r) - \mathbb{E}(a(X) = 1 \mid G = b) = 0$$

The loss function

$$\ell(d, y, g) = \begin{cases} 1 & \text{if } d = 1 \\ 0 & \text{otherwise} \end{cases}$$

returns a relaxed version of this criterion, since

$$e_g(a) = \mathbb{E}\left[\ell(a(X), Y, g) \mid G = g\right] = \mathbb{E}\left[a(X) = 1 \mid G = g\right]$$

so $|e_r(a) - e_b(a)|$ is the absolute difference in the probability that a group-r individual and a group-b individual receive the decision $d = 1$.

A.2. **False Positives.** Another common fairness criterion is equality of false positives across two groups (Angwin and Larson, 2016; Chouldechova, 2017; Kleinberg et al., 2017). For example, among borrowers who would not have defaulted on their loan if approved, prediction of default should be equal across the two groups. Formally, an algorithm a satisfies equality of false positive rates if

$$\mathbb{E}(a(X) = 1, Y = 0 \mid G = r) - \mathbb{E}(a(X) = 1, Y = 0 \mid G = b) = 0$$

The loss function

$$\ell(d, y, g) = \begin{cases} 1 & \text{if } (d, y) = (1, 0) \\ 0 & \text{otherwise} \end{cases}$$

returns a relaxed version of this criterion, since

$$e_g(a) = \mathbb{E}\left[\ell(a(X), Y, g) \mid G = g\right] = \mathbb{E}\left[a(X) = 1, Y = 0 \mid G = g\right]$$

is the false-positive rate for group g, and so $|e_r(a) - e_b(a)|$ is the absolute difference in false positive rates. A fairness criterion based on the difference in false negative rates can be accommodated similarly.

A.3. **Equalized Odds.** Another popular fairness criterion asks for equalized odds (Hardt et al., 2016), which an algorithm a satisfies if

$$\mathbb{E}_Y[\mathbb{E}_X[a(X) \mid G=r, Y] - \mathbb{E}_X[a(X) \mid G=b, Y]] = 0 \tag{A.1}$$

The inner difference compares the average decision for group-r and group-b individuals who share the same type Y, and the outer expectation averages over those values of Y.

The loss function

$$\ell(d,y,g) = \begin{cases} \frac{P(Y=y)}{P(Y=y|G=g)} & \text{if } d=1 \\ 0 & \text{otherwise} \end{cases}$$

returns a relaxed version of this criterion, since

$$\begin{aligned}
\mathbb{E}[\ell(d,y,g) \mid G=r] = P(Y=0 \mid G=r) \times \mathbb{E}\left[\frac{P(Y=0)}{P(Y=0 \mid G=r)} \times \mathbb{1}(d=1) \mid G=r, Y=0\right] \\
+ P(Y=1 \mid G=r) \times \mathbb{E}\left[\frac{P(Y=1)}{P(Y=1 \mid G=r)} \times \mathbb{1}(d=1) \mid G=r, Y=1\right] \\
= P(Y=0) \times \mathbb{E}[\mathbb{1}(d=1) \mid G=r, Y=0] \\
+ P(Y=1) \times \mathbb{E}[\mathbb{1}(d=1) \mid G=r, Y=1]
\end{aligned}$$

so $|\mathbb{E}[\ell(a(X),Y,G) \mid G=r] - \mathbb{E}[\ell(a(X),Y,G) \mid G=b]|$ is exactly the LHS of (A.1).

Appendix B. Proofs for Main Text Results

B.1. Characterization of the Feasible Set.

Lemma B.1. *The full-design feasible set $\mathcal{E}(X)$ is a closed and convex polygon.*

Proof. Given algorithm a, we slightly abuse notation to let $a(x)$ denote the probability of choosing decision $d=1$ at covariate vector x. We further let $x_{y,g}$ denote the conditional probability that $Y=y$ and $G=g$ given $X=x$. Finally, let p_x denote the probability of $X=x$. Then the group errors can be written as follows:

$$\begin{aligned}
e_g(a) &= \mathbb{E}\left[a\left(X\right)\ell\left(1,Y,g\right) + \left(1-a\left(X\right)\right)\ell\left(0,Y,g\right) \mid G=g\right] \\
&= \sum_x \left(a\left(x\right) \sum_y \frac{x_{y,g}}{p_g} \ell\left(1,y,g\right) + \left(1-a\left(x\right)\right) \sum_y \frac{x_{y,g}}{p_g} \ell\left(0,y,g\right) \right) \cdot p_x,
\end{aligned}$$

where p_g is the prior probability that $G=g$. The set of all feasible errors is given by

$$\mathcal{E}\left(X\right) = \left\{ \left(e_r\left(a\right), e_b\left(a\right)\right) \,:\, a(x) \in [0,1] \; \forall x \in \mathcal{X} \right\}.$$

If we let

$$E\left(x\right) := \left\{ \lambda \left(\sum_y \frac{x_{y,r}}{p_r} \ell\left(1,y,r\right), \sum_y \frac{x_{y,b}}{p_b} \ell\left(1,y,b\right) \right) \right.$$

$$+ (1-\lambda)\left(\sum_y \frac{x_{y,r}}{p_r}\ell(0,y,r), \sum_y \frac{x_{y,b}}{p_b}\ell(0,y,b)\right) : \lambda \in [0,1]\Bigg\}$$

represent a line segment in $\mathbb{R}^2$, then we see that

$$\mathcal{E}(X) = \sum_{x \in \mathcal{X}} E(x) \cdot p_x.$$

This is a (weighted) Minkowski sum of line segments, which must be a closed and convex polygon. □

B.2. **Proof of Theorem 1.** First observe that the FA frontier must be part of the boundary of the feasible set $\mathcal{E}(X)$, because any interior point (e_r, e_b) is FA-dominated by $(e_r - \epsilon, e_b - \epsilon)$ which is feasible when ϵ is small.

Consider the group-balanced case, where R_X lies weakly above the 45-degree line and B_X lies weakly below. If $R_X = B_X$, then this point simultaneously achieves minimal error for both groups, as well as minimal unfairness since it must be on the 45-degree line. In this case it is clear that the fairness-accuracy frontier consists of that single point, which FA-dominates every other feasible point. Another degenerate case is when the entire feasible set $\mathcal{E}(X)$ consists of the line segment $R_X B_X$. Here again it is easy to see that the entire line segment is FA-undominated, and the result also holds.

Next we show that the upper boundary of $\mathcal{E}(X)$ connecting R_X to B_X (excluding R_X and B_X) is FA-dominated. One possibility is that the upper boundary consists entirely of the line segment $R_X B_X$. Take any point Q on this line segment, and through it draw a line parallel to the 45-degree line. Then this line intersects the boundary of $\mathcal{E}(X)$ at another point Q' (otherwise we return to the degenerate case above). By our current assumption about the upper boundary, this point Q' must be strictly below the line segment $R_X B_X$. It follows that Q' reduces both group errors compared to Q, by the same amount. Thus $Q' >_{FA} Q$. If instead the upper boundary is strictly above the line segment $R_X B_X$, then through any such boundary point Q we can still draw a line parallel to the 45-degree line. But now let Q^* be the intersection of this line with the extended line $R_X B_X$. If Q^* lies between R_X and B_X, then it is feasible and FA-dominates Q because both groups' errors are reduced by the same amount. Suppose instead that Q^* lies on the extension of the ray $B_X R_X$ (the other case being symmetric), then we claim that R_X itself FA-dominates Q. Indeed, by definition Q must have weakly larger e_r than R_X. And because in this case Q^* is farther away from the 45-degree line than R_X (this is where we use the assumption that R_X is already above that line), Q^* and thus Q also induce strictly larger group error difference $e_b - e_r$ than R_X. Hence Q has larger e_r, $e_b - e_r$ as well as e_b when compared to R_X, as we desire to show.

To complete the proof for the group-balanced case, we need to show that the lower boundary connecting R_X to B_X is *not* FA-dominated. R_X (and symmetrically B_X) cannot be FA-dominated, because it minimizes e_r and conditional on that further minimizes e_b uniquely. Take any other point Q on the lower boundary. If Q lies on the line segment $R_X B_X$, then the lower boundary consists entirely of this line segment. In this case Q minimizes a certain weighted average of group errors $\alpha e_r + \beta e_b$ across all feasible points, where $\alpha, \beta > 0$ are such that the vector (α, β) is orthogonal to the line segment $R_X B_X$ (which necessarily has a negative slope). Any such point Q cannot be FA-dominated, since a dominant point would have smaller $\alpha e_r + \beta e_b$. Finally suppose Q is a boundary point strictly below the line segment $R_X B_X$. Then it minimizes some weighted sum of group errors $\alpha e_r + \beta e_b$, and it will suffice to show that the weights α, β must be positive. Indeed, $\alpha, \beta \leq 0$ cannot happen because Q induces smaller e_r, e_b than Q^* (Q^* defined in the same way as before but now to the top-right of Q) and thus larger $\alpha e_r + \beta e_b$. $\alpha > 0 \geq \beta$ cannot happen because Q induces larger e_r and smaller e_b than R_X, and thus also larger $\alpha e_r + \beta e_b$. Symmetrically $\beta > 0 \geq \alpha$ cannot happen either. So we indeed have $\alpha, \beta > 0$, which implies that Q is FA-undominated. This proves the result for the group-balanced case.

This argument can be adapted to the group-skewed case as follows. Suppose X is r-skewed, so that R_X and B_X are both above the 45-degree line. To show that the upper boundary connecting R_X to F_X is FA-dominated, we choose any boundary point Q and (similar to the above) let Q^* be on the extended line $R_X F_X$ such that QQ^* is parallel to the 45-degree line. If Q^* is on the line segment $R_X F_X$ then it is a feasible point that FA-dominates Q. If Q^* lies on the extension of the ray $F_X R_X$, then as before it can be shown that $R_X >_{FA} Q$. Finally if Q^* lies on the extension of the ray $R_X F_X$, then it must be the case that F_X lies on the 45-degree line (otherwise it will not minimize $|e_r - e_b|$ as defined). In this case Q is a point that is below the 45-degree line, but also above the extended line $B_X F_X$ by convexity of the feasible set. Since F_X already has larger e_b than B_X, we see that Q must in turn have larger e_b than F_X. But then it follows that Q is FA-dominated by F_X because it has larger e_b, larger $e_r - e_b$ (being below the 45-degree line where F_X belongs to), and thus also larger e_r.

It remains to show that the lower boundary connecting R_X to F_X is FA-undominated. By essentially the same argument, we know that the lower boundary from R_X to B_X is FA-undominated. As for the lower boundary from B_X to F_X, note that if some point Q here is FA-dominated by another boundary point $\widehat{Q}$, then $\widehat{Q}$ must induce smaller $|e_b - e_r|$. Since $e_b - e_r$ is positive at Q, this means that $\widehat{Q}$ induces smaller $e_b - e_r$ than Q, without the absolute value applied to the difference. So either $\widehat{Q}$ lies on the lower boundary from Q to F_X, or $\widehat{Q}$ belongs to the other side of the 45-degree line (i.e., below it). Either way the alternative point $\widehat{Q}$ must be farther away from B_X than Q on the lower boundary, so that by convexity $\widehat{Q}$ lies above the extended line $B_X Q$. Given that Q already has larger e_b than

B_X, this implies that $\widehat{Q}$ has even larger e_b than Q. Hence $\widehat{Q}$ cannot in fact FA-dominate Q, completing the proof.

B.3. **Proof of Corollary 1.** Suppose X is group-balanced, then by Theorem 1 the fairness-accuracy frontier is the lower boundary from R_X to B_X. Let L_X be the group error pair that consists of the e_r in R_X and the e_b in B_X (geometrically, L_X is such that the line segments $R_X L_X$ and $B_X L_X$ are parallel to the axes). Then because R_X, B_X have respectively minimal group errors in the feasible set, and because we are considering the lower boundary, any point on this lower boundary $\mathscr{F}(X)$ must belong to the triangle with vertices R_X, B_X and L_X. This implies by convexity that each edge of this lower boundary has a negative slope (just note that the first and final edges must have negative slopes). Because of this, if we start from R_X and traverse along this lower boundary, it must be the case that e_r continuously increases while e_b continuously decreases. Thus in the group-balanced case there does not exist any strong fairness-accuracy conflict along the fairness-accuracy frontier.

On the other hand, suppose X is r-skewed. Then we claim that B_X and F_X (which are assumed to be distinct) present a strong fairness-accuracy conflict. Indeed, by assumption of r-skewness, B_X is weakly above the 45-degree line. F_X must also be weakly above the 45-degree line because otherwise it would be less fair compared to the point on the line segment $B_X F_X$ that also belongs to the 45-degree line. Thus, the fact that F_X is weakly more fair than B_X implies that F_X entails smaller $e_b - e_r$ than B_X. By definition of B_X, F_X entails larger e_b than B_X. Combining the above two observations, we know that F_X also entails larger e_r than B_X. Hence F_X induces larger group errors than B_X for both groups, but reduces the difference in group errors. This is a strong fairness-accuracy conflict as we desire to show.

B.4. **Proof of Proposition 1.** We recall the proof of Lemma B.1, where we showed that the feasible set $\mathcal{E}(X)$ can be written as $\sum_x E(x) \cdot p_x$, with $E(x)$ representing the line segment connecting the two points

$$\left(\sum_y \frac{x_{y,r}}{p_r} \ell(1, y, r), \sum_y \frac{x_{y,b}}{p_b} \ell(1, y, b)\right)$$

and

$$\left(\sum_y \frac{x_{y,r}}{p_r} \ell(0, y, r), \sum_y \frac{x_{y,b}}{p_b} \ell(0, y, b)\right).$$

If X reveals G, then for each realization x, either $x_{y,r} = 0$ for all y or $x_{y,b} = 0$ for all y. Thus each $E(x)$ is a horizontal or vertical line segment, implying that $\mathcal{E}(X)$ must be a rectangle with $R_X = B_X$ being its bottom-left vertex.

Suppose without loss of generality that $R_X = B_X$ lies above the 45-degree line. If the rectangle $\mathcal{E}(X)$ does not intersect the 45-degree line, then it is easy to see that F_X must be the bottom-right vertex of $\mathcal{E}(X)$. In this case the fairness-accuracy frontier is the entire bottom edge of the rectangle, which is a horizontal line segment. If instead the rectangle $\mathcal{E}(X)$ intersects the 45-degree line, then F_X is the intersection between the bottom edge of $\mathcal{E}(X)$ and the 45-degree line. Again the fairness-accuracy frontier is the horizontal line segment from $R_X = B_X$ to F_X. This proves the result.

B.5. **Proof of Lemma 1.** We first characterize the input-design feasible set, and later study the input-design fairness-accuracy frontier. It is clear that regardless of what garbling the designer gives the agent, the agent's payoff will be weakly better than what can be achieved under no information. Thus any error pair that is implementable by input-design must belong to the halfspace H. Such an error pair must also belong to the feasible set $\mathcal{E}(X)$, so we obtain the easy direction $\mathcal{E}^*(X) \subseteq \mathcal{E}(X) \cap H$ in the lemma.

Conversely, we need to show that a feasible error pair $(e_r, e_b) \in \mathcal{E}(X)$ that satisfies $\alpha_r e_r + \alpha_b e_b \leq e_0$ can be implemented by some garbling T. Consider a garbling T that maps X to $\Delta(\mathcal{D})$, with the interpretation that the realization of $T(x)$ is the recommended decision for the agent. If we abuse notation to let $a(x)$ denote the probability that the recommendation is $d = 1$ at covariate vector x, then this algorithm a needs to satisfy the following obedience constraint for $d = 1$:[39]

$$\sum_{y,g} \frac{\alpha_g}{p_g} \sum_x p_{x,y,g} \cdot a(x) \cdot \ell(1, y, g) \leq \sum_{y,g} \frac{\alpha_g}{p_g} \sum_x p_{x,y,g} \cdot a(x) \cdot \ell(0, y, g).$$

The above is just equation (1) adapted to the current setting with the observation that given the recommendation $T = 1$, the conditional probability of $Y = y$ and $G = g$ is proportional to the recommendation probability $\sum_x p_{x,y,g} \cdot a(x)$, where we use $p_{x,y,g}$ as a shorthand for $\mathbb{P}(X = x, Y = y, G = g)$.

Let us rewrite the above displayed equation as

$$\sum_{x,y,g} p_{x,y,g} \frac{\alpha_g}{p_g} \cdot a(x)\ell(1, y, g) \leq \sum_{x,y,g} p_{x,y,g} \frac{\alpha_g}{p_g} \cdot a(x)\ell(0, y, g).$$

If we add $p_{x,y,g} \frac{\alpha_g}{p_g}(1 - a(x))\ell(0, y, g)$ to each summand above, we obtain

$$\sum_{x,y,g} p_{x,y,g} \frac{\alpha_g}{p_g} \cdot (a(x)\ell(1, y, g) + (1 - a(x))\ell(0, y, g)) \leq \sum_{x,y,g} p_{x,y,g} \frac{\alpha_g}{p_g} \cdot \ell(0, y, g). \tag{B.1}$$

Now, the LHS above can be rewritten as $\sum_{x,y,g} p_{x,y,g} \frac{\alpha_g}{p_g} \cdot \mathbb{E}_{D \sim a(x)}[\ell(D, y, g) \mid X = x, Y = y, G = g]$, which is also equal to $\sum_g \alpha_g \cdot \mathbb{E}_{D \sim a(x)}[\ell(D, Y, g) \mid G = g]$. This is precisely the agent's expected loss when following the designer's recommended decisions.

On the other hand, the RHS in (B.1) can be seen to be the agent's expected loss when taking the decision $d = 0$ regardless of the designer's recommendation. Thus, we deduce

[39]By a version of the revelation principle, such garblings together with the following obedience constraints are without loss for studying the feasible decisions, in a general setting.

that the obedience constraint for the recommendation $d = 1$ is equivalent to (B.1), which simply says that the agent's payoff under the designer's recommendation should be weakly better than the constant decision $d = 0$ ignoring the recommendation. Symmetrically, the other obedience constraint for the recommendation $d = 0$ is equivalent to the agent's payoff being better than the constant decision $d = 1$. Put together, these obedience constraints thus reduce to the requirement that the designer's recommendation gives the agent a payoff that exceeds what can be achieved with no information.

For any error pair (e_r, e_b) that is feasible under unconstrained design, we can construct a garbling T that implements it by recommending the desired decision. If (e_r, e_b) belongs to the halfspace H, then by the previous analysis we know that obedience is satisfied. Thus (e_r, e_b) is implementable under input-design, showing that $\mathcal{E}(X) \cap H = \mathcal{E}^*(X)$ as desired.

Finally we turn to the fairness-accuracy frontier and argue that $\mathscr{F}^*(X) = \mathscr{F}(X) \cap H$. In one direction, if an error pair is FA-undominated in $\mathcal{E}(X)$ and implementable under input design, then it is also FA-undominated in the smaller set $\mathcal{E}^*(X)$. This proves $\mathscr{F}(X) \cap H \subseteq \mathscr{F}^*(X)$. In the opposite direction, suppose for contradiction that a certain point $(e_r, e_b) \in \mathscr{F}^*(X)$ does not belong to $\mathscr{F}(X) \cap H$. Since $\mathscr{F}^*(X) \subseteq \mathcal{E}^*(X) \subseteq H$, we know that (e_r, e_b) must not belong to $\mathscr{F}(X)$. Thus by definition of $\mathscr{F}(X)$, (e_r, e_b) is FA-dominated by some other error pair $(\widehat{e_r}, \widehat{e_b}) \in \mathcal{E}(X)$. In particular, we must have $\widehat{e_r} \leq e_r$ and $\widehat{e_b} \leq e_b$, which implies $\alpha_r \widehat{e_r} + \alpha_b \widehat{e_b} \leq \alpha_r e_r + \alpha_b e_b \leq e_0$ (the first inequality uses $\alpha_r, \alpha_b \geq 0$ and the second uses $(e_r, e_b) \in \mathscr{F}^*(X) \subseteq \mathcal{E}^*(X)$). It follows that the FA-dominant point $(\widehat{e_r}, \widehat{e_b})$ also belongs to H and thus $\mathcal{E}^*(X)$. But this contradicts the assumption that (e_r, e_b) is FA-undominated in $\mathcal{E}^*(X)$. Such a contradiction completes the proof.

B.6. **Proof of Proposition 3.** We now deduce Proposition 3 from Lemma 1. If X is group-balanced, then by Theorem 1 we know that $\mathscr{F}(X)$ is the part of the boundary of $\mathcal{E}(X)$ that connects R_X to B_X from below. Clearly, $\mathscr{F}^*(X) = \mathscr{F}(X)$ can only hold if $R_X, B_X \in \mathscr{F}^*(X) \subseteq H$, so we focus on the "if" direction of the result. Suppose $R_X, B_X \in H$, then we claim that the entire lower boundary of $\mathcal{E}(X)$ from R_X to B_X belongs to H. Indeed, let L_X be the error pair that consists of the e_r in R_X and the e_b in B_X. Geometrically, L_X is such that the line segments $R_X L_X$ and $B_X L_X$ are parallel to the axes. Because R_X, B_X have respectively minimal group errors in the feasible set $\mathcal{E}(X)$, and because we are considering the lower boundary, any point on this lower boundary $\mathscr{F}(X)$ must belong to the triangle with vertices R_X, B_X and L_X. Since R_X, B_X, L_X all belong to the halfspace H ($L_X \in H$ because the agent's payoff weights α_r, α_b are non-negative), we deduce that $\mathscr{F}(X) \subseteq H$. Hence whenever $R_X, B_X \in H$, we have by Lemma 1 that $\mathscr{F}^*(X) = \mathscr{F}(X) \cap H = \mathscr{F}(X)$. This argument proves Proposition 3 in the group-balanced case.

Suppose instead that X is r-skewed (a symmetric argument applies to the b-skewed case). To generalize the above argument, we need to show that whenever R_X, F_X belong to H, then so does the entire lower boundary connecting these points. To see this, note that by the definition of B_X and F_X, the lower boundary connecting these two points consists of positively sloped edges.[40] So across all points on this part of the lower boundary, F_X maximizes $\alpha_r e_r + \alpha_b e_b$. Thus the assumption $F_X \in H$ implies that the lower boundary from B_X to F_X belongs to H. In particular $B_X \in H$, which together with $R_X \in H$ implies that the lower boundary from R_X to B_X also belongs to H (by the same argument as in the group-balanced case before). Hence the entire lower boundary from R_X to F_X belongs to H, as we desire to show.

B.7. **Proof of Proposition 4.** We first present a simple lemma which conveniently restates the property of "uniform worsening of frontier":

Lemma B.2. *Excluding covariate X' over X uniformly worsens the frontier if and only if $\mathscr{F}^*(X)$ does not intersect with $\mathscr{F}^*(X, X')$.*

The proof of this lemma is straightforward: If there exists a point in $\mathscr{F}^*(X)$ that also belongs to $\mathscr{F}^*(X, X')$, then this point is not FA-dominated by any point in $\mathscr{F}^*(X, X')$, so that the frontier does not uniformly worsen when excluding X'. On the other hand, suppose no point in $\mathscr{F}^*(X)$ belongs to $\mathscr{F}^*(X, X')$. Note that any point in $\mathscr{F}^*(X)$ is implementable via a garbling of X and thus implementable via a garbling of X, X'. Thus any such point belongs to $\mathcal{E}^*(X, X')$, and since it is not FA-optimal in this set, it must be FA-dominated by some FA-optimal point in this (compact) set. In this case we do have uniform worsening of the frontier, as we desire to show.

Below we use Lemma B.2 to deduce Proposition 4. The key observation is that whether or not G is excluded does not affect the minimal (or maximal) feasible error for either group. This is because if we want to minimize the error of a particular group g using an algorithm that depends on X, then we essentially condition on $G = g$ anyways.

With this observation, suppose X is strictly group-balanced. Then R_X lies strictly above the 45-degree line and B_X lies strictly below. Since we assume $R_X, B_X \in H$, Proposition 3 tells us that the input-design fairness-accuracy frontier $\mathscr{F}^*(X)$ is the same as the unconstrained fairness-accuracy frontier $\mathscr{F}(X)$, and by Theorem 1 this frontier is the lower boundary of the feasible set $\mathcal{E}(X)$ connecting R_X to B_X. By Lemma B.2, we just need to show that in this case the lower boundary of $\mathcal{E}(X)$ from R_X to B_X does not intersect with the input-design fairness-accuracy frontier $\mathscr{F}^*(X, G)$ given (X, G). To characterize the latter frontier, let $L_X = R_{X,G} = B_{X,G}$ denote the error pair that has the same e_r as R_X and the same e_b as B_X. Without loss of generality assume L_X lies weakly above the 45-degree line.

[40]If we start from B_X and traverse the lower boundary to the right until F_X, then the first edge of this boundary must be positively sloped because B_X has minimum e_b. The final edge of this boundary must also be positively sloped, since otherwise the starting vertex of this edge would be closer to the 45-degree line than F_X. It follows by convexity that the entire boundary from B_X to F_X has positive slopes.

Then from Proposition 1 we know that the unconstrained fairness-accuracy frontier $\mathscr{F}(X,G)$ is the horizontal line segment from L_X to $F_{X,G}$. This point $F_{X,G}$ is the intersection between the line segment $L_X B_X$ and the 45-degree line (here we use the fact that L_X lies above the 45-degree line and B_X lies below). As $B_X \in H$, the points L_X and $F_{X,G}$ also belong to H because they have equal e_b and smaller e_r compared to B_X. Hence the input-design fairness-accuracy frontier $\mathscr{F}^*(X,G)$ is also the line segment from L_X to $F_{X,G}$. To see that this horizontal line segment does not intersect the boundary of $\mathcal{E}(X)$ from R_X to B_X, just note that B_X is the only point on that boundary with the same (minimal) e_b as any point on the horizontal line segment. But B_X does not belong to that line segment because it is strictly below the 45-degree line. This proves the result when X is strictly group-balanced.

Now suppose X is not strictly group-balanced. Then R_X and B_X lie weakly on the same side of the 45-degree line, and without loss of generality let us assume they lie weakly above. It is still the case that the unconstrained fairness-accuracy frontier $\mathscr{F}(X,G)$ is the horizontal line segment from L_X to $F_{X,G}$. But in the current setting $F_{X,G}$ must be weakly closer to the 45-degree line than B_X, which means that B_X now lies in between L_X and $F_{X,G}$. In other words, $B_X \in \mathscr{F}(X)$ and $B_X \in \mathscr{F}(X,G)$. But by assumption, B_X also belongs to H. So Lemma 1 tells us that B_X belongs to the input-design fairness-accuracy frontiers $\mathscr{F}^*(X)$ and $\mathscr{F}^*(X,G)$. This shows that the two frontiers $\mathscr{F}^*(X)$ and $\mathscr{F}^*(X,G)$ intersect, which completes the proof by Lemma B.2.

B.8. **Proof of Proposition 5.** Let $\underline{e}_g = \min\{e_g \mid (e_r, e_b) \in \mathcal{E}(X,G)\}$ and $\overline{e}_g = \max\{e_g \mid (e_r, e_b) \in \mathcal{E}(X,G)\}$ be the minimal and maximal feasible errors for group g given covariate vector (X,G), and define $\underline{e}^*_g = \min\{e_g \mid (e_r, e_b) \in \mathcal{E}(X,G,X')\}$ and $\overline{e}^*_g = \max\{e_g \mid (e_r, e_b) \in \mathcal{E}(X,G,X')\}$ to be the corresponding quantities given (X,G,X'). The following lemma says that additional access to X' reduces the minimal feasible error for group g relative to (X,G) if and only if X' is decision-relevant over X for group g.

Lemma B.3. *$\underline{e}^*_g < \underline{e}_g$ if X' is decision-relevant over X for group g, and $\underline{e}^*_g = \underline{e}_g$ if it is not.*

Proof. Let $a_g : \mathcal{X} \to \{0,1\}$ be any strategy mapping each realization of X into an optimal outcome for group g, i.e.,

$$a_g(x) \in \operatorname*{arg\,min}_{d \in \{0,1\}} \mathbb{E}\left[\ell(d,Y,g) \mid G=g, X=x\right] \quad \forall x \in \mathcal{X}.$$

Likewise let $a^*_g : \mathcal{X} \times \mathcal{X}' \to \{0,1\}$ satisfy

$$a^*_g(x,x') \in \operatorname*{arg\,min}_{d \in \{0,1\}} \mathbb{E}\left[\ell(d,Y,g) \mid G=g, X=x, X'=x'\right] \quad \forall x \in \mathcal{X},\ \forall x' \in \mathcal{X}'.$$

By optimality of a^*_g,

$$\mathbb{E}\left[\ell(a^*_g(x,x'),Y,g) \mid G=g, X=x, X'=x'\right]$$

$$\leq \mathbb{E}\left[\ell(a_g(x), Y, g) \mid G = g, X = x, X = x'\right] \quad \forall x \in \mathcal{X}, \forall x' \in \mathcal{X}'. \tag{B.2}$$

Suppose X' is decision-relevant over X for group g. Then there exist $x \in \mathcal{X}$ and $x', \tilde{x}' \in \mathcal{X}'$ such that the optimal assignment for group g is uniquely equal to 1 at (x, x') and 0 at $(x, \tilde{x}')$, where both (x, x') and $(x, \tilde{x}')$ have positive probability conditional on $G = g$. But then (B.2) must hold strictly at either (x, x') or $(x, \tilde{x}')$. By taking the expectation of (B.2) conditional on $G = g$, we obtain

$$\underline{e}_g^* = \mathbb{E}\left[\ell(a_g^*(X, X'), Y, g) \mid G = g\right] < \mathbb{E}\left[\ell(a_g(X), Y, g) \mid G = g\right] = \underline{e}_g.$$

If X' is not decision-relevant over X for group g, then (B.2) holds with equality at every x, x', and the equivalence $\underline{e}_g^* = \underline{e}_g$ follows. □

We now use Lemma B.2 and B.3 to prove Proposition 5. First suppose (X, G) is r-skewed, in which case $R_X = B_X$ lies strictly above the 45-degree line. By Proposition 1, the unconstrained fairness-accuracy frontier $\mathscr{F}(X, G)$ is then the horizontal line segment from $R_{X,G} = B_{X,G}$ to $F_{X,G}$.

If X' is not decision-relevant over X for group b, then from Lemma B.3 we know that the minimal feasible error for group b is the same given (X, G, X') as given (X, G). By assumption that (X, G) is r-skewed, group b's minimal error given (X, G) exceeds group r's minimal error given (X, G). Since group b's minimal error is the same given (X, G) and (X, G, X'), while group r's minimal error is weakly smaller given (X, G, X') compared to (X, G), it must be that group b minimal error given (X, G, X') also exceeds the group r minimal error given (X, G, X'). In other words, $R_{X,G,X'} = B_{X,G,X'}$ also lies strictly above the 45-degree line, and the fairness-accuracy frontier $\mathscr{F}(X, G, X')$ is the horizontal line segment from $R_{X,G,X'} = B_{X,G,X'}$ to $F_{X,G,X'}$. Crucially, this line segment shares the same e_b as the line segment from $R_{X,G} = B_{X,G}$ to $F_{X,G}$. In addition, as $R_{X,G,X'}$ must have weakly smaller e_r than $R_{X,G}$, and $F_{X,G,X'}$ must be weakly closer to the 45-degree line than $F_{X,G}$, we deduce that the unconstrained fairness-accuracy frontier $\mathscr{F}(X, G, X')$ is a horizontal line segment that is a superset of the line segment $\mathscr{F}(X, G)$. Thus, in particular, $R_{X,G} = B_{X,G}$ belongs to both of these frontiers. Lemma 1 thus imply that $R_{X,G} = B_{X,G}$ also belongs to the input-design fairness-accuracy frontiers $\mathscr{F}^*(X, G)$ and $\mathscr{F}^*(X, G, X')$ ($R_{X,G} = B_{X,G}$ belongs to H because this point can be implemented by giving (X, G) to the agent, who will then minimize both groups' errors given this information). By Lemma B.2, uniform worsening of the frontier does not occur when excluding X', as we desire to show.

If X' is decision-relevant over X for group b, then Lemma B.3 tells us that $\underline{e}_b^* < \underline{e}_b$ with strict inequality. There are two cases to consider here. One case involves $\underline{e}_b^* > \underline{e}_r^*$, so that (X, G, X') is r-skewed just as (X, G) is. Then the unconstrained fairness-accuracy frontier $\mathscr{F}(X, G, X')$ is again a horizontal line segment, but with e_b equal to $\underline{e}_b^*$. Since $\underline{e}_b^* < \underline{e}_b$, this frontier is parallel but lower than the fairness-accuracy frontier $\mathscr{F}(X, G)$. Thus $\mathscr{F}(X, G)$

does not intersect $\mathscr{F}(X, G, X')$. As their subsets, the input-design fairness-accuracy frontiers $\mathscr{F}^*(X, G)$ and $\mathscr{F}^*(X, G, X')$ also do not intersect. Thus by Lemma B.2, there is uniform worsening of the frontier. In the remaining case we have $\underline{e}_b^* \leq \underline{e}_r^*$, so that (X, G, X') is b-skewed. Then the unconstrained fairness-accuracy frontier $\mathscr{F}(X, G, X')$ is now a *vertical* line segment with $e_r = \underline{e}_r^*$. The points on this frontier have varying e_b, but any of the e_b does not exceed $\underline{e}_r^*$ because these points are below the 45-degree line. Because $\underline{e}_r^* \leq \underline{e}_r < \underline{e}_b$, we thus know that any point on the frontier $\mathscr{F}(X, G, X')$ has strictly smaller e_b compared to any point on $\mathscr{F}(X, G)$. Once again these two unconstrained frontiers do not intersect, and nor do the input-design frontiers. This proves Proposition 5 when (X, G) is r-skewed.

A symmetric argument applies when (X, G) is b-skewed, so below we focus on the case where (X, G) is group-balanced. That is, $R_{X,G} = B_{X,G}$ lies on the 45-degree line. In this case the fairness-accuracy frontiers $\mathscr{F}(X, G)$ and $\mathscr{F}^*(X, G)$ are both this singleton point. If X' is not decision-relevant over X for group b, then Lemma B.3 tells us that $\underline{e}_b^* = \underline{e}_b = \underline{e}_r \geq \underline{e}_r^*$. When equality holds the fairness-accuracy frontiers $\mathscr{F}(X, G, X')$ and $\mathscr{F}^*(X, G, X')$ are also the singleton point $R_{X,G} = B_{X,G}$, and uniform worsening does not occur. If we instead have strict inequality $\underline{e}_b^* = \underline{e}_b > \underline{e}_r^*$, then (X, G, X') is r-skewed and the unconstrained fairness-accuracy frontier $\mathscr{F}(X, G, X')$ is a horizontal line segment with one of the endpoints being $F_{X,G,X'} = R_{X,G} = B_{X,G}$. Thus $R_{X,G} = B_{X,G}$ belongs also to the input-design fairness-accuracy frontier $\mathscr{F}^*(X, G, X')$, showing that $\mathscr{F}^*(X, G)$ and $\mathscr{F}^*(X, G, X')$ intersect. Uniform worsening of the frontier does not occur either way.

B.9. **Details of Example 12.** In this section, we compute the input-design feasible set and fairness-accuracy frontier for Example 12. Since X is a null signal, garblings of (X, X') are the same as garblings of X'. Without loss, we can restrict attention to garblings of X' that take two values, $d = 1$ and $d = 0$, which correspond to the designer's decisions for the agent. Any such garbling can be identified with a pair (α, β), where α is the probability with which $X' = 1$ is mapped into $d = 1$, and β is the probability with which $X' = 0$ is mapped into $d = 1$. It is easy to check that the agent's obedience constraint reduces to the simple inequality $\alpha \geq \beta$, which intuitively requires the agent to choose $d = 1$ more often when $X' = 1$.

For any pair (α, β), the two groups' errors can be calculated as

$$e_r(\alpha, \beta) = \frac{1}{2}(1 - \alpha) + \frac{1}{2}\beta = 0.5 - 0.5(\alpha - \beta),$$

$$e_b(\alpha, \beta) = \frac{1}{2} \cdot 0.6(1 - \alpha) + \frac{1}{2} \cdot 0.4(1 - \beta) + \frac{1}{2} \cdot 0.4\alpha + \frac{1}{2} \cdot 0.6\beta = 0.5 - 0.1(\alpha - \beta).$$

So as $\alpha - \beta$ ranges from 0 to 1, the implementable group errors constitute the line segment connecting $(0, 0.4)$ with $(0.5, 0.5)$. This entire line segment is also the fairness-accuracy frontier $\mathscr{F}^*(X, X')$, as illustrated in Figure 9 in the main text.

For an Egalitarian designer, sending the null signal X leads to the point $(0.5, 0.5)$ and yields a payoff of 0. In contrast, we say that the designer "makes use of X' over X" if the garbling T is *not* independent of X' conditional on X (in this example the conditioning is irrelevant since X is null). Whenever T is not independent of X', then for some realizations of T the agent believes $X' = 1$ is more likely, which makes $d = 1$ strictly optimal. Thus, whenever the designer makes use of X' in the garbling, the agent is strictly better off compared to the null signal, and the resulting error pair must be distinct from $(0.5, 0.5)$. But given the shape of the implementable set, this means that the designer is strictly worse off when any information about X' is provided to the agent.

Conversely, suppose X' is decision-relevant over X for both groups. Then by Proposition 1, the unconstrained frontier $\mathscr{F}(X, X')$ is either a horizontal line segment with $e_b = \underline{e}_b^* < \underline{e}_b = \underline{e}_b$, or a vertical line segment with $e_r = \underline{e}_r^* < \underline{e}_r = \underline{e}_b$. Either way the point $R_X = B_X$ does not belong to this frontier, showing that $\mathscr{F}(X)$ does not intersect with $\mathscr{F}(X, X')$. Hence $\mathscr{F}^*(X)$ and $\mathscr{F}^*(X, X')$ also do not intersect, and by Lemma B.2 we know that there is uniform worsening of the frontier. This completes the entire proof of Proposition 5.

References

Agan, A. and S. Starr (2018): "Ban the Box, Criminal Records, and Racial Discrimination: A Field Experiment," *The Quarterly Journal of Economics*, 133, 191–235.

Agarwal, A., A. Beygelzimer, M. Dudík, J. Langford, and H. Wallach (2018): "A Reductions Approach to Fair Classification," in *ICML*.

Alonso, R. and O. Câmara (2016): "Persuading Voters," *American Economic Review*, 106, 3590–3605.

Andreoni, J. and J. Miller (2002): "Giving According to GARP," *Econometrica*, 70, 737–753.

Angwin, J. and J. Larson (2016): "Machine bias," ProPublica.

Arnold, D., W. Dobbie, and P. Hull (2021): "Measuring Racial Discrimination in Algorithms," *AEA Papers and Proceedings*, 111, 49—54.

Bergemann, D. and S. Morris (2019): "Information Design: A Unified Perspective," *Journal of Economic Literature*, 57, 44–95.

Bertrand, M. and E. Kamenica (2020): "Coming apart? Cultural distances in the United States over time," Working Paper.

Blattner, L., S. Nelson, and J. Spiess (2022): "Unpacking the Black Box: Regulating Algorithmic Decisions," Working Paper.

Bolton, G. E. and A. Ockenfels (2000): "ERC: A Theory of Equity, Reciprocity, and Competition," *American Economic Review*, 90, 166–193.

Caplin, A., D. Martin, and P. Marx (2023): "Modeling Machine Learning," Working Paper.

Chan, J. and E. Eyster (2003): “Does Banning Affirmative Action Lower College Student Quality?” *American Economic Review*, 93, 858–872.

Charness, G. and M. Rabin (2002): “Understanding Social Preferences with Simple Tests,” *The Quarterly Journal of Economics*, 117, 817–869.

Che, Y.-K., K. Kim, and W. Zhong (2019): “Statistical Discrimination in Ratings-Guided Markets,” Working Paper.

Chohlas-Wood, A., M. Coots, E. Brunskill, and S. Goel (2021): “Learning to be Fair: A Consequentialist Approach to Equitable Decision-Making,” Working Paper.

Chouldechova, A. (2017): “Fair Prediction with Disparate Impact: A Study of Bias in Recidivism Prediction Instruments.” *Big Data*, 5, 153–163.

Corbett-Davis, S., E. Pierson, A. Feller, S. Goel, and A. Huq (2017): “Algorithmic decision-making and the cost of fairness,” in *Proceedings of the 23rd Conference on Knowledge Discovery and Data Mining*.

Cowgill, B. and M. T. Stevenson (2020): “Algorithmic Social Engineering,” *AEA Papers and Proceedings*, 110, 96–100.

Cowgill, B. and C. E. Tucker (2020): “Algorithmic Fairness and Economics,” Working Paper.

Curello, G. and L. Sinander (2022): “The Comparative Statics of Persuasion,” Working Paper.

Dessein, W., A. Frankel, and N. Kartik (2022): “Test-Optional Admissions,” Working Paper.

Diana, E., T. Dick, H. Elzayn, M. Kearns, A. Roth, Z. Schutzman, S. Sharifi-Malvajerdi, and J. Ziani (2021): “Algorithms and Learning for Fair Portfolio Design,” in *Proceedings of the 22nd ACM Conference on Economics and Computation*.

Dworczak, P., S. Kominers, and M. Akbarpour (2021): “Redistribution Through Markets,” *Econometrica*, 89, 1665–1698.

Dwork, C., M. Hardt, T. Pitassi, O. Reingold, and R. Zemel (2012): “Fairness through awareness,” in *Proceedings of the 3rd Innovations in Theoretical Computer Science Conference*, 214–226.

Dwork, C. and A. Roth (2014): “The Algorithmic Foundations of Differential Privacy,” *Found. Trends Theor. Comput. Sci.*, 9, 211–407.

Ellison, G. and P. A. Pathak (2021): “The Efficiency of Race-Neutral Alternatives to Race-Based Affirmative Action: Evidence from Chicago’s Exam Schools,” *American Economic Review*, 111, 943–75.

Fehr, E. and K. M. Schmidt (1999): “A Theory of Fairness, Competition, and Cooperation,” *The Quarterly Journal of Economics*, 114, 817–868.

Feigenberg, B. and C. Miller (2021): "Would Eliminating Racial Disparities in Motor Vehicle Searches have Efficiency Costs?*," *The Quarterly Journal of Economics*, 137, 49–113.

Ferry, J., U. Aïvodji, S. Gambs, M.-J. Huguet, and M. Siala (2022): "Improving Fairness Generalization Through a Sample-Robust Optimization Method," *Machine Learning*.

Fisman, R., S. Kariv, and D. Markovits (2007): "Individual Preferences for Giving," *American Economic Review*, 97, 1858–1876.

Fuster, A., P. Goldsmith-Pinkham, T. Ramadorai, and A. Walther (2021): "Predictably Unequal? The Effects of Machine Learning on Credit Markets," *Journal of Finance*.

Garfinkel, S. L., J. M. Abowd, and S. Powazek (2018): "Issues Encountered Deploying Differential Privacy," in *Proceedings of the 2018 Workshop on Privacy in the Electronic Society*, New York, NY, USA: Association for Computing Machinery, WPES'18, 133–137.

Garg, N., H. Li, and F. Monachou (2021): "Standardized Tests and Affirmative Action: The Role of Bias and Variance," in *Proceedings of the 2021 ACM Conference on Fairness, Accountability, and Transparency*, New York, NY, USA: Association for Computing Machinery, FAccT '21, 261.

Hansen, V. P. B., A. T. Neerkaje, R. Sawhney, L. Flek, and A. Søgaard (2022): "The Impact of Differential Privacy on Group Disparity Mitigation," *ArXiv*, abs/2203.02745.

Hardt, M., E. Price, and N. Srebro (2016): "Equality of Opportunity in Supervised Learning," in *Advances in Neural Information Processing Systems*, 3315–3323.

Harsanyi, J. (1953): "Cardinal Utility in Welfare Economics and in the Theory of Risk-Taking," *Journal of Political Economy*, 61, 434–435.

——— (1955): "Cardinal Welfare, Individualistic Ethics, and Interpersonal Comparison of Utility: Comment," *Journal of Political Economy*, 63, 309–321.

Ichihashi, S. (2019): "Limiting Sender's Information in Bayesian Persuasion," *Games of Economic Behavior*, 117, 276–288.

——— (2023): "Privacy, Transparency, and Policing," Working Paper.

Jung, C., S. Kannan, C. Lee, M. M. Pai, A. Roth, , and R. Vohra (2020): "Fair Prediction with Endogenous Behavior," Working Paper.

Kamenica, E. (2019): "Bayesian Persuasion and Information Design," *Annual Review of Economics*, 11, 249–272.

Kamenica, E. and M. Gentzkow (2011): "Bayesian Persuasion," *American Economic Review*, 101, 2590–2615.

Kasy, M. and R. Abebe (2021): "Fairness, Equality, and Power in Algorithmic Decision-Making," in *ACM Conference on Fairness, Accountability, and Transparency*.

Klare, B. F., M. J. Burge, J. C. Klontz, R. W. Vorder Bruegge, and A. K. Jain (2012): "Face Recognition Performance: Role of Demographic Information," *IEEE Transactions on Information Forensics and Security*, 7, 1789–1801.

Kleinberg, J., J. Ludwig, S. Mullainathan, and A. Rambachan (2018): "Algorithmic Fairness," *AEA Papers and Proceedings*, 108, 22–27.

Kleinberg, J. and S. Mullainathan (2019): "Simplicity Creates Inequity: Implications for Fairness, Steretotypes, and Interpretability," Working Paper.

Kleinberg, J., S. Mullainathan, and M. Raghavan (2017): "Inherent Trade-Offs in the Fair Determination of Risk Scores," in *8th Innovations in Theoretical Computer Science Conference (ITCS 2017)*, vol. 67, 43:1–43:23.

Lipton, Z. C., A. Chouldechova, and J. McAuley (2018): "Does mitigating ML's impact disparity require treatment disparity," in *32nd Conference on Neural Information Processing Systems*.

Little, C. O., M. Weylandt, and G. I. Allen (2022): "To the Fairness Frontier and Beyond: Identifying, Quantifying, and Optimizing the Fairness-Accuracy Pareto Frontier," .

Loewenstein, G. F., L. Thompson, and M. H. Bazerman (1989): "Social utility and decision making in interpersonal contexts," *Journal of Personality and Social Psychology*, 57, 426–441.

Ludwig, J. and S. Mullainathan (2021): "Algorithmic Behavioral Science: Machine Learning as a Tool for Scientific Discovery," Working Paper.

Lundberg, S. J. (1991): "The Enforcement of Equal Opportunity Laws Under Imperfect Information: Affirmative Action and Alternatives," *The Quarterly Journal of Economics*, 106, 309–326.

Mehrabi, N., F. Morstatter, N. Saxena, K. Lerman, and A. Galstyan (2022): "A Survey on Bias and Fairness in Machine Learning," *ACM Computing Surveys*, 54, 1–35.

Menon, A. K. and R. C. Williamson (2018): "The cost of fairness in binary classification," in *Proceedings of the 1st Conference on Fairness, Accountability and Transparency*, ed. by S. A. Friedler and C. Wilson, PMLR, vol. 81 of *Proceedings of Machine Learning Research*, 107–118.

Obermeyer, Z., B. Powers, C. Vogeli, and S. Mullainathan (2019): "Dissecting racial bias in an algorithm used to manage the health of populations," *Science*, 366, 447–453.

Parfit, D. (2002): "Equality or Priority?" in *The Ideal of Equality*, ed. by M. Clayton and A. Williams, New York: Palgrave Macmillan, 81–125.

Persico, N. (2002): "Racial Profiling, Fairness, and Effectiveness of Policing," *American Economic Review*, 92, 1472–1479.

Rabin, M. (1993): "Incorporating Fairness into Game Theory and Economics," *American Economic Review*, 83, 1281–1302.

Rambachan, A., J. Kleinberg, S. Mullainathan, and J. Ludwig (2021): "An Economic Approach to Regulating Algorithms," Working Paper.

Rawls, J. (1971): *A Theory of Justice*, Harvard University Press.

Rose, E. K. (2021): "Who Gets a Second Chance? Effectiveness and Equity in Supervision of Criminal Offenders," *The Quarterly Journal of Economics*, 136, 1199–1253.

Roth, A. and M. Kearns (2019): *The Ethical Algorithm: The Science of Socially Aware Algorithm Design*, Oxford University Press.

Rothstein, J. M. (2004): "College performance predictions and the SAT," *Journal of Econometrics*, 121, 297–317, higher education (Annals issue).

Saez, E. and S. Stantcheva (2016): "Generalized Social Marginal Welfare Weights for Optimal Tax Theory," *American Economic Review*, 106, 24–45.

Sagawa, S., P. W. Koh, T. B. Hashimoto, and P. Liang (2020): "Distributionally Robust Neural Networks," in *International Conference on Learning Representations*.

Sullivan, C. (2022): "Eliciting Preferences over Life and Death: Experimental Evidence from Organ Transplantation," Working Paper.

Systemwide Academic Senate (2020): *Report of the UC Academic Council Standardized Testing Task Force (STTF)*, University of California, Oakland, CA.

Viviano, D. and J. Bradic (2023): "Fair Policy Targeting," *Journal of the American Statistical Association*.

Wei, S. and M. Niethammer (2020): "The Fairness-Accuracy Pareto Front," .

Yang, C. S. and W. Dobbie (2020): "Equal Protection Under Algorithms: A New Statistical and Legal Framework," *Michigan Law Review*, 119.

Online appendix to the paper

Algorithmic Design: The Fairness-Accuracy Frontier

Annie Liang Jay Lu Xiaosheng Mu

February 27, 2023

O.1. **Different Loss Functions.** In this section, we generalize Theorem 1 when fairness and accuracy are evaluated using loss functions that are possibly different but not "directly opposed."

As in the main text, let $a : \mathcal{X} \to \Delta(D)$ describe a generic algorithm and let $\mathscr{A}_X$ be the set of all algorithms. Different from the main text, we have two loss functions—an accuracy loss function $\ell^A : \mathcal{D} \times \mathcal{Y} \times \mathcal{G} \to \mathbb{R}$ and a fairness loss function $\ell^F : \mathcal{D} \times \mathcal{Y} \times \mathcal{G} \to \mathbb{R}$. For either group $g \in \{r, b\}$, let

$$e_g^A(a) = \mathbb{E}_{D\sim a(X)}[\ell^A(D, Y, g) \mid G = g] \qquad \forall a \in \mathscr{A}_X$$
$$e_g^F(a) = \mathbb{E}_{D\sim a(X)}[\ell^F(D, Y, g) \mid G = g] \qquad \forall a \in \mathscr{A}_X$$

be group errors defined using the respective loss functions. We use $e^A(a) \equiv (e_r^A(a), e_b^A(a))$ to denote the error pairs evaluated by the accuracy loss function, and

$$\mathcal{E}(X) = \{e^A(a) : a \in \mathscr{A}_X\}$$

to denote the set of feasible (accuracy) error pairs. Also define

$$\Delta(a) = |e_r^F(a) - e_b^F(a)| \qquad \forall a \in \mathscr{A}_X$$

to be the gap between group errors evaluated by the fairness loss function, i.e., the "unfairness" of algorithm a. The function $u : \mathcal{E}(X) \to \mathbb{R}$ satisfying

$$u(e) = \min_{a\in\mathcal{A}_X} \{\Delta(a) : e^A(a) = e\}$$

maps each (accuracy) error pair to the minimal achievable unfairness value. This function is well-defined as $\Delta(\cdot)$ is continuous and $e^A(\cdot)$ is linear.

We now extend the definitions of FA-dominance and the fairness-accuracy frontier.

Definition O.1. Let $>_{FA}$ be the partial order on $\mathcal{E}(X)$ satisfying $(e_r, e_b) >_{FA} (e_r', e_b')$ if $e_r \leq e_r'$, $e_b \leq e_b'$, and $u(e) \leq u(e')$, with at least one of these inequalities strict.

Definition O.2. $\mathscr{F}(X)$ is the set of all pairs $e \in \mathcal{E}(X)$ that are FA-undominated, i.e. no $e' \in \mathcal{E}(X)$ exists that satisfies $e' >_{FA} e$.

When $\ell^F = \ell^A$ then we can express u directly as a function of the (accuracy) error-pairs, $u(e) = |e_r - e_b|$, and so these definitions reduce to Definitions 2, 3 and 5.

Lemma O.4. *$u(\cdot)$ is piecewise linear and convex.*

Proof. Since $\mathcal{X}$ is finite, $e^A(\cdot)$ is linear and $\Delta(\cdot)$ is piecewise linear, $u(\cdot)$ must be piecewise linear. We now prove convexity. Fix any $e_1, e_2 \in \mathcal{E}(X)$. Since these error pairs are feasible, there exist algorithms $a_1, a_2 \in \mathcal{A}_X$ that implement them, i.e., $u(e_i) = u\left(e^A(a_i)\right) = \Delta(a_i)$ for each $i = 1, 2$. Let $a = \lambda a_1 + (1-\lambda) a_2$ for $\lambda \in [0,1]$ and note that since $e^A(\cdot)$ is linear, $e^A(a) = \lambda e_1 + (1-\lambda) e_2$. Thus,

$$\begin{aligned} u(\lambda e_1 + (1-\lambda) e_2) &= u\left(e^A(a)\right) \leq \Delta(a) = \left|e_r^F(a) - e_b^F(a)\right| \\ &= \left|\lambda e_r^F(a_1) + (1-\lambda) e_r^F(a_2) - \left(\lambda e_b^F(a_1) - (1-\lambda) e_b^F(a_2)\right)\right| \\ &\leq \lambda \left|e_r^F(a_1) - e_b^F(a_1)\right| + (1-\lambda)\left|e_r^F(a_2) - e_b^F(a_2)\right| \\ &\leq \lambda u(e_1) + (1-\lambda) u(e_2) \end{aligned}$$

as desired. □

Given Lemma O.4, the directional derivatives of u are well-defined in the interior of $\mathcal{E}$. We generalize Theorem 1 under the following assumption.

Assumption 1. *There does not exist $e \in \mathcal{E}(X)$ such that $D_{(1,0)}u(e) < 0$ and $D_{(0,1)}u(e) < 0$.*

This assumption says that, for at least one group, increasing error must hurt fairness. It rules out the case when fairness and accuracy are directly opposed, in the sense that increasing errors in both groups improves fairness. Since we are primarily interested in the tradeoffs between fairness and accuracy due to informational constraints rather than the definitions of fairness and accuracy being intrinsically in conflict, we view this assumption as a natural one for our purposes. In the case when both loss functions are the same so $u(e) = |e_r - e_b|$, this assumption is always satisfied.[41]

We now define the fairness-optimal set. First, let

$$\underline{\Delta} := \min_{e \in \mathcal{E}(X)} u(e)$$

be the minimal achievable level of unfairness.

Definition O.3 (Pareto Frontier). For any set $E \subseteq \mathbb{R}^2$, let $\mathcal{P}(E)$ denote the usual Pareto frontier of E, i.e., all points $(e_r, e_b) \in E$ where no $(e_r', e_b') \in E$ is weakly smaller in each entry and strictly smaller in at least one.

[41] In the case where $Y \in \{0,1\}$, the accuracy loss function is the misclassification rate $\ell^A(d,y) = \mathbb{1}(d \neq y)$, and the fairness loss function is $\ell^F(d,y) = \mathbb{1}(d = 1)$, a sufficient condition for Assumption 1 to hold is existence of $x, x' \in \mathcal{X}$ such that $\mathbb{E}(Y = 1 \mid X = x, G = g) < 1/2$ and $\mathbb{E}(Y = 1 \mid X = x', G = g) < 1/2$ for both g (so that the Bayes-optimal assignment at both x and x' is 0 for members of either group), and also $\mathbb{P}(X = x \mid G = r) > \mathbb{P}(X = x \mid G = b)$ while $\mathbb{P}(X = x' \mid G = r) < \mathbb{P}(X = x' \mid G = b)$. Details are available upon request. We leave to future work the derivation of other conditions on primitives for specific (ℓ^A, ℓ^F) pairings.

Definition O.4. The fairness-optimal set is

$$F_X \equiv \mathcal{P}\left\{e \in \mathcal{E}\left(X\right) \,:\, u\left(e\right) = \underline{\Delta}\right\}$$

It's easy to see that F_X is always a subset of the fairness-accuracy frontier.

Theorem O.1. *Under Assumption 1, the following hold:*

(1) If $F_X \subseteq \mathcal{P}\left(\mathcal{E}\left(X\right)\right)$, then $\mathscr{F}\left(X\right) = \mathcal{P}\left(\mathcal{E}\left(X\right)\right)$

(2) If $F_X \nsubseteq \mathcal{P}\left(\mathcal{E}\left(X\right)\right)$, then F_X is a singleton and $\mathscr{F}\left(X\right)$ is the union of $\mathcal{P}\left(\mathcal{E}\left(X\right)\right)$ and a connected sequence of positively-sloped line segments ending at F_X

Thus, the condition that the fairness-maximizing point F_X belongs to the Pareto frontier generalizes group-balance. That is, when this condition is satisfied, we can restrict attention to the usual Pareto frontier without loss. Moreover, no two points on the fairness-accuracy frontier can be Pareto-ranked. When (generalized) group-balance fails, then the frontier consists of two parts: the Pareto frontier, and a positively-sloped sequence of lines, along which every pair of points has the property that one point involves higher errors for both groups but greater fairness. Corollary 1 thus extends directly under this generalized notion of group-balance.

O.2. **Proof of Theorem O.1.** To save on notation we suppress dependence on X in what follows, using $\mathscr{F}$ for the fairness-accuracy frontier and $\mathcal{E}$ for the feasible set. We first show that the fairness-accuracy frontier is the union of the Pareto frontiers of the unfairness sublevel sets.

Definition O.5. For any $\Delta \in \mathbb{R}$, let $\mathcal{E}_{\leq\Delta} = \{e \in \mathcal{E} \mid u(e) \leq \Delta\}$ be u's Δ-sublevel set.

Lemma O.5. $\mathscr{F} = \bigcup_{\Delta} \mathcal{P}(\mathcal{E}_{\leq\Delta})$.

Proof. Fix any unfairness level Δ and point $e \in \mathcal{P}(\mathcal{E}_{\leq\Delta})$. We will show that e must belong to the fairness-accuracy frontier $\mathscr{F}$. Suppose to the contrary that there exists $e' \in \mathcal{E}$ such that $e'_r \leq e_r$, $e'_b \leq e_b$, and $u(e') \leq u(e)$ with at least one inequality strict. Since $u(e') \leq u(e) \leq \Delta$, the error pair e' must belong to $\mathcal{E}_{\leq\Delta}$. But since e belongs to the Pareto frontier $\mathcal{P}(\mathcal{E}_{\leq\Delta})$, there cannot exist a point $e' \in \mathcal{E}_{\leq\Delta}$ satisfying $e'_r \leq e_r$ and $e'_b \leq e_b$ with either inequality strict. Thus

$$e'_r = e_r \quad e'_b = e_b \quad u(e') < u(e)$$

in contradiction of the definition of $u(e)$.

In the other direction, consider any $e \in \mathscr{F}$ and set $\Delta \equiv u(e)$ so that $e \in \mathcal{E}_{\leq\Delta}$. We will show that $e \in \mathcal{P}(\mathcal{E}_{\leq\Delta})$. Suppose not. Then there exists $e' \in \mathcal{E}_{\leq\Delta}$ such that $e'_r \leq e_r$ and $e'_b \leq e_b$ with at least one inequality strict. But since also $u(e') \leq u(e) = \Delta$, it must be that $e' >_{FA} e$, and we have the desired contradiction. □

Assumption 1 implies that the Pareto frontiers $\mathcal{P}(\mathcal{E}_{\leq\Delta})$ takes either of two forms:

Lemma O.6. *For every $\Delta \geq \underline{\Delta}$,*

(a) *If $\mathcal{E}_{\leq\Delta} \cap \mathcal{P}(\mathcal{E}) \neq \emptyset$, then $\mathcal{P}(\mathcal{E}_{\leq\Delta}) = \mathcal{E}_{\leq\Delta} \cap \mathcal{P}(\mathcal{E})$*

(b) *If $\mathcal{E}_{\leq\Delta} \cap \mathcal{P}(\mathcal{E}) = \emptyset$, then $\mathcal{P}(\mathcal{E}_{\leq\Delta})$ is a singleton.*

That is, if the sublevel set $\mathcal{E}_{\leq\Delta}$ has nonempty intersection with the Pareto frontier $\mathcal{P}(\mathcal{E})$, then the Pareto frontier of $\mathcal{E}_{\leq\Delta}$ is precisely this intersection. Otherwise, the Pareto frontier $\mathcal{P}(\mathcal{E}_{\leq\Delta})$ must be a singleton.

Proof. Suppose $\mathcal{E}_{\leq\Delta}$ has nonempty intersection with the accuracy frontier $\mathcal{P}(\mathcal{E})$. This intersection $\mathcal{E}_{\leq\Delta} \cap \mathcal{P}(\mathcal{E})$ must be part of the frontier $\mathcal{P}(\mathcal{E}_{\leq\Delta})$, since if a point e is Pareto undominated within $\mathcal{E}$, it must also be Pareto undominated within the smaller set $\mathcal{E}_{\leq\Delta}$.

Suppose $\mathcal{P}(\mathcal{E}_{\leq\Delta})$ includes a point that does not belong to $\mathcal{P}(\mathcal{E})$. Since the sublevel sets are nested convex polygons (by Lemma O.4), $\mathcal{P}(\mathcal{E}_{\leq\Delta})$ must include an entire line segment not included in $\mathcal{P}(\mathcal{E})$. This line segment must further be negatively sloped, since any Pareto frontier consists exclusively of negatively sloped lines. Choose any point e in the interior of this line segment. Since e is not in $\mathcal{P}(\mathcal{E})$, it must be Pareto dominated by some other point $e' \in \mathcal{E}$. Consider a point e^* between e' to e and arbitrarily close to e. Since the line segment must have negative slope, it must be that $D_{(1,0)}u(e^*) < 0$ and $D_{(0,1)}u(e^*) < 0$. But this contradicts Assumption 1. So $\mathcal{P}(\mathcal{E}_{\leq\Delta})$ cannot include any points outside of $\mathcal{P}(\mathcal{E})$, and we conclude that $\mathcal{P}(\mathcal{E}_{\leq\Delta}) = \mathcal{E}_{\leq\Delta} \cap \mathcal{P}(\mathcal{E})$ as desired.

Now suppose $\mathcal{E}_{\leq\Delta} \cap \mathcal{P}(\mathcal{E}) = \emptyset$. Suppose towards contradiction that $\mathcal{P}(\mathcal{E}_{\leq\Delta})$ is not a singleton. Then $\mathcal{P}(\mathcal{E}_{\leq\Delta})$ consists of negatively sloped line segments. Choose some point e in the interior of one such line segment. Since $\mathcal{P}(\mathcal{E}_{\leq\Delta})$ does not intersect with $\mathcal{P}(\mathcal{E})$, e must be Pareto dominated by some point $e' \in \mathcal{E}$. By the same argument above, this contradicts Assumption 1. □

Now we can complete the proof of Theorem O.1. First suppose F_X belongs to the accuracy frontier $\mathcal{P}(\mathcal{E})$. Since every $\mathcal{E}_{\leq\Delta}$ for $\Delta \geq \underline{\Delta}$ includes F_X, each sublevel set must have nonempty intersection with $\mathcal{P}(\mathcal{E})$. Applying Lemma O.5 and Part (a) of Lemma O.6, each $\mathcal{P}(\mathcal{E}_{\leq\Delta})$ is a subset of $\mathcal{P}(\mathcal{E})$, and we recover all of $\mathcal{P}(\mathcal{E})$ as we vary over Δ. So $\mathscr{F} = \mathcal{P}(\mathcal{E})$.

Next suppose F_X does not belong to the accuracy frontier $\mathcal{P}(\mathcal{E})$. Define

$$U \equiv \{\Delta \mid \mathcal{E}_{\leq\Delta} \cap \mathcal{P}(\mathcal{E}) = \emptyset\}$$

be the unfairness levels Δ for which the sublevel set $\mathcal{E}_{\leq\Delta}$ have empty intersection with $\mathcal{P}(\mathcal{E})$. For any $\Delta \in U^c$, the previous arguments apply and show that the full accuracy frontier $\mathcal{P}(\mathcal{E})$ is again recovered as part of the fairness-accuracy frontier $\mathscr{F}$.

For any $\Delta \in U$, Part (b) of Lemma O.6 implies that the accuracy frontier in this sublevel set is a singleton, and hence can be characterized as the point $A_\Delta = \arg\min_{e \in \mathcal{E}_{\leq\Delta}} e_r$, where the choice of group r is arbitrary. The sublevel set $\mathcal{E}_{\leq\Delta}$ is convex and compact for each $\Delta \in U$, and $\mathcal{E}_{\leq\Delta}$ is continuous at each $\Delta \in U$ by continuity of $u(e)$. By the theorem of the maximum, A_Δ is continuous in Δ, so the set $\{A_\Delta\}_{\Delta\geq 0}$ is connected. By Lemma O.4, this path consists of a sequence of line segments. Moreover, since the sets $\mathcal{E}_{\leq\Delta}$ are nested, and the point A_Δ simultaneously minimizes e_r and e_b within the set $\mathcal{E}_{\leq\Delta}$, these points must move weakly down and left as Δ increases, so the path consists of a sequence of positively sloped line segments. Thus the fairness-accuracy frontier $\mathscr{F}$ is the union of the accuracy frontier $\mathcal{P}(\mathcal{E})$ and a sequence of positively sloped line segments connecting $\mathcal{P}(\mathcal{E})$ to F_X, as desired.

O.3. **General Fairness Criteria.** In this section, we consider the general case where fairness is evaluated using $|\phi(e_r) - \phi(e_b)|$ for some strictly increasing continuous function ϕ. For instance, if ϕ is log, then this reduces to using the ratio of error rates as a measure of fairness. The characterization of the fairness-accuracy frontier remains the same except the fairness optimal point F_X may now be different. Whether it expands or contracts depends on the curvature of ϕ as the following Proposition demonstrates.[42]

Proposition O.1. *Let $\mathscr{F}'(X)$ denote the fairness-accuracy frontier where fairness is evaluated using*

$$|\phi(e_r) - \phi(e_b)|$$

for strictly increasing $\phi : \mathbb{R} \to \mathbb{R}$. Then

(1) $\mathscr{F}(X) = \mathscr{F}'(X)$ if X is group-balanced
(2) $\mathscr{F}(X) \subset \mathscr{F}'(X)$ if X is group-skewed and ϕ in concave
(3) $\mathscr{F}(X) \supset \mathscr{F}'(X)$ if X is group-skewed and ϕ in convex

Proof. Let $\mathcal{E}(X)$ and $\mathcal{E}'(X)$ denote the feasible sets where fairness is defined using $|e_r - e_b|$ and $|\phi(e_r) - \phi(e_b)|$ respectively. Let F_X and F'_X denote the corresponding fairness optimal points. First, note that if X is group-balanced, then by the same argument as Theorem 1, $\mathscr{F}(X) = \mathscr{F}'(X)$ is the lower boundary from $R_X = R'_X$ to $B_X = B'_X$.

Now, suppose X is r-skewed without loss. Let e and e' correspond to F_X and F'_X so

$$e_b - e_r \leq e'_b - e'_r$$
$$\phi(e'_b) - \phi(e'_r) \leq \phi(e_b) - \phi(e_r)$$

First, suppose ϕ is concave. We will show that $e'_r \geq e_r$. Suppose by contradiction that $e'_r < e_r$ so $\phi(e'_r) < \phi(e_r)$. Thus,

$$\phi(e'_b) - \phi(e_b) \leq \phi(e'_r) - \phi(e_r) < 0$$

so $e'_b < e_b$. Thus, we have $e'_r \leq e'_b < e_b$. Note that

$$e'_b = \lambda e_b + (1-\lambda)\, e'_r$$

[42]We assume that the accuracy and fairness loss functions are the same but can generalize the results in this section via the same methodology as in Section O.1.

where

$$\lambda := \frac{e'_b - e'_r}{e_b - e'_r}$$

We thus have

$$\begin{aligned}
\phi(e_b) - \phi(e_r) + \phi(e'_r) &\geq \phi(e'_b) = \phi(\lambda e_b + (1-\lambda)e'_r) \\
&\geq \lambda\phi(e_b) + (1-\lambda)\phi(e'_r) \\
(1-\lambda)(\phi(e_b) - \phi(e'_r)) &\geq \phi(e_r) - \phi(e'_r) \\
(e_b - e'_b)\frac{\phi(e_b) - \phi(e'_r)}{e_b - e'_r} &\geq \phi(e_r) - \phi(e'_r)
\end{aligned}$$

where the second inequality follows from the fact that ϕ is concave. Since $e_r - e'_r \geq e_b - e'_b$, this implies

$$\frac{\phi(e_b) - \phi(e'_r)}{e_b - e'_r} \geq \frac{\phi(e_r) - \phi(e'_r)}{e_r - e'_r}$$

Since X is r-skewed, $e_b \geq e_r > e'_r$. Since ϕ is concave, the above inequality must be satisfied with equality. This means that

$$(e_b - e'_b)\frac{\phi(e_b) - \phi(e'_r)}{e_b - e'_r} \geq \phi(e_r) - \phi(e'_r) = (e_r - e'_r)\frac{\phi(e_b) - \phi(e'_r)}{e_b - e'_r}$$

so $e_b - e'_b = e_r - e'_r$ or $e_b - e_r = e'_b - e'_r$. But e corresponds to F_X and since e' achieves the same fairness as e, it must be that $e_r \leq e'_r$. This contradicts our assumption that $e'_r < e_r$. Thus, $e'_r \geq e_r$ and by the same argument characterizing the FA frontier as in Theorem 1, $\mathscr{F}(X) \subset \mathscr{F}'(X)$. The case for when ϕ is convex is symmetric. □

O.4. **Adversarial Agents.** We now consider the problem outlined in Section 4, when one of the weights α_r, α_b is negative.[43] Without loss, let $\alpha_r > 0 > \alpha_b$, reflecting an adversarial agent who prefers for group b's error to be higher. The first half of Lemma 1 extends fully.

Lemma O.7. *For every covariate vector X, $\mathcal{E}^*(X) = \mathcal{E}(X) \cap H$.*

But the analogous equivalence for the FA frontier does not extend. Instead, similar to the development of R_X, B_X, and F_X, define

$$G^*_X \equiv \operatorname*{arg\,min}_{(e_r,e_b)\in\mathcal{E}^*(X)} e_g$$

to be the feasible point in $\mathcal{E}^*(X)$ that minimizes group g's error (breaking ties by minimizing the other group's error), and define

$$F^*_X \equiv \operatorname*{arg\,min}_{(e_r,e_b)\in\mathcal{E}^*(X)} |e_r - e_b|$$

[43]It is straightforward also to consider the case where both weights are negative, but we do not consider this setting to be practically relevant.

to be the point that minimizes the absolute difference between group errors (breaking ties by minimizing either group's error).

Definition O.6. Covariate vector X is:

- *input-design r-skewed* if $e_r < e_b$ at R^*_X and $e_r \leq e_b$ at B^*_X
- *input-design b-skewed* if $e_b < e_r$ at B^*_X and $e_b \leq e_r$ at R^*_X
- *input-design group-balanced* otherwise

The proof for Theorem 1 applies for any compact and convex feasible set, and so directly implies:

Theorem O.2. *The input-design fairness-accuracy (FA) frontier $\mathscr{F}^*(X)$ is the lower boundary of the input-design feasible set $\mathcal{E}^*(X)$ between*

(a) *R^*_X and B^*_X if X is input-design group-balanced*
(b) *G^*_X and F^*_X if X is input-design g-skewed*

We can use this characterization to extend our result from Section 4.2.1.

Definition O.7. X is *strictly input-design-group-balanced* if $e_r < e_b$ at R^*_X and $e_b < e_r$ at B^*_X.

Proposition O.2. *Suppose $\alpha_r > 0 > \alpha_b$ and X is strictly input-design group-balanced. Then excluding G over X uniformly worsens the frontier.*

This result says that, perhaps surprisingly, even if the agent choosing the algorithm has adversarial motives against one of the groups, the designer may still prefer to send information about group identity. The notion of group-balanced covariate vectors, suitably adapted to the input design setting, again serves as a sufficient condition for uniform worsening of the frontier when excluding G.

Proof. By assumption that X is strictly input-design group-balanced, the input-design FA frontier given X is the lower boundary of $\mathcal{E}^*(X)$ from R^*_X to B^*_X, which consists of negatively sloped edges. We will show that every point on this frontier is FA-dominated by some point in $\mathcal{E}^*(X,G)$.

If this point (e_r, e_b) is distinct from B^*_X and R^*_X, then we claim that for sufficiently small positive ϵ, the point $(e_r - \epsilon, e_b - \epsilon)$ belongs to $\mathcal{E}^*(X,G)$. Indeed, $(e_r - \epsilon, e_b - \epsilon)$ belongs to the unconstrained feasible set $\mathcal{E}(X,G)$ because this feasible set is a rectangle, and $e_r - \epsilon$, $e_b - \epsilon$ are within the minimal and maximal group errors achievable given X. Moreover, (e_r, e_b) must have smaller group-r error and larger group-b error compared to B^*_X, which means the same is true for $(e_r - \epsilon, e_b - \epsilon)$. Since $\alpha_r > 0 > \alpha_b$, the point $(e_r - \epsilon, e_b - \epsilon)$ must belong to H given that B^*_X does. Hence when (e_r, e_b) differs from B^*_X and R^*_X, it is FA-dominated by $(e_r - \epsilon, e_b - \epsilon) \in \mathcal{E}^*(X,G)$.

Suppose now that $(e_r, e_b) = B_X^*$. Then by similar argument it is FA-dominated by $(e_r - \epsilon, e_b) \in \mathcal{E}^*(X, G)$. Finally if $(e_r, e_b) = R_X^*$, then it is FA-dominated by $(e_r, e_b - \epsilon) \in \mathcal{E}^*(X, G)$. In all these cases the FA frontier uniformly worsens when excluding G, completing the proof. □

O.5. **Supplementary Material to Section 3.3.**

O.5.1. *Proof of Proposition 2.* We will show that $B_X = R_X$ under conditional independence. Recall from the proof of Lemma B.1 that

$$\mathcal{E}\left(X\right) = \sum_{x \in \mathcal{X}} E\left(x\right) p_x$$

where

$$\begin{aligned} E\left(x\right) = \Bigg\{ & \lambda \left(\sum_y \frac{x_{y,r}}{p_r} \ell\left(1, y\right), \sum_y \frac{x_{y,b}}{p_b} \ell\left(1, y\right) \right) \\ & + \left(1 - \lambda\right) \left(\sum_y \frac{x_{y,r}}{p_r} \ell\left(0, y\right), \sum_y \frac{x_{y,b}}{p_b} \ell\left(0, y\right) \right) \; : \; \lambda \in [0,1] \Bigg\} \end{aligned}$$

Under conditional independence, $x_{y,g} = x_y x_g$ so we have

$$E\left(x\right) = \left\{ \left(\lambda \sum_y x_y \ell\left(1, y\right) + \left(1 - \lambda\right) \sum_y x_y \ell\left(0, y\right) \right) \left(\frac{x_r}{p_r}, \frac{x_b}{p_b} \right) \; : \; \lambda \in [0,1] \right\}$$

This means that for each realization $x \in \mathcal{X}$, the outcome that gives the lower error for group r also gives the lower error for group b. In other words, when $\sum_y x_y \ell\left(1, y\right) \leq \sum_y x_y \ell\left(0, y\right)$, then outcome $Y = 1$ is optimal for both groups (and vice-versa for the other outcome). Consider the following algorithm:

$$f\left(x\right) = \begin{cases} 1 & \text{if } \sum_y x_y \ell\left(1, y\right) \leq \sum_y x_y \ell\left(0, y\right) \\ 0 & \text{if } \sum_y x_y \ell\left(1, y\right) > \sum_y x_y \ell\left(0, y\right) \end{cases}$$

This algorithm will deliver the lowest error for both groups and

$$\left(e_r\left(f\right), e_b\left(f\right)\right) = R_X = B_X$$

as desired.

O.5.2. *Strong Independence.* We consider here another special case of conditional independence when covariate vectors satisfy the following strong independence condition:

Definition O.8. Say that X satisfies *strong independence* if for both groups g,

$$\mathbb{P}(G = g \mid Y = y, X = x) = p_g \quad \forall x, y.$$

In this case, the feasible set turns out to be a line segment on the 45-degree line, and the fairness-accuracy frontier is a single point, as depicted in Figure 10.

Proposition O.3. *Suppose X is strongly independent. Then the fairness-accuracy frontier is a single point on the 45-degree line.*

Proof. We continue to follow the notation laid out in the proof of Lemma B.1. Note that under strong independence,

$$\begin{aligned} \frac{x_{y,r}}{x_{y,b}} &= \frac{\mathbb{P}(Y=y, G=r \mid X=x)}{\mathbb{P}(Y=y, G=b \mid X=x)} \\ & \frac{\mathbb{P}(Y=y, G=r, X=x)}{\mathbb{P}(Y=y, G=b, X=x)} \\ &= \frac{\mathbb{P}(G=r \mid Y=y, X=x)}{\mathbb{P}(G=b \mid Y=y, X=x)} = \frac{p_r}{p_b}. \end{aligned}$$

Thus $\frac{x_{y,r}}{p_r} = \frac{x_{y,b}}{p_b}$ for all x, y. It follows that the line segment $E(x)$, which connects the two points $\left(\sum_y \frac{x_{y,r}}{p_r}\ell(1,y), \sum_y \frac{x_{y,b}}{p_b}\ell(1,y)\right)$ and $\left(\sum_y \frac{x_{y,r}}{p_r}\ell(0,y), \sum_y \frac{x_{y,b}}{p_b}\ell(0,y)\right)$, lies on the 45-degree line. Therefore $\mathcal{E}(X) = \sum_x E(x) \cdot p_x$ is also on the 45-degree line. □

The FA frontier consists of the single point that is achieved by conditioning on all of the available information in X. Since this point is on the 45-degree line, both groups have the same error. Thus, this point is simultaneously optimal for Rawlsian, Utilitarian, and Egalitarian designers—indeed, fairness-accuracy preferences are completely irrelevant here: All designers who agree on the basic FA-dominance principle outlined in Definition 2 prefer the same policy.

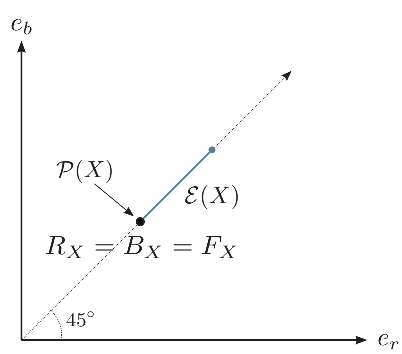


FIGURE 10. Depiction of the fairness-accuracy frontier under assumption of strong independence

O.6. **Microfoundations for the FA frontier.** We now provide a foundation of our FA frontier as the optimal points for different classes of FA preferences.[44] First, consider the following utility over errors

$$w(e_r, e_b) = \alpha_r e_r + \alpha_b e_b + \alpha_f |e_r - e_b|$$

where $\alpha_r, \alpha_b < 0$ and $\alpha_f \leq 0$. Call the corresponding preference of this utility *simple.* Simple preferences are FA preferences. For example, both the Utilitarian and Rawlsian preferences are simple. To see this for the Utilitarian designer, set $\alpha_r = -p_r$, $\alpha_b = -p_b$ and $\alpha_f = 0$. To see this for the Rawlsian designer, set $\alpha_r = \alpha_b = \alpha_f = -1$.

Given any FA preference $\succeq$, let

$$\mathscr{F}_{\succeq}(X) = \{e \in \mathcal{E}(X) : e \succeq e' \text{ for all } e' \in \mathcal{E}(X)\}$$

denote the set of $\succeq$-optimal points. We now provide the following characterizations of the FA frontier.[45]

Proposition O.4. *The following are equivalent:*

(1) $e \in \mathscr{F}(X)$
(2) $e \in \mathscr{F}_{\succeq}(X)$ *for some FA preference* $\succeq$
(3) $\{e\} = \mathscr{F}_{\succeq}(X)$ *for some FA preference* $\succeq$
(4) $e \in \mathscr{F}_{\succeq}(X)$ *for some simple FA preference* $\succeq$

The above result shows that our FA frontier is the set of all optimal points for all FA preferences. Moreover, $\mathscr{F}(X)$ is minimal in the sense that we cannot exclude any points from $\mathscr{F}(X)$ without hurting some designer. This is because for every point $e \in \mathscr{F}(X)$, there exists some FA preference $\succeq$ such that e is the *unique* optimal error pair given $\succeq$ within the feasible set $\mathcal{E}(X)$. Finally, our FA frontier also corresponds to the optimal points for all simple FA preferences.

Proof. We will first show that (3) implies (2) implies (1) implies (3). Note that (3) implies (2) is trivial. To see why (2) implies (1), suppose $e \in \mathscr{F}_{\succeq}(X)$ for some FA preference $\succeq$ but $e \notin \mathscr{F}(X)$. Thus, there exists some $e' >_{FA} e$ so $e' \succ e$ yielding a contradiction.

We now prove that (1) implies (3). Fix some $e^* \in \mathscr{F}(X)$ and let $h : \mathbb{R} \to (0,1)$ be a strictly decreasing function. Define

$$w(e) = \begin{cases} 1 + h(e_r + e_b) & \text{if } e = e^* \text{ or } e >_{FA} e^* \\ h(e_r + e_b) & \text{otherwise} \end{cases}$$

[44]Note that we could have alternatively defined FA preferences to be weakly decreasing in e_r, e_b and $|e_r - e_b|$. The equivalence of (1), (3) and (4) in Proposition O.4 would still hold.
[45]The proof of the equivalence of (1) and (4) in Proposition O.4 relies on finite X. The other parts do not.

and let $\succeq$ be the corresponding preference. We will show that $\succeq$ is an FA preference. Suppose $e >_{FA} e'$ so $h(e_r + e_b) > h(e'_r + e'_b)$. If both points FA-dominate e^* or neither do, then $w(e) > w(e')$. The only remaining case is when $e >_{FA} e^*$ but e' does not FA-dominate e^*, in which case

$$w(e) = 1 + h(e_r + e_b) > 1 > h(e'_r + e'_b) = w(e')$$

Thus, $\succeq$ is an FA preference. Now, since $e^* \in \mathscr{F}(X)$, there exists no other $e \in \mathcal{E}(X)$ such that $e >_{FA} e^*$. That means that for all $e \in \mathcal{E}(X) \setminus \{e^*\}$, $w(e^*) > w(e)$ so $\{e^*\} = \mathscr{F}_{\succeq}(X)$. This proves (3).

Finally, we show the equivalence of (1) and (4). Note that (4) implies (2) which implies (1) from above. We now show that (1) implies (4). Fix some $e^* \in \mathscr{F}(X)$, so by Theorem 1, e^* must either belong to the lower boundary from R_X to B_X or the lower boundary from B_X to F_X, where the latter case only happens when X is r-skewed (we omit the symmetric situation when X is b-skewed). If e^* belongs to the boundary from R_X to B_X, then from the proof of Theorem 1 we know that e^* belongs to an edge of this boundary that has negative slope. Thus there exists a vector (α_r, α_b) that is normal to this edge, such that e^* maximizes $\alpha_r e_r + \alpha_b e_b$ among all feasible points. Since this edge has negative slope, it is straightforward to see that $\alpha_r, \alpha_b < 0$. So e maximizes the simple utility $\alpha_r e_r + \alpha_b e_b$ as desired.

If instead X is r-skewed and e^* belongs to the boundary from B_X to F_X, then again e^* belongs to an edge of this boundary. But now this edge must have weakly positive slope (since the edge starting from B_X has weakly positive slope by the definition of B_X, and since the boundary is convex). In addition, this slope must be strictly smaller than 1 because otherwise F_X would be farther away from the 45-degree line compared to its adjacent vertex on this boundary. It follows that the outward normal vector (β_r, β_b) to the edge that e^* belongs to satisfies $\beta_r \geq 0 \geq -\beta_r > \beta_b$. The point e^* of interest maximizes $\beta_r e_r + \beta_b e_b$ among all feasible points. Now let us choose any α_f to belong to the interval $(\beta_b, -\beta_r)$, which is in particular negative. Further define $\alpha_r = \beta_r + \alpha_f < 0$ and $\alpha_b = \beta_b - \alpha_f < 0$. Then $\beta_r e_r + \beta_b e_b$ can be rewritten as $\alpha_r e_r + \alpha_b e_b + \alpha_f(e_b - e_r)$. If we consider the simple utility $\alpha_r e_r + \alpha_b e_b + \alpha_f |e_b - e_r|$, then for any other feasible point e^{**} it holds that

$$\begin{aligned}
\alpha_r e_r^{**} + \alpha_b e_b^{**} + \alpha_f |e_b^{**} - e_r^{**}| &\leq \alpha_r e_r^{**} + \alpha_b e_b^{**} + \alpha_f (e_b^{**} - e_r^{**}) \\
&= \beta_r e_r^{**} + \beta_b e_b^{**} \\
&\leq \beta_r e_r^{*} + \beta_b e_b^{*} \\
&= \alpha_r e_r^{*} + \alpha_b e_b^{*} + \alpha_f (e_b^{*} - e_r^{*}) \\
&= \alpha_r e_r^{*} + \alpha_b e_b^{*} + \alpha_f |e_b^{*} - e_r^{*}|,
\end{aligned}$$

where the first inequality holds since $\alpha_f \leq 0$ and the last equality holds because $e^* \in \mathscr{F}(X)$ must be weakly above the 45-degree line. Hence the above inequality shows that e^* maximizes the simple utility we have constructed, completing the proof. $\square$