LOW SAFE INTEREST RATES: A CASE FOR DYNAMIC INEFFICIENCY?

GAETANO BLOISE AND PIETRO REICHLIN

Current version: April 20, 2023

Abstract

We reexamine the tests for dynamic inefficiency in productive overlapping-generations economies with stochastic growth. Contrary to certain claims in the recent literature, we argue that the size of real, long-term, safe interest rates relative to average GDP growth is an inconclusive test for dynamic inefficiency. A more accurate test should take into account the correlation between growth and the marginal utility of wealth. We provide an exhaustive criterion based on the growth-adjusted *dominant root* of the stochastic discount factor emerging at the competitive equilibrium. Surprisingly, a preliminary rough empirical application of this criterion uncovers *dynamic inefficiency* of the US economy for any reasonable degree of risk aversion. We also distinguish capital overaccumulation from an inefficient distribution of consumption risk. The refined test for capital overaccumulation is rather stringent: Capital is not overaccumulated if the net dividend remains positive *with some probability*, as opposed to *always*, as in the original Abel et al. [1]'s formulation.

JEL Classification Numbers: D60, G1, E21, E62, H2, H21.

1. INTRODUCTION

Real yields on safe bonds have been persistently low relative to GDP growth for most of the last seventy years, and especially since the late 1980s. According to the evidence documented by Blanchard [10], the 10-year rate on US T-bills has averaged 5.6%, while nominal GDP growth has averaged 6.3% from 1950 onward (see also Del Negro et al. [19], Lunsford and West [28] and Rogoff et al. [32]). This phenomenon has generated an intense debate about *dynamic inefficiency* and the social benefits of government debt rollover. Using Blanchard [10]'s own words, 'the signal sent by low rates is not only that debt may not have a substantial fiscal cost, but also that it may have limited welfare costs'. In a deterministic environment, the assessment reduces to a straight comparison of the safe interest rate with the rate of growth of the economy. In an uncertain world, however, a welfare evaluation is more controversial because returns and growth rates are time-varying, affected by uncertain events, and historical averages are only unreliable statistics. The purpose of this paper is to clarify which relation between safe rates, risky returns and

We are grateful to Herakles Polemarchakis, Paolo Siconolfi and Yiannis Vailakis for comments and observations. We also thank two anonymous reviewer and the editor for their suggestions.

Gaetano Bloise, Department of Economics and Finance, Tor Vergata University of Rome, Rome (Italy).

Pietro Reichlin, Department of Economics and Finance, LUISS 'Guido Carli', Rome (Italy); CEPR; EIEF.

GDP growth rates is relevant for assessing dynamic inefficiency in an economy with truly stochastic growth.

In their influential paper, Abel et al. [1] observed that the historical experience of a safe interest rates below an economy's average growth rate is not sufficient to establish dynamic inefficiency. Indeed, due to capital risk, a low safe rate can coincide permanently with a high rate of profit in a dynamically efficient economy. They provided an alternative sufficient criterion guaranteeing dynamic efficiency on the economy, the so-called *net dividend criterion*. However, this criterion is inconclusive when capital income exceeds investment or falls short of it some of the time, and this limitation appears to be rather relevant in empirical applications. Although Abel et al. [1] found that their sufficient criterion is verified based on US data between 1929 and 1985, Geerolf [20] documented a failure of the criterion for a variety of advanced economies using more recent observations.

A resurgence of interest for dynamic inefficiency is testified by a flourishing recent literature on stochastic overlapping-generations economies with capital accumulation. Abel and Panageas [2] claim that a welfare-improving debt rollover is feasible even in a dynamically efficient economy whenever the *safe* rate falls short of the growth rate of the economy. Hellwig [22] contends that the assessment of dynamic inefficiency must consider the *safe* rate of return and that '[c]ontrary claims in the literature are based on misunderstandings'. Kocherlakota [25] asserts that equilibrium is dynamically inefficient when the yield of a long-term discount bond (a sort of long-term *safe* rate) is dominated by the population growth rate, even when the short-term safe rate itself exceeds growth. Altogether, and somewhat in contrast with Abel et al. [1]'s earlier findings, these contributions advocate a determinant role of safe rates for dynamic inefficiency. We shall instead argue that average short- and long-term safe rates are only unreliable statistics and propose a more accurate criterion.

To investigate these issues, we study a conventional overlapping-generations economy with capital accumulation where uncertainty derives from productivity shocks affecting the GDP growth rate.[1] We distinguish between two potential sources of inefficiency: conditional Pareto inefficiency, defined as the occurrence of feasible Pareto improvements conditional on the state at which generations are born, and capital overaccumulation, defined as the possibility of increasing aggregate consumption at all contingencies through progressive capital reductions. It is known that, under uncertainty, conditional Pareto inefficiency might occur without capital overaccumulation, due to a misallocation of consumption risk (Barbie et al. [8]). The distinction is not a merely scholastic exercise, and it is relevant both for the assessment and for the implied policy prescriptions.

[1]The nature of uncertainty is not relevant for our analysis, and we can straightforwardly encompass stochastic population growth. We take expositional advantage from the fact the growth rates are exogenously determined. Otherwise our approach would preliminarily need the identification of a maximum sustainable growth path for the economy, and then an adjustment all the arguments consistently.

To ascertain conditional Pareto inefficiency involves judgements on individuals' preferences for intertemporal substitution and attitudes towards consumption risk, whereas capital overaccumulation is exhaustively reflected by capital returns relative to growth rates, independently of individuals' preferences. Comparatively more information is to be extracted from market prices in order to establish conditional Pareto inefficiency and, as we shall argue extensively in this paper, a persistently low safe interest rate might well be a misguided sufficient statistic. Furthermore, the schemes of transfers correcting conditional Pareto inefficiency are in general highly state-dependent, require calibrated compensations across generations, and do not reduce to a straight reallocation of uncontingent consumption from young to old individuals.

Drawing on the established literature, we present exhaustive and operational criteria both for conditional Pareto inefficiency and for capital overaccumulation. For heuristic purposes, inspired by Kocherlakota [25], we preliminarily consider a simplified framework in which a stochastically growing endowment can be stored yielding an uncertain return. We assume that both the growth of endowments and the rate of return on storage follow a Markov chain and that utility is homothetic, so that competitive equilibrium inherits the Markov property. In such an environment, we argue that conditional Pareto inefficiency is fully characterized by the *dominant root* of the matrix of *growth-adjusted state prices*, extending Aiyagari and Peled [3].[2] This allows us to draw certain implications of low interest rates for dynamic inefficiency.

The necessity of a dominant root approach is a natural consequence of a time-varying environment. Interest rates need be compounded over time, so as to estimate the welfare-effects of consumption reallocations propagating across periods, and the dominant root serves to extrapolate long-term tendencies. Understated by the previous literature was the role of stochastic growth rates. A straight comparison of long-term interest rate with the average growth is highly deceptive. Indeed, the same physical transfer entails different implications for social welfare when growth is low rather than high. The interest rate reflects the first-order effect for a single individual, but the need of intergenerational compensations imposes further discipline in terms of feasibility of a perpetual scheme of transfers. As a consequence, *safe* interest rates are not really *safe* when growth is stochastic and have to be upward corrected by the negative correlation between growth and the marginal utility of wealth: low interest rates might well be consistent with conditional Pareto efficiency of a competitive equilibrium. In an hypothetical world in which all variables are identically and independently distributed, a test for conditional Pareto efficiency would reduce to

$$r > \mathbb{E}g + (1+r)\operatorname{cov}(g, m),$$

where r is the safe rate, g is the growth rate and m is the marginal utility of wealth, which is typically *negatively* correlated with capital returns and output growth.

[2] A growth-adjustment in a stochastic environment is also studied by Kocherlakota [26, Section 4].

We further argue that, in line with the previous literature (*e.g.*, Barbie et al. [8]), conditional Pareto inefficiency might occur even absent capital overaccumulation. A criterion for capital overaccumulation necessarily requires marginal productivity of capital falling short of growth in all states of the world and in all periods over the infinite horizon. This is substantially more demanding than Abel et al. [1]'s net dividend criterion established in the theoretical and empirical literature. The amended criterion might help to dissipate the mentioned empirical ambiguities exhibited by historical time series.

Unfortunately, competitive equilibria are only fortuitously Markovian in an overlapping-generations economy with capital accumulation, and this complicates our analysis substantially. However, our theory extends, and all intuitions remain, by means of a more sophisticated approach. The method is based on locating the *spectral radius* of a sort of (growth-adjusted) valuation operator commonly used in macroeconomic finance theory, an extension of the dominant root approach for finite Markov chains. As the criterion is grounded on the conventional stochastic discount factor, it is operational and suitable of an empirical application, given the recent progresses on long-term risk and asset prices pioneered by Alvarez and Jermann [5], Christensen [17] and Hansen and Scheinkman [21]. In the context of a nonparametric recursive utility model of risk, for instance, Christensen [17] provides empirical estimates of the spectral radius of the stochastic discount factor for the US economy.

The spectral radius of the valuation operator is related to the yield of long-term discount bonds. Kocherlakota [25] argues that the long-term yield, relative to growth, is the relevant statistic for dynamic inefficiency. We find that, under stochastic growth, this characterization requires a major amendment, as the long-term yield might fall short of growth even when the economy is dynamically efficient. The relevant statistic is the yield of an *hypothetical* long-term discount bond *indexed* to long-term growth. Therefore, the historical observation of low long yields on government bonds, documented in Blanchard [10], cannot be taken as a persuasive evidence of dynamic inefficiency.

Without aim at being exhaustive, we provide a preliminary empirical application of our criterion to the US economy. We postulate a CES-type stochastic discount factor and, assuming a conventional autoregressive form for output growth, we manage to derive an explicit formula for the growth-adjusted spectral radius. We then calibrate the formula on the empirical averages of interest and growth rates for the US economy. It turns out that, for plausible degrees of risk-aversion, the spectral radius falls outside the unit circle, thus providing evidence of dynamic inefficiency. We should stress, however, that, in general, this finding does not imply that government Ponzi schemes are feasible or that social security transfers are welfare improving. In fact, social security schemes and debt Ponzi games hinge on the stronger requirement that intergenerational transfers from young to old, or a debt rollover, are sustainable along all histories of GDP shocks, and this condition may not be verified even if the economy is dynamically inefficient.

The paper is organized as follows. In section 3 we present a simple, and unambitious, application of our criterion. In section 4 we illustrate our criterion in a simplified Markov setting. In sections 5-6 we describe a competitive equilibrium of an overlapping-generations economy with stochastic growth and we establish necessity and sufficiency of our criterion.

2. Related literature

Capital overaccumulation was initially studied by Cass [14] in a deterministic environment. He proved that dynamic inefficiency can be characterized in terms of the sequence of gross interest rates which, absent uncertainty, are equal to the gross marginal products of capital. Too much capital implies a small marginal productivity and small interest rates. The Cass Criterion for testing dynamic inefficiency asserts that the sum of future values of capital units over the infinite time, net of population or GDP growth, diverges.

Balasko and Shell [6] showed that the divergence of the infinite sum of the reciprocals of the contingent commodity prices (the criterion due to Cass) completely characterizes competitive Pareto optimal allocations in deterministic pure-exchange overlapping-generations economies. Similar characterizations were obtained by Peled [31] and Manuelli [29] under stationary uncertainty, whereas Chattopadhyay and Gottardi [16] developed a more general Cass Criterion in a pure endowment economy based on the convergence of the weighted sum of the reciprocals of present value prices.

Assuming stationary uncertainty, Zilcha [35] provided a test for dynamic inefficiency based on the expected value of the log of a function representing the asymptotic value of compounded marginal products of capital. The intuition is that, under uncertainty, a relatively large marginal productivity of capital does not necessarily imply a large value of the long-run safe rate (as it would be the case in a deterministic setting).

Turning to the more recent literature, our paper is also related to Abel and Panageas [2] and Hellwig [22] Both argue that conditional Pareto inefficiency arises if the safe rate falls short of the growth rate of the economy.[3] Differently from our findings, these assessments are not related to the returns on risky assets. As our characterization of dynamic efficiency is (almost) exhaustive, this discrepancy can only be justified by a misalignment of the assumptions on the primitives of the economy. In fact, their characterizations crucially rely on the absence of a proper stochastic growth.

[3]Literally, Abel and Panageas [2] assert that a welfare-improving debt rollover is feasible in a dynamically efficient economy. However, dynamic efficiency in their analysis is to be understood as the absence of capital overaccumulation, as implied by their explicit reference to Zilcha [35]'s criterion. Thus, for our purposes of comparison, they prove that equilibrium is conditionally Pareto inefficient when the safe rate is dominated by the growth rate.

3. A SIMPLE TEST

We provide a simple empirical test for dynamic inefficiency based on the dominant root (spectral radius) of the implicit stochastic discount factor determined at a competitive equilibrium. The theory justifying this criterion will be developed and explained in all details in the rest of the paper. In the literature in macroeconomic finance, this sort of Perron-Frobenius approach has been recently developed by Christensen [17] and Hansen and Scheinkman [21] with the aim at separating the permanent from the transitory component of the stochastic discount factor, thus evaluating long-run risk. In their application it is unnecessary to relate the spectral radius to growth. In our analysis, instead, this comparison is the ultimate purpose, as the dominant root serves to estimate the long-term discrepancy between interest rate and growth. In fact, in a time-varying environment, the dominant root (spectral radius) might be suggestively interpreted as the ratio of long-term growth into long-term interest rate,

$$\rho = \left(\frac{1 + g_{\infty}}{1 + r_{\infty}} \right),$$

with dynamic inefficiency occurring whenever growth exceeds the interest rate ($\rho > 1$).

As in Blanchard and Weil [11, Section 2], we consider a conventional overlapping-generations economy with Epstein-Zin utility,

$$U_t \left(c_t^y, c_{t+1}^o \right) = (1 - \beta) \log c_t^y + \beta \log \left(\left(\mathbb{E}_t \left(c_{t+1}^o \right)^{1-\gamma} \right)^{\frac{1}{1-\gamma}} \right),$$

where $\gamma > 0$ is a coefficient of relative risk aversion. Output is produced by a Cobb-Douglas technology using labor and capital, according to $y_t = a_t k_{t-1}^{\alpha}$, where a_t is a stochastic TFP variable. Introducing growth rates, $y_{t+1} = (1 + g_{t+1}) y_t$, we obtain a *stochastic discount factor* of the form

$$m_{t,t+1} = \delta \left(\frac{(1 + g_{t+1})^{-\gamma}}{\mathbb{E}_t (1 + g_{t+1})^{1-\gamma}} \right),$$

where $\delta = \beta (1 - \alpha) / \alpha > 0$ collects all the other primitive parameters except for the degree of risk aversion γ. We compute the growth-adjusted spectral radius of this stochastic discount factor as

$$\rho = \lim_{n \to \infty} \sqrt[n]{\frac{1}{y_t} \mathbb{E}_t m_{t,t+n} y_{t+n}} = \delta \lim_{n \to \infty} \sqrt[n]{\mathbb{E}_t \prod_{j=1}^{n} \frac{(1 + g_{t+j})^{1-\gamma}}{\mathbb{E}_{t+j-1} (1 + g_{t+j})^{1-\gamma}}},$$

where $m_{t,t+n} = m_{t,t+1} \cdots m_{t+n-1,t+n}$ is the compound stochastic discount factor. As we mentioned earlier, dynamic inefficiency holds for $\rho > 1$.

To verify the empirical implications of our theory, we assume that output growth stochastically evolves according to

$$\log (1 + g_{t+1}) = (1 - \varphi) \mu + \varphi \log (1 + g_t) + \log \zeta_{t+1},$$

where the autoregressive coefficient satisfies $|\varphi| < 1$ and the innovation $\log \zeta$ is normally distributed with mean $\mathbb{E} \log \zeta = 0$ and standard deviation $\sigma > 0$. Under these maintained assumptions, it can be shown that $\rho = \delta$. Hence, a peculiar feature of this specification is that efficiency is unrelated to the degree of risk aversion. Furthermore, observe that the time-varying short-term safe interest rate is given by

$$\left(\frac{1}{1+r_t}\right) = \mathbb{E}_t m_{t,t+1} = \delta \frac{\mathbb{E}_t \left(1+g_{t+1}\right)^{-\gamma}}{\mathbb{E}_t \left(1+g_{t+1}\right)^{1-\gamma}}.$$

Computing the unconditional means of safe interest and growth rates, this finally yields

$$\rho = \left(\frac{1+\mathbb{E}g}{1+\mathbb{E}r}\right) \exp\left(-\gamma\sigma^2\right). \tag{*}$$

We evaluate this condition using the time series for the US economy in the time interval 1950-2018. To compute the average real GDP growth we use the GDP deflator, whereas the average safe real rate is identified with the average nominal 3-month Treasury bill secondary market rate, net of inflation. The values are $\mathbb{E}r = 1.22\%$, $\mathbb{E}g = 3.17\%$, $\varphi = 20.77\%$ and $\sigma = 2.11\%$. Plugging these values in (*), we find that $\rho < 1$ holds for $\gamma > 43$, which is an implausibly large value for the relative degree of risk aversion.

It may be useful to compare our estimates with those that can be derived from the Lucas [27]'s traditional consumption-based asset-pricing model with power utility. In this case, the stochastic discount factor is

$$m_{t,t+1} = \delta \left(1+g_{t+1}\right)^{-\gamma},$$

where $\gamma > 0$ is the coefficient of relative risk-aversion and $\delta > 0$ can be interpreted as a time discount factor. Following Jiang et al. [23], we further assume that growth shocks are purely transitory ($\varphi = 0$). As a consequence, the safe interest rate is constant and satisfies

$$\log\left(1+r_t\right) = -\log\delta + \gamma\mu - \frac{\gamma^2\sigma^2}{2}.$$

In turn, the spectral radius is given by

$$\rho = \delta \exp\left(\left(1-\gamma\right)\mu + \frac{\left(1-\gamma\right)^2\sigma^2}{2}\right).$$

Computing unconditional means, we obtain that this spectral radius has the same characterization provided in (*). Jiang et al. [23] calibrate $\gamma = 10$ and $\sigma = 5\%$. We instead estimate $\sigma = 2.24\%$ for the US economy in the time interval 1950-2018 and calibrate $\gamma = 23$, thus accommodating their insistence on matching a maximum Sharpe ratio of $\gamma\sigma = 0.50$. Given the estimate of σ, condition (*) implies that $\rho < 1$ holds for $\gamma > 39$. The thought experiment unveils that, even when risk-aversion is chosen unplausibly large to match the empirically observed equity premium, this correction is not sufficient to restore dynamic efficiency.

The spectral radius is essentially determined by the ratio of average growth into average interest rate, because the variance of the innovation is comparatively small relative to the gap between such rates. These qualitative findings are unaltered if we consider alternative stochastic discount factors. Hence, according to this assessment, the US economy is dynamically *inefficient*. Admittedly, the validity of our test is limited by some key simplifying assumptions. Most prominently, the utility function exhibits a unitary intertemporal elasticity of substitution, which delivers equilibrium investment as a constant fraction of labor income. Adapting our approach to more general preferences would necessitate more sophisticated computational methods, and this is beyond the scope of our (mostly theoretical) analysis. Hopefully, some of the techniques could be borrowed from Christensen [17].

As mentioned previously, a conventional assessment of dynamic inefficiency is grounded on Abel et al. [1]'s net dividend criterion. They argue that its application to the US economy and the economies of other major developed countries provides evidence for dynamic efficiency. As we shall clarify later on (see Proposition 7.1), when the net dividend criterion for efficiency is satisfied, it turns out that $\rho < 1$ necessarily. Therefore, our empirical findings conflict with the conventional wisdom established in the literature.

Recently, Geerolf [20] provided a reassessment of dynamic inefficiency by means of the net divided criterion. Contrary to Abel et al. [1], he found that the criterion for dynamic efficiency is not verified for any advanced economy, leaving a substantial room for ambiguity. Furthermore, it should be noticed that the net dividend criterion for dynamic efficiency lacks an appropriate empirical foundation. Indeed, its validity should be ascertained on all possible paths, rather than only on the empirically observed path. In fact, as noticed by Barbie et al. [7], not even an *almost-surely* verification would be sufficient! On the contrary, our proposed criterion is properly grounded on statistical inference.

Although our test provides evidence of a market failure, and suggests that there is scope for policy interventions, it would be precipitous to identify these policies with those most commonly considered in the macroeconomic literature. For instance, as clarified by Blanchard and Weil [11, Section 2], Ponzi schemes with uncontingent debt are infeasible in this economy: The debt-to-income ratio will exceed any finite threshold, such as the ratio of saving into output, with positive probability, so that the Ponzi game will prove infeasible. This also implies that, although dynamically inefficient, a scheme of uncontingent transfers from young to old individuals would not yield a Pareto improvement, because young individuals could not be compensated eventually along a path of persistently low output growth.

4. Heuristic illustration

4.1. **A Markov setting.** To illustrate our criterion, we consider a simple overlapping-generations economy of two-period lived individuals born at all $t = 0, 1, 2, \ldots$. We denote by y_t the aggregate endowment of the unique consumption good available at period t, and we suppose that the good can be stored for one period yielding an uncertain return R. The aggregate endowment y_t grows at rate g. We assume that both g and R follow a simple Markov stochastic process. Namely, we let (g_{ij}, R_{ij}) be the realizations of capital return R and growth rate g from state i to state j in some finite state space, with $\mu_{ij} > 0$ being the transition probability.

The utility of a young agent born at time t is

$$U_t\left(c_t^y, c_{t+1}^o\right) = u\left(c_t^y\right) + \mathbb{E}_t u\left(c_{t+1}^o\right),$$

where c_t^y and c_{t+1}^o are the young and old age consumption and utility exhibits constant elasticity of substitution, namely, marginal utility satisfies $u'(c) = c^{-\sigma}$ for some $\sigma > 0$. Each individual maximizes this lifetime utility subject to budget constraints,

$$c_t^y \leq y_t - k_t \text{ and } c_{t+1}^o \leq R_{t+1} k_t,$$

where k_t is the capital stored at time t. At a competitive equilibrium, the first-order condition for the optimality of capital investment requires

$$u'\left(c_t^y\right) = \mathbb{E}_t u'\left(c_{t+1}^o\right) R_{t+1}.$$

By constant elasticity of substitution, along with the simple demographic and productive structure, a competitive equilibrium is fully determined by $c_t^y = \phi_i y_t$, $k_t = (1 - \phi_i)\, y_t$ and $c_{t+1}^o = R_{ij}\,(1 - \phi_i)\, y_t$ for some constant $0 < \phi_i < 1$, where i is the state when young and j is the state when old. Importantly, equilibrium marginal rate of substitution depends only on current and future states,

$$m_{ij} = \left(\frac{u'\left(c_{t+1}^o\right)}{u'\left(c_t^y\right)}\right) = \left(\frac{c_t^y}{c_{t+1}^o}\right)^\sigma = \left(\frac{\phi_i}{R_{ij}\,(1 - \phi_i)}\right)^\sigma.$$

As usual, m_{ij} can be interpreted as the traditional stochastic discount factor in asset pricing theory.

4.2. **Pareto optimality.** We argue that the test for conditional Pareto optimality reduces to locating the dominant root ρ of the positive matrix Q of implicit *growth-adjusted state prices* given as

$$q_{ij} = (1 + g_{ij})\, \mu_{ij} m_{ij},$$

where q_{ij} is the implicit price in state i of a share of output in the next period, conditional on state j, in terms of an equal share of current output. Competitive equilibrium is conditionally Pareto efficient when $\rho < 1$, and inefficient when $\rho > 1$. This criterion is perfectly consistent with the dominant root characterization provided by Aiyagari and Peled [3] for a non-growing economy. It is worth noticing that, when returns and growth rates are *identically and independently distributed* over time, conditional Pareto efficiency obtains if

$$\rho = \mathbb{E}\left(1+g\right)m = \frac{1+\mathbb{E}g}{1+r} + \text{cov}\left(g,m\right) < 1,$$

where the safe interest rate satisfies $r = \left(\mathbb{E}m\right)^{-1} - 1$. Therefore, at an efficient competitive equilibrium, the safe rate can be substantially smaller than the expected growth rate of output, provided that the latter is sufficiently positively correlated with the rate of return on capital (and, hence, negatively correlated with the stochastic discount factor).

By Perron-Frobenius Theorem, the positive matrix Q admits a strictly positive dominant eigenvector v satisfying the eigenvalue equation

$$\rho v = Qv.$$

With some abuse of notation, we denote with v itself a stochastic process taking value v_i when i is the current state.[4] In this alternative notation, the eigenvalue equation becomes

$$\rho v_t = \mathbb{E}_t\left(1+g_{t+1}\right)m_{t+1}v_{t+1},$$

which corresponds to $\rho v_i = \left(Qv\right)_i$ where i is the current state. We will use the dominant eigenvector process to estimate directions of welfare-improving changes in consumptions relative to aggregate endowment.

It is immediate to verify that, when $\rho > 1$, a competitive equilibrium is conditionally Pareto inefficient. Consider a small reallocation of consumptions given by

$$\hat{c}_t^y = c_t^y - \epsilon v_t y_t \qquad \text{and} \qquad \hat{c}_{t+1}^o = c_{t+1}^o + \epsilon v_{t+1} y_{t+1}.$$

Evaluating the welfare impact for a sufficiently small $\epsilon > 0$, we obtain

$$\begin{aligned} \Delta U_t &\approx \epsilon\left(-u'\left(c_t^y\right)v_t + \mathbb{E}_t u'\left(c_{t+1}^o\right)\left(1+g_{t+1}\right)v_{t+1}\right)y_t \\ &= \epsilon u'\left(c_t^y\right)\left(\mathbb{E}_t\left(1+g_{t+1}\right)m_{t+1}v_{t+1} - v_t\right)y_t \\ &= \epsilon u'\left(c_t^y\right)\left(\rho-1\right)v_t y_t > 0. \end{aligned}$$

As consumption of the initial old generation increases, this reallocation is Pareto improving, thus confirming our claim.

We now turn to the sufficient condition for conditional Pareto efficiency. To this end, suppose that $\rho < 1$ and assume that a planner is able to Pareto improve upon equilibrium by reallocating consumptions and capital over time. Furthermore, define

$$\begin{aligned} \tilde{c}_t^y - c_t^y &= -\tau_t y_t + \left(k_t - \tilde{k}_t\right), \\ \tilde{c}_{t+1}^o - c_{t+1}^o &= \tau_{t+1} y_{t+1} - R_{t+1}\left(k_t - \tilde{k}_t\right), \end{aligned}$$

where we conveniently measure the implicit transfer from young to old individuals, net of the readjustment in capital investment, as a share of current endowment. We assume that $\tau_t \leq 1$ and, at no loss of generality, we postulate that $\tau_0 > 0$, as the consumption of

[4] More precisely, given a history of states, $v_t = v_t\left(i_0, \ldots, i_{t-1}, i_t\right) = v_{i_t}$.

the initial old individual cannot decrease in a Pareto improving reallocation and the initial stock of capital is inherited from the past. Estimating the impact on welfare, by decreasing marginal utility, we obtain

$$
\begin{aligned}
\Delta U_t &\leq u'\left(c_t^y\right)\left(\tilde{c}_t^y - c_t^y\right) + \mathbb{E}_t u'\left(c_{t+1}^o\right)\left(\tilde{c}_{t+1}^o - c_{t+1}^o\right) \\
&= \left(\mathbb{E}_t\left[u'\left(c_{t+1}^o\right)\left(1+g_{t+1}\right)\tau_{t+1}\right] - u'\left(c_t^y\right)\tau_t\right) y_t \\
&\quad + \left(\mathbb{E}_t\left[u'\left(c_{t+1}^o\right) R_{t+1}\right] - u'\left(c_t^y\right)\right)\left(\tilde{k}_t - k_t\right).
\end{aligned}
$$

The last term vanishes because capital investment fulfills individual first-order conditions. As $\Delta U_t \geq 0$ by Pareto dominance, rearranging terms leads to

$$\tau_t^+ \leq \mathbb{E}_t\left(1+g_{t+1}\right) m_{t+1} \tau_{t+1}^+,$$

where $\tau^+ = \max\{\tau, 0\}$. Since the eigenvector is determined up to a scalar factor, we can assume that $\tau_{t+1}^+ \leq v_{t+1}$, so obtaining

$$\tau_t^+ \leq \mathbb{E}_t\left(1+g_{t+1}\right) m_{t+1} \tau_{t+1}^+ \leq \mathbb{E}_t\left(1+g_{t+1}\right) m_{t+1} v_{t+1} = \rho v_t.$$

As $\tau_t^+ \leq \rho v_t$, reproducing the same logic leads to

$$\tau_{t-1}^+ \leq \mathbb{E}_{t-1}\left(1+g_t\right) m_t \tau_t^+ \leq \mathbb{E}_{t-1}\left(1+g_t\right) m_t \rho v_t = \rho^2 v_{t-1}.$$

Proceeding by backward induction, we finally obtain $\tau_0^+ \leq \rho^t v_0$ which, as $\lim_{t\to\infty} \rho^t = 0$, reveals that the redistribution was never initiated, thus contradicting conditional Pareto inefficiency. Intuitively, a Pareto improving reallocation would be exploding along some path of states, thus violating feasibility.

4.3. **Capital inefficiency.** In a stochastic environment conditional Pareto optimality might fail without implying any capital overaccumulation. Capital is overaccumulated whenever aggregate consumption can be increased in some period without requiring any contraction in future periods. In our maintained example, letting

$$\gamma = \max_i \min_j \frac{1+g_{ij}}{R_{ij}},$$

capital is not overaccumulated if $\gamma < 1$. The argument is almost immediate: as capital is sufficiently productive with some probability, any reallocation preserving aggregate consumption would require an increasing contraction of the capital stock to compensate for the output losses, eventually violating feasibility.

More formally, to the purpose of contradiction, assume that aggregate consumption can be increased. By feasibility, current capital contraction needs to exceed output losses due to previous capital decumulation. Therefore, capital readjustments necessarily satisfy

$$R_{t+1}\left(k_t - \hat{k}_t\right) \leq \left(k_{t+1} - \hat{k}_{t+1}\right),$$

where we assume that capital is reduced over time, so that $k_t - \hat{k}_t \geq 0$. Evaluating relative to output, this yields

$$\left(\frac{R_{t+1}}{1+g_{t+1}}\right)\epsilon_t \leq \epsilon_{t+1},$$

where $k_t - \hat{k}_t = \epsilon_t y_t$. Along a path in which the ratio of growth into returns is bounded by $\gamma < 1$, which occurs with positive probability over any horizon of length t, feasibility implies $\gamma^{-t}\epsilon_0 \leq \epsilon_t$. As $\gamma < 1$, the dynamics is explosive, violating feasibility, $\epsilon_t \leq 1$.

4.4. **Why does growth adjustment matter?** The upshot of the above discussion is that a low interest rate might not be a symptom of inefficiency, as the pattern of output growth should be taken into account. We provide here a heuristic explanation of this result and comment on a possible relevant implication for policies. To this end, we consider a situation in which safe interest rate r is constant.

In the absence of uncertainty, the gap between r and g is the relevant sufficient statistic for dynamic efficiency and the failure of efficiency implies that positive transfers from young to old (a sort of *social security policy*) would Pareto improve upon a competitive equilibrium. Both of these properties fail in a stochastic environment. To see this, suppose that $r < \mathbb{E}g$ and evaluate the effect of transferring a certain amount $\epsilon_t > 0$ from young to old individuals at time t, with transfers growing at the average rate, $\epsilon_{t+1} = (1+\mathbb{E}g)\,\epsilon_t$. The first-order effect of this consumption adjustment on the young individual is always welfare improving, as

$$\Delta U_t \approx -u'\left(c_t^y\right)\epsilon_t + \mathbb{E}_t\left[u'\left(c_{t+1}^o\right)\epsilon_{t+1}\right] = u'\left(c_t^y\right)\left(\frac{\mathbb{E}g - r}{1+r}\right)\epsilon_t > 0.$$

This first-order approximation, however, is only a misguided intuition.

The transfers are intergenerational and, in order to be welfare-improving, they have to satisfy, in all periods and for all realizations of uncertainty, the condition

$$U_t\left(c_t^y - \epsilon_t, c_{t+1}^o + \epsilon_{t+1}\right) \geq U_t\left(c_t^y, c_{t+1}^o\right).$$

Considering a log-utility, simple manipulations yield

$$\sum_j \mu_{ij}\log\left(\frac{R_{ij} + (1+\mathbb{E}g)\,(2\epsilon_t/y_t)}{R_{ij}}\right) \geq \log\left(\frac{1}{1-(2\epsilon_t/y_t)}\right).$$

The left hand-side is the benefit from additional consumption in the old-age, whereas the right is the utility loss due to less consumption when young. Using Jensen's inequality, a Pareto improvement requires

$$\frac{\epsilon_t}{y_t} \leq \frac{\mathbb{E}g - r}{2(1+\mathbb{E}g)}.$$

And, evaluating along a path with low growth rate $g < \mathbb{E}g$,

$$\left(\frac{1+\mathbb{E}g}{1+g}\right)^t \frac{\epsilon_0}{y_0} \leq \frac{\mathbb{E}g - r}{2(1+\mathbb{E}g)},$$

a condition that cannot be verified for t large enough. The transfer scheme entails a perpetual commitment to compensate old individuals at the average growth rate $\mathbb{E}g$ for the consumption contraction when young, but this might turn unsustainable because output might grown at lower rate $g < \mathbb{E}g$ for a prolonged phase with some small probability. Along this path, the transfer ϵ_t grows faster than income y_t, thus becoming large relative to *status quo* consumption, and the first-order effect loses any informative content: the transfer might be actually welfare depressing. In other terms, the scheme of transfers cannot be implemented without compromising the welfare of certain future generations with small (however positive) probability.

In an economy without growth, in order to ensure the feasibility of uncontingent transfers, Hellwig [22] assumes that each generation receives an additional small income $\epsilon > 0$. This might admittedly seem an innocuous assumption. In an economy with stochastic growth, however, an analogous experiment would require an additional income growing according to

$$\epsilon_{t+1} = (1 + \mathbb{E}g)\,\epsilon_t.$$

This is instead a dramatic alteration of the growth pattern of the economy: the added resources would become dominant along some paths and the overall income would be eventually growing at least at the previous average growth rate $\mathbb{E}g$ of the economy. Under stochastic growth, the infeasibility of uncontingent transfers is the natural implication of uncertain growth prospects.

We conclude with the observation that the described scheme of transfers might not be welfare improving even though competitive equilibrium is conditional Pareto inefficient, with $\rho > 1$. Assuming shocks are identically and independently distributed, and considering a $\log$-utility for computations,

$$\left(\frac{1}{1+r}\right) = \mathbb{E}\left(\frac{1}{R}\right) \text{ and } \rho = \mathbb{E}\left(\frac{1+g}{R}\right).$$

By direct inspection, it is immediate to verify that these conditions are certainly consistent with $r < \mathbb{E}g$ and $\rho > 1$.

5. Competitive equilibrium

We study a canonical overlapping-generations economy with capital accumulation. To simplify our analysis, we assume that growth is only determined by an exogenous technological progress. All other elements are conventional, and shared with Abel et al. [1] and the related literature. Unconventional is the spectral radius condition we propose to assess dynamic inefficiency.

We assume that uncertainty is governed by an irreducible Markov transition $\mu : S \rightarrow \Delta(S)$, where S is a finite state space and $\Delta(S)$ is the space of probability measures on S. This process will be affecting productivity and technological progress. For notational parsimony, we describe all relevant variables as stochastic processes. In particular, we let $\mathcal{L}$ be the space of stochastic processes with values in $\mathbb{R}$, that is, an element f of $\mathcal{L}$ is a sequence $(f_t)_{t\in\mathbb{T}}$ of $\mathcal{F}_t$-measurable random variables $f_t : \Omega \to \mathbb{R}$, where $\mathcal{F}_t$ is the algebra generated by partial histories of Markov states in S and $\mathbb{T} = \{0, 1, \ldots, t, \ldots\}$ is the infinite sequence of periods. All our statements will be understood as relative to histories occurring with positive probability.

Production is described by a smooth, concave, strictly increasing, bounded (reduced-form) production function $f : \mathbb{R}^+ \to \mathbb{R}^+$. Production is subject to a stochastic process a in $\mathcal{L}^+$ affecting productivity exogenously. Thus, given capital stock k_t in $\mathbb{R}^+$, the output in the next period is $y_{t+1} = a_{t+1} f(k_t)$ in $\mathbb{R}^+$. We assume that the technology exhibits constant returns to scale and factor prices are determined by competitive markets. Consistently, capital return $R_{t,t+1}$ in $\mathbb{R}^+$ and wage w_t in $\mathbb{R}^+$ satisfy

$$R_{t,t+1} = a_{t+1} f'(k_t) \text{ and } w_t = a_t f(k_{t-1}) - a_t f'(k_{t-1}) k_{t-1}.$$

In particular, capital return equates the marginal product of capital.

The utility of a young agent is

$$U_t\left(c_t^y, c_{t+1}^o\right) = u\left(c_t^y\right) + \delta \mathbb{E}_t u\left(c_{t+1}^o\right),$$

where $u : \mathbb{R}^+ \to \mathbb{R}$ is a smooth, strictly increasing, concave Bernoulli utility and δ in $\mathbb{R}^{++}$ is a discount factor. Notice that this utility is evaluated *interim*, that is, conditional on information available at the birth of the generation. This lifetime utility is maximized subject to budget constraints

$$k_t + c_t^y \leq w_t \text{ and } c_{t+1}^o \leq R_{t,t+1} k_t,$$

where c_t^y and c_{t+1}^o in $\mathbb{R}^+$ are consumption when young and when old, and k_t in $\mathbb{R}^+$ is the investment in capital, the only asset available to transfer wealth over time.[5] At an interior optimal plan, the first-order condition imposes

$$u'\left(c_t^y\right) = \delta \mathbb{E}_t R_{t,t+1} u'\left(c_{t+1}^o\right).$$

Adhering to a common practice in this literature, we interpret marginal rates of substitution as *state prices*, or as a *stochastic discount factor*, that is,

$$m_{t,t+1} = \delta \frac{u'\left(c_{t+1}^o\right)}{u'\left(c_t^y\right)}.$$

In other terms, $m_{t,t+1}$ in $\mathbb{R}^+$ is the price at time t in $\mathbb{T}$ of one unit of output to be delivered in the next period conditional on the occurrence of some Markov state.

Given an initial stock of capital k_{-1} in $\mathbb{R}^+$, a *competitive equilibrium* is defined by a capital accumulation path k in $\mathcal{L}^+$, consumption plans (c^y, c^o) in $\mathcal{L}^+ \times \mathcal{L}^+$ and factor

[5] Completing the asset market with a full set of elementary Arrow securities would be immaterial in this framework, because young and old individuals cannot share risk due to the simple demographic structure.

prices (R, w) in $\mathcal{L}^{+} \times \mathcal{L}^{+}$ such that (a) the consumption plan maximizes lifetime utility subject to budget constraints of each generation, given prices, and (b) all markets clear, that is, at every t in $\mathbb{T}$,

$$k_t + c_t^o + c_t^y = a_t f\left(k_{t-1}\right).$$

We impose certain regularity conditions on the competitive equilibrium. These restrictions are minimal and will be maintained throughout the analysis.

First, we assume that consumption is always strictly positive, so that the first-order condition applies. Second, we postulate that expected output growth rate is bounded uniformly, that is, for some sufficiently large λ in $\mathbb{R}^{+}$, at every t in $\mathbb{T}$,

$$\mathbb{E}_t a_{t+1} \leq \lambda a_t.$$

The technological progress might sustain any arbitrarily large, though bounded, growth rate. Third, the marginal rate of substitution is also bounded uniformly, that is, for some sufficiently large λ in $\mathbb{R}^{+}$,

$$\delta u'\left(c_{t+1}^o\right) \leq \lambda u'\left(c_t^y\right).$$

Fourth, we assume that the marginal product of capital is uniformly strictly positive, relative to growth, that is, for some sufficiently large λ in $\mathbb{R}^{+}$,

$$a_{t+1} \leq \lambda a_t R_{t,t+1}.$$

These conditions will allow us to assess dynamic inefficiency by only evaluating first-order effects.

An allocation is *feasible* if, given initial capital stock k_{-1} in $\mathbb{R}^{+}$, for every t in $\mathbb{T}$,

$$\tilde{k}_t + \tilde{c}_t^o + \tilde{c}_t^y = a_t f\left(\tilde{k}_{t-1}\right).$$

A competitive equilibrium is conditional Pareto inefficient if there exists an alternative feasible allocation such that $\tilde{c}_0^o > c_0^o$ and, for every t in $\mathbb{T}$, $U_t\left(\tilde{c}_t^y, \tilde{c}_{t+1}^o\right) \geq U_t\left(c_t^y, c_{t+1}^o\right)$. It is only at no loss of generality that we assume that the welfare of the initial old generation increases (if not, we could reinitiate the economy at some future contingency when the first change takes place). Notice that, in evaluating inefficiency, the planner is entitled to modify the path of capital accumulation *and* lifetime consumption of all generations. Therefore, we do not distinguish inefficiencies arising from an overaccumulation of capital from those arising because of a misallocation of consumption.

6. DYNAMIC INEFFICIENCY

6.1. **Overview.** In order to assess conditional Pareto inefficiency, we introduce a spectral radius approach. This method is a generalization of the dominant root criterion in the established literature (*e.g.*, Aiyagari and Peled [3]), and it has recently been applied by Christensen [17] and Hansen and Scheinkman [21] to long-run risk. The spectral radius is related to the yield of an hypothetical long-run discount bond delivering consumption indexed to growth. A competitive equilibrium is Pareto efficient when this yield is positive, and Pareto inefficient when negative. We also clarify that Pareto inefficiency might occur in the absence of capital overaccumulation, and propose an amended spectral radius approach to overaccumulation.

6.2. **Spectral radius.** To verify conditional Pareto efficiency of a competitive equilibrium, we introduce

$$\mathcal{L}(a) = \left\{ v \in \mathcal{L} : |v_t| \leq \lambda a_t \text{ for some } \lambda \in \mathbb{R}^+ \right\}.$$

This is the space of stochastic processes growing no faster than output over time. We then consider the operator $T : \mathcal{L}(a) \to \mathcal{L}(a)$ given by

$$(Tv)_t = \mathbb{E}_t m_{t,t+1} v_{t+1} = \delta \mathbb{E}_t \left(\frac{u'\left(c^o_{t+1}\right)}{u'\left(c^y_t\right)} \right) v_{t+1}.$$

This operator is well-defined because of the maintained assumptions on equilibrium. As in [5, 17, 21], we introduce the *spectral radius* defined as

$$\rho(T) = \lim_{n \to \infty} \sqrt[n]{\|T^n\|},$$

where the underlying supremum norm is

$$\|v\| = \inf \left\{ \lambda \in \mathbb{R}^+ : |v| \leq \lambda a \right\}.$$

The spectral radius coincides with the Perron-Frobenius dominant root when a corresponding eigenprocess v in the interior of $\mathcal{L}^+(a)$ exists, that is, $\rho(T) v_t = (Tv)_t$ at every t in $\mathbb{T}$. To illustrate the nature of the spectral radius, we provide examples and a heuristic interpretation. As the spectral radius method is not fully conventional, we present a basic theory in a dedicated Appendix A.

Example 6.1 (Markov framework)**.** Assume the stochastic discount factor is Markov with respect to finite state space S. In other terms, $q_{ij} = \mu_{ij} m_{ij}$ in $\mathbb{R}^+$ is the price in state i in S of one unit of output delivered in the next period conditional on the occurrence of state j in S. We can arrange all such state prices in a positive matrix Q in $\mathbb{R}^{S \times S}$. By Perron-Frobenius Theorem there exists a positive vector v in $\mathbb{R}^S$ such that, for some ρ in $\mathbb{R}^+$,

$$\rho v = Qv.$$

Alternatively,

$$\rho v_i = (Tv)_i = \sum_{j \in S} q_{ij} v_j.$$

By a well-know theorem of analysis, this is the spectral radius (see Appendix A).

Example 6.2 (Abel et al. [1])**.** This example is taken from Abel et al. [1, Section III]. The exogenous stochastic process a in $\mathcal{L}^+$ satisfies

$$a_{t+1} = (1 + g + \nu_{t+1}) a_t,$$

where g lies in $\mathbb{R}^{++}$ and ν in $\mathbb{R}$ is an identically and independently distributed shock with $1+g+\nu>0$, $\mathbb{E}\nu=0$, and utility takes the log-form. The stochastic discount factor is

$$m_{t,t+1}=\frac{\delta}{1+g+\nu_{t+1}}.$$

By direct computation,

$$(Ta)_t=\mathbb{E}_t m_{t,t+1}a_{t+1}=\delta\mathbb{E}_t\frac{1}{1+g+\nu_{t+1}}\left(1+g+\nu_{t+1}\right)a_t=\delta a_t.$$

Therefore, δ in $(0,1)\subset\mathbb{R}^{+}$ is an eigenvalue, $\delta a=(Ta)$, and, as a lies in the interior of $\mathcal{L}^{+}(a)$, it is the spectral radius $\rho(T)=\delta$ (see Claim A.3 in Appendix A).

Remark 6.1 (Long-term *versus* short-term rate)**.** Kocherlakota [26, Proposition 4] argues that long yield falls short of the expected short yield. More precisely, in our notation, he proves that

$$\mathbb{E}\log(1+r)\geq-\log\rho,$$

where the unconditional expectation is taken with respect to the invariant measure. As a consequence, in an economy without growth, equilibrium is dynamically inefficient ($\rho>1$) whenever the expected short-term interest rate is negative. This feature is illustrated by Example 6.3. Despite first appearance, this is not inconsistent with our previous claims. In our streamlined example, the short-term interest rate is constant and, thus, coincides with the long-term interest rate. The spectral radius, instead, is given as

$$\rho=\mathbb{E}m(1+g)=\left(\frac{1+\mathbb{E}g}{1+r}\right)+\operatorname{cov}(m,g).$$

Equilibrium might be dynamically efficient ($\rho<1$) even though the short-term interest rate falls short of expected growth.

Example 6.3 (Long-term *versus* short-term rate)**.** Consider a simple Markov setting with two states and deterministic cyclic transitions ($\mu_{ii}=0$). There is no growth of the endowment. Let $R_1>0$ and $R_2>0$ be distinct safe returns in the two states, and assume asset-pricing under risk-neutrality ($m_{ij}=\mu_{ij}R_i^{-1}$). We argue that

$$\rho=\sqrt{\frac{1}{R_1R_2}}.$$

Indeed, the eigenvalue equation writes as

$$\rho v_i=\frac{1}{R_i}v_j,$$

where the deterministic transition is from state i to the other state j. Setting $v_i=\sqrt{R_i^{-1}}$, straightforward calculations confirm our claim. We compare this dominant root with the average short-term rate in this simple setting, where the only invariant measure is $\mu=(1/2,1/2)$. As shown in Figure 1, a positive expected short yield is consistent with dynamic inefficiency.

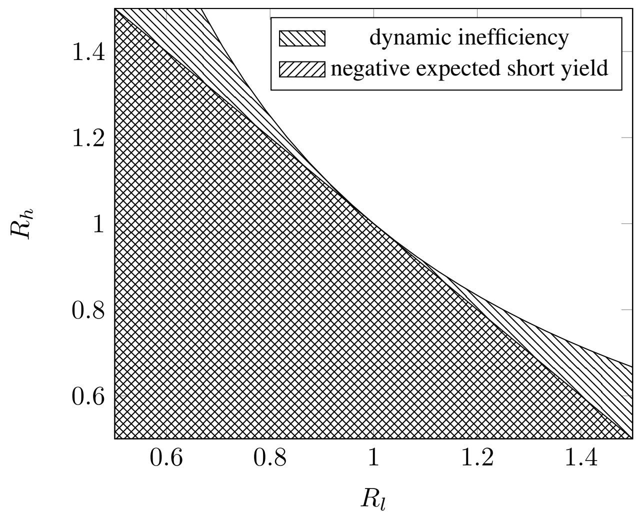

FIGURE 1. Short *versus* long yield

6.3. **Pareto inefficiency.** We are now ready to present a sufficient condition for Pareto efficiency. In particular, efficiency occurs when long-term interest rate exceeds growth, as precisely expressed by the condition $\rho(T) < 1$.

Proposition 6.1 (Sufficiency)**.** *A competitive equilibrium is efficient if* $\rho(T) < 1$.

Proof. Consider a planner improving upon the equilibrium allocation. Feasibility imposes

$$\hat{k}_t + \hat{c}_t^y + \hat{c}_t^o = a_t f\left(\hat{k}_{t-1}\right),$$

given the initial stock of capital k_{-1} in $\mathbb{R}^+$. The competitive equilibrium instead satisfies

$$k_t + c_t^y + c_t^o = a_t f\left(k_{t-1}\right).$$

We define

$$\tau_t = \tilde{c}_t^o - c_t^o - a_t\left(f\left(\tilde{k}_{t-1}\right) - f\left(k_{t-1}\right)\right),$$

and notice that by feasibility

$$\tilde{c}_t^y - c_t^y = -\tau_t - \left(\tilde{k}_t - k_t\right).$$

It worth remarking that $\tau_0 > 0$ at no loss of generality, as welfare increases for initial old individual and the initial capital stock is inherited from the past. We now derive the implication in terms of operator $T : \mathcal{L}^+(a) \to \mathcal{L}^+(a)$.

Concavity of the utility implies

$$0 \leq U_t\left(\tilde{c}_t^y, \tilde{c}_{t+1}^o\right) - U_t\left(c_t^y, c_{t+1}^o\right) \leq u'\left(c_t^y\right)\left(\tilde{c}_t^y - c_t^y\right) + \delta \mathbb{E}_t u'\left(c_{t+1}^o\right)\left(\tilde{c}_{t+1}^o - c_{t+1}^o\right).$$

Using the definition of process τ in $\mathcal{L}(a)$,

$$\begin{aligned} u'\left(c_t^y\right)\tau_t &\leq \delta\mathbb{E}_t u'\left(c_{t+1}^o\right)\tau_{t+1} \\ &\quad -u'\left(c_t^y\right)\left(\tilde{k}_t - k_t\right) + \delta\mathbb{E}_t u'\left(c_{t+1}^o\right)a_{t+1}\left(f\left(\tilde{k}_t\right) - f\left(k_t\right)\right). \end{aligned}$$

By concavity of the production function, we finally obtain

$$\begin{aligned} u'\left(c_t^y\right)\tau_t &\leq \delta\mathbb{E}_t u'\left(c_{t+1}^o\right)\tau_{t+1} \\ &\quad -u'\left(c_t^y\right)\left(\tilde{k}_t - k_t\right) + \delta\mathbb{E}_t u'\left(c_{t+1}^o\right)R_{t.t+1}\left(\tilde{k}_t - k_t\right). \end{aligned}$$

As capital is optimally chosen by the young individual at equilibrium, the last term vanishes, so establishing that $\tau \leq (T\tau)$. Finally, possibly replacing it with $\max\{0, \tau_t\}$, we can assume that process τ lies in $\mathcal{L}^+(a)$. We now derive a contradiction.

At no loss of generality, we can assume that $\tau \leq a$. Notice that, for every n in $\mathbb{N}$, as linear operator $T : \mathcal{L}^+(a) \to \mathcal{L}^+(a)$ is monotone,

$$\tau \leq (T^n\tau) \leq (T^n a) \leq \rho^n a,$$

where ρ lies in $(\rho(T), 1) \subseteq \mathbb{R}^+$ and n in $\mathbb{N}$ is chosen sufficiently large. The middle inequality is due to monotonicity. To prove the extreme right hand-side inequality, we use the fact that, for any sufficiently large n in $\mathbb{N}$, $\sqrt[n]{\|T^n\|} \leq \rho$, so that

$$\|(T^n a)\| \leq \|T^n\| \leq \rho^n.$$

Moreover, by the supremum norm, $(T^n a) \leq \|(T^n a)\| a$. Hence, combining the last two inequalities, $(T^n a) \leq \rho^n a$. Finally, going to the limit,

$$0 < \tau \leq \lim_{n\to\infty} \rho^n a = 0,$$

a contradiction. □

We now turn to dynamic inefficiency, thus proving that equilibrium is inefficient whenever $\rho(T) > 1$. This involves two complications of independent nature. First, as the spectral radius capture first-order effects on welfare, some restrictions are needed to ensure that first-order welfare increases translate into actual welfare increases. This is not obvious because the economy involves *infinitely many* individuals. Second, the spectral radius might not be an eigenvalue of the operator and, thus, no eigenprocess can be associated to the spectral radius. As the eigenprocess identifies the direction of consumption changes yielding the Pareto improvement, the approach might fail. We repair to these potential issues by adding further assumptions on equilibrium, and on the induced operator $T : \mathcal{L}^+(a) \to \mathcal{L}^+(e)$. The first group of restrictions are shared with the established literature (see, for instance, the curvature conditions in Chattopadhyay and Gottardi [16]). The existence of an eigenprocess is specific to our approach, and can be weakened by

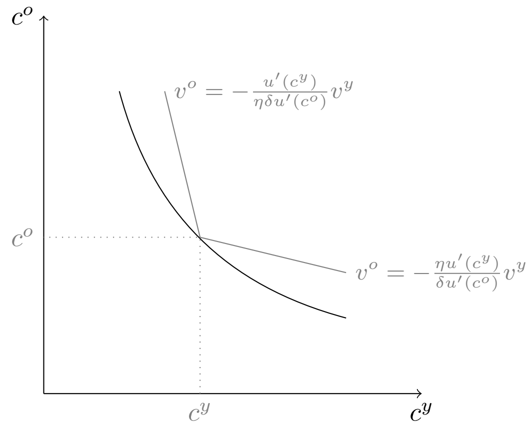

FIGURE 2. Uniformly smooth preferences

adding restrictions on equilibrium processes, so as to ensure the applicability of a general Perron-Frobenius Theorem (as in Christensen [17] and Hansen and Scheinkman [21]).[6]

Non-vanishing consumption. *For some λ in $\mathbb{R}^{++}$, $c_t^y \geq \lambda a_t$ for every t in $\mathbb{T}$.*

Uniformly smooth preferences. *Given any process v in $\mathcal{L}^+(a)$, and given any η in $(0,1) \subset \mathbb{R}^+$,*

$$u'\left(c_t^y\right) v_t \leq \eta\delta \mathbb{E}_t u'\left(c_{t+1}^o\right) v_{t+1}$$

implies, for some sufficiently small ϵ in $\mathbb{R}^{++}$, uniformly over all periods t in $\mathbb{T}$,

$$U_t\left(c_t^y - \epsilon v_t, c_{t+1}^o + \epsilon v_{t+1}\right) \geq U_t\left(c_t^y, c_{t+1}^o\right).$$

The nature of this last condition is to prevent the insurgence of kinks in the limit indifference curves. For finitely many generations, this restriction is always satisfied: the first-order utility increase translates into an actual utility increase in an open neighborhood. With infinitely many generations, this is not granted anymore without further assumptions ensuring some uniformity of the open neighborhood. The geometric intuition is illustrated by Figure 2: On some uniform open neighborhood, the curvature of the indifference curves remains bounded.

Existence of an eigenprocess. *A dominant eigenprocess exists, that is, there exists v in $\mathcal{L}^+(a)$ such that*

$$\rho(T)\, v = (Tv).$$

Proposition 6.2 (Necessity)**.** *A competitive equilibrium is inefficient if $\rho(T) > 1$, provided that all the previous assumptions are satisfied.*

[6] Assumption 1 in Kocherlakota [26] implies the existence of some sort of eigenprocess.

Proof. For a given η in $\left(\rho\left(T\right)^{-1},1\right) \subset \mathbb{R}^{+}$, we have $v \leq \eta\left(Tv\right)$. Consider the reallocation of consumption only given by $\tilde{c}_t^y = c_t^y - v_t$ and $\tilde{c}_t^o = c_t^o + v_t$. Up to an innocuous rescaling, because consumption of the young individual is not vanishing relative to output, this is feasible. Welfare increases for all generations by the assumption of uniformly smooth preferences. □

We add two remarks: first, on the gap between necessary and sufficient conditions; second, on the separation of consumption from production inefficiency. We do not assess efficiency when $\rho\left(T\right) = 1$. This is a situation in which the long-term interest rate exactly balances growth. In such a situation, the first-order effect is ambiguous and an evaluation of second-order effects becomes necessary. Under appropriate curvature assumptions, in line with the established literature (*e.g.*, Chattopadhyay and Gottardi [16]), a competitive equilibrium is efficient whenever $\rho\left(T\right) = 1$. We omit the proof because it would be laborious, reproducing steps in the previous literature and without adding any further insight. Therefore, up to technicalities, and subject to some appropriate curvature restrictions, we submit an educated conjecture.

Conjecture 6.1 (Necessity and sufficiency)**.** *A competitive equilibrium is Pareto efficient if and only if $\rho\left(T\right) \leq 1$, provided that appropriate curvature conditions are satisfied in addition to the maintained assumptions.*

As a last observation, it is worth remarking that, whenever $\rho\left(T\right) > 1$, a planner is able to induce a welfare improvement by a mere reallocation of consumption, without any alteration of capital accumulation. This, of course, does not imply that a readjustment of production plans would not permit further welfare gains. Rather, it implies that a failure of Pareto efficiency is always revealed by a misallocation of consumption, and a supplemental investigation of the production side of the economy is unnecessary.

Conjecture 6.2 (Consumption inefficiency)**.** *A competitive equilibrium is Pareto inefficient if and only if a welfare improvement is feasible by a mere reallocation of consumptions, provided that appropriate curvature conditions are satisfied in addition to the maintained assumptions.*

6.4. **Capital overaccumulation.** Pareto efficiency depends of individual preferences, and such preferences might not be observable through prices when markets are incomplete. Mostly for this reason, the previous literature studied *capital overaccumulation*. This is a situation in which aggregate consumption might be increased in some period without being decreased in any other period. To ascertain overaccumulation of capital requires no evaluation of trade-offs and, hence, no knowledge of individual preferences. We now identify conditions ensuring the absence of capital overaccumulation.

Define the operator $D : \mathcal{L}\left(a\right) \to \mathcal{L}\left(a\right)$ as

$$\left(Dv\right)_t = \sup z_t \text{ subject to } R_{t,t+1} z_t \leq v_{t+1} \ \mathcal{F}_t\text{-almost surely,}$$

where $\mathcal{F}_t$ is the information available at t in $\mathbb{T}$. This operator is monotone superlinear. As in our previous analysis, we define the spectral radius $\gamma(D)$ in $\mathbb{R}^+$. The (reciprocal of) the spectral radius might be interpreted as an estimation of the long-term return to capital along the most optimistic path. We provide an explicit computation in a simple Markov framework (Example 6.4).

Proposition 6.3 (Capital overaccumulation)**.** *Capital is not overaccumulated if* $\gamma(D) < 1$.

Proof. Supposing not, there exists an alternative capital accumulation path such that

$$\tilde{k}_t - k_t \leq a_t \left(f\left(\tilde{k}_{t-1}\right) - f\left(k_{t-1}\right)\right).$$

Indeed, this condition ensures that aggregate consumption does not decrease at any contingency. Exploiting concavity of the utility function, we obtain

$$R_{t-1,t}\left(k_{t-1} - \tilde{k}_{t-1}\right) \leq k_t - \tilde{k}_t.$$

Setting $v_t = \tilde{k}_t - k_t$, this implies

$$R_{t,t+1} v_t \leq v_{t+1} \text{ if and only if } v \leq (Dv).$$

We can assume that v lies in $\mathcal{L}^+(a)$ at no loss of generality, and argue as in the proof of our first proposition. □

Example 6.4 (Capital overaccumulation)**.** How to interpret the spectral radius condition for the absence of capital overaccumulation? It is worth considering a simplified framework in which the return to capital is governed by a Markov transition with strictly positive probabilities on a finite state space S. Consistently, we let R_{ij} in $\mathbb{R}^+$ be the return to capital invested in previous state i in S when state j in S occurs. We claim that

$$\gamma(D) \leq \gamma = \max_{i \in S} \min_{j \in S} \frac{1}{R_{ij}}.$$

Indeed, letting $\mathbf{1}$ in $\mathbb{R}^S$ be the unit vector, we see that

$$(D\mathbf{1})_i = \min_{j \in S} \frac{1}{R_{ij}} \leq \gamma \mathbf{1}_i,$$

thus implying that $\|D\| \leq \gamma$. It is then immediate to conclude that

$$\gamma(D) = \lim_{n \to \mathbb{N}} \sqrt[n]{\|D^n\|} = \inf_{n \in \mathbb{N}} \sqrt[n]{\|D^n\|} \leq \|D\| \leq \gamma.$$

In other terms, with no output growth, as long as the net return to capital remains strictly positive in every state with some probability, capital is not overaccumulated.

7. Comments

7.1. **A comparison with Abel et al. [1].** We compare our characterization with the traditional net dividend criterion proposed by Abel et al. [1]. In particular, we relate that criterion to conditional Pareto efficiency and capital overaccumulation, and argue that our refinements permit an assessment even when the net dividend criterion remains ambiguous.[7]

The *net divided criterion* for Pareto efficiency is the requirement that, for some ϵ in $\mathbb{R}^{++}$,

$$\epsilon v_t \leq d_t,$$

where d in $\mathcal{L}^+$ is the net dividend and v in $\mathcal{L}^+$ is the value of the market portfolio. Identifying terms in our (as well as in Abel et al. [1]'s) framework, the value of the market portfolio is $v_t = k_t$, and the dividend is determined as

$$d_{t+1} = a_{t+1} f' (k_t) k_t - k_{t+1}.$$

In this notation, consumptions of young and old individuals are

$$c_t^y = w_t - v_t \text{ and } c_{t+1}^o = v_{t+1} + d_{t+1},$$

exactly as in Abel et al. [1, Equations (1.2)-(1.3)]. As noticed by Chattopadhyay [15], the net dividend criterion is incomplete, and its amendment requires that the value of the market portfolio does not vanish relative to output. For this reason, we also postulate that, for some λ in $\mathbb{R}^{++}$, $v_t \geq \lambda a_t$ uniformly over all periods t in $\mathbb{T}$.

We argue that our criterion for conditional Pareto efficiency is in fact a refinement of Abel et al. [1]'s net dividend criterion.

Proposition 7.1 (Net dividend, I)**.** *The net dividend criterion is satisfied only if* $\rho(T) < 1$.

Proof. By no arbitrage, a necessary condition at a competitive equilibrium,

$$v_t = \mathbb{E}_t m_{t,t+1} (v_{t+1} + d_{t+1}).$$

Invoking the net dividend criterion,

$$v_t \geq (1 + \epsilon) \mathbb{E}_t m_{t,t+1} v_{t+1}.$$

Therefore, setting $\rho(1 + \epsilon) = 1$, we obtain $\rho v \geq (Tv)$, thus delivering $\rho(T) < 1$ (see Claim A.2 in Appendix A). $\square$

Turning to the overaccumulation of capital, we prove that a substantially weaker criterion rules out this inefficiency, as illustrated by Example 7.1. To the purpose of comparison, we say that the net dividend criterion is satisfied *with some probability* if

$$\mu (\{\epsilon v_{t+1} \leq d_{t+1}\} | \mathcal{F}_t) > 0,$$

[7] Abel et al. [1]'s notion of dynamic efficiency coincides *stricto sensu* with conditional Pareto efficiency. However, their narrative unfolds along the idea of a socially inefficient stock of capital. The intuition provided after their Proposition 1 is based on a comparison between the total return on the aggregate stock of capital and the new investment and, as a matter of fact, their proof of inefficiency mirrors Cass [14]'s argument for capital overaccumulation.

where $\mathcal{F}_t$ denotes the information available at t in $\mathbb{T}$. In other terms, whenever the net dividend will exceed a constant share of the market portfolio value with positive probability at every contingency. In fact, our spectral radius characterization is related to a probabilistic version of the net dividend criterion.

Proposition 7.2 (Net dividend, II)**.** *The net divided criterion for Pareto efficiency is satisfied* with some probability *only if* $\gamma(D) < 1$.

Proof. The probabilistic net dividend criterion imposes, for some γ in $(0,1) \subset \mathbb{R}^+$,

$$k_{t+1} \leq \gamma R_{t,t+1} k_t \text{ with positive } \mathcal{F}_t\text{-conditional probability.}$$

By definition of $D : \mathcal{L}^+(a) \to \mathcal{L}^+(a)$, there exists z_t in $\mathbb{R}^+$ such that

$$(Dk)_t = z_t \text{ and } R_{t,t+1} z_t \leq k_{t+1}\ \mathcal{F}_t\text{-almost surely.}$$

Comparing with the net dividend criterion, we obtain

$$(Dk)_t = z_t \leq \gamma k_t.$$

As $k \geq \lambda a$ for some λ in $\mathbb{R}^{++}$, and $(Dk) \leq \gamma k$, this immediately implies $\gamma(D) < 1$ □

Example 7.1 (Net dividend)**.** Here is a simple example in which our criterion is satisfied, whereas the net dividend criterion fails. States $S = \{l, h\}$ can occur with equal probability in each period. Assume that capital stock is constant at level k in $\mathbb{R}^{++}$. Capital returns satisfy $R_l < 1 < R_h$. Hence, the net dividend criterion cannot be satisfied. By the characterization in Example 6.4, $\gamma(D) < 1$.

7.2. **Ponzi schemes.** We show that a government debt rollover (without primary surpluses) is feasible only when our test for conditional Pareto efficiency fails. Hence, Blanchard and Weil [11, Section 4]'s and Abel and Panageas [2]'s findings that a debt rollover policy is possible in a dynamically efficient economy necessarily entail a violation of our criterion for efficiency and, hence, of Abel et al. [1]'s net dividend criterion. Quite clearly, debt rollovers may be feasible in the absence of capital overaccumulation.

A government debt rollover plan, or a Ponzi scheme, is a (non-trivial) stochastic process b in $\mathcal{L}^+(a)$ such that

$$b_t \leq \left(\frac{1}{1+r_t}\right) b_{t+1},$$

with b_{t+1} being $\mathcal{F}_t$-measurable. This latter constraint implies that the government is restricted to issue uncontingent debt only (*i.e.*, a safe one-period bond). In a rollover plan, the government is able to issue new debt as a repayment of previous debt, and accumulated debt does not explode relative to output (because b is an element of $\mathcal{L}^+(a)$).

Proposition 7.3 (Ponzi schemes)**.** *A government debt rollover plan is unfeasible if* $\rho(T) < 1$.

Proof. It is immediate to verify that the existence of a government debt rollover plan implies $b \leq (Tb)$. Assuming that $b \leq a$ at no loss of generality, this yields

$$b \leq (T^n b) \leq (T^n a) \leq \rho^n a,$$

where ρ lies in $(0,1) \subset \mathbb{R}^+$. The left hand side inequality is verified by monotonicity of $T : \mathcal{L}^+(a) \to \mathcal{L}^+(a)$, and the extreme right hand side inequality holds due to the same arguments in the proof of Proposition 6.1. Thus, $b \leq \lim_{n\to\infty} \rho^n a = 0$, a contradiction. $\square$

7.3. **Representative indvidual.** In an economy with a representative individual, equilibrium is obviously dynamically efficient. More interestingly, any stochastic discount factor consistent with the representative individual necessarily involves $\rho(T) < 1$. Therefore, any of such stochastic discount factors is unsuitable to separate a dynamically inefficient from a dynamically efficient regime. We shortly elaborate on this issue.

Consider a representative individual with a monotone concave utility function $U_0 : \mathcal{L}^+(a) \to \mathbb{R}$. The stochastic discount factor m in $\mathcal{L}^+$ is generated by the marginal evaluation of this representative individual, that is,

$$U_0(c+v) - U_0(c) \leq \mathbb{E}_0 \sum_{t=0}^{\infty} m_{0,t} v_t.$$

Notice that utility is not restricted by further assumptions beyond monotone concavity. Consistently, our analysis encompasses conventional recursive utilities such as Epstein-Zin and risk-sensitive preferences.

Existence of an interior eigenprocess. *A dominant interior eigenprocess exists, that is, there exists v in the interior of $\mathcal{L}^+(a)$ such that*

$$\rho(T)\, v = (Tv).$$

Claim 7.1 (Representative individual)**.** *Under the maintained assumptions,*[8] *the stochastic discount factor m in $\mathcal{L}^+$ is consistent with a representative individual only if $\rho(T) < 1$.*

Proof. Given any λ in $(0,1) \subset \mathbb{R}^+$, we obtain

$$U_0(\lambda y) - U_0(y) \leq -(1-\lambda)\, \mathbb{E}_0 \sum_{t=0}^{\infty} m_{0,t} y_t,$$

thus implying

$$\mathbb{E}_0 \sum_{t=0}^{\infty} m_{0,t} y_t \leq \left(\frac{1}{1-\lambda}\right)(U_0(y) - U_0(\lambda y)).$$

[8]To simplify our exposition, we also assume that Markov transition $\mu : S \to \Delta(S)$ is strictly positive, that is, every state s' in S is reached from state s in S with positive probability in the next period. Without such a restriction, our arguments would hold true subject to an appropriate almost-surely qualification.

We conclude that

$$\sum_{n=0}^{\infty} (T^n y) \text{ is finite.}$$

Using the interior dominant eigenprocess, and assuming that $y \geq v$ at no loss of generality,

$$\sum_{n=0}^{\infty} (T^n y) \geq \sum_{n=0}^{\infty} (T^n v) = \sum_{n=0}^{\infty} \rho(T)^n v.$$

The value is finite only if $\rho(T) < 1$. □

7.4. **Uninsurable risks.** The presence of uninsurable risks introduces major conceptual complications in our analysis of dynamic inefficiency.[9] Market incompleteness represents an independent source of static inefficiency and, as a consequence, competitive equilibrium will typically be conditional Pareto inefficient, even when the interest rate unambiguously exceeds growth. In addition to incomplete insurance, the inefficiency might be caused by an intertemporal misallocation of resources, as it happens with a full set of contingent claims. To disentangle dynamic from static inefficiency requires some sort of counterfactual experiments.

An educated intuition suggests to consider reallocations of *insurable* risks only, thus treating market incompleteness as an ineludible constraint. Thus, for instance, in an economy when only the risk-free bond is traded, the planner might be restricted to uncontingent transfers from young to old individuals (as in Hellwig [22, Section 4]). More generally, reallocations could be permitted only along the span generated by certain available assets, so as to assess efficiency *conditional* on the limited insurance permitted by the market. This approach bears a relevant limit: the extent of insurable risk depends on prices and, thus, the planner's restrictions are not independent of equilibrium. With these unavoidable limitations understood, we propose a method to assess dynamic inefficiency with uninsurable risks.

Consider a planner able to implement certain contingent transfers only, as in in Hellwig [22, Section 4]. In particular, we assume the existence of a finite set J of assets, each with a contingent return $\tilde{R}^j_{t,t+1}$ in $\mathbb{R}^+$. All such assets are traded in a competitive market and priced subject to no arbitrage at a competitive equilibrium, that is, price q^j_t in $\mathbb{R}^+$ of asset j in J satisfies

$$q^j_t = \mathbb{E}_t m_{t,t+1} \tilde{R}^j_{t,t+1}.$$

The planner's ability to redistribute consumptions is restricted by

$$\tilde{c}^o_{t+1} - c^o_{t+1} = \sum_{j \in J} \lambda^j_t \tilde{R}^j_{t,t+1}, \tag{*}$$

[9]For uninsurable risks be effective at equilibrium, we need to consider a richer demographic structure of overlapping generations, with many heterogeneous individuals in each generation or longer finite horizons for each generation.

where λ_t^j is a weight in $\mathbb{R}$. A competitive equilibrium is *constrained* conditional Pareto inefficient if it is Pareto dominated by an alternative feasible allocation subject to restriction (*). Similarly, a *constrained* government debt rollover plan is a (non-trivial) process b in $\mathcal{L}^+(a)$ such that

$$b_t \leq \mathbb{E}_t m_{t,t+1} b_{t+1}, \quad \text{with } b_{t+1} = \sum_{j \in J} \lambda_t^j \tilde{R}_{t,t+1}^j.$$

As a complement to our previous analysis, we provide a characterization of constrained inefficiency.

Proposition 7.4 (Restricted transfers)**.** *A competitive equilibrium is* constrained *inefficient only if a* constrained *government debt rollover plan is feasible.*

Proof. Similar to the initial part of the proof of Proposition 6.1. □

8. Government debt valuation

Dynamic inefficiency is pervasively related to the debate on government debt sustainability. In a dynamically efficient environment, absent further liquidity or solvency constraints, government debt equals the present value of government primary surpluses, as required by the government debt valuation equation, or intertemporal budget constraint. Instead, a safe interest rate persistently lower than the GDP growth rate seems to suggest a failure of this valuation equation and the ability of the government to run permanent positive net transfers to the private sector. These issues have been recently discussed, among others, by Bassetto and Cui [9], Blanchard [10] and Jiang et al. [23, 24].

In a stochastic environment, despite first appearances, the government valuation equation may hold even if the average safe rate is systematically lower than the average GDP growth rate. Furthermore, as clarified by Bohn [12], and recently reasserted by Jiang et al. [24], persistent government deficits *on average* are feasible, provided that government debt entails an implicit insurance premium, due to primary government surpluses moving countercyclically and, thus, being positively correlated with the marginal utility of wealth. This interpretation assumes the validity of the government debt valuation equation which, in turn, requires to satisfy the government transversality condition. Jiang et al. [24] argue that, when the government commits to a stationary debt-output policy, the transversality condition for debt is *naturally* satisfied in a world with permanent output shocks. We observe that this statement is equivalent to satisfy our criterion for dynamic efficiency. In addition, consistently with our findings in this paper, a comparison between average interest rate and growth in neither sufficient nor necessary for the transversality condition to be satisfied in an economy with growth risk.

Consider a government issuing debt b in $\mathcal{L}^+$ subject to a budget constraint of the form

$$\mathbb{E}_t m_{t,t+1} b_{t+1} + s_t = b_t,$$

where s in $\mathcal{L}$ is a process describing the government's primary surplus (a deficit whenever negative). The transversality condition [24, Equation (1)] holds if

$$\lim_{n\to\infty} \mathbb{E}_t m_{t,t+n} b_n = 0,$$

where the compounded stochastic discount factor is computed as $m_{0,0} = 1$ and $m_{0,t+1} = m_{0,1} \cdots m_{t,t+1}$. The transversality condition enforces a canonical government debt valuation equation of the form

$$b_0 = \sum_{t=0}^{\infty} \mathbb{E}_0 m_{0,t} s_t.$$

More permissively than Jiang et al. [24], we assume that the government commits to a uniformly bounded debt-output ratio such that

$$\lambda_l y_t \leq b_t \leq \lambda_u y_t,$$

where λ_l and λ_u are in $\mathbb{R}^{++}$. Thus, perpetual fluctuations of the debt-into-output ratio are permitted as long as debt does not explode or vanish relative to output. We also postulate a reinforced regularity condition for the spectral radius. This assumption is satisfied in Jiang et al. [24]'s environment (see Example 8.1) as well as in many other conventional frameworks.

Existence of an interior eigenprocess. *A dominant interior eigenprocess exists, that is, there exists v in the interior of $\mathcal{L}^+(a)$ such that*

$$\rho(T) v = (Tv).$$

Claim 8.1 (Transversality condition)**.** *Under the maintained assumptions, the transversality condition is satisfied if and only if $\rho(T) < 1$.*

Proof. By the existence of an interior eigenprocess, and given the bounds on the debt-output ratio, at no loss of generality, we can assume that

$$\lambda_l v_t \leq b_t \leq \lambda_u v_t.$$

Furthermore, notice that

$$\mathbb{E}_t m_{t,t+n} b_n = (T^n b)_t.$$

Using monotonicity of the valuation operator, jointly with the invariance of the dominant eigenspace, it follows that

$$\lambda_l \rho(T)^n v = \lambda_l (T^n v) \leq (T^n b) \leq \lambda_u (T^n v) = \lambda_u \rho(T)^n v.$$

This suffices to establish our claim. □

Example 8.1 (Jiang et al. [24])**.** Assume that income is governed by

$$\log y_{t+1} = \mu + \log y_t + \sigma \epsilon_{t+1},$$

where the innovation is normally distributed with mean $\mathbb{E}\epsilon = 0$ and normalized standard deviation $\sqrt{\mathbb{E}\epsilon^2} = 1$. Furthermore, the process for the stochastic discount factor is

$$\log m_{t,t+1} = -\varrho - \frac{1}{2}\gamma^2 - \gamma\epsilon_{t+1}.$$

Under these specifications, by direct computation,

$$\mathbb{E}_t m_{t,t+1} y_{t+1} = \mathbb{E}_t \exp\left(-\varrho - \frac{1}{2}\gamma^2 + (\sigma - \gamma)\,\epsilon_{t+1} + \mu\right) y_t.$$

This immediately yields $\rho\,(T)\,y = (Ty)$, with the dominant root given by

$$\rho\,(T) = \exp\left(-\varrho + \frac{\sigma^2}{2} - \gamma\sigma + \mu\right).$$

We conclude that

$$\rho\,(T) < 1 \text{ if and only if } \mu + \frac{\sigma^2}{2} - \gamma\sigma < \varrho,$$

consistently with Jiang et al. [24].

To uncover the insurance-premium adjustment implied by the valuation equation, we consider equivalent probabilities with density given by $\pi_{0,0} = 1$ and

$$\pi_{0,t+1} = \left(\frac{1}{\mathbb{E}_t m_{t,t+1}}\right) m_{t,t+1}\pi_{0,t}.$$

Assuming that the transversality condition is satisfied, the valuation equation takes the form

$$(*) \quad b_0 = s_0 + \sum_{t=0}^{\infty} \mathbb{E}_0 \prod_{j=0}^{t} \left(\frac{1}{1+r_j}\right) s_{t+1} + \sum_{t=0}^{\infty} \operatorname{cov}_0 \left(\pi_{0,t+1}, \prod_{j=0}^{t} \left(\frac{1}{1+r_j}\right) s_{t+1}\right).$$

Debt value reflects two components: the first component is the risk-neutral present value of future primary surpluses, whereas the second component identifies a sort of insurance premium, namely, the gap between the market value of debt and the risk-neutral present value of primary surpluses. Consistently with Bohn [12, Equation (16)], and more recently with Jiang et al. [24], persistent primary deficits on average do not violate the intertemporal budget constraint as long as discounted primary surpluses are positively correlated with the equivalent probabilities. Notice that, when the safe interest rate is constant, this requires a positive correlation of the marginal utility of wealth with government primary surpluses and, therefore, the tax burden must increase under poor economic conditions.

An assessment of government debt sustainability by means of the valuation equation presupposes the government's ability to issue contingent debt. When government debt is uncontingent, sustainability requirements are more stringent. Santos and Woodford [33, Proposition 2.2] provide the basic principles for the intertemporal accounting when only limited assets are available. Whenever the government issues a one-period discount bond only (and government debt cannot be negative), sustainability of fiscal plans requires

$$b_0 \geq \limsup_{T\to\infty} s_0 + \sum_{t=0}^{T} \prod_{j=0}^{t} \left(\frac{1}{1+r_j}\right) s_{t+1}, \tag{**}$$

where the inequality is understood to be satisfied almost surely and the equality holds true along some sample path. In other terms, government debt equals the most optimistic valuation of primary surpluses.[10] Historical experience suggests a negative covariance between the marginal utility of wealth and primary surpluses, as taxes decline during recessions and fiscal consolidation occurs under recovery. Under a constant safe interest rate, the covariance term in (**) is negative under empirically plausible conditions and safe debt is unsustainable if the government debt valuation equation is to be satisfied.

Jiang et al. [23] find that, though the US tax and spending levels are close to each other, a calibrated and estimated valuation of the tax claim is largely below the valuation of the spending claim. As a result, the market valuation of government primary surpluses is negative and falls short of the market value of government debt, thus violating the valuation equation. This valuation gap can be interpreted as a failure of the transversality condition and, hence, by Claim 8.1, as evidence of dynamic inefficiency. Jiang et al. [23, Section 7.1] reject this interpretation as economically implausible. Their major argument relies on the observation the risk-adjusted discount rate exceeds growth ($\rho < 1$). Furthermore, they argue that rational bubbles are hard to sustain absent other frictions (*e.g.*, Santos and Woodford [33]). However, both the risk-correction of the discount rate and ruling out speculative bubbles reflect the hypothesis that $\rho < 1$, whereas the violation of government debt valuation might well be taken as evidence that $\rho > 1$.

9. Conclusion

We have provided a characterization of dynamic inefficiency in overlapping generations economy with stochastic growth. Our major insight is that a meaningful assessment of dynamic efficiency cannot be based on a comparison between interest rates and average growth rate. Furthermore, even when equilibrium is dynamically inefficient, this does not imply that a social security scheme will be welfare improving. However, once growth uncertainty is properly accounted for, the experience of historically low interest rates appears being associated with some evidence of dynamic inefficiency.

[10] This seems counterintuitive, because a proper solvency requirement for the government would impose a bound associated with the most pessimisti valuation. However, notice that condition (**) arises mechanically from the fact that, with uncontigent debt, the government budget constraint reduces to

$$\left(\frac{1}{1+r_t}\right) b_{t+1} + s_t = b_t.$$

Furthermore, when condition (**) fails in an economy with a representative agent, holding safe government bonds over time, so as to meet fiscal obligations, would be more costly than trading in contingent claims and, hence, suboptimal for the representative individual.

REFERENCES

[1] Abel, A. B., Mankiw, G. N., Summers, L. H. and Zeckhauser, R. J.: 1989, Assessing dynamic efficiency: Theory and evidence, *Review of Economic Studies* **56**, 1–20.

[2] Abel, A. B. and Panageas, S.: 2022, Running primary deficits forever in a dynamically efficient economy: Feasibility and optimality, *Working Paper 30554*, National Bureau of Economic Research.

[3] Aiyagari, S. R. and Peled, D.: 1991, Dominant root characterization of Pareto optimality and the existence of optimal equilibria in stochastic overlapping generations models, *Journal of Economic Theory* **54**, 69–83.

[4] Aliprantis, C. D. and Border, K. C.: 2006, *Infinite Dimensional Analysis*, third edn, Berlin: Springer.

[5] Alvarez, F. and Jermann, U.: 2005, Using asset prices to measure the persistence of the marginal utility of wealth, *Econometrica* **73**, 1977–2016.

[6] Balasko, Y. and Shell, K.: 1980, The overlapping generations model, i: The case of pure exchange without money, *Journal of Economic Theory* **23**, 281–306.

[7] Barbie, M., Hagedorn, M. and Kaul, A.: 2004, Assessing aggregate tests of efficiency for dynamic economies, *Topics in Macroeconomics, The Berkeley Electronic Press* **4**(1), 1–15.

[8] Barbie, M., Hagedorn, M. and Kaul, A.: 2007, On the interaction between risk-sharing and capital accumulation in a stochastic OLG model with production, *Journal of Economic Theory* **137**, 568–579.

[9] Bassetto, M. and Cui, W.: 2018, The fiscal theory of the price level in a world of low interest rates, *Journal of Economic Dynamics and Control* **89**(5-22).

[10] Blanchard, O.: 2019, Public debt and low interest rates, *American Economic Review* **109**, 1197–1229.

[11] Blanchard, O. and Weil, P.: 2001, Dynamic efficiency, the riskless rate, and debt ponzi games under uncertainty, *Advances in Macroeconomics* **1**, 1–21.

[12] Bohn, H.: 1995, The sustainability of budget deficits in a stochastic economy, *Journal of Money, Credit and Banking* **27**, 257–271.

[13] Campbell, J. Y.: 2018, *Financial decisions and markets*, Princeton U. P., Princeton, NJ.

[14] Cass, D.: 1972, On capital overaccumulation in the aggregative, neoclassical model of economic growth: A complete characterization, *Journal of Economic Theory* **4**, 200–223.

[15] Chattopadhyay, S.: 2008, The cass criterion, the net dividend criterion, and optimality, *Journal of Economic Theory* **139**, 335–352.

[16] Chattopadhyay, S. and Gottardi, P.: 1999, Stochastic olg models, market structure, and optimality, *Journal of Economic Theory* **89**, 21–67.

[17] Christensen, T.: 2017, Nonparametric stochastic discount factor decomposition, *Econometrica* **85**, 1501–1536.

[18] Cochrane, John, H.: 2001, *Asset Pricing*, Princeton Univesity Press.

[19] Del Negro, M., Giannone, D., Giannoni, M. and Tambalotti, A.: 2019, Global trends in interest rates, *Journal of International Economics* **118**, 248–262.

[20] Geerolf, F.: 2018, Reassessing dynamic efficiency. Mimeograph, UCLA.

[21] Hansen, L. P. and Scheinkman, J. A.: 2009, Long-term risk: An operator apporach, *Econometrica* **77**(1), 177–234.

[22] Hellwig, M. F.: 2021, Safe assets, risky assets, and dynamic inefficiency in overlapping-generations economies. Discussion Papers of the Max Planck Institute for Research on Collective Goods, No. 2021/10.

[23] Jiang, Z., Lustig, H., Van Nieuwerburgh, S. and Xiaolan, M. Z.: 2021, The us public debt valuation puzzle. NBER Working Paper 16092.

[24] Jiang, Z., Lustig, H., Van Nieuwerburgh, S. and Xiaolan, M. Z.: 2022, Manufacturing risk-free government debt, *Working Paper 27786*, National Bureau of Economic Research.

[25] Kocherlakota, N.: 2022a, Capital overaccumulation in a risky world: The long yield as a sufficient statistic. Mimeograph, University of Rochester.

[26] Kocherlakota, N. R.: 2022b, Infinite debt rollover in stochastic economies. Mimeograph, University of Rochester.

[27] Lucas, R. E.: 1978, Asset prices in an exchane economy, *Econometrica* **46**, 1429–1445.

[28] Lunsford, K. G. and West, K. D.: 2019, Some evidence on secular drivers of us safe real rates, *American Economic Journal: Macroeconomics* **11**, 113–139.

[29] Manuelli, R. E.: 1990, Existence and optimality of currency equilibrium in stochastic overlapping generations models: The pure endowment case, *Journal of Economic Theory* **51**, 268–294.

[30] Mehra, R. and Prescott, E. C.: 1985, The equity premium: A puzzle, *Journal of Monetary Economics* **15**, 145–161.

[31] Peled, D.: 1982, Informational diversity over time and the optimality of monetary equilibria, *Journal of Economic Theory* **28**, 255–274.

[32] Rogoff, K. S., Rossi, B. and Schmelzing, P.: 2022, Long-run trends in long-maturity real rates 1311-2021, *Working Paper 30475*, National Bureau of Economic Research.

[33] Santos, M. S. and Woodford, M.: 1997, Rational asset pricing bubbles, *Econometrica* **65**(1), 19–57.

[34] Stock, J. H. and Watson, M. W.: 2002, Has the business cycle changed and why?, *NBER Macroeconomic Annual* **17**, 160–218.

[35] Zilcha, I.: 1991, Characterizing efficiency in stochastic overlapping generations models, *Journal of Economic Theory* **55**, 1–16.

APPENDIX A. SPECTRAL RADIUS

A mathematical treatment of the spectral radius of positive linear operators can be found in Aliprantis and Border [4, Chapter 20]. For completeness we present a short illustration of the relevant theory for the linear monotone operator $T : \mathcal{L}(a) \to \mathcal{L}(a)$ used throughout our analysis.

The *spectral radius* is given by

$$\rho(T) = \lim_{n \in \mathbb{N}} \sqrt[n]{\|T^n\|},$$

where the operator norm is as usual defined as

$$\|T^n\| = \sup_{v \in \mathcal{L}(a)} \{\|(T^n v)\| : \|v\| \leq 1\}.$$

We preliminarily clarify that, by monotonicity, the operator norm is easily determined.

Claim A.1 (Operator norm). *For every n in $\mathbb{N}$,*

$$\|T^n\| = \|(T^n a)\|.$$

Proof. The operator norm is given by

$$\|T^n\| = \sup_{v \in [-a,a]} \|(T^n v)\|.$$

Notice that, by monotonicity,

$$|(T^n v)| \leq (T^n |v|).$$

Indeed,

$$-(T^n |v|) \leq (T^n (-|v|)) \leq (T^n v) \leq (T^n |v|).$$

It follows that

$$\|(T^n a)\| \leq \|T^n\| \leq \sup_{|v| \leq a} \|(T^n |v|)\| = \|(T^n a)\|,$$

so proving the claim. □

We next provide a characterization of the spectral radius.

Claim A.2 (Spectral radius). *Spectral radius $\rho(T)$ lies in $(0,1) \subset \mathbb{R}^+$ if and only if there exists ρ in $(0,1) \subset \mathbb{R}^+$ such that, for some f in the interior of $\mathcal{L}^+(a)$,*

$$\rho f \geq (Tf). \qquad (*)$$

Proof of necessity. As f lies in the interior of $\mathcal{L}^+(a)$, there exist λ_h and λ_l in $\mathbb{R}^{++}$ such that $\lambda_h a \geq f \geq \lambda_l a$. When condition (*) is satisfied, by iterating, we obtain

$$\rho^n f \geq T^n(f).$$

Using the bounds, and exploiting monotonicity, this yields

$$\rho^n \left(\frac{\lambda_h}{\lambda_l}\right) a \geq (T^n a),$$

which in turn implies

$$\rho^n \left(\frac{\lambda_h}{\lambda_l}\right) \geq \|T^n\| .$$

Taking the root,

$$\rho \sqrt[n]{\frac{\lambda_h}{\lambda_l}} \geq \sqrt[n]{\|T^n\|},$$

thus proving that $\rho \geq \rho(T)$. $\square$

Proof of sufficiency. Pick any ρ in $(\rho(T), 1) \subset \mathbb{R}^+$. For any sufficiently large n in $\mathbb{N}$,

$$(T^n a) \leq \|T^n\| a \leq \rho^n a. \tag{**}$$

Defining

$$f^n = a + (Ta) + \cdots + (T^n a),$$

notice that, by linearity,

$$a + (Tf^n) = f^{n+1}.$$

The process monotonically (pointwise) converges to f in $\mathcal{L}^+(a)$ because condition (**) provides a geometric upper bound eventually. We so obtain that, in the limit,

$$a + (Tf) = f.$$

Furthermore, for some sufficiently large ρ in $(0,1) \subset \mathbb{R}^+$,

$$(1-\rho) f \leq a.$$

We so conclude that

$$(1-\rho) f + (Tf) \leq f \text{ if and only if } (Tf) \leq \rho f.$$

This proves our claim. $\square$

We conclude with a sort of Perron-Frobenius Theorem

Claim A.3 (Dominant root)**.** *If root ρ in $\mathbb{R}^+$ satisfies the eigenvalue equation $\rho f = (Tf)$ for some eigenprocess f in the interior of $\mathcal{L}^+(a)$, then it is the spectral radius, $\rho = \rho(T)$.*

Proof. At no loss of generality, we can assume that $\|f\| = \|a\| = 1$. By linearity, iterating the eigenvalue equation implies $\rho^n f = (T^n f)$. Using the definition of the operator norm,

$$\|T^n\| \geq \|(T^n f)\| = \|\rho^n f\| = \rho^n \|f\| = \rho^n.$$

This yields $\rho(T) \geq \rho$. Let λ in $\mathbb{R}^{++}$ be such that $f \leq a \leq \lambda f$. This is possible because f lies in the interior of $\mathcal{L}^+(a)$. By monotonicity, we obtain

$$(T^n a) \leq \lambda (T^n f) \leq \lambda \rho^n f \leq \lambda \rho^n a,$$

thus implying $\|T^n\| \leq \lambda \rho^n$ and, hence, $\sqrt[n]{\|T^n\|} \leq \sqrt[n]{\lambda}\rho$. As $\lim_{n\to\infty} \sqrt[n]{\lambda} = 1$, this proves the claim. $\square$