Economic distributions, primitive distributions, and demand recovery in monopolistic competition

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Abstract

We link fundamental technological and taste distributions to endogenous economic distributions of prices and firm size (output, profit) generated under monopolistic competition with heterogeneous productivities as per recent Trade and IO models. We derive full equivalence properties for monopoly mark-ups, demand shape, marginal revenue, and profits to match distributions of cost, price, output, and profit under monopolistic competition. Demand and one distribution determine the rest. We provide constructive proofs to recover demand and all distributions from just two (e.g., price and cost distributions uncover demand form), and derive restrictions on distribution pairs. We extend to mark-up distributions.

JEL Classification: L13, F12

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1 Introduction

Distributions of economic variables have attracted the interest of economists at least since Pareto (1896). In industrial organization, firm size distributions (measured by output, sales, or profit) have been analyzed. Other studies have looked at the distribution of prices within an industry and across industries (Kaplan and Menzio, 2015, and Hitsch, Hortacsu, and Lin, 2017); recent research has focussed on mark-up distributions (De Loecker, Eeckhout, and Unger, 2020). Firm sizes within industries are wildly asymmetric, and frequently involve a long-tail of smaller firms (e.g., Anderson, 2006, Gabaix, 2016). Particular distributions – mainly the Pareto and log-normal – seem to fit the data well. Much work in international trade looks at the size distribution of firms (e.g., Melitz, 2003, Eaton, Kortum, and Kramarz, 2011, Head, Mayer, and Thoenig, 2014).

We show how the distributions of these "economic" variables (prices, output, profit, and mark-ups) are jointly determined by the fundamental underlying distributions of tastes and technologies, and we determine the links between the various distributions. We link the economic ones to each other and to the primitive (cost) distribution and consumer tastes (as encapsulated in demand). More surprisingly, the primitives can be uncovered from the observed economic distributions.

The idea of linking demand to distributions is analyzed in two recent papers which complement the present study. Mrázová, Neary, and Parenti (2021) study the relations between parameterized equilibrium distributions of sales and relative mark-ups and the (primitive) productivity distribution for a cleverly parameterized demand form. They are mainly interested in when distributions are in the same ("self-reflecting") class (e.g., when both productivity and sales are log-normal or Pareto). They provide some empirical analysis of log-normal and Pareto distributions. In parallel, Anderson and de Palma (2020) start with the special (and central) case of the CES and the "Pareto circle" that all relevant distributions are Pareto if one is, and they extend to find the distribution classes associated to other key parameterized distribution forms.[1] They extend the analysis to generalized logit-CES demand forms, which are not covered by the MNP parameterization. These papers provide useful schematic links, but they do not address *general* demand functions nor how *arbitrary* distribution shapes can be used to recover demand nor which distribution combinations are consistent with the monopolistic competition model.

We start by deploying a general monopolistic competition model with a continuum of firms (see Thisse and Ushchev, 2018, for a review of this literature).[2] We focus on productivity (cost) differences across firms, in line with much recent work in Trade, although theoretical IO models have almost exclusively looked at symmetric settings. The Trade literature is mainly based on CES demand, while we take general demand functions as our starting point. We show how the demand function delivers a mark-up function, and then we prove our key converse result that the mark-up (or "pass-through" function of Weyl and Fabinger, 2013) determines the form of the demand function. We then engage these results and analogous ones with constructive proofs to determine how (and under which conditions) cost distribution and demand suffice to determine the shape of the economic price, profit, and output distributions. Along broader lines, we determine when and how any two distributions suffice to deliver all the missing pieces.

We contribute several results to the theory of pass-through and monopoly, and engage them as key relations for the monopolistic competition analysis. One is that a continuously differentiable and strictly monotonic mark-up (pass-through) implies a strictly (-1)-concave demand. We construct the demand function from the mark-up function and we prove the equivalence of the following properties: strictly (-1)-concave demand; strictly increasing mark-

[1] They also introduce heterogeneous product qualities to break the Pareto circle for the CES.

[2] Ironically, Chamberlin (1933) is best remembered for his symmetric monopolistic competition analysis. Yet he went to great length to point out that he believed asymmetry to be the norm, and that symmetry was a very restrictive assumption.

up function; strictly decreasing marginal revenue. The counterpart to Hotelling's Lemma for monopolistic competition establishes that profit is strictly convex in unit cost and its derivative is the inverse marginal revenue. Another relation is that optimized profit is strictly decreasing in optimal price.

We use these results to uncover restrictions on distributions to be compatible with monopolistic competition. First, we determine how the properties of demand and a single distribution (such as cost, price, output, or profit) suffice to determine all the other distributions. Our next contribution is perhaps more surprising because it works in the opposite direction and theoretically identifies demand. We reconstruct demand from any distribution pair. For example, if both the profit distribution and the price distribution are strictly monotonic then there exists a strictly (-1)-concave demand function which renders them consistent with the monopolistic competition model whatever their shapes. We consider mark-up distributions in a fully separate Section. The next Section describes the model and its back-drop and outlines a detailed road-map of the paper.

2 The Model: overview and roadmap

There is a continuum of firms. Each has constant unit production costs, c, but these differ across firms with domain $[\underline{c}, \bar{c}]$, where $\underline{c} \geq 0$. Within this continuum, each firm effectively faces a monopoly problem where its price choice is independent of the actions of rivals. We allow for a general common demand function.

2.1 Demand side

Assumption 1 *Suppose that demand $h(p)$ for a firm charging p is a positive, strictly decreasing, strictly (-1)-concave, and $\mathcal{C}^2$ function on its domain $[\underline{c}, \infty)$, with $h(0) > \bar{c}$.*

This is equivalent to $\frac{1}{h(.)}$ strictly convex, and is a minimal condition ensuring profit strict quasi-concavity: see Caplin and Nalebuff (1991) and Anderson, de Palma, and Thisse (1992, p.164) for more on ρ-concave functions and see Weyl and Fabinger (2013) for the properties of pass-through as a function of demand curvature.[3] We suppress the impact of other firms' actions on demand, which would be expressed as aggregate variables in the individual demand function (as illustrated next paragraph). Under monopolistic competition with a continuum of firms, each firm's individual action has no measurable impact on the aggregate variables (for example, the "price index" in the CES model, or the Logit denominator). Because we look at the cross-section relation between equilibrium distributions, the actions of other firms are held constant across the comparison, and therefore are not changing.

One micro-foundation for the function $h(p)$ comes from a continuous-discrete choice model (Hanemann, 1984). Consumers make a discrete choice of which product to buy but their conditional demand is price-sensitive. Concretely, suppose that consumer conditional indirect utility is $u_i = y + v(p_i) + \varepsilon_i$ where y is income, $v(.)$ has the properties of a conditional indirect utility function (decreasing, convex in p_i) and the ε_i are i.i.d. Gumbel distributed. Then conditional demand is $v'(p_i)$ and demand for product i is (by Roy's identity) $x_i = (-v'(p_i)) \frac{\exp(y+v(p_i))}{D}$, where $D = \int_0^1 \exp(y + v(p_l))\, dl$ is a constant for individual firms under monopolistic competition. Setting $x_i(p)$ equal to $h(p)$ and integrating delivers the conditional indirect sub-utility function $v(.)$ that generates demand $h(p)$ from this continuous-discrete consumer approach. When $v(p_i) = -p_i$ we have logit (with unit conditional demand and hence log-linear $h(.)$); CES has $v(p_i) = -\ln p_i$ and hence constant elasticity demand.[4]

While we can construct a monopolistic competition model with discrete choice roots from

[3] A1 is actually not very restrictive. If there were segments of demand that were strictly log-convex then marginal revenue slopes up in such regions and so first-order conditions cannot constitute even a local maximum and so are irrelevant. For such situations there would be an effective gap in demand and so corresponding outputs would have zero density in the monopolistic competition analysis. We deal with strict (-1)-concavity for simplicity, for then corresponding distributions of economic variables are strictly increasing on their domains. See the CEPR DP 15731 for details.

[4] Aggregate consumer surplus is $V = \int_0^1 \exp v(p_l)\, dl + y$. Applying Roy's Lemma delivers the demands. This is analogous to a representative consumer for the population with heterogeneous tastes over varieties.

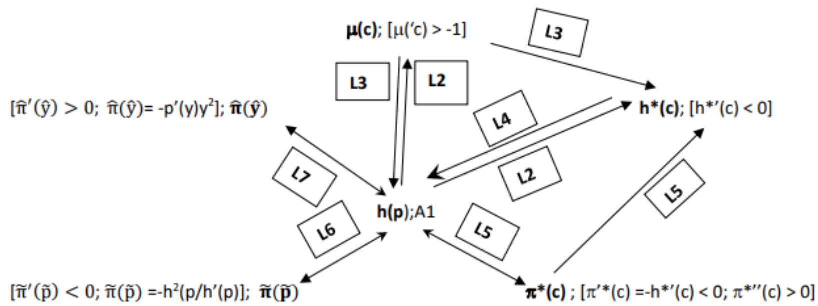
$h(p)$ as the constructive proof above shows, the converse is not true. Perloff and Salop (1985) show that the tail of the density $f(\varepsilon)$ must satisfy $\lim_{\varepsilon\to\infty} (\ln f(\varepsilon))'$ finite, in order for the limit price equilibrium to strictly exceed marginal cost and therefore constitute monopolistic competition. Otherwise, with a bounded density or too thin a tail (such as normal and hence probit) the limit is perfect competition and so cannot deliver a function $h(p)$. The Gumbel (and hence Logit and CES) satisfy the requisite tail property.

Another approach builds on representative consumer models of product differentiation (see e.g. Spence, 1976 and Dixit and Stiglitz, 1977). The canonical CES model was extended by Kimball (1995).[5] We provide an analogous version for quasilinear utility by defining the utility function implicitly by $\int_0^1 \Upsilon(x_l - u)\, dl = 1$, so the corresponding quasi-linear utility is $U = u + X_0$ where X_0 is consumption of numeraire, u is the sub-utility function, and $\Upsilon'(.) > 0$. We derive the corresponding demand functions by maximizing U under the budget constraint, which leads to *inverse* demands given by $\frac{\partial u}{\partial x_i} = p_i = \frac{\Upsilon'(x_i - u)}{D}$, where $D = \int_0^1 \Upsilon'(x_l - u)\, dl$ is constant under monopolistic competition so that i's demand is given by inversion (up to a positive constant) as $x_i = \Upsilon'^{-1}(p_i) + u$. Setting x_i equal to $h(p)$ and integrating back delivers the (modified) Kimball function $\Upsilon(.)$ that generates demand $h(p)$ from this representative consumer approach.

2.2 Monopoly (and monopolistic competition) properties of demand

Our focus is on cost and the endogenous economic variables: price/mark-up, profit, and output. We establish in Section 3 key properties relating these variables - we show that A1 implies strictly monotonic and continuously differentiable relations between any pair of them and that

[5] Kimball (1995) considers a single sector (no outside good) and postulates a utility function, U, implicitly defined by $\int_0^1 \Upsilon\left(\frac{x_l}{U}\right) dl = 1$, where $\Upsilon(1) = 1$ and $\Upsilon'(.) > 0$. The special case when $\Upsilon(\xi) = \xi^\rho$ delivers the CES ($U = \left(\int_0^1 x_l^\rho\right)^{\frac{1}{\rho}}$) and constant elasticity demand. As Kimball argues the function $\Upsilon(.)$ allows any demand shape. We thanks a referee for alerting us to Kimball's work.



knowing any relation determines all the others. These results constitute the full set of links of main variables for monopoly and monopolistic competition, showing how (for example) a pass-through function (the element derived by Weyl and Fabinger, 2013, from demand) suffices to determine all the satellite elements. The diagram below provides the map to the Lemmas and the summary Theorem of Section 3 that establish these links.

One key set of relations allow us to circle between demand, mark-ups, and output /marginal revenue. Another key relation involves strict monotonicity for profits: A1 implies that *equilibrium profits are convex in unit cost with derivative equal to output* (Lemma 5, and the counterpart to Hotelling's Lemma), equilibrium *profits strictly decrease with equilibrium prices* (Lemma 6) *and strictly increase with equilibrium output* (Lemma 7).

2.3 Equilibrium distribution relations for monopolistic competition

The result above that A1 implies that each variable is a function of any of the others enables us to back out the equilibrium distribution relations for monopolistic competition. These equilibrium relations are written in the following form. Let z denote the fraction of firms with profit below some level π. Given the strict monotonic relations between variables, the same set of firms have output *below* some corresponding level y, and these same firms have costs and prices *above* corresponding levels c and p. Thus the firms with costs strictly higher than some value c are the same ones that have prices strictly higher than p, an output strictly below y and a profit strictly below π, where the specific values satisfy $\pi = (p - c)\, h\,(p)$, where $h\,(p) = y$ and the mark-up $(p - c)$ satisfies the first-order condition (3) below (see Anderson and de Palma, 2001, for analogous properties for oligopoly). Writing $F_C\,(c)$ as the cumulative cost distribution function, etc., gives the following key ranking property:[6]

$$1 - F_C\,(c) = 1 - F_P\,(p) = F_Y\,(y) = F_\Pi\,(\pi) = z. \tag{1}$$

For the reader who wishes to look ahead to how this all fits together graphically, we refer them to Figure 1 below. The technical analysis of monopolistic competition equilibrium begins with Theorem 2 in Section 4. Here we show that the primitives of the model, the demand function $h\,(p)$ and the cost distribution $F_C\,(c)$, tie down the other distributions. This analysis uncovers several relations between equilibrium densities that must hold in equilibrium. These therefore generate potentially testable implications of the model in terms of densities. Theorem 3 assumes the demand function is known (a common assumption in many empirical studies) and shows that knowing any one of the endogenous distributions (price, output, or profit) suffices to identify all the other distributions, including costs.

In Section 5 we turn to the theoretical identification exercise proper where we do not *a priori* assume a demand function but we instead identify it. To do so, we need any two distributions, under the restrictions uncovered after Theorem 2 that they be compatible with the monopolistic competition model. We therefore consider each pair of distributions (F_C, F_P, F_Y, and F_Π). If two distributions are compatible and both strictly increasing and $\mathcal{C}^2$, we construct the implied demand function satisfying A1. A guide to the content of the Theorems is given in this Table:

[6] Mark-ups will be treated separately because they are not necessarily monotone increasing in c.

	Seed CDFs	Restrictions	Testable predictions	Unique demand satisfying (A1)
Thm 4	F_C F_P	$F_C(c) > F_P(p(c))$	$\mu'(c) > 0$ iff $f_C(c) > f_P(p(c))$	up to mult. factor
Thm 5	F_Y F_P	$f_P/f_Y \downarrow$ in p	$f_Y \downarrow \Longrightarrow f_P \downarrow$; $f_P \uparrow \Longrightarrow f_Y \uparrow$	up to add. factor
Thm 6	F_Y F_C	No restriction	Any $\{F_Y, F_C\}$ is compatible	up to add. factor
Thm 7	F_Π F_P	No restriction	Any $\{F_\Pi, F_P\}$ is compatible	up to add. factor
Thm 8	F_Π F_C	$f_C/f_\Pi \downarrow$ in c	$f_\Pi \downarrow \Longrightarrow f_C \downarrow$; $f_C \uparrow \Longrightarrow f_\Pi \uparrow$	up to add. factor
Thm 9	F_Π F_Y	None	Any $\{F_\Pi, F_Y\}$ is compatible	up to add. factor

Table 1. Guide to Distribution Pair Theorems

Where pertinent we give further testable properties implied from elasticity relations between densities. We also draw out the difference between log-concave and log-convex demands when discussing pass-through.

Section 6 gives results specific to mark-ups. Section 7 concludes with some extensions.

3 Monopoly demands, mark-ups, and profits

This section proves the results of the diagram above. In what follows, whenever we use a "prime" symbol on a variable, we shall understand the function to be continuously differentiable ($\mathcal{C}^1$).

3.1 Demand to mark-up and output

The profit for a firm with per unit cost c is $\pi = (p-c)\,h(p) = mh(m+c)$, where $m = p - c$ is its mark-up. We will make extensive use of the following result on demand, which follows because strict (-1)-concavity is by definition that $1/h(u)$ is strictly convex.

Lemma 1 *A $\mathcal{C}^2$ function $h(u)$ is strictly (-1)-concave iff $[h(u)/h'(u)]' > -1$. Equivalently, $h(u)\,h''(u) - 2\,[h'(u)]^2 < 0$.*

With a continuum of firms, the equilibrium mark-up, m, satisfies the first-order condition

$$m + \frac{h(m+c)}{h'(m+c)} = 0, \qquad c \in [\underline{c}, \bar{c}]\,. \tag{2}$$

The solution to (2), denoted $\mu(c)$, is uniquely determined (and strictly positive) by A1 via Lemma 1. It constitutes a maximum to profit because profit is rising for all $m < \mu(c)$ and falling for all $m < \mu(c)$.

Applying the implicit function theorem to (2) shows that

$$\mu'(c) = \frac{-\left[h(\mu+c)/h'(\mu+c)\right]'}{1+\left[h(\mu+c)/h'(\mu+c)\right]'} > -1, \tag{3}$$

where the denominator is strictly positive under A1 by Lemma 1. Continuity of $\mu'(c)$ implies equilibrium price is $\mathcal{C}^1$. Because $\mu'(c) > -1$, price strictly increases in cost.

Denote the value of equilibrium demand by $h^*(c) = h(\mu(c)+c)$. Now, $h^*(c)$ is strictly decreasing given that $\mu'(c) > -1$:

$$h^{*\prime}(c) = (\mu'(c)+1)\, h'(\mu(c)+c) < 0. \tag{4}$$

Because $h^*(c)$ is a continuous and strictly decreasing function, marginal revenue (which is $h^{*-1}(c)$) is also continuous and strictly decreasing. To summarize:

Lemma 2 *For given demand $h(p)$ satisfying A1: (i) the equilibrium mark-up, $\mu(c) > 0$ is the unique $\mathcal{C}^1$ solution to (2), with $\mu'(c) > -1$; (ii) the equilibrium price, $p(c) > c$, is a $\mathcal{C}^1$ function with $p'(c) > 0$; (iii) the associated equilibrium demand, $h^*(c) \equiv h(\mu(c)+c)$, is $\mathcal{C}^1$ with $h^{*\prime}(c) < 0$.*

Our converse result to Lemma 2 indicates how the mark-up function $\mu(c)$ implies the form of inverse marginal revenue, $h^*(c)$, and hence determines the form of $h(p)$ satisfying A1.

3.2 From mark-ups to output and demand

Here we show how any mark-up function $\mu(c)$ (with $\mu'(c) > -1$) can be used to find the associated equilibrium demand and demand function, $h(p)$. (Equivalently, we can start with a

$\mathcal{C}^1$ and strictly increasing relation between equilibrium price and cost, $p(c)$.)

Lemma 3 *Consider any positive mark-up function $\mu(c) > 0$ for $c \in [\underline{c}, \bar{c}]$ with $\mu'(c) > -1$, i.e., $p'(c) > 0$. Then there exists an equilibrium output function $h^*(c)$ with $h^{*\prime}(c) < 0$, $c \in [\underline{c}, \bar{c}]$ and given by (6), which is unique up to a positive multiplicative factor. The associated primitive demand function, $h(p)$ (unique up to a positive multiplicative factor) is given by (7) and satisfies A1 on its domain $[\mu(\underline{c}) + \underline{c}, \mu(\bar{c}) + \bar{c}]$.*

Proof. First note from (2) and (4) that

$$\frac{dh^*(c)/dc}{h^*(c)} = \frac{(\mu'(c)+1)\,h'(\mu(c)+c)}{h(\mu(c)+c)} = -\frac{\mu'(c)+1}{\mu(c)} \equiv g(c) < 0, \tag{5}$$

because $\mu'(c) > -1$ by assumption. Thus $[\ln h^*(c)]' = g(c)$, and so $\ln\left(\frac{h^*(c)}{h^*(\underline{c})}\right) = \int_{\underline{c}}^{c} g(v)\,dv$, or

$$h^*(c) = h^*(\underline{c}) \exp\left(\int_{\underline{c}}^{c} g(v)\,dv\right), \quad c \geq \underline{c}, \tag{6}$$

which determines $h^*(c)$ up to the positive factor $h^*(\underline{c})$; it is strictly decreasing because $g(c) < 0$.

We can now use the output function, $h^*(c)$ (which is inverse marginal revenue), to back out the demand function, $h(m+c)$, via the following steps. First, define $u \equiv p(c) = \mu(c) + c$, which is strictly increasing because $\mu'(c) + 1 > 0$, so the inverse function $p^{-1}(\cdot)$ is strictly increasing. Now, $h(u) = h^*(p^{-1}(u))$ and thus the function $h(\cdot)$ is recovered on the domain $u \in [\mu(\underline{c}) + \underline{c}, \mu(\bar{c}) + \bar{c}]$. Using (6) with $h(u) = h^*(p^{-1}(u))$,

$$h(u) = h^*(\underline{c}) \exp\left(\int_{\underline{c}}^{p^{-1}(u)} g(v)\,dv\right), \tag{7}$$

and so we recover the pricing first-order condition (2):

$$\frac{h(u)}{h'(u)} = \frac{1}{g(p^{-1}(u))\,[p^{-1}(u)]'} = \frac{p'(c)}{-\frac{\mu'(c)+1}{\mu(c)}} = -\mu(c) < 0, \tag{8}$$

where first and second step follow from (5) with $u = p(c)$ and the last step follows because $p'(c) = \mu'(c) + 1$. Thus, since $h(u) = h(\mu(c) + c) = h^*(c)$. So,

$$\left[\frac{h(u)}{h'(u)}\right]' = -\frac{\mu'(c)}{\mu'(c)+1} > -1, \tag{9}$$

and so $h(u)$ is strictly (-1)-concave (using Lemma 1). Note that $h(.)$ is $\mathcal{C}^2$ because $\mu(.)$ was assumed differentiable. ■

Recalling that $\mu(c) = p(c) - c$ for $c \in [\underline{c}, \bar{c}]$, the restriction used in the Lemma ($\mu'(c) > -1$) is that $p'(c) > 0$ so that *any* arbitrary (differentiable) increasing price function of costs can be associated to a unique demand function that could generate it (up to the multiplicative factor).

The reason that demand is only determined up to a positive factor is simply that multiplying demand by a positive constant does not change the optimal mark-up (when marginal costs are constant, as here). The mark-up function can only determine the demand shape, but not its scale. The steps in the proof are readily confirmed for the $\rho-$linear example given at the end of this Section.

Notice that the function $h(\cdot)$ is tied down only on the domain for which we have information about the equilibrium mark-up value in the market. Outside that domain, we know only that $h(\cdot)$ must be consistent with the maximizer $\mu(c)$, which restricts $h(\cdot)$ to be not "too" convex.

The results so far indicate that knowing either $h(.)$ or $\mu(c)$ suffices to determine the other and $h^*(c)$ (up to constants in the first case). We next show that knowing $h^*(c)$ strictly decreasing determines $h(p)$ satisfying A1.

3.3 From strictly decreasing MR to strictly (-1)-concave demand

First note that $h^*(c)$ is strictly decreasing if and only if marginal revenue, $h^{*-1}(y) \equiv MR(y) > 0$, is strictly decreasing, with both $\mathcal{C}^1$. This is because these are inverse functions. Next, integrating $MR(y)$ yields total revenue, $TR(y)$, which is therefore $\mathcal{C}^2$ (and it is strictly quasiconcave, and monotone increasing for $MR(y) > 0$). Inverse demand, $p(y)$, is then $TR(y)/y$, and this is a $\mathcal{C}^2$ function. Inverting it yields $h(p)$ as a $\mathcal{C}^2$ function. It remains to show that $h(p)$ is strictly (-1)-concave. The next result concludes the issue.

Lemma 4 *Consider any inverse marginal revenue function, $h^*(c) > 0$, strictly decreasing and $\mathcal{C}^1$. Then there exists demand, $h(p)$ satisfying A1, which can be recovered up to a constant.*

Proof. First note that $h(p)$ is strictly (-1)-concave if and only if $h''h - 2(h')^2 < 0$. Write the inverse demand as $p(y)$ so that $h'(p) = \frac{1}{p'(y)}$ and $h''(p) = -\frac{p''(y)}{(p'(y))^3}$. Then the strict (-1)-concavity condition we are to show becomes:

$$p''y + 2p' < 0. \tag{10}$$

Now we want to find $p(y)$, using the steps explained before the Lemma. Let $MR(y)$ denote $h^{*-1}(c)$, i.e., marginal revenue. So then Total Revenue, $TR(y)$ is the integral of $MR(y)$ and equilibrium inverse demand, $p(y)$, is

$$p(y) = \frac{TR(y)}{y} = \frac{\int_0^y MR(u)\,du}{y} = \frac{\int_0^y h^{*-1}(u)\,du}{y} = \frac{\int_0^y c(u)\,du}{y},$$

and its inverse is $h(p)$: note demand $h(p)$ is only determined up to a constant (from the step where $MR(.)$ is integrated). Hence $p'(y) = \frac{yc(y) - \int_0^y c(u)du}{y^2}$ and $p''(y) = \frac{c'(y)y^2 - 2\left(yc(y) - \int_0^y c(u)du\right)}{y^3}$. Using these expressions in (10) gives $MR'(y) = p''(y)\,y + 2p'(y) < 0$ (where we used $c'(y) < 0$, i.e. marginal revenue slopes down). ■

Intuitively, one can always add a rectangular hyperbola to any inverse demand (the rectangular hyperbola has a zero Marginal Revenue) and get the same Marginal Revenue function.

3.4 Hotelling's Lemma

Call the equilibrium profit $\pi^*(c) = \mu(c) h^*(c)$. The next result shows its properties under A1 and how demand satisfying A1 can be constructed from any such function.

Lemma 5 *The equilibrium profit function, $\pi^*(c) = \mu(c) h^*(c)$, is strictly convex and $\mathcal{C}^2$ with $\pi^{*\prime}(c) = -h^*(c) < 0$ if and only if A1 holds. Consider any demand function $h(p)$ satisfying A1. Then the equilibrium profit function, $\pi^*(c) = \mu(c) h^*(c)$, is strictly convex and $\mathcal{C}^2$ with $\pi^{*\prime}(c) = -h^*(c) < 0$. Conversely, for any positive profit function $\pi^*(c)$ which is strictly decreasing and strictly convex and $\mathcal{C}^2$ there exists a demand function, $h(p)$ satisfying A1, which can be recovered up to a constant.*

Proof. Applying the envelope theorem to the profit function $\pi(p; c) = (p - c) h(p)$ implies that $\pi^{*\prime}(c) = -h(p(c)) = -h^*(c) < 0$. Because $h^*(c)$ is $\mathcal{C}^1$ and strictly decreasing by Lemma 2, $\pi^*(c)$ is $\mathcal{C}^2$ and is strictly convex. Conversely, the derivative of any purported convex and decreasing profit function $\pi^*(c)$ represents a valid function $h^*(c)$ and Lemma 4 shows that there exists a corresponding demand function $h(p)$ satisfying A1. ■

This is the monopolistic competition (and monopoly) counterpart to Hotelling's Lemma for competitive firms (that the derivative of profit with respect to price is minus demand). While the result is straightforward, we do not know any statement of it for monopolistic competition.

Because it specifies optimal output as a function of marginal cost, $h^*(c)$ is the inverse marginal revenue curve.

3.5 Equilibrium prices and profits, and output and profit

Another new characterization result (used in Theorem 7) concerns the properties of equilibrium profit when written as a function of equilibrium price, denoted by $\tilde{p}$. Call this relationship $\tilde{\pi}(\tilde{p})$. Inserting the mark-up first-order condition (2) into the profit function $\pi(p) = (p - c) h(p)$ gives the desired relation as $\tilde{\pi}(\tilde{p}) = -h^2(\tilde{p})/h'(\tilde{p})$. This relation defines a strictly decreasing and continuous function if and only if $h^2(p)/h'(p)$ is strictly increasing. But this is the condition for $1/h(p)$ to be strictly convex: equivalently, this is A1. Furthermore, along the lines of our earlier demand construction results, it is straightforward to argue that for any purported $\mathcal{C}^2$ and strictly decreasing function $\tilde{\pi}(\tilde{p})$ there exists a demand function, $h(p)$ satisfying A1, which can be recovered (by integration) up to a constant. In summary:

Lemma 6 *Equilibrium profit as a function of equilibrium price, $\tilde{\pi}(\tilde{p}) = -h^2(\tilde{p})/h'(\tilde{p})$, is a strictly decreasing and continuous function if A1 holds. Conversely, for any profit function $\tilde{\pi}(\tilde{p})$ which is positive, strictly decreasing, and $\mathcal{C}^2$ there exists a demand function, $h(p)$ satisfying A1, which can be recovered up to a constant.*

The intuition for the relation between equilibrium profit and equilibrium price is as follows. Suppose that some price is optimally chosen on the demand curve $h(p)$. Then it must be that the price satisfies the condition that marginal revenue equals marginal cost, with marginal revenue downward-sloping locally: as shown above, marginal revenue is strictly decreasing if and only if demand satisfies A1. The optimal profit is continuously decreasing with the optimal price because both are driven in a continuous way by costs: higher costs entail both higher prices and lower profits, as Lemma 2 and Lemma 5 attest.

Analogously, we can describe the relation between equilibrium output and equilibrium profit. Denote the former by $\hat{y}$ and denote the relation $\hat{\pi}(\hat{y})$. This is quickest to derive using the inverse demand, $p(y)$, for which the standard $MR = MC$ condition writes as $p'(\hat{y})\hat{y} + p(\hat{y}) - c = 0$, and hence $\hat{\pi}(\hat{y}) = -p'(\hat{y})\hat{y}^2$. Then: $\hat{\pi}'(\hat{y}) = -\hat{y}\left(2p'(\hat{y}) + p''(\hat{y})\hat{y}\right)$. Noting that $p'(\hat{y}) = 1/h'(\tilde{p})$ and $p''(\hat{y}) = -h''(\tilde{p})/(h'(\tilde{p}))^3$, this gives $\hat{\pi}'(\hat{y}) = -\frac{h(\tilde{p})}{(h'(\tilde{p}))^3}\left\{2(h'(\tilde{p}))^2 - h''(\tilde{p})h(\tilde{p})\right\}$, which is strictly negative if and only if A1 holds.

Lemma 7 *Equilibrium profit as a function of equilibrium output, $\hat{\pi}(\hat{y})$, is a positive, strictly increasing, and continuous function if A1 holds. Conversely, for any profit function $\hat{\pi}(\hat{y}) = -p'(\hat{y})\hat{y}^2$ which is strictly increasing and $\mathcal{C}^2$ there exists a demand function, $h(p)$ satisfying A1, which can be recovered up to a constant.*

The intuition again comes from thinking about higher costs delivering both lower output and lower profit, so that these variables move together.

3.6 Demand relations synthesis

We summarize the results obtained so far in this Section as follows.

Theorem 1 *Consider the set of elements $h(p)$, $\mu(c)$, $h^*(c)$, $\pi^*(c)$, $\tilde{\pi}(\tilde{p})$, and $\hat{\pi}(\hat{y})$. Any demand function $h(p)$ satisfying A1 implies the existence of a mark-up function with $\mu(c)$ with $\mu'(c) > -1$, an output (inverse marginal revenue) function $h^*(c)$ with $h^{*\prime}(c) < 0$, a convex and decreasing equilibrium profit function $\pi^*(c)$ with $\pi^{*\prime}(c) = -h^*(c)$, a decreasing equilibrium profit function $\tilde{\pi}(\tilde{p})$, and an increasing equilibrium profit function $\hat{\pi}(\hat{y})$. Likewise, the knowledge of any one of the other elements, $\mu(c)$, $h^*(c)$, $\pi^*(c)$, $\tilde{\pi}(\tilde{p})$, and $\hat{\pi}(\hat{y})$ with the required properties delivers the others with the required properties.*

The demand assumption A1 (that demand is strictly (-1)-concave and $\mathcal{C}^2$) implies various properties and conversely, as detailed in the Lemmas, including:

Equilibrium price strictly increases with cost.

Marginal revenue is strictly decreasing.

Equilibrium profit is strictly convex in cost and its derivative is minus the equilibrium demand, which strictly decreases in cost.

Equilibrium profit is strictly decreasing in equilibrium price.

Equilibrium profit is strictly increasing in equilibrium output.

The diagram in Section 2.2 shows these relations and the constituent Lemmas. The key implication of this Theorem is that we have strictly monotonic relations between variables. We use this to twin strictly monotonic distributions below.

3.7 Decreasing or increasing mark-ups

Some characterization results rely on a delineation of the degree of curvature of demand.

Corollary 1 *Under A1, if demand is strictly log-concave (resp. strictly log-convex), higher cost firms have lower (resp. higher) equilibrium markups $\mu'(c) < 0$, (resp. $\mu'(c) > 0$). Equivalently, $p'(c) \in (0,1)$ for strictly log-concave demand, and $p'(c) > 1$ for strictly log-convex demand. Conversely, $h(p)$ is strictly log-convex if $\mu'(c) > 0$ and strictly log-concave if $\mu'(c) < 0$.*

Proof. First, the numerator of (3), $-[h(\mu + c)/h'(\mu + c)]'$ is (weakly) positive for h log-convex and (weakly) negative for h log-concave. The last result follows from (8). ■

In the log-concave case, low-cost firms use their advantage in both mark-up and output dimensions. Under log-convexity, low-cost firms exploit the opportunity to capitalize on much larger demand by setting small mark-ups. In both cases though, as per Lemma 2, profits are higher with lower costs. For $h(\cdot)$ strictly log-concave, $\mu'(c) < 0$, so firms with higher costs have lower mark-ups in the cross-section of firm types (price pass-through is less than 100%). They also have lower equilibrium outputs. The only demand function with constant (absolute) mark-up is the exponential (associated to the Logit), which has $h(\cdot)$ log-linear in p (i.e., $h(p) \propto \exp\left(\frac{-p}{\sigma}\right)$ where σ is a positive constant), and so $\frac{h(m+c)}{h'(m+c)}$ is constant. When $h(\cdot)$ is strictly log-convex, the mark-up *increases* with c, so cost pass-through is greater than 100%, which is a hall-mark of CES demands. They have constant elasticity and hence constant relative mark-up (and $\mu'(c) > 0$) so a 1% cost rise causes a 1% equilibrium price rise.

3.8 An illustrative example: $\rho-$linear demand

An important special case is when demand is $\rho-$linear (which means that h^ρ is linear):

$$h(p) = (1 + \rho(k - p))^{1/\rho}, \tag{11}$$

where k is a constant satisfying $1 + \rho(k - c) > 0$, and $\rho > -1$ as required for A1. Then

$$\mu(c) = \frac{1 + \rho(k - c)}{1 + \rho} > 0, \tag{12}$$

which is linear in c, with $\mu'(c) > -1$, since $1 + \rho > 0$. Moreover, $\mu'(c) < 0$ if $h(p)$ is log-concave, for $-1 < \rho < 0$, and $\mu'(c) > 0$ if $h(p)$ is log-convex for $\rho > 0$. For $\rho = 1$ demand is linear and the standard property is apparent that mark-ups fall fifty cents on the dollar with cost. Log-linearity is $\rho = 0$ (note that $\lim_{\rho\to 0} h(.) = \exp(k - p)$) and delivers a constant mark-up (see Anderson and de Palma, 2020). A constant elasticity of demand (which the CES model delivers) results from the parameter restriction $\rho = -1/k \in (-1, 0)$ and $h(p) \propto p^{-1/\rho}$.[7]

For $\rho-$linear demands, equilibrium demand is $h^*(c) = \left(\frac{1+\rho(k-c)}{1+\rho}\right)^{1/\rho}$ and then (by (5)), $\frac{dh^*(c)/dc}{h^*(c)} = \frac{-1}{1+\rho(k-c)} = -\frac{\mu'(c)+1}{\mu(c)} < 0$. Notice that $h^*(c)$ is also ρ-linear.

From (11) and (12) we have $\pi^*(c) = \mu(c) h^*(c) = \left(\frac{1+\rho(k-c)}{1+\rho}\right)^{(1+\rho)/\rho}$, which is indeed decreasing in c, and convex for $\rho > -1$, as anticipated (Lemma 5). Finally, the expression for $\tilde{\pi}(\tilde{p})$ is $(1 + \rho(k - \tilde{p}))^{(1+\rho)/\rho}$, decreasing in $\tilde{p}$ for $\rho > -1$, which concurs with Lemma 6.

4 Distributions for monopolistic competition

The relations above in Section 3.7 already determine some links between the equilibrium price distribution, the cost distribution, and the demand. We now show how the other economic distributions are determined and linked in the model. That is, how is one distribution "passed

[7] Unfortunately, the symbol ρ is also traditionally used to parameterize the CES function: call the CES version ρ^c ($\in (0, 1)$ for substitutes) and then the current usage says that $\rho = \rho^c - 1$.

through" to the others via the demand function and the corresponding equilibrium links between variables shown in Section 3.

Figure 1 illustrates the equilibrium links between demand, output and cost distributions under monopolistic competition. The upper right panel gives the demand curve, from which we determine the corresponding marginal revenue function. The latter is the key to finding the output distribution from the cost distribution. Notice that $h^* (c)$ defined above determines the equilibrium output (for a firm with per unit cost c) as a function of its cost. As earlier noted, the inverse function, $c = h^{*-1} (y)$ therefore traces out the marginal revenue curve.

Figure 1: Construction of marginal revenue, output, and price from demand, and cost distribution to price and output distribution

The distribution of costs is given in the upper left panel. The negative linear relation between the cost and output distributions is given in the lower left panel: as noted in Lemma 2, higher costs are associated to lower outputs. Therefore, the $z\%$ of firms with costs below c are the $z\%$ of firms with output above $y = h^* (c)$. We hence choose some arbitrary level $z \in (0, 1)$ (see (1)). This means that all firm types with cost levels above $c (z) = F_C^{-1} (1 - z)$ are the firms with outputs and profits below y and π. That is, $1 - F_C (c) = F_Y (h^* (c))$ $(= z)$. The lower right panel therefore connects this relation as the output distribution, $F_Y (y)$. (Notice that in the above argument, only the marginal revenue curve was used from the demand side: as we show later in Section 5, the cost and output distribution determine the marginal revenue, but we then need to integrate up to find demand).

Figure 1 also provides the information to determine the price distribution. The upper right panel gives the vertical distance between the marginal revenue and demand, which is the mark-up (which can be expressed as $\mu (c)$), and is thus the vertical shift between cost and price distributions in the upper left panel. It can be constructed simply from the information in the top two panels[8] by drawing across the demand price associated to a marginal revenue - marginal cost intersection. We could also draw in the mark-up distribution in the upper left panel, but have avoided the extra clutter here. Notice that (as drawn) the price and cost distributions diverge, as is consistent with Lemma 2 for increasing $\mu(c)$, i.e., log-concave demand.

In summary, the marginal revenue curve $h^{*-1}(y)$ together with the cost distribution tie down the output distribution (and conversely, by reversing the analysis). The demand function then finds the price distribution, and therefore relates price and output distributions.

One relation that is missing in the Figure is the profit distribution. But, as Lemma 2 shows, analogous arguments apply: $\pi^*(c)$ is a decreasing function and so the relation $1 - F_C(c) = F_\Pi(\pi^*(c)) \ (= z)$ can be used to construct the profit distribution.

The following result establishes the existence of a unique equilibrium for the monopolistic competition model. Consequently, equilibrium distributions are tied down from the primitives on costs and demand.

Theorem 2 *Let there be a continuum of firms, with demand satisfying A1. Suppose that F_C is known and is strictly increasing and $\mathcal{C}^2$ on its domain. Then the distributions F_P, F_Y, and F_Π are strictly increasing and $\mathcal{C}^2$ on their domains. They are given by $F_P(p) = F_C(c(p))$; $F_Y(y) = 1 - F_C(h^{*-1}(y))$; and $F_\Pi(\pi) = 1 - F_C(\pi^{*-1}(\pi))$, where $c(p)$ inverts $p(c)$, $h^{*-1}(y)$ inverts $h^*(c)$, and $\pi^{*-1}(\pi)$ inverts $\pi^*(c)$.*

Proof. Let $p(c)$ denote the equilibrium price for a firm with cost c; from (3) we have $\mu'(c) > -1$ so that $p(c)$ is strictly increasing, and define the inverse relation as $c(p)$, which is strictly increasing. The relation $p(c)$ (and hence its inverse) is determined from $h(.)$ by Lemma 2.

Given F_C, then $F_P(p)$ is determined by $F_P(p) = F_C(c(p))$. Next, consider $F_Y(y)$. By result (4) we know that output $y = h^*(c)$ is a monotonic decreasing function, and so the

[8] Hence we were able to give results on the relationships between cost and price distributions in Section 3.7 without reference to the output distribution.

fraction of firms with output below $y = h^* (c)$ is the fraction of firms with cost above c, so $F_Y (h^* (c)) = 1 - F_C (c)$, or indeed

$$F_Y (y) = \Pr (h^* (C) < y) = \Pr \left(C > h^{*-1} (y)\right) = 1 - F_C \left(h^{*-1} (y)\right). \tag{13}$$

Finally, by Lemma 5 we know that profit $\pi^* (c) = \mu (c) h^* (c)$ is a strictly decreasing function, and so the fraction of firms with profit below $\pi^* (c)$ is the fraction of firms with costs above c, so $F_\Pi (\pi^* (c)) = 1 - F_C (c)$. That is

$$F_\Pi (\pi) = \Pr (\Pi < \pi) = \Pr (\pi^* (C) < \pi) = \Pr \left(C > \pi^{*-1} (\pi)\right) = 1 - F_C \left(\pi^{*-1} (\pi)\right). \tag{14}$$

■

The key relations underlying the twinning of distributions are the strictly monotonic relations between cost, output, profit, and price (see Lemma 2. A specific cost distribution generates specific output, profit, and price distributions. Conversely, as we show in the next result, this output, profit, or price distribution could only have been generated from the initial cost distribution.

The relations above already impose three key restrictions between the equilibrium distributions emanating from the monopolistic competition model. We elaborate upon them further below, once we discuss the individual distribution pairs explicitly in Section 5. But, for the moment we underscore them as necessary conditions on densities that the monopolistic competition model must satisfy. First, the positive mark-up for monopolistic competition, $\mu (c) > 0$ implies $F_C (c) > F_P (p)$. We discuss this further following Theorem 5, where we also break out the implications of demand log-concavity for density relations. Second, the relation between price and output distributions, $F_P (p) = 1 - F_Y (h (p))$, inverts to give $\frac{1}{h(p)} = \frac{1}{F_Y^{-1}(1-F_P(p))}$. The need for the LHS to be convex under A1 implies f_Y / f_P must be increasing in p (or that its reciprocal be decreasing in y). We discuss this further after Theorem 4. Third, the relation between cost and profit distributions, $F_C(c) = 1 - F_\Pi(\pi^*(c))$, inverts to give $\pi^*(c) = 1 - F_\Pi^{-1}(1 - F_C(c))$. The need for the LHS to be convex (by Lemma 5) implies f_C/f_Π must be increasing in c. We discuss this further after Theorem 8.

Interestingly, the other relations involve no pairwise restrictions (modulo those discussed in the next sub-section where we invoke a density elasticity analysis to put on further structure). This means that in the sequel (Section 5) there are *no restrictions on the shapes of the other pairs of distributions that can be compatible with monopolistic competition.*

Researchers often impose specific demand functions (such as CES, or logit). Here we forge the (potentially testable) empirical links that are imposed by so doing: Theorem 2 shows that when a specific functional form is imposed for h (as is done in most of the literature), then all the relevant distributions can be found from $F_C(c)$. Furthermore, all distributions can be found from any one of them.

Theorem 3 *Let there be a continuum of firms with demand satisfying A1. Consider the set of 3 distributions, $\{F_P, F_Y, F_\Pi\}$. Suppose that demand and any one distribution is known and is strictly increasing and $\mathcal{C}^2$ on its domain. Then F_C and all other distributions in the set are explicitly recovered and all are strictly increasing and $\mathcal{C}^2$ on their domains.*

Proof. Consider F_P. Then $F_C(c) = F_P(p(c))$, where $p(c)$ is the equilibrium price relation, which we showed in Lemma 2 to be $\mathcal{C}^2$, and both the other distributions are determined from the steps in the proof of Theorem 2 earlier.

Next start with F_Y. Because $h(p)$ is strictly decreasing, then F_P is determined by $F_P(p) = 1 - F_Y(h(p))$ and is $\mathcal{C}^2$. By the argument above, F_C is then determined, and hence so is $F_\Pi(\pi)$.

Finally, start with F_Π. By Lemma 2 we know that profit $\pi^*(c) = \mu(c)\, h^*(c)$ is a strictly decreasing function. Therefore $F_C(c)$ is recovered from $F_C(c) = 1 - F_\Pi(\pi^*(c))$ and is $\mathcal{C}^2$. From

Theorem 2, F_P is recovered, and so is F_Y. ■

The Theorem says that for any (-1)-concave demand function and any economic distribution, there is only one cost distribution that is consistent with them. The other economic distributions are likewise pinned down.

4.1 Density elasticity relations

There are clean and useful conditions that show which elasticities connect the densities. They are all different aspects of the demand side. For example, the profit density elasticity is related to the cost density elasticity via the elasticities of profit and (inverse) marginal revenue (with respect to unit cost, c), both of which are derived from the demand form.

Lemma 8 *Consider two distributions $F_{X_1}(x_1)$ and $F_{X_2}(x_2)$, which are continuous and strictly increasing on their respective domains. Let x_1 and x_2 be related by a monotone function $x_1 = \xi(x_2)$. Then $\eta_{f_{X_2}} = \eta_{f_{X_1}}\eta_\xi + \eta_{\xi'}$, where $\eta_{\xi'}$ is the elasticity of $\xi'(x_2)$ and $\eta_{f_{X_2}}$ is the elasticity of f_{X_2}: $\eta_{f_{X_2}} = x_2 f'_{X_2}(x_2)/f_{X_2}(x_2)$, etc.*

A1 imposes several restrictions on the various demand-side elasticities that appear in the density elasticity relations below. In particular, $\eta_h < -1$ is the property that demand must be elastic at equilibrium in a monopolistic competition setting, mirroring the standard monopoly property. Furthermore, $\eta_{h^*} < 0$ is the property that marginal revenue slopes down. The elasticity of the demand curve slope, $\eta_{h'} = \frac{h''p}{h'}$, has the sign of $-h''$ and so is positive for concave demand, and negative for convex demand. The elasticity of the output function slope, $\eta_{h^{*\prime}}$, involves third derivatives of demand, though notable benchmarks are that it is zero for linear demand (because marginal revenue is linear) and for constant elasticity. The elasticity of maximized profit (with respect to c), η_π, is particularly interesting. Write this as

$$\eta_\pi = \frac{\pi'(c)}{\pi(c)}c = -\frac{ch^*(c)}{\mu(c)\,h^*(c)} = -\frac{c}{\mu(c)} = -\frac{1}{\ell - 1} < 0. \tag{15}$$

The third expression is the ratio of total cost to total profit;[9] the fourth one is the reciprocal of mark-up over cost; the last one writes this in terms of relative mark-up $\ell = p/c$, which we examine in Section 6. We describe other elasticity relations in the sequel.

5 Rationalizability of distributions via demand

An old question in consumer theory is whether a demand system can be generated from a set of underlying preferences (see Antonelli, 1896, and the discussion in Mas-Collel, Whinston, and Green, 1995, pp. 70-75). Here we look at when any arbitrary pair of economic/primitive distributions (F_C, F_P, F_Y, F_Π) could be consistent with the monopolistic competition model with demand satisfying A1.

5.1 Deriving demand and all distributions from price and cost ones

We now determine demand when there are strict monotone relations between two variables. Suppose first that price and cost distributions, F_P and F_C, are known and are strictly monotonic. Because mark-ups are necessarily positive, it must be that the price distribution first-order stochastically dominates the cost one. However, we will show that this is the *only* restriction on the distributions. The demand function will ensure that they are compatible, though the only restriction on it is that it be strictly (-1)-concave.

Because price strictly increases with cost, the price and cost distributions are matched: the fraction of firms with costs below some level c equals the fraction of firms with prices below the price charged by a firm with cost c. This enables us to back out the corresponding mark-up function $\mu(c)$ and then access Lemma 2.

Theorem 4 *Let the cost and price distributions, F_C and F_P be two arbitrary strictly increasing and $\mathcal{C}^2$ functions on their domains with $F_C(c) > F_P(p(c))$. Then there exists a strictly (-1)-*

[9] Else $\eta_\pi = -\frac{TC}{TR-TC} = \frac{-1}{(TR-TC)-1}$.

concave demand function (unique up to a positive multiplicative factor) that rationalizes these distributions in the monopolistic competition model.

Proof. Consider a distribution of costs, F_C and a distribution of prices, F_P satisfying $F_C(c) > F_P(c)$ (so that the price distribution is right of the cost one: note that $F_C(c) > F_P(c) = 0$ for c below the lower bound of the domain of the price distribution). We wish to find a demand function satisfying A1. Define $p(c) = F_P^{-1}(F_C(c))$, which is a strictly increasing function. It satisfies $p(c) > c$ given that $F_C(c) > F_P(p(c))$. Then Lemma 3 implies that there exists an $h(.)$ satisfying A1 up to a positive multiplicative factor. ■

We can then determine the other economic relations (see also Theorem 2):

Corollary 2 *Let the cost and price distributions, F_C and F_P be two arbitrary strictly increasing and $\mathcal{C}^2$ functions on their domains with $F_C(c) > F_P(p(c))$. Then the mark-up function $\mu(c)$ is found from (16) and $\mu'(c) > 0$; inverse marginal revenue is found from (6) and the demand function is given from (7), up to a positive multiplicative factor, $h^*(\underline{c})$. The output and profit distributions are determined, up to $h^*(\underline{c})$, by (13) and (14).*

Proof. We can write the price-cost margin, as a function of c, as (from (1))

$$\mu(c) = F_P^{-1}(F_C(c)) - c, \tag{16}$$

with $\mu(c) > 0$ because $F_C(c) > F_P(p(c))$ and so $\mu'(c) > -1$. Hence a unique such mark-up function $\mu(c)$ exists given the cost and price distributions. With the function $\mu(c)$ thus determined, we can invoke Lemma 3 to uncover the equilibrium demand function $h^*(\cdot)$ (unique up to a positive multiplicative factor) as given by (5) and (6), and the demand function is given from (7). By Lemma 3, this demand function satisfies A1, as postulated. ■

From (16) we write

$$\mu'(c) = \frac{f_C(c)}{f_P(p(c))} - 1, \tag{17}$$

which shows that $\mu'(c) > 0$ iff $f_C(c) > f_P(p(c))$. As we know from Corollary 1, log-convex demand begets increasing mark-ups. Thus the equilibrium prices "spread out" vis-a-vis the costs, and hence engender a more spread price density than cost density when we take the price induced from a given cost. Conversely, log-concave demand delivers a decreasing mark-up and so prices tend to pile up, meaning the price density is more peaked than the cost density. Put another way, the price distribution is a *compression* of the cost distribution when h is log-concave, and a *magnification* when h is log-convex, in the simple sense that prices are closer together (or, respectively, farther apart) than costs. The border case (Logit / log-linear demand) has constant mark-ups, so the price distribution mirrors the cost one.[10]

The construction of the demand function is illustrated in Figure 1. The only restriction we use here is that the cost distribution first-order stochastically dominates the price one. Given this property, any pair of ($\mathcal{C}^2$) price and cost functions is consistent with the monopolistic competition model. In the next section, we show that the price and output distributions are restricted if they are to be consistent.

5.2 Price and output distributions

Now suppose that price and output distributions, F_P and F_Y, are known.

Theorem 5 *Let the price and output distributions, F_P and F_Y be two arbitrary strictly increasing and $\mathcal{C}^2$ functions on their domains. Then there exists a unique strictly (-1)-concave demand function, $h(p) = F_Y^{-1}(1 - F_P(p))$, that rationalizes these distributions in the monopolistic competition model if and only if $f_P(p)/f_Y(y)$ is strictly decreasing in p.*

[10] One parameterized example is the flexible CES-Logit demand function model introduced in Anderson and de Palma (2020): $h(p) = kp^{b-1}\exp\left(\frac{(p^b-1)/b}{\mu}\right)$ where $k > 0$ is constant for monopolistic competition and $\mu > 0$ is a measure of product differentiation. The CES corresponds to the limit $b \to 0^+$, while the Logit corresponds to the limit $b \to 1^-$. From the f.o.c. $\left(1 - \frac{c}{p}\right)p^b + (1-b)\mu = 1$, $\mu'(c) < 0$ for $b < 1$ (strict log-concavity) and $\mu'(c) > 0$ for $b > 1$ (strict log-convexity), while $\frac{d(p/c)}{dc} > 0$ iff $b < 0$ $b \neq 0$.

Proof. From the two distributions $y = F_Y^{-1}(1 - F_P(p)) = h(p)$ is the unique candidate demand function. While this is decreasing in p, as desired, we also require that the function $F_Y^{-1}(1 - F_P(p))$ is strictly (-1)-concave to be consistent with the monopolistic competition model. This condition holds if and only if $f_P(p)/f_Y(h(p))$ is strictly decreasing in p. ■

The other distributions and relations are determined analogously to Corollary 2. If the implied demand shape does not satisfy the (-1)-concavity condition, the purported demand relation would not have a downward-sloping marginal revenue curve everywhere, and any price-output pair with an upward sloping marginal revenue could not be consistent with profit maximization by a firm.

The condition that $f_P(p)/f_Y(y)$ be strictly decreasing rules out various combinations. For example, if $f_p(p)$ is increasing (locally, say), then we cannot have $f_Y(y)$ (locally) increasing too. But both decreasing is fully consistent. Indeed, the required consistency condition is $f_P'(p) f_Y(y) - f_Y'(y) f_P(p) h'(p) < 0$, or $\eta_P < \eta_Y \eta_h$, so the price density elasticity should be negative if the output one is positive. Conversely, if the price density elasticity is positive then the output one should be negative. If such necessary (empirically testable) conditions do not hold the market cannot be described by the proposed monopolistic competition approach.

The density elasticity relation between price and output is given by applying Lemma 8 to $F_P(p) = 1 - F_Y(h(p))$:

$$\eta_{f_P} = \eta_{f_Y}\eta_h + \eta_{h'}. \tag{18}$$

The elasticity of the demand slope has shown up elsewhere in pricing formulae (e.g., in Helpman and Krugman, 1985). On the RHS, the demand elasticity, η_h, is negative, while the slope elasticity $\eta_{h'} = \frac{h"p}{h'}$ is positive for concave demand and negative for convex demand. For linear demand we have a benchmark that the price and output density elasticities have *opposite* signs. Concave demand implies that decreasing output density drives increasing price density. For convex demand, increasing price density drives decreasing output density. To interpret the negative relation in the benchmark, recall that the low price firms are the high output ones, so we are looking at opposite ends of the distributions/densities effectively. Think about an increasing price density. Then there are more firms with higher prices: translating to the output density, there are more firms with lower outputs.

5.3 Cost and output distributions

Although price and output distributions are jointly restricted, surprisingly, cost and output distributions are not. Suppose that F_C and F_Y are known.

Theorem 6 *Let the cost and output distributions, F_C and F_Y be two arbitrary strictly increasing and $\mathcal{C}^2$ functions on their domains. Then there exists a strictly (-1)-concave demand function (unique up to a positive constant) that rationalizes these distributions in the monopolistic competition model.*

Proof. From the two distributions $y = F_Y^{-1}(1 - F_C(c)) = h^*(c)$ is the candidate function for optimized demand. The only restriction is that it slope down, which is satisfied, and that it be continuous, which is also immediately satisfied. Hence it is rationalizable, and we can use Lemma 4 to back up to the implied demand function, $h(p)$, which is therefore determined up to a positive constant. ■

The defining relation for elasticity densities for this pair is $F_Y(h^*(c)) = 1 - F_C(c)$. Then[11]

$$\eta_{f_Y}\eta_{h^*} = \eta_{f_C} - \eta_{h^{*\prime}}.$$

This is directly comparable to the price-output relation (18) (namely $\eta_{f_Y}\eta_h = \eta_{f_P} - \eta_{h'}$). Drawing on that analysis, a linear marginal revenue is a useful benchmark,[12] for which output

[11] Write the density relation $-h^{*\prime}(c) f_Y(h^*(c)) = f_C(c)$ in log form: the elasticity relation follows directly.

[12] This comes from linear demand, but is not limited to that – we can add a rectangular hyperbola to demand and still get a linear marginal revenue.

and cost densities necessarily go in opposite directions. Constant elasticity of demand is just like for price-output, given that the parameters are the same for both cases.

5.4 Deriving demand from price and profit distributions

We now use Lemma 3 to find a unique demand function satisfying A1 from *any* pair of distributions. This is quite a surprising result. For example, there exists a demand function that squares Pareto distributions for both prices and profits, or normal and log-normal, etc. All other distributions are then determined.

Theorem 7 *Let the price and profit distributions, F_P and F_Π, be two arbitrary strictly increasing and $\mathcal{C}^2$ functions on their domains. Then there exists a strictly (-1)-concave demand function (unique up to a positive constant) that rationalizes these distributions in the monopolistic competition model.*

Proof. From (1), first write $1-F_P(p) = F_\Pi(\pi) = z$. Then we can write $\pi = F_\Pi^{-1}(1 - F_P(p)) = h(p)\mu(p) \equiv \tilde{\pi}(p)$, where we recall that $\tilde{\pi}(p)$ denotes the relation between the observed maximized profit level observed and the corresponding maximizing price. As shown in Lemma 6, $\tilde{\pi}(p) = -h^2(p)/h'(p)$. Integrating $(1/h(p))' = \frac{1}{\tilde{\pi}(p)}$ gives:

$$h(p) = \frac{1}{\int_{\underline{p}}^{p} \frac{dr}{F_\Pi^{-1}(1-F_P(r))} + k}. \tag{19}$$

This determines the demand form up to the positive constant $k = 1/h\left(\underline{p}\right)$ (in the position in the above formula): (19) is $\mathcal{C}^2$ and decreasing in p. Furthermore,

$$\left(\frac{1}{h(p)}\right)' = \frac{1}{F_\Pi^{-1}(1 - F_P(p))},$$

which is strictly increasing because both distributions are strictly increasing. That is $1/h(p)$ is convex and so, equivalently, $h(p)$ is (-1)-concave.

By Theorem 3 all the other distributions are determined. ■

Therefore, the price and profit distributions define the function (19) and the resulting demand function satisfies A1 *without any further restrictions.* This means, for example, that a decreasing price density is consistent with an increasing profit density (very many high profit firms and yet very few high price ones). The underlying cost distribution along with demand is what renders these features compatible. As regards the constant k, knowing the demand level at any one point ties down the whole demand function.

We have just shown that there are no restrictions on price and profit distribution shapes, though we have restrictions on some other pairs of distribution functions that can be combined in order to be consistent with the monopolistic competition model.

5.5 Cost and profit distributions

This is another case where monopolistic competition restricts the distribution pair.

Theorem 8 *Let the cost and profit distributions, F_C and F_Π be two arbitrary strictly increasing and $\mathcal{C}^2$ functions on their domains. Then there exists a demand function $h(p)$ satisfying A1 (unique up to a constant) that rationalizes these distributions in the monopolistic competition model if and only if $f_C(c)/f_\Pi(\pi^*(c))$ is strictly increasing in c, with $\pi^*(c) = F_\Pi^{-1}(1 - F_C(c))$.*

Proof. From the two distributions, $\pi^*(c) = F_\Pi^{-1}(1 - F_C(c))$ is the candidate profit function. This is decreasing in c, as desired, but it also needs to be strictly convex, by Lemma 5, in order to be consistent with the monopolistic competition model. The convexity condition is that $h^*(c) = f_C(c)/f_\Pi(\pi^*(c))$ is strictly increasing in c. Using Lemma 4, there exists a demand function $h(p)$ satisfying A1, which is unique up to a constant. ■

Applying Lemma 8 to the case $1 - F_C(c) = F_\Pi(\pi^*(c))$, we get $\eta_{f_\Pi}\eta_\Pi = \eta_{f_C} - \eta_{h^*}$. Recall that $\pi^{*\prime}(c) = -h^*(< 0)$ and $h^{*\prime} < 0$, so that $\eta_\Pi < 0$ and $\eta_{h^*} < 0$. If the profit density is increasing, then the cost density is necessarily decreasing, but the reverse is not true: a strong enough decreasing cost density is needed for an increasing profit density. Conversely, $\eta_{f_C} > 0 \Rightarrow \eta_{f_\Pi} < 0$: an increasing cost density implies a decreasing profit one.

Finally, distribution elasticities uncover relations. From $1 - F_C(c) = F_\Pi(\pi^*(c))$, we can write $\frac{f_C}{1-F_C} + \frac{f_\pi}{F_\pi}\pi'(c) = 0$, which in elasticity form (recalling $\eta_\Pi = \frac{-1}{\ell-1}$ from (15)) becomes

$$\eta_{F_\Pi} = -(\ell - 1)\,\eta_{S_C} \tag{20}$$

where the subscript S_C denotes the survivor function of costs and the corresponding elasticity $\eta_{S_C} < 0$. Thus the profit distribution is more elastic the bigger the relative mark-up, ℓ, as the survivor cost parlays into more profit distribution response. (20) also indicates how mark-ups can be estimated directly from the two elasticities.

5.6 Output and profit distributions

The final case returns to no restrictions.

Theorem 9 *Let the output and profit distributions, F_Y and F_Π be two arbitrary strictly increasing and $\mathcal{C}^2$ functions on their domains. Then there exists a strictly (-1)-concave demand function (unique up to a constant) that rationalizes these distributions in the monopolistic competition model. This unique net demand function, and the other distributions, are determined explicitly in the proof.*

The proof is in the Appendix (and illustrated in the ρ-linear demand example below). It is based on the relation, $\Psi(z)$, between the counter z and the cost level (or any economic variable): $\Psi(z) = \int_0^z \left[F_\Pi^{-1}(r)\right]' / F_Y^{-1}(r)\,dr = \bar{c} - c$ (see (24)).

What the Theorem ties down is net demand (inverse demand minus cost): if both inverse demand and cost shift by the same amount then equilibrium quantity (output) and mark-up are unaffected, so profit is unchanged too. Thus output and profit distributions tie down the shape of the net inverse demand and the shape of the other distributions, but not the inverse demand curve height. As we saw above, price and cost distributions alone do not tie down the demand scale, and nor do price and profit distributions. But the other pairs of distribution combinations fully determine the demand function and all distributions.

5.7 Examples

We illustrate the Theorems above with distributions that generate ρ-linear demand. Details are in the Appendix.

Recovering ρ-linear demand. *Suppose that* $F_Y(y) = \frac{(1+\rho)y^\rho - 1}{\rho}$, $y \in \left[\frac{1}{(1+\rho)^{1/\rho}}, 1\right]$, *and* $F_\Pi(\pi) = \frac{(1+\rho)\pi^{\rho/(1+\rho)} - 1}{\rho}$, $\pi \in \left[\frac{1}{(1+\rho)^{(1+\rho)/\rho}}, 1\right]$, *with* $\rho > -1$. *Then demand is* ρ*-linear (see (11) with* $k = \bar{c}$*) and the cost distribution is uniform.*

Uniform costs give a useful benchmark for some important properties relating cost and profit distributions. For the example above, we have $f_\Pi(\pi) = \pi^{-1/(1+\rho)}$, so that the density of the profit distribution is decreasing, despite the underlying cost distribution that generates it being flat. This property indicates how profit density "piles up" at the low end. The output density shape is also interesting. For linear demand ($\rho = 1$) it is clearly flat – equilibrium quantity is a linear function of cost. For convex demand ($\rho < 1$) it is decreasing, but for concave demand it is *increasing*, despite the property just noted that the profit density is decreasing. This suggests that (for concave demand), a decreasing output density requires an increasing cost density, which *a fortiori* entails a decreasing profit density. The Appendix also illustrates that knowing the profit and cost distributions ties down the full model, using the same parameters, and gives the steps involved.

6 Mark-up distribution

Recent work has delivered mark-up distributions from several different methodologies, most notably the recent production function approach. We now analyze how mark-up distributions interact with the other ones, and how they help retrieve demand. Thus far in the analysis, via A1 and the subsequent Lemmas it entails, variables are always either positively or negatively linked (e.g. prices and costs, or prices and outputs respectively). Mark-ups though either go up or down with the other variables depending on the degree of concavity of demand. This is true for both absolute and relative mark-ups, which we consider in turn. As we show, this means that each pair of a mark-up distribution and another distribution and can yield two solutions, depending on whether or not costs are fully passed through.

6.1 Absolute mark-ups

Our first result with this distribution, $F_M(m)$ with $m = p - c$, extends and modifies Theorem 3. We claim that knowing the demand function and $F_M(m)$ suffices to tie down the other distributions if the demand is either strictly log-concave or strictly log-convex (see Corollary 1). We divide the analysis into two cases, depending on the log-concavity or log-convexity of $h(.)$.[13] Under log-concavity, we know that $\mu(c)$ is decreasing and cost pass-through is less than 100%. This entails higher (per unit) mark-ups at firms with lower costs, so that $z = F_M(m) = 1 - F_C(c)$. Rewriting, we recover the cost distribution from $F_C(c) = 1 - F_M(\mu(c))$, and we can then find all the other distributions from the relations in Theorem 3.

On the other hand, $\mu(c)$ is increasing when $h(.)$ is log-convex, and cost pass-through is more than 100%. Then higher (per unit) mark-ups are set at lower firm outputs, and in that case we recover the cost distribution (and hence all others) from $F_C(c) = F_M(\mu(c))$.

When the demand has both log-concave and log-convex segments, then some values of mark-

[13] The mixed case is discussed briefly below, with more in the CEPR version of the paper.

up could have been delivered by two (or more) values of c. Thus $\mu(c)$ is not invertible and so the cost distribution cannot be recovered without supplementary information (at least on the domain for which $\mu(c)$ is not invertible and if $f_M(m) > 0$ for such values).

Likewise, knowing $F_M(m)$ and one other distribution does not tell us all other distributions and demand without further qualification. If we knew in addition that $h(.)$ were log-concave (resp. log-convex), then we can pin down the demand form, and the other distributions using analogues to Theorems 4, 5, 7: we discuss the procedure in the next sub-section. However, without knowing a priori which side of log-linearity (the Logit) $h(.)$ falls, we get two candidate solutions. Indeed, if $[\ln(h)]'$ changed sign over its domain then $h(.)$ cannot be tied down.

We now give a fuller treatment of relative mark-ups since these are more commonly derived empirically.

6.2 Relative mark-ups

A growing recent literature (see e.g. De Loecker, Eeckhout, and Unger, 2020), has been estimating markups from a production function approach. The mark-ups estimated are not the absolute mark-ups with which we started the paper ($m = p - c$), but a unit-free version. This has been expressed in various ways, like the Lerner index $\frac{p-c}{p}$, or else $\frac{p-c}{c}$, or, most commonly, by $\ell = \frac{p}{c} \geq 1$. The other variants can all be expressed in terms of ℓ, and we retain this last version, which we term relative mark-up.

We first recall the classic Inverse Elasticity Rule (IER), which applies to our monopolistic competition formulation. It writes the Lerner index as

$$\frac{(p-c)}{p} = \frac{1}{\varepsilon},$$

with $\varepsilon \equiv -\frac{ph'(p)}{h(p)}$, the elasticity of demand (in absolute terms) and $\varepsilon > 1$ so firms produce where demand is elastic. From the IER we have the equilibrium relation $\ell = \frac{\varepsilon}{\varepsilon-1} > 0$, which is decreasing in ε with domain $(1, \infty)$ and domain $(1, \infty)$ (see e.g., Melitz, 2018).

If we know the demand form $h(p)$ then we know the corresponding expressions for ε and ℓ. If we know the equilibrium prices we know the equilibrium mark-up relation $\ell(p)$ of those equilibrium prices too. We distinguish two cases.

First, Marshall's Second Law of Demand (henceforth M2L) is that ε is strictly increasing in p, equivalently, ε is strictly decreasing with output, $y = h(p)$. Then $\ell'(p) < 0$. Second, we say that the "Converse Law" (to M2L) holds if ε is strictly decreasing in p and $\ell'(p) > 0$.[14] The CES forms the boundary case in which ε is constant, and so too is then equilibrium $\ell(p)$. Note that strict log-concavity of $h(p)$ implies M2L and the Converse Law implies strict log-convexity of demand.[15]

We briefly discuss relative pass-through as measured from the price and cost distributions. Recall $F_C(c) = F_P(p)$ and so $\frac{dp}{dc} = \frac{f_C(c)}{f_P(p)}$ and $\mu'(c) > 0$ (log-convexity) entails $f_C(c) > f_P(p)$, which we can interpret as the cost density driving more spread in the price density. The Converse Law, being a stronger property, entails a stronger condition. Indeed, since the Converse Law implies strictly increasing $\ell(c)$, then it implies $p'(c) > \ell$ and hence $cf_C(c) > pf_P(p)$. Equivalently, the elasticity of the cost distribution should exceed that of the price distribution. M2L implies the opposite elasticity relation, which in turn is implied by the condition $f_C(c) < f_P(p)$ for $\mu'(c) < 0$ (strict log-concavity).

In the sequel, we treat the case when M2L holds; the converse case follows analogously but flips the distributional relations as indicated below. When M2L holds, $\ell'(p) < 0$, knowing the price distribution ties down the distribution of ℓ from the relation $F_L(\ell(p)) = 1 - F_P(p)$.[16] Conversely the mark-up distribution ties down the price distribution. We can leverage this ar-

[14] We restrict attention to when the derivative of ε is monotone, so we rule out cases which switch between the Law and its converse.

[15] That is, $\left[\frac{h'(p)}{h(p)}\right]' < 0$ (log-concavity) $\Rightarrow \left[\frac{ph'(p)}{h(p)}\right]' < 0$ (M2L) because $\left[\frac{ph'(p)}{h(p)}\right]' = p\left[\frac{h'(p)}{h(p)}\right]' + \frac{h'(p)}{h(p)}$.

[16] When the converse law holds, $F_L(\ell(p)) = F_P(p)$.

gument to provide the equilibrium relations with the other distributions. For example, because prices rise with costs we have $F_C(c) = 1 - F_L(\ell(c))$ (and we expand below on the relation $\ell(c)$, which implies $\ell'(c) < 0$ when M2L holds); because equilibrium profits fall with equilibrium prices by Lemma 6 we have $F_\Pi(\pi) = F_L(\ell(p(\pi)))$ as big mark-ups are associated to big profits when M2L holds (and conversely under the Converse Law).

Therefore *knowing the demand form and the mark-up distribution delivers all the other distributions too.* Conversely, *any distribution along with demand form delivers the mark-up distribution.* This means that Theorem 3 extends to include the mark-up distribution when M2L applies or when its converse applies.

We now turn to the question of demand recoverability from the mark-up distribution and one other. Our existing results enable us to find the demand (assuming it obeys M2L). We illustrate with the cost distribution. As noted above, $\ell(c) = F_L^{-1}(1 - F_C(c))$, with $\ell'(c) = -\frac{f_C(c)}{f_L(\ell)}$ is negative, as required. Rewrite this as $p(c) = cF_L^{-1}(1 - F_C(c))$, from which we have $\mu(c) = c\left(F_L^{-1}(1 - F_C(c)) - 1\right)$, so that we recover the absolute mark-up function. Now, as shown in Lemma 2, from the function $\mu(c)$ we can recover the demand function up to a multiplicative factor. However, we need $\mu'(c) > -1$ for A1 to hold and to apply Lemma 2. The required condition is

$$\ell f_L(\ell) > c f_C(c),$$

which can equivalently be written in elasticity form with the interpretation that the elasticity of the relative mark-up distribution should exceed that of the cost survivor distribution. If the Converse Law holds, $p(c) = cF_L^{-1}(F_C(c))$ and necessarily $p'(c) > 0$ without conditions. Hence $\mu(c) = p - c = c\left(F_L^{-1}(F_C(c)) - 1\right)$ and Lemma 2 then delivers the demand form from $\mu(c)$.[17] The next result summarizes.

[17] The demand is necessarily log-convex because $\mu'(c) = (\ell - 1) + c\frac{f_C(c)}{f_L(\ell)} > 0$.

Theorem 10 *Let $F_L(\ell)$ and $F_C(c)$ be given distributions. If $\ell f_L(\ell) > c f_C(c)$ there exists a demand satisfying Marshall's Second Law (M2L), which is unique up to a positive factor. There always exists a demand (unique up to a positive factor) satisfying the Converse Law.*

Demand recovery from the price distribution follows a similar procedure, and also accesses our key recovery result in Lemma 2, but yet with some surprise, so these are not completely sibling results. Under M2L, we have $\ell(p) = F_L^{-1}(1 - F_P(p))$, with $\ell'(p) = -\frac{f_P(p)}{f_L(\ell)} < 0$, as the mark-up which is recovered empirically from the two distributions. Rewrite this in terms of the supporting cost $c(p)$ as $c(p) = p/\left(F_L^{-1}(1 - F_P(p))\right)$ and we require this is increasing in p for A1 to hold and price to rise with cost, and this also implies that the relation $c(p)$ is invertible so the corresponding $p(c)$ is increasing. This condition necessarily holds because $c'(p) > 0$ (it has the sign of $\ell f_L + p f_P$). Therefore there is no restriction on the distributions for this case (contrast the analogous cost case). To deploy Lemma 2 we need $\mu(c) = p(c) - c$ to have derivative greater than -1, or $p'(c) > 0$, which we have just argued to be true, and so demand is recovered. For demand to satisfy the Converse Law, we have the mark-up delivered from the two distributions as $\ell(p) = F_L^{-1}(F_P(p))$, with $\ell'(p) = \frac{f_P(p)}{f_L(\ell)} > 0$. Then the supporting cost is $c(p) = p/F_L^{-1}(F_P(p))$ and we need this increasing for the same reason as above. Again $c'(p) > 0$ implies the desired invertibility of $p(c)$ and that Lemma 2 can be applied. The condition for $c'(p) > 0$ is $\ell f_L > p f_P$ (so the elasticity of the mark-up distribution should exceed the elasticity of the survivor function of the price distribution). We summarize as:

Theorem 11 *Let $F_L(\ell)$ and $F_P(p)$ be given distributions. If $\ell f_L(\ell) > p f_P(p)$ there exists a demand satisfying the Converse Law, which is unique up to a positive factor. There always exists a demand (unique up to a positive factor) satisfying Marshall's Second Law.*

It is interesting here that the price and cost distributions deliver restrictions in opposite cases for demand. Pairing the mark-up distribution to the output and profit distributions also bear some similar conclusions. In both cases, there is always a solution under the Converse Law without restriction; but the M2L case has (similar) restrictions for both.[18]

7 Conclusions

The basic ideas here are simple. Market performance depends on the economic fundamentals of tastes and technologies, and how these interact in the market-place. The fundamental distribution of tastes and technologies feeds through the economic process to generate the endogenous distribution of economic variables, such as prices, outputs, and profits. The assumption of a monopolistically competitive market structure delivers the clean and tractable feed-through from fundamental distributions to performance distributions.[19]

As we show, any pair of the (endogenous) economic distributions can be reverse engineered to back out the model's primitives. If two distributions can be estimated from a data-set, then they can be checked with respect to the consistency conditions of the model. If so, demand can be recovered and compared to the commonly-used forms (like CES and Logit). The empirical density elasticities also yield relations that can be evaluated in the light of the model.

We have focused on demand recovery. Surprisingly, demand can be recovered just from profit and price distributions (for example). We show what restrictions on the distributions the model entails, and we provide constructive proofs to find demand. We have chosen to present the details for all distribution pairs because each pair yields different patterns in terms of the distribution restrictions, underlying demand construction, or constants not identified in demand.

The simplest case is output and profit, for which the distributions immediately deliver

[18] The restrictions come from the condition $p'(c) > 0$. In elasticity form these are respectively $\eta_{F_L} + \eta_{F_Y}\eta_\Pi < 0$ (recall that $\eta_\Pi = \frac{-1}{\ell-1} < 0$ from (15)) and $\eta_{F_L} + \eta_{F_Y}\eta_Y < 0$, with $\eta_Y = \frac{ch^{*\prime}(c)}{h^*(c)} < 0$.

[19] An oligopoly analysis would be hugely more cumbersome because then firms' types would be realized draws of costs from an underlying distribution and we would need to track outcomes across all possible draw combinations.

demand, although there is a restriction on densities consistent with the model. Cost and price distributions deliver a mark-up function, from which we can use our new result (Lemma 3, which goes in the converse direction from Weyl and Fabinger, 2013) on deriving demand from pass-through. Two other cases - output and profit - entail a sort of cost pass-through, and we first deliver new pass-through results on these before again going in the reverse direction and deriving demand from the distribution pairs. Both proofs involve our constructive result (Lemma 4) how to find demand from marginal revenue: the common ground between output and profit is determined from our new result (Lemma 5) on the link that $\pi^{*\prime}(c) = -y$. However, while the cost-profit distribution pair involves a restriction (from the convexity of $\pi^*(c)$), the cost-output distribution pair is not restricted (as we show in the proof). The last two distribution pairs are also not restricted, but need separate proofs. These pairs are profit-price and profit-output, for which variable pairs we also deliver new results showing how their optimized values vary together (they are orchestrated by underlying cost variation).

The distribution of relative mark-ups is also interesting. For each distribution with which the mark-up distribution is paired, we find two cases for demand, depending on whether demand obeys Marshall's Second Law or its converse, and we find either two solutions or just one alone. Interestingly, the assured solution can be in either demand class depending on the distribution.

We have assumed that firms differ by marginal costs but that these are constant irrespective of output. We here briefly address marginal costs that are not constant. To retain a simple cost heterogeneity across firms, assume that *marginal* costs are $c + \hat{c}(y)$, with c being idiosyncratic to firms with distribution $F_C(c)$ while $\hat{c}(y)$ is common to all firms with $\hat{c}(0) = 0$. Our device to parallel the earlier analysis is to work with the inverse demand. We had before the primitive demand $h(p)$ with inverse $p(y)$. For the present inquiry, we now define $\hat{p}(y) = p(y) - \hat{c}(y)$ as the net inverse demand (gross of the idiosyncratic cost component, c). We hence define its inverse function $\hat{h}(\hat{p})$. Analogous to A1, we assume this function is strictly (-1)-concave. Note that $\hat{h}(\hat{p})$ is the demand as a function of $p(y) - \hat{c}(y)$, which is the demand price net of the common marginal cost component. Note that an increasing and convex marginal cost suffices to render $\hat{h}(\hat{p})$ (-1)-concave from $h(p)$ (-1)-concave, but this is not necessary.[20]

Now though we have exactly the same model as before, and with the same distributions except that we have replaced the price distribution with the distribution of $\hat{p}$. This means that all of the results hold modulo this transformation. In particular, the results of Theorems 2 and 3 that knowing $\hat{h}(\hat{p})$ and one distribution (any of c, π, y, and $\hat{p}$) suffices to find them all, modulo restrictions on allowable distributions akin to those before. Likewise, Theorems 4 through 9 hold, *mutatis mutandis*. Note that if we know the demand and cost function, Theorems 2 and 3 allow recovery of the price distribution too from any other distribution. However, for the counterparts to Theorems 4 through 9, we need more information (such as a third distribution) in order to recover the price distribution and to identify the breakdown between marginal cost and demand function.

One future research direction is to investigate more the inheritance properties of distributions both theoretically and empirically. For example, curvature properties such as ρ-concavity translate from one distribution to another. Also, the moments of different economics distributions are related through the economic relations: for example, the modes of the various distributions follow a simple relation via the elasticity analysis.

Finally, we here considered only one dimensional heterogeneity across firms. Multi-dimensional heterogeneity could also be analyzed with a generalized version of the methods described here. For example, suppose that firms differed with respect to both product quality (ζ) and cost (c) according to a joint fundamental distribution $F(\zeta, c)$, and we wrote demand as a function

[20] We show that $1/\hat{h}(\hat{p})$ convex if $1/h(p)$ convex and the inverse marginal cost function $y(\hat{c})$ is (-1)-convex, where the latter condition is satisfied for increasing and convex marginal cost functions, $\hat{c}(y)$. First, $1/h(p)$ convex is equivalent to $2p' + p''y < 0$, so using $\hat{p}(y) = p(y) - \hat{c}(y)$, it suffices that $2p' - \hat{c}' + (p'' - \hat{c}'')y < 0$ which is ensured for $-2\hat{c}' - \hat{c}''y < 0$, which is the same as $y(\hat{c})$ being (-1)-convex. Hence increasing and convex marginal cost suffices.

$h(\zeta - p)$.[21] Naturally, firms with higher ζ and lower c would have higher equilibrium profit and output. Then the set of firms with profit above any particular level of profit would be those below some critical locus $c(\zeta)$. Likewise for those with the highest outputs. However, the trade-off would be different for output and profit criteria. The economic fundamentals $F(\zeta, c)$ and $h(\zeta - p)$ would determine equilibrium distributions of the economics variables (profit, price, output, etc.) Now, as long as the iso-profit and iso-output loci (to take one pair as an example) satisfy a monotonicity condition, then we can determine the primitive distribution $F(\zeta, c)$ from the profit or output distributions by varying costs and quality across the feasible space and then matching the distributions to uncover it. In this way, we can extend the conceptual idea expounded here. It remains to determine what restrictions on primitives are needed to ensure full invertibility. Notice that we would need demand and (at least) two other distributions to determine the primitive distribution (and hence all the others.) Conversely, we would need at least three economic distributions to find the demand and the whole system. More generally, the informational requirements would increase with the number of dimensions of heterogeneity of primitive variables.

[21] Anderson and de Palma (2020) look at a one-dimensional version of this model for logit-CES with cost depending monotonically on quality.

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Appendix

Proof of Theorem 9

The assumption that F_Y and F_Π are $\mathcal{C}^1$ distributions means that we can invert them and write each of them as a function of the counter z. Both output and profit are increasing functions of cost, c. Therefore we can match the distributions: the firms with the highest $z\%$ of the costs are those with the lowest $z\%$ of the outputs and profits. Furthermore, because the distribution functions are differentiable, then z is a differentiable function of the underlying cost, and we can invert it. Call this inverted relation $c(z)$, with $c'(z) < 0$.

Choose some arbitrary level $z \in (0,1)$ such that $1 - F_C(c) = F_Y(y) = F_\Pi(\pi) = z$. Then the firms with cost levels above $c(z) = F_C^{-1}(1-z)$ are the firms with outputs and profits below y and π. For this proof, we introduce z as an argument into the various outcome variables to track the dependence of the variables on the level of $z(c) = 1 - F_C(c)$. However, the expression for $F_C(c)$ is not known at this point.

Write $y(z) = F_Y^{-1}(z)$ and equilibrium demand is

$$h^*(c) = y(z(c)) = F_Y^{-1}(1 - F_C(c)) > 0. \tag{21}$$

Because $\pi^*(z(c)) = m(z(c))\, y(z(c)) = F_\Pi^{-1}(z(c))$, then

$$m(z(c)) = \frac{F_\Pi^{-1}(z(c))}{F_Y^{-1}(z(c))} = \mu(c), \tag{22}$$

and equilibrium profit is $\pi^*(z((c))) = \mu(z(c))\, h^*(z(c)) = F_\Pi^{-1}(z(c))$.

From Lemma 5 we have $\pi^{*\prime}(z(c)) = -h^*(z(c))$, so the relation between the counter z and the cost level c with $z(\underline{c}) = 1$,$z(\overline{c}) = 0$ is $dz/dc = -h^*(c) / [\pi^*(z(c))]'$, and hence

$$z'(c) = -\frac{F_Y^{-1}(z(c))}{\left[F_\Pi^{-1}(z(c))\right]'} < 0. \tag{23}$$

Thus $\Psi(z) = -\int_{\bar{c}}^{c} dv = \bar{c} - c$,or $c(z) = \bar{c} - \Psi(z)$, where $\Psi(z)$ is the key transformation between z and c :

$$\Psi(z) = \int_0^z \frac{\left[F_\Pi^{-1}(r)\right]'}{F_Y^{-1}(r)} dr, \tag{24}$$

with $\Psi(0) = 0$, $\Psi(1) = \bar{c} - \underline{c}$. Because $\Psi'(z) = \frac{\left[F_\Pi^{-1}(z)\right]'}{F_Y^{-1}(z)} > 0$, the required relation between z and c is $z(c) = \Psi^{-1}(\bar{c} - c)$.

Since $p = h(\mu(c) + c)$, the inverse demand is $p = \frac{F_\Pi^{-1}(z(c))}{F_Y^{-1}(z(c))} + c = \frac{F_\Pi^{-1}\left(\Psi^{-1}(\bar{c}-c)\right)}{F_Y^{-1}(\Psi^{-1}(\bar{c}-c))} + c$. This makes clear that a shift up in all costs by Δ and a corresponding shift up in the inverse demand by Δ (so the domain of the cost distribution shifts up by Δ, i.e., $\bar{c}$ becomes $\bar{c} + \Delta$) keeps both the firm's output choice and mark-up constant so output and profit are not changed. This means that these two distributions can only pin down *net* (inverse) demand.

This allows us to uncover the distribution of cost, which is thus given by

$$F_C(c) = 1 - z(c) = 1 - \Psi^{-1}(\bar{c} - c). \tag{25}$$

The remaining unknowns can be backed out now knowing $z(c)$: equilibrium demand is $h^*(c) = F_Y^{-1}(\Psi^{-1}(\bar{c} - c))$ from (21). Therefore, since $h^*(c)$ is strictly decreasing in c, by Lemma 4 we can claim there exists a demand function $h(p)$ which satisfies A1 up to a constant. Note that this is exactly the property that the (21) delivers because $\Psi^{-1}(\bar{c} - c)$ is strictly decreasing in c (from (25)). Finally, the mark-up function is recovered from $\mu(c) = \frac{F_\Pi^{-1}\left(\Psi^{-1}(\bar{c}-c)\right)}{F_Y^{-1}(\Psi^{-1}(\bar{c}-c))}$ from (22).

Details for ρ-linear examples

First suppose that we know $F_Y(y) = \frac{(1+\rho)y^\rho - 1}{\rho}$, $y \in \left[\frac{1}{(1+\rho)^{1/\rho}}, 1\right]$, *and* $F_\Pi(\pi) = \frac{(1+\rho)\pi^{\rho/(1+\rho)} - 1}{\rho}$, $\pi \in \left[\frac{1}{(1+\rho)^{(1+\rho)/\rho}}, 1\right]$, *with* $\rho > -1$.

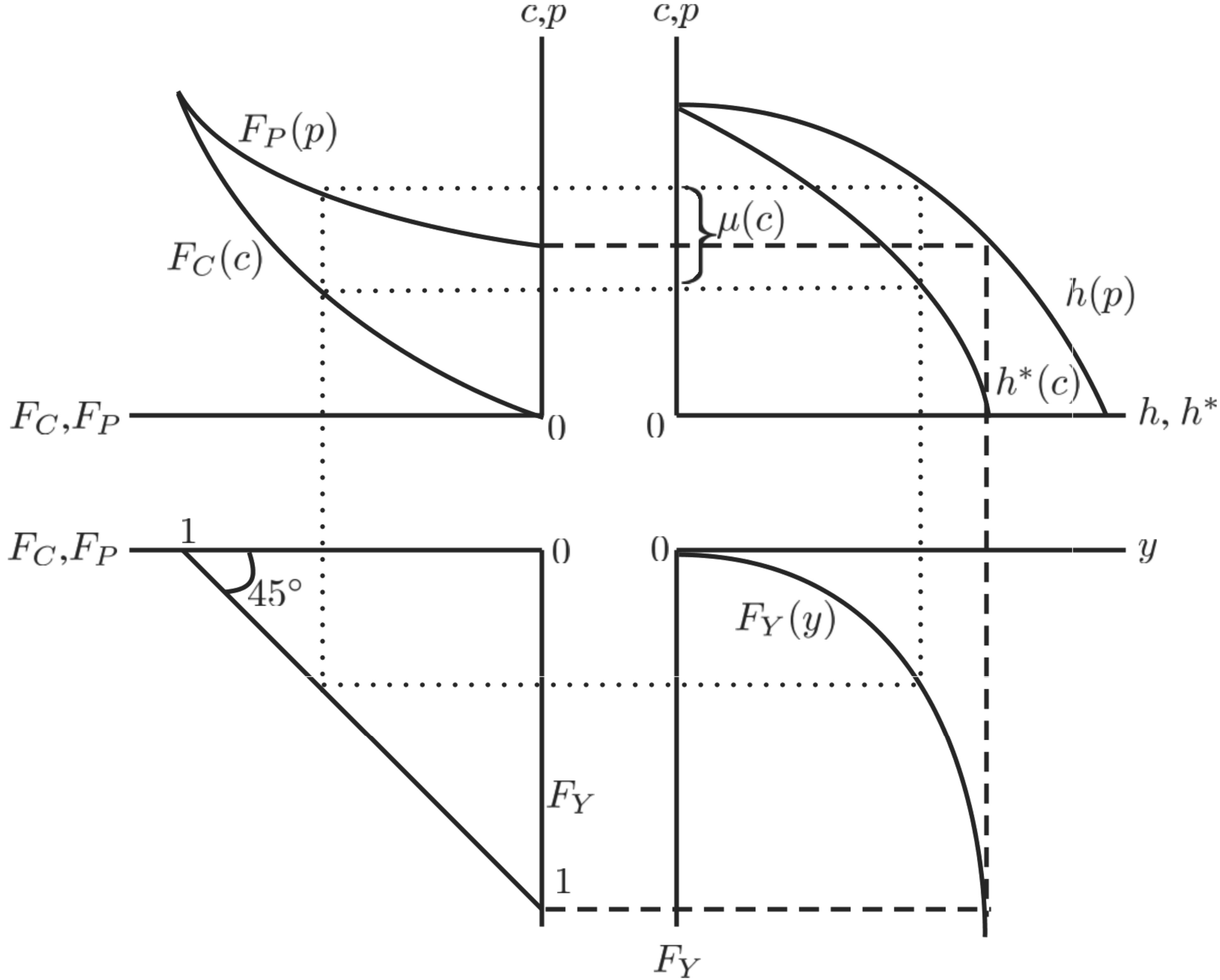
Hence $F_Y^{-1}(z) = \left(\frac{\rho z + 1}{1+\rho}\right)^{1/\rho}$ and $F_\Pi^{-1}(z) = \left(\frac{\rho z + 1}{1+\rho}\right)^{(1+\rho)/\rho}$. By (22), the ratio of these two yields the mark-up, $m(z) = \frac{\rho z + 1}{1+\rho} > 0$. Because $\left[F_\Pi^{-1}(z)\right]' = \left(\frac{\rho z + 1}{1+\rho}\right)^{1/\rho}$, we can write $\Psi(z) = \int_0^z \frac{\left[F_\Pi^{-1}(r)\right]'}{F_Y^{-1}(r)} dr = z = \bar{c} - c$, so $c(z) = \bar{c} - z$ $(c'(z) = -1)$. Now, $F_C(c) = 1 - \Psi^{-1}(\bar{c} - c) = c - \underline{c}$

(Uniform cost.) Hence $\mu(c) = \frac{\rho(\bar{c}-c)+1}{1+\rho}$ (from (22). Then $y(c) = F_Y^{-1}(z(c)) = \left(\frac{\rho(\bar{c}-c)+1}{1+\rho}\right)^{1/\rho}$, and $h^*(c) = y(c)$. We now want to find the associated demand, $h(p)$. We use the fact that $p = \mu(c) + c = \frac{1+c+\rho\bar{c}}{1+\rho}$ to write $h(p) = (1+\rho(\bar{c}-p))^{1/\rho}$, which is therefore a $\rho-$linear demand function with the parameter k set at $k = \bar{c}$, and $\rho > -1$ implies $h(.)$ is (-1)-concave.

Note that $y(\bar{c}) = \left(\frac{1}{1+\rho}\right)^{1/\rho}$, as verified by the upper bound, $\bar{c}$, while the lower bound condition $\underline{c} = \bar{c} - 1$ implies that $y(\underline{c}) = 1$, so costs are uniformly distributed on $[\underline{c}, \bar{c}]$. Lastly, $\lim_{\rho\to 0} y(c) = \exp(\bar{c} - c - 1)$ gives the logit equilibrium demand.

Suppose now that it is known that $F_C(c) = c$ *for* $c \in [0,1]$ *and* $F_\Pi(\pi) = \frac{(1+\rho)\pi^{\rho/(1+\rho)}-1}{\rho}$.

We first write $\pi^*(c)$ to find $h^*(c) = -\pi^{*\prime}(c)$. Matching the distribution levels, $1 - c = \frac{(1+\rho)\pi^{\rho/(1+\rho)}-1}{\rho}$, or $\pi^*(c) = \left(\frac{\rho(1-c)+1}{1+\rho}\right)^{(1+\rho)/\rho}$ and hence the domain of the profit function is $\pi \in \left[\frac{1}{(1+\rho)^{(1+\rho)/\rho}}, 1\right]$. Hence $y(c) = h^*(c) = \left(\frac{\rho(1-c)+1}{1+\rho}\right)^{1/\rho}$, so both output and profit are power functions. Then we use $c = 1 - F_Y(y)$ with $F_Y(y) = \frac{(1+\rho)y^\rho-1}{\rho}$ to get , $\mu(c) = \frac{\pi^*(c)}{h^*(c)} = \left(\frac{\rho(1-c)+1}{1+\rho}\right) = [h^*(c)]^\rho$ (consistent with: $y(c)^\rho = \left(\frac{\rho(1-c)+1}{1+\rho}\right)$). Now use $p = \mu(c) + c$ to find $h(p) = (1+\rho(1-p))^{1/\rho}$ and hence the (ρ-linear) demand form is tied down, including the value of the constant ($k = 1$: see (11), and consistent with the specification $\bar{c} = 1$). Finally, we use (16) to write $\mu'(c) = \frac{f_C(c)}{f_P(p)} - 1$; so $f_P(p) = 1+\rho$ and $F_P(p) = (1+\rho)(p-1)$ for $p \in \left[1, \frac{2+\rho}{1+\rho}\right]$.



c,p
c,p
F_P(p)
μ(c)
F_C(c)
h(p)
h*(c)
F_C,F_P
0
0
h, h*
1
F_C,F_P
0
0
y
45°
F_Y(y)
F_Y
1
F_Y