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THE CORE OF AN N PERSON GAME

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Sufficient conditions are given for a general n person game to have a nonempty core. The conditions are a consequence of convexity of preferences if the game arises from an exchange economy. The proof of sufficiency is based on a finite algorithm, and makes no use of fixed point theorems.

1. INTRODUCTION

THE PROBLEMS of distribution in an economic system may be analysed either by means of the behavioral assumptions of a competitive model or by the more flexible techniques of n person game theory. In the competitive model, consumers are assumed to respond to a set of prices by maximizing utility subject to a budget constraint and producers by maximizing profit. Consistent production decisions and an allocation of commodities are obtained by the determination of a set of prices at which all markets are in equilibrium.

The analysis of these problems by means of n person game theory requires us to specify the production and distribution activities that are available to an arbitrary coalition of economic agents. It is frequently sufficient to summarize the detailed strategic possibilities open to a coalition by the set of possible utility vectors that can be achieved by the coalition. For example, in a pure exchange economy each coalition will have associated with it the collection of all utility vectors that can be obtained by arbitrary redistributions of the resources of that coalition.

The core of an n person game is a generalization of Edgeworth's contract curve. A vector of utility levels is suggested which is feasible for all of the players acting collectively, and an arbitrary coalition is examined to see whether it can provide higher utility levels for *all* of its members. If this is possible, the utility vector which was originally suggested is said to be blocked by the coalition. The core of the n person game consists of those utility vectors which are feasible for the entire group of players and which can be blocked by no coalition.

As we have seen during the last several years, there is an intimate connection between these two methods of analysis. If the conventional assumptions of the competitive model are made, such as convexity of preferences and convexity and constant returns to scale for the production set, then there will be a price system at which all markets are in equilibrium and a resulting assignment of commodity bundles to consumers. The utility vector associated with this competitive equilibrium may be shown to be in the core. Even further, if the number of consumers tends

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to infinity in a suitable way, the set of possible utility vectors in the core becomes smaller and tends, in the limit, to those utility vectors associated with competitive equilibria [3].

We do not, of course, expect a competitive equilibrium if the classical assumptions of the competitive model are not made. On the other hand the formulation of the problem of distribution by means of n person game theory is sufficiently flexible to accommodate any number of departures from the classical model. The set of possible utility vectors achievable by a coalition can be discussed in the presence of increasing returns to scale in production, public ownership of some commodities, and social rather than exclusively private goods, to name only a few departures from the classical model. This raises the question of determining conditions which are sufficient to guarantee the existence of a utility vector in the core, and which are described directly in terms of the structure of an n person game rather than appealing indirectly to the existence of competitive equilibria.

In order to see the form that such conditions might take, let us begin by examining a game with three players. In this case there are seven possible coalitions; the three one player coalitions, the three two player coalitions, and the coalition of all three players. Each such coalition will be able to obtain a set of utility vectors depending on the strategies available to its members. It will be useful to denote by V_S the set of those vectors achievable by the coalition S. $V_{(123)}$ will be represented geometrically by a set of vectors in three space, $V_{(12)}$ will lie in the plane determined by the coordinate axes 1 and 2, and in general V_S will lie in that linear subspace of three space whose coordinates correspond to the members of S. The sets V_S will be assumed to have several technical properties such as being closed and containing any point whose coordinates are less than or equal to those of a point in V_S.

For this game to have a core which is not empty, $V_{(123)}$ must be sufficiently large so as to contain a vector which cannot be blocked by any coalition. One meaning of the term "sufficiently large" can be obtained by assuming that it is to the advantage of a disjoint collection of coalitions to combine. For example if $u_1 \in V_{(1)}$ and $(u_2, u_3) \in V_{(23)}$ then I will assume that $(u_1, u_2, u_3) \in V_{(123)}$, and similarly for all other partitions of the set of three players. The assumption that the game is superadditive, in this sense, is quite natural for most economic models. It is, however, not sufficient to guarantee the existence of a vector in the core, and one additional relationship is required.

Let us assume for a moment that the game derives from a market model in which the three players exchange the commodities which they initially own. The preferences of the ith player will be represented by a utility function $u_i(x^i)$, with x^i the commodity bundle received by this player. The commodity bundle initially owned by the ith player will be denoted by ω^i. With this notation the set $V_{(123)}$ is described by

$$V_{(123)} = \{(u_1, u_2, u_3) \mid u_j \leqslant u_j(x^j) \text{ for some } (x^1, x^2, x^3) \text{ with } x^1 + x^2 + x^3 = \omega^1 + \omega^2 + \omega^3\},$$

the set $V_{(12)}$ by

$$V_{(12)}=\{(u_1, u_2)\,|\,u_j \leqslant u_j(x^j) \text{ for some } (x^1, x^2) \text{ with } x^1+x^2=\omega^1+\omega^2\},$$

with a similar definition for every set V_S.

This game is clearly superadditive in the sense given above, even in the absence of convex preferences. We know, however, that a market game without convex preferences need not have a core [3], and we should therefore look for some way of translating the convexity of preferences into a relationship that can be stated solely in terms of the sets V_S, in order to find the missing condition. Let us proceed in the following way. Assume that we are given a vector (u_1, u_2, u_3) which is arbitrary except that it satisfies the following three conditions:

$$(u_1, u_2)\in V_{(1,2)}\,,$$
$$(u_2, u_3)\in V_{(2,3)}\,,$$
$$(u_1, u_3)\in V_{(1,3)}\,.$$

In the market economy this means that there are commodity bundles (x^1, x^2), (y^2, y^3) and (z^1, z^3) with

$$\begin{aligned} x^1+x^2 \qquad &= \omega^1+\omega^2 \qquad , \\ y^2+y^3 &= \qquad \omega^2+\omega^3\,, \\ z^1 \qquad +z^3 &= \omega^1 \qquad +\omega^3\,, \end{aligned}$$

and

$$\begin{aligned} u_1(x^1)&\geqslant u_1\,, & u_1(z^1)&\geqslant u_1\,, \\ u_2(x^2)&\geqslant u_2\,, & u_2(y^2)&\geqslant u_2\,, \\ u_3(y^3)&\geqslant u_3\,, & u_3(z^3)&\geqslant u_3\,. \end{aligned}$$

But then

$$\frac{x^1+z^1}{2}\,,\quad \frac{x^2+y^2}{2}\,,\quad \frac{y^3+z^3}{2}$$

represents a feasible trade for all three players since these vectors total to $\omega^1+\omega^2+\omega^3$. If the preferences of the three consumers are convex then the utility levels associated with this trade can be described quite easily, since convexity implies that

$$u_1\left(\frac{x^1+z^1}{2}\right) \geqslant \min\,[u_1(x^1)\,,\quad u_1\,(z^1)]\geqslant u_1\,,$$

$$u_2\left(\frac{x^2+y^2}{2}\right) \geqslant \min\,[u_2(x^2)\,,\quad u_2(y^2)]\geqslant u_2\,,$$

$$u_3\left(\frac{y^3+z^3}{2}\right) \geqslant \min\,[u_3(y^3)\,,\quad u_3(z^3)]\geqslant u_3\,.$$

In other words the vector (u_1, u_2, u_3) is obtainable by the three player coalition and is therefore in $V_{(1,2,3)}$.

This rather curious translation of convexity connecting the three two player coalitions to the coalition of all players is, in conjunction with the assumptions of superadditivity, sufficient for the existence of a vector in the core of a three person game. In order to discuss the appropriate generalization of these conditions let us turn to a more formal definition of an n person game following Aumann and Peleg [1].

The set of n players will be denoted by N and an arbitrary coalition by S. For each set S, E^S will mean the Euclidean space of dimension equal to the number of players in S and whose coordinates have as subscripts the players in S. If u is a vector in E^N then u^S will be its projection onto E^S.

We shall associate with each coalition S a set V_S, in E^S, which represents the set of possible utility vectors that can be obtained by that coalition. The members of S may have to engage in a variety of activities, depending on the nature of the n person game, in order to obtain a particular vector in V_S. For our purposes, however, all that is required is a summary of the utility vectors achievable by each coalition.

It will be useful to make the following assumptions about the sets V_S.

1. For each S, V_S is a closed set.
2. If $u \in V_S$ and $y \in E^S$ with $y \leqslant u$, then $y \in V_S$.
3. The set of vectors in V_S in which each player in S receives no less than the maximum that he can obtain by himself is a nonempty, bounded set.

These conditions are all quite mild and need no particular comment. They are slightly different from the conditions assumed by Aumann and Peleg; in particular the sets V_S need not be convex.

We have already seen how a three person exchange model gives rise to a game in this form. In an exchange economy with n consumers, where the ith player's utility function is given by $u_i(x^i)$ and his initial holdings by ω^i, a vector $u \in E^S$ will be in V_S if we can find commodity bundles x^i with $\Sigma_{i \in S} x^i = \Sigma_{i \in S} \omega^i$ and $u_i(x^i) \geqslant u_i$ for all $i \in S$. Production may be introduced by assuming that each coalition has the ability to transform commodities according to some production set, though this is by no means the only way to incorporate production into an n person game theory model.

As another example consider the classical case of an n person game with transferable utility described by a number f_S associated with each coalition. What this means is that a vector $u \in E^S$ may be obtained by the coalition S if $\Sigma_{i \in S} u_i \leqslant f_S$, so that the sets V_S consist of half spaces defined by hyperplanes whose normal vectors have components either zero or one.

Let u be a point in V_N and u^S its projection onto E^S. The vector u is blocked by the set S if we can find a point $y \in V_S$ with $y > u^S$, or in other words if the coalition S can obtain a higher utility level for each of its members than that given by the vector u. A point $u \in V_N$ will be in the core if it cannot be blocked by any set S.

In order to determine the appropriate generalization of our extra condition in the three person case, we must have recourse to the concept of a balanced collection of coalitions studied by Shapley [8], Peleg [7], and Bondareva [2] in the context of a game with transferable utility.

DEFINITION: Let $T=\{S\}$ be a collection of coalitions in an n person game. T is said to be a *balanced collection* if it is possible to find nonnegative weights δ_S, for each coalition in T, such that

$$\sum_{\substack{S\in T\\ S\supset\{i\}}} \delta_S=1 \quad \text{for each } i\,.$$

In other words, the weights δ_S are to have the property that if any individual is selected, the sum of the weights corresponding to those coalitions in T which contain the individual, must be equal to one. Another way to phrase the definition is by saying that the characteristic function of the set of all players is a nonnegative linear combination of the characteristic functions of the coalitions in a balanced collection.

Balanced collections of coalitions do represent a generalization of the collection of all two player coalitions studied in the three person game, since $\delta_{(12)}=\delta_{(13)}=\delta_{(23)}=\frac{1}{2}$ will serve as an appropriate system of weights. It is somewhat unfortunate, given the importance of balanced collections in the study of the core, that we have no really intuitive definition for determining when a given collection is balanced.

This concept permits us to extend to an n person game the additional requirement imposed in the three person case.

DEFINITION: An n person game is *balanced* if for every balanced collection T, a vector u must be in V_N if $u^S\in V_S$ for all $S\in T$.

The main theorem of this paper may now be stated.

THEOREM 1: *A balanced n person game always has a nonempty core.*

It should be noticed that all of the concepts that have been introduced are purely ordinal in character; they are invariant if a continuous monotonic transformation is applied to the utility of any individual. In fact the discussion could be carried out on an abstract level with the outcomes for each individual represented by arbitrary ordered sets.

2. SOME EXAMPLES OF BALANCED GAMES

The condition that a game be balanced is undoubtedly quite obscure and it will be useful to examine some examples. A market game with convex preferences will always be balanced, for let T be an arbitrary balanced collection and u a vector with $u^S\in V_S$ for each S in T. This means that for each such coalition there is a way of redistributing its assets so as to obtain the vector u^S. If the redistribution gives player i (assuming that he is a member of S) the commodity bundle x_s^i, then

$$\sum_{i\in S} x_S^i = \sum_{i\in S} \omega^i \quad \text{and} \quad u_i(x_S^i) \geqslant u_i .$$

In order to show that the game is balanced we need to construct an allocation $x^1, ..., x^n$ with $\Sigma_1^n x^i = \Sigma_1^n \omega^i$ and $u_i(x^i) \geqslant u_i$ for all i. This allocation may be constructed in terms of the weights δ_S used in the definition of a balanced collection.

For each player i, let us define x^i as $\Sigma_{S\in T,\, S\supset\{i\}}\, \delta_S x_S^i$. By the definition of δ_S, each x^i is a *convex* combination of x_S^i with S ranging over those sets in T which contain the ith player. If the preferences are assumed to be convex then $u_i(x^i)$ is greater than or equal to the smallest of the numbers $u_i(x_S^i)$, and is therefore greater than or equal to u_i. We have, by this device, constructed an assignment of commodity bundles which provides a utility level for each player no less than his corresponding component of u. In order to show that $u \in V_N$ we need only verify that $\Sigma_1^n x^i = \Sigma_1^n \omega^i$. But

$$\begin{aligned}
\sum_1^n x^i &= \sum_1^n \sum_{S\supset\{i\}} \delta_S x_S^i \\
&= \sum_{S\in T} \delta_S \sum_{i\in S} x_S^i \\
&= \sum_{S\in T} \delta_S \sum_{i\in S} \omega^i \\
&= \sum_1^n \omega^i \sum_{\substack{S\in T \\ S\supset\{i\}}} \delta_S = \sum_1^n \omega^i .
\end{aligned}$$

This argument demonstrates that an exchange economy with convex preferences will always give rise to a balanced n person game, and assuming the validity of the main result of this paper, such a game will always have a nonempty core. It is interesting that no additional assumptions are required such as strict monotonicity of the preferences, or strict positivity of the initial holdings. (Of course, in an exchange model, some additional assumptions are required in order to pass to the limit and obtain the existence of competitive equilibria.)

In our second example, an n person game with transferable utility, it is quite easy to verify that a balanced game has a nonempty core without using the more subtle techniques to be developed later. In fact, a game with transferable utility has a core if and only if it is balanced.

If the sets V_S consist of those vectors in E^S with $\Sigma_{i\in S} u_i \leqslant f_S$, the vector $(u_1, ..., u_n)$ will be in the core if

$$\sum_{i=1}^{n} u_i \leqslant f_N \quad \text{and}$$

$$\sum_{i \in S} u_i \geqslant f_S \quad \text{for all subsets } S.$$

The first inequality implies that $u \in V_N$ and the second set that u cannot be blocked by any coalition S. In other words the game will have a core if the linear programming problem

$$\min \sum_{1}^{n} u_i$$

$$\sum_{i \in S} u_i \geqslant f_S, \quad \text{for all } S,$$

has a solution in which the objective function is equal to f_N. The dual variables may be denoted by δ_S, one for each subset, and the dual linear programming problem is

$$\max \sum_{S} \delta_S f_S$$

$$\sum_{S \supset \{j\}} \delta_S = 1, \quad \text{and} \quad \delta_S \geqslant 0.$$

(We have equality here, since the variables in the primal problem are unrestricted in sign.) Let $\{\hat{\delta}_S\}$ be a solution of the dual problem and $\{\hat{u}_i\}$ a solution to the primal problem. Then the collection T of those coalitions for which $\hat{\delta}_S > 0$ is, by definition, a balanced collection, and the solution of the primal problem provides us with a vector $\hat{u}$ such that $\Sigma_{i \in S} \hat{u} = f_S$ for all $S \in T$, since the positivity of a dual variable forces the primal constraints to be equalities. But then $\hat{u}^S \in V_S$ for all $S \in T$ and if the game is balanced, this implies that $\hat{u} \in V_N$ or $\Sigma_1^n \hat{u}_i \leqslant f_N$. This shows that a balanced game has a nonempty core in the case of transferable utility. As we shall see, the general n person game, without the assumption of transferable utility, requires more elaborate techniques than those of linear programming.

3. A COMBINATORIAL PROBLEM WHICH IMPLIES THEOREM 1

The proof of Theorem 1 will be divided into two parts. We begin by selecting from each set V_S (with S a *proper* subset of N), a finite number of vectors $u^{1,S}$, $u^{2,S}$, ..., $u^{k_S,S}$, that gives no player in S less than the maximum he can achieve by himself. If the game is balanced, the algorithm will then calculate a vector in V_N which cannot be blocked by any proper coalition using a vector from this finite list to block. The passage to the limit, which involves selecting an infinite, dense sequence of vectors from each V_S will be discussed in a later section.

Essentially we are approximating each set V_S by a set with a finite number of "corners," as illustrated by Figure 1. Since this approximation does not modify the property that the game is balanced, we may, for the moment, restrict our attention to games in which the sets V_S do, in fact, have a finite numbers of corners for each proper subset S.

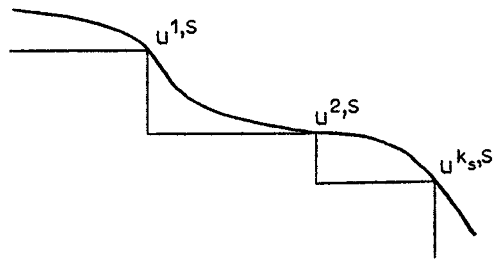

FIGURE 1

It is useful to summarize the data of a finite game by means of a matrix C with n rows, the number of players in the game, and $\Sigma_S k_S$ columns, one column for each of the vectors involved in defining the game. The rows of C will be indexed by i and the columns will require a pair of subscripts (j, S), so that a typical entry in C will be denoted by c_{ijS}. If player i is contained in the coalition S, then c_{ijS} is defined to be the ith component of the vector $u^{j,S}$. It will be useful to define c_{ijS} to be equal to some large number M if player i is not a member of S. The particular choice of M is irrelevant to the actual calculation as long as it is selected to be larger than any of the components of the vectors $u^{j,S}$.

Let us also define a matrix A with n rows and $\Sigma_S k_S$ columns by $a_{ijS}=1$ if player i is in coalition S, and zero otherwise. A is the incidence matrix of players versus sets, with the column representing S appearing as many times as there are corners in V_S.

Let me begin with an example of a three person game in order to illustrate the problem. In this example, the set V_S for a typical two player coalition will be assumed to have two corners. The matrix C is given by

$$\begin{array}{ccccccccc} (1) & (2) & (3) & (1,12) & (2,12) & (1,13) & (2,13) & (1,23) & (2,23) \end{array}$$
$$\begin{bmatrix} 0 & M & M & 6 & 2 & 12 & 3 & M & M \\ M & 0 & M & 6 & 8 & M & M & 7 & 2 \\ M & M & 0 & M & M & 2 & 8 & 5 & 9 \end{bmatrix}.$$

In this example each player, by himself, can obtain a maximum utility of zero. In general, information about V_N need not be included in the matrix C.

It is useful to examine the problem from a geometric point of view. I have drawn, in Figure 2, the set, which may be called V, of those points in the nonnegative orthant which are necessarily contained in $V_{(123)}$ if the game is balanced. V contains those points on the coordinate planes which are achievable by the two player coalitions, since a two player coalition and its complementary one player coalition form a balanced collection. V also contains those vectors (u_1, u_2, u_3) with $(u_1, u_2) \in V_{(12)}$, $(u_1, u_3) \in V_{(13)}$ and $(u_2, u_3) \in V_{(23)}$. From the assumption that the game is balanced we know only that $V_{(123)}$ contains V; it may be considerably larger.

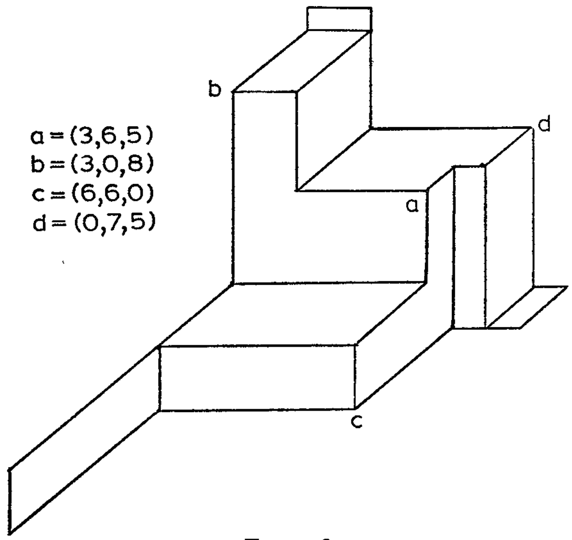

FIGURE 2

No point in the nonnegative orthant can be blocked by a one player coalition. Is there a point in V which can be blocked by no two player coalition? In other words is there a point in V whose three projections are on the boundaries of the sets of utility levels achievable by the two player coalitions? The reader may verify that there is only one such point, $u=(3, 6, 5)$. Of course this vector need not be in the core if it is not Pareto optimal, but then any Pareto optimal point in $V_{(123)}$ which is greater than or equal to u, is in the core.

The vector (3, 6, 5) is generated by the points (6, 6, 0), (3, 0, 8), and (0, 7, 5) in the sense that if we form the square submatrix of C, corresponding to these points,

$$\begin{bmatrix} 6 & 3 & M \\ 6 & M & 7 \\ M & 8 & 5 \end{bmatrix},$$

and define u_i to be the minimum of the ith row of this square submatrix, then $(u_1, u_2, u_3)=(3, 6, 5)$. The fact that u cannot be blocked is clear from the C matrix, for if u were blocked by S, then there would be a column j, S with $c_{ijS}>u_i$ for all i, and the reader may verify that no such column exists.

Analytically the argument that u is in $V_{(123)}$, if the game is balanced, depends on the observation that

$$(3, 6, 5)^{(12)}=(3, 6)\leqslant(6, 6)\in V_{(12)}$$
$$(3, 6, 5)^{(13)}=(3, 5)\leqslant(3, 8)\in V_{(13)}$$
$$(3, 6, 5)^{(23)}=(6, 5)\leqslant(7, 5)\in V_{(23)},$$

and that the three two element sets form a balanced collection.

In the general case we also consider a square submatrix of C, and define u_i to be the minimum of the entries in the ith row of this submatrix. For the vector u to lead us to a point in the core two properties are required. First of all we want u to be blocked by no coalition, and this means that for every column in the C matrix at least one entry must be less than or equal to the corresponding entry in the u vector. Of course not every square submatrix of C will produce a u vector with this property, and part of the algorithm will be concerned with determining submatrices of this sort.

In order to conclude that the vector u is in V_N a second condition will have to be imposed on the columns defining the submatrix of C.

Let T be the collection of those coalitions S which appear in at least one column of the square submatrix. For each S in T the vector u^S is surely in V_S, since it is less than or equal to one of the corners appearing in V_S. In order to conclude that $u \in V_N$ it is sufficient that the collection T be balanced, or in other words that there exist nonnegative numbers δ_S, zero for those S not in T, and such that

$$\sum_{S \supset \{i\}} \delta_S = 1 \,, \quad \text{for} \quad i = 1, \dots, n \,.$$

But this is equivalent to saying that the equations $Ax = e$ (with e the vector all of whose components are 1) have a nonnegative solution, with $x_{jS} = 0$, for any column j, S not appearing in the square submatrix of C, since we may take $\delta_S = \Sigma_j x_{jS}$.

In other words we look for a feasible basis in the sense used in linear programming for the equations $Ax = e$. The n columns of this feasible basis give rise to a square submatrix of C, and u_i is defined to be the minimum of the ith row of this submatrix. The feasible basis is to be selected so that for every column in the C matrix at least one entry is less than or equal to the corresponding entry in the u vector.

It is very useful to generalize the problem by considering an arbitrary matrix A, with n rows and m columns, rather than a repeated incidence matrix, a C matrix of the same dimensions as A, and an arbitrary vector b. In this more general case the columns of both the A and C matrix will have the subscript j rather than the more cumbersome subscript (j, S) appropriate to a repeated incidence matrix. We look for a feasible basis for the equations $Ax = b$, and for each such basis we define $u_i = \min \{c_{ij} \mid \text{for all } j \text{ appearing in the feasible basis}\}$. Will there be a basis, so that for every column k, there is at least one i with $u_i \geqslant c_{ik}$?

In order to guarantee an affirmative answer to this more general question, the matrices A and C will be assumed to have the properties described in the following definition.

Definition: Let A and C be two n by m matrices of the form:

$$A = \begin{bmatrix} 1 \dots 0 & a_{1,n+1} \cdots a_{1,m} \\ \vdots & \vdots \\ 0 \dots 1 & a_{n,n+1} \cdots a_{n,m} \end{bmatrix},$$

$$C=\begin{bmatrix} c_{1,1}\cdots c_{1,n} & c_{1,n+1}\cdots c_{1,m} \\ \vdots & \vdots \\ c_{n,1}\cdots c_{n,n} & c_{n,n+1}\cdots c_{n,m}\end{bmatrix}.$$

We say that A and C are in standard form if

1. for each row i, c_{ii} is the minimum of the elements in its row, and
2. for each nondiagonal element c_{ij} in the square submatrix of C consisting of the first n columns, and for each column $k>n$, we have $c_{ij}\geqslant c_{ik}$.

The reader may easily verify that the matrices A and C arising from a finite game are in standard form if the vectors $u^{j,S}$ representing the corners, give each player in S no less than the maximum that he can achieve by himself, and if the constant M is selected to be larger than any of the components of these vectors.

THEOREM 2: *Let A and C be two n by m matrices in standard form. Let b be a nonnegative vector such that the set $\{x \mid x\geqslant 0$ and $Ax=b\}$ is bounded. Then there is a feasible basis for the equations $Ax=b$, so that if we define $u_i=\min c_{ij}$, for all columns j in this basis, then for every column k, there is an index i with $u_i\geqslant c_{ik}$.*

From the previous discussion it is clear that Theorem 2 implies that a balanced game has a nonempty core if each V_S has a finite number of corners.

4. AN ALGORITHM FOR THE PROBLEM OF THEOREM 2

The problem posed in Theorem 2 is not remotely a linear programming problem even though only linear inequalities are involved. Any attempt to cast this problem in a linear programming form would run into the difficulty that not all of the relevant inequalities are to be satisfied simultaneously. An attempt to use integer programming methods would neither provide an existence theorem nor take advantage of the special structure of the problem. The algorithm of this paper, which is based on the ingenious procedure discovered by Lemke and Howson [4, 5] for the solution of a two person nonzero sum game, provides a method for calculating a solution to this problem, and since the algorithm terminates in a finite number of steps, the existence of at least one solution is guaranteed.

Theorem 2 may be demonstrated directly by an examination of the two person nonzero sum game in which one player has a payoff matrix A and has the columns for his pure strategies. The second player, whose strategies are the rows, has a payoff matrix given by $B=(b_{ij})$ where $b_{ij}=-1/c_{ij}^{\eta}$ (we may assume, without loss of generality, that $c_{ij}>0$, since the theorem is unchanged if each entry in C is increased by the same amount). Theorem 2 may then be obtained by letting the exponent η tend to infinity and considering a convergent subsequence of equilibrium points for these games. The proof is quite simple, but since it involves the selection of a convergent subsequence it is basically nonconstructive, even though the

Lemke-Howson argument may be used to calculate an equilibrium point for each value of η. The remainder of this section will be devoted to a modification of this algorithm which is applicable to the limiting case directly.

The terminology introduced in the following definition will be useful in discussing the limiting algorithm.

DEFINITION: An *ordinal basis* for the matrix C consists of a set of n columns $j_1, j_2, ..., j_n$ so that if $u_i = \min(c_{ij_1}, c_{ij_2}, ..., c_{ij_n})$, then for every column k, there is at least one i with $u_i \geqslant c_{ik}$.

The term *ordinal basis* used in this definition is meant to be suggestive of an analogy with linear programming, as we shall see from some of the properties described below. But first of all it should be noticed that our theorem will be demonstrated if we can exhibit a feasible basis for A which is simultaneously an ordinal basis for C.

The algorithm for the determination of such a set of columns alternates between pivot steps for the linear equations, and a related operation on the matrix C. We make the standard nondegeneracy assumption that all of the variables associated with the n columns of a feasible basis for the equations $Ax=b$ are strictly positive. The nondegeneracy assumption for C takes the rather novel form that no two elements in the same row are equal. Both of these assumptions can be brought about by perturbations of the corresponding matrices.

Before turning to a discussion of pivot steps it is useful to note that the nondegeneracy assumption for C implies that each column of an ordinal basis has precisely one row minimizer to be used in forming the vector u. This may be seen if the column k, in the definition of an ordinal basis, is taken to be a column in the basis.

LEMMA 1: *Let $j_1, j_2, ... j_n$ be the columns of a feasible basis for the equations $Ax=b$, and let j^* be an arbitrary column not in this collection. Then, if the problem is nondegenerate and the convex set $\{x \mid x \geqslant 0, Ax=b\}$ is bounded, there is a unique feasible basis consisting of column j^* and $n-1$ columns of the original feasible basis.*

This is, of course, a standard result in linear programming, which says that if the constraint set is bounded and if the problem is nondegenerate, then any column outside of a basis may be introduced and as a result of a pivot step precisely one column will be eliminated. Something very much like taking a pivot step may be applied to an ordinal basis of the matrix C. With one exception, a specific column in an ordinal basis may be removed and a unique column introduced from outside so that the new set of columns is also an ordinal basis.

LEMMA 2: *Let $j_1, j_2, \ldots j_n$ be an ordinal basis of C, and j_1 an arbitrary one of these columns. Assume that $j_2, \ldots, j_n$ are not all selected from the first n columns of C. Then if no two elements in the same row of C are equal and if C is in standard form, there is a unique column $j^* \neq j_1$ such that $j^*, j_2, \ldots, j_n$ is an ordinal basis.*

The steps involved in removing a specific column from an ordinal basis and replacing it by a column outside the basis will be called an ordinal pivot step. We shall first give a definition of an ordinal pivot step, then show that the new set of columns is an ordinal basis, and finally demonstrate that the introduction of no other column will lead to an ordinal basis.

DEFINITION: Consider an ordinal basis for C and a specific column to be removed. In the $n \times n-1$ matrix of remaining columns precisely one column will contain two row minimizers, one of which is new and the other a row minimizer for the original basis. Let the row associated with the latter have an index i^*. Examine all columns in C for which $c_{ik} > \min\{c_{ij} \mid j$ remains in the basis$\}$ holds for all i not equal to i^*. Of these columns, select the one which maximizes c_{i^*k}. An ordinal pivot step introduces this column into the basis.

It is possible to carry out an ordinal pivot step if there is a column in C for which $c_{ik} > \min\{c_{ij} \mid j$ remains in the basis$\}$ for all $i \neq i^*$. But since the matrix C is in *standard form*, the column

$$\begin{bmatrix} c_{1i^*} \\ \vdots \\ c_{ni^*} \end{bmatrix}$$

will surely satisfy this condition unless the $n-1$ columns remaining in the basis come from the first n columns of the matrix C.

If j^* is the column brought into the basis, then the new u vector is given by $u'_i = \min\{c_{ij} \mid j$ remains in the basis$\}$ for $i \neq i^*$, and $u'_{i^*} = c_{i^*j^*}$. The way in which j^* is selected implies that there is no column in C all of whose components are strictly larger than those of u', so that the new collection of columns is an ordinal basis.

In order to see that the introduction of no column other than j^* will lead to an ordinal basis, let us consider the square submatrix obtained from the original ordinal basis:

$$\begin{bmatrix} c_{1j_1} & c_{1j_2} & \cdots & c_{1j_n} \\ c_{2j_1} & c_{2j_2} & \cdots & c_{2j_n} \\ \vdots & & & \\ c_{nj_1} & c_{nj_2} & \cdots & c_{nj_n} \end{bmatrix}.$$

To be specific let us assume that the row minimizers occur along the diagonal, that column j_1 is to be removed from the basis, and that the second smallest element in the first row is c_{1j_2}. This implies $i^*=2$.

When a new column is brought in to replace the first column, c_{3j_3} must still be the row minimum for the third row, since otherwise there would be no row minimum in the third column, and similarly for $c_{4j_4}, ..., c_{nj_n}$. We know therefore that if column j^* is brought into the basis, then $c_{3j^*}>c_{3j_3}$, $c_{4j^*}>c_{4j_4}$, ..., $c_{nj^*}>c_{nj_n}$.

Two cases occur for the row minima of the first two rows, either $c_{1j^*}<c_{1j_2}$ and $c_{2j^*}>c_{2j_2}$ or the reverse. The first case leads back to the original basis. To see this we notice that if the first case does take place, the new u vector will be given by

$$\begin{bmatrix} c_{1j^*} \\ c_{2j_2} \\ \vdots \\ c_{nj_n} \end{bmatrix},$$

and if the new set of columns are to be an ordinal basis, then for any column k we must have $u_i \geqslant c_{ik}$ for at least one i. But if k is the column j_1, this means $c_{1j^*} \geqslant c_{1j_1}$. On the other hand the old set of columns was assumed to be an ordinal basis, and so for any k, $c_{ij_i} \geqslant c_{ik}$ for some i. But here we may take $k=j^*$ and we see that $c_{1j_1} \geqslant c_{1j^*}$, so that $c_{1j_1}=c_{1j^*}$. By the nondegeneracy assumption, no two elements of the same row are equal and therefore $j^*=j_1$ and we are back to the original set of columns.

It is in the second variant, in which the minimizing elements in the first two rows are reversed, that we move to a new basis. In this case we look for a column j^* in which $c_{1j^*}>c_{1j_2}$, $c_{3j^*}>c_{3j_3}$, ..., $c_{nj^*}>c_{nj_n}$, or $c_{ij^*}>\min\{c_{ij} \mid j$ remains in the basis$\}$ for all $i \neq 2$, and in order for the new basis to be feasible we must select from these columns so as to maximize c_{2j^*}. But this is the column described in the definition of an ordinal pivot step, and Lemma 2 has therefore been demonstrated.

It is useful to note that ordinal pivot steps are reversible; if j_1 is eliminated from a basis and j^* brought in, then j^* may be eliminated from the new basis and the original basis will be obtained. Ordinal pivot steps are remarkably easy to carry out. They involve only ordinal comparisons of elements in the same row, and therefore the entries in the matrix C can be selected from arbitrary ordered sets, one for each row, rather than being real numbers. For example the entries in a row may be commodity bundles ordered by a preference ordering.

We are now ready to discuss the algorithm for determining a set of columns that is simultaneously a feasible basis for the equations $Ax=b$, and an ordinal basis for the matrix C. It is quite easy to find a pair of bases, one a feasible basis for the matrix A, and the other an ordinal basis for the matrix C which, while not identical, are quite close, and we shall use such a pair of bases as a starting point in the algorithm. The columns $(1, 2, ..., n)$ form a feasible basis for the matrix A, and the columns

$$\begin{bmatrix} c_{1j} & c_{12} & \cdots & c_{1n} \\ c_{2j} & c_{22} & & c_{2n} \\ \vdots & \vdots & & \vdots \\ c_{nj} & c_{n2} & \cdots & c_{nn} \end{bmatrix}$$

form an ordinal basis for the matrix C if j is selected from all of the columns $k>n$ so as to maximize c_{1k}. The columns in the C basis are given by $(j, 2, ..., n)$. The relationship between the two bases may be described by saying that the A basis contains column 1 and $n-1$ remaining columns. The $n-1$ remaining columns are also contained in the C basis along with one additional column other than the first. The ingenious idea introduced by Lemke and Howson is to insist that this relationship be maintained between the two bases.

In other words we will always be in a position where the feasible A basis can be described by the columns $(1, j_2, j_3, ..., j_n)$ and the ordinal basis for C by $(j_1, j_2, j_3, ..., j_n)$, with $j_1 \neq 1$. What steps can be taken so as to preserve this property? There are only two possible steps, one a pivot step for the matrix A and the other an ordinal pivot step for the matrix C.

A pivot step on the matrix A will leave this relationship unchanged only if column j_1 is introduced into the A basis. It is of course possible that column 1 will be eliminated from the A basis when column j_1 is brought in; the problem would be solved if this were to occur since the same basis $(j_1, j_2, ..., j_n)$ would then be obtained for both matrices. If column 1 is not eliminated by the pivot step then some other column, say j_i, will be. The two bases will still stand in the same relation with j_i being the column in the C basis which does not appear in the A basis.

The other possible continuation is to do an ordinal pivot step on the C matrix, eliminating one of its columns. The mutual relation between the A and C bases will be retained only if column j_1 is eliminated from the C basis. If j_1 is eliminated and column 1 is introduced into the C basis, the problem is solved since the same basis $(1, j_2, ..., j_n)$ will then be obtained for both matrices. On the other hand if column $j^* \neq 1$ is brought into the C basis when j_1 is eliminated, the two bases again stand in the same relationship with j^* being the column in the C basis which does not appear in the A basis.

As we have seen, the ordinal pivot step on the matrix C can always be carried out, except in the case where the columns other than j_1 in the ordinal basis for C are selected from the first n columns in the C matrix. For this case to occur the A basis must be given by $(1, 2, ..., n)$ and the C basis by $(j, 2, ..., n)$ so that we are in the starting position described above. From the starting position there is only one pivot step to be taken, namely to introduce column j into the A basis. From all other positions in which the A and C bases stand in the correct relationship, two pivot steps are available.

These considerations suggest the following algorithm. Start with the bases described above and take the one pivot step that is available. At any other point one of the two possible pivot steps will have been used in reaching that position, so that the only continuation is by means of the one remaining pivot step. There is a unique continuation at each step and the process can only terminate when we pivot into a solution of the problem. There are a finite number of possible positions, and if we never return to the same position the process will inevitably terminate at a set of columns which is simultaneously a feasible basis for $Ax=b$, and an ordinal basis for C.

Cycling is impossible, for if the first position to be repeated is the initial position, this would imply that there are two possible pivot steps from the initial position, which we have seen to be false. On the other hand if the process first repeats at some point other than the initial position, then there would be *three* possible pivot steps proceeding from that point, which is again impossible. The algorithm must therefore terminate in a finite number of steps with a solution to the problem.

This concludes the proof of Theorem 2, and as a consequence, Theorem 1, in the case in which the sets V_S have a finite number of corners for each proper subset S.

5. AN EXAMPLE OF THE ALGORITHM

Let us consider an example of the algorithm applied to a balanced three person game. I will assume that each player by himself can achieve a zero utility level and nothing larger. Each two player V_S will be assumed to have three corners, and since the game is balanced $V_{(123)}$ need not be explicitly considered. The matrix C will consist of twelve columns, three for each two player coalition, and one column for each single player. Each column will have a large arbitrary number entered in those rows corresponding to players not in the coalition, and in order to avoid degeneracy these numbers will be different for different columns:

$$C = \begin{bmatrix} 0 & M_2 & M_3 & 12 & 3 & 2 & 9 & 5 & 4 & M_{10} & M_{11} & M_{12} \\ M_1 & 0 & M_3 & 6 & 7 & 9 & M_7 & M_8 & M_9 & 5 & 2 & 8 \\ M_1 & M_2 & 0 & M_4 & M_5 & M_6 & 3 & 8 & 10 & 6 & 9 & 4 \end{bmatrix}.$$

The M's are arbitrary other than satisfying the inequalities $M_1 > M_2 > \ldots > M_{12} > 12$. The matrix A is the incidence matrix of players versus sets suitably repeated.

In order to avoid degeneracy in the matrix A, the last column has been perturbed by small ε's, subject to $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_3$.

Step 1. We begin with a basis for the matrix A consisting of columns (1, 2, 3) and for C columns (10, 2, 3), so that the u vector associated with this C basis is $u=(M_{10}, 0, 0)$. The first step is to bring the tenth column into the A basis, and the pivot element for this step is printed in bold:

$$A = \left[\begin{array}{cccccccccccc|c} 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1+\varepsilon_1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1+\varepsilon_2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1+\varepsilon_3 \end{array}\right].$$

Column two has been removed from the A basis, and therefore it must be removed from the C basis. In the remaining two columns, column ten has two row minimizers, with the new one in the second row. We therefore examine all columns k, with $c_{2k}>5$, $c_{3k}>0$ and select the one which maximizes c_{1k}. This is column twelve.

Step 2. The A matrix which appears after the first pivot step is

$$\left[\begin{array}{cccccccccccc|c} 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1+\varepsilon_1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1+\varepsilon_2 \\ 0 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & 1 & 0 & 0 & 0 & \varepsilon_3-\varepsilon_2 \end{array}\right].$$

The A basis is (1, 3, 10) and the C basis (12, 3, 10), with a u vector given by $u=(M_{12}, 5, 0)$. We continue by bringing column twelve in the A basis. No calculation is required since column ten, which is identical with column twelve, will be eliminated. Column ten must therefore be eliminated from the C basis. If we consider the submatrix of columns (12, 3, 10),

$$\begin{bmatrix} M_{12} & M_3 & M_{10} \\ 8 & M_3 & 5 \\ 4 & 0 & 6 \end{bmatrix},$$

we see that when column ten is eliminated, column twelve has two row minima, with the new one appearing in the second row. We therefore maximize c_{1k} subject to $c_{2k}>8$, $c_{3k}>0$, and obtain column seven.

The algorithm then proceeds through the following pivot steps.

Step 3. The A basis is (1, 3, 12), the C basis (7, 3, 12) and $u=(9, 8, 0)$. Column seven is brought into the A basis and column three removed. Column three is then removed from the C basis and column eight is introduced.

Step 4. The A basis is (1, 7, 12), the C basis (8, 7, 12) and $u=(5, 8, 3)$. Column eight is brought into the A basis and column seven removed. Column seven is then removed from the C basis and column four is introduced.

Step 5. The A basis is (1, 8, 12), the C basis (4, 8, 12) and $u=(5, 6, 4)$. Column four is introduced into the A basis, and column one eliminated, so that the solution is obtained. Columns (4, 8, 12) form a feasible basis for A and an ordinal basis for C. The utility vector $u=(5, 6, 4)$ cannot be blocked by any two or one player set, and if the game is balanced this vector must be in $V_{(123)}$. Any Pareto optimum point in $V_{(123)}$ which is greater than or equal to (5, 6, 4) must be in the core.

There is a substantial amount of arbitrariness in initiating the algorithm: in the ordering of the M's, in the lexicographical ordering used in the A matrix, and in the determination of a pair of bases which differ in at most one column. The question of which points in the core will be determined by these variations is an intriguing one but one which I would prefer to postpone.

6. SOME GENERAL REMARKS

We have completed the proof of Theorem 1 in the case in which each V_S has a finite number of corners. The general case may be studied by selecting a finite list of vectors from each set V_S, and applying the algorithm to obtain a vector in V_N which cannot be blocked by any vector in the list. If the number of vectors in each V_S is systematically increased, becoming everywhere dense in the limit, we will obtain a sequence of vectors in V_N, and any limit point of this sequence will be blocked by no coalition, thereby demonstrating Theorem 1.

In Section 2 it was shown that in an exchange economy, convexity of preferences implies that the game is balanced and therefore has a nonempty core. If the number of players tends to infinity in an appropriate way, and some additional assumptions are made, the core approaches the set of competitive equilibria. The existence of competitive equilibria has therefore been demonstrated by an argument based on an algorithm rather than the use of fixed point theorems. An alternative procedure, discovered by Mr. Rolf Mantel is to apply Theorem 2 directly to the existence of equilibrium prices without any reference to n person game theory. Mr. Mantel's work appears in his thesis [6].

I would like to suggest two motives for attempting to avoid the abstract proofs of the existence of equilibria that have previously been given. First, the modern treatment of competitive models has tended to focus on problems of existence rather than problems of computation, and this second aspect of the problem is worth examination. Of course a prohibitive amount of information would be required to calculate equilibrium prices for an actual economy, and this will surely continue to be true even if the capabilities of electronic computers were to increase at a fanciful rate in the future. On the other hand experimental calculations on small economic models will, I think, be quite useful, and it is now possible to perform such calculations on models involving the consumer side of the economy.

To be sure, fixed point theorems do provide a method of calculating equilibrium prices in the following sense. Let $f_j(\pi)$ be the excess demand for the jth commodity at prices π. These functions are assumed to be continuous, satisfy the Walras Law $\{\pi \cdot f(\pi)=0\}$ and be homogeneous of degree zero. At an equilibrium system of prices each excess demand is less than or equal to zero, and the property of homogeneity permits us to restrict our attention to prices lying on the simplex $\pi_j \geqslant 0$, $\Sigma \pi_j = 1$. Let the price simplex be divided into a simplicial subdivision and label each vertex of the subdivision with some commodity whose excess demand is less than or equal to zero at that system of prices. Then Sperner's Lemma, the central argument in the proof of fixed point theorems, tells us that there will be at least one small simplex all of whose vertices are labeled differently.

If the subdivision is sufficiently fine, the price system at the center of the distinguished simplex will have excess demands which are less than or equal to zero or if positive the excess demands will be quite small. The price obtained in this way will be close to an equilibrium price in a functional sense though not necessarily in terms of Euclidean distance. The functional sense of distance in which each market is approximately in equilibrium is probably the relevant one, and since the excess demands for a finite number of price vectors can be calculated if consumer preferences and the production set are known, we do have a computing procedure based on fixed point theorems for calculating equilibrium prices with a given degree of accuracy.

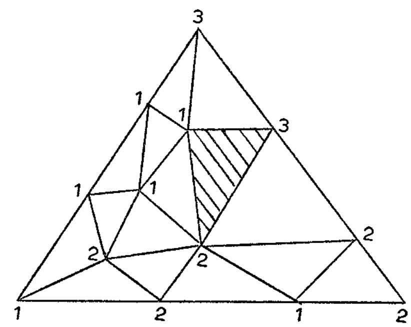

FIGURE 3

When the sets V_S are approximated by sets with a finite number of corners our algorithm will calculate, in a similar functional sense, a point arbitrarily close to the core. Its advantage over the procedure based on Sperner's Lemma seems to me to be that the latter technique is essentially an exhaustive search, and not a systematic algorithm. This is admittedly a vague distinction which can only be clarified by computational experience, but the reader should compare the situation with that arising in linear programming. An exhaustive search of all vertices of a convex polyhedron in order to determine the maximum of a linear function is less efficient as a computational device than the orderly sequence of steps prescribed by the simplex method.

The second advantage of the method discussed in this paper, is in the possibility of its economic interpretation. Since the conventional Walrasian adjustment of prices does not always converge to equilibrium, we have at present no uniformly valid computational procedure for problems involving the consumer side of an economy which is at the same time economically suggestive. An exhaustive search based on Sperner's Lemma has no economic interpretation as a possible adjustment mechanism for arriving at an equilibrium state. On the other hand the method of this paper may be capable of such an interpretation. It proposes at each iteration a utility vector which cannot be blocked by any set of players. If the utility vector is feasible for the set of all players the problem is finished. If not a vector is proposed which differs from the old vector for only two players, one of whom receives more and the other less. The reader will have to judge for himself whether the algorithm has any economic interpretation as an adjustment mechanism, but my feeling is that the possibility of interpretation is distinctly higher than that arising from an exhaustive search.

Brouwer's fixed point theorem itself may be demonstrated quite simply by taking the matrix A as a repeated unit matrix in Theorem 2. Our algorithm can therefore replace Sperner's Lemma in providing the combinatorial basis for fixed point theorems. There is at present, however, insufficient evidence to know whether the algorithm seriously avoids an exhaustive search in approximating the fixed points of continuous mappings.

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REFERENCES

[1] AUMANN, R. J., AND B. PELEG: "Von Neumann–Morgenstern Solutions to Cooperative Games Without Side Payments," *Bulletin of the American Mathematical Society*, 66, (1960) pp. 173–179.

[2] BONDAREVA, O.: "The Core of an N Person Game," *Vestnik Leningrad University*, 17, (1962) No. 13, pp. 141–142.

[3] DEBREU, G., AND H. SCARF: "A Limit Theorem on the Core of an Economy," *International Economic Review*, Vol. 4, No. 3, Sept., 1963.

[4] LEMKE, C. E.: "Bimatrix Equilibrium Points and Mathematical Programming," *Management Science*, Vol. 11, No. 7, May, 1965.

[5] LEMKE, C. E., AND J. T. HOWSON, "Equilibrium Points of Bi-matrix Games," *SIAM Journal*, Vol. 12, July, 1964.

[6] MANTEL, ROLF: "Toward a Constructive Proof of the Existence of Equilibrium in a Competitive Economy," submitted as a thesis to the Department of Economics, Yale University, 1965.

[7] PELEG, B.: *An Inductive Method for Constructing Minimal Balanced Collection of Finite Sets*, Research Program in Game Theory and Mathematical Economics, Memorandum No. 3, February, 1964, Department of Mathematics, Hebrew University, Jerusalem.

[8] SHAPLEY, L. S.: *On Balanced Sets and Cores*, RAND Corp. Memorandum, RM-4601-PR, June, 1965.