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Ascending Auctions with Package Bidding

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Abstract

A family of ascending package auction models is introduced in which bidders may determine their own packages on which to bid. In the proxy auction (revelation game) versions, the outcome is a point in the core of the exchange economy for the reported preferences. When payoffs are linear in money and goods are substitutes, sincere reporting constitutes a Nash equilibrium and the outcome coincides with the Vickrey auction outcome. Even when goods are not substitutes, ascending proxy auction equilibria lie in the core with respect to the true preferences. Compared to the Vickrey auction, the proxy auctions generate higher equilibrium revenues, are less vulnerable to shill bidding and collusion, can handle budget constraints much more robustly, and may provide better ex ante investment incentives.

KEYWORDS: auction theory, FCC auctions, package bidding, combinatorial bidding, activity rule, bid Improvement rule, e-commerce, electronic commerce

1. Introduction

Asset sales and procurement auctions often begin with an evaluation of how to structure the lots that are sold or bought. In asset sales, a farm can be sold either as a single entity or it might be sold in pieces such as the house and barn, the arable land, other land, and perhaps water rights; radio spectrum licenses can cover an entire nation or be split among smaller geographic areas and the available set of frequencies in each area sold as a single unit or in smaller pieces; a large company can be sold intact to new owners or broken up into individual product divisions. Similar decisions are made in procurement auctions, in which a manager might decide to buy needed products and services from a single supplier or to break the purchase into smaller parts. The regulators, brokers, investment bankers, auctioneers and industrial buyers who conduct these auctions commonly consult potential bidders in an effort to identify which packages are most attractive to bidders and best serve the auction designer's revenue, cost or efficiency goals.[1]

Some auction designs allow bidders a choice of packages on which to bid. For example, Cassady (1967) describes a sale of five buildings of a bankrupt real estate firm in which three buildings were defined to constitute a "complex." An ascending auction was used for the sale, with bids accepted for either the individual buildings or for the complex.[2] More subtly, Dutch flower auctions allow winning bidders to take as many lots as they wish at the winning price.[3] Both of these designs encourage direct competition among bidders who seek to buy differently sized packages, which can increase prices compared to auction formats in which the lot size is predetermined.

In recent years, there has been growing interest in allowing bidders much greater flexibility to name the packages on which they bid. The London Transportation authority procures bus service from private operators in a sealed-bid auction that allows bids on all combinations of routes within a particular auction, and 46% of winning bids involve such combinations (Cantillon and Pesendorfer, 2001). The ascending auction planned for Federal Communications Commission (FCC) Auction No. 31, which will sell spectrum licenses in the 700 MHz band, permits bids for any of the 4095 possible packages of the twelve licenses on offer. Other examples include proposed auctions for paired airport take-off and landing slots (Rassenti, Smith and Bulfin, 1982) and for industrial procurement, in which individual sellers may offer all or part of a bill of materials and services sought by buyers (Milgrom, 2000b).

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[1] The revenue and efficiency criteria can lead to quite different choices; see Palfrey (1983). Milgrom (2000a) reports examples in which the sum of total value and auction revenue is constant across packaging decisions, so that there is a dollar for dollar trade-off between creating value and raising revenue.

[2] Sometimes, bidders for large packages are required to specify bids on certain smaller packages as well. An example is an auction one author designed for selling the power portfolio of the Portland General Electric Company (PGE), which was adopted by the company and the Oregon Public Utility Commission. The auction design requires that bidders for the whole package of plants and contracts must also name "decrements" for individual power supply contracts on which there are competing individual bids.

[3] Katok and Roth (2001) report experiments to assess the performance of Dutch auctions in the presence of scale economies.

Package auctions can be either static sealed bid auctions, such as the Vickrey (1961) auction or the Bernheim-Whinston (1986) menu auction, or dynamic (multi-round or continuous) auctions. In the latter case, bidders may bid on a multiplicity of packages (including single-item packages) and may improve their bids or add new packages during the course of the auction. The eventual winning bids are the ones that maximize the total selling price of the goods.

In Section 2, we review the reasons for the recent interest in ascending package auctions. Subsections discuss advances in technology, changes in spectrum regulation, shortcomings of the generalized Vickrey auction, theory and experience with the FCC's simultaneous ascending auction (SAA), and evidence from economics laboratories regarding the performance of an ascending package auction design.

Our formal analysis begins in Section 3 with descriptions of the package exchange problem and our model of the ascending package auction. That section also introduces a particular myopic strategy, which we call "straightforward bidding," which may account for the performance of bidders in certain laboratory settings.[4]

Section 4 explores an associated direct revelation mechanism called the "*ascending proxy auction*." We intend this auction to be both a possible account of bidding in certain experiments and a new auction design in its own right. The insight guiding our analysis is that the ascending proxy auction is a new kind of "deferred acceptance algorithm," having some properties similar to the algorithms studied in two-sided matching theory.[5] To develop this idea in detail, we introduce the coalitional game that corresponds to the package economy. We show that the auction algorithm terminates at a core allocation of the package exchange economy relative to the reported preferences. This result, which corresponds to a familiar one in two-sided matching theory, can be extended to models with budget constraints, using the insight that deferred acceptance algorithms apply equally well to cooperative games without transferable utility.

In section 5, we study bidding incentives and equilibrium in the proxy auction. We define *semi-sincere reports* to be reports that reduce the valuation of all packages by an equal amount relative to the truth (except that negative values are reported as zero). We show that for any pure profile of opposing strategies, each bidder always has a best reply in semi-sincere strategies and that equilibria in such strategies exist. If all bidders adopt such strategies and if losing bidders bid sincerely, then the set of possible equilibrium payoff vectors is identical to the set of bidder-Pareto-optimal points in the core of the cooperative game among the bidders and the seller.

[4] An ascending package auction design with "myopic bidding," similar to our "straightforward bidding," was studied by Parkes and Ungar (2000a). Their work emphasizes the computational advantages of such a design. The same authors (Parkes and Ungar (2000b)) also study proxy agents with a specification that differs substantially from ours.

[5] See Roth and Sotomayor (1990) for an excellent review and survey of two-sided matching theory. Our new algorithm and its analysis differ from all previous matching algorithms and analyses in several important ways. First, our model involves package offers, rather than offers for the individual items. Second, in our algorithm, an offer that is tentatively "rejected" at one round may nevertheless be accepted later as part of the final allocation. Third, much of the analysis of model does not require the assumption of "substitutes," which has been required for all previous matching analyses. Finally, ours is the first model of many-to-one matching in which Nash equilibrium strategies have been explicitly characterized.

Sections 6 and 7 compare the outcomes of the ascending proxy auction and the Vickrey auction. One result identifies the conditions on preferences under which the Vickrey outcome is in the core. Provided that all additive valuations are possible, the Vickrey outcome is guaranteed to be in the core if and only if the goods are substitutes for every possible bidder preference ordering. Under the same conditions, truthful reporting is a Nash equilibrium of the ascending proxy auction and its outcome coincides with the Vickrey outcome. Thus, the most important differences between the designs emerge when goods are not substitutes. In that case, the Vickrey auction outcome entails seller revenues so low that the payoff vector falls outside the core.

Section 8 explores generalizations of the ascending proxy auction that work even when there are limits on monetary transfers. The Vickrey auction generally loses its desirable properties when such limits are imposed. By way of contrast, the ascending proxy auction has extensions that preserve many of its desirable properties. Section 9 concludes.

2. Background

A variety of developments have contributed to the present drive to develop and implement package auctions. These can be grouped into three general categories: rapid advances in technology, favorable developments in regulated spectrum markets and unregulated Internet exchange markets, and research that highlights the potential benefits of package auctions.

2.1 Changing Technology and Markets

The most important group factors contributing to the new package bidding designs is associated with the rapid advance of technology, which enables certain new auction designs. To understand the technical challenge, suppose that bidders submit bids for overlapping packages. Given these bids, the first step of finding the sets of “consistent” bids in which each individual item is included in just one package (“sold just once”) is a hard computational problem. Then, the total bid associated with each such package must be computed and the revenue-maximizing set of “consistent” bids must be found. All this must be done quickly, while bidders sit in front of their individual computer terminals.

To get an idea of the size of the problem, consider the plan for FCC Auction No. 31 of spectrum licenses. The twelve licenses on offer allow for 4095 distinct combinations involving between one and twelve licenses. A decade ago, this number of combinations might have overwhelmed users and posed serious computational problems. Now, however, there are processors, interfaces, algorithms, and communications systems that make it practical for users to identify and bid for many combinations, for auctioneers to compute optimal bid combinations, and for all to verify and track the progress of the auction, even from remote locations.[6]

Even as technology was advancing, markets were changing in ways that facilitate the adoption of sophisticated auction designs. The adoption in 1993 of legislation authorizing spectrum auctions in the US and the bold decision by the FCC the following year to adopt

[6] Some computational aspects of package bidding are surveyed by deVries and Vohra (2001).

the computerized simultaneous ascending auction (SAA) gave an important boost to advocates of more sophisticated auction designs. The perceived successes of spectrum auctions have led some to propose even more ambitious designs.

In Australia, spectrum regulators eager to "let the market decide" all details of the allocation initiated a serious discussion about the sale of "postage stamp" sized licenses. These would entail very small geographic areas and narrow slivers of bandwidth to be licensed and ultimately recombined as desired by spectrum buyers. Those proposals were shelved because of concerns that the complementarity among the licenses might make the auction and subsequent resale markets perform poorly.

Shortly afterward, another area of high-tech applications began to develop as Internet-based businesses raced to develop electronic markets that could serve the needs of business customers. Often, industrial buyers seek to purchase not just single components but all the materials and services for a large project, such as a construction project. Multiple suppliers may each supply part of the buyer's needs on terms that may involve quantity discounts, which make the buyer's procurement optimization problem a non-convex one. If such procurements are to be managed by competitive bidding, then some form of package auction will be needed, and supporting software for these auctions has recently become available.

These developments and others have inspired new research by economists, operations researchers and computer scientists into the theory and practice of package bidding.

2.2 Vickrey Auctions: Advantages and Disadvantages

The theory of package bidding, like so much of auction theory, can be traced to the seminal paper by William Vickrey (1961), which focused on sales involving a single type of good. Vickrey's mechanism can be described as follows. Each bidder is asked to report to the auctioneer its entire demand schedule for all possible quantities. The auctioneer uses that information to select the allocation that maximizes the total value. It then requires each buyer to pay an amount equal to the *lowest* total bid the buyer could have made to win its part of the final allocation, given the other bids. Vickrey showed that, with this payment rule, it is in each bidder's interest to make its "bid" correspond to its actual demand schedule, regardless of the bids made by others. Subsequent work by Clarke (1971) and Groves (1973) demonstrated that a generalization of the Vickrey mechanism leads to the same "*dominant strategy property*" in a much wider range of applications. In particular, Vickrey's conclusion holds even when there are many types of goods, provided the original requirement to make bids for "all possible quantities" is replaced by the requirement to make bids on "all possible packages." This extension has come to be known as the "generalized Vickrey auction" or as the "Vickrey-Clarke-Groves (VCG) mechanism."

The advantages of the VCG mechanism are further confirmed by certain uniqueness and equivalence results. Results by Green and Laffont (1979) and Holmstrom (1979) imply that any efficient mechanism with the dominant strategy property and in which losers have zero payoffs is *equivalent* to the VCG mechanism, in the sense of always leading to identical equilibrium outcomes. Williams (1999) finds that all Bayesian mechanisms that yield efficient equilibrium outcomes and in which losers have zero payoffs lead to the same expected equilibrium payments as the VCG mechanism.

These discoveries had wide ramifications. For some operations researchers, they seemed to reduce the economic problem of auction design to a computational problem. If only one could describe and compute values and allocations quickly, it seemed, then the generalized Vickrey auction would become a practical solution to a wide range of resource allocation problems.

For economists, Vickrey's findings raised expectations about the possibility of designing effective auctions using economic analysis. As we shall see, however, the VCG mechanism has profound defects that make it unlikely that it could be useful for sales of publicly owned assets, even when efficiency is the auction designer's goal.[7]

Efficiency in auction design has several aspects. One is efficiency of the allocation itself, given the preferences of participants when the auction is run. In auctions of public assets, higher revenues also improve efficiency, since auction revenues can displace distortionary tax revenues. A third efficiency issue is the cost of the auction to the various parties. A fourth is the incentives that the auction design creates for related choices, such as pre-auction investments or choices of technology and organization.

Probably the most important disadvantage of the Vickrey auction is that the revenues it yields can be very low or zero, even when the items being sold are quite valuable. For example, consider a hypothetical auction of two spectrum licenses to three bidders. Bidder 1 wants only the package of two licenses, for which it is willing to pay \$2 billion, while bidders 2 and 3 are each willing to pay \$2 billion for a single license. A quick calculation establishes that the Vickrey outcome allocates the licenses to bidders 2 and 3, each at a price of zero! This defect of the VCG mechanism is, by itself, decisive for most practical applications, but there are other important defects as well.

The second and third disadvantages of the Vickrey design are its vulnerability to shill bidding[8] and collusion, even by losing bidders. For example, consider an auction of two identical licenses to two bidders. Bidder 1 is willing to pay \$2 billion for the pair of licenses and nothing for a single license. Bidder 2 is willing to pay \$1 billion for the pair and \$0.5 billion for a single license. In a Vickrey auction, bidder 1 should win both items, for a payment of \$1 billion, and bidder 2 should win nothing. However, suppose that bidder 2 misrepresents both his value and his identity by submitting bids under two separate names. Bidder 2 bids \$2 billion for one item, while his shill bidder 3 separately bids \$2 billion for one item. As in the previous example, bidders 2 and 3 together win both items, paying zero in the Vickrey auction. The shill strategy profitably converts losing bidder 2 into a winning bidder. If we modify this example to replace bidder 2 by

[7] Parts of the following discussion of disadvantages of the Vickrey auction are drawn from a report to the FCC by Charles River Associates and Market Design Inc (1997). The reports to the FCC and related papers were presented at a conference sponsored by the FCC, the National Science Foundation, and the Stanford Institute for Economic Policy Research. See http://wireless.fcc.gov/auctions/conferences/combin2000/papers.html.

[8] Sakurai, Yokoo and Matsubara (1999) and Yokoo, Sakurai and Matsubara (2000) refer to bids by a buyer shill as "false-name bids." They argue that in situations (such as Internet auctions) where the auctioneer cannot completely determine the identities of bidders, the auctioneer must be wary of whether bidders can profit by submitting additional bids under false identities.

two bidders each of whom is willing to pay \$0.5 billion for one license, then those bidders could profitably collude by both raising their bids to \$2 billion, becoming winners and paying prices of zero.[9]

Notice that these first three weaknesses are illustrated in examples in which the licenses are not all substitutes, because bidder 1 has value only for the entire package. If we modified the examples to make the licenses substitutes, for example, by specifying that bidder 1 was willing to pay \$1 billion for each license rather than just \$2 billion for the pair, then the identified problems would vanish. The seller would get a total payment of \$2 billion in the first example, which is a *competitive* payment in the sense that the payoff outcome lies in the core.[10] In the shill bidding and collusion examples, changing bidder 1's values in this way forces the others to pay at least \$2 billion in total to win, which makes shill bidding and loser collusion unprofitable. The distinction between valuations that are substitutes and the remaining cases plays a decisive role in the general theory we develop.

A fourth weakness of the Vickrey auction is that its dominant strategy property depends on unlimited bidder budgets.[11] To illustrate, consider an auction in which a certain bidder A has values of \$1 billion for a single license or \$2 billion for two licenses based on the profits it can earn, but can spend only \$1.2 billion to buy licenses. Its two competitors, bidders B and C, each want only one license. Bidder B is willing to pay \$800 million for a license. Bidder C may be willing to pay \$1.1 billion or it may have access to a substitute license from another source, in which case it will bid zero. In a Vickrey auction, bidder A should win either two licenses or one, depending on the last bidder's decision. In either case, its total payment will be \$800 million, so its budget is always adequate to make its Vickrey payment. Yet, if A's budget limit constrains it to bid no more than \$1.2 billion for any package, then it has no dominant strategy. For example, suppose that bidder B plays its dominant strategy, bidding \$800 million for a single license. Then, if bidder C bids zero, A's best reply requires that it bid less than \$400 million for a single license (and, say, \$1.2 billion for the package). If instead bidder C bids high, then A's best reply entails bidding more than \$800 million for a single license. Since these are inconsistent, A has no dominant strategy.

The first three weaknesses described above are all remedied by our basic ascending proxy auction, and the fourth is remedied by a variation on that auction. The remaining weaknesses, described below, are mitigated in various degrees by the ascending proxy auction, but they are not completely resolved.

In the first example above, the Vickrey auction creates a significant bias in decisions about technology and organization. For suppose bidders 2 and 3 could merge, adopt a coordinated technology, and increase the total value of the licenses from \$4 billion to \$5 billion. Despite the value created by the merger, the parties would find this merger

[9] This collusion example is based on a suggestion by Jeremy Bulow.

[10] In the unmodified example, the Vickrey outcome was not in the core. The coalition consisting of the seller and bidder 1 could "block" the Vickrey outcome since, by themselves, they earn a coalition payoff of \$2 billion, but the Vickrey outcome gave them only a payoff of zero.

[11] Che and Gale (1998) analyze revenue differences among first-price and second-price auctions in the presence of budget constraints.

unprofitable, because the amount they would have to pay in the Vickrey auction would rise from zero to $2 billion. Total profits would therefore fall from $4 billion to $3 billion, discouraging the efficient reorganization.

Another potentially significant issue is the cost of determining valuations. A traditional assumption in auction theory analyses is that each bidder knows all its values or can compute them at a zero cost. For package auctions, the sheer number of combinations that a bidder must evaluate makes that assumption especially dubious. Compared to most of the other costs involved in conducting combinatorial auctions, bidder valuation costs are relatively less affected by advancing technologies, particularly when the asset valuation process requires substantial human inputs.[12] Potential buyers who find it too expensive to investigate every packaging alternative will instead choose a few packages to evaluate fully. Ideally, auction design should account for the way those choices are made as well as the evaluation costs that bidders incur.

When package evaluation is costly, the interacting choices that bidders make about which packages to evaluate further complicate the analysis. For example, suppose the items offered in an auction are {ABCD}. It is unlikely to be an effective strategy to bid for package AB unless someone else is bidding either on CD or on C and D separately. In a Vickrey auction or any sealed bid auction, if it is too costly to evaluate all the packages, then bidders must guess about which packages are most relevant and how to allocate their limited evaluation resources. In comparison, a multi-round ascending auction economizes on the need to guess because bidders can adapt their plans based on observations made during the auction.[13]

Another characteristic of the Vickrey auction that is sometimes considered a drawback is its use of *explicit* price discrimination: two bidders may pay different prices for identical allocations, even when both have made the same bids for those allocations.[14] Such discriminatory pricing may be illegal and is often regarded as "unfair."

[12] The nature of human inputs to the valuation process is a matter of controversy among auction consultants. In our experience, valuing significant business assets involves both investigating the asset itself and creating business plans showing how they will be used. For example, a bidder hoping to purchase parts of an electrical generating portfolio might investigate the physical condition of each plant, the availability of land and water for cooling to allow plant expansion, actual and potential transmission capacity, and other physical variables. In addition, it will consider labor and contractual constraints, zoning and other regulatory constraints, the condition of markets in which power might be sold, partnerships that might enhance the asset value, and so on. The final valuation is the result of an optimization over business plans using all this information, and tempered by human judgment. When the assets in the collection interact in complex ways that affect the optimal business plan, then significant extra costs must be incurred to evaluate each package.

[13] This topic has recently attracted the attention of several researchers. See Parkes, Ungar and Foster (1999), Compte and Jehiel (2000), and Rezende (2000). See also the discussion of advantages of ascending auctions over sealed-bid auctions in the Introduction of Ausubel (1997a).

[14] To illustrate the price discrimination problem, suppose there are two bidders—A and B—and two items—X and Y. A valuation for a bidder is a triple (x,y,z), specifying how much the bidder would be willing to pay for item X alone, item Y alone, and the package XY. Suppose the parties report valuations of (12,12,13) and (12,12,20). The result is that A and B will each be awarded an item (at an efficient allocation, either bidder may get either item) at prices of 8 and 1 respectively, even though the items are perfect substitutes and bidders A and B made identical bids for the individual items.

Most favorable evaluations of Vickrey auctions depend sensitively on assumptions that each bidder knows its own values (the "private values" model) and that participation is exogenous. For sales of a single good, the revenues from second price auctions can be nearly zero when there is "common value" or "almost common value" uncertainty (Milgrom (1981), Klemperer (1998)) or endogenous entry decisions (Bulow, Huang and Klemperer (1999)). Standard sealed bid auctions do not share this fault.

Finally, the fact that Vickrey auctions require bidders to report their valuations has been regarded as problematic (Rothkopf, Teisberg and Kahn (1990)), at least for non-computerized auctions in which secure encryption technologies are not available. Winning bidders may fear that information revealed by their bids will be used by auctioneers to cheat them or by third parties to disadvantage them in subsequent negotiations. Similarly, the public has sometimes been outraged when bidders for government assets are permitted to pay significantly less than their announced maximum prices in a Vickrey auction (McMillan (1994)). A bidder's motive to conceal its information can destroy the dominant strategy property that accounts for much of the appeal of the Vickrey auction.[15]

These various defects of the Vickrey auction have led to increased interest in exploring alternative designs, such as multiple round pay-as-bid auctions. The *multiple rounds* feature provides feedback to bidders about relevant packages, economizes on bidder evaluation efforts, and conceals the winning bidder's maximum willingness to pay. The *pay-as-bid* feature avoids the low revenue outcomes of the Vickrey auction and discourages shill bidding and some kinds of collusive strategies. The two features combined turn out to also alleviate problems associated with budget constraints. The next sections consider two such designs: the simultaneous ascending auction (SAA), which entails *no* package bidding, and the simultaneous ascending auction with package bidding (SAAPB).

2.3 Simultaneous Ascending Auctions

The simultaneous ascending auction, which has been employed by the FCC in almost all U.S. radio spectrum auctions (and, with minor variation, in most other spectrum auctions worldwide), differs from the Vickrey auction in two ways that have made it an attractive practical alternative to the Vickrey auction: it is a *pay-as-bid*, *multiple round* auction design.

The auction design has an iterative structure, with the "state of the auction" after each round described by the identities of the standing high bidders and the amounts of the standing high bids for each item. Initially, the standing high bid for each item is zero[16] and the standing high bidder is the seller. During each round, bidders may raise the bid by

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When the items are not identical, the price discrimination is not so obvious, but the auction outcome is not generally "envy free": a bidder may prefer the price and allocation assigned to another bidder and may complain on that basis.

[15] A similar case can be made against ordinary first-price auctions, since the theoretical bid functions are invertible to reveal bidders' values. In this respect, ascending auctions are theoretically superior to both kinds of sealed bid auctions because they better conceal the winning bidder's valuation.

[16] A reserve price may also be used.

an integer number of increments on any items that they wish, which determines new bidders and standing high bids. The process repeats itself until there is a round with no new bids on any item. At that point, bidding on all items is closed and the standing high bids determine the prices. As described earlier, there is also an activity rule designed to ensure that bidding activity starts out high and declines during the auction as prices rise far enough to discourage some bidders from continuing.

Although early experimental testing of the SAA demonstrated that it performed well in some environments possibly resembling the FCC environment (Plott (1997)), it has a variety of theoretical limitations. One important issue with the SAA is the incentive that it provides for *demand reduction* (Ausubel and Cramton (2002)). For example, consider an auction of two identical licenses to two bidders. Bidder 1 values a single license at $3 billion and the pair of licenses at $6 billion. Bidder 2 values a single license at $2 billion and has no additional value for the pair. The seller might hope that the SAA would lead to a price of $2 billion (+ ε) for each license, with bidder 1 winning both. However, winning one license at a price of zero (which yields $3 billion in profits) is preferred by bidder 1 to winning two licenses at a price of $2 billion each (which yields $2 billion in profits). Thus, bidder 1 strategically withholds his demand for the second license and bids for only a single license, causing the auction to conclude at prices of zero. This outcome is unsatisfactory both in its inefficiency and low revenues. By contrast, the ascending proxy auction of this paper and the Vickrey auction both avoid the inefficiency of demand reduction in this example by allowing bidder 1 to win the package of two licenses for a price of $2 billion. The associated revenue is high enough that the outcome is a *core outcome*, that is, the seller cannot do better by reneging on the auction terms and instead negotiating with the losing bidder.

Another important limitation of the SAA is its degraded performance in laboratory experiments in which the substitutes condition fails (Ledyard, Porter and Rangel (1997)), a condition that may have applied to the radio spectrum auctions (Ausubel, Cramton, McAfee and McMillan (1997)). To explain the role of the substitutes condition in theoretical terms, we compare two different situations.

In the first situation, the items for sale are substitutes[17] for all the bidders. In addition, the bid increment is "small" and the initial prices are low enough to attract at least one bid during the auction for every item. In such cases, suppose bidders bid "straightforwardly" at each round for the items in a package they most prefer at the current prices. Then, the final allocation is efficient and the final prices are competitive

[17] Generally, goods are substitutes when increasing the price of one does not reduce demand for the other. The modified terms "gross substitutes" and "net substitutes" are often used to distinguish between substitutes for uncompensated and compensated demand, respectively. Since bidders in our auction model (and most others) have quasilinear utilities, there is no distinction between "gross" and "net" so we refer simply to "substitutes."

The substitutes condition has figured prominently in multi-good auction theory since its use by Kelso and Crawford (1982) to evaluate an auction model of the labor market. Gul and Stacchetti (1999) characterize the preference orderings satisfying the condition. Lehmann (private communication) has shown that in models with at least three discrete goods, the substitutes condition holds on a lower-dimensional subspace of the space of possible valuation vectors. In particular, that establishes that the condition is non-generic.

Table 1: Bidder Values

	Item A	Item B	Package AB
Bidder 1	a	b	$a+b+c$
Bidder 2	$a+\alpha c$	$b+\alpha c$	$a+b$

The example uses $c>0$ and $0<\alpha<1$.

equilibrium prices for an economy with "almost" the same values as the actual economy, differing by at most the relevant bid increment (Milgrom, 2000a).

The preceding result entails several interesting conclusions for the case when goods are substitutes. First, market-clearing prices do exist, despite the indivisibility of the items offered for sale. Second, the information communicated during the course of the SAA is rich enough to allow the auction algorithm to discover equilibrium prices and allocations. Third, the auction algorithm can recover from some kinds of anomalous bidding behavior early in the auction. Starting from *any* prices that are below the minimum equilibrium price vector, if the bidders bid straightforwardly from that point forward, then the sequence of prices and allocations converges to the equilibrium prices and allocation.

The preceding conclusions change drastically in the second situation, where some items may be complements.[18] Indeed, let S denote the set of valuations in which bidders regard the items as substitutes. If T is *any strict superset* of S and provided that there are at least two bidders, there exists a profile of valuations drawn from T such that no competitive equilibrium exists (Milgrom, 2000a).

Intuition for this result is provided by Table 1, which tabulates bidder values. In the table, bidder 1's values are any ones for which the two licenses are complements. Bidder 2's values are then constructed so that (1) the items are substitutes for bidder 2 and (2) the unique efficient outcome is for bidder 1 to win both licenses. By the first welfare theorem, if a competitive equilibrium exists, then the outcome is efficient. Suppose it is so and let p_A and p_B be the equilibrium prices. Since bidder 2 does not demand either good at equilibrium, it must be true that $a+\alpha c \le p_A$ and $b+\alpha c \le p_B$. On the other hand, since bidder 1 purchases both goods at equilibrium, we may infer that $a+b+c \ge p_A + p_B$. For $\alpha>½$, these inequalities are inconsistent: the contradiction establishes that no market clearing prices exist.

In the example just described, if bidder 1 does not know whether $\alpha>½$, then it may face a difficult bidding problem. Suppose that its competitor, bidder 2, bids straightforwardly. To win both items, bidder 1 will have to bid more than a for item A and more than b for item B. Consider the situation at any time after the prices exceed a

[18] The idea that price formation processes behave drastically differently in the cases of substitutes and complements has a long history in economics. Arrow, Block and Hurwicz (1959) first established the global stability of *tatonnement* in the case of "gross" substitutes (substitutes using uncompensated demand functions). Milgrom and Roberts (1991) showed that the same sort of stability holds over a vast family of discrete and continuous time, synchronous and asynchronous, backward- and forward-looking price-setting processes. Scarf (1960) provided examples of global instability in the case when the goods are complements, sharply contrasting with the case for substitutes.

and b and bidder 2 places a bid on, say, item A. If bidder 1 stops now, it will wind up acquiring only item B for a price greater than its value b to the bidder. If $\alpha > ½$ and if it continues to bid, it will eventually find that the total price exceeds $a+b+c$. At that point, it has no hope of avoiding a loss. For an efficient outcome always to emerge, bidder 1 must always take this risk and always decide in this situation to stop bidding and accept the loss from acquiring a single item, but this is rarely the optimal strategy. Thus, one might expect to find inefficient outcomes and bidder losses as common occurrences in circumstances like this one.

The problem facing a bidder for whom goods are sometimes complements has come to be called the "exposure problem." In 1994, the FCC adopted a rule permitting bid withdrawals under some circumstances to mitigate it. Experimental evidence (Porter, 1997) suggests that withdrawals do mitigate the problem, but they do not solve it completely.

2.4 Experimental Evaluation of Ascending Auction Designs

Economic experimentation, as well as theoretical analysis, has played a crucial role in the growing interest in ascending auctions with package bidding. There is a venerable line of experimental research suggesting that auctions may enable bidders to solve relatively complex multi-item assignment problems.

Rassenti, Smith and Bulfin (1982) are often credited with the first experimental study of package auctions. They studied the problem of allocating airport time slots, a natural application for package auctions given that landing and takeoff slots are strong complements. The experimental subjects were assigned valuations for packages of items and were instructed to participate in a sealed-bid auction in which package bids could be submitted. The winning bids were selected so as to maximize the sum of the prices bid, and bidders were charged uniform prices intended to approximate shadow prices. While sincere bidding is not incentive compatible in this mechanism, more than 90% of the available gains from trade were typically realized in the experiments.

Banks, Ledyard and Porter (1989) conducted an early and influential study of ascending package auctions. These authors developed two dynamic procedures in which bidders submit package bids and the winning bids are selected so as to maximize the sum of the prices bid: an iterative Vickrey-Groves mechanism, in which payments are second-price in the same sense as in the Vickrey auction; and an adaptive user selection mechanism, in which payments are first-price in the sense that winning bidders pay their final bids. Experimental subjects were assigned valuations for packages of items and participated in these ascending package auctions, as well as some alternative procedures meant to represent administrative processes and markets. The ascending package auctions outperformed the alternatives, on average realizing 80% of the available efficiencies.

Brewer and Plott (1996) investigated a problem of allocating the usage of railroad tracks. While their experiments involved a dynamic mechanism that did not utilize package bids, the mechanism was similar in flavor (in that it required the auctioneer to calculate the revenue-maximizing feasible allocation at each iteration) and it was also aimed at solving a complex fitting problem. Their experiments on average realized over 97% of the available efficiencies.

Table 2: Findings of the Cybernomics Experiment

Complementarity Condition:	None	Low	Medium	High
<u>Efficiency</u>				
SAA (No packages)	97%	90%	82%	79%
SAAPB	99%	96%	98%	96%
<u>Revenues</u>				
SAA (No packages)	4631	8538	5333	5687
SAAPB	4205	8059	4603	4874
<u>Rounds</u>				
SAA (No packages)	8.3	10	10.5	9.5
SAAPB	25.9	28	32.3	31.8

Recently, a large experimental study was sponsored by the FCC and conducted by Cybernomics (2000), comparing the performance of the SAA to that of a particular combinatorial auction called the simultaneous ascending auction with package bidding (SAAPB). The major findings of that study are summarized in Table 2.

The study was conducted under four experimental conditions. In the first, a bidder's value for any package was equal to the sum of its values for the individual items in the package. This condition involves no complementarities. The remaining three conditions involved increasing amounts of complementarity, labeled low, medium and high. Bidder values were drawn at random for each experimental condition and were used twice, once for a group of subjects participating in the non-package auction—the SAA—and once for a group participating in the package auction—the SAAPB. Efficiency in the study was measured by the ratio of the total value of the allocation resulting from the auction to the maximum of that total over all possible allocations.

The experimental results show several prominent features. First, the measured efficiency of the SAA falls off markedly as complementarities increase, but the efficiency of the package auction is largely unaffected by complementarity.[19] Second, the SAAPB took roughly three times as many rounds to reach completion, compared to the SAA. In addition, revenues are higher in all conditions for the SAA compared to the SAAPB.

All experiments require making implementation choices that may affect the experimental outcome. For that reason, experimental results are most convincing when similar results are obtained under a variety of relevant conditions. In the present case, the Cybernomics experiment was constructed to involve a simple kind of complementarities that make it relatively easy to compute optimal allocations. This is problematic for two reasons: the selected complementarities appear unlikely to reflect those in the structure of the actual FCC auction and, viewing the package auction as an optimization algorithm, the relative simplicity of the package optimization problem might have influenced the experimental outcome.

[19] In the experiments, the ascending package auction generates higher efficiency than the SAA even when there are no complementarities, although the reported difference is small. The theoretical analysis for the case of substitutes provides a possible explanation: the straightforward bidding that supports efficient allocations is incentive-compatible in the package auction, but not in the SAA.

The Cybernomics experimental environment may also have offered less scope for strategic manipulation of the rules than the real-world FCC auction environment. There are several reasons to suspect this. First, the experimental subjects' lack of information about other bidders' values is not typical of FCC spectrum auctions and makes it harder for them to exploit the strategic opportunities that the auction affords. Compounding this is the fact that rounds were relatively short, affording subjects little opportunity to evaluate others' bids and assess the strategic opportunities. Third, the relatively long training sessions that subjects required seemed to highlight their difficulty in understanding the rules, further limiting their ability to exploit gaps in the rules. Long as these sessions were, they fall far short of the preparation undertaken by bidders in the FCC auctions, where the stakes are also very much higher. Finally, unlike bidders in the FCC auction, subjects in the experiments had no access to expert assistance or to analyses that could pinpoint opportunities for strategic bidding.

Despite these limitations, the history of successes of various package auctions in laboratory settings from Rassenti, Smith and Bulfin (1982) to Cybernomics (2000), is striking. In the next several sections, we provide a theoretical analysis that seeks to account for the success of the package auction experiments and to explore more generally the strategic opportunities that such auctions create.

3. An Ascending Package Auction

Let there be finite number N of types of items to be sold and let $M=(M_1,\ldots,M_N)$ denote the number of items of each type. A *package* or *bundle* $z=(z_1,\ldots,z_N)$ is a vector of integers whose components indicate the number of units of each type in the package. The relevant packages in the auction are those for which $0 \le z \le M$; let $[0,M]$ denote this set.

An important special case arises when M is a vectors of 1's, meaning that the auction treats each item as unique. This condition has been common in the FCC auctions, including the planned package Auction No. 31. Although there is no loss of generality in limiting attention to this case, there is a practical reason to group together identical items. Setting $M_k > 1$ reduces the number of possible bids—sometimes drastically so. For example, if there is only one type of item, then a bidder can completely describe its values by reporting just M_1 bids, instead of the 2^{M_1} bids needed to specify a separate value for each possible package.

The set of auction participants L consists of the seller, designated by $l = 0$, and the buyers, designated by $l = 1,\ldots,|L|-1$. Each buyer l has a valuation vector $v_l = \left(v_l(z) : z \in [0,M]\right)$, where $v_l(z)$ specifies the "value" of package z to bidder l. We limit and simplify our analysis by the following assumptions:

(i) *Private values*: each bidder l knows its own value vector v_l; it does not change its value when it learns about what others are willing to pay.

(ii) *Quasilinear utility without externalities*:

a. A bidder l that acquires package z and pays price $b_l(z)$ earns a net payoff of $v_l(z)-b_l(z)$, which does not depend on what l's competitors acquire.

b. A bidder l that acquires nothing and pays nothing earns a net payoff of zero: $v_l(0)=0$.

(iii) *Monotonicity* (*Free Disposal*): For all l and $z \le z'$, $v_l(z) \le v_l(z')$.

(iv) *Zero seller values*: $v_0(z) \equiv 0$.

Assumption (i) rules out various kinds of "value interdependencies," including especially possible "common value" elements (Milgrom and Weber, 1982), in which the factors that affect value are the same for various bidders but in which bidders have different estimates of those values. In the spectrum auctions, the relevant factors include technology and demand forecasts. Rational bidders should often respect common value estimates made by competitors' analysts as much as or more than the estimates made by their own analysts—a fact that can have a profound impact on bidding strategy, but we omit that issue in the present analysis.

Assumption (ii) limits bidders' payoffs to be linear in money, as is standard in auction theory. In addition, it excludes a class of externality issues that was first emphasized by Jehiel and Moldovanu (1996). Particularly, in spectrum auctions, buyers do interact after the auction and these interactions could influence bidding behavior.[20] For the most part, we abstract from these issues to make progress on other important aspects of the auction design; however, the generalized ascending package auction formulated in Section 8 may enable the relaxation of Assumption (ii).[21]

The last two assumptions are innocuous and standard.

One very important modeling/design issue is the form that package bidding may take. In our analysis, we assume *full mutual exclusivity*: bidders are free to make mutually exclusive bids on as many packages as they wish. This is essentially the same rule as is used in the generalized Vickrey auction, and it is the most flexible form of package bidding. Such a rule imposes no restrictions on what packages the bidder may name or on what amounts it may bid for different packages.

In practice, other rules governing packages have sometimes been adopted in which certain bids from a single bidder are not taken to be mutually exclusive. Allowing such bids *in addition to* all mutually exclusive bids would have no consequences for our theoretical analysis, for allowing such bids merely enriches the language in which the complete bid vector can be expressed.[22] However, allowing such bids *in lieu of* mutually exclusive bids could restrict the bidders' options and prevent the theorems of this paper

[20] See also Jehiel and Moldovanu (2001) and the references therein, as well as Das Varma (2000a&b).

[21] Our auction can be adapted to apply to public decisions in much the same fashion as the Bernheim-Whinston (1986) "menu auctions."

[22] Nissan (1999) investigates the expressive power of various "languages" for package bidding, supposing that the objective of a bidding language is to express the richest possible set of plausible preferences as succinctly as possible.

from applying. A bidder who wished to buy package *A* or package *B*, *but not both*, might find its options limited.[23]

Many more details, including some that are left unspecified in standard game theoretic analyses,[24] are needed to complete the rules of a practical auction design. The rules that figure into our analysis are the following ones.[25]

First, all bids are firm offers. A bidder can never reduce or withdraw a bid it has made on any package. Any new bid a bidder makes on any package z must be positive and must exceed the bidder's best previous bid on z by some integer number of bid increments.

Second, after each round, the auctioneer identifies a set of "provisionally winning bids." This is the set of bids that maximizes the total price, subject to two kinds of constraints: at most M_n items of each type n may be sold, and each bidder may be associated with at most one provisionally winning bid. We further suppose that, as in the FCC design, the auctioneer announces the full history of winning and losing bids after each round (although straightforward bidders will not utilize all that information).

Third, the auction proceeds in a sequence of bidding rounds until a round elapses in which no new bids are submitted by any bidder.[26] The auction then ends and the provisionally winning bids at that time become the winning bids in the auction.[27]

A final modeling decision is whether bidders are free to bid directly in the auction or whether their bidding is required to be intermediated through "proxy agents" that may place constraints on the form that bids may take. Sections 3.1 – 3.3, below, will allow bidders to bid directly, while Section 3.4 *et seq*. will introduce proxy bidding constraints.

Several differences between the ascending package auction and the SAA merit highlighting. First, the minimum bids can differ *among bidders* on any item or package.[28] Second, a bid that was a losing bid at round t can become a provisional winner at a *later* round, such as round t+1. This is illustrated in Table 3 below by the bid of 5 by bidder Y, which becomes a provisional winner in round R+1 even though it was not one in round R. Third, the price of an item or package can *decrease* from round to round. This is

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[23] The package bidding rules for FCC Auction No. 31 include a combination of exclusive and non-exclusive bids. Bids made in different rounds are treated as mutually exclusive, but bids made in the same round are not. A bidder who prefers its bids made in different rounds to be treated as non-exclusive can accomplish that by "renewing" old bids in the current round. In our model in this paper, an ability to make mutually exclusive bids in different rounds is sufficient to imply all of our results (provided that there are no activity rules requiring multiple bids within a round).

[24] The details omitted in conventional game theoretic analyses include how long each bidder has to submit its bid, the design of the user interface, how much discretion the auctioneer has to make exceptions, and many more.

[25] Some aspects of the ascending package auction technology are described in greater detail in Ausubel (1999, 2000) and Ausubel and Milgrom (2001).

[26] The rules for FCC Auction No. 31 specify that the auction proceeds in a sequence of bidding rounds until *two consecutive rounds* elapse with no new bids.

[27] In contrast to the rules for both the simultaneous ascending auction and FCC Auction No. 31, no "activity rules" are included in the present model. The discussion following Theorem 3 indicates the need for either activity rules or proxy bidders to quicken the pace of ascending package auctions

[28] In this respect, the theory of package auctions resembles the theory of public goods, in which there may not exist anonymous prices that support an efficient allocation.

Table 3: Sample Rounds in a Package Auction

	Item A	Item B	Package AB
Round *R*			
Bidder X	4	0	0
Bidder Y	5	0	0
Bidder Z	0	0	6
Provisional Winning Bids	-	-	6
Round *R*+1			
Bidder X	4	2	0
Provisional Winning Bids	5	2	-
Round *R*+2			
Bidder Y	5	6	0
Provisional Winning Bids	4	6	-

illustrated in the table by the fall in the price of Item *A* from 5 in round *R*+1 to 4 in round *R*+2. In the SAA, prices for individual items can never fall. The fact that bids may change from losing to winning explains why, in the SAAPB, it is necessary to specify whether bids from previous rounds remain binding on bidders.

These may appear to be complicating features: they make the auction less transparent for onlookers and might seem to create certain new strategic bidding issues. Nevertheless, without these features, straightforward bidding would not generally lead to such nearly efficient outcomes, as described in the following sub-sections.

3.1 Bidding Strategies

Let H_t denote the list, or "history" of bids made by all bidders up to and including round t (t = 1, 2, …). Let $B_l^t(z) = B_l(H_t, z)$ denote the highest bid made by bidder l for package z up to and including time t and let $B_l^0(z) \equiv 0$. We assume for simplicity that the seller sets reserve prices of zero for all packages.

A bidding strategy b_l for any bidder l is a map from histories to new bids that satisfies the minimum bid restriction that, for every package z, $b_l^t(z) > B_l^{t-1}(z) \Rightarrow b_l^t(z) \geq B_l^{t-1}(z) + I_l^{t-1}(z)$, where $I_l^{t-1}(z) \equiv I_l(H_{t-1}, z)$ is the bid increment applicable to bidder l for package z at round t. One may equivalently describe l's strategy in terms of the function B_l, requiring that $B_l^t(z) \geq B_l^{t-1}(z)$ and that $B_l^t(z) > B_l^{t-1}(z) \Rightarrow B_l^t(z) \geq B_l^{t-1}(z) + I_l^{t-1}(z)$.

3.2 Round by Round Optimization

An *allocation* is a vector $x = \left(x_l \in \mathbb{R}_+^N; l \in L \setminus 0\right)$, where x_l denotes the package or bundle assigned to bidder l. The allocation is "*feasible*" if:

$$\begin{aligned} &\textstyle\sum_{l \in L \setminus 0} x_l \leq M \\ &x_l \in \mathbb{Z}_+^N, \text{ for all } l \in L \setminus 0. \end{aligned} \tag{1}$$

The first constraint is the usual resource constraint limiting the quantities available. The second constraint says that each bidder's package is described by a vector of non-negative integers. Let $X = \{x|x$ satisfies (1)$\}$ be the set of feasible allocations.

The provisional winning allocation for round t, x^{*t}, maximizes the sum of the provisionally accepted bids:

$$x^{*t} \in \arg\max_{x \in X} \sum\nolimits_{l \in L \setminus 0} B_l^t(x_l). \tag{2}$$

Let us assume that in case there are multiple optima in (2), there is some fixed tie-breaking rule that depends only on the vector of best bids $B^t = \left(B_l^t(x_l); l \in L, x_l \in [0, M]\right)$.

3.3 Straightforward Bidding and Limited Straightforward Bidding

We now investigate a strategy in which bidders bid "straightforwardly" at each round on the package that has the highest profit potential. Professor Charles Plott has called this strategy "bidding the gradient" and observed that it is consistent with the behavior of some bidders in his package auction experiments.

The lowest price that l can bid for any package x_l at round t is l's highest bid from the previous round if l was the provisional winner, or otherwise that bid plus one increment:

$$\underline{B}_l^t(x_l) = \begin{cases} B_l^{t-1}(x_l) & \text{if } x_l = x_l^{*t-1} \\ B_l^{t-1}(x_l) + I_l^{t-1}(x_l) & \text{otherwise} \end{cases} \tag{3}$$

Bidder l's potential profit from bidding on x_l at round t is $\hat{\pi}_l^t(x_l) = v_l(x_l) - \underline{B}_l^t(x_l)$ and its "optimal potential profit" is $\pi_l^t = \max(0, \max_{x_l} \hat{\pi}_l^t(x_l))$. The *straightforward bidding strategy* $\hat{B}_l(\bullet | v_l)$ is the strategy in which l makes new bids only on the packages with the optimal potential profit and makes the minimum bid $\underline{B}_l^t(x_l)$ on each such package. The strategy can be described mathematically by:

$$\hat{B}_l^t(x_l \mid v_l) = \begin{cases} \underline{B}_l^t(x_l) & \text{if } \hat{\pi}_l^t(x_l) = \pi_l^t \\ B_l^{t-1}(x_l) & \text{otherwise} \end{cases} \tag{4}$$

<u>Theorem 1</u>. Let $\varepsilon > 0$ be an upper bound on the bid increments used during the auction. If bidder l plays the straightforward strategy throughout the auction, then

(i) for all rounds t and packages x_l, $| B_l^t(x_l) - \max(0, v_l(x_l) - \pi_l^t) | \le \varepsilon$, and

(ii) at the final round T, either l is a provisionally winning bidder or $\pi_l^T = 0$.

Properties (i) and (ii) of Theorem 1 characterize the main implications of straightforward bidding and provide the means to evaluate how well the model fits behavior found in experiments. It holds that, approximately, bidders bid for the same profit on every package within an auction round and that losing bidders stop only when that "target profit" is nearly zero. Empirically, one may measure the distance between the strategies actually played and straightforward strategies by finding the smallest ε for which statements (i) and (ii) are true, and then use that to make predictions about how well the auction will replicate the other properties derived below.

It will prove convenient to embed the straightforward strategies in a larger class of *limited straightforward bidding strategies*—ones in which bidder l bids straightforwardly except that it makes no bids that have a potential profit less than some target amount $\tilde{\pi}_l \geq 0$. The limited straightforward bidding strategy $\hat{B}_l(\bullet | v_l, \tilde{\pi}_l)$ is described similarly to straightforward bidding, using the equation:

$$\hat{B}_l^t(x_l \mid v_l, \tilde{\pi}_l) = \begin{cases} \underline{B}_l^t(x_l) & \text{if } \hat{\pi}_l^t(x_l) = \max\left(\tilde{\pi}_l, \max_{x_l} \hat{\pi}_l^t(x_l)\right) \\ B_l^{t-1}(x_l) & \text{otherwise} \end{cases} . \tag{5}$$

3.4 The Ascending Proxy Auction

In much of the remainder of this paper, we focus attention on a particular version of the model, in which each buyer l instructs a "proxy agent" that bids on its behalf. The agent for bidder l accepts as input a valuation profile $\tilde{v}_l$ and bids straightforwardly according to that profile. [29] Observe that a bidder can utilize this proxy agent to implement either a straightforward bidding strategy or a limited straightforward bidding strategy. (Straightforward bidding is implemented simply by the "sincere strategy," i.e., truthfully instructing the proxy: $\tilde{v}_l(x_l) = v_l(x_l)$. Limited straightforward bidding is implemented by the "semi-sincere strategy" in the proxy auction, i.e., shading the proxy instructions accordingly: $\tilde{v}_l(x_l) = \max\{v_l(x_l) - \tilde{\pi}_l , 0\}$.) Using a proxy converts the game into a direct revelation mechanism.

We further limit attention to the case in which bid increments are negligibly small. In that case, the highest bids made by l's proxy agent up to any time t are characterized by a number π_l^t with $B_l^t(x_l) = \max\left(0, \tilde{v}_l(x_l) - \pi_l^t\right)$, where $\tilde{v}_l$ is the valuation reported to the proxy agent. With this specification, when the auction ends at some round t, the winning packages are among those that maximize the total reported value. To see this, let T denote the set of bidders whose bids maximize total revenue at the final round. At the final round, bidders $l \notin T$ will have set $\pi_l^t = 0$, so:

$$\max_{x \in X} \sum_{l \in L \setminus 0} B_l^t(x_l) = \max_{x \in X} \sum_{l \in L \setminus 0} [\tilde{v}_l(x_l) - \pi_l^t] = \max_{x \in X} \sum_{l \in L \setminus 0} \tilde{v}_l(x_l) - \sum_{l \in T} \pi_l^t . \tag{6}$$

While the use of negligible increments enables clean representations like (6), it has the unfortunate byproduct of introducing the possibility of ties into the bidding. We treat ties by augmenting the standard notion of Nash equilibrium in the manner suggested by

[29] In this paper, each proxy agent can only receive instructions once. We explore elsewhere (Ausubel and Milgrom, 2001) versions of this model in which the instructions to the proxy agent can be changed periodically. Observe that if bidders were able to revise their proxy instructions at every round, then this description would entail no restrictions on how a bidder might bid. However, if bidders can change their proxy instructions at only limited times, the design may be a useful restriction on continuously-changing dynamic strategies, while still maintaining many advantages of a dynamic auction design. For example, it may allow bidders to reduce their planning costs or to make inferences in a common value setting.

Simon and Zame (1990). In our formal treatment of the ascending proxy auction, an equilibrium comprises not only a vector of strategies $\{\tilde{v}_l\}_{l\in L\backslash 0}$ for every bidder, but also an allocation x of the items. The allocation x is required to be *consistent* with the strategy profile $\{\tilde{v}_l\}_{l\in L\backslash 0}$, in the sense that x maximizes the total bid: $x \in \arg\max_{y\in X} \sum_{l\in L\backslash 0} \tilde{v}_l(y_l)$. However, in the event that there are multiple maximizers of the total bid, the tie-breaking rule is specified as part of the solution.[30]

Let $\pi_l(\tilde{v};x)$ denote the payoff to bidder l from strategy profile $\tilde{v} = \{\tilde{v}_l\}_{l\in L\backslash 0}$ and allocation x. The strategy $\tilde{v}_l$ is said to be a *best reply to* $\tilde{v}_{-l}$ *at* x if the allocation x is consistent with $(\tilde{v}_l, \tilde{v}_{-l})$ and if there are no alternative strategy v_l' and allocation x' such that:

(i) x' is consistent with $(v_l', \tilde{v}_{-l})$;

(ii) x is not consistent with $(v_l', \tilde{v}_{-l})$; and

(iii) $\pi_l(v_l', \tilde{v}_{-l}; x') > \pi_l(\tilde{v}_l, \tilde{v}_{-l}; x)$.

The strategy profile and allocation $(\tilde{v}, x)$ is said to be an *augmented Nash equilibrium* if $\tilde{v}_l$ is a best reply to $\tilde{v}_{-l}$ at x for every bidder l. We shall say that $\tilde{v}$ is a *Nash equilibrium* if there is some x such that $(\tilde{v}, x)$ is an augmented Nash equilibrium.

Informally, an augmented equilibrium is a strategy profile and tie-breaking rule such that no bidder can, by changing its bid, change the outcome in its favor, while a "Nash" equilibrium consists just of the corresponding strategy profile. Much of the remainder of this paper characterizes Nash equilibria of this *ascending proxy auction*.

4. Package Auctions, Matching Theory and the Core

Our approach in this section is to analyze the ascending proxy auction as a kind of *deferred acceptance algorithm* in the sense that term is understood in the theory of two-sided matching.

All deferred acceptance algorithms begin with a report of ordinal preferences over outcomes by the participants. These reports are processed in a series of rounds that mimic an imaginary market process. At each round, players on one side of a market—let us call them the "buyers"—make offers to the "sellers." Each seller compares the offers it has in hand, including any offers it may be holding from the previous round, and rejects all but its most preferred offer. At the next round, each rejected buyer who wishes to buy more makes the offer that ranks next highest on its preference list. The process iterates until no new offers are made. When the process ends, any still unrejected offers become finally

[30] Readers unfamiliar with the Simon-Zame procedure should reflect on the example of a sealed bid auction for a single item worth 5 and 10 to the two bidders, respectively. With discrete but small bid units ε, there is a unique pure strategy equilibrium in undominated strategies. At that equilibrium, bidder 1 bids 5-ε and bidder 2 bids 5. With "negligible" bid increments, the unique equilibrium in undominated strategies specifies that both bidders bid 5 and that bidder 2 wins. The outcome specified in this way is the limit of the discrete increment case.

accepted. This class of algorithms acquires its name from the fact that even the best offers are not finally accepted until after the last round.

Deferred acceptance algorithms arise both in models where there is no exchange of money—the so-called "marriage problem" and "college admissions problem" are examples—and in ones where money is exchanged. The ordinary ascending "English" auction is an example of the latter kind, with buyers shouting offers and the seller holding onto its best offer only until a better offer is made. Most closely related to our analysis is one by Kelso and Crawford (1982), who analyzed a kind of simultaneous ascending auction in which the bidders are firms and the sellers are workers with preferences over both the employer's identity and income.

A recurring result in matching models with and without money is that the outcome of the deferred acceptance algorithm is a "stable match" or core allocation of the matching game. That is, the outcome has the property that there is no coalition of players that can match or trade among themselves in a way that coalition members all prefer to the outcome proposed by the algorithm. In particular, this result implies that the core of the matching game is non-empty.

A second recurring result in matching models is that the core allocation to which the deferred acceptance algorithm converges is the most preferred point in the core for each buyer and the least preferred point in the core for each seller. In particular, this implies that such a core point exists.

A third result found in some matching models is that it is a dominant strategy for each *buyer* to report its preferences truthfully. That is, for any specification of the preferences of the others, the result of the algorithm when a buyer reports truthfully is at least as favorable as when it reports untruthfully, and any given untruthful strategy leads to strictly worse outcomes for some profile of preferences of the other players. However, it is not generally a dominant strategy for the *sellers* to report their preferences truthfully.

Each of these results relies on certain assumptions about preferences. One is that the players on one side of the market—the buyers or the sellers—wish to have just one trading partner. A second is sometimes interpreted to mean that, from each player's point of view, the parties on the other side of the market are "substitutes" in a limited sense, to be discussed below.

In this section, we explore the ascending proxy auction as a kind of deferred acceptance algorithm. Straightforward bidding by the proxy bidders plays an essential role in our analysis: it enforces the condition found in all deferred acceptance algorithms that the buyers first make the offers they prefer most and proceed monotonically to the offers they prefer least, stopping at last when no unmade offer is preferred to the no-trade outcome. As part of the development, we find analogues for the results described above and other typical results of matching theory.

To make formal sense of these claims, we begin by defining a coalitional form game (L,w) that is associated with the package economy. The coalitional value function w is zero if the seller is not a member of the coalition (i.e., $w(S) = 0$ if $0 \notin S$) and otherwise is defined by the following expression:

$$w(S) = \max_{x \in X} \sum\nolimits_{l \in S} v_l(x_l)\,. \tag{7}$$

Thus, the value of a coalition that includes the seller is the maximum total value the players can create by trading among themselves. We then define the core by:

$$Core(L,w) = \left\{ \pi : w(L) = \sum\nolimits_{l \in L} \pi_l,\, w(S) \le \sum\nolimits_{l \in S} \pi_l \text{ for all } S \subset L \right\}. \tag{8}$$

Thus, the core is the set of profit allocations that are feasible for the coalition of the whole and unblocked by any coalition *S*.

Given a vector of reported values, the seller's revenue at any round *t* of the ascending proxy auction is given by:

$$\begin{aligned} \pi_0^t &= \max_{x \in X} \sum\nolimits_{l \in L \setminus 0} \hat{B}_l^t(x_l \mid v_l) \\ &= \max_{x \in X} \sum\nolimits_{l \in L \setminus 0} \max\left(0, v_l(x_l) - \pi_l^t\right) \\ &= \max_{x \in X} \max_{S \subset L} \sum\nolimits_{l \in S \setminus 0} [v_l(x_l) - \pi_l^t] \\ &= \max_{S \subset L} \max_{x \in X} \sum\nolimits_{l \in S \setminus 0} [v_l(x_l) - \pi_l^t] \\ &= \max_{S \subset L} \left\{ w(S) - \sum\nolimits_{l \in S \setminus 0} \pi_l^t \right\}. \end{aligned} \tag{9}$$

The five steps in (9) follow from (i) the definition of π_0^t, (ii) the characterization of the proxy bidder behavior given above, (iii) the observation that $S = \left\{ l \mid v_l(x_l) - \pi_l^t \ge 0 \right\}$ achieves the maximum, (iv) maximization, and (v) the definition of *w*.

The characterization that $\pi_0^t = \max_{S \subset L} \left\{ w(S) - \sum\nolimits_{l \in S \setminus 0} \pi_l^t \right\}$ suggests an interpretation of the package auction as a coalitional bargaining process, in which each buyer at round *t* makes a profit demand $\pi_l^t \ge 0$. According to the process, coalition *S* then offers the seller revenue equal to the coalition value *w*(*S*) minus the sum of the members' profit demands. The provisionally winning coalition is the one that offers the highest price. Buyers who are not part of the provisionally winning coalition at *t* but who have demanded a positive profit then reduce their profit demands at the next round. The process ends when all non-winning buyers are demanding profits of zero and so do not reduce their demands any further.

This coalitional-bargaining interpretation of the auction process leads to the following characterization of the outcome.

<u>Theorem 2</u>. In the ascending proxy auction, given the reported preferences, the final payoff allocation is in the core: $\pi^T \in Core(L,w)$.

<u>Proof</u>. We must show that π^T is feasible and not blocked by any coalition.

An immediate consequence of (9) is that for all coalitions *S* that include the seller, $w(S) \le \sum\nolimits_{l \in S} \pi_l^t$, and the same result holds trivially for coalitions excluding the seller. Hence, for all *t*, the payoffs vector π^t is unblocked; in particular, π^T is unblocked.

Let S^* be the final (round T) winning coalition, including the seller. Then by (9) and since $\pi_l^T = 0$ for any "losing bidder" $(l \notin S^*)$,

$$w(L) \le \sum_{l \in L} \pi_l^T = \sum_{l \in S^*} \pi_l^T = w(S^*) \le w(L) \tag{10}$$

Hence, $w(L) = \sum_{l \in L} \pi_l^T$, which establishes that π^T is feasible. ■

The power of restricting players to bid through proxy agents can be seen immediately in the next theorem. If a player's opponents are allowed to use arbitrary pure strategies in a dynamic package auction, then the player's best reply may need to be arbitrarily complex. However, if a player's opponents are restricted to bidding through proxy agents as described in Section 3.4 above, then the player necessarily has a best reply in limited straightforward strategies, which itself can be implemented with the same form of proxy agent. In turn, this provides a justification for restricting players to bid through proxy agents. We have:

Theorem 3. Suppose that bidders in $L \setminus l$ use proxy bidders and report the valuation profile $\tilde{v}_{-l}$, and let $x^* \in \arg\max_{x \in X} \left\{ v_l(x_l) + \sum_{k \ne l} \tilde{v}_k(x_k) \right\}$. Then:

(a) in the ascending proxy auction, there is a semi-sincere strategy $\tilde{v}_l$ that is a best reply for bidder l to $\tilde{v}_{-l}$ at x^*; and

(b) in the ascending package auction with small positive bid increments, no strategy B_l can give bidder l an appreciably better payoff than he earns from $\tilde{v}_l$ in the ascending proxy auction, i.e., for every $\varepsilon > 0$, there exists $\delta > 0$ such that if δ is an upper bound on the bid increments used in the auction, then the payoff to l from every strategy B_l is bounded by: $\pi_l(B_l, \tilde{v}_{-l}) < \pi_l(\tilde{v}_l, \tilde{v}_{-l}; x^*) + \varepsilon$.

Proof. (a) Define $X^*(\tilde{\pi}_l, \tilde{v}_{-l}) = \arg\max_{x \in X} \left\{ \max\left\{ v_l(x_l) - \tilde{\pi}_l, 0 \right\} + \sum_{k \ne l} \tilde{v}_k(x_k) \right\}$ and define $X_l^*(\tilde{\pi}_l, \tilde{v}_{-l})$ to be the projection of $X^*(\tilde{\pi}_l, \tilde{v}_{-l})$ on bidder l. Let $\pi_l^* = \sup\left\{ \tilde{\pi}_l : X_l^*(\tilde{\pi}_l, \tilde{v}_{-l}) \ne \varnothing \text{ or } \tilde{\pi}_l = 0 \right\}$. (Thus, π_l^* is the largest positive amount that bidder l can uniformly shade his valuation while still being efficiently allocated some items or zero if no such positive amount exists.)

We claim that $\tilde{v}_l(x_l) \equiv \max\left\{ v_l(x_l) - \pi_l^*, 0 \right\}$ is a best reply for bidder l to $\tilde{v}_{-l}$ at x^*. By Theorem 2, the payoff allocation resulting from any play of the proxy agents is an element of the core (defined with respect to the proxy instructions), and hence yields an allocation $x \in \arg\max_{y \in X} \sum_{l \in L \setminus 0} \tilde{v}_l(y_l)$. Thus, any bid by bidder l that is shaded by more than π_l^* cannot win, meaning that bidder l's profit cannot exceed π_l^*. If $\pi_l^* = 0$, the proof is complete. If $\pi_l^* > 0$, then x^* awards a package to l which, if l reports $\tilde{v}_l$, earns l a profit of exactly π_l^*.

(b) Given $\delta > 0$, let $\tilde{V}_{-l}(\delta)$ denote the set of all profiles of valuations for bidder l's opponents satisfying $|v_k(x) - \tilde{v}_k(x)| \le \delta$, for all $k \ne l$ and for all $x \in [0, M]$. Let $\pi_l^*(\delta) = \sup\left\{\tilde{\pi}_l : \tilde{\pi}_l = 0 \text{ or } X_l^*(\tilde{\pi}_l, v_{-l}) \ne \varnothing \text{ for some } v_{-l} \in \tilde{V}_{-l}(\delta)\right\}$. Bidder l's payoff from any strategy B_l played in the ascending package auction against proxies with reports of $\tilde{v}_{-l}$ is bounded by $\pi_l^*(\delta)$. The conclusion then follows from the fact that $\lim_{\delta \to 0} \pi_l^*(\delta) = \pi_l^*$. ■

Theorem 3 identifies a highly desirable property of the ascending proxy auction, which differentiates it from many other existing auction formats. In the FCC's simultaneous ascending auction, for example, we have already described the incentive for large bidders to distort their reported preferences, withholding demand in order to reduce prices. According to theorem 3, no such incentive exists in the ascending proxy or package auctions. In the ascending package auction, there could be "threat equilibria," in which a bidder is deterred from bidding for the packages it most wants to acquire by the threat that its competitors will retaliate to drive up its prices. According to theorem 3, the ascending proxy auction makes threats such as those impossible: a bidder *always* has a best reply in which it does not distort its relative valuation among packages. This means that regardless of others' strategies, the bidder can bid the full incremental value for any additional package it wishes to acquire, besides the package it expects to win, without fear that doing so will reduce its payoff in the game.

An analysis using Theorem 3 also explains the need for either activity rules or proxy bidders to pace the ascending package auction. To illustrate the potential problem, suppose all bidders except bidder l play limited straightforward strategies. According to theorem 3, bidder l has a best reply that is a limited straightforward strategy as well, but to know that strategy it may need to know the corresponding profit level π_l. Without that information, the bidder could be tempted to adopt the following "maximal delay strategy" σ instead: place no bid until all bidding stops and thereafter bid straightforwardly.[31] We now argue that the temptation is substantial: σ is always a best reply, regardless of the values or profit targets of the other bidders.

Suppose that the winning coalition is S when strategy σ calls for l to begin bidding. Then, the bidder playing σ would place new bids only when the auction would otherwise end with S winning and, if $\pi_l > 0$, l would eventually bid just enough so that the payoff profile is unblocked by $S \cup \{0\}$. Treating bid increments as negligible, it is evident that no lower bids could win against the opposing profile, so strategy σ is a best reply, regardless of which limited straightforward strategies the others may play.

Auctions using proxy bidders thus have a natural speed advantage over less automated alternatives. They rule out certain kinds of delay, allow the use of smaller (more efficient) bid increments, and avoid the need for activity rules that are necessary to speed an unstructured auction but can distort the auction outcome.

[31] But for the activity rule, this would be feasible in FCC Auction No. 31, because bidding does not close until there are *two* rounds with no new bids.

5. Equilibrium of the Ascending Proxy Auction

Let us say that a payoff profile $\pi \in Core(L,w)$ is a *bidder-Pareto-optimal* point in the core if there is no $\pi' \in Core(L,w)$ with $\pi' \neq \pi$ and $\pi'_l \geq \pi_l$ for every bidder *l*. We begin by observing that every such bidder-Pareto-optimal point is the payoff vector associated with a Nash equilibrium of the ascending proxy auction.

Theorem 4. Suppose $\pi \in Core(L,w)$ is a bidder-Pareto-optimal point in the core. Then, there exists a Nash equilibrium of the ascending proxy auction with associated payoff vector π. One strategy profile that supports π has each bidder *l* playing a semi-sincere strategy, reporting the valuation function of $\tilde{v}_l(\bullet) \equiv \max\{v_l(\bullet) - \pi_l\ , 0\}$. Moreover, if $\tilde{v}$ is a Nash equilibrium in semi-sincere strategies at which losing bidders bid sincerely, then $\pi(\tilde{v})$ is a bidder-Pareto-optimal point in $Core(L,w)$.

Proof. Suppose π is a bidder-Pareto-optimal point in $Core(L,w)$. We would like to show that the reports $\tilde{v}_l(\bullet)$ are a Nash equilibrium of the ascending proxy auction. Suppose not. Then there is some player *l* and some unilateral deviation for that player that leads to a winning coalition T $(T \ni l)$ and profit outcome vector $\hat{\pi}$. For bidder *l*, $\hat{\pi}_l > \pi_l$, and for all bidders $k \in T$, the proxy strategies imply that $\hat{\pi}_k \geq \pi_k$.

Since π is bidder Pareto optimal, there is a coalition S such that $l \notin S$ and $w(S) = \sum_{k \in S} \pi_k$ (Otherwise, for some $\varepsilon > 0$, there would be a point in the core at which *l* gets $\pi_l + \varepsilon$, the seller gets $\pi_0 - \varepsilon$, and others payoffs are as specified by π, contradicting bidder Pareto optimality.) Let $\beta(S)$ and $\beta(T)$ denote the highest total revenue associated with bids by the bidders in coalitions *S* and *T* during the proxy auction, given the specified deviation by bidder *l*. We show that $\beta(S) > \beta(T)$, contradicting the hypothesis that *T* is the winning coalition. Indeed, using (9):

$$\begin{aligned}
\beta(S) &\geq w(S) - \sum_{k \in S \setminus 0} \max(\pi_k, \hat{\pi}_k) \\
&> w(S) - \sum_{k \in S \setminus 0} \pi_k - \sum_{k \in T \setminus 0} \max(0, \hat{\pi}_k - \pi_k) \\
&= \pi_0 - \sum_{k \in T \setminus 0} \max(0, \hat{\pi}_k - \pi_k) \\
&\geq w(T) - \sum_{k \in T \setminus 0} \pi_k - \sum_{k \in T \setminus 0} \max(0, \hat{\pi}_k - \pi_k) \\
&= w(T) - \sum_{k \in T \setminus 0} \hat{\pi}_k \\
&= \beta(T)
\end{aligned} \tag{11}$$

The first step in (11) follows from the proxy rules: any losing bidders in *S* stop bidding only when their potential profits reach the specified levels. The strict inequality in the second step follows because $l \in T \setminus S$ and $\hat{\pi}_l > \pi_l$. The third step follows by selection of *S*, the fourth because $\pi \in Core(L,w)$, and the fifth and sixth by the definitions of *T*, $\hat{\pi}$ and $\beta(T)$.

If π is not bidder-Pareto-optimal in the core, then there exists l and $\pi_l' > \pi_l$ such that $(\pi_l', \pi_{-l}) \in Core(L,w)$. Bidder l can deviate by reporting $\max\{\tilde{v}_l - \pi_l + \pi_l', 0\}$ instead of reporting $\tilde{v}_l$, thereby increasing its profits to π_l'. ■

Theorem 4 is related to theoretical results about Nash equilibrium behavior in some other deferred acceptance algorithms. In the matching theory "marriage" model, for example, using the version of the algorithm in which the men make the offers, there is a Nash equilibrium in which all men report truthfully and each woman plays a "semi-sincere" strategy—moving the *no-match* outcome up in her rank order list to just below that woman's most preferred outcome in the core (Roth and Sotomayor, 1990). If one regards the reports $\tilde{v}_l$ as specifying ordinal preferences over allocation-payment pairs, then the semi-sincere strategies described in the theorem are similar: they merely move the no-trade point up in the bidders rank order list without changing the relative ranking of any other pair of outcomes. Other studies of many-to-one matching have found no equilibrium of this sort; it is the use of package bidding that gives the proxy auction its unique strategic structure.

The theorem is also closely related to the Bernheim-Whinston (1986) equilibrium strategies in their menu auction game. In the environment studied here, the menu auction is simply a sealed-bid package auction in which the auctioneer takes the collection of bids that maximizes its total revenue and allocates the goods accordingly. That paper employs coalition-proofness to single out equilibrium strategy profiles at which payoffs are bidder-Pareto-optimal points in the core, leading to equilibrium strategies that coincide exactly with the equilibrium reports to proxy bidders identified in theorem 4.

Although individual bidders suffer no loss by limiting themselves to semi-sincere strategies,[32] there do exist Nash equilibria whose allocations are different from those identified in theorem 4. We describe some of those here and compare them with the corresponding Nash equilibrium in the Vickrey auction.

For a first example, suppose there is just one good and two bidders, A and B, with values of 5 and 10 respectively. Then, in addition to the familiar equilibrium in which the parties bid 5 and 10, respectively, there is a Nash equilibrium (of both the Vickrey and ascending proxy auctions) in which A bids 12 and B bids 4, and another in which A bids 6 and B bids 10. A related example arises when there are two identical items and three bidders, A, B and C. Bidders A and B each want one item only and have a value of 10, while bidder C values only the pair and has a value of 15. Then, there is a Nash equilibrium of both the Vickrey and ascending proxy auction in which A and B bid zero for every package and C bids 0 for a single item and 15 for the pair. All of the equilibria described here are supported by the insincere bidding of losing bidders, which is ruled out in the second half of theorem 4.

[32] Provided that the others are playing a pure strategy profile. In a previous version of this paper, we showed that if the opposing strategy profile is uncertain, then a bidder can benefit from a "parking" strategy, in which it delays making serious bids until late in the auction. Parking strategies can sometimes be best implemented by avoiding semi-sincere strategies. Indeed, the incentives for parking is an important reason to prefer proxy implementations over more free-form package auctions.

In models with multiple goods, it is not just losing bidders whose behavior can lead to these odd-seeming outcomes: insincere *incremental* bids by winning bidders for items and packages they do *not* win have consequences much like insincere bidding by losing bidders. For example, suppose there are two identical items and two bidders, A and B. A's values are 10 for a single item or 20 for a pair, while B's corresponding values are 9 and 18. Then there is a Nash equilibrium (of the ascending and Vickrey auctions) in which A and B both bid 12 for one item and 12 for the pair.[33] To identify equilibria that we believe are most likely to be played, we focus the second half of theorem 4 on equilibria using semi-sincere strategies.

6. Vickrey Outcomes and the Core

We saw in Section 5 that the ascending package auction may be interpreted as a deferred acceptance algorithm and that—similar to deferred acceptance algorithms in two-sided matching models—the outcomes will tend to be core allocations. However, in matching models, the analysis can often be further refined to conclude that the outcome is the point in the core that is unanimously most preferred by the buyers and unanimously least preferred by the sellers. In general, such a point need not exist in the package auction environment, but as we shall now see, the existence is fully characterized by a simple condition relating two of the central concepts of this paper: a bidder-Pareto-optimal core outcome exists if and only if the Vickrey outcome is contained in the core.

Let $\bar{\pi}$ denote the *Vickrey payoff vector*, i.e., the payoffs associated with the dominant-strategy equilibrium of the generalized Vickrey auction. For bidders $(l \in L \setminus 0)$, $\bar{\pi}_l = w(L) - w(L \setminus l)$, while the seller's payoff is $\bar{\pi}_0 = w(L) - \sum_{l \in L \setminus 0} \bar{\pi}_l$. We have:

Theorem 5. A bidder's Vickrey payoff $\bar{\pi}_l$ is l's highest payoff over all points in the core. That is, for all $l \in L \setminus 0$: $\bar{\pi}_l = w(L) - w(L \setminus l) = \max\{\pi_l \mid \pi \in Core(L,w)\}$.

Proof. By inspection, the payoff vector defined by $\pi_0 = w(L \setminus l)$, $\pi_l = w(L) - w(L \setminus l)$, and $\pi_j = 0$, for $j \neq 0, l$, satisfies $\pi \in Core(L,w)$. Hence, $\bar{\pi}_l \leq \max\{\pi_l \mid \pi \in Core(L,w)\}$.

Now suppose that π is a feasible profit allocation with $\pi_l > \bar{\pi}_l$ for some $l \neq 0$. Then $\sum_{k \in L \setminus l} \pi_k = w(L) - \pi_l < w(L \setminus l)$, so $L \setminus l$ blocks the allocation. So, $\pi \notin Core(L,w)$. ■

Theorem 6. The core contains a bidder Pareto-dominant point if and only if the Vickrey payoff vector $\bar{\pi}$ is in the core. If $\bar{\pi}$ is in the core, then it is bidder Pareto-dominant.

Proof. By Theorem 5, $\bar{\pi}_l \geq \pi_l$ for all $\pi \in Core(L,w)$ and $l \in L \setminus 0$. Hence, if $\bar{\pi}$ is in the core, it is bidder Pareto-dominant.

[33] This equilibrium is even coalition proof, leading to profits of 10 and 9, upon which the coalition cannot improve.

For the converse, we show that no other core point can be bidder Pareto-dominant. For suppose $\bar{\pi} \neq \hat{\pi} \in Core(L, w)$. Then by theorem 5, there is some bidder j for whom $\hat{\pi}_j < \bar{\pi}_j$ and some other core point at which j receives $\bar{\pi}_j$. ■

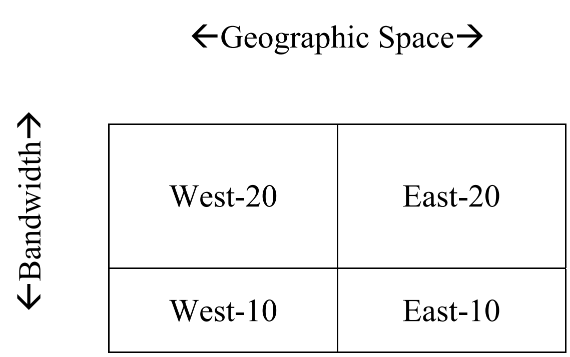
In situations where the condition of Theorem 6 holds, there is effectively a coincidence of interests among bidders in the ascending package auction. No opposing bidder k benefits if some bidder j bids a higher amount than would yield a payoff of π_j. Conversely, in situations where the condition of Theorem 6 does not hold, the proof has identified bidders j and k for which there is a nondegenerate bargaining problem. Bidders j and k, together, must bid some requisite amount of money in order to block some other bidder; to the extent that bidder j bids more, bidder k can get away with bidding less.

Theorems 4 and 6 together provide some insight into the geometry of the adjustment process of the ascending package auction, viewed in the space of payoffs. At every round, the payoff profile is unblocked. Once losing bidders have no more profitable bids, the profile lies in the core and no bidder can gain by continuing to bid. Furthermore, if the condition of Theorem 6 holds, no bidder has any incentive to manipulate the adjustment path to enter the core at any point other than the bidder-Pareto-preferred point. However, if the condition of Theorem 6 does not hold, then individual bidders do have an incentive to manipulate where the adjustment path enters the core. Indeed, bids in this situation are similar to concessions in a bargaining game. Just as a bargainer may wish to avoid making concessions to see how much others are willing to concede, a bidder in the package auction may bid slowly, hoping to avoid making unnecessary concessions. Using semi-sincere strategies in the proxy auction amounts to deciding how far to concede in a bargaining game. According to Theorem 4, the outcome of such bargaining can be any point that is bidder-Pareto-optimal in the core.

7. When Sincere Bidding is an Equilibrium

We found in Section 6 that, if the Vickrey payoff vector is an element of the core, then there exists no fundamental bargaining problem among the bidders. It might then seem that, under exactly this condition, each bidder could simply use the straightforward bidding strategy and this would yield an equilibrium. However, this conclusion is not quite right, as the following example demonstrates.

Suppose that there are four spectrum licenses. In order to understand that the following bidder valuations are sensible, it is helpful to depict the licenses as follows:



There are five bidders. Bidder 1 desires a 10-MHz band of spectrum covering both East and West. Bidders 2 and 3 desire a 20-MHz band of spectrum covering both East and West. Bidder 4 wants the full 30-MHz of spectrum in the East. Bidder 5 wants the full 30-MHz of spectrum in the West. Thus:

v_1(West-10, East-10) = 10,
v_2(West-20, East-20) = 20,
v_3(West-20, East-20) = 25,
v_4(East-20, East-10) = 10, and
v_5(West-20, West-10) = 10,

with all singletons and all other doubletons valued at zero.

Observe that the Vickrey payoff vector, (20, 10, 0, 5, 0, 0), is an element of the core, corresponding to Bidder 3 paying 20 for his desired licenses and Bidder 1 paying 0 for his desired licenses. Nevertheless, straightforward bidding is likely to lead Bidder 1 to pay a positive price.[34]

7.1 Buyer-Submodular Values

In this section, we develop a stronger condition that renders truthful reporting an equilibrium of the ascending proxy auction. We will require not only that the Vickrey payoff vector is an element of Core(L,w), but that the analogous condition holds for all subcoalitions S that include the seller.

<u>Definition</u>. The coalitional value function w is *buyer-submodular* if for all $l \in L \backslash 0$ and all coalitions S and S' satisfying $0 \in S \subset S'$, $w(S \cup \{l\}) - w(S) \geq w(S' \cup \{l\}) - w(S')$.[35]

The next two theorems apply to the case in which the coalitional value function is buyer-submodular. For the next theorem, we define the "Vickrey payoff vectors" for the cooperative game restricted to the members of coalition S, which we denote $\bar{\pi}(S)$, by $\bar{\pi}_l(S) \equiv w(S) - w(S \setminus l)$ for $l \in S \setminus 0$ and $\bar{\pi}_0(S) \equiv w(S) - \sum_{l \in S \backslash 0} \bar{\pi}_l(S)$.

<u>Theorem 7</u>. The following three statements are equivalent:

(1) The coalitional value function w is buyer-submodular.

[34] For example, if each bidder raises his bid by the same bid increment whenever he is not a provisional winner, we see that (so long as Bidder 2 remains in the auction), Coalition {1, 2} is a provisional winner ¼ of the time, Coalition {1, 3} is a provisional winner ¼ of the time, and Coalition {4, 5} is a provisional winner ½ of the time. With starting prices of zero, straightforward bidding would lead Bidder 1 to bid 10, Bidders 2 and 3 to bid 15 each, and Bidders 4 and 5 to bid 10 apiece. At this point, Bidders 4 and 5 drop out of the auction, and so there is nothing further to induce Bidder 1 to raise his bid. But he has already, irrevocably, reached a bid of 10; when his bidder-Pareto-optimal core payment equals 0. If Bidder 1 had instead limited his bidding to 0, he still would have won the West-10 and East-10 licenses.

[35] This condition (which is the opposite of the "convexity" condition introduced in Shapley (1971)) and a version of theorem 7 first appeared in Ausubel (1997b), which this paper supercedes. That predecessor also included the following necessary condition for the conclusions of theorem 7: $w(L) - w(L \setminus S) \geq \sum_{l \in S \backslash 0} \left(w(L) - w(L \setminus l) \right)$, for all coalitions S $(0 \in S \subset L)$. Bikhchandani and Ostroy (2002) subsequently developed the implications of these conditions for dual problems to the package assignment problem.

(2) For every coalition S that includes the seller, the Vickrey payoff vector is in the core: $\bar{\pi}(S) \in Core(S,w)$.

(3) For every coalition S that includes the seller, there is a unique core point that is Pareto-best for the buyers and, indeed:

$$Core(S,w) = \{\pi_S \mid \sum_{l \in S} \pi_l = w(S),\ 0 \le \pi_l \le \bar{\pi}_l(S) \text{ for all } l \in S \setminus 0\} \tag{12}$$

Proof. Define $\Pi_S = \{\pi_S \mid \sum_{l \in S} \pi_l = w(S),\ 0 \le \pi_l \le \bar{\pi}_l(S) \text{ for all } l \in S \setminus 0\}$. Suppose that (1) holds. It follows from Theorem 5 that $Core(S,w) \subset \Pi_S$. For the reverse inclusion, suppose $\pi_S \in \Pi_S$ and that $S = \{1,\ldots,|S|\}$. Let S' be a subcoalition of S including the seller, say, $S' = \{0,\ldots,k\}$. We show that the blocking inequality associated with coalition S' is satisfied. This follows because:

$$\begin{aligned}
\sum_{l \in S'} \pi_l &= w(S) - \sum_{l=k+1}^{|S|} \pi_l \\
&\ge w(S) - \sum_{l=k+1}^{|S|} \bar{\pi}_l(S) \\
&= w(S) - \sum_{l=k+1}^{|L|} [w(S) - w(S \setminus l)] \\
&\ge w(S) - \sum_{l=k+1}^{|L|} [w(\{0,\ldots,l\}) - w(\{0,\ldots,l-1\})] \\
&= w(S) - [w(S) - w(S')] \\
&= w(S')
\end{aligned} \tag{13}$$

The first step in (13) follows from feasibility of π, the second from $\pi \in \Pi$, the third from the definition of $\bar{\pi}$, and the fourth from the condition that bidders are substitutes. Hence, (1)⇒(3).

It is immediate that (3)⇒(2).

For (2)⇒(1), suppose (1) fails, that is, w is not buyer-submodular. Then, there exists a player i such that $w(S) - w(S \setminus i)$ is not weakly decreasing in S. Hence, there is a coalition S' including the seller and players $i, j \in S' \setminus 0$ such that. $w(S') - w(S' \setminus i) > w(S' \setminus j) - w(S' \setminus ij)$. So, $\sum_{l \in S' \setminus ij} \bar{\pi}_l(S') = w(S') - \bar{\pi}_i(S') - \bar{\pi}_j(S') = w(S' \setminus i) + w(S' \setminus j) - w(S') < w(S' \setminus ij)$, so the payoff allocation $\bar{\pi}_S$ is blocked by coalition $S \setminus ij$, that is, (2) also fails. ■

Theorem 8. Suppose that the coalitional value function is buyer-submodular. Then, truthful reporting is a Nash equilibrium strategy profile of the ascending proxy auction and leads to the generalized Vickrey outcome: $\pi^T = \bar{\pi}$.

Proof. We first establish that truthful reporting leads to the Vickrey payoff vector. Suppose there is some round t at which $\pi_l^t < \bar{\pi}_l$. We show that l is necessarily part of the winning coalition at that round. Let S be any coalition including the seller but not bidder l. Then,

$$\begin{aligned} w(S)-\sum\nolimits_{k\in S}\pi_k^t &< w(S)-\sum\nolimits_{k\in S}\pi_k^t+\left(\bar{\pi}_l-\pi_l^t\right) \\ &= w(S)-\sum\nolimits_{k\in S\cup\{l\}}\pi_k^t+w(L)-w(L\setminus l) \\ &\le w(S)-\sum\nolimits_{k\in S\cup\{l\}}\pi_k^t+w(S\cup l)-w(S) \\ &= w(S\cup\{l\})-\sum\nolimits_{k\in S\cup\{l\}}\pi_k^t \end{aligned} \tag{14}$$

So, l's profit π_l^t is at least $\bar{\pi}_l$ minus one bid increment ε. Taking the bid increment to zero for the ascending proxy auction proves that for $l\neq 0$, $\pi_l^T\ge\bar{\pi}_l$, and the reverse inequality follows from theorem 5.

Second, we show that truthful reporting is a best response to all other bidders reporting truthfully. For any bidder l and any report by that bidder, theorem 5 implies that the payoff to coalition $L\setminus l$ is at least $w(L\setminus l)$. Since the total payoff to all players is at most $w(L)$, l's payoff to any strategy is bounded above $\bar{\pi}_l=w(L)-w(L\setminus l)$, which is the payoff that results from truthful reporting. ■

Theorem 9. Suppose there is a single good of each type. Let V_{add} denote the set of additive valuation functions and suppose that $V_l\supset V_{add}$ for all $l\in L\setminus 0$.[36] Suppose that for all valuation profiles $v\in\times_{l\in L}V_l$, the corresponding coalitional value function is buyer-submodular. Consider the *restricted ascending proxy auction* games in which each buyer l may announce only preferences satisfying $v_l\in V_l$. If bidder l's actual preferences are in V_l and there are at least two buyers (including l), then it is a *dominant strategy* for l to report its preferences truthfully.

Proof. By theorem 8, truthful reporting is always a best reply. It remains to show that there is no other uniform best reply.

Let l have actual preferences v_l and let $\hat{v}_l\neq v_l$. In particular, there is some package z and item m such that $\hat{v}_l(z)-\hat{v}_l(z-1_m)\neq v_l(z)-v_l(z-1_m)$. Then, by our assumptions, we may specify that there is just one other buyer, say buyer j, who has strictly positive values, as follows. Bidder j's values are additive, so we may specify that by specifying a value for each good individually. For all elements of the package $M-z$, j's values are so high that j is assigned at least $M-z$ for either report by buyer l. Also, j's value for good m is $v_j(M-z+1_m)-v_j(M-z)=\frac{1}{2}\left(\hat{v}_l(z)-\hat{v}_l(z-1_m)+v_l(z)-v_l(z-1_m)\right)$. Finally, j's value for each other good is zero. Then, l's report determines only the allocation of good m, and reporting honestly pays higher than reporting $\hat{v}_l$ by an amount equal to $\left|\frac{1}{2}\left(\hat{v}_l(z)-\hat{v}_l(z-1_m)+v_l(z)-v_l(z-1_m)\right)\right|$. Hence, when l's value is v_l, reporting honestly can possibly pay strictly more than reporting $\hat{v}_1$. ■

[36] When there may be multiple goods of each kind, a similar result can be obtained using the set V_{add} of valuations of the form $v(z)=\lambda\bullet(z\wedge\bar{z})$, where the vectors λ and $\bar{z}$, respectively, designate the marginal values and maximum desired quantities for each kind of good.

7.2 When Goods are Substitutes

In models for which the coalitional game (L,w) is the primitive, assumptions about the coalitional value function are not subject to further analysis. In the present model, however, coalition values are not primitive—they are derived from individual package values. It is natural to ask: what conditions on bidder valuations *imply* that bidders are substitutes in the coalition game? We answer that question with two theorems. Theorem 11 holds that if the goods are substitutes for all bidders, then the coalitional value function is bidder-submodular. According to theorem 12, other kinds of valuations are generally inconsistent with this conclusion. Indeed, if the set of individual valuations contains (i) the additive valuations and (ii) any valuation for which the goods are not substitutes, then there exist valuation profiles such that the coalitional value function is not bidder-submodular. In view of Theorems 7-9, this has implications about the structure of the core, payoffs in the Vickrey auction, and bidder incentives in the proxy auction.

Since demand in discrete models is not generally single-valued, we shall say that goods are substitutes if the demand for each good is nondecreasing in the prices of other goods when attention is restricted to the set of prices for which demand is single-valued. For present purposes, we shall suppose that the bidders distinguish individual goods so that for all k, $M_k = 1$. This allows express the substitutes condition conveniently using the indirect utility function, which expresses the buyer's maximum utility at a given price vector p: $u_l(p) = \max_z \{v_l(z) - p \bullet z\}$.

Theorem 10. Goods are "*substitutes*" for bidder l if and only if the indirect utility function $u_l(\cdot)$ is submodular.

Proof. By an envelope theorem,[37] the following conclusions are true: First, the indirect utility function is absolutely continuous and partially differentiable almost everywhere in each good's price p_m. Second, the partial derivative exists for precisely those price vectors at which l's demand for good m is single-valued. Third, at those prices p, the partial derivative is equal to $-x_{lm}$, where x_{lm} is the quantity demanded of good m at price vector p. By definition, the substitutes condition is satisfied if and only if $x_{lm}(p)$ is nondecreasing in each p_j for $j \neq m$. Thus, goods are substitutes if and only if $\partial u_l / \partial p_m$ is nonincreasing in each p_j for $j \neq m$, that is, if and only if u_l is submodular. ■

Theorem 11. If goods are substitutes for all bidders, then the coalition value function is bidder-submodular.

Remark. The proof outline reflects the following intuition: When goods are substitutes, starting from any given package, the opportunity cost to coalition S of any good is increasing in the size (inclusiveness) of the coalition. (Establishing this with indirect utility functions occupies the main part of the proof.) Consequently, for any fixed package z that might be assigned to a new coalition member, the cost of giving away that package is increasing in the coalition size. So, the incremental value of an additional member, which is the new member's value of the package it receives minus the coalition's opportunity cost of that package, is decreasing in the coalition size.

[37] See Milgrom and Segal (2002), Corollary 4.

Proof. Let S be a coalition that includes the seller. Coalition S's value for a package z is given by $v_S(z) = \max_{x \in X(z)} \sum_{l \in S} v_l(x_l)$. The feasible set $X(z)$ incorporates two restrictions: that only the quantity vector z is available and that each buyer may acquire only zero or one good of each type.

The corresponding coalition indirect utility function is given by:

$$u_S(p) = \max_z \{v_S(z) - p \bullet z\} = \sum_{l \in S} u_l(p), \tag{15}$$

so $u_S(\bullet)$ is also submodular.

Fix any package z and let B be a large number that exceeds the incremental value of any good to any coalition. For all p, $u_S(p) \geq v_S(z) - p \bullet z$, so $v_S(z) \leq \min_{p \in [0,B]^N} \{u_S(p) + p \bullet z\}$. Then, taking $p_m = 0$ for all m such that $z_m = 1$ and $p_m = B$ otherwise leads to $u_S(p) = v_S(z) - p \bullet z$. So, $v_S(z) \geq \min_{p \in [0,B]^N} \{u_S(p) + p \bullet z\}$. Hence,

$$v_S(z) = \min_{p \in [0,B]^N} \{u_S(p) + p \bullet z\}. \tag{16}$$

The objective function in (16) is continuous, antitone ("weakly decreasing") and submodular in p and has weakly decreasing differences in (p,S). The feasible set of prices is a closed interval. Hence, applying the Topkis monotonicity theorem, the set of minimizers has a maximum element $p(S \mid z)$, which is an isotone ("weakly increasing") function of S. By inspection, for each m such that $x_m = 0$, $p_m(S \mid z) = v_L(M)$.

We claim that if $z_m = 1$, then $p_m(S \mid z) = v_S(z) - v_S(z - 1_m)$. To see this, note first that by (15) and (16), $z \in \arg\max_{z'} v_S(z') - p(S \mid z) \bullet z'$. Suppose $z_m = 1$ and letting $\varepsilon > 0$, set $p'_\varepsilon = p(S \mid z) + \varepsilon 1_m$. By definition of $p(S \mid z)$, demand for good m at price vector p'_ε is zero. By construction, demand for goods j for which $z_j = 0$ is zero is zero at price vector p'_ε, because each such good is priced at $v_L(M)$. By the condition of substitutes, demand for the remaining goods is undiminished. Hence, $z - 1_m \in \arg\max_{z'} v_S(z') - p'_\varepsilon \bullet z'$ for all $\varepsilon > 0$. By the theorem of the maximum, the same must hold for $\varepsilon = 0$, that is, $z - 1_m \in \arg\max_{z'} v_S(z') - p(S \mid z) \bullet z'$. So, $v_S(z - 1_m) - p(S \mid z) \bullet (z - 1_m) = v_S(z) - p(S \mid z) \bullet z$, and hence $p_m(S \mid z) = v_S(z) - v_S(z - 1_m)$.

Let $z^n = (1,...,1,0,...,0)$ denote a vector with n initial 1's. We may suppose without loss of generality that $z = z^m$ for some m. Then,

$$v_S(M) - v_S(z^m) = \sum_{j=m+1}^{n} \left(v_S(z^j) - v_S(z^{j-1}) \right) = \sum_{j=m+1}^{n} p_j(S \mid z^j). \tag{17}$$

This difference is nondecreasing in S since each term is so.

Notice that $w(S \cup \{l\}) - w(S) = \max_z v_S(z) + v_l(M - z) - v_S(M)$. By the preceding paragraph, the right-hand expression is a maximum of nonincreasing functions of S, so $w(S \cup \{l\}) - w(S)$ is itself nonincreasing in S. ■

Given the additional condition that the possible bidder valuations include the additive values, the goods-are-substitutes condition is necessary, as well as sufficient, to conclude that the coalitional value function is bidder-submodular. To state this result clearly, we introduce three sets of goods valuation functions. Let V denote the set of valuations from which the bidders in the auction may draw values, V_{add} the set of additive valuation functions, and V_{sub} the set of valuation functions for which the goods are substitutes.

Theorem 12. Suppose that there are at least four possible bidders. Further suppose that there is a single unit of each kind and that $V_{add} \subset V$. Then the following three conditions are equivalent:

(1) $V \subset V_{sub}$

(2) For every profile of bidder valuations drawn for each bidder from V, the coalitional value function is bidder-submodular.

(3) For every profile of bidder valuations drawn for each bidder from V, $\overline{\pi} \in Core(L, w)$.

Proof. By Theorems 11 and 7, (1)⇒(2)⇒(3). It remains to show that (3)⇒(1).

Suppose that the substitutes condition fails for some valuation $v_1 \in V$, which we may take to be the valuation of buyer 1. Then, there exist two goods, m and n, and a price vector, p, with $p_n, p_m > 0$, such that for $0 \le \hat{p}_m < p_m$, there is a unique maximizer x' of $v_1(x) - (\hat{p}_m, p_{-m}) \bullet x$ satisfying $x'_n = x'_m = 1$, and for $p_m < \hat{p}_m$, there is a unique maximizer x'' satisfying $x''_n = x''_m = 0$.[38] By Berge's theorem, it follows that at the price vector p, both x' and x'' are optimal. Moreover, any alternative bundle, x''', with the property that $x'''_m = 0$ and $x'''_n = 1$ is strictly suboptimal at price vector p.[39]

Since $V_{add} \subset V$, we may take buyers 2, 3 and 4 to have additive valuations as follows: $v_2(x) = \sum_{k \ne n,m} p_k x_k$; $v_3(x) = p_m x_m + p_n x_n$; and $v_4(x) = \hat{p}_m x_m$ where $\hat{p}_m > p_m$. Since x' is optimal for buyer 1 at price vector p above, $w(0123) = w(012)$. Since x'' is the unique optimum for buyer 1 at price vector $(p_{-m}, \hat{p}_m)$ and since $p_n > 0$, $w(01234) > w(0124)$. At the Vickrey payoffs, $\overline{\pi}_0 + \overline{\pi}_1 + \overline{\pi}_2 = w(01234) - \left(\overline{\pi}_3 + \overline{\pi}_4\right)$ = $w(01234)$ – $\big([$ $w(01234)$ –

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[38] The detailed construction begins by noting that, if the substitutes condition fails, then for any ε>0, there exist two goods, m and n, and a price vector, $(\overline{p}_m, p_{-m})$, such that: (i) buyer 1 has unique demands at price vector $(\overline{p}_m, p_{-m})$ and $x_{1n}(\overline{p}_m, p_{-m}) = 1$; and (ii) buyer 1 has unique demands at price vector $(\overline{p}_m + \varepsilon, p_{-m})$ and $x_{1n}(\overline{p}_m + \varepsilon, p_{-m}) = 0$. Hence, there exists $p_m \in (\overline{p}_m, \overline{p}_m + \varepsilon)$ at which the demand for good n changes from 1 to 0. Furthermore, $x_{1m}(\overline{p}_m, p_{-m}) \ne x_{1m}(\overline{p}_m + \varepsilon, p_{-m})$, for otherwise (given the quasilinear preferences), the change in the price of good m would have no effect on buyer 1's demand for good n.

[39] Otherwise, by inspection, x''' would also be optimal at any price vector $(\hat{p}_m, p_{-m})$ where $\hat{p}_m > p_m$, contradicting the uniqueness of the optimum at such prices.

$w(0124)] + [w(01234) - w(0123)]\big) = w(0123) + w(0124) - w(01234) < w(0123) = w(012)$. So, coalition 012 blocks the Vickrey payoff allocation and $\bar{\pi} \notin Core(L, w)$. ■

Theorem 12 is closely related to theorems about the existence of competitive equilibrium goods prices in models like this one. Indeed, Milgrom (2000a) shows that if $V_{sub} \subset V$, then a competitive equilibrium exists for every profile of bidder valuations drawn from *V* if and only if $V = V_{sub}$. Thus, possible failures of the substitutes condition are problematic for traditional market mechanisms as well as for the Vickrey mechanism.

The failure of the substitutes condition is also closely connected to some extreme possibilities for manipulation in the Vickrey auction, including the shill bidding and collusion by losers described in the introduction. In turn, it should be recognized that these possibilities for manipulation are intimately related to a failure of monotonicity of revenues in the set of bidders.

We define an auction to exhibit *bidder monotonicity* if there is no preference profile such that adding another bidder reduces the seller's equilibrium revenues. Bidder monotonicity formalizes the familiar property of ordinary single-item private-values auctions that increasing bidder participation can only benefit the seller. The next theorem shows that the substitutes condition is sufficient for bidder monotonicity in the Vickrey auction, and for shill bidding and loser collusion to be unprofitable. Moreover, the substitutes condition is also necessary for these conclusions if the set of bidder values is otherwise sufficiently inclusive.

Theorem 13. Suppose that there is a single unit of each kind and that $V_{add} \subset V$. Then the following four conditions are equivalent:[40]

(1) $V \subset V_{sub}$.

(2) For every profile of bidder valuations drawn for each bidder from *V*, adding bidders can never reduce the seller's total revenues in the Vickrey auction.

(3) For every profile of bidder valuations drawn for each bidder from *V*, any shill bidding is unprofitable in the Vickrey auction.

(4) For every profile of bidder valuations drawn for each bidder from *V*, any joint deviation by losing bidders is unprofitable in the Vickrey auction.

Proof. (1)⇒(2): Let L, L' $(\{0\} \subset L \subset L')$ be any nested sets of players that include the seller. As before, $\bar{\pi}(L)$ and $\bar{\pi}(L')$ denote the associated vectors of Vickrey auction payoffs. By Theorem 11, if every bidder has substitutes preferences, then the coalitional value function is bidder-submodular. Hence, for every bidder $l \in L \setminus \{0\}$, we have: $\bar{\pi}_l(L') = w(L') - w(L' \setminus l) \leq w(L) - w(L \setminus l) = \bar{\pi}_l(L)$. Meanwhile (see the inequality in footnote 35): $\sum_{l \in L' \setminus L} \bar{\pi}_l(L') \leq w(L') - w(L)$. Summing these inequalities, we conclude: $\bar{\pi}_0(L') = w(L') - \sum_{l \in L' \setminus \{0\}} \bar{\pi}_l(L') \geq w(L) - \sum_{l \in L \setminus \{0\}} \bar{\pi}_l(L) = \bar{\pi}_0(L)$, as required.

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[40] For models starting with a fixed number of bidders *N*, the proof establishes that (1)⇒(2), (1)⇒(3), (1)⇒(4), (2)⇒(1) and (3)⇒(1), provided *N*≥3, and (4)⇒(1), provided *N*≥4.

(1)⇒(3): By Theorem 11, if the substitutes condition holds for all bidders, then the coalitional value function is bidder-submodular. Then by Proposition 3 of Yokoo, Sakurai and Matsubara (2000), shill bidding is unprofitable in the Vickrey auction.

(1)⇒(4): Let v be the maximum value function for the coalition of winning bidders. Since goods are substitutes for the winning bidders, v is submodular. Suppose a coalition S of losing bidders deviates and acquires the bundles $(\hat{x}_l)_{l\in S}$. The Vickrey price paid by any losing bidder l to acquire its bundle is at least $v(M-\sum_{i\in S\setminus l}\hat{x}_i)-v(M-\sum_{i\in S}\hat{x}_i)\geq v(M)-v(M\setminus\hat{x}_l)$. The right-hand side is the Vickrey price for a lone deviator, so a losing bidder's price for any bundle is less if it deviates alone than if it participates in a joint deviation by coalition S. Hence, no losing bidder can benefit from such a deviation.

(3)⇒(1): Suppose that the substitutes condition fails for some valuation $v_1\in V$, which we may take to be the valuation of buyer 1. Then exactly as in the proof of Theorem 12, there exist two goods, m and n, a price vector, p, and two bundles x' and x'', such that at price vector p, $\{x',x''\}=\arg\max\{v_1(x)-p\cdot x\}$, where $x'_n=x'_m=1$ and $x''_n=x''_m=0$. Let $\Delta=v(x''_{-n},1)-v(x'')$ be the incremental value of good n to buyer 1 at x''. Following the proof of Theorem 12, we may take buyers 2 and 3 to have the additive valuations of $v_2(x)=\sum_{k\neq n,m}p_k x_k$ and $v_3(x)=p_m x_m+p_n x_n$. Observe that, in the sincere bidding equilibrium of the Vickrey auction, buyer 3 receives a payoff of zero. However, suppose that buyer 3 can enter an additional bid, $v_4(x)=(p_m+p_n-\Delta)x_m$, under the false name of buyer 4. With this bid, the shill buyer 4 wins good m for payment of p_m. Also, buyer 3 then wins good n for payment of $\Delta<p_n$, so the shill bidding is profitable for buyer 3.

(2)⇒(1): The construction in the preceding paragraph also provides a violation of bidder monotonicity. Adding buyer 4, now a real bidder, reduces the seller's revenues.

(4)⇒(1): The preceding construction also applies if buyer 4 is a real bidder but has valuation $v_4(x)=p_m x_m$. In that case, it illustrates profitable collusion among losing bidders. ■

These failings of the Vickrey auction contrast sharply with the properties of the ascending proxy auction. For the latter, to check bidder monotonicity, we need to be identify the "equilibrium revenues." We focus attention on the equilibria of the proxy auction that are consistent with the selection in theorem 4 and minimize revenues. Then, the equilibrium revenue is $\min\pi_0$ subject to $\sum_{l\in S}\pi_l\geq v(S)$ for every coalition $S\subset L$. It is obvious that introducing additional bidders in this formulation simply adds constraints to the minimization problem, increasing π_0. In this sense, the ascending proxy auction satisfies bidder monotonicity.[41]

[41] The argument in this paragraph together with the earlier characterization, under substitutes preferences, that the Vickrey payoffs are the unique core point that is Pareto-best for the buyers (theorems 7 and 11) provide an alternative proof that (1)⇒(2) in Theorem 13.

The precise statement that shill bidding is unprofitable in the proxy auction is the following: Given any pure strategy profile for opposing bidders –*l*, there is a best reply for bidder *l* that does not use shill bids. In particular, at any pure strategy Nash equilibrium, there is no gain to any deviation using shill bids. This result follows from the same reasoning as in Theorem 3(a). Given the opposing bidders' strategies, there exists a maximum profit target, π_l^*, attainable by bidder *l*—singly, or with additional shill bids. Similar reasoning shows that there are never profitable joint deviations for losing bidders in the proxy auction.

8. Generalized Ascending Package Auctions

Our analysis in this paper has focused on the case in which bidder preferences are "quasi-linear" (utility is representable as the value of packages minus any cash paid), the constraints are the "packaging constraints" described by (1), and the auctioneer cares only about the total price offered. These assumptions can fail in practice for a variety of reasons.

Looking first at bidders, one reason is the possibility of binding budget constraints. A buyer, despite its preferences, may be unable to finance the purchase of the package it wants. A second possibility is that a buyer's preference among packages may depend on the price it pays, for example because the price a spectrum buyer pays may affect its ability to make complementary investments in wireless infrastructure. A third reason, relevant for procurement auctions, is that bidders may have policies or contracts that limit price discounts but allow other inducements. A fourth possibility is that some of the bidders may be taking both sides of the market—making offers to sell and to buy—requiring the description of feasible bids to be modified accordingly. On the seller/auctioneer's side, preferences and constraints may also be more complex than represented in our model. In the analysis of Brewer and Plott (1996), the auctioneer sells rights of passage to Swedish trains. Feasibility and safety constraints limit, for example, the use of fast trains too soon after slow ones or the use of trains heading in opposite directions on the same track. Similarly, when the bid-taker is a buyer procuring inputs, it might have complex preferences that give weight to such varied factors as the quality of the input, the proportion of minority suppliers, the distance of the nearest supplier to a certain facility, or the total number of suppliers used.

To accommodate all these possibilities and more, we introduce the idea of a generalized ascending package auction. The set of offers that can be made by bidder *l* is formulated as a general finite set, X_l, which includes the "null offer" $\varnothing_l$. Any strict preference ordering of bidder *l* over the set X_l can be described by a utility function u_l. At the null offer, $\varnothing_l$, bidder *l* does not transact and earns zero utility. Let $X = \times_{l \in L \setminus 0} X_l$ be the set of all possible offer profiles for bidders. Let $F \subseteq X$ be the set of all feasible offer profiles, and for any coalition of bidders *S*, let $F_S = \{x \in F : x_l = \varnothing_l \text{ for all } l \notin S \setminus 0\}$ be the set of offer profiles that are feasible for *S*. The auctioneer's strict preference ordering over offer profiles in *F* is represented by the utility function u_0; the null offer profile $\vec{\varnothing} = (\varnothing_1, ..., \varnothing_{|L|-1})$ is assumed always to be feasible and $u_0(\vec{\varnothing}) = 0$.

A *generalized ascending proxy auction* uses this generalized formulation as intermediated through proxy agents. Let P_l be the set of strict preference orderings over X_l. Each bidder *l*'s strategy set is a subset of P_l (or a corresponding set of utility functions u_l). The mechanism, which is determined by the triple (X, F, u_0), operates in a series of rounds. At the first round, the proxy agent for bidder *l* sets $x_l^1 = \arg\max_{X_l} u_l(x_l)$, and the corresponding *profit demand* is $\pi_l^1 = \max_{X_l} u_l(x_l)$. The null bid $\varnothing_l$ is also treated as one of bidder *l*'s offers, so $B_l^1 = \{\varnothing_l\} \cup \{x_l^1\}$. As for the auction without proxy bidders, $F^1 = \{x \in F : x_l \in B_l^1 \text{ for all } l \in L \setminus 0\}$: this is the feasible set of bids available to the auctioneer after round 1. The mechanism chooses $\hat{x}^1 = \arg\max_{x \in F^1} \{u_0(x)\}$, and S^1 denotes the round 1 *winning coalition*.

Proceeding iteratively, let $t > 1$ denote the current round, and let the round t–1 *winning coalition* be $S^{t-1} = \left\{l \in L \setminus 0 : \hat{x}_l^{t-1} \neq \varnothing_l\right\}$. Let A_l^t identify the individually rational bids not yet made by bidder *l* when round *t* begins: $A_l^t = \{x_l \mid 0 \leq u_l(x_l) < \pi_l^{t-1}\}$ if $\pi_l^{t-1} > 0$, and $A_l^t = \{\varnothing_l\}$ if $\pi_l^{t-1} = 0$. Round *t* proceeds as follows:

- Members of the winning coalition make no new offers: If $l \in S^{t-1}$ then $B_l^t = B_l^{t-1}$ and $\pi_l^t = \pi_l^{t-1}$.
- Bidders with no remaining profitable bids make no new offers: if $A_l^t = \{\varnothing_l\}$, then $B_l^t = B_l^{t-1}$ and $\pi_l^t = 0$.
- Each other bidder makes the next most profitable bid on its list: if $A_l^t \neq \{\varnothing_l\}$ and $l \notin S^{t-1}$, then $x_l^t = \arg\max_{x_l \in A_l^t} u_l(x_l)$, $B_l^t = B_l^{t-1} \cup \{x_l^t\}$, and $\pi_l^t = u_l(x_l^t)$.
- The auctioneer selects its most preferred feasible offer profile according to the objective u_0 from the accumulated list of offers:

$$\hat{x}^t = \arg\max_{F^t} \{u_0(x)\} \text{ and } \pi_0^t = u_0(\hat{x}^t), \text{ where } F^t = \left\{x \in F : x_l \in B_l^t \text{ for all } l \in L \setminus 0\right\}. \quad (18)$$

The process continues iteratively, terminating after the first round in which no new offers are made.

One example of a generalized ascending proxy auction is the discretized version of the ascending proxy auction studied earlier. This corresponds to the case in which each bidder's strategy set is the set of preferences describable by a value function v_l (but with offer amounts limited to discrete units); the auctioneer ranks offer profiles according to total revenues, and the feasible set F is the set of all compatible profiles of packages. Our equilibrium analysis in the preceding sections employed the transferable utility core and assumed that the actual preferences could also be found in this family.

A second example of such a process arises from the earlier environment augmented by a budget constraint that limits bids. Preferences can then be completely described by a valuation function and a budget limit.

We have proved the following result:

Theorem 14. The final outcome of any generalized ascending proxy auction is an NTU-core allocation with respect to the reported preferences.

A sequel paper will contain the proof of this theorem and will develop further results concerning generalized ascending package auctions.

9. Conclusion

Our analysis of the ascending package auction may explain the efficient outcomes that sometimes emerge in experiments with package bidding and entail new predictions as well. If bidders bid "straightforwardly," the auction outcome is not only efficient, but also a core allocation of the exchange game.

The ascending proxy auction is a new kind of deferred acceptance algorithm, related to the algorithms studied in matching theory. This new auction design has significant advantages compared to the Vickrey auction. It avoids the very low revenues possible in the Vickrey auction by always selecting core allocations. By contrast, the Vickrey auction is assured to select such allocations only when goods are substitutes. The ascending proxy auction also avoids some of the extreme vulnerability of the Vickrey auction to shill bids and to collusion by coalitions of losing bidders. When technology choice affects the organization of the bidders, the ascending proxy is neutral about the choice of technology, but the Vickrey auction is not. Finally, in its multi-stage form, the ascending proxy auction economizes on the bidders' costs of evaluating packages by allowing them to focus their efforts on the packages that they have a reasonable chance to win based on the bids made by competitors earlier in the auction.

Besides the basic ascending proxy auction, we have also introduced a generalized version that applies for much more general preferences and constraints than are permitted in the standard Vickrey model. For example, it applies to models in which bidders are budget-constrained and to procurement problems with their characteristic multi-faceted selection criteria. In such circumstances, the Vickrey auction may not even apply, or it may lead to inefficient outcomes, but the generalized ascending proxy auction selects allocations in the NTU-core. We believe that this family of auction designs holds considerable promise for a variety of practical applications.

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