Labor Market Dynamics and Teacher Spatial Sorting*

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Abstract

This paper provides a unifying explanation for the lack of supply of skilled teachers in remote locations. I build an empirical model of dynamic two-sided matching to link teachers' and schools' preferences with equilibrium sorting and job-to-job flows. I show that this mapping is invertible such that preferences can be identified and estimated from observed matches. Taking these tools to panel data on the assignment of public teachers in Peru, I show that the spatial disaggregation of labor demand coupled with the concentration of labor supply in cities imply the existence of a *spatial job ladder*. Low quality teachers get displaced in remote schools and move toward urban schools by climbing up the ladder once they have accumulated experience and skills. Labor mobility thus magnifies the urban-rural gap in teacher quality by one third. Dynamic wage contracts that foster retention can largely mitigate this effect.

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1 Introduction

Many public and private services are provided locally and require the presence of a skilled workforce on-site. In such labor markets, the distribution of workers across locations has important welfare consequences. Unequal access to essential services such as education, childcare or healthcare directly contributes to spatial inequalities. Moreover, geographical differences in the overall quality of local services and amenities are key drivers of the spatial distribution of human capital, creating a feedback loop that would reinforce existing inequalities (Diamond and Gaubert, 2022). Understanding what drives worker sorting and mobility across locations is thus a first-order concern.

This paper studies this question in the context of the provision of an essential local public service: education. Teachers are key inputs of school quality (Rivkin et al., 2005) and strong predictors of students' later outcomes (Chetty et al., 2014b). Evidence of heterogeneous teacher effects further reveals that low ability students can potentially benefit more from being exposed to good teachers (Ahn et al., 2021; Bobba et al., 2021). This implies that an unequal access to skilled teachers can harm both equity and efficiency. However, analyzing sorting and mobility in teachers' labor markets is challenging as (i) wages are often set through collective bargaining and do not adjust to local labor market conditions and (ii) positions are often allocated through frictionless centralized clearinghouses. Job search or spatial equilibrium models are not tailored to such settings as they rely on wages to clear local labor markets or search frictions to rationalize sorting and job-to-job flows (Diamond, 2016; Moscarini and Postel-Vinay, 2018). Instead, an emerging literature has relied on empirical models of two-sided matching to study the role of workers' idiosyncratic preferences over job attributes in shaping sorting when prices are fixed (Agarwal, 2015; Bobba et al., 2021; Bates et al., 2022). Yet, these papers abstract away from labor market dynamics, making the analysis of sorting and mobility incomplete.

This paper bridges these literatures by incorporating dynamics into an empirical model of two-sided matching. It then applies these novel tools to study the causes of teacher spatial sorting and mobility and their consequences on spatial inequalities in access to skilled teachers.

I make several methodological and empirical contributions. First, I build a model of dynamic two-sided matching with non-transferable utility where forward-looking agents repeatedly meet in a single market and form matches according to their idiosyncratic preferences and expectations about their future matching opportunities. I propose a tractable large market approximation yielding an analytical solution to the model which directly maps agents' preferences into sorting and job-to-job transitions. Second, I show that this mapping is invertible such that the preferences of participating agents can be nonparametrically identified from data on realized matches. Third, I take this methodology to panel data on the allocation of public teachers in Peru and show that (i) the spatial disaggregation of labor demand, (ii) the concentration of labor supply in cities and (iii) the presence of home bias in teachers' preferences, lead to the existence of a *spatial job ladder*. As a result, low quality teachers get displaced in remote locations, creating a wide urban-rural gap in teacher quality, and move toward urban schools once they have accumulated experience and skills, which further magnifies this gap by one third. Finally, I show that dynamic wage contracts can reduce inequalities in access to skilled teachers by incentivizing teacher retention.

I start the analysis by leveraging countrywide panel data on the centralized allocation of public teachers in Peru. I document that remoteness is highly predictive of teacher sorting as high-skilled teachers concentrate in urban schools, while low-skilled teachers mostly work in remote locations. Teachers working in remote locations switch from job-to-job at a high rate to get closer to urban centers. This implies that teacher attrition rates in remote villages are three times greater than in cities. Movers are, on average, of higher quality than those who replace them. Labor market dynamics thus seem to largely reinforce spatial inequalities in teacher quality and student achievement.

To understand what drives local labor demand and supply and how they translate into equilibrium sorting and career paths, I develop an empirical model of dynamic two-sided matching without transfers. Teachers and schools meet repeatedly in a single market over several time periods. The observed characteristics of both sides evolve endogenously according to their matching decisions. Agents are forward-looking and form preferences over observed and unobserved job/teacher attributes. I impose few assumptions on preferences and beliefs: (i) the systematic and unobserved part of the payoff functions are additively separable, (ii) the unobserved taste shocks are *iid* with a type-I upper tail and (iii) agents have rational expectations about their future match payoffs. I extend the concept of stability, widely used in static empirical models of two-sided matching, to this dynamic setting. I assume that the observed match in each period is stable with respect to teachers' and schools' lifetime utility and that beliefs about future aggregate states are consistent with their realizations.

To map preferences into sorting, I build on the static framework of Menzel (2015) and leverage the implications of stability in a large market setting where the number of agents on both sides grows to infinity. Stability implies that, in each period, each teacher is matched to her preferred job among the set of jobs that would be willing to hire her and vice versa. We can thus reinterpret the realized matches as the outcome of two dynamic discrete choice models with unobserved and endogenous choice sets. Under the assumption that shocks have a type-I upper tail, I show that the information contained in choice sets, that is necessary to characterize conditional choice probabilities, can be summarized into sufficient statistics called *inclusive values*. In the limit economy, inclusive values converge to the unique solution of a fixed-point problem, which explicitly models the dependence between preferences and choice sets. This allows us to derive an analytical expression for the equilibrium conditional choice probabilities and map preferences into sorting.

I show that the mapping between preferences and observed sorting is invertible. The joint surplus function can be nonparametrically identified from data on realized matches. Under appropriate exclusion restrictions or with the availability of additional data, preferences can be separately identified from the joint surplus. I provide these results in two settings: (i) finite horizon and nonstationarity of preferences and aggregate states and (ii) infinite horizon and stationarity. I then propose a maximum likelihood estimator that can be tractably used for a parametric version of this framework.

Equipped with this methodology, I identify and estimate teachers' and schools' preferences from data on observed matches within the centralized assignment procedure in Peru. To separately identify preferences from the joint surplus, I use additional data on how schools rank the applicants they interview. The estimated preference parameters indicate that (i) geographical proximity to home is highly predictive of teachers' preferences and (ii) schools highly value observed measures of teacher quality, such as experience. This results in the existence of a spatial job ladder. As labor demand is widely scattered while teachers' home location is concentrated in cities, fact (i) implies that teachers have a strong distaste for remote locations putting rural schools at the bottom of the ladder and urban schools at the top. As the number of jobs located in urban centers is limited, fact (ii) implies that excess supply is rationed based on quality such that high-skilled teachers concentrate in cities while low-skilled teachers are matched to remote schools. The spatial job ladder also has important consequences on labor market dynamics. Teachers accumulate experience and human capital throughout their career and climb up the ladder by matching closer to home. As a consequence, rural schools fail to retain skilled teachers and sustain disproportionately low levels of teaching experience and quality. Overall, I estimate that teacher mobility along the spatial job ladder explains one third of the urban-rural gap in teacher quality.

I then investigate the effectiveness of dynamic wage contracts aimed at slowing down labor mobility and mitigating its adverse effects on spatial inequalities through retention bonuses. To do so, I simulate the equilibrium response to a policy that would impose a minimum contract length in exchange for appropriate compensation to prevent teachers from moving up the ladder. If compensation is too low, this policy creates large shortages as it forces teachers to commit and prevents them from rematching ex-post. This highlights a key trade off between recruitment and retention in the presence of a job ladder. Bonuses that would negate this adverse sorting effect amount to a 20-40% wage increase depending on the contract length.

I conclude the analysis with a thought experiment simulating the equilibrium in a counterfactual scenario where teachers' home locations would be scattered across the country instead of being concentrated in cities. As proximity to home is no longer associated with proximity to cities, the spatial job ladder collapses. Teachers still aim to match close to home but face little competition for these positions. Consequently, high quality teachers are no longer disproportionately matched to urban schools. The rate at which teachers switch jobs drops by half. Job-to-job flows are no longer directed from rural schools toward urban schools which shuts down urban-rural inequalities in attrition. This suggests that designing policies targeting the root causes of the existence of the spatial job ladder, such as investing in training local teachers, might be more effective than aiming at slowing down its symptoms through recruitment or retention policies.

Related literature

This paper relates and contributes to several strands of the literature. First, I contribute to a growing literature at the intersection of industrial organization and econometrics studying the empirical content of two-sided matching models with non-transferable utility (NTU).[1] Several papers investigate, in a static setting, how preferences of participating agents can be identified from reported preferences (Fack et al., 2019; Agarwal and Somaini, 2020) or realized matches (Menzel, 2015; Diamond and Agarwal, 2017; He et al., 2022; Agarwal and Somaini, 2022; Ederer, 2022). Yet, there are few equivalent results for models of dynamic two-sided matching, despite being increasingly studied in the matching theory literature.[2] A handful of papers study waitlist mechanisms (Agarwal et al., 2021; Waldinger, 2021; Verdier and Reeling, 2022) or include dynamics in college admissions/school choice models (Larroucau and Rios, 2020). However, these papers study priority-based assignment mechanisms where the preferences of one side of the market are known ex-ante. This paper contributes to this literature by building an empirical model of dynamic two-sided matching where the preferences of both sides of the market are unknown. It extends the concept of stability to a dynamic setting to map preferences into sorting and show that preferences can be nonparametrically identified from data on realized matches.

Second, I contribute to a large literature in labor and urban economics studying the causes and welfare consequences of spatial skill sorting (Moretti, 2013; Diamond, 2016; Diamond and Gaubert, 2022). I provide a unifying explanation for the lack of access to local services requiring skilled labor in remote areas. As labor demand is inherently spatially scattered in these markets while human capital concentrates in cities, the presence of home bias generates the existence of a spatial job ladder, which has drastic consequences on spatial sorting and mobility. The tools provided in this paper could help understand the causes and welfare

[1] Following the seminal work of Choo and Siow (2006), a large literature on empirical models of two-sided matching with transferable utility (TU) has evolved separately (Fox, 2010; Galichon and Salanié, 2022; Gualdani and Sinha, Forthcoming).

[2] See Baccara and Yariv (2021) for a survey of this rapidly growing literature.

consequences of important phenomenons such as the existence of medical deserts. I also contribute to the literature studying sorting and labor mobility through on-the-job search models (Moscarini and Postel-Vinay, 2018) by showing that labor market dynamics can alternatively be rationalized by a frictionless dynamic two-sided matching model.[3]

Third, I contribute to a recent literature on equilibrium models of the teachers' labor market (Tincani, 2021; Biasi et al., 2021; Bates et al., 2022; Bobba et al., 2021). These papers study teacher sorting through static models of two-sided matching. I provide a general framework nesting the existing approaches and derive conditions under which preferences are nonparametrically identified from realized matches. I also show the importance of labor market dynamics in shaping teacher sorting, which is typically ignored in this literature.

Fourth, I relate to a large body of work in the economics of education studying the causes and consequences of teacher attrition (Boyd et al., 2005; Falch and Strøm, 2005; Falch, 2011; Hanushek et al., 2016; Bonhomme et al., 2016). This paper provides a unifying framework to study teacher sorting and mobility. I show that attrition is mostly caused by teachers leaving rural schools by climbing up the spatial job ladder. I then provide new evidence on the costs of attrition by quantifying its role in shaping urban-rural inequalities in access to skilled teachers.

Finally, this paper relates to a literature in public economics studying the design of incentives to recruit and retain civil servants in underprivileged areas. Several papers explored the role of wage incentives on recruitment, effort and retention but found mixed results on retention (Deserranno, 2019; Leaver et al., 2021; Bobba et al., 2021). Instead, I explore the effect of dynamic wage contracts designed to increase retention. I show that these policies can have strong adverse effects on recruitment if teachers are not properly compensated for the implied lack of flexibility. This highlights a trade off between recruiting and retaining workers in the presence of a job ladder.

[3]This raises the question of whether search frictions and idiosyncractic preferences over job attributes can be separately identified from matched employer-employee data in typical search models. I plan to investigate this in future work.

Overview

Section 2 briefly describes the institutional setting and the data. Section 3 presents relevant descriptive evidence. Section 4 introduces the equilibrium model and characterizes the mapping between preferences and sorting. Section 5 states the main identification results. Sections 6 and 7 discuss the empirical strategy and the results. Section 8 concludes.

2 Context and Data

In this section, I briefly describe the different types of contracts under which teachers can be employed and how the centralized clearinghouse allocating teaching positions is organized. I then give a short summary of the different sources of data used throughout the paper.

2.1 Institutional Setting

Public teachers in Peru can be hired under two types of contracts. *Temporary* contracts last at least one year and can be renewed up to a second year. *Permanent* contracts can last indefinitely and are akin to usual civil servant contracts. Temporary contracts are paid a fixed rate that does not vary with experience. Permanent teachers can get promoted throughout their career to higher ranks in the civil servant scale system to get higher wages.[4] On the lowest scale, permanent teachers are paid the same wage as temporary teachers. On the highest scale, permanent teachers are paid 75% more. In an effort to make remote schools and schools with difficult teaching conditions more attractive, the Ministry of Education provides wage bonuses to teachers working in schools belonging to a predetermined set of categories (see Appendix B.1 for more details). However, the overall spatial variation in wages induced by this bonus scheme remains very limited.[5]

Since 2015, the allocation of new teaching positions is organized through a biennial centralized clearinghouse. All teachers without a permanent contract seeking a position have

[4]Promotions are awarded through a national standardized evaluation and a decentralized evaluation made by a committee evaluating teachers' performance and professional career.

[5]Table A.2 shows summary statistics on various job characteristics. One standard deviation in wages corresponds to only 16% of the minimum wage.

to go through this process.[6] The allocation is organized into three steps that take place at the end of the academic year from November to January. First, all applicants participate in the *national competency exam*, which assesses their skills and curricular knowledge. If their score falls above a given threshold, teachers become eligible for permanent contracts. Second, eligible applicants participate in the allocation of permanent positions. Teachers form an unconstrained list of choices within the same province and are then interviewed by their three top schools.[7] Schools then make offers to their preferred candidates. Finally, all remaining teachers participate in the allocation of temporary contracts. In this step, schools are passive and cannot express their preferences. Teachers are ranked according to their test score and choose among the set of available positions by order of priority. Finally, schools which did not manage to recruit anyone can resort to hiring non-certified teachers through temporary contracts. More details about the test and the timing of the allocation mechanism are available in Appendix B.1.

2.2 Data

I combine several sets of administrative data provided by the Ministry of Education in Peru to create a unique record of teachers' movements across schools throughout their careers. Most importantly, I observe teachers repeatedly applying through the centralized assignment platform, allowing for a deeper investigation of the causes of these movements.[8] I briefly describe these data sources below.[9]

Teacher assignment data: I observe a panel including all positions and teachers employed in the public sector in Peru from 2015 to 2021. For each teacher in each year, I know in which position they work, which type of contract they hold and which wage they receive. I supplement these data with additional sources of information on jobs and teachers (see

[6] Permanent teachers seeking to get transferred to another school need to go through a separate decentralized procedure.

[7] As schools cannot interview more than ten applicants, capacity constraints are rationed using test scores as priorities.

[8] Bobba et al. (2021) use similar data but do not exploit the panel dimension of the data and abstract away from the role of labor dynamics.

[9] I restrict the analysis to public primary education. Primary schools are evenly distributed across the country while secondary schools are sometimes missing in remote locations. Teachers' spatial sorting is thus a more salient concern for primary education.

Appendix B.2 for details on the data construction). First, I link teachers national ID to the Household Targeting System (SISFOH) data containing information about their poverty status, education level and, most importantly, their home location. It also allows me to link each teacher to other members of their household and know their marital status, whether they have children and whether they live with their parents. Second, I link each job to the School Census containing a wide set of locality and school characteristics. I observe whether a given locality has access to basic amenities such as water and electricity. I also have information about the precise geolocalization of the school and the level of poverty and rurality of its locality.

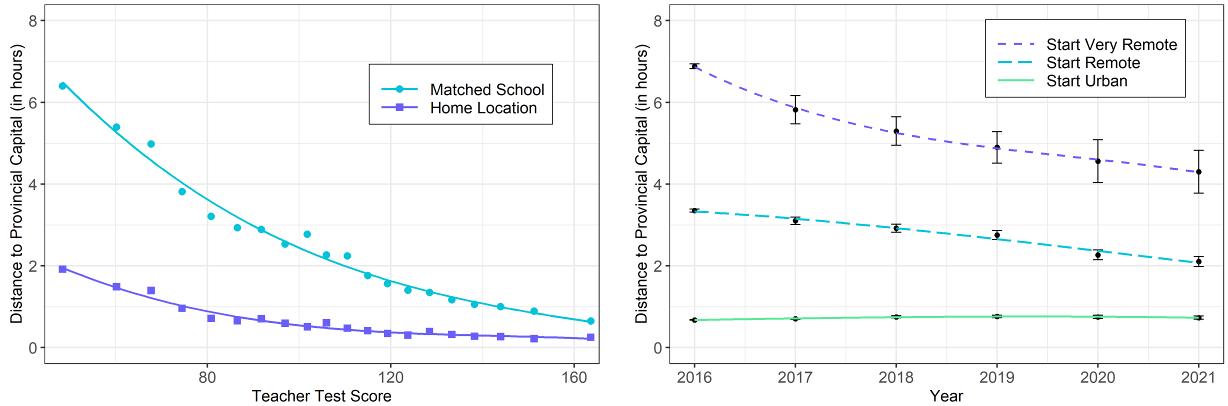
Centralized assignment data: I have access to detailed information about the biennial countrywide centralized assignment of new teaching positions from 2015 to 2019. I observe the universe of participating applicants and positions in each step of the mechanism. The dataset contains information on applicants' test scores at the national competency exam. I also have access to detailed information on the allocation of permanent positions. In particular, I observe the set of applicants each school interviews and how they rank them. The dataset also records the final match for both temporary and permanent contracts in each year. Finally, and key to my analysis, this dataset can also be linked to the teacher assignment data in order to track applicants and positions across years.

Note that the teacher assignment data and centralized assignment data do not necessarily overlap. The centralized assignment data contains information about the set of applicants and vacancies that end up staying unmatched and thus do not appear in the teacher assignment data. The teacher assignment data contains information about applicants already holding a permanent contract and non-certified teachers who are not allowed to participate in the centralized allocation mechanism.

3 Descriptive Evidence

Jobs are geographically scattered across locations which greatly differ in their level of remoteness and amenities (see Table A.2). One quarter of positions are located more than four hours away from the provincial capital. One third of the available positions are located in

Figure 1: Sorting and Movements Across Locations



a) Sorting **b) Job-to-Job Transitions**

Notes. This figure uses the teacher assignment data. Panel A plots binned averages of the distance (in hours) between applicants' home location and the provincial capital as well as applicants' matched location and the provincial capital. Each bin is equally spaced using vigintiles of the distribution of teachers' test scores. Panel B plots the evolution of the distance between teachers' matched schools and the provincial capital over the period 2016-2021 for three groups of teachers starting at different levels of remoteness in 2016.

schools that have no access to electricity or water. In contrast, teachers' home locations are concentrated in cities: 82% of applicants live in a provincial capital (see Table A.3). In this section, I provide suggestive evidence that this creates an imbalance between local supply and demand, which shapes teacher spatial sorting and mobility and translates into spatial inequalities in teaching quality.

3.1 Spatial Sorting and Mobility

I first leverage data on the centralized assignment mechanism to document how teachers sort across locations, in the cross-section, based on observed measures of teacher quality. Panel A of Figure 1 plots the relationship between teachers' test scores and the distance between their matched school and the provincial capital. I find that high scoring teachers are disproportionately matched to schools located close to urban centers. Specifically, teachers in the top decile of the score distribution work on average 45 minutes away from the provincial capital, while teachers in the bottom decile work 6 hours away. This pattern is not driven by spatial disparities in the quality of local workers as low scoring teachers live close to urban centers, on average.

I then document how teachers move across locations throughout their careers using the panel structure of the data. Among the set of teachers who started a new job in 2016, 40% switched jobs at least twice over the period 2016-2021. This number decreases to 25% for teachers starting in urban areas in 2016 while it increases to 60% for teachers starting in remote locations. Panel B of Figure 1 plots the time trend of the remoteness of teachers' matched schools. I find that as teachers switch jobs, they also switch locations and progressively move closer to urban centers. The rate at which they move increases with the remoteness of their starting job. Teachers who start in remote locations get closer to the provincial capital by almost three hours. In contrast, teachers who already start in proximity to urban centers do not get closer by switching jobs.

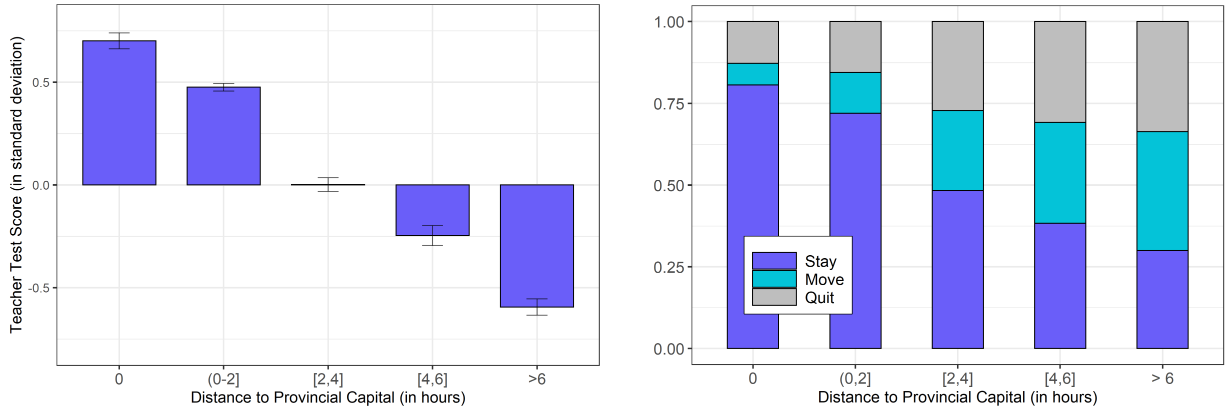
These patterns suggest that teachers have a distaste for remoteness, potentially creating an imbalance between local labor supply and demand. Excess supply in urban locations seems to be rationed through observed measures of teacher quality such as test scores. As a result, low-quality teachers work temporarily far from urban centers and switch from job-to-job at a high rate to move closer to cities.

3.2 Spatial Inequalities

Teacher spatial sorting and movements across locations have direct consequences on the distribution of teaching quality across space. The sorting patterns described in Figure 1 directly imply that teachers working in remote schools are less qualified than teachers working in cities. Panel A of Figure 2 shows the resulting urban-rural gap in teacher test scores. Teachers working in the provincial capital score on average 1.3 standard deviations higher than teachers working in very remote schools located more than 6 hours away from the provincial capital. Similarly, the magnitude and direction of the job-to-job flows described in Figure 1 imply that schools located in rural areas face high attrition rates. Panel B of Figure 2 shows that between 2016 and 2018, the teacher attrition rate in schools located in remote villages is 50 percentage points higher than in schools located in the provincial capital.

It has been widely documented that teacher attrition negatively affects student learning through disruption and the resulting loss of experience (Hanushek et al., 2016). I provide descriptive evidence in line with these results. I compare movers with the teachers who replaced them in 2018 over several dimensions. Table 1 shows that movers are significantly

Figure 2: Spatial Inequalities



a) Teacher Test Score **b) Transition 2016-2018**

NOTES. This figure uses the teacher assignment data and documents urban-rural inequalities in teacher test scores and in the type of job transitions between 2016 and 2018. Panel A shows the average test score of matched teachers for several bins of the distance to the provincial capital. Panel B shows the share of teachers that stayed in the same school, moved to another school or quit teaching in the public sector for several bins of the schools' distance to the provincial capital.

more experienced than newcomers. Eleven percent of newcomers have no prior experience. Newcomers are 6 percentage points more likely to be non-certified. I also find that movers score on average 0.16 standard deviations higher at the national exam compared to newcomers. This is quite substantial as this corresponds to 12% of the urban-rural gap in teacher test scores.

As the literature points out that observable measures of teacher quality can be poor predictors of teacher value added (Rockoff, 2004), I also provide additional evidence in Appendix C that movers are of significantly higher value added than newcomers. To do so, I follow Chetty et al. (2014a) and estimate teacher value added using matched teacher-classroom data. I find that movers' value added is 0.10 standard deviations higher than newcomers. This corresponds to 50% of a standard deviation in value added which is quite substantial. This result is consistent with evidence of large value added gains through experience in the early stages of teachers' careers (Rockoff, 2004; Rivkin et al., 2005; Araujo et al., 2016).

Overall, these findings suggest that teacher sorting and mobility have important consequences on spatial inequalities. Schools located in remote areas fail to attract high-quality teachers and face high attrition rates. As movers are replaced by teachers of lower experience and quality, labor market dynamics sustain and exacerbate spatial inequalities in teaching

Table 1: Movers vs. Newcomers

	Movers	Newcomers	Difference
Competency Score	0.545	0.389	0.156 (0.017)
Non-certified	0.125	0.181	0.056 (0.006)
Value Added	0.057	-0.042	0.099 (0.031)
Experience			
No Experience	0	0.109	-0.109 (0.004)
Between 1 and 2 years	0.166	0.190	-0.024 (0.006)
Between 3 and 5 years	0.308	0.250	0.058 (0.007)
Between 6 and 10 years	0.271	0.187	0.084 (0.007)
Above 10 years	0.123	0.083	0.040 (0.005)

NOTES. This table uses the centralized assignment data to compare the temporary teachers that moved to a different school between 2016 and 2018 to the teachers that were hired to replace them in 2018 over several dimensions. Details on how value added is estimated are in Appendix C.

quality and student achievement.

The suggestive evidence presented in this section highlights the need for further investigation on the causes of teacher spatial sorting and mobility. More specifically, it is crucial to understand (i) how teachers trade off geographical proximity against other job/locality characteristics and (ii) how schools ration excess labor supply. To do so, I develop next a general model of dynamic two-sided matching mapping teachers' and schools' preferences into equilibrium sorting and job-to-job flows.[10]

4 Empirical Model of Dynamic Two-Sided Matching

In this section, I build on Menzel (2015) and develop a general model of dynamic two-sided matching with non-transferable utility incorporating the following features. First, an empirical model of teachers' and schools' preferences able to quantify how agents trade off a potentially large set of job and teacher attributes. Second, state variables that evolve over time depending on agents' matching decisions. Third, forward-looking agents that anticipate the effect of their current action on the future. Finally, an equilibrium concept mapping

[10] As reallocation entails costly migration decisions, embedding these decisions within a dynamic framework is crucial to disentangle moving costs from taste for specific locality characteristics such as amenities or remoteness (Kennan and Walker, 2011).

these elements into sorting and job-to-job flows.

This section is divided into two parts. I first describe the environment, the preference model and introduce the equilibrium concept. Then, I characterize the mapping between preferences and realized sorting.

4.1 Model

Throughout this section, I refer to one side of the market as teachers and the other side as schools. I assume that matching is one-to-one meaning that each school only opens one vacancy. Alternatively, we can consider jobs as separate entities such that matching is one-to-one by design. To simplify the analysis, I use a large market approximation to obtain a tractable analytical expression linking primitives to equilibrium sorting. I start by introducing the relevant parts of the model in the finite economy before defining the asymptotic sequence that characterizes the limit economy.

4.1.1 Timing

I consider a repeated matching game where a set of schools and teachers meet in a single market in each period. An extension considering the opposite polar case where matches are irreversible is in Appendix F. Time is discrete and indexed by $t = 1, ..., T$. I assume that $T \in [1, \infty]$ meaning that the model nests both the static case $T = 1$, which corresponds to Menzel (2015), and the infinite horizon case $T = \infty$. For simplicity, I assume that the set of participating agents and schools is fixed over time. However, this framework can be extended to settings where agents enter and exit the market sequentially in an exogenous way. Teachers are indexed by $i \in \mathcal{I} = \{1, ..., n_w\}$ and schools are indexed by $j \in \mathcal{J} = \{1, ..., n_m\}$. In each period t, a matching is formed summarized by the functions μ_{wt}, which maps $\mathcal{I}$ to $\mathcal{J} \cup \{0\}$ and μ_{mt}, which maps $\mathcal{J}$ to $\mathcal{I} \cup \{0\}$ where 0 is the option of staying unmatched. The resulting matching is summarized in $\boldsymbol{\mu} = (\mu_{wt}, \mu_{mt})_{t=1}^{T}$.

Teacher i and school j are characterized in each period t by a set of observed characteristics which are collected into two vectors $\boldsymbol{x}_{it}$ and $\boldsymbol{z}_{jt}$. I fix the probability distribution functions of their initial value $\boldsymbol{x}_{i1}$ and $\boldsymbol{z}_{j1}$ as $w_1(\boldsymbol{x})$ and $m_1(\boldsymbol{z})$ with support $\mathcal{X}_1$ and $\mathcal{Z}_1$ and assume that they are exogenous. Individual states evolve stochastically over time depending on agents' matching decisions $\boldsymbol{\mu}$ through the Markov transition probability distribution functions $w_{t+1}(\boldsymbol{x}_{it+1}|\boldsymbol{x}_{it}, \boldsymbol{z}_{\mu_{wt}(i)t})$ and $m_{t+1}(\boldsymbol{z}_{jt+1}|\boldsymbol{x}_{\mu_{mt}(j)t}, \boldsymbol{z}_{jt})$. I denote separately $m_{0t+1}(\boldsymbol{z}_{jt+1}|\boldsymbol{z}_{jt})$ and $w_{0t+1}(\boldsymbol{x}_{it+1}|\boldsymbol{x}_{it})$ the transition probability distribution functions for agents choosing to stay unmatched. Throughout the rest of the paper, I drop the index t from the functions w, w_0, m and m_0 for simplicity. Finally, individual matching decisions in period t aggregate into the probability distribution functions of observed states w_{t+1} and m_{t+1} as follows:

$$w_{t+1}(\boldsymbol{x}, \boldsymbol{\mu}) = \int_{\mathcal{X}_t} \int_{\mathcal{Z}_t} w(\boldsymbol{x}|s, h) f_t(s, h) dh ds + \int_{\mathcal{X}_t} w_0(\boldsymbol{x}|s) f_t(s, *) ds$$

$$m_{t+1}(\boldsymbol{z}, \boldsymbol{\mu}) = \int_{\mathcal{X}_t} \int_{\mathcal{Z}_t} m(\boldsymbol{z}|s, h) f_t(s, h) dh ds + \int_{\mathcal{Z}_t} m_0(\boldsymbol{z}|h) f_t(*, h) dh$$

where $f_t(x, z)$, $f_t(x, *)$ and $f_t(*, z)$ are, respectively, the joint probability distribution function of the characteristics of matched teachers and schools, of unmatched teachers and of unmatched schools in period t. A formal definition of these functions is in the next subsection.

4.1.2 Preferences and Beliefs

Agents are forward looking and anticipate how their current decision affects their lifetime utility. I define the lifetime utility that teacher i gets from being matched with school j in period t as:

$$U_{ijt} = U_t(\boldsymbol{x}_{it}, \boldsymbol{z}_{jt}) + \sigma\eta_{ijt} + \beta_w \int \overline{U}_{it+1}(\boldsymbol{x}_{it+1}) w(\boldsymbol{x}_{it+1}|\boldsymbol{x}_{it}, \boldsymbol{z}_{jt}) d\boldsymbol{x}_{it+1}$$

whereas the lifetime utility that school j gets from being matched with teachers i in period t is defined as:

$$V_{ijt} = V_t(\boldsymbol{x}_{it}, \boldsymbol{z}_{jt}) + \sigma\epsilon_{ijt} + \beta_m \int \overline{V}_{jt+1}(\boldsymbol{z}_{jt+1}) m(\boldsymbol{z}_{jt+1}|\boldsymbol{x}_{it}, \boldsymbol{z}_{jt}) d\boldsymbol{z}_{jt+1}$$

Agents' lifetime utility is first composed of a *flow utility*, which agents enjoy from their match in period t. It includes a systematic part $U_t(\boldsymbol{x}_{it}, \boldsymbol{z}_{jt})$ and $V_t(\boldsymbol{x}_{it}, \boldsymbol{z}_{jt})$, where the functions (U_t, V_t) are unknown, and unobserved shocks $(\eta_{ijt}, \epsilon_{ijt})$ which are assumed to enter additively.

σ is a normalizing sequence which is defined later. I impose the following assumptions on these objects.

Assumption 1 *(i) U_t and V_t are uniformly bounded in absolute value and $p \geq 1$ times differentiable with uniformly bounded partial derivatives in $\mathcal{X} \times \mathcal{Z}$ for all t.*
(ii) ϵ_{ijt} and η_{ijt} are drawn independently from $\boldsymbol{x}_{it}$ and $\boldsymbol{z}_{jt}$ from a distribution with absolutely continuous c.d.f. $G(s)$ and density $g(s)$. The upper tail of the distribution $G(s)$ is of type I with auxiliary function $a(s) = \frac{1-G(s)}{g(s)}$.

Assumption 1.(i) is a standard regularity condition which ensures that the functions U_t and V_t are well-behaved. Assumption 1.(ii) imposes restrictions on the upper tail of the distribution of ϵ_{ijt} and η_{ijt} but leaves the lower tail unrestricted. As the number of teachers and schools grows to infinity, the number of independent draws of ϵ_{ijt} and η_{ijt} also grows. All draws of ϵ_{ijt} and η_{ijt} from the lower tail of their distribution thus become inconsequential in determining which is the most preferred school or teacher. As in Menzel (2015), I assume that G belongs to a class of distributions which has a type I extreme value distributed upper tail.[11] Note that this class of functions encompasses most of the parametric distributions traditionally used in discrete choice models. For the Gamma distribution or the Gumbel distribution, Assumption 1.(ii) holds for $a(s) = 1$. For the standard normal distribution, it holds for $a(s) = \frac{1}{s}$.

Agents' lifetime utility is then composed of a *continuation value*. Teachers and schools internalize that their matching decisions affect their future states and thus their future payoffs. This continuation value is the discounted sum of future expected payoffs. I assume that teachers discount future utility at a rate β_w, while schools discount at a rate β_m. I define $\overline{U}_{it+1}$ and $\overline{V}_{jt+1}$ as agents' expectations about $U_{i\mu_{t+1}(i),t+1}$ and $V_{\mu_{t+1}(j)j,t+1}$ conditional on their future state variables. As agents only observe their current states, I integrate this object over the transition distribution functions m and w. I impose the following assumptions on $\overline{U}_{it+1}$ and $\overline{V}_{jt+1}$.

[11]This class of distribution is also called the domain of attraction of the Gumbel distribution (Resnick (1987))

Assumption 2 *For each period t, each teacher $i = 1, ..., n_w$ and each school $j = 1, ..., n_m$:*

$$\overline{U}_{it+1}(x) = \mathbb{E}_{\mathcal{S}_t}[U_{i\mu_{t+1}(i),t+1}|x_{i,t+1} = x] \quad \text{and} \quad \overline{V}_{jt+1}(z) = \mathbb{E}_{\mathcal{S}_t}[V_{\mu_{t+1}(j)j,t+1}|z_{j,t+1} = z]$$

where $\mathcal{S}_t$ is the information set of participating agents in period t:

$$\mathcal{S}_t = \{(\tilde{m}_s)_{s=t}^T, (\tilde{w}_s)_{s=t}^T, G, (U_s)_{s=t}^T, (V_s)_{s=t}^T\}$$

Assumption 2 states that agents have rational expectations about the lifetime utility they will get from their future match conditional on their future state. Agents have incomplete information about the exact realization of the future observed and unobserved states of other participants. Instead, I assume that they know the distribution of taste shocks and the payoff functions for all subsequent periods. I also assume that they form beliefs $(\tilde{m}_s)_{s=t}^T, (\tilde{w}_s)_{s=t}^T$ about the probability distribution functions of future aggregate states $(m_s(\boldsymbol{\mu}))_{s=t}^T, (w_s(\boldsymbol{\mu}))_{s=t}^T$. I assume that individual agents are atomistic and internalize that their decisions only influence their own future state and not the future aggregate states.

4.1.3 Normalizations

For the limit economy to predict sorting patterns that are consistent with the finite economy, I make a few technical assumptions. First, I specify the utility of the outside option as follows:

$$U_{i0t} = \sigma \max_{k=1,\ldots,J} \eta_{i0,k} + \beta_w \int \overline{U}_{it+1}(\boldsymbol{x}_{it+1}) w_0(\boldsymbol{x}_{it+1}|\boldsymbol{x}_{it}) d\boldsymbol{x}_{it+1}$$

$$V_{0jt} = \sigma \max_{k=1,\ldots,J} \epsilon_{0j,k} + \beta_m \int \overline{V}_{jt+1}(\boldsymbol{z}_{jt+1}) m_0(\boldsymbol{z}_{jt+1}|\boldsymbol{z}_{jt}) d\boldsymbol{z}_{jt+1}$$

I then assume that the size of the market is denoted by n and impose the following normalizations on the asymptotic sequence:

Assumption 3 *The asymptotic sequence is controlled by $n = 1, 2, ...$ and we define:*

(i) $n_w = [\exp(\gamma_w)n]$, $n_m = [\exp(\gamma_m)n]$

(ii) $J = [n^{1/2}]$

(iii) $\sigma = \frac{1}{a(b_n)}$ *where* $b_n = G^{-1}(1 - n^{-1/2})$

Assumption 3.(i) allows to flexibly control the relative sizes of each side of the market through the parameters γ_w and γ_m. Assumption 3.(ii) guarantees that, in each period t, the probability that teachers or schools stay unmatched does not degenerate to zero in the limit. If the size of the outside option does not grow with the size of the market, the probability that it becomes dominated by an alternative option will tend to one given that taste shocks have unbounded support. Assumption 3.(iii) controls the scale of the unobserved shocks such that both the unobserved and systematic parts of the payoffs jointly determine agents' choices in the limit. Given that U_t and V_t are bounded and that the support of taste shocks is unbounded, U_t and V_t would become irrelevant in the limit without this restriction. More specifically, if G is Gumbel, then $b_n \asymp \frac{1}{2}\log(n)$ and $\sigma_n = 1$. If taste shocks are standard normal, $b_n \asymp \sqrt{\log n}$ and $\sigma_n \asymp b_n$ and for Gamma distributed taste shocks, $b_n \asymp \log(n)$ and $\sigma_n = 1$.

4.1.4 Equilibrium

To rationalize the observed matching and link it to the primitives of the model, I impose the following equilibrium assumptions.

Assumption 4 *The match $\boldsymbol{\mu}$ is such that, for all $i = 1, ..., n_w$ and $j = 1, ..., n_m$ in each period t:*

(i) Individually rational in period t: $U_{i\mu_{wt}(i)t} \geq U_{i0t}$ and $V_{\mu_{mt}(j)jt} \geq V_{0jt}$.

(ii) No blocking pairs in period t: There exists no pair (i, j) such that $U_{ijt} > U_{i\mu_{wt}(i)t}$ and $V_{ijt} > V_{\mu_{mt}(j)jt}$.

(iii) Consistent beliefs about aggregate states:

$$\tilde{w}_{t+1}(\boldsymbol{x}) = w_{t+1}(\boldsymbol{x}, \boldsymbol{\mu}) = \int_{\mathcal{X}_t}\int_{\mathcal{Z}_t} w(\boldsymbol{x}|s,h)f_t(s,h)dhds + \int_{\mathcal{X}_t} w_0(\boldsymbol{x}|s)f_t(s,*)ds$$

$$\tilde{m}_{t+1}(\boldsymbol{z}) = m_{t+1}(\boldsymbol{z}, \boldsymbol{\mu}) = \int_{\mathcal{X}_t}\int_{\mathcal{Z}_t} m(\boldsymbol{z}|s,h)f_t(s,h)dhds + \int_{\mathcal{Z}_t} m_0(\boldsymbol{z}|h)f_t(*,h)dh$$

Assumption 4 (i) and (ii) impose that the outcome of the match is stable in each period t given agents' lifetime utility. This means that there should exist no teacher-school pair that would prefer to break their current match to rematch together instead. Note that I impose no restriction on preferences such that within a single period there could exist many different stable outcomes (Roth and Sotomayor, 1992). I define the teacher-optimal stable match in period t as μ_t^W and the firm-optimal stable match in period t as μ_t^M. Assumption 4 (iii) imposes that agents' beliefs about the distribution of future aggregate states are consistent with the actual realized equilibrium distributions.

4.2 Linking Primitives to Equilibrium Sorting

Equilibrium sorting and job-to-job transitions are summarized by the joint distributions of matched characteristics in each period t. I define this distribution for a given random matching μ_t from a finite economy indexed by n as follows:

$$F_{nt}(x_{it}, z_{jt}|\mu_t) = \frac{1}{n}\sum_{i=0}^{n_w}\sum_{j=0}^{n_m}\mathbb{P}(x_{it} \leq x, z_{jt} \leq z, \mu_{wt}(i) = j)$$

I then denote F_t the limit of the distribution function F_{nt} as the size of the market n grows to infinity. I also define the joint density of matched characteristics as f_t. The goal of this section is to express f_t as a function of the primitives of the model.

The proof is divided in four steps. First, I show that stability implies that the realized matches in each period can be interpreted as the outcome of two dynamic discrete choice models with endogenous and unobserved choice sets called *opportunity sets*. Second, I consider a simplified economy with observed and exogenous choice sets and derive the limit of conditional choice probabilities. Third, I show that the information contained in opportunity sets which is necessary to characterize conditional choice probabilities can be summarized into sufficient statistics called *inclusive values*. Finally, I show that, in the limit, these inclusive values converge to the unique solution of a fixed point problem. This allows to characterize conditional choice probabilities and, in turn, f_t as a function of agents' payoff functions.

4.2.1 Opportunity Sets

Given an arbitrary match μ, I define the opportunity set of a teacher in period t as the set of schools that would be willing to hire her instead of its currently matched employee in the same period. Similarly, the opportunity set of a school is the set of teachers that would be willing to quit their current employer to work there. Formally, I define the opportunity set faced by a given teacher $i \in \mathcal{I}$ in period t given a match μ as:

$$M_{it}(\mu) = \{j \in \mathcal{J} : V_{ijt} \geq V_{\mu_{mt}(j)jt}\}$$

Similarly, I define the opportunity set of school $j \in \mathcal{J}$ as:

$$W_{jt}(\mu) = \{i \in \mathcal{I} : U_{ijt} \geq U_{i\mu_{mt}(i)t}\}$$

I state the first important result:

Proposition 1 *Consider a match μ^* satisfying Assumption 4, for all $i = 1, ..., n_w$ and $j = 1, ..., n_m$:*
(i) For all $t = 1, ..., T$:

$$U_{i\mu^*_{wt}(i)t} = \max_{k \in M_{it}(\mu^*) \cup \{0\}} U_{ikt} \quad \textit{and} \quad V_{\mu^*_{mt}(j)jt} = \max_{l \in W_{jt}(\mu^*) \cup \{0\}} V_{ljt}$$

(ii) Under Assumption 2, for all $t < T$:

$$\overline{U}_{it+1}(x) = \mathbb{E}_{\mathcal{S}_t}\left[\max_{k \in M_{it+1}(\mu^*) \cup \{0\}} U_{ikt+1} | x_{it+1} = x\right]$$

$$\overline{V}_{jt+1}(z) = \mathbb{E}_{\mathcal{S}_t}\left[\max_{l \in W_{jt+1}(\mu^*) \cup \{0\}} V_{ljt+1} | z_{jt+1} = z\right]$$

See Appendix D.1 for a proof of this result. Proposition 1.(i) states that a match μ^*_t is stable if and only if each teacher $i \in \mathcal{I}$ is matched to her preferred school among her opportunity set and each school $j \in \mathcal{J}$ is matched to its preferred teacher among its opportunity set. Proposition 1.(ii) thus follows immediately from (i). This result implies that an equilibrium match μ^* can be rewritten as the outcome of two dynamic discrete choice models where each agent's choice set is its opportunity set. The characterization of optimal choices within dynamic discrete choice models has been extensively studied and used in a variety of settings. However, existing results cannot be transposed to this problem as opportunity sets (i) depend on agents' preferences and are thus unobserved and (ii) depend on the overall equilibrium match and are thus potentially endogenous. The rest of the proof shows that these two issues can be circumvented thanks to a large market approximation.

4.2.2 Conditional Choice Probabilities

To simplify the analysis, I start by characterizing the limit of conditional choice probabilities (CCPs) and expected future payoffs under arbitrary exogenous choice sets and by fixing the aggregate states distributions. I assume that $M_{it} = \{1, ..., J\}$ and $W_{jt} = \{1, ..., J\}$ for all t and I fix m_t and w_t for all t.

Proposition 2 *Consider a given teacher $i \in \mathcal{I}$. Under Assumption 1-3 we have:*
(i) For all t, as $J \to \infty$:

$$J\mathbb{P}(U_{ijt} \geq U_{ikt}, k = \{0, 1, ..., J\}|\boldsymbol{x}_{it}, \boldsymbol{z}_{jt}) \longrightarrow \frac{\exp\left\{U_t(\boldsymbol{x}_{it}, \boldsymbol{z}_{jt}) + \beta_w \int \overline{U}_{t+1}(s)w(s|\boldsymbol{x}_{it}, \boldsymbol{z}_{jt})ds\right\}}{\exp\left\{\beta_w \int \overline{U}_{t+1}(s)w_0(s|\boldsymbol{x}_{it})ds\right\} + \int \exp\left\{U_t(\boldsymbol{x}_{it}, h) + \beta_w \int \overline{U}_{t+1}(s)w(s|\boldsymbol{x}_{it}, h)ds\right\}m_t(h)dh}$$

$$\mathbb{P}(U_{i0t} \geq U_{ikt}, k = \{0, 1, ..., J\}|\boldsymbol{x}_{it}) \longrightarrow \frac{\exp\left\{\beta_w \int \overline{U}_{t+1}(s)w_0(s|\boldsymbol{x}_{it})ds\right\}}{\exp\left\{\beta_w \int \overline{U}_{t+1}(s)w_0(s|\boldsymbol{x}_{it})ds\right\} + \int \exp\left\{U_t(x_{it}, h) + \beta_w \int \overline{U}_{t+1}(s)w(s|\boldsymbol{x}_{it}, h)ds\right\}m_t(h)dh}$$

(ii) For all t:

$$\begin{aligned}\overline{U}_{t+1}(x) = \log\Bigg(&\exp\left\{\beta_w \int \overline{U}_{t+2}(s)w_0(s|x)ds\right\} \\ &+ \int \exp\left\{U_{t+1}(x, h) + \beta_w \int \overline{U}_{t+2}(s)w(s|x, h)\right\}m_{t+1}(h)dh\Bigg) + \log(J) + \gamma + o(1)\end{aligned}$$

where $\gamma \approx 0.5772$ is Euler's constant. See Appendix D.2 for a proof of this result. The same result holds symmetrically for the school side. Proposition 2 shows that, under the assumption that unobservables have a type-I upper tail, CCPs converge to the usual Logit formula when the number of alternatives grow to infinity. Similarly, expectations about future payoffs can be computed using the logsum formula commonly used in dynamic discrete choice models with type-I errors.[12]

Note that the conditional choice probability of choosing a particular alternative j would converge to zero if we do not weight it by J, the rate at which the total number of alternatives increases. Lemma 1 in Appendix D.4 establishes that the size of opportunity sets increases at a rate $\sqrt{n}$ which justifies Assumption 3.(ii).

4.2.3 Inclusive Values

I now introduce that opportunity sets are unobserved and endogenous and show that the implications of Proposition 2 allow us to tackle both of these issues.

Endogeneity arises as shifting teacher i's unobserved preferences in a given period t could affect her own opportunity set by triggering a chain of rematches. As in Menzel (2015), I find that, as the size of the market increases, the probability for such an event to occur vanishes to zero. This result stems from two implications of Proposition 2: (i) the probability that school j rematches with a specific teacher i vanishes to zero as the size of opportunity sets increases to infinity and (ii) the probability of choosing the outside option instead, which would terminate such a chain of rematches, is nondegenerate in the limit. This implies that the dependence between taste shocks and opportunity sets vanishes in the limit. Note that this claim can only be proven for the opportunity sets derived from the school-optimal and teacher-optimal stable matchings μ_t^M and μ_t^W. The distribution of taste shocks conditional on opportunity sets is only well defined for the extremal matchings, given that they are the only stable matchings that always exist irrespective of the size of the market. This result is formalized in Lemmas 2 and 3 in Appendix D.4.

I now consider a sequence of school-optimal stable matches μ^M. As opportunity sets' endogeneity vanishes in the limit for extremal matchings, we can then use Proposition 2 (i) to bound[13] teachers' CCPs in period t, assuming that we would observe the corresponding

[12]These CCPs exhibit the independence of irrelevant alternatives (IIA) property which limits the model's ability to allow for flexible substitution patterns. Introducing unobserved discrete types or random coefficients to relax this assumption is possible.

[13]Note that I only provide bounds given that there are several potential stable matches μ_t^* such that $M_{it}(\mu_t^*) = M_{it}(\mu_t^M)$ and $W_{jt}(\mu_t^*) = W_{jt}(\mu_t^M)$.

opportunity set $M_{it}(\mu_t^M)$ and future expected payoff function $\overline{U}_{it+1}^M$:

$$n^{1/2}\mathbb{P}(U_{ijt} \geq \max_{k\in M_{it}(\mu_t^M)\cup\{0\}} U_{ikt}|x_{it}, z_{jt}, (z_{kt})_{k\in M_{it}(\mu_t^M)}, M_{i\tau}(\mu_t^M), \overline{U}_{it+1}^M) \leq \tag{1}$$
$$\frac{\exp\left\{U_t(\boldsymbol{x}_{it}, \boldsymbol{z}_{jt}) + \beta_w \int \overline{U}_{it+1}^M(s)w(s|\boldsymbol{x}_{it}, \boldsymbol{z}_{jt})ds\right\}}{\exp\left\{\beta_w \int \overline{U}_{it+1}^M(s)w_0(s|\boldsymbol{x}_{it})ds\right\} + n^{-1/2}\sum_{k\in M_{it}(\mu_t^M)} \exp\left\{U_t(\boldsymbol{x}_{it}, \boldsymbol{z}_{kt}) + \beta_w \int \overline{U}_{it+1}^M(s)w(s|\boldsymbol{x}_{it}, \boldsymbol{z}_{kt})ds\right\}} + o(1)$$

Similar bounds can be computed for a sequence of teacher-optimal stable match μ^W where the direction of the inequality is reversed. The same result also holds for the school side with the direction of the inequality reversed. Using Proposition 2 (ii), we can also bound agents' expectations about their match payoff under a sequence of school-optimal stable matches μ^M as follows:

$$\overline{U}_{it}^M(x) \geq \log\left(\exp\left\{\beta \int \overline{U}_{it+1}^M(s)w_0(s|x)ds\right\} + n^{-1/2}\sum_{k\in M_{it}(\mu_t^M)} \exp\left\{U_t(x, \boldsymbol{z}_{kt}) + \beta \int \overline{U}_{it+1}^M(s)w(s|x, \boldsymbol{z}_{kt})ds\right\}\right) + \frac{1}{2}\log(n) + \gamma + o(1) \tag{2}$$

where again similar bounds can be computed for the teacher-optimal stable match and for the school side with the direction of the inequality reversed.

In Equations (1) and (2), $n^{-1/2}\sum_{k\in M_{it}(\mu_t^M)} \exp\left\{U_t(\boldsymbol{x}_{it}, \boldsymbol{z}_{kt}) + \beta_w \int \overline{U}_{it+1}^M(s)w(s|\boldsymbol{x}_{it}, \boldsymbol{z}_{kt})ds\right\}$ serves as a sufficient statistic that collapses all the information contained in opportunity sets which is needed to approximate CCPs and expectations about future payoffs. These objects are called *inclusive values*. More generally, I define teacher i's inclusive value given a sequence of realized matches μ^* as:

$$I_{wit}^* = n^{-1/2}\sum_{k\in M_{it}(\mu_t^*)} \exp\left\{U_t(\boldsymbol{x}_{it}, \boldsymbol{z}_{kt}) + \beta_w \int \overline{U}_{it+1}^*(s)w(s|\boldsymbol{x}_{it}, \boldsymbol{z}_{kt})ds\right\}$$

Similarly, I define school j's inclusive value given μ^* as:

$$I_{mjt}^* = n^{-1/2}\sum_{l\in W_{jt}(\mu_t^*)} \exp\left\{V_t(\boldsymbol{x}_{lt}, \boldsymbol{z}_{jt}) + \beta \int \overline{V}_{jt+1}^*(s)m(s|\boldsymbol{x}_{lt}, \boldsymbol{z}_{jt})ds\right\}$$

I also define I_{wit}^{M} and I_{mjt}^{M} as the inclusive values that would arise under a sequence of school-optimal stable matches μ^{M} in period t and I_{wit}^{W} and I_{mjt}^{W} as the inclusive values that would arise under a sequence of teacher-optimal stable matches μ^{W} in period t.

4.2.4 Fixed point characterization

Inclusive values are unobserved as we do not observe opportunity sets, we do not know which stable match is selected and we do not know agents' expectations about future match payoffs. The rest of the proof shows that the inclusive values arising from an equilibrium match μ^{*} can be approximated by the solution of a fixed point problem.

I first show that, as in the static case (Menzel (2015)), inclusive values arising from a sequence of school-optimal and teacher-optimal stable matches in a given period t can be approximated by *expected inclusive value functions*. I rewrite I_{wit}^{M} as:

$$\begin{aligned} I_{wit}^{M} &= \frac{1}{n}\sum_{k=1}^{n_m} \exp\left\{U(\boldsymbol{x}_{it}, \boldsymbol{z}_{kt}) + \beta_w \int \overline{U}_{it+1}^{M}(s) w(s|\boldsymbol{x}_{it}, \boldsymbol{z}_{kt}) ds\right\} \times \sqrt{n}\mathbb{1}\{k \in M_{it}(\mu_t^{M})\} \\ &= \frac{1}{n}\sum_{k=1}^{n_m} \exp\left\{U(\boldsymbol{x}_{it}, \boldsymbol{z}_{kt}) + \beta_w \int \overline{U}_{it+1}^{M}(s) w(s|\boldsymbol{x}_{it}, \boldsymbol{z}_{kt}) ds\right\} \sqrt{n}\mathbb{1}\{V_{ikt} \geq \max_{l \in W_{kt}(\mu_t^{M}) \cup \{0\}} V_{lkt}\} \end{aligned}$$

The inclusive value of a given teacher is determined by the set of schools that would accept her, which in turn depends on the preferences of all schools as well as their opportunity sets. Using the school analogous of Equation (1), I thus show that:

$$I_{wit}^{M} \geq \hat{\Gamma}_{wt}^{M}(x_{it}) + o_p(1) \quad \text{and} \quad I_{mjt}^{M} \leq \hat{\Gamma}_{mt}^{M}(z_{jt}) + o_p(1)$$

where $\hat{\Gamma}_{wt}^{M}$ and $\hat{\Gamma}_{mt}^{M}$ are the school-optimal expected inclusive value function of teachers and schools in period t which are defined as:

$$\hat{\Gamma}_{wt}^{M}(x_{it}) = \frac{1}{n}\sum_{k=1}^{n_m} \frac{\exp\left\{U_t(\boldsymbol{x}_{it}, \boldsymbol{z}_{kt}) + V_t(\boldsymbol{x}_{it}, \boldsymbol{z}_{kt}) + \beta \int \overline{U}_{it+1}^{M}(s) w(s|\boldsymbol{x}_{it}, \boldsymbol{z}_{kt}) ds + \beta \int \overline{V}_{kt+1}^{M}(s) m(s|\boldsymbol{x}_{it}, \boldsymbol{z}_{kt}) ds\right\}}{\exp\left\{\beta \int \overline{V}_{kt+1}^{M}(s) m_0(s|\boldsymbol{z}_{kt}) ds\right\} + I_{mkt}^{M}}$$

$$\hat{\Gamma}^M_{mt}(z_{jt}) = \frac{1}{n}\sum_{l=1}^{n_w} \frac{\exp\left\{U_t(\boldsymbol{x}_{lt}, \boldsymbol{z}_{jt}) + V_t(\boldsymbol{x}_{lt}, \boldsymbol{z}_{jt}) + \beta\int \overline{U}^M_{lt+1}(s)w(s|\boldsymbol{x}_{lt}, \boldsymbol{z}_{jt})ds + \beta\int \overline{V}^M_{jt+1}(s)m(s|\boldsymbol{x}_{lt}, \boldsymbol{z}_{jt})ds\right\}}{\exp\left\{\beta\int \overline{U}^M_{lt+1}(s)w_0(s|\boldsymbol{x}_{lt})ds\right\} + I^M_{wlt}}$$

where I define $\overline{U}^M_{it+1}$ and $\overline{V}^M_{jt+1}$ as follows:

$$\overline{U}^M_{it+1}(x) = \log\left(\exp\left\{\beta\int \overline{U}^M_{it+2}(s)w_0(s|x)ds\right\} + I^M_{wit+1}\right)$$

$$\overline{V}^M_{jt+1}(z) = \log\left(\exp\left\{\beta\int \overline{V}^M_{jt+2}(s)m_0(s|z)ds\right\} + I^M_{mjt+1}\right)$$

Note that similar bounds can be established for the inclusive values that would arise under the teacher-optimal stable match:

$$I^W_{wit} \leq \hat{\Gamma}^W_{wt}(x_{it}) + o_p(1) \quad \text{and} \quad I^W_{mjt} \geq \hat{\Gamma}^W_{mt}(z_{jt}) + o_p(1)$$

A formal exposition and proof of this result can be found in Lemma 4 in Appendix D.4. The inclusive value of a given teacher can be approximated by a function of schools' preferences and inclusive values. Similarly, the inclusive value of a given school can be approximated by a function of teachers' preferences and inclusive values. Hence, the two-sided nature of the problem gives rise naturally to a fixed point problem characterizing these inclusive values. Dynamics add a layer of complexity as expectations about future payoffs depend on future inclusive values. There is thus dependence between inclusive values within and across periods.

The rest of the proof entails characterizing this fixed point problem and showing that inclusive values arising from an equilibrium match μ^* can be approximated by its solution. I define the fixed point mappings as follows:

$$\hat{\Psi}_{wt}[\boldsymbol{\Gamma}](x) = \frac{1}{n}\sum_{k=1}^{n_m} \frac{\exp\left\{U_t(x, \boldsymbol{z}_{kt}) + V_t(x, \boldsymbol{z}_{kt}) + \beta\int \overline{U}_{t+1}[\boldsymbol{\Gamma}](s)w(s|x, \boldsymbol{z}_{kt})ds + \beta\int \overline{V}_{t+1}[\boldsymbol{\Gamma}](s)m(s|x, \boldsymbol{z}_{kt})ds\right\}}{\exp\left\{\beta\int \overline{V}_{t+1}[\boldsymbol{\Gamma}](s)m_0(s|\boldsymbol{z}_{kt})ds\right\} + \Gamma_{mt}(z_{kt})}$$

$$\hat{\Psi}_{mt}[\boldsymbol{\Gamma}](z) = \frac{1}{n}\sum_{l=1}^{n_w} \frac{\exp\left\{U_t(\boldsymbol{x}_{lt}, z) + V_t(\boldsymbol{x}_{lt}, z) + \beta\int \overline{U}_{t+1}[\boldsymbol{\Gamma}](s)w(s|\boldsymbol{x}_{lt}, z)ds + \beta\int \overline{V}_{t+1}[\boldsymbol{\Gamma}](s)m(s|\boldsymbol{x}_{lt}, z)ds\right\}}{\exp\left\{\beta\int \overline{U}_{t+1}[\boldsymbol{\Gamma}](s)w_0(s|\boldsymbol{x}_{lt})ds\right\} + \Gamma_{wt}(x_{lt})}$$

$$\overline{U}_{t+1}[\mathbf{\Gamma}](x) = \log\left(\exp\left\{\beta\int\overline{U}_{t+2}[\mathbf{\Gamma}](s)w_0(s|x)ds\right\} + \Gamma_{wt+1}(x)\right)$$

$$\overline{V}_{t+1}[\mathbf{\Gamma}](z) = \log\left(\exp\left\{\beta\int\overline{V}_{t+2}[\mathbf{\Gamma}](s)m_0(s|z)ds\right\} + \Gamma_{mt+1}(z)\right)$$

I then show that for a given equilibrium match μ^*, for any $x \in \mathcal{X}$ and $z \in \mathcal{Z}$ in each period t:

$$\hat{\Gamma}^*_{wt}(x) = \hat{\Psi}_{wt}[\hat{\mathbf{\Gamma}}^*](x) + o_p(1) \quad \text{and} \quad \hat{\Gamma}^*_{mt}(z) = \hat{\Psi}_{mt}[\hat{\mathbf{\Gamma}}^*](z) + o_p(1) \tag{3}$$

meaning that inclusive values in period t arising from an equilibrium match μ^* can be approximated by fixed points of the mappings $\hat{\Psi}_{wt}, \hat{\Psi}_{mt}$. To characterize the limit of inclusive values, I then consider the limit version of this fixed point problem:

$$\Gamma_{wt} = \Psi_{wt}[\mathbf{\Gamma}] \quad \text{and} \quad \Gamma_{mt} = \Psi_{mt}[\mathbf{\Gamma}] \quad \forall t \tag{4}$$

where Ψ_{wt} and Ψ_{mt} are defined in Appendix D.3. The final step of the proof shows that this population fixed point problem has a unique solution and that the approximate solution of the finite sample fixed point problem converges to it. This is stated in the following result:

Theorem 1 *Under Assumption 1-4:*
(i) The mapping $(\log\mathbf{\Gamma}_{\boldsymbol{w}}, \log\mathbf{\Gamma}_{\boldsymbol{m}}) \mapsto (\log\mathbf{\Psi}_{\boldsymbol{m}}[\mathbf{\Gamma}], \log\mathbf{\Psi}_{\boldsymbol{w}}[\mathbf{\Gamma}])$ *is a contraction.*
(ii) The fixed point problem described in Equation (4) always has a unique solution $\mathbf{\Gamma}^*_{\boldsymbol{m}}, \mathbf{\Gamma}^*_{\boldsymbol{w}}$.
(iii) For any equilibrium μ^*, $I^*_{wit} \longrightarrow \Gamma^*_{wt}(x_{it})$ *and* $I^*_{mjt} \longrightarrow \Gamma^*_{mt}(z_{jt})$ *for all* i, j *and* t.

The complete proof of this result can be found in Appendix D.4. Theorem 1 has several implications. First, it implies that for any arbitrary equilibrium match μ^*, inclusive values converge to the same limit. Consequently, even if there might exist several matches which satisfy the equilibrium conditions in Assumption 4, all are observationally equivalent in the limit. Second, it implies that we can easily characterize conditional choice probabilities as inclusive value functions can be derived by iterating a contraction mapping.

4.2.5 Main result

From Theorem 1 and Proposition 2, we can fully characterize analytically the equilibrium of the model as a function of teachers' and schools' payoff functions. The limit joint density of matched characteristics f_t can be derived from the limit of conditional choice probabilities and has the following expression:

$$\frac{f_t(x,z)}{w_t(x)m_t(z)} = \frac{\exp\left\{U_t(x,z) + V_t(x,z) + \beta\int \overline{U}^*_{t+1}(s)w(s|x,z)ds + \beta\int \overline{V}^*_{t+1}(s)m(s|x,z)ds + \gamma_w + \gamma_m\right\}}{\left(\exp\left\{\beta\int \overline{U}^*_{t+1}(s)w_0(s|x)ds\right\} + \Gamma^*_{wt}(x)\right)\left(\exp\left\{\beta\int \overline{V}^*_{t+1}(s)m_0(s|z)ds\right\} + \Gamma^*_{mt}(z)\right)}$$

$$\frac{f_t(x,*)}{w_t(x)} = \frac{\exp\left\{\beta\int \overline{U}^*_{t+1}(s)w_0(s|x)ds + \gamma_w\right\}}{\left(\exp\left\{\beta\int \overline{U}^*_{t+1}(s)w_0(s|x)ds\right\} + \Gamma^*_{wt}(x)\right)}$$

$$\frac{f_t(*,z)}{m_t(z)} = \frac{\exp\left\{\beta\int \overline{V}^*_{t+1}(s)m_0(s|z)ds + \gamma_m\right\}}{\left(\exp\left\{\beta\int \overline{V}^*_{t+1}(s)m_0(s|z)ds\right\} + \Gamma^*_{mt}(z)\right)}$$

where $f_t(x,*)$ and $f_t(*,z)$ are, respectively, the density of the characteristics of unmatched teachers and unmatched schools. I define the equilibrium expected future payoff functions $\overline{U}^*_{t+1}$ and $\overline{V}^*_{t+1}$ recursively as:

$$\overline{U}^*_{t+1}(x) = \log\left(\exp\left\{\beta\int \overline{U}^*_{t+2}(s)w_0(s|x)ds\right\} + \Gamma^*_{wt+1}(x)\right)$$

$$\overline{V}^*_{t+1}(z) = \log\left(\exp\left\{\beta\int \overline{V}^*_{t+2}(s)m_0(s|z)ds\right\} + \Gamma^*_{mt+1}(z)\right)$$

and the equilibrium aggregate states distribution w^*_t and m^*_t as:

$$w^*_t(x) = \int_{\mathcal{X}_t}\int_{\mathcal{Z}_t} w(x|s,h)f_{t-1}(s,h)dhds + \int_{\mathcal{X}_t} w_0(x|s)f_{t-1}(s,*)ds$$

$$m^*_t(z) = \int_{\mathcal{X}_t}\int_{\mathcal{Z}_t} m(z|s,h)f_{t-1}(s,h)dhds + \int_{\mathcal{Z}_t} w_0(x|s)f_{t-1}(*,h)dh$$

To simulate the equilibrium in practice, one first needs to solve for inclusive values given the specified payoff functions, the initial aggregate distribution of states m_1 and w_1 and the transition distribution functions. From there, it is then possible to construct U_{ijt} and V_{ijt} given a simulated set of taste shocks ϵ and η. To reach a stable match and simulate the equilibrium in a given period t, any version of the Deferred Acceptance algorithm can be used as they are observationally equivalent. Monte Carlo simulations testing the validity of the convergence results derived in this section can be found in Appendix E.

5 Identification and Estimation

The previous section built an equilibrium model of dynamic two-sided matching and provided a tractable way to map preferences into sorting. This section shows that this mapping can be inverted such that one can identify and estimate preferences from observed sorting.

5.1 Sampling Process

I assume that the available data is a random sample of a panel of individuals from the population regardless of whether they are schools or teachers. One observation in a given period t is thus composed of this individual alone, in the case where it is unmatched, or along with its matched partner otherwise. The probability that a matched individual is selected by this sampling process is thus twice the probability that an unmatched individual is selected. The joint density function of matched characteristics h_t arising from this sampling process relates to f_t in the following way:

$$h_t(x,z) = \frac{2f_t(x,z)}{\exp\{\gamma_{wt}\} + \exp\{\gamma_{mt}\}}$$

where $h_t(x,z)$ is the mass of schools with observed characteristics z matched with teachers with observed characteristics x in period t arising from the sampling scheme defined above and $\exp\{\gamma_w\} + \exp\{\gamma_m\}$ is the total mass of teachers and schools available in this economy. Similarly, I define:

$$h_t(x,*) = \frac{f_t(x,*)}{\exp\{\gamma_w\} + \exp\{\gamma_m\}}$$

$$h_t(*, z) = \frac{f_t(*, z)}{\exp\{\gamma_w\} + \exp\{\gamma_m\}}$$

where $h_t(*, z)$ is the mass of unmatched schools with observed characteristics z and $h_t(x, *)$ is the mass of unmatched teachers.

I also assume that we observe the aggregate distribution of observed states m_t and w_t as this can be easily recovered from f_t as follows:

$$\int_{\mathcal{Z}_t} f_t(x, z)dz + f_t(x, *) = w_t(x) \exp\{\gamma_w\}$$

$$\int_{\mathcal{X}_t} f_t(x, z)dx + f_t(*, z) = m_t(z) \exp\{\gamma_m\}$$

Finally, I assume that the Markov transition density functions m, m_0, w and w_0 can be directly identified from data on observed state transitions.

5.2 Identification

The primitives of the model that we do not observe and wish to identify and estimate from the data are the payoff functions $(U_t)_{t=1}^T$ and $(V_t)_{t=1}^T$ and the discount factors β_w and β_m. We know from the literature on dynamic discrete choice models that intertemporal preferences cannot be identified from observed choices without further assumptions (Magnac and Thesmar (2002)).[14] I thus fix the value of the discount factors from now onward. Similarly, I cannot allow for $T = \infty$ while having a nonstationary setting. I thus consider two polar cases: (i) $T < \infty$ and nonstationarity and (ii) $T = \infty$ and stationarity.

5.2.1 Finite horizon

Given the recursive structure of the problem, the identification argument in the finite horizon case can be done by backward induction. Starting from the last period T, we can identify the joint surplus as follows:

$$U_T(x, z) + V_T(x, z) = \log\left(\frac{f_T(x, z)}{f_T(x, *)f_T(*, z)}\right)$$

[14] Similarly, the flow utility of one alternative needs to be fixed in each period. This is already done through normalizing the option of staying unmatched.

We can also identify Γ^*_{wT} and Γ^*_{mT} from the distribution of unmatched teachers and schools:

$$\Gamma^*_{wT}(x) = \frac{w_T(x)\exp(\gamma_{wT})}{f_T(x,*)} - 1$$

$$\Gamma^*_{mT}(z) = \frac{m_T(z)\exp(\gamma_{mT})}{f_T(*,z)} - 1$$

$\overline{U}_T$ and $\overline{V}_T$ can then be computed by backward induction:

$$\overline{U}_T(x) = \log(1 + \Gamma^*_{wT}(x)) + \gamma$$

$$\overline{V}_T(z) = \log(1 + \Gamma^*_{mT}(z)) + \gamma$$

From there, we can then repeat the same steps to identify the inclusive value functions and the joint surplus in period $T-1$. Finally, we iterate the procedure to identify the joint surplus and the inclusive value functions in all periods t. This results in the following proposition.

Proposition 3 *Under Assumption 1-4 and for $T < \infty$:*
(i) The joint surplus function $U_t + V_t$ and the inclusive value functions Γ^*_{wt} and Γ^*_{mt} are identified for all t from f_t, the limiting joint distribution of matched characteristics in period t.
(ii) Without further restrictions, we cannot separately identify U_t and V_t for all t.

We face a similar identification challenge as in the static case (Menzel, 2015) as preferences are not separately identified from the joint surplus. However, note that this is not necessarily a negative result. Given that the joint distribution of matched characteristics is solely driven by the joint surplus, knowing the joint surplus is enough to perform counterfactuals where we would change the distribution of teachers' and schools' observed attributes. Nevertheless, we might be interested in identifying and estimating preferences as these might be objects of interest. Exclusion restrictions might be useful to disentangle preferences from the joint surplus, as in the static case (Ederer, 2022). In the empirical analysis, I use additional data on how schools rank the applicants they interview to disentangle teachers' and schools' preferences from the joint surplus.

5.2.2 Infinite horizon

To allow for $T = \infty$, I impose the following assumptions.

Assumption 5 *(i) Stationarity of preferences:* $U_t = U$ *and* $V_t = V$ *for all* t.
(ii) Stationarity of aggregate states distribution: $m_t = m$ *and* $w_t = w$ *for all* t.

Assumption 5 has the direct implication that inclusive value functions are also stationary $\Gamma_{mt} = \Gamma_m$ and $\Gamma_{wt} = \Gamma_w$ for all t. As a consequence, $\overline{U}_t = \overline{U}$ and $\overline{V}_t = \overline{V}$. However, Assumption 5 (ii) is fairly restrictive as it forces aggregate states to remain on a predetermined stationary path which might not be consistent with what the model predicts. Showing existence of a stationary equilibrium which would satisfy consistency requirements is left for future work. Assumption 5 then implies that we can write:

$$\frac{f(x,*)}{w(x)} = \frac{\exp\left\{\beta \int \overline{U}^*(s) w_0(s|x) ds\right\}}{\left(\exp\left\{\beta \int \overline{U}^*(s) w_0(s|x) ds\right\} + \Gamma_w^*(x)\right)}$$
$$= \frac{\exp\left\{\beta \int \overline{U}^*(s) w_0(s|x) ds\right\}}{\exp\left\{\overline{U}^*(x) - \gamma\right\}} = \exp\left\{\beta \int \overline{U}^*(s) w_0(s|x) ds - \overline{U}^*(x) + \gamma\right\}$$

From there, we can invert this mapping to recover $\overline{U}^*$. We can follow the same steps to recover $\overline{V}$ from $f(*, z)$. It is then immediate to see that we can identify $U + V$ from $f(x, z)$.

Proposition 4 *Under Assumption 1-5 and for* $T = \infty$*:*
(i). The joint surplus function $U + V$ *and the inclusive value functions* Γ_w^* *and* Γ_m^* *are identified from the limiting joint distribution of matched characteristics in each period* f.
(ii). Without further restrictions, we cannot separately identify U *and* V.

Note that in the stationary case, a single cross section is sufficient to identify and estimate $U + V$ as the joint distribution of matched characteristics does not depend on t anymore. However, this does not mean that dynamics do not play a role as agents still make forward looking decisions.

5.3 Estimation

I consider a parametric version of this framework where I define the payoff functions as $U(x, z; \boldsymbol{\theta_t})$ and $V(x, z; \boldsymbol{\theta_t})$ such that U and V are known for all (x, z) up to a vector of unknown parameters $\boldsymbol{\theta_t}$. I assume that we observe a random sample of K individuals over each period t, drawn from the sampling scheme described in Section 5.1, along with their respective matches. For a given observation k in period t, we observe a vector $(x_t(k), z_t(k))$ which is encoded differently depending on the type of match we observe. For an unmatched teacher, indexed by $w_t(k) = 0$, I record its characteristics in $x_t(k)$ and encode $z_t(k)$ as missing. Similarly, for an unmatched school, indexed by $m_t(k) = 0$, I record its characteristics in $z_t(k)$ and encode $x_t(k)$ as missing. For a matched teacher or school, indexed by $m_t(k) = w_t(k) = 1$, I record their characteristics in $(x_t(k), z_t(k))$. We can then construct the following sample average log-likelihood:

$$L(\boldsymbol{x}, \boldsymbol{z}; \boldsymbol{\theta}) = \frac{1}{KT} \sum_{t=1}^{T} \sum_{k=1}^{K} \log \big[\mathbb{1}\{w_t(k) = 0\} h_t(x_t(k), *, \boldsymbol{\theta_t}) + \mathbb{1}\{m_t(k) = 0\} h_t(*, z_t(k), \boldsymbol{\theta_t}) \\ + \mathbb{1}\{m_t(k) = 1, w_t(k) = 1\} h_t(x_t(k), z_t(k), \boldsymbol{\theta_t}) \big]$$

Calculating the likelihood function for a given parameter vector $\boldsymbol{\theta}$ first involves solving the fixed point problem described in Equation 4 to derive the inclusive values. This can be achieved by setting up an inner loop which will apply the contraction mapping until convergence. The estimator proposed is then defined as:

$$\hat{\boldsymbol{\theta}} = \underset{\boldsymbol{\theta} \in \Theta}{\arg\max}\, L(\boldsymbol{x}, \boldsymbol{z}; \boldsymbol{\theta})$$

Asymptotic inference for $\hat{\boldsymbol{\theta}}$ is then standard if the size of the sample is not too large relative to the size of the overall economy. As noted in Menzel (2015) and Diamond and Agarwal (2017), the inherent structure of matching markets could introduce dependence between observations. A bootstrap procedure could then be used for inference otherwise (Diamond and Agarwal, 2017; Menzel, 2021). Monte Carlo simulations testing the validity of the proposed estimation strategy can be found in Appendix E.

6 Empirical Strategy

The rest of the paper leverages the above general methodology to identify and estimate teachers' and schools' preferences and investigate the determinants of the observed spatial sorting and job-to-job flows. Before showing the results of the empirical analysis, I briefly describe how I adapt the model to the context under study by defining the estimation sample, how model primitives are parameterized and discussing the identification strategy.

Estimation Sample: Throughout the empirical analysis, I consider one side as being teachers and the other side as jobs such that matching is one-to-one.[15] I use several parts of the centralized assignment data for identification and estimation. First, I use information on the universe of applicants and positions that participate in the centralized allocation for the academic years 2016, 2018 and 2020. This allows me to identify directly from the data the distribution of aggregate states m_t and w_t for $t = \{2016, 2018, 2020\}$. I then use data on realized matches following the sampling process described in Section 5 in order to identify the joint distribution of matched characteristics f_t for $t = \{2016, 2018, 2020\}$. Finally, I supplement the analysis with additional data on how schools rank the applicants they interview. This allows me to overcome the negative result highlighted in Proposition 3 by separately identifying preferences from the joint surplus. As the horizon of the data is limited, I fix the distribution of aggregate states and the payoff functions to be stationary from 2020 onward and set the horizon of the model to $T = \infty$. This avoids assuming that the continuation value of a match in 2020 is zero.

Permanent vs. Temporary Contracts: I consider the joint allocation of permanent and temporary positions. Permanent contracts have several non-standard features that the model needs to account for. Teachers are forced to stay at least three years in the first permanent job they accept. Once a teacher accepts a permanent position, it can no longer participate in the centralized allocation mechanism (see Appendix B for more details). This has several

[15] In a static setting, observing the same school making several choices brings additional identification power to pin down schools' unobserved preference heterogeneity (Ederer, 2022). Investigating whether this result also holds in a dynamic setting is left for future work.

implications. First, this implies that choosing a permanent contract is a commitment to stay at least three years in the same location. It is thus crucial to model how agents sort between these two types of contracts in order to explain teachers movements across locations.[16] Second, this means that choosing a permanent position is a terminating action as teachers exit the market if they do so. I thus specify the lifetime utility that teacher i gets from choosing a permanent position j as follows:

$$U_{ijt} = U(x_{it}, z_{jt}, \boldsymbol{\theta}_{\text{perm}}) + \sigma\eta_{ijt}$$

while the utility that teacher i gets from choosing a temporary position k is defined as:

$$U_{ikt} = U(\boldsymbol{x}_{it}, \boldsymbol{z}_{kt}, \boldsymbol{\theta}_{\text{temp}}) + \sigma\eta_{ikt} + \beta_w \int \overline{U}_{it+1}(\boldsymbol{x}_{it+1}) w(\boldsymbol{x}_{it+1}|\boldsymbol{x}_{it}, \boldsymbol{z}_{kt}) d\boldsymbol{x}_{it+1}$$

Similarly, on the school side, I assume that accepting to match with a teacher with a permanent contract is a terminating action. I thus define the utility that a school with a permanent vacancy j gets from being matched with teacher i as:

$$V_{ijt} = V(\boldsymbol{x}_{it}, \boldsymbol{z}_{jt}, \boldsymbol{\gamma}) + \sigma\epsilon_{ijt}$$

As for temporary jobs, the allocation mechanism is priority-based and schools cannot express their preferences, I assume that the utility that a school with a temporary vacancy j gets from being matched with teacher l is:

$$V_{ljt} = s_{lt}$$

where s_{lt} is teacher l's test score in period t. This slightly simplifies the problem as we can directly observe which temporary jobs are in teachers' opportunity sets.

Parametrization Payoffs: As one time period spans two academic years, I first set the discount factor β_w to 0.9. The model aims to capture (i) how teachers trade off geographical

[16] Figure A.2 shows that sorting across permanent and temporary contracts explains a large part of the observed attrition patterns.

proximity with other job characteristics such as wages and amenities and (ii) how schools value observed measures of teacher quality. I thus parametrize teachers' payoff function as follows:

$$U(\boldsymbol{x_{it}}, \boldsymbol{z_{jt}}; \boldsymbol{\theta}) = \theta_0 + \theta_1 w_{jt} + \boldsymbol{a'_{jt}\theta_2} + \boldsymbol{d'_{ij}\theta_3} + \boldsymbol{m'_{ijt}\theta_4} + \boldsymbol{z'_{jt}\theta_5} + \boldsymbol{x'_{it}\theta_6}$$

Where w_{jt} is the monthly wage offered in school j in year t, $\boldsymbol{a_{jt}}$ is a vector of indicators measuring the local level of amenities through the availability of a range of services such as electricity, sewage, medical centers, internet and libraries, $\boldsymbol{d_{ij}}$ is a spline of the distance between school j and teacher i's home location and $\boldsymbol{m_{ijt}}$ is a set of dummies indicating if teacher i's current location is in the same region or province as school j. I also include other teacher characteristics $\boldsymbol{x_{it}}$ such as experience, marital status, gender and age as well as other school/locality characteristics $\boldsymbol{z_{jt}}$ part of the bonus scheme driving the variation in wages. I then parametrize schools' payoff function as:

$$V(\boldsymbol{x_{it}}, \boldsymbol{z_{jt}}; \gamma) = \gamma_0 + \boldsymbol{s'_{it}\gamma_1} + \boldsymbol{e'_{it}\gamma_2} + \boldsymbol{z'_{jt}\gamma_3} + \boldsymbol{x'_{it}\gamma_4}$$

Where s_{it} is a vector of the different components of teacher i's test score in period t, e_{it} is a vector of dummies dividing the experience level of teacher i in period t in discrete categories. I also include various additional teacher and school characteristics in $\boldsymbol{x_{it}}$ and z_{jt}. Note that I exclude wages, amenities and geographical proximity from schools' preferences. I directly test for these exclusion restrictions by estimating schools' preferences separately and find that we cannot reject that the parameters associated to these characteristics are jointly equal to zero.

The main parameters of interest on the teacher side are $(\theta_1, \theta_2, \boldsymbol{\theta_3}, \boldsymbol{\theta_4})$. They quantify the trade offs between wages, amenities and geographical proximity which drive how mobile labor supply is. On the school side, the main parameters of interest are $\boldsymbol{\gamma_1}, \boldsymbol{\gamma_2}$ as they are likely to explain how the demand side rations excess supply and thus how teacher quality is distributed across locations.

Transition processes: I separate state variables evolving over time in several groups. I assume that age and experience evolve deterministically and exogenously by getting incremented by one every year. The competency score s_{it}, which is contained in x_{it}, evolves stochastically and exogenously. I assume that the transition distribution function of s_{it} is conditionally normal such that:

$$s_{it+1}|\boldsymbol{x_{it}}, \boldsymbol{z_{jt}} \sim \mathcal{N}(\boldsymbol{x'_{it}}\boldsymbol{\beta_x} + \boldsymbol{z'_{jt}}\boldsymbol{\beta_z}, \sigma^2)$$

I estimate $(\boldsymbol{\beta_x}, \boldsymbol{\beta_z})$ via an auxiliary linear regression and report the estimates in Table A.4. Finally, $\boldsymbol{m_{ijt}}$ evolves deterministically and endogenously. Each move across provinces or regions updates teachers' current location such that they internalize that moving again in the next period might be costly.

Additional identifying variation: To overcome the negative result of Proposition 3 and separately identify teachers' and schools' preferences from the joint surplus, I use additional data on how schools rank the applicants they interview. As the set of interviewees in each school is determined by teachers' rank-ordered list and priority index, schools' choice sets are independent of schools' unobserved preferences by construction. Schools do not have incentives to misreport their preferences at this stage as job offers are automatically sent in order of the reported ranks. I thus assume that these rankings are truthful and use them to construct the corresponding exploded logit conditional choice probabilities. The log of these CCPs then enters additively in the log-likelihood derived in Section 5.[17]

Discussion of the stability assumption: While I cannot directly test whether the observed matching is stable, several properties of the allocation mechanism, described in Section 2, limit the presence of frictions that could lead to the existence of blocking pairs. First, the allocation of temporary positions is implemented via serial dictatorship by sequentially asking teachers to choose their preferred position by order of priority. This procedure is equivalent to Deferred Acceptance and thus leads to a stable allocation. Second, the allocation of interviews for permanent positions is also done via serial dictatorship, which ensures that teachers get interviewed by their preferred schools by order of priority, if they reveal their true prefer-

[17] As this likelihood is not exact, I correct for standard errors using the standard formula for the asymptotic variance of QMLE.

ences. Still, as the number of interviews per teacher is limited to three, two issues might arise: (i) teachers might end up unmatched because they failed all their interviews while schools with unfilled positions might be willing to hire them and (ii) teachers might anticipate this possibility and try to avoid it by being strategic when forming their rank-ordered lists.[18]

To mitigate the first concern, the Ministry implemented an aftermarket such that all unassigned permanent positions and teachers can meet and match in order to minimize justified envy. Modeling potential mismatches generated by the second concern would be challenging as this would entail having access to data on teachers' beliefs about their chances to succeed at the interviews and developing a dynamic model of strategic reporting where preferences of both sides of the market are unknown. This is beyond the scope of the available data and the proposed methodology and is left for future research. Instead, I propose a test assessing the validity of the estimated parameters by leveraging the cutoffs determining eligibility to permanent positions in a regression discontinuity design.[19] Teachers just above the cutoff have both permanent and temporary positions in their choice sets while teachers just below the cutoff only have temporary positions in their choice sets. Comparing the matching outcomes of these two groups allows to pin down how teachers trade off job attributes depending on whether the position is permanent or temporary. I thus verify whether the estimated model can replicate the threshold crossing effect on the characteristics of teachers' matched schools. Figure A.4 shows that the model predictions match the observed responses at the threshold. Eligible teachers are more likely to choose a permanent position and are willing to trade off the benefits of permanent contracts with geographical proximity.

[18]Additionally, teachers might have ex-post justified envy if they refuse a permanent position and realize ex-post that the available temporary positions are worse. To solve this issue, one could instead model the allocation of permanent and temporary contracts sequentially and not jointly. However, the results derived in Appendix F show that these two models are observationally equivalent in the limit if teachers have rational expectations about their future match payoffs when choosing a permanent contract.

[19]Using an external source of data to separately identify schools' preferences using truthful rankings also allows to mitigate these concerns.

7 Results

7.1 Preferences and the Spatial Job Ladder

I report in Panel A of Table 2 the estimated willingness to pay of teachers for amenities, proximity to home and moving away from their current location. Consistently with the migration literature, I find a large distaste for moving (Kennan and Walker, 2011). Teachers would be willing to give up 309 USD from their monthly wage to avoid moving 10 kilometers away from home. This is quite substantial as this corresponds to 61 % of the base monthly teacher wage. Similarly, teachers' willingness to pay to avoid moving out of their current location is large. The cost of moving out of their current province is estimated at 677 USD while the cost of changing regions is estimated at 1,146 USD. In comparison, the willingness to pay for local amenities is quite small and ranges from 16 USD to 98 USD.[20]

To quantify how much these attributes explain the variation in teachers' preferences, I simulate teachers' lifetime utility by drawing random Gumbel shocks and by using the estimated parameters to compute their flow utility and continuation value for each job. I then plot the ranking of each job according to its predicted lifetime utility against its ranking in terms of distance, amenities and wages. Panel A of Figure 3 shows that distance very strongly predicts how teachers rank jobs. The correlation between the ranking with respect to utility and the ranking with respect to distance is 0.68. On the other hand, I find that wages and amenities are poor predictors of how teachers rank jobs. This implies that labor markets are very local as labor supply is not mobile, which is consistent with the findings of Manning and Petrongolo (2017).

Panel B of Table 2 shows the results of the estimation of schools' preferences. I find that schools highly value observed measures of teacher quality such as test scores and experience. I investigate how much of the ranking of teachers with respect to schools' utility can be explained by their ranking with respect to test scores. Panel B of Figure 3 plots the relationship between the two for both permanent and temporary jobs. Mechanically, the

[20] I report how the willingness to pay for different job attributes differ depending on whether the contract is permanent or temporary, under the assumption that agents are myopic, in Table A.5. I strongly reject that $\boldsymbol{\theta}_{\text{perm}} = \boldsymbol{\theta}_{\text{temp}}$ which is equivalent to rejecting that agents are myopic, as permanent and temporary contracts do not differ in the first years of employment.

Table 2: Selected Preference Estimates

Panel A: Teachers' Preferences (in monthly USD)	
Amenities	-
Electricity	97.67 (34.71)
Sewage	16.76 (12.79)
Library	29.41 (15.78)
Internet	15.96 (19.86)
Spline Distance from Home Location	-
Slope $<$ 20km	-30.94 (1.96)
Slope $\in$ [20km,100km]	-7.11 (0.50)
Slope $\geq$ 100km	-1.38 (0.10)
Moving Costs	-
$\neq$ Province	-676.95 (51.43)
$\neq$ Region	-469.62 (57.21)
Panel B: Schools' Preferences	
Constant	-0.488 (0.082)
Experience	-
$<$ 3 years	-0.737 (0.041)
$>$ 10 years	0.057 (0.040)
Competency Score	-
Reading	0.669 (0.023)
Logic	0.571 (0.020)
Curricular Knowledge	1.397 (0.022)

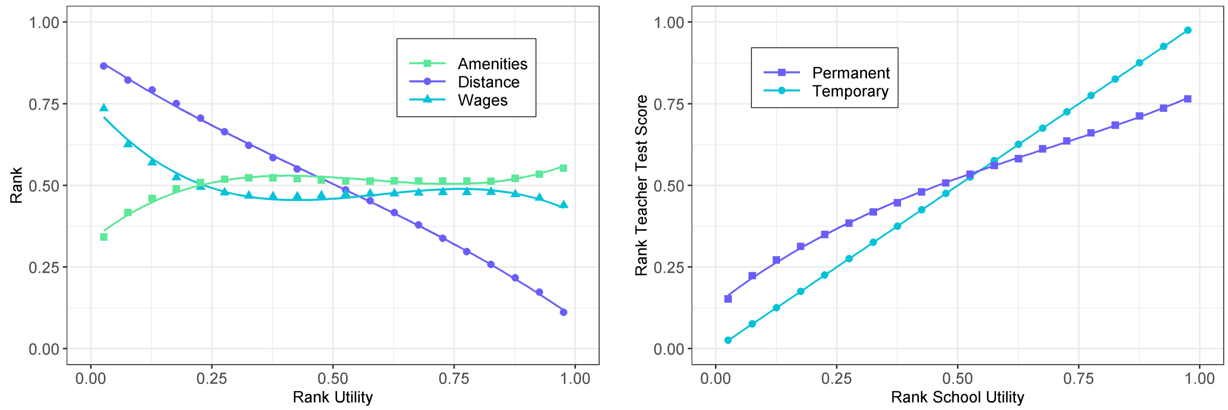
NOTES. This table shows selected estimates of $\boldsymbol{\theta}$ and $\boldsymbol{\gamma}$ from the specification of teachers and schools preferences. θ_1 is normalized to one such that teachers' preference estimates are expressed in terms of monthly willingness to pay in USD.

relationship is one-to-one for temporary jobs, as test scores are used as priorities to allocate seats. The relationship is also very strong for permanent jobs as the correlation between the utility ranking and the test score ranking is 0.6.

Overall, the estimated preference parameters indicate that (i) geographical proximity is highly predictive of how teachers rank the available jobs and (ii) schools mostly value observed measures of teacher quality such as teachers' test scores. These two facts have strong implications for spatial sorting and inequalities.

I first show that the combination of fact (i) with the concentration of teachers' home location in cities and the dispersion of jobs across the country, documented in Section 3, implies the existence of a *spatial job ladder*. As teachers' home location is concentrated in

Figure 3: Spatial Job Ladder



a) Supply **b) Demand**

Notes. Panel A plot the relationship between the rank of teachers' lifetime utility U_{ijt} estimated using the results displayed from Table 2 and the ranks of various job attributes such as: the distance between teachers' home location and the school's locality, the wage offered by the schools and the level of local amenities. Panel B performs the same exercise and plot the ranks of schools' estimated utility V_{ijt} against the ranks of teachers' test scores for both temporary and permanent positions.

cities, teachers' distaste for working far from their home location implies a strong distaste for schools located in remote areas. Additionally, as schools are geographically scattered, cities offer very few positions compared to the total number of applicants. Overall, this implies that remoteness becomes the main driver of how teachers rank the available jobs. This results in the existence of a spatial job ladder where jobs located in remote areas are at the bottom whereas jobs located in cities are at the top. As a result, schools in cities face excess supply and are free to hire the teachers they prefer from the set of new applicants or poach their preferred teachers from schools which are on a lower rung of the ladder.[21] This has direct implications on labor market dynamics, as teachers which start at the bottom of the ladder switch jobs at a higher rate to climb up toward urban areas. Consequently, schools located in remote areas face higher attrition rates than schools located in cities.

The extent to which the spatial job ladder translates into spatial inequalities in education provision depends on which teacher attributes schools value. If schools would select teachers at random, lucky teachers would be able to move to urban schools but this would not generate

[21]This contrasts with traditional models of the job ladder where productive firms offer higher wages to poach skilled workers from unproductive firms (Moscarini and Postel-Vinay, 2018). In this setting, where wage differentiation is very limited, the job ladder is instead determined by non-pecuniary factors such as geographical location.

unequal sorting with respect to teaching quality.[22] Fact (ii) implies that urban schools ration excess supply using observed measures of teacher quality such as test scores and experience. Consequently, the spatial job ladder creates large spatial inequalities in teaching quality through two channels. First, among the set of new applicants, urban schools systematically select the highest scoring teachers while rural schools are left with the lowest scoring teachers. Second, urban schools poach teachers who have accumulated sufficient experience and human capital throughout their career from rural schools. The latter thus fail to retain skilled teachers and sustain disproportionately low levels of teaching experience and quality.

Reducing spatial inequalities in teaching quality thus requires shutting down the mechanisms through which the spatial job ladder operates or directly targeting the causes of the existence of the spatial job ladder. Next, I use the tools developed in Section 4 along with the estimated teachers' and schools' payoff functions to perform several counterfactual experiments aiming at achieving these goals. Before doing so, I assess the credibility of the equilibrium predictions generated by the model by testing its ability to predict patterns consistent with the data.

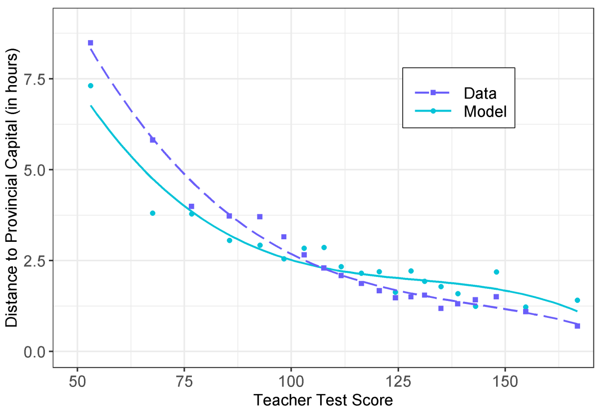
7.2 Model Fit

I perform several checks to assess how well the model predicts the patterns generated by the spatial job ladder. I first test whether the cross-sectional spatial sorting patterns predicted by the model match the ones observed in the data. I then simulate job-to-job flows and check whether the model can replicate the observed movements of teachers from rural to urban areas.

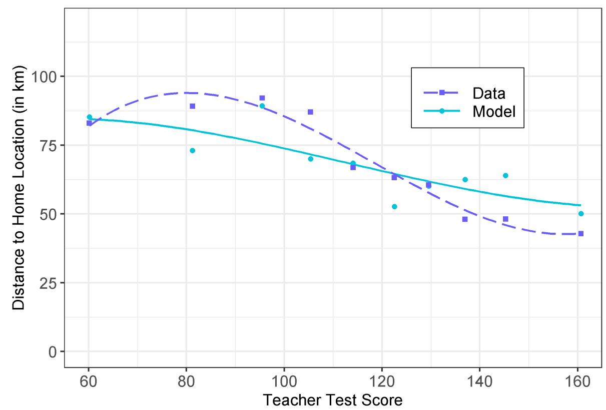
To simulate the status quo equilibrium matching, I first derive $U(\boldsymbol{x_{it}}, \boldsymbol{z_{jt}}, \hat{\boldsymbol{\theta}})$ and $V(\boldsymbol{x_{it}}, \boldsymbol{z_{jt}}, \hat{\boldsymbol{\gamma}})$ for all t. I then randomly draw Extreme Value Type I taste shocks ϵ_{ijt} and η_{ijt} for all (i, j, t) to construct the flow utilities. I solve the fixed point problem described in Equation 4 by fixing the aggregate distributions of observables in 2016 as the baseline m_1 and w_1 to obtain the equilibrium inclusive values for each agent. Given the inclusive values, I then compute teachers' continuation value from choosing a temporary contract. I then simulate forward

[22] Still, attrition would be higher in rural schools which could have a disruptive effect on student learning and imply a net efficiency loss in teaching quality due to the loss of school specific experience. See Appendix C for estimates of the net loss in teacher value added implied by a move from one school to another.

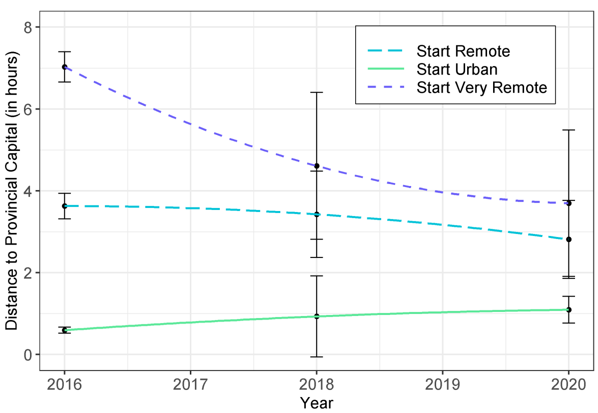
Figure 4: Model Fit



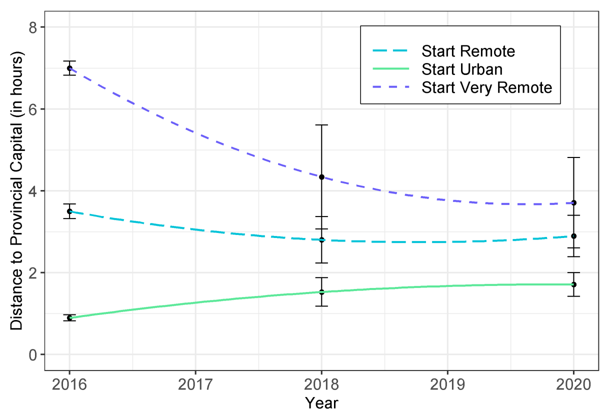
a) Sorting: Remoteness



b) Sorting: Distance



c) Job-to-Job Transitions: Data



d) Job-to-Job Transitions: Model

Notes. This figure uses the centralized assignment data from 2016 to 2020 and compares realized sorting patterns and job-to-job transitions with model predictions. Panel A plots averages of the remoteness of teachers' matched schools based on equally spaced bins of the distribution of teachers' test scores both in the actual data and in the simulated equilibrium. Panel B shows the result of a similar exercise using the distance between teachers' matched schools and their home location. Panel C plots the evolution over time of teachers' matched schools observed in the data depending on the remoteness of the school in which they started in 2016. Panel D plots the same trend using the job-to-job transitions simulated by the model.

by constructing the lifetime utilities U_{ij1} and V_{ij1}, deriving the teacher-optimal stable match using the teacher-proposing Deferred Acceptance algorithm and updating teachers' location, experience, age and test scores using the estimated transition process. I then iterate this procedure to simulate the entire non-stationary equilibrium path.

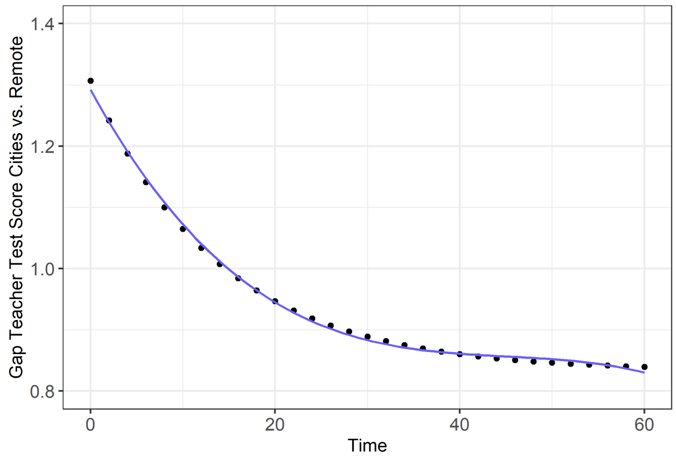
Panel A of Figure 4 shows a binned scatter plot of the relationship between teachers' test scores and the remoteness of their matched school. The model is able to replicate the rationing of excess supply through test scores and generate the strong negative relationship between remoteness and test scores. Panel B of Figure 4 also shows that the model is able to replicate sorting with respect to geographical proximity. Overall, this indicates that the main drivers of spatial sorting are well captured by the estimated preferences.

Panel C and D of Figure 4 compare the career paths of teachers depending on where they started on the spatial job ladder with their simulated counterparts. Specifically, I compare the job-to-job flows from rural to urban areas as teachers climb up the ladder. I find that the model captures the trend that teachers originally matched to rural schools climb the job ladder by moving toward urban areas. The model is thus able to replicate the important labor market dynamics that characterize the spatial job ladder.

7.3 Counterfactuals

In this section, I first quantify the gains of shutting down labor mobility along the job ladder to isolate the role of labor market dynamics in explaining the observed urban-rural gap in teaching quality. I then explore the effectiveness of retention policies that would prevent teachers from climbing up the ladder. More specifically, I simulate the effect of imposing a minimum contract length, which is a commonly used retention policy in the public sector. Finally, I explore the equilibrium effects of tackling directly the root causes of the existence of the spatial job ladder. To do so, I simulate equilibrium sorting and mobility under the scenario where teachers' home location would be scattered across the country instead of being concentrated in cities.

Figure 5: Labor Market Dynamics and Spatial Inequalities



NOTES. This figure shows the result of artificially increasing teachers utility for their matched school in 2016 in order to quantify the share of the urban rural gap in teacher quality explained by teacher mobility. It plots the difference between the average teacher score (in standard deviations) in schools located in cities and schools located more than six hours away from the provincial capital along the transition path triggered by this counterfactual exercise over 60 years.

7.3.1 Labor Market Dynamics and Spatial Inequalities

Teacher mobility likely contributes to a large extent to the urban-rural gap in teacher quality as teachers matched to rural areas leave toward urban areas once they have accumulated skills and experience. As shown in Section 3, movers are of significantly higher quality that those who replace them as the job ladder rewards teachers with higher test scores and more experience. To quantify how much of the urban-rural gap in teaching quality is explained by labor mobility, I start by simulating the equilibrium path under a counterfactual scenario where agents would have a very high preference for staying in their current job. Assuming that μ^*_{2016} is the equilibrium match under the status quo in 2016, I thus artificially increase $U_{i\mu^*_{2016}(i)t}$ for all teachers i and all subsequent years $t > 2016$ and simulate the long-run equilibrium paths. This counterfactual exercise shuts down voluntary moves away from rural areas such that rural schools no longer lose their most qualified and experienced teachers and can benefit from the accumulation of human capital on-the-job.

Figure 5 plots the evolution of the urban-rural gap in teacher quality from 2016 onward under this counterfactual. I find that shutting down labor mobility makes the urban-rural gap in teacher test score sharply drops from 1.3 to 0.8 standard deviation in the long run. This decline is stronger in the short run as the gap decreases by 0.1 standard deviation after four years only.[23]

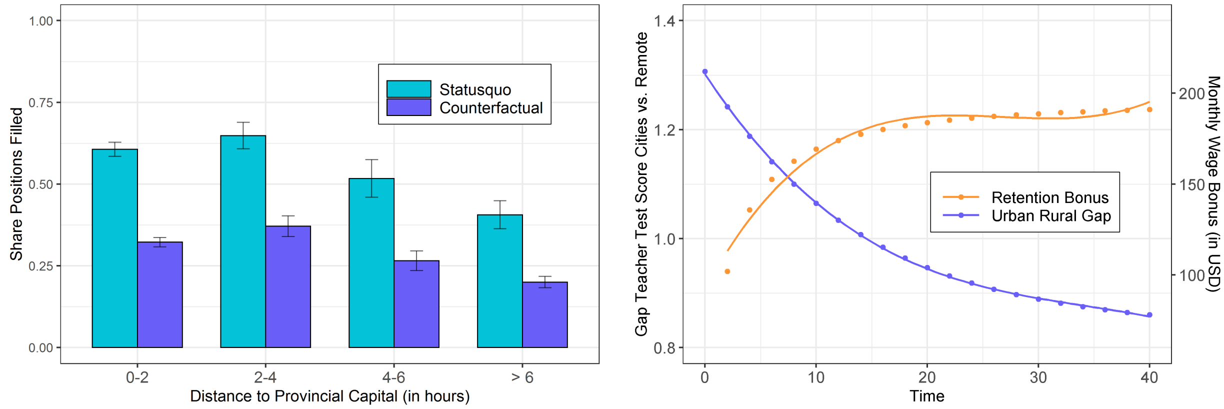
This exercise allows us to decompose the channels through which the spatial job ladder fuels spatial inequalities in teaching quality by shutting down labor market dynamics. Schools at the bottom of the ladder can now retain their highest skilled teachers while schools at the top of the ladder, on the contrary, can no longer poach skilled teachers from rural schools. Overall, this exercise shows that labor market dynamics explain 38% of the existing urban-rural gap in teacher quality. The remaining 62% are explained by initial unequal sorting in 2016 that cannot be offset by human capital accumulation on-the-job. This result highlights the importance of labor market dynamics in explaining spatial sorting and inequalities, even in a frictionless setting with rigid wages. It also suggests that there might be important benefits in investing in retaining existing teachers rather than aiming at recruiting higher quality teachers.

7.3.2 Evaluating Retention Policies

I then investigate the effectiveness of retention policies aiming at shutting down labor mobility and its adverse effects on spatial inequalities. Using the estimated model, I simulate the effects of removing teachers' option to rematch by enforcing a minimum contract length. If agents were myopic, I find that this policy would reach the same results as described in Figure 5 and close the urban-rural gap in teaching quality by 38% in the long-run by stopping skilled teachers from leaving rural schools. However, as agents are forward looking, teachers react ex-ante to this policy and their labor market participation plunges creating large shortages. Panel A of Figure 6 shows the share of filled vacancies under the status quo and under the policy which would enforce a minimum contract length of four years. As this policy forces teachers to commit and does not allow them to rematch and climb the ladder, they prefer to wait until they get better matching opportunities in the future. This results in a sharp drop in the share of filled vacancies. This finding highlights a key trade off between recruitment and retention. In the presence of a job ladder, retention policies that make rematching more difficult imply a significant decrease in the continuation value of accepting a job and generate

[23] This is driven by the fact that teachers test scores evolve more rapidly when they start from lower initial values, as suggested by the estimates in Table A.4.

Figure 6: Compulsory Service Policy



a) Adverse Sorting **b) Retention Bonuses**

NOTES. Panel A of this figure plots the share of filled vacancies for different bins of schools' remoteness in the status quo and under the counterfactual scenario where we would enforce a minimum contract length of four years. Panel B plots the effect of enforcing this policy on the urban rural gap in teacher test scores (in standard deviations) along with the monthly wage bonuses that would offset the adverse sorting effect shown in Panel A. The x-axis represents the minimum length of the contract.

strong adverse sorting responses. To avoid the latter, such policies should compensate workers for preventing them to improve their matching outcomes through job switching.

I then compute the amount that should be given as compensation to avoid this adverse sorting effect. I define the status quo match as μ^* and compute the monthly wage bonuses b_i for each teacher i which solve the following equation:

$$U_{i\mu_t^*(i)t} = U(x_{it}, z_{jt}, \hat{\boldsymbol{\theta}}) + \hat{\theta}_1 b_i + \eta_{ijt} + 0.9\Bigg(\int\int U(x, z, \hat{\boldsymbol{\theta}}) w(x|x_{it}, z_{jt}) m(z|x_{it}, z_{jt}) + \hat{\theta}_1 b_i + \gamma \\ + 0.9 \int \overline{U}_{t+2}(s) w(s|x, z) ds dx dz\Bigg)$$

The bonus b_i solving this equation makes teacher i indifferent between matching to a school for at least two years and matching to the same school for at least four years. Implementing this retention bonus scheme thus avoids the adverse sorting effect documented in Panel A of Figure 6. A similar equation can be formulated to compute the bonuses necessary to retain teachers for an additional τ years. I denote the solution to these equations for teacher i b_i^τ.

I compute b_i^τ for $\tau \in \{2, 4, 6, ..., 40\}$ and plot its average for each τ in Panel B of Figure 6. I also report the effect of this policy on the urban rural gap in teacher test scores measured in standard deviations on the same figure. Imposing a minimum contract length of four years would entail compensating teachers by a 100 USD bonus on their monthly salary on average. This number gradually increases as the minimum contract length increases before reaching a plateau of approximately 200 USD.[24]

Overall, this result shows that retention policies have large potential benefits in the long run but come at a cost which should be benchmarked against other alternatives. I find that this policy would reduce the urban-rural gap in teacher quality by 38% for an average monthly cost of 200 USD per teacher, which corresponds to a 40% increase in their monthly salaries.

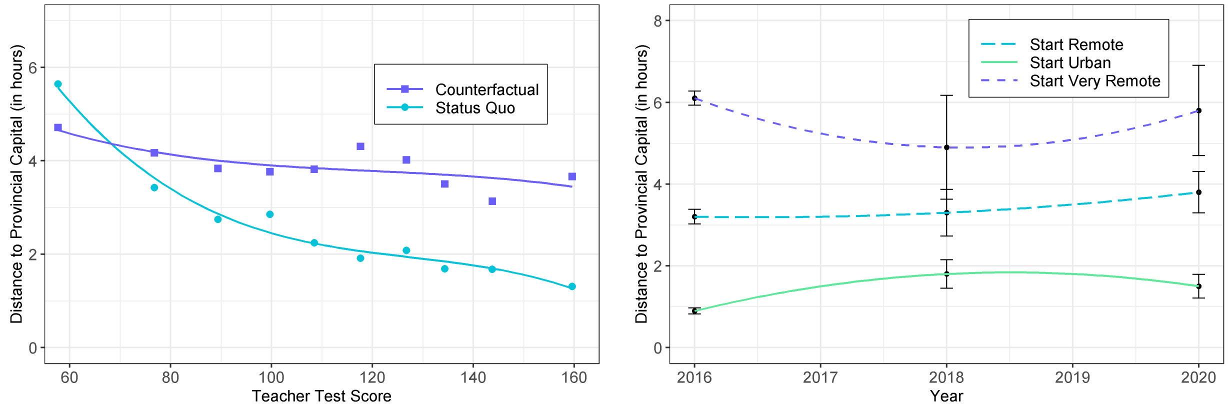
7.3.3 Shutting Down the Spatial Job Ladder

The existence of the spatial job ladder is mainly caused by three factors: (i) teachers' distaste for moving far from home, (ii) the concentration of teachers' home location in cities and (iii) the geographical dispersion of schools. As the spatial job ladder is responsible for the observed spatial inequalities, the most effective way of reducing inequalities would be to target its fundamental causes. In this section, I take (i) and (iii) as given and explore, as a thought experiment, what would be the consequences of shutting down (ii). To do so, I perform a counterfactual exercise that randomly changes teachers' home locations such that they are scattered across the country. I randomly draw teachers' new home location from the set of localities in which schools are situated. I then simulate equilibrium sorting and movements across locations.

Panel A of Figure 7 plots teacher sorting with respect to test scores and schools' remoteness under this counterfactual exercise. I find that the spatial job ladder collapses. As labor supply is scattered across the country, competition for local jobs disappears. Teachers match overall close to home and no longer have a systematic distaste for remote schools. High skilled teachers are thus no longer disproportionately matched to schools located in cities. Labor market dynamics are also strongly affected. The rate at which teachers move throughout the period 2016-2020 drops by half as low quality teachers are no longer sent far from home. Panel B of Figure 7 shows that the direction of the flows also changes as

[24]The concavity of the wage bonus scheme comes from the fact that agents discount the future at an increasing rate.

Figure 7: Random Home Location



a) Sorting **b) Job-to-Job Flows**

NOTES. This figure shows the results of a counterfactual experiment which would reallocate teachers' home location randomly across Peru. I randomly draw the location of each teacher from the list of localities in which school are situated and recompute the equilibrium. Panel A plots binned averages of teachers matched school's remoteness in the status quo and under this counterfactual. Panel B plots the counterfactual evolution of the remoteness of teachers matched schools from 2016 to 2020 starting from different initial levels of remoteness.

teacher no longer leave rural schools to get closer to urban centers. As a result, urban-rural inequalities in teacher attrition disappear and rural schools can benefit from experience and skill accumulation on-the-job. These results shows that designing policies targeting the root causes of the existence of the spatial job ladder, such as investing in training local teachers, might be more effective than aiming at slowing down its symptoms through recruitment or retention policies.

8 Conclusion

This paper investigates the causes of teacher spatial sorting and mobility and their consequences on spatial inequalities in teaching quality. To this end, I develop an empirical framework of dynamic matching without transfers. I assume that agents make forward-looking matching decisions and that their payoff functions depend on a various set of job and teacher attributes. I provide a tractable way to map teachers' and schools' preferences into sorting and job-to-job flows. I then show that one can invert this mapping and identify agents' preferences from data on realized sorting.

Using this methodology, I then show the existence of a spatial job ladder. Teachers concentrate in cities while jobs are scattered geographically. As teachers have a strong distaste for moving, this creates excess supply in cities which is rationed using observed measures of teacher quality. As a consequence, teacher quality is highly unequally distributed and teachers working in remote areas leave toward urban areas as soon as they have accumulated enough experience. Overall I find that labor mobility magnifies inequalities in teaching quality by one third. Finally, I assess the effectiveness of retention policies aimed preventing teachers from rematching along the job ladder. I find that this triggers a massive flow out of the teaching profession such that the positive effects of retention are largely outweighed by the losses incurred through teacher shortages. This highlights a key trade off between recruitment and retention in the presence of a job ladder and shows that retention policies should compensate for the implied lack of flexibility.

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Appendix

A Additional Tables and Figures

Table A.1: Data Description

	2016	2017	2018	2019	2020	2021
Panel A: Teacher Assignment Data						
# Teachers	116,559	116,939	116,128	115,358	115,233	116,024
Permanent	94,162	89,604	91,683	90,889	89,106	87,507
Temporary	22,397	27,361	24,466	24,505	26,174	28,516
Panel B: Centralized Allocation Mechanism						
# Test Takers	77,594	-	78,758	68,301	71,586	-
in Permanent Position Alloc.	6,770	-	9,777	5,905	4,005	-
in Temporary Position Alloc.	60,853	-	66,280	-	60,294	-
in Both	3,436	-	4,195	-	2,517	-
in None	13,407	-	6,896	-	9,804	-
# Vacancies	18,493	-	36,113	9,818	17,858	-
in Permanent Position Alloc.	6,460	-	13,620	9,818	5,014	-
in Temporary Position Alloc.	15,372	-	30,645	-	16,481	-
in Both	3,339	-	8,152	-	3,637	-

Notes. Panel A shows the total number of employed teachers in each year depending as well as the number of teachers holding a temporary or a permanent contract. Panel B displays the number of participants to the national competency test in each year it took place. It also shows the number of applicants and vacancy which participated in the allocation of temporary and permanent positions.

Table A.2: Summary Statistics: Job Characteristics

	Mean	Std. Deviation	Min	25% Pctile	75% Pctile	Max
Job Characteristics						
Baseline Monthly Wage (USD)	537.286	50.234	507.614	507.614	532.995	799.492
Temporary	0.186	0.389	0.000	0.000	0.000	1.000
Multigrade	0.207	0.405	0.000	0.000	0.000	1.000
Single Teacher	0.0462	0.210	0.000	0.000	0.000	1.000
Bilingual	0.107	0.309	0.000	0.000	0.000	1.000
School Characteristics						
Distance Prov. Capital (hours)	1.406	4.073	0.000	0.0609	1.144	72.000
Population	1143.708	2593.554	0.001	0.382	350.766	7567.716
Altitude (meters)	1506.684	1501.124	1.000	120.000	3104.000	5002.000
Local Amenities						
Electricity	0.952	0.215	0.000	1.000	1.000	1.000
Water	0.853	0.354	0.000	1.000	1.000	1.000
Sewage	0.726	0.446	0.000	0.000	1.000	1.000
Medical Center	0.770	0.421	0.000	1.000	1.000	1.000
Internet	0.582	0.493	0.000	0.000	1.000	1.000
Bank	0.388	0.487	0.000	0.000	1.000	1.000
Library	0.303	0.460	0.000	0.000	1.000	1.000

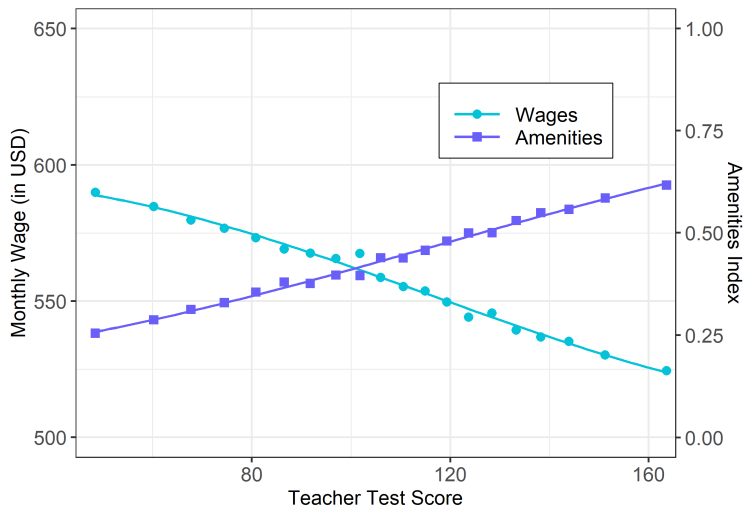
NOTES. This table uses the teacher assignment data to show summary statistics on the characteristics of the jobs filled in 2016. The baseline monthly wage does not contain experience bonuses.

Table A.3: Summary Statistics: Teacher Characteristics

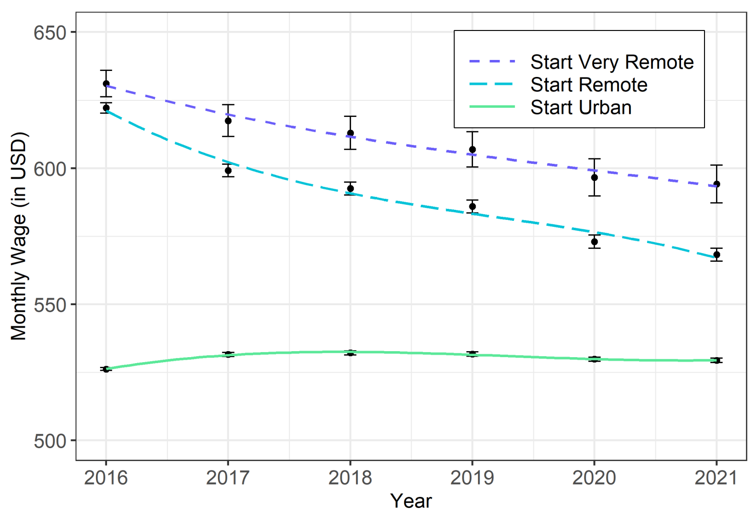
	Mean	Std. Deviation	Min	25% Pctile	75% Pctile	Max
Age	33.676	7.458	19.000	32.000	42.000	78.000
Female	0.707	0.455	0.000	0.000	1.000	1.000
Lives in Provincial Capital	0.819	0.385	0.000	1.000	1.000	1.000
Married	0.471	0.499	0.000	0.000	1.000	1.000
Total Test Score	97.672	29.895	0.000	74.500	119.500	191.500
Score 1: Reading	29.626	9.710	0.000	22.000	38.000	50.000
Score 2: Logical Reasoning	21.963	9.323	0.000	14.000	28.000	50.000
Score 3: Curricular Knowledge	46.083	15.881	0.000	35.000	57.500	97.500
Experience $<$ 3 years	0.237	0.426	0.000	0.000	0.000	1.000
Experience $>$ 10 years	0.130	0.337	0.000	0.000	0.000	1.000

Notes. This table shows summary statistics on the characteristics of the applicants to the centralized assignment platform in 2017.

Figure A.1: Sorting and Movements Across Locations: Wages



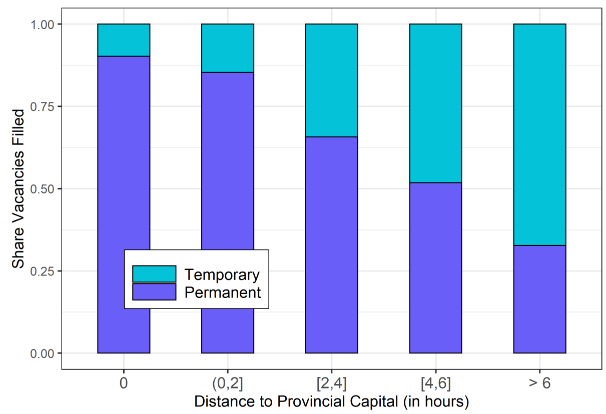
a) Sorting



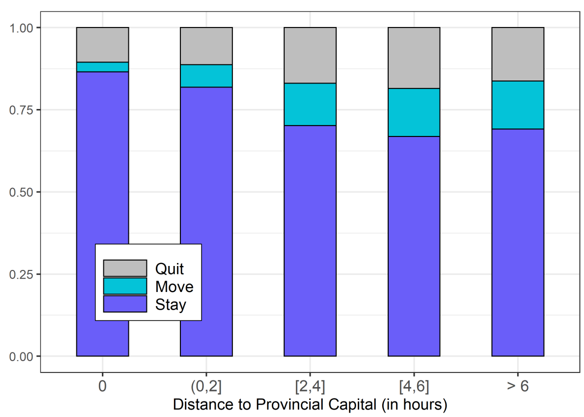
b) Job-to-Job Transitions

Notes. This figure uses the teacher assignment data to document teacher sorting and movements along other dimensions such as wages and amenities. Panel A plots binned averages of the monthly wage teachers receive as well as the level of amenities in the locality of their matched school based on their test scores. Bins are equally spaced based on vigintiles of the test score distribution. Panel B plots the evolution of the wage received by teachers over the period 2016-2021 depending on where they start in 2016. The purple line corresponds to teachers which start in schools located between 6 and 8 hours away from the provincial capital. The blue line corresponds to teachers which start in schools located between 4 and 6 hours. The green line corresponds to teachers which start in schools located between 2 and 4 hours.

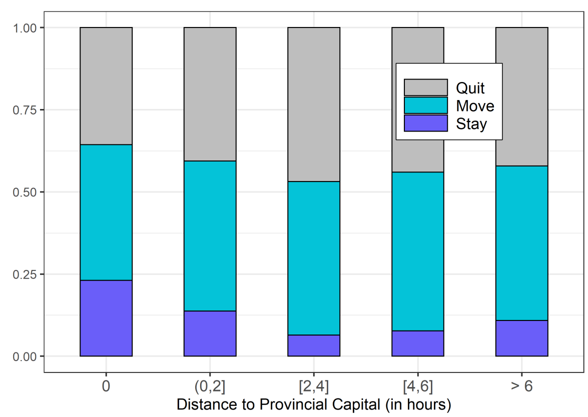
Figure A.2: Temporary vs. Permanent Contracts



a) Sorting Temporary vs. Permanent



b) Transition 2016-2018: Permanent



c) Transition 2016-2018: Temporary

Notes. Panel A of this figure uses the teacher assignment data to document how teachers sort across types of contract depending on the distance of their matched school to the provincial capital. Panel B and C show, for both permanent and temporary contracts, the share of teachers that stayed in the same school, moved to another school or quit teaching in the public sector for several bins of the schools' distance to the provincial capital.

Table A.4: Transition Process Test Scores

	Estimate	Std. Error
Constant	0.104	0.104
Teacher Characteristics		
Test Score t	0.868	0.007
Female	-0.008	0.011
Age < 30	0.102	0.011
Age > 50	-0.040	0.023
Experience < 3	-0.037	0.016
Exerience > 10	0.016	0.014
Married with kids	-0.030	0.011
School Characteristics		
Wage	-0.040	0.042
Frontier	-0.017	0.020
Bilingual	0.008	0.015
VRAEM	-0.017	0.021
log(Population)	-0.000	0.015
log(Population)^2	0.001	0.002
log(Population)^3	-0.000	0.000
Distance to Capital	-0.003	0.008
Distance to Capital^2	0.000	0.000
Distance to Capital^3	-0.000	0.000
log(pop) $\times$ Distance	0.000	0.001

Notes. This table displays the estimates of the coefficients of the following linear regression: $s_{it+1} = \boldsymbol{x}'_{\boldsymbol{it}}\boldsymbol{\beta_x} + \boldsymbol{z}'_{\boldsymbol{\mu_{wt}(i)t}}\boldsymbol{\beta_z} + \epsilon_{it}$. The estimation sample is composed of the set of teachers who took the test both in 2015 and 2017 as well as the set of teachers who took the test both in 2017 and 2019. Test scores are standardized to have mean 0 and standard deviation 1.

Table A.5: Teachers Preferences: Temporary vs. Permanent

	Temporary	× Permanent
Constant	356.22 (94.29)	4188.93 (504.84)
School/Locality Characteristics		
Amenities	23.40 (7.02)	50.24 (35.34)
Bilingual	-60.32 (10.51)	-721.86 (55.22)
Frontera	20.01 (11.36)	-15.71 (75.54)
VRAEM	-91.25 (14.99)	144.13 (101.53)
Preference for Home		
Dist (< 20km)	-18.18 (0.50)	-13.75 (2.50)
20km ≤ Dist < 100km	-6.02 (0.14)	-0.49 (0.62)
Dist ≥ 100km	-0.63 (0.02)	-0.56 (0.10)
Moving Costs		
≠ Province	-520.44 (14.39)	425.44 (50.34)
≠ Region	-121.98 (14.34)	-1006.74 (52.66)
Other Wage Determinants		
log(Pop)	83.30 (20.42)	-161.06 (125.39)
$\log(\text{Pop})^2$	-14.48 (2.12)	19.70 (13.78)
$\log(\text{Pop})^3$	0.54 (0.07)	-0.81 (0.47)
Distance to Capital	90.43 (12.89)	-277.86 (62.11)
Dist^2	-0.66 (1.76)	0.42 (5.44)
Dist^3	0.01 (0.08)	0.01 (0.24)
Dist × log(Pop)	-9.40 (1.46)	25.52 (6.66)
Teacher Characteristics		
Female	-58.77 (5.75)	-173.92 (40.28)
Urban	-62.59 (7.83)	-194.23 (68.31)
Married with kids	-11.96 (5.36)	99.66 (34.66)
Age < 30	18.32 (5.65)	396.84 (35.24)
Age > 50	-47.80 (15.64)	-141.33 (101.07)
Exp. < 3	14.68 (7.65)	20.88 (56.92)
Exp. > 10	-101.81 (8.55)	106.56 (52.92)

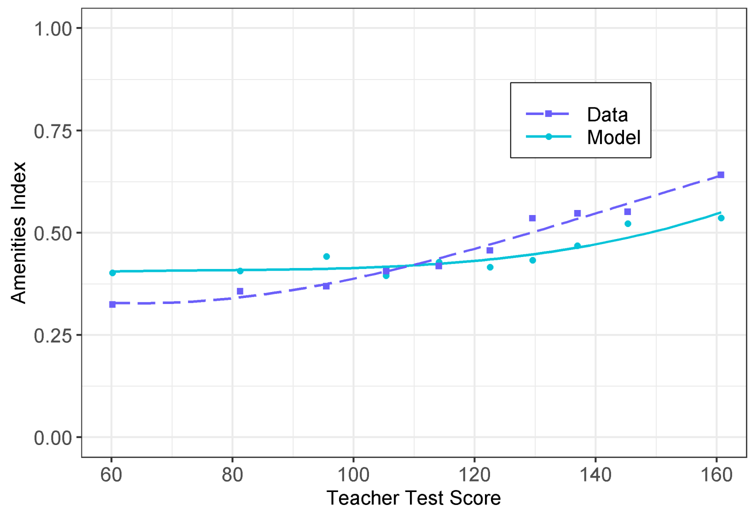
NOTES. This table displays the estimates of $\boldsymbol{\theta}_{\text{temp}}$ and $\boldsymbol{\theta}_{\text{perm}} - \boldsymbol{\theta}_{\text{temp}}$ assuming that $\beta = 0$, meaning that agents are myopic. The wage coefficient is normalized to 1 such that estimates are expressed in monthly willingness to pay in USD. Standard errors are in parenthesis.

Table A.6: Preference Estimates: Schools

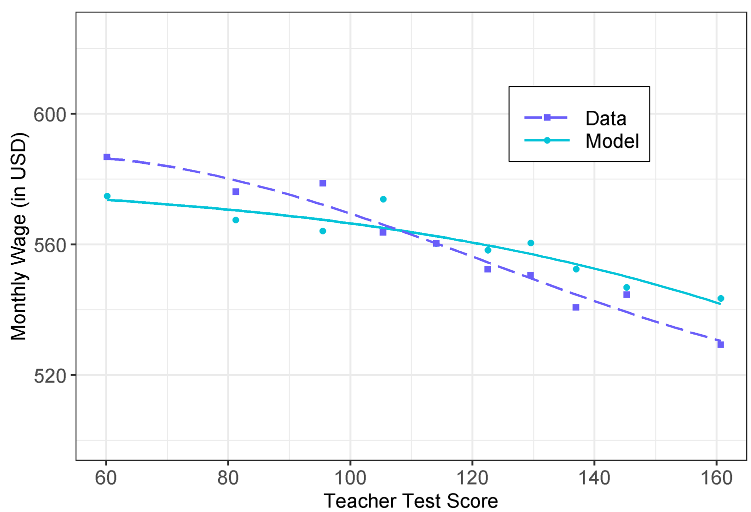
	(1)
Constant	-0.488 (0.082)
Female	-0.349 (0.022)
Married with kids	-0.038 (0.022)
Age < 30	0.128 (0.023)
Age > 50	-0.067 (0.089)
Experience < 3	-0.737 (0.041)
Experience > 10	0.057 (0.040)
Score 1: Reading	0.669 (0.023)
Score 2: Logic	0.571 (0.020)
Score 3: Curricular Knowledge	1.397 (0.022)

NOTES. This table displays the estimates of $\boldsymbol{\gamma}$ which are schools' preference parameters defined in Section 6. Standard errors are in parentheses.

Figure A.3: Model Fit Spatial Sorting: Additional Figures



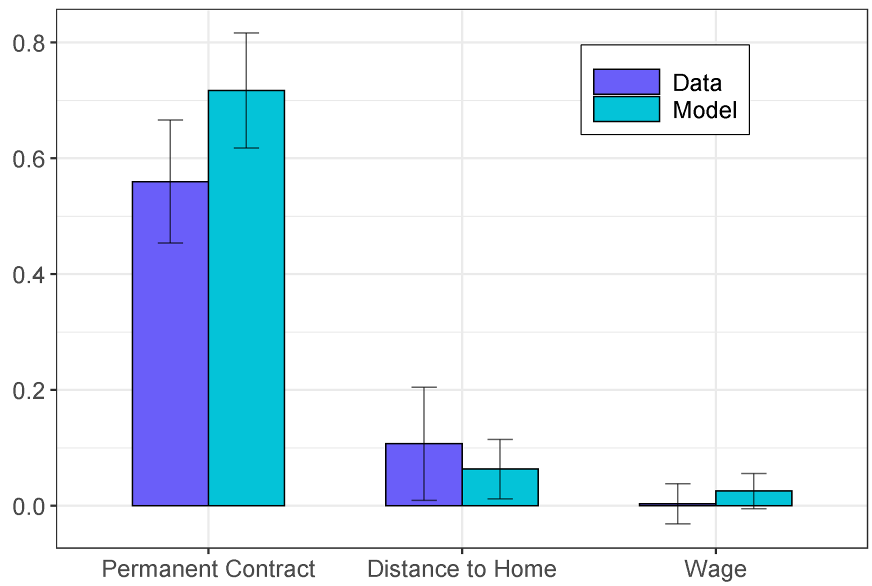
a) Amenities



b) Wages

Notes. This figure uses the centralized assignment data in 2018 and compares realized sorting patterns with model predictions. Panel A plots averages of level of amenities of teachers' matched schools based on equally spaced bins of the distribution of teachers' test scores both in the actual data and in the simulated equilibrium. Panel B shows the result of a similar exercise with wages instead.

Figure A.4: Validation RDD: Eligibility Cutoff



Notes. This figure displays the effect of crossing the test score threshold determining eligibility to permanent contracts on the probability to choose a permanent contract, the distance between teachers' matched schools and their home location, and the wage received from their matched schools. It computes these threshold crossing effects both in the actual data and in the equilibrium match simulated by the model and compares them.

B Context & Data: Details

B.1 Additional Institutional Details

B.1.1 Contracts and Wages

Public teachers in Peru can be hired under two types of contract. *Temporary* contracts last at least one year and can be renewed up to a second year, if both the school and the teacher agree. After two years, the position is either destroyed, if the allocated budget was fixed, or proposed again on the labor market. The same teacher could eventually teach in the same position but would have to apply again to get hired. *Permanent* contracts can last indefinitely. The coexistence of these two types of contracts is a common feature of civil servants' labor markets around the world. Permanent contracts are akin to usual civil servants contracts which make the profession attractive by insuring teacherss against unemployment. Temporary contracts are more precarious and are usually meant for schools to get a flexible access to a larger pool of applicants and react to unexpected transfers and/or the creation of new classrooms.

Wages are set by the government at the country level and vary along several dimensions. Temporary contracts are paid a fixed rate which does not vary with experience. To make the profession more attractive and keep up with inflation, the base monthly wage increased gradually from S/1,396 (363.19$) in 2016 to S/2,000 (520.33$) in 2017 and S/2,200 (572.36$) in 2019 to finally reach S/2,400 (624.39$) in 2021. Regarding permanent contracts, the pay scale is divided in six categories and teachers can apply once a year for a promotion through a centralized platform.[25] At the highest scale, the wage is 75% higher than the starting wage. Note that, at the exception of 2016 where it was S/1,550 (403.25$), the starting wage is exactly similar to the base wage for temporary contracts and followed the same time trend.

A wage bonus scheme was implemented by the Ministry of Education in order to make schools located in distressed areas or with worse teaching conditions more attractive. Teachers handling several grades receive a monthly wage bonus of S/140, schools with a single teacher provide a bonus of S/200, schools located in guerilla zones (VRAEM) provide a bonus of

[25] Promotions are awarded through a national standardized evaluation and a decentralized evaluation made by a committee which evaluates teachers' performance and professional career.

S/300, schools located close to the country borders provide a bonus of S/100 and schools which teach in several languages provide a bonus of S/50. Finally a set of wage bonuses ranging from S/70 to S/500 based on arbitrary cutoff rules compensates teachers based on the remoteness of the school's locality. Bobba et al. (2021) use these threshold in a regression discontinuity design to estimate the causal impact of increasing wages on recruitment and student achievement.

B.1.2 Allocation Mechanism

To make the allocation process of teaching positions more transparent, the Ministry of Education switched from a decentralized to a centralized application system in 2015. The use of centralized clearinghouses to allocate public sector jobs is becoming increasingly common (Roth, 2018) as they allow to reduce search frictions by regrouping all offers and applicants on the same market. The allocation of both temporary and permanent contracts is organized sequentially between November and March. Note that once teachers get awarded a permanent contract, they need to go through a separate procedure in order to be transferred to another school.[26]

National Competency Test: Before teaching positions are allocated, all applicants take a test evaluating their teaching competency. They get graded on three skills: (i) reading comprehension, (ii) logic reasoning and (iii) curricular knowledge. To be eligible for a permanent position, a teacher should get a score of at least 30/50 in part (i) and (ii) of the test and a score of a least 60/100 in part (iii) of the test. These are stringent requirements since only 9% of applicants end up being eligible (see Table A.1).

Allocation of permanent positions: The Ministry first publishes the list of available positions. Teachers eligible for a permanent position then form a list of choices within the same province.[27] Applicants are then assigned for interviews to their preferred three schools, with a total of 10 available slots per school.[28] For schools that are oversubscribed, test scores are used as priorities. Schools then interview and rank each applicant. Finally, they make offers

[26] Transfers are handled every year in a decentralized way. Priority in the transfer application system depends on seniority and other criterias which are not made public by the Ministry of Education.

[27] The maximum length of the list went from five in 2015 to being unrestricted from 2017 onwards.

[28] In 2015, applicants were assigned to two schools maximum and there were 20 slots per school.

sequentially to their preferred applicants. All unassigned applicants can then participate to an exceptional stage that allocates the remaining unfilled slots. At the end of this round, unassigned teachers can decide to participate in the allocation of temporary positions which takes place shortly after.

Allocation of temporary positions: All ineligible applicants along with eligible applicants which did not choose a permanent position participate in the allocation of temporary contracts. Teachers choose first a province. Within each province, serial dictatorship is used to assign teachers to schools using test scores as priorities. Schools do not have any role in the allocation process and cannot express their preferences over applicants. As in the allocation of permanent positions, unfilled vacancies are proposed to unassigned teachers from a different province in an exceptional stage.

This mechanism took place every year from 2015 to 2021 except in 2016. Note that in 2018 and 2020, only permanent positions were proposed.

B.2 Data Construction

I combine several sources of data provided by the Ministry of Education in Peru to construct the teacher assignment data and centralized assignment data described in Section 2.

Teacher occupation and payroll system (NEXUS): This dataset records annually each teacher and its matched position over the period 2012-2021. I restrict the data to primary school teachers which hold either a permanent or temporary contract. I exclude teachers working in several jobs by acting as a temporary replacement for other teachers on leave. Each teacher and position are identified by a unique ID which can be linked to other data sources. Each position is linked to the corresponding school which is also identified by a unique ID.

School census: This dataset contains information on a wide range of schools' and localities' characteristics. I observe detailed information on access to a wide range of services at the locality level such as electricity, water, sewage, medical centers, libraries or internet. I also observe the travel time between the locality and the closest provincial capital. I observe the number of inhabitants in the locality. I know whether the school has a second language of instruction, whether it has a single classroom. I also have access to the precise geocoordinates of the locality.

Household Targeting System (SISFOH): This dataset comes from Bobba et al. (2021) and contains information on the socio-economic status of the population of Peru in order to better target social benefits. It regroups individuals into households and records their home location, highest level of education, gender and their poverty status. I also observe their role with respect to the head of the household meaning that I can identify if individuals have children, are married or live with their parents.

Survey Centralized Allocation: The Ministry of Education surveys all the applicants that participate in the centralized allocation mechanism. I have thus additional information about applicants' level of experience in the public and private sector. I know which languages they speak and in which university or institute they went.

Centralized Allocation Mechanism: This dataset contains all the details of each step of the centralized allocation mechanism over the period 2015-2019. I observe the results of the national competency test for each applicant. I observe the set of applicants and positions participating in the allocation of permanent positions. I know where teachers apply, which schools interview them, how schools rank them and the final match. Finally, I observed the set of applicants and positions participating in the allocation of temporary positions. I do not observed teachers' final decision but I infer their match using the teacher assignment data.

The teacher assignment data combines the NEXUS with the school census and the SISFOH. The centralized assignment data combines the centralized allocation mechanism with the survey, the SISFOH and the school census.

C Value Added Model

I use data on the national evaluation of students in 2nd and 4th grade. I observe standardized test scores in math and in Spanish and I can match each classroom to its corresponding teacher. I can also match students to the SISFOH data to recover parental characteristics such as their education level or their poverty status.

Following closely Chetty et al. (2014a), I assume that each student i in year t is assigned to classroom $c = c(i,t)$ and that each teacher $j(c)$ is assigned to a classroom c. I restrict the analysis to primary schools meaning that teachers only teach one class per year. I denote μ_{jt} the value added of teacher j in year t normalized to have mean 0 and measured in student test scores standard deviations. I allow value added to drift over time. Finally, I assume that student i's test score in year t A^*_{it} relates to value added in the following way:

$$A^*_{it} = \boldsymbol{X}'_{it}\boldsymbol{\beta} + \mu_{jt} + \theta_c + \epsilon_{it}$$

where $\boldsymbol{X}_{it}$ includes a set of student, classroom and school characteristics, θ_c is an exogenous shock at the classroom level and ϵ_{it} is an idiosyncratic shock at the student-year level. I assume that the stochastic processes μ_{jt} and ϵ_{it} are stationary meaning that $\mathbb{E}[\mu_{jt}|t] = \mathbb{E}[\epsilon_{it}|t] = 0$, $\text{Cov}(\mu_{jt}, \mu_{jt+s}) = \sigma_{\mu s}$, $\text{Cov}(\epsilon_{it}, \epsilon_{it+s}) = \sigma_{\epsilon s}$ and $\text{Var}(\mu_{jt}) = \sigma^2_\mu$ for all t.

I estimate μ_{jt} using the following procedure. First, I estimate $\boldsymbol{\beta}$ by regressing test scores A^*_{it} on $\boldsymbol{X}_{it}$ and teacher fixed effects α_j. Estimating $\boldsymbol{\beta}$ using within-teacher variation avoids attributing the teacher effect to variation in $\boldsymbol{X}_{it}$.[29] I then construct the following residualized test scores:

$$A_{it} = A^*_{it} - \boldsymbol{X}'_{it}\hat{\boldsymbol{\beta}}$$

and average them at the teacher-year level to construct $\overline{A}_{jt} = \frac{1}{n}\sum_{i\in\{i:j=j(c(i,t))\}} A_{it}$ for all j,t. Finally, I shrink these estimates by projecting $\overline{A}_{jt}$ on past residualized test scores $\boldsymbol{A}_j^{-t} =$

[29] Table C.1 shows that not including teacher FE introduces severe bias in the coefficients associated with classroom and school characteristics.

$(\overline{A}_{j1}, ..., \overline{A}_{jt-1})$. The estimator of VA $\hat{\mu}_{jt}$ can thus be written as:

$$\hat{\mu}_{jt} = \sum_{s=1}^{t-1} \psi_s \overline{A}_{js}$$

where $\boldsymbol{\psi} = (\psi_1, ..., \psi_{t-1})$ are the coefficients of the OLS regression of $\overline{A}_{jt}$ on $\boldsymbol{A}_j^{-t}$. Note that I only use $t-1$ to predict VA such that $\psi = \frac{\sigma_{A,1}}{\sigma_A^2} = \frac{\text{Cov}(\overline{A}_{jt}, \overline{A}_{jt-1})}{\text{Var}(\overline{A}_{jt})}$.

The results of the estimation of $\hat{\mu}_{jt}$ are displayed in Table C.2. The auto-correlation ψ is estimated at 0.466. To get a proper estimate for the standard deviation of value added $\sigma_\mu = \sigma_{A,0}$ in elementary schools, Chetty et al. (2014a) perform a non-linear extrapolation from their estimates of $\sigma_{A,s}$ for $1 \leq s \leq 7$. However, I do not have access to test score data prior to 2016 making the replication of this exercise impossible. As pointed out in Chetty et al. (2014a), $\sigma_{A,0} \geq \sigma_{A,1}$ making $\sqrt{\hat{\sigma}_{A,1}}$ an estimator of a lower bound on σ_μ. I estimate this lower bound to be 0.3 which is substantially larger than previous estimates.[30]

I then perform the usual checks for forecast unbiasedness of $\hat{\mu}_{jt}$ following Chetty et al. (2014a). I first regress A_{it} on $\hat{\mu}_{jt}$ and find a coefficient of 1.030 with 95% confidence interval $[0.944, 1.116]$. Standard errors are clustered at the school level. This regression should give us a coefficient of 1 which is not rejected by the data. I then project A_{it} on parental characteristics that are excluded from X_{it} such as socio-economic status and regress μ_{jt} on this projection. I find a coefficient of 0.008 with a tight 95% confidence interval $[0.002, 0.014]$ meaning that we can rule out any substantial sorting of students across teachers based on parental characteristics.[31]

To estimate the cost of attrition, I use teacher switching as a quasi-experiment as in Chetty et al. (2014a) to test two hypotheses: (i) skills are not perfectly transferable across schools and (ii) attrition implies a net loss for the origin school. If skills are perfectly transferable across schools, switching to a different school after a long employment spell should have no effect on value added. I assume that switching decisions are independent of unobserved

[30] Chetty et al. (2014a) estimate $\sigma_\mu = 0.163$ and Bates et al. (2022) estimate $\sigma_\mu = 0.249$. This could be explained by the fact that most other studies use data on urban districts while the data used in this paper covers the universe of teachers in Peru. If high value added teachers are concentrated in cities, estimating σ_μ in urban districts could understate its population value.

[31] I explore the relationship between $\hat{\mu}_{jt}$ and A_{it} as well as between $\hat{\mu}_{jt}$ and predicted scores using parental SES non parametrically in Figure C.1. To do this, I construct averages for 20 equal sized bins of value added to get an approximation of the conditional expectation function.

factors that could affect drift in value added. This rules out scenarios where teachers decide to switch to a different school because they anticipate that they will have a higher value added there. I then compare the difference in value added between 2016 and 2018 for teachers that stayed in the same school with the same difference for teachers that moved to a different school. To do so, I estimate β in the following two-way fixed effects regression:

$$\mu_{gt} = \alpha_g + \delta_t + \beta D_{gt} + \epsilon_{gt} \tag{5}$$

where $\mu_{gt} = \frac{1}{N_{gt}} \sum_{j \in g} \hat{\mu}_{jt}$, α_g is a group fixed effect, δ_t is a time fixed effect and D_{gt} is group g's treatment status in period t. In this simple setting $g \in \{\text{Movers}, \text{Stayers}\}$ and $t \in \{2016, 2018\}$ and I assume that $D_{\text{Stayers},t} = 0$ for all t and $D_{\text{Movers},2016} = 0$ and $D_{\text{Movers},2018} = 1$. In this setting, β corresponds to the ATT.

Table C.3 shows the results of the estimation of β with standard errors clustered at the teacher level. In Panel A, I estimate β conditional on movers having more than one year of experience in the school they taught in before switching. I find that moving implies a net loss of value added of 0.056σ corresponding to 26% of a standard deviation of teacher value added. This is consistent with the hypothesis that skills are not perfectly transferable across schools. As a placebo test, I consider movers with no prior experience in the schools they were before switching in Panel B. They should not have accumulated school specific skills prior to moving which is consistent with the zero effect found in Table C.3. These results show that job-to-job transitions imply a sizeable aggregate loss in value added.

I then perform a second exercise quantifying the loss in productivity following a move at the school level. I find that leavers are substantially of higher quality than the teachers who replace them. Using value added prior to moving I find a difference of 0.10 standard deviation which corresponds to around 50% of a standard deviation in value added.

Table C.1: Value Added: Estimation of $\boldsymbol{\beta}$

	(1)	(2)
Constant	0.781 (0.047)	0.559 (0.133)
$t = 2018$	0.020 (0.005)	0.023 (0.005)
Student Level Controls		
Lagged Math Score	0.431 (0.006)	0.444 (0.005)
Lagged Math Score^2	0.010 (0.002)	0.012 (0.002)
Lagged Math Score^3	-0.027 (0.002)	-0.029 (0.001)
Lagged Spanish Score	0.236 (0.005)	0.444 (0.005)
Lagged Spanish Score^2	0.010 (0.002)	0.012 (0.002)
Lagged Spanish Score^3	-0.012 (0.001)	-0.029 (0.001)
Female	-0.112 (0.005)	-0.111 (0.005)
Age	-0.071 (0.005)	-0.052 (0.005)
Ethnicity: Quechua	0.045 (0.027)	0.058 (0.025)
Ethnicity: Native	-0.022 (0.026)	-0.036 (0.023)
Classroom Level Controls		
Ethnicity: Quechua	1.657 (0.094)	0.379 (0.160)
Ethnicity: Native	-1.303 (0.093)	-0.478 (0.151)
Size	0.003 (0.000)	-0.001 (0.001)
School Level Controls		
Lagged Math Score	-0.052 (0.018)	-0.384 (0.073)
Lagged Math Score^2	0.004 (0.016)	-0.213 (0.058)
Lagged Math Score^3	0.074 (0.016)	0.031 (0.045)
Lagged Spanish Score	0.245 (0.019)	0.483 (0.073)
Lagged Spanish Score^2	-0.047 (0.015)	0.140 (0.057)
Lagged Spanish Score^3	-0.001 (0.012)	0.038 (0.037)
Teacher FE	✗	✓

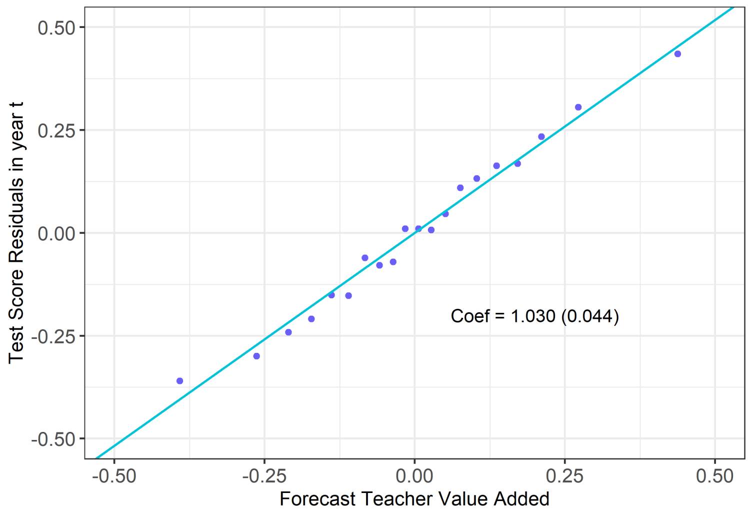
NOTES. This table displays the estimates of $\boldsymbol{\beta}$ from the linear regression of student test scores on student, classroom and school characteristics described in Section D. Column 1 shows the results of this regression without teacher fixed effects. Column 2 includes teacher fixed effects. Standard errors are in parentheses

Table C.2: Value Added: Structural Parameters

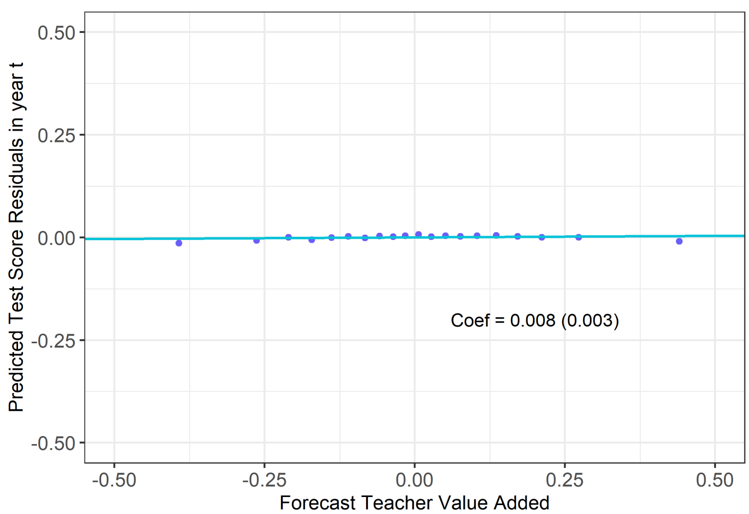
Parameter	Estimate	Std. Error	95% CI
$\sigma_{A,1}$	0.089	0.005	$[0.079, 0.100]$
σ_A	0.192	0.007	$[0.179, 0.205]$
ψ	0.466	0.019	$[0.446, 0.485]$
Lower Bound σ_μ	0.300	0.009	$[0.282, 0.316]$

NOTES. This Table displays the estimates of the structural parameters of the teacher value added model described in Section D.

Figure C.1: Value Added: Robustness Checks



a) Actual Score



b) Predicted Score using Parental SES

Notes. Panel A of this figure plot averages of the test score residuals A_{it} for 20 equally spaced bins of the forecasted teacher value added. Panel B of plot averages of the test score residuals A_{it} projected onto parental socio-economic status for 20 equally spaced bins of the forecasted teacher value added. The reported coefficients correspond to the slope of the blue line. Standard errors are in parentheses.

Table C.3: Imperfectly Transferable Skills

	Estimate	Std. Errors	95% CI
Panel A: Past Tenure			
ATE Movers: β	-0.056	0.023	$[-0.102, -0.011]$
α_{Stayers}	-0.005	0.004	$[-0.014, 0.003]$
α_{Movers}	-0.026	0.020	$[-0.066, 0.024]$
δ_{2018}	0.004	0.005	$[-0.005, 0.013]$
Panel B: No Past Tenure			
ATE Movers: β	0.004	0.018	$[-0.031, 0.038]$
α_{Stayers}	-0.004	0.004	$[-0.012, 0.005]$
α_{Movers}	-0.021	0.014	$[-0.048, 0.007]$
δ_{2018}	0.004	0.005	$[-0.006, 0.013]$

Notes. This table displays the results of the estimation of Equation (5). Panel A restricts the sample to teachers which have been in the same school prior to 2016 for more than three years. Panel B restricts the sample to teachers which have been in the same school prior to 2016 for less than three years. Standard errors are clustered at the teacher level.

D Proofs

D.1 Proof of Proposition 1

I first show that part (i) of Proposition 1 is a direct implication of Assumption 4 (i) and (ii), i.e that the match is stable in period t.

Consider a match μ_t and suppose first that either Assumption 4 (i) or (ii) is violated such that μ_t is not stable. First, suppose that (i) does not hold meaning that there exists a teacher-school pair (i, j) such that $U_{ijt} > U_{i\mu_{wt}(i)t}$ and $V_{ijt} > V_{\mu_{mt}(j)jt}$. This would mean that $j \in M_{it}(\mu_t)$ and $U_{ijt} > U_{i\mu_{wt}(i)t}$ which contradicts that $U_{i\mu_{wt}(i)t} = \max_{k\in M_{it}(\mu_t)\cup\{0\}} U_{ikt}$. Now, suppose that (ii) does not hold meaning that $U_{i0t} > U_{i\mu_{wt}(i)t}$ or $V_{0jt} > V_{\mu_{mt}(j)jt}$. In both cases, this would contradict that $U_{i\mu_{wt}(i)t} = \max_{k\in M_{it}(\mu_t)\cup\{0\}} U_{ik}$ or $V_{\mu_{mt}(j)jt} = \max_{l\in W_{jt}(\mu_t)\cup\{0\}} V_{ljt}$.

Now, suppose that for a given i, $U_{i\mu_{wt}(i)t} < \max_{k\in M_{it}(\mu_t)\cup\{0\}} U_{ikt}$. This means that there exists a school $k' \in M_{it}(\mu) \cup \{0\}$ such that $U_{ik't} > U_{i\mu_{wt}(i)t}$. If $k' = 0$ this immediately contradicts stability. If $k' \in M_{it}(\mu)$ this implies that $V_{ik't} \geq V_{\mu_{mt}(k')k't}$ and $U_{ik't} > U_{i\mu_{wt}(i)t}$. If $V_{ik't} = V_{\mu_{mt}(k')k't}$ this implies that $k' = \mu_w(i)$ and we reach a contradiction. Otherwise we have that $U_{ik't} > U_{i\mu_{wt}(i)t}$ and $V_{ik't} > V_{\mu_m(k')k't}$ which contradicts stability. The argument is symmetric for the school's side.

Part (ii) of Proposition 1 is a direct consequence of part (i) and Assumption 2.

D.2 Proof of Proposition 2

As $\overline{U}_{t+1}$ is independent of η_{ijt} under Assumption 1 and with exogenous choice sets, I treat it as fixed and rewrite $U_{ijt} = u_{ijt} + \sigma\eta_{ijt}$ for simplicity. The proof of part (i) of Proposition 2 is then identical to the proof of Lemma 3.1 in Menzel (2015).

$$\begin{aligned}\mathbb{P}(U_{ijt} \geq \max_{k=0,1,\ldots,J} U_{ikt}|(u_{ikt})_{k=1}^J) &= \int \mathbb{P}(U_{ijt} \geq U_{ikt}, k \in \mathcal{I} - \{j\}|(u_{ikt})_{k=1}^J, \eta_{ijt} = s)g(s)ds \\ &= \int \prod_{k\in\mathcal{I}-\{j\}} G(\sigma^{-1}(u_{ijt} - u_{ikt}) + s)g(s)ds\end{aligned}$$

$$= \int \prod_{k=1}^{2J} G(\sigma^{-1}(u_{ijt} - u_{ikt}) + s) \frac{g(s)}{G(s)} ds$$

As in Menzel (2015), I then do the change of variables $s = a_J h + b_J$ where $a_J = a(b_J)$ and $b_J = G^{-1}(1 - J^{-1/2})$ and multiply by J on both sides:

$$J\mathbb{P}(U_{ijt} \geq \max_{k=0,1,\ldots,J} U_{ikt} | (u_{ikt})_{k=1}^J) = \int \exp\left(\frac{1}{J}\sum_{k=1}^{2J} J \log G(a_J(u_{ijt} - u_{ikt} + h) + b_J)\right) \frac{Ja_J g(a_J h + b_J)}{G(a_J h + b_J)} dh$$

Following Resnick (1987) and under Assumption 1 we can show that:

$$J \log G(a_J(u_{ijt} - u_{ilt} + h) + b_J) \to -e^{-(u_{ijt} - u_{ikt} + h)}$$

$$\frac{Ja_J g(a_J h + b_J)}{G(a_J h + b_J)} \to e^{-h}$$

We thus have under Assumption 1:

$$\begin{aligned} J\mathbb{P}(U_{ijt} \geq \max_{k=0,1,\ldots,J} U_{ikt} | (u_{ikt})_{k=1}^J) &= \int \exp\left(-\frac{1}{J}\sum_{k=1}^{2J} e^{-(u_{ijt} - u_{ikt} + h)}\right) e^{-h} dh + o(1) \\ &= \int \exp\left(-\frac{1}{J}\sum_{k=1}^{2J} e^{-h} e^{(u_{ikt} - u_{ijt})}\right) e^{-h} e^{-h} dh + o(1) \end{aligned}$$

I then do a final change of variable $s = e^{-h}$ such that we get:

$$\begin{aligned} J\mathbb{P}(U_{ijt} \geq \max_{k=0,1,\ldots,J} U_{ikt} | (u_{ikt})_{k=1}^J) &= \int_0^{+\infty} \exp\left(-\frac{1}{J}\sum_{k=1}^{2J} s e^{(u_{ikt} - u_{ijt})}\right) s ds + o(1) \\ &= \frac{\exp(u_{ijt})}{\frac{1}{J}\sum_{k=1}^{2J} \exp(u_{ikt})} + o(1) \end{aligned}$$

From this we can finally show that:

$$J\mathbb{P}(U_{ijt} \geq \max_{k=0,1,\ldots,J} U_{ikt} | \boldsymbol{x}_{it}, (\boldsymbol{z}_{kt})_{k=1}^J) =$$

$$\frac{\exp\left\{U_t(\boldsymbol{x}_{it}, \boldsymbol{z}_{jt}) + \beta_w \int \overline{U}_{t+1}(s) w(s|\boldsymbol{x}_{it}, \boldsymbol{z}_{jt}) ds\right\}}{\exp\left\{\beta_w \int \overline{U}_{t+1}(s) w_0(s|\boldsymbol{x}_{it}) ds\right\} + \frac{1}{J}\sum_{k=1}^J \exp\left\{U_t(\boldsymbol{x}_{it}, \boldsymbol{z}_{kt}) + \beta_w \int \overline{U}_{t+1}(s) w(s|\boldsymbol{x}_{it}, \boldsymbol{z}_{kt}) ds\right\}} + o(1)$$

which implies that:

$$J\mathbb{P}(U_{ijt} \geq \max_{k=0,1,\dots,J} U_{ikt}|\boldsymbol{x}_{it}, \boldsymbol{z}_{jt}) \longrightarrow$$
$$\frac{\exp\left\{U_t(\boldsymbol{x}_{it}, \boldsymbol{z}_{jt}) + \beta_w \int \overline{U}_{t+1}(s)w(s|\boldsymbol{x}_{it}, \boldsymbol{z}_{jt})ds\right\}}{\exp\left\{\beta_w \int \overline{U}_{t+1}(s)w_0(s|\boldsymbol{x}_{it})ds\right\} + \int \exp\left\{U_t(\boldsymbol{x}_{it}, h) + \beta_w \int \overline{U}_{t+1}(s)w(s|\boldsymbol{x}_{it}, h)ds\right\}m_t(h)dh}$$

which finishes the proof of part (i) of Proposition 2.

Using similar steps as in McFadden et al. (1973), we can then show that:

$$\begin{aligned}\mathbb{E}(\max_{k=0,1,\dots,J} U_{ikt}|\boldsymbol{x}_{it}, (\boldsymbol{z}_{kt})_{k=1}^J) = \log\Bigg(&\exp\left\{\beta_w \int \overline{U}_{t+1}(s)w_0(s|\boldsymbol{x}_{it})ds\right\} \\ &+\frac{1}{J}\sum_{k=1}^{J} \exp\left\{U_t(\boldsymbol{x}_{it}, \boldsymbol{z}_{kt}) + \beta_w \int \overline{U}_{t+1}(s)w(s|\boldsymbol{x}_{it}, \boldsymbol{z}_{kt})ds\right\}\Bigg) + \log(J) + \gamma + o(1)\end{aligned}$$

where γ is Euler's constant. Under Assumption 1, we can finally apply the law of large numbers to show that:

$$\begin{aligned}\mathbb{E}(\max_{k=0,1,\dots,J} U_{ikt}|\boldsymbol{x}_{it}, (\boldsymbol{z}_{kt})_{k=1}^J) = \log\Bigg(&\exp\left\{\beta_w \int \overline{U}_{t+1}(s)w_0(s|\boldsymbol{x}_{it})ds\right\} \\ &+ \int \exp\left\{U_t(\boldsymbol{x}_{it}, h) + \beta_w \int \overline{U}_{t+1}(s)w(s|\boldsymbol{x}_{it}, h)ds\right\}m_t(h)dh\Bigg) + \log(J) + \gamma + o(1)\end{aligned}$$

which concludes the proof of part (ii) of Proposition 2.

D.3 Definition Ψ_{wt} and Ψ_{mt}

$$\Psi_{wt}[\boldsymbol{\Gamma}](x) = \int \frac{\exp\left\{U_t(x,h) + V_t(x,h) + \beta \int \overline{U}_{t+1}[\boldsymbol{\Gamma}](s)w(s|x,h)ds + \beta \int \overline{V}_{t+1}[\boldsymbol{\Gamma}](s)m(s|x,h)ds\right\}}{\exp\left\{\beta \int \overline{V}_{t+1}[\boldsymbol{\Gamma}](s)m_0(s|h)ds\right\} + \Gamma_{mt}(h)} m_t[\boldsymbol{\Gamma}](h)dh$$

$$\Psi_{mt}[\boldsymbol{\Gamma}](z) = \int \frac{\exp\left\{U_t(h,z) + V_t(h,z) + \beta \int \overline{U}_{t+1}[\boldsymbol{\Gamma}](s)w(s|h,z)ds + \beta \int \overline{V}_{t+1}[\boldsymbol{\Gamma}](s)m(s|h,z)ds\right\}}{\exp\left\{\beta \int \overline{U}_{t+1}[\boldsymbol{\Gamma}](s)w_0(s|h)ds\right\} + \Gamma_{wt}(h)} w_t[\boldsymbol{\Gamma}](h)dh$$

$$\overline{U}_{t+1}[\boldsymbol{\Gamma}](x) = \log\left(\exp\left\{\beta \int \overline{U}_{t+2}[\boldsymbol{\Gamma}](s)w_0(s|x)ds\right\} + \Gamma_{wt+1}(x)\right)$$

$$\overline{V}_{t+1}[\mathbf{\Gamma}](z) = \log\left(\exp\left\{\beta \int \overline{V}_{t+2}[\mathbf{\Gamma}](s) m_0(s|z) ds\right\} + \Gamma_{mt+1}(z)\right)$$

$$w_t[\mathbf{\Gamma}](x) = \int_{\mathcal{X}_t}\int_{\mathcal{Z}_t} w(x|s,h) f_{t-1}[\mathbf{\Gamma}](s,h) dh ds + \int_{\mathcal{X}_t} w_0(x|s) f_{t-1}[\mathbf{\Gamma}](s,*) ds$$

$$m_t[\mathbf{\Gamma}](z) = \int_{\mathcal{X}_t}\int_{\mathcal{Z}_t} m(z|s,h) f_{t-1}[\mathbf{\Gamma}](s,h) dh ds + \int_{\mathcal{Z}_t} m_0(z|s) f_{t-1}[\mathbf{\Gamma}](*,h) dh$$

D.4 Proof of Theorem 1

I start by proving part (i) and (ii) of Theorem 1. Throughout the rest of the proof I set WLOG $\gamma_w = \gamma_m = 0$ and $\beta_w = \beta_m = \beta$ for simplicity. I first restrict the space of functions to which the solutions to the fixed point problem described in Equation (4) can belong. Namely, I show that we can restrict ourselves to a Banach space of continuous functions. Assume that there exists a set of $2 \times T$ functions Γ^*_{wt} and Γ^*_{mt} for all $t = 1, ..., T$ that solve the fixed point problem. I start by showing that these solutions are bounded from above. By definition of Ψ_{wt} and $\overline{U}_t$ and using that $\Gamma^*_{mt} \geq 0$ for all t, we can proceed by backward induction and show:

$$\begin{aligned}
\Gamma^*_{wT}(x) = \Psi_{wT}[\mathbf{\Gamma}^*](x) &= \int \frac{\exp\{U_T(x,s) + V_T(x,s)\}}{1 + \Gamma^*_{mT}(s)} m_T(s) ds \\
&\leq \int \exp\{U_T(x,s) + V_T(x,s)\} m_T(s) ds \\
&\leq \exp\{\bar{U} + \bar{V}\}
\end{aligned}$$

where $\bar{U}$ and $\bar{V}$ are the upper bounds of the functions U_t and V_t for all t, respectively. From there we can bound $\overline{U}^*_T$ as follows:

$$\begin{aligned}
\overline{U}^*_T(x) &= \log\left(1 + \Gamma^*_{wT}(x)\right) \\
&\leq \log\left(1 + \exp\{\bar{U} + \bar{V}\}\right)
\end{aligned}$$

Similar bounds can be derived on the school side. We can then iterate this procedure and bound Γ_{wT-1} and $\overline{U}_{T-1}$:

$$\begin{aligned}\Gamma^*_{wT-1}(x) &= \int \frac{\exp\{U_{T-1}(x,h) + V_{T-1}(x,h) + \beta\int \overline{U}_T(s)w(s|x,h)ds + \beta\int \overline{V}_T(s)m(s|x,h)ds\}}{1+\Gamma^*_{mT-1}(h)} m_{T-1}(h)dh \\ &\leq \exp\left\{\overline{U} + \overline{V} + 2\beta\log(1+\exp\{\overline{U}+\overline{V}\})\right\}\end{aligned}$$

$$\begin{aligned}\overline{U}^*_{T-1}(x) &= \log\left(\exp\left\{\beta\int \overline{U}^*_T(s)w_0(s|x)ds\right\} + \Gamma^*_{wT-1}(x)\right) \\ &\leq \log\left(\exp\left\{\beta\log\big(1+\exp\{\overline{U}+\overline{V}\}\big)\right\} + \exp\left\{\overline{U}+\overline{V}+2\beta\log\big(1+\exp\{\overline{U}+\overline{V}\}\big)\right\}\right)\end{aligned}$$

Boundedness of Γ^*_{wt} is thus implied by boundedness of $\overline{U}_{t+1}$ which is in itself implied by boundedness of Γ^*_{wt+1} and $\overline{U}_{t+2}$. By induction we can thus show that boundedness of Γ^*_{wT} implies boundedness of Γ^*_{wt} for all $t = 1, ..., T$. The same argument applies to Γ^*_{mt}. Continuity of Γ^*_{wt} and Γ^*_{mt} follows from continuity of U and V and that the integrals are nonnegative. Differentiability of Γ^*_{wt} and Γ^*_{mt} also follows from differentiability of U and V which is stated in Assumption 1. We can thus restrict the spaces in which Γ^*_{wt} and Γ^*_{mt} belong to a Banach space of nonnegative bounded continuous functions which I call $\mathcal{C}$.

I now turn to the proof that the mapping $(\log\boldsymbol{\Gamma_w}, \log\boldsymbol{\Gamma_m}) \mapsto (\log\boldsymbol{\Psi_m}[\boldsymbol{\Gamma}], \log\boldsymbol{\Psi_w}[\boldsymbol{\Gamma}])$ is a contraction. Consider two sets of functions $\boldsymbol{\Gamma} = (\Gamma_{mt}, \Gamma_{wt})_{t=1}^T$ and $\tilde{\boldsymbol{\Gamma}} = (\tilde{\Gamma}_{mt}, \tilde{\Gamma}_{wt})_{t=1}^T$ belonging to $\mathcal{C}^{2T}$. I show that there always exists a constant $\lambda < 1$ such that:

$$\Big|\Big|\log\boldsymbol{\Psi_w}[\boldsymbol{\Gamma_m}] - \log\boldsymbol{\Psi_w}[\tilde{\boldsymbol{\Gamma}}_{\boldsymbol{m}}]\Big|\Big|_\infty \leq \lambda\Big|\Big|\log\boldsymbol{\Gamma_m} - \log\tilde{\boldsymbol{\Gamma}}_{\boldsymbol{m}}\Big|\Big|_\infty$$

The mean value inequality for vector valued functions defined on Banach spaces implies that:

$$\begin{aligned}&\Big|\Big|\log\boldsymbol{\Psi_w}[\boldsymbol{\Gamma_m}](x) - \log\boldsymbol{\Psi_w}[\tilde{\boldsymbol{\Gamma}}_{\boldsymbol{m}}](x)\Big|\Big|_\infty \leq \\ &\qquad \sup_{a\in[0,1]}\Big|\Big|D\log\boldsymbol{\Psi_w}[a\log\boldsymbol{\Gamma_m} + (1-a)\log\tilde{\boldsymbol{\Gamma}}_{\boldsymbol{m}}](x)\Big|\Big|_\infty\Big|\Big|\log\boldsymbol{\Gamma_m}(x) - \log\tilde{\boldsymbol{\Gamma}}_{\boldsymbol{m}}(x)\Big|\Big|_\infty\end{aligned}$$

where $D\log\boldsymbol{\Psi_w}$ are the Gateaux derivatives of $\log\boldsymbol{\Psi_w}$. The rest of the proof consists in showing that these derivatives are strictly bounded below 1.

Starting with $t=1$, I rewrite $\log\Psi_{w1}$ such that:

$$\log\Psi_{w1}[\log\mathbf{\Gamma}](x) =$$
$$\log\int\frac{\exp\left\{U_1(x,h)+V_1(x,h)+\beta\int\overline{U}_2[\log\mathbf{\Gamma}](s)w(s|x,h)ds+\beta\int\overline{V}_2[\log\mathbf{\Gamma}](s)m(s|x,h)ds\right\}}{\exp\{\beta\int\overline{V}_2[\log\mathbf{\Gamma}](s)m_0(s|h)ds\}+\exp\{\log\Gamma^*_{m1}(h)\}}m_1(h)dh$$

where $\overline{U}_2$ and $\overline{V}_2$ are defined as:

$$\overline{U}_2[\log\mathbf{\Gamma}](x)=\log\left(\exp\left\{\beta\int\overline{U}_3[\log\mathbf{\Gamma}](s)w_0(s|x)ds\right\}+\exp\{\log\Gamma_{w2}(x)\}\right)$$

$$\overline{V}_2[\log\mathbf{\Gamma}](z)=\log\left(\exp\left\{\beta\int\overline{V}_3[\log\mathbf{\Gamma}](s)m_0(s|z)ds\right\}+\exp\{\log\Gamma_{m2}(z)\}\right)$$

The Gateaux derivative of $\log\Psi_{w1}$ with respect to $\log\Gamma_{m1}$ can be bounded in absolute value as:

$$\begin{aligned}
&\Bigg|-\frac{1}{\Psi_{w1}[\mathbf{\Gamma}](x)}\int\frac{\Gamma_{m1}(h)}{\exp\{\beta\int\overline{V}_2[\mathbf{\Gamma}](s)m_0(s|h)ds\}+\Gamma_{m1}(h)}\times\\
&\quad\frac{\exp\left\{U_1(x,h)+V_1(x,h)+\beta\int\overline{U}_2[\mathbf{\Gamma}](s)w(s|x,h)ds+\beta\int\overline{V}_2[\mathbf{\Gamma}](s)m(s|x,h)ds\right\}}{\exp\{\beta\int\overline{V}_2[\mathbf{\Gamma}](s)m_0(s|h)ds\}+\Gamma_{m1}(h)}m_1(h)dh\Bigg|\\
\leq&\quad\frac{\lambda_1}{\Psi_{w1}[\mathbf{\Gamma}](x)}\int\frac{\exp\left\{U_1(x,h)+V_1(x,h)+\beta\int\overline{U}_2(s)w(s|x,h)ds+\beta\int\overline{V}_2(s)m(s|x,h)ds\right\}}{\exp\{\beta\int\overline{V}_2(s)m_0(s|h)ds\}+\Gamma_{m1}(h)}m_1(h)dh\\
=&\quad\lambda_1
\end{aligned}$$

where λ_1 is an upper bound of the ratio

$$\frac{\Gamma^*_{m1}(h)}{\exp\{\beta\int\overline{V}_2[\log\mathbf{\Gamma}](s)m_0(s|h)ds\}+\Gamma^*_{m1}(h)}\leq\frac{\Gamma^U}{\exp\{\beta\overline{U}^U\}+\Gamma^U}=\lambda_1<1$$

A similar bound can be computed for the Gateaux derivative of $\log\Psi_{m1}$ with respect to $\log\Gamma_{w1}$.

I use a similar argument to show that the Gateaux derivative of $\log\Psi_{w1}$ with respect to

$\log \Gamma_{mt}$ for $t > 1$ can be bounded in absolute value by the upper bound of the following expression:

$$\beta \int D_{mt}\overline{V_2}[\log \mathbf{\Gamma}](s)m(s|x,h)ds - \beta \int D_{mt}\overline{V_2}[\log \mathbf{\Gamma}](s)m_0(s|h)ds \frac{\exp\{\beta \int \overline{V}_2(s)m_0(s|h)ds\}}{\exp\{\beta \int \overline{V}_2(s)m_0(s|h)ds\} + \Gamma^*_{m1}(h)}$$

where I define $D_{mt}\overline{V_2}$ as the Gateaux derivative of $\overline{V_2}$ with respect to $\log \Gamma_{mt}$. From there, we can show that for all $1 < t < T$:

$$D_{mt}\overline{V_t}[\log \mathbf{\Gamma}](z) = \frac{\Gamma^*_{mt}(z)}{\exp\{\beta \int \overline{V}_{t+1}(s)m_0(s|z)ds\} + \Gamma^*_{mt}(z)} \leq \lambda_1 \tag{6}$$

From this result, we proceed by induction and show that for all $1 < t < T$:

$$\begin{aligned} D_{mt+1}\overline{V_t}[\log \mathbf{\Gamma}](z) = & \quad \beta \int D_{mt+1}\overline{V}_{t+1}[\log \mathbf{\Gamma}](s)m_0(s|z)ds \frac{\exp\{\beta \int \overline{V}_{t+1}(s)m_0(s|z)ds\}}{\exp\{\beta \int \overline{V}_{t+1}(s)m_0(s|z)ds\} + \Gamma^*_{mt}(z)} \\ \leq & \quad \beta\lambda_1 \frac{\exp\{\beta \overline{U}^U\}}{\exp\{\beta \overline{U}^U\} + \Gamma^U} = \beta\lambda_1\lambda_2 \end{aligned}$$

We can iterate this procedure to show that for all $1 < t \leq t' < T$:

$$D_{mt'}\overline{V_t}[\log \mathbf{\Gamma}](x) \leq \beta^{t'-t}\lambda_1\lambda_2^{t'-t} < 1 \tag{7}$$

For $t' = T$ we can easily verify that:

$$D_{mT}\overline{V_t}[\log \mathbf{\Gamma}](x) \leq \beta^{T-t}\lambda_1^2\lambda_2^{T-t-1} < 1$$

This implies that we can bound from above the first term of the derivative of $\log \Psi_{w1}$ with respect to $\log \Gamma_{mt}$ for all $1 < t < T$ by:

$$\beta^{t-1}\lambda_1\lambda_2^{t-2} < 1$$

while the second term can be bounded by:

$$\beta^{t-1}\lambda_1\lambda_2^{t-1} < 1$$

This implies that the difference between the two is strictly below 1. Similarly for $t = T$, we can bound the first term from above by

$$\beta^{T-1}\lambda_1^2\lambda_2^{T-3} < 1$$

while the second term can be bounded by:

$$\beta^{T-1}\lambda_1^2\lambda_2^{T-2} < 1$$

Again, this holds symetrically for Ψ_{m1}. This finishes to show that the Gateaux derivatives of $\log \Psi_{w1}$ and $\log \Psi_{m1}$ are strictly bounded below 1.

I now consider $\log \Psi_{wt}$ such that $1 < t < T$. I rewrite $\log \Psi_{wt}$ such that:

$$\log \Psi_{wt}[\log \mathbf{\Gamma}](x) =$$
$$\log \int \frac{\exp\left\{U_t(x,h) + V_t(x,h) + \beta \int \overline{U}_{t+1}[\log \mathbf{\Gamma}](s)w(s|x,h)ds + \beta \int \overline{V}_{t+1}[\log \mathbf{\Gamma}](s)m(s|x,h)ds\right\}}{\exp\{\beta \int \overline{V}_{t+1}[\log \mathbf{\Gamma}](s)m_0(s|h)ds\} + \exp\{\log \Gamma_{mt}(h)\}} m_t[\log \mathbf{\Gamma}](h)dh$$

where $\overline{U}_{t+1}$, $\overline{V}_{t+1}$ and m_t are defined as:

$$\overline{U}_{t+1}[\log \mathbf{\Gamma}](x) = \log\left(\exp\left\{\beta \int \overline{U}_{t+2}[\log \mathbf{\Gamma}](s)w_0(s|x)ds\right\} + \exp\{\log \Gamma_{wt+1}(x)\}\right)$$

$$\overline{V}_{t+1}[\log \mathbf{\Gamma}](z) = \log\left(\exp\left\{\beta \int \overline{V}_{t+2}[\log \mathbf{\Gamma}](s)m_0(s|z)ds\right\} + \exp\{\log \Gamma_{mt+1}(z)\}\right)$$

$$m_t[\log \mathbf{\Gamma}](z) = \int_{\mathcal{X}_{t-1}} \int_{\mathcal{Z}_{t-1}} m(z|s,h) f_{t-1}[\log \mathbf{\Gamma}](s,h)dhds + \int_{\mathcal{Z}_{t-1}} m_0(z|s) f_{t-1}[\log \mathbf{\Gamma}](*,h)dh$$

and f_{t-1} can be expressed as follows:

$$f_{t-1}(x,z) = \frac{\exp\left\{U_{t-1}(x,z) + V_{t-1}(x,z) + \beta \int \overline{U}_t(s)w(s|x,z)ds + \beta \int \overline{V}_t(s)m(s|x,z)ds\right\} w_{t-1}[\log \mathbf{\Gamma}](x) m_{t-1}[\log \mathbf{\Gamma}](z)}{\left(\exp\left\{\beta \int \overline{U}_t(s)w_0(s|x)ds\right\} + \exp\{\log \Gamma_{wt-1}(x)\}\right)\left(\exp\left\{\beta \int \overline{V}_t(s)m_0(s|z)ds\right\} + \exp\{\log \Gamma_{mt-1}(z)\}\right)}$$

$$f_{t-1}(*,z) = \frac{\exp\left\{\beta \int \overline{V}_t(s) m_0(s|z) ds\right\}}{\left(\exp\left\{\beta \int \overline{V}_t(s) m_0(s|z) ds\right\} + \exp\{\log \Gamma_{mt-1}(z)\}\right)} m_{t-1}[\log \mathbf{\Gamma}](z)$$

I first consider the derivative of $\log \Psi_{wt}$ with respect to $\log \Gamma_{mt-1}$ and write is as:

$$-\frac{1}{\Psi_{wt}[\mathbf{\Gamma}](x)} \int \frac{\exp\left\{U_t(x,h) + V_t(x,h) + \beta \int \overline{U}_{t+1}[\mathbf{\Gamma}](s) w(s|x,h) ds + \beta \int \overline{V}_{t+1}[\mathbf{\Gamma}](s) m(s|x,h) ds\right\}}{\exp\{\beta \int \overline{V}_{t+1}[\mathbf{\Gamma}](s) m_0(s|h) ds\} + \Gamma_{mt}(h)}$$
$$\times \frac{D_{mt-1} m_t[\mathbf{\Gamma}](h)}{m_t[\mathbf{\Gamma}](h)} m_t[\mathbf{\Gamma}](h) dh$$

where I define $D_{mt-1} m_t$ as the derivative of m_t with respect to $\log \Gamma_{mt-1}$ which can be written as:

$$\begin{aligned}
D_{mt-1} m_t[\mathbf{\Gamma}](z) = & \int_{\mathcal{X}_{t-1}} \int_{\mathcal{Z}_{t-1}} m(z|s,h) f_{t-1}(s,h) \left[- \frac{\Gamma_{mt-1}(h)}{\exp\{\beta \int \overline{V}_t(s) m_0(t|h) dt\} + \Gamma_{mt-1}(h)} \right. \\
& \left. + \frac{D_{mt-1} m_{t-1}[\mathbf{\Gamma}](h)}{m_{t-1}(h)} \right] dh ds \\
& + \int_{\mathcal{Z}_1} m_0(z|h) f_1(*,h) \left[- \frac{\Gamma_{mt-1}(h)}{\exp\{\beta \int \overline{V}_t(s) m_0(t|h) dt\} + \Gamma_{mt-1}(h)} \right. \\
& \left. + \frac{D_{mt-1} m_{t-1}[\mathbf{\Gamma}](h)}{m_{t-1}(h)} \right] dh \\
& \leq m_t(z) \left[-\lambda_1 + \frac{D_{mt-1} m_{t-1}[\mathbf{\Gamma}](h)}{m_{t-1}(h)} \right]
\end{aligned}$$

Similarly, we can iterate once more and write using Equation 6

$$\begin{aligned}
D_{mt-1} m_{t-1}[\mathbf{\Gamma}](z) = & \int_{\mathcal{X}_{t-2}} \int_{\mathcal{Z}_{t-2}} m(z|s,h) f_{t-2}(s,h) \left[\beta \int D_{mt-1} \overline{V}_{t-1}(s) m(t|s,h) dt \right. \\
& - \beta \int D_{mt-1} \overline{V}_{t-1}(s) m_0(t|h) dt \frac{\exp\{\beta \int \overline{V}_{t-1}(s) m_0(t|h) dt\}}{\exp\{\beta \int \overline{V}_{t-1}(s) m_0(t|h) dt\} + \Gamma^*_{mt-2}(h)} \\
& \left. + \frac{D_{mt-1} m_{t-2}[\mathbf{\Gamma}](h)}{m_{t-2}(h)} \right] dh ds \\
& + \int_{\mathcal{Z}_1} m_0(z|h) f_1(*,h) \left[\beta \int D_{mt-1} \overline{V}_{t-1}(s) m_0(t|s,h) dt \right. \\
& - \beta \int D_{mt-1} \overline{V}_{t-1}(s) m_0(t|h) dt \frac{\exp\{\beta \int \overline{V}_{t-1}(s) m_0(t|h) dt\}}{\exp\{\beta \int \overline{V}_{t-1}(s) m_0(t|h) dt\} + \Gamma^*_{mt-2}(h)} \\
& \left. + \frac{D_{mt-1} m_{t-2}[\mathbf{\Gamma}](h)}{m_{t-2}(h)} \right] dh \\
& \leq m_{t-1}(z) \left[\beta\lambda_1 - \beta\lambda_1\lambda_2 + \frac{D_{mt-1} m_{t-2}[\mathbf{\Gamma}](h)}{m_{t-2}(h)} \right]
\end{aligned}$$

Using Equation 7, we then iterate further:

$$
\begin{aligned}
D_{mt-1}m_{t-1}[\mathbf{\Gamma}](z) &\leq m_{t-1}(z)\left[\beta\lambda_1 - \beta\lambda_1\lambda_2 + \beta^2\lambda_1\lambda_2 - \beta^2\lambda_1\lambda_2^2 + \frac{D_{mt-1}m_{t-3}[\mathbf{\Gamma}](h)}{m_{t-3}(h)}\right] \\
&\leq m_{t-1}(z)\left[\beta\lambda_1 - \beta^2\lambda_1\lambda_2^2 + \frac{D_{mt-1}m_{t-3}[\mathbf{\Gamma}](h)}{m_{t-3}(h)}\right]
\end{aligned}
$$

Given that $D_{mt-1}m_1 = 0$ by definition and that $\lambda_1 < 1$,λ_2 and $\beta < 1$, we can thus conclude by induction that:

$$
\frac{D_{mt-1}m_{t-1}[\mathbf{\Gamma}](h)}{m_{t-1}(h)} < 1
$$

which directly implies that:

$$
\frac{D_{mt-1}m_{t}[\mathbf{\Gamma}](h)}{m_{t}(h)} < 1
$$

and that the derivative of $\log \Psi_{wt}$ with respect to $\log \Gamma_{mt-1}$ is strictly bounded from above in absolute value by 1. Similar steps can be used to show the same result for the Gateaux derivative of $\log \Psi_{wt}$ with respect to any $\log \Gamma_{mt'}$ or $\log \Gamma_{wt'}$ with $t \neq t'$. Symmetrical results apply for $\log \Psi_{mt}$.

Overall this implies that:

$$
\sup_{a\in[0,1]} \left|\left| D \log \mathbf{\Psi}_{\boldsymbol{w}}[a \log \mathbf{\Gamma}_{\boldsymbol{m}} + (1-a)\log \tilde{\mathbf{\Gamma}}_{\boldsymbol{m}}](x) \right|\right|_{\infty} < 1
$$

which finishes to prove that there exists a constant $\lambda < 1$ such that:

$$
\left|\left| \log \mathbf{\Psi}_{\boldsymbol{w}}[\mathbf{\Gamma}_{\boldsymbol{m}}] - \log \mathbf{\Psi}_{\boldsymbol{w}}[\tilde{\mathbf{\Gamma}}_{\boldsymbol{m}}] \right|\right|_{\infty} \leq \lambda \left|\left| \log \mathbf{\Gamma}_{\boldsymbol{m}} - \log \tilde{\mathbf{\Gamma}}_{\boldsymbol{m}} \right|\right|_{\infty}
$$

I thus conclude that the mapping $(\log \mathbf{\Gamma}_{\boldsymbol{w}}, \log \mathbf{\Gamma}_{\boldsymbol{m}}) \mapsto (\log \Psi_w[\mathbf{\Gamma}], \log \Psi_m[\mathbf{\Gamma}])$ is a contraction which proves claim (i) of Theorem 1. The proof of part (ii) is a direct implication of the Banach fixed point theorem.

Before proving part (iii) of Theorem 1, intermediary steps are needed. In what follows, I follow Menzel (2015) and first prove that the size of opportunity sets grow at a rate $\sqrt{n}$. From this, I then show that the dependence between opportunity sets and taste shocks under the extremal matchings vanishes as n grows to infinity. I then use this result to show that we can approximate inclusive values arising from any stable match by inclusive value functions which have an approximate fixed point representation. I then finally prove that the solution to the finite sample fixed point problem converges to the unique solution of the population fixed point problem which concludes the proof of Theorem 1.(iii).

D.4.1 Rate of Size of Feasible Choice Sets

Define, for a given stable matching μ_t^*, the number of schools feasible to teacher i and the number of teachers feasible to school j in period t as:

$$J_{wit}^* = \sum_{j=1}^{n_m} \mathbb{1}\{V_{ijt} \geq \max_{l \in W_{jt}(\mu_t^*) \cup \{0\}} V_{ljt}\} \quad \text{and} \quad J_{mjt}^* = \sum_{i=1}^{n_w} \mathbb{1}\{U_{ijt} \geq \max_{k \in M_{it}(\mu_t^*) \cup \{0\}} U_{ikt}\}$$

Similarly, define the number of school that teacher i would accept and the number of teachers that school j would accept:

$$L_{wit}^* = \sum_{j=1}^{n_m} \mathbb{1}\{U_{ijt} \geq \max_{k \in M_{it}(\mu_t^*) \cup \{0\}} U_{ikt}\} \quad \text{and} \quad L_{mjt}^* = \sum_{i=1}^{n_w} \mathbb{1}\{V_{ijt} \geq \max_{l \in W_{jt}(\mu_t^*) \cup \{0\}} V_{ljt}\}$$

I now state the following result:

Lemma 1 *Under Assumptions 1-3 and for any stable matching μ_t^*, we have:*

$$n^{1/2} \frac{\exp(-\bar{V} - \beta \overline{V}^U + \gamma_m)}{1 + \exp(\bar{U} + \bar{V} + \beta \overline{U}^U + \beta \overline{V}^U + \gamma_w)} \leq J_{wi}^* \leq n^{1/2} \exp(\bar{V} + \beta \overline{V}^U + \gamma_m)$$

$$n^{1/2} \frac{\exp(-\bar{U} - \beta \overline{U}^U + \gamma_w)}{1 + \exp(\bar{U} + \bar{V} + \beta \overline{U}^U + \beta \overline{V}^U + \gamma_m)} \leq J_{mj}^* \leq n^{1/2} \exp(\bar{U} + \beta \overline{U}^U + \gamma_w)$$

$$n^{1/2} \frac{\exp(-\bar{U} - \beta \overline{U}^U + \gamma_m)}{1 + \exp(\bar{U} + \bar{V} + \beta \overline{U}^U + \beta \overline{V}^U + \gamma_m)} \leq L_{wi}^* \leq n^{1/2} \exp(\bar{U} + \beta \overline{U}^U + \gamma_m)$$

$$n^{1/2} \frac{\exp(-\bar{V} - \beta \overline{V}^U + \gamma_w)}{1 + \exp(\bar{U} + \bar{V} + \beta \overline{U}^U + \beta \overline{V}^U + \gamma_w)} \leq L_{mj}^* \leq n^{1/2} \exp(\bar{V} + \beta \overline{V}^U + \gamma_w)$$

for each $i = 1, ..., n_w$ and $j = 1, ..., n_m$ with probability approaching 1 as $n \to \infty$.

Proof: As in Menzel (2015), we can define exogenous sets $\overline{W}_{jt} = \{i : U_{ijt} \geq U_{i0t}\}$ and $\overline{M}_{it} = \{j : V_{ijt} \geq V_{0jt}\}$ such that $W_{jt}(\mu_t^*) \subset \overline{W}_{jt}$ and $M_{it}(\mu_t^*) \subset \overline{M}_{it}$ as well as $W_{jt}^\circ =$

$\{i : U_{ijt} \geq \max_{k \in \overline{M}_{it}(\mu_t^*) \cup \{0\}} U_{ikt}\}$ and $M_{it}^{\circ} = \{j : V_{ijt} \geq \max_{l \in \overline{W}_{jt}(\mu_t^*) \cup \{0\}} V_{ljt}\}$ such that $W_{jt}^{\circ} \subset W_{jt}(\mu_t^*)$ and $M_{it}^{\circ} \subset M_{it}(\mu_t^*)$.

From this, I construct the following bounds on J_{wi}^*:

$$J_{wit}^{\circ} = \sum_{j=1}^{n_m} \mathbb{1}\{j \in M_{it}^{\circ}\} \leq \sum_{j=1}^{n_m} \mathbb{1}\{j \in M_{it}(\mu^*)\} \leq \sum_{j=1}^{n_m} \mathbb{1}\{j \in \overline{M}_{it}\} = \overline{J}_{wit}$$

from there, using Proposition 2, we can show that:

$$\begin{aligned} \mathbb{E}[\overline{J}_{wit}|x_{it}, z_{1t}, ..., z_{n_m t}] &= \frac{1}{J} \sum_{j=1}^{n_m} \frac{\exp\{V(x_{it}, z_{jt}) + \beta \int \overline{V}_{jt+1}(s) m(s|x_{it}, z_{jt}) ds\}}{1 + \frac{1}{J} \exp\{V(x_{it}, z_{jt}) + \beta \int \overline{V}_{jt+1}(s) m(s|x_{it}, z_{jt}) ds\}} + o(1) \\ &\leq \frac{n_m}{J} \exp\{\bar{V} + \beta \overline{V}^U\} + o(1) \end{aligned}$$

which implies under Assumption 3 that:

$$\mathbb{E}[\overline{J}_{wit}] \leq n^{1/2} \exp\{\bar{V} + \beta \overline{V}^U + \gamma_m\} + o(1)$$

Following the same steps as Menzel (2015) we can then show that the variance of $\overline{J}_{wit}$ converges to zero which implies that:

$$n^{-1/2}(\overline{J}_{wit} - \mathbb{E}[\overline{J}_{wit}]) \to 0$$

We have thus established that $J_{wit}^* \leq n^{1/2} \exp\{\overline{V} + \beta \overline{V}^U + \gamma_m\}$ with probability approaching 1 as $n \to \infty$. Following the same steps, we can show symmetrically that:

$$J_{mjt}^* \leq n^{1/2} \exp\{\bar{U} + \beta \overline{U}^U + \gamma_w\}$$

$$L_{wit}^* \leq n^{1/2} \exp\{\bar{V} + \beta \overline{V}^U + \gamma_m\}$$

$$L_{mjt}^* \leq n^{1/2} \exp\{\bar{U} + \beta \overline{U}^U + \gamma_w\}$$

with probability approaching 1 as $n \to \infty$. We now consider the lower bound J_{wit}°. We can again use Proposition 2 to show that:

$$\mathbb{E}[J^\circ_{wit}|(x_{lt})_{l\in\overline{W}_{jt}},(z_{kt})_{k=1}^{n_m}] = \frac{1}{J}\sum_{j=1}^{n_m}\frac{\exp\{V_t(x_{it},z_{jt})+\beta\int\overline{V}_{jt+1}(s)m(s|x_{it},z_{jt})ds\}}{1+\frac{1}{J}\sum_{l\in\overline{W}_{jt}}\exp\{V(x_{lt},z_{jt})+\beta\int\overline{V}_{jt+1}(s)m(s|x_{lt},z_{jt})ds\}}+o(1)$$
$$\geq \frac{n_m}{J}\frac{\exp\{-\bar{V}-\beta\overline{V}^U\}}{1+\frac{\overline{J}_{mjt}}{J}\exp\{\bar{V}+\beta\overline{V}^U\}}+o(1)$$

Using the higher bound for J^*_{mj} derived just above and Jensen's inequality, we can finally show that:

$$\mathbb{E}[J^\circ_{wit}] \geq n^{1/2}\frac{\exp\{-\bar{V}-\beta\overline{V}^U+\gamma_m\}}{1+\exp\{\bar{V}+\bar{U}+\beta\overline{V}^U+\beta\overline{U}^U+\gamma_w\}}+o(1)$$

Following Menzel (2015) we can then also show that the variance of J°_{wit} converges to zero which implies that:

$$n^{-1/2}(J^\circ_{wit}-\mathbb{E}[J^\circ_{wit}])\to 0$$

This establishes that $J^*_{wit}\geq n^{1/2}\frac{\exp\{-\bar{V}-\beta\overline{V}^U+\gamma_m\}}{1+\exp\{\bar{V}+\bar{U}+\beta\overline{V}^U+\beta\overline{U}^U+\gamma_w\}}$ with probability approaching 1 as $n\to\infty$. Following the same steps, we can show that symmetrically, we have:

$$J^*_{mjt}\geq n^{1/2}\frac{\exp\{-\bar{U}-\beta\overline{U}^U+\gamma_w\}}{1+\exp\{\bar{V}+\bar{U}+\beta\overline{V}^U+\beta\overline{U}^U+\gamma_m\}}$$

$$L^*_{wit}\geq n^{1/2}\frac{\exp\{-\bar{U}-\beta\overline{U}^U+\gamma_m\}}{1+\exp\{\bar{V}+\bar{U}+\beta\overline{V}^U+\beta\overline{U}^U+\gamma_m\}}$$
$$L^*_{mjt}\geq n^{1/2}\frac{\exp\{-\bar{V}-\beta\overline{V}^U+\gamma_w\}}{1+\exp\{\bar{V}+\bar{U}+\beta\overline{V}^U+\beta\overline{U}^U+\gamma_w\}}$$

with probability approaching 1 as $n\to\infty$. This concludes the proof of Lemma 1.

D.4.2 Exogeneity of Feasible Choice Sets

We now need to show that as $n\to\infty$, the dependence between agents taste shocks and opportunity sets vanishes. As there taste shocks are independent across periods under Assumption 1, the dependence between unobserved preferences and opportunity sets can only arise within period. The proof thus mirrors very closely Menzel (2015) which proves the same result in the static case.

For the rest of the proof, I define the following set of indicator functions $E^*_{ijt} = \mathbb{1}\{i \in W_{jt}(\mu^*_t)\}$ and $D^*_{ijt} = \mathbb{1}\{j \in M_{it}(\mu^*_t)\}$ for all teachers $i = 1, ..., n_w$ and schools $j = 1, ..., n_m$. The first result to establish is that the probability that changing one availability indicator affects another agents' opportunity set converges to zero as $n \to \infty$. I first prove the following result:

Lemma 2 *Suppose Assumption 1-3 hold and suppose we change one availability indicator* E^*_{ijt} *exogenously to* $\tilde{E}_{ijt} = 1 - E^*_{ijt}$ *and then iterate the deferred acceptance algorithm from this point until convergence. Denote the resulting availability indicators* $\{\tilde{E}_{lkt}, \tilde{D}_{lkt} : l = 1, ..., n_w, k = 1, ..., n_m\}$. *We have for any teacher* l *and school* k*:*

(i). $\mathbb{P}(\tilde{D}_l \neq D^*_l | D^*_l, D^*_{ij} = 0) = \mathbb{P}(\tilde{E}_k \neq E^*_k | E^*_l, D^*_{ij} = 0) = 0$

(ii). There exist constants $\overline{a} < \infty$ *and* $0 < \lambda < 1$ *such that:*

$$\mathbb{P}(\tilde{D}_l \neq D^*_l | D^*_l, D^*_{ijt} = 1) \leq n^{-1/2} \frac{\overline{a}}{1 - \lambda}$$

$$\mathbb{P}(\tilde{E}_k \neq E^*_k | E^*_l, E^*_{ijt} = 1) \leq n^{-1/2} \frac{\overline{a}}{1 - \lambda}$$

The same result holds for an exogenous change of D_{ijt} *to* $\tilde{D}_{ijt} = 1 - D_{ijt}$.

Proof: Suppose we change E^*_{jit} exogenously to $\tilde{E}_{jit} = 1 - E_{jit}$ and that we iterate the deferred acceptance algorithm from this stage. This will only trigger a chain of rematches if this affects the indirect utility of either i or j. Suppose $D^*_{ijt} = 0$ and that $E^*_{ijt} = 0$ meaning that school j is not feasible to teacher i and vice versa. Suppose now that $\tilde{E}_{jit} = 1 - E^*_{ijt} = 1$, meaning that suddenly teacher i's preference for school j increase such that teacher i becomes feasible for school j. This will not affect the indirect utility of school j nor teacher i given that school j is not feasible to teacher i. This change will thus not trigger a chain of rematches. A similar argument can be used in the case where E^*_{ijt} changes from 1 to $\tilde{E}_{jit} = 1 - E^*_{ijt} = 0$. This establishes part (i) of Lemma 2.

Now suppose that $D^*_{ijt} = 1$ such that if $\tilde{E}_{ijt} = 1 - E^*_{ijt} = 1$, now school j and teacher i will want to rematch together or if $\tilde{E}_{ijt} = 1 - E^*_{ijt} = 0$ school j and teacher i will break their current match. This will trigger a chain of rematches than can potentially cycle back to teacher i or school j's opportunity set. I start by showing that, at each step s of these subsequent rematches, there is at most one indicator in the vector $D_l^{(s)}$ corresponding to a school k with $E_{lkt}^{(s)} = 1$ that will change. The idea of the proof is the following: suppose that a given teacher l matched to school k in step $(s-1)$ becomes unavailable to school k in step s. This school will then replace this teacher by its most preferred feasible applicant, which will only change the availability indicator of this school to this newly hired teacher. On the other hand, if a given teacher becomes available to a school while this school prefers this teacher to its matched employee, then it will replace them by this new employee, making this school unavailable to the kicked out employee. In both cases, this will only change at most one availability indicator among the teachers who are willing to match with this school. Note that at each of these steps, there is a chance that the chain is terminated if the next preferred feasible option is the outside option. A similar argument can be used symmetrically from the teachers perspective.

The rest of the proof now consists in bounding the probability that the chain is terminated by either (a) school k or teacher l preferring the outside option to any other option in their opportunity set or (b) a change in availability indicators of teacher k D_k. I define μ_t^s the state of the match in iteration s of the deferred acceptance algorithm following an exogenous change of E_{ijt} to $\tilde{E}_{ijt} = 1 - E_{ijt}$. The first step bounds the probability that the chain is terminated by the outside option at stage s.

I start from the following observation: given that $\mathbb{P}(V_{lkt} > V_{k,(q)}(W_k(\mu^s))|x_l, z_k) \geq \mathbb{P}(V_{lk} > V_{k,(1)}(W_k(\mu^s))|x_l, z_k)$ and that $W_{k,(1)}^{\circ} \subset W_k^* \subset \overline{W}_k$, we have from Proposition 2 and Lemma 1 that for any school k and teacher l:

$$
\begin{aligned}
&\mathbb{P}(V_{lkt} > \max_{l \in W_{kt}(\mu_t^s) \cup \{0\}} V_{lkt}|x_{lt}, z_{kt}) \\
&\geq \mathbb{P}(V_{lkt} > \max_{l \in \overline{W}_{kt} \cup \{0\}} V_{lkt}|x_{lt}, z_{kt}) \\
&= n^{-1/2} \frac{\exp(V(z_{kt}, x_{lt}) + \beta \int \overline{V}_{kt+1}(s) m(s|x_{lt}, z_{kt}) ds)}{1 + \frac{1}{J} \sum_{i \in \overline{W}_{kt}} \exp(V(z_{kt}, x_{it}) + \beta \int \overline{V}_{kt+1}(s) m(s|x_{it}, z_{kt}) ds)} + o(1) \\
&\geq n^{-1/2} \frac{\exp(V(z_{kt}, x_{lt}) + \beta \int \overline{V}_{kt+1}(s) m(s|x_{lt}, z_{kt}) ds)}{1 + \exp(\bar{U} + \bar{V} + \beta \overline{V}^U + \beta \overline{U}^U + \gamma_w)} + o(1)
\end{aligned}
$$

This implies that, conditional on D_i^* and as n approaches infinity:

$$\mathbb{P}(V_{0kt} > \max_{l\in W_{kt}(\mu_t^s)\cup\{0\}} V_{lkt}|D_i^*, x_{it}, z_{kt}) \geq \frac{1}{1+\exp(\bar{U}+\bar{V}+\beta\overline{V}^U+\beta\overline{U}^U+\gamma_w)} =: p_s$$

Following now the same steps as Menzel (2015), we have, by Bayes law that:

$$\mathbb{P}(V_{0kt} > \max_{l\in W_{kt}(\mu_t^s)\cup\{0\}} V_{ljt}|D_l^*, \tilde{D}_{lkt}^{(s)}=1, x_{lt}, z_{kt}) \geq \frac{\underline{L}p_s}{\overline{L}(1-p_s)+\underline{L}p_s}$$

where $\overline{L}$ and $\underline{L}$ are respectively the upper and lower bounds on L_{mj}^* taken from Lemma 1. From there, we finally get that:

$$1-\mathbb{P}(V_{0kt} > \max_{l\in W_{kt}(\mu_t^s)\cup\{0\}} V_{ljt}|D_l^*, \tilde{D}_{lkt}^{(s)}=1, x_{lt}, z_{kt}) \leq \frac{\overline{L}\exp(\bar{U}+\bar{V}+\beta\overline{V}^U+\beta\overline{U}^U+\gamma_w)}{\overline{L}\exp(\bar{U}+\bar{V}+\beta\overline{V}^U+\beta\overline{U}^U+\gamma_w)+\underline{L}} =: \lambda < 1$$

This essentially means that the probability that the chain is not terminated at stage s is bounded away from 1.

Now we bound the probability that the chain leads to a change in D_l at stage s. We can thus bound the following probability using Proposition 2 and Lemma 1:

$$\begin{aligned}
&\mathbb{P}(V_{lkt} > \max_{l\in W_{kt}(\mu_t^s)\cup\{0\}} V_{lkt})|x_{lt}, z_{kt}) \\
&\leq \mathbb{P}(V_{lkt} > \max_{l\in W_{kt}^{\circ}\cup\{0\}} V_{lkt}|x_{lt}, z_{kt}) \\
&= n^{-1/2}\frac{\exp(V(z_{kt},x_{lt})+\beta\int\overline{V}_{kt+1}(s)m(s|x_{lt},z_{kt})ds)}{1+\frac{1}{J}\sum_{i\in W_{kt}^{\circ}}\exp(V(z_{kt},x_{it})+\beta\int\overline{V}_{kt+1}(s)m(s|x_{it},z_{kt})ds)} + o(1) \\
&\leq n^{-1/2}\exp(\bar{V}+\beta\overline{V}^U) + o(1)
\end{aligned}$$

This implies that for n sufficiently large, we have:

$$\begin{aligned}
&\mathbb{P}(\tilde{D}_l^{(s)} \neq D_l^*|D_l^*, \tilde{D}_{lkt}^{(s)}=1, x_l, z_k) \\
&\leq \frac{n^{-1/2}\exp(\bar{V}+\beta\overline{V}^U)\overline{L}}{n^{-1/2}\exp(\bar{V}+\beta\overline{V}^U)\overline{L}+\underline{L}} \leq n^{-1/2}\exp(\bar{V}+\beta\overline{V}^U)\frac{\overline{L}}{\underline{L}} = n^{-1/2}\overline{a}
\end{aligned}$$

Using the law of total probability, we can thus bound as $n\to\infty$ the conditional probability that $\tilde{D}_l \neq D_l^*$

$$\mathbb{P}(\tilde{D}_l \neq D_l^* | D_l^*) \leq \sum_{s=1}^{\infty} \lambda^s n^{-1/2} \overline{a} \leq \frac{n^{-1/2}\overline{a}}{1-\lambda}$$

which proves part (b) of Lemma 2.

From there, I state the main result that the dependence between taste shocks and agents' opportunity sets vanishes as $n \to \infty$. I first define the joint distribution of $\eta_i = (\eta_{i1}, ..., \eta_{in_m})'$, $\epsilon_j = (\epsilon_{1j}, ..., \epsilon_{n_w j})'$ and the availability indicators D_i^W, E_j^W, D_i^M, E_j^M corresponding to the teacher-optimal and the school-optimal stable matches. Note that I consider these two specific matches since the teacher-optimal and school-optimal stable matches are defined with probability 1 conditional on the realization of the taste shocks η_i and ϵ_j. Indeed, the distribution of availability indicators arising from an arbitrary stable match D_i^* would not be well defined. I also define: $D_{i,-j}^W = (D_{i1}^W, ..., D_{i(j-1)}^W, D_{i(j+1)}^W, ..., D_{in_m}^W)$ and $E_{-i,j} = (E_{1j}^W, ..., E_{(i-1)j}^W, E_{(i+1)j}^W, ..., E_{n_w j}^W)$ with analogous notations for the school optimal match. I then define the conditional c.d.f.s:

$$G_{\eta|D}^W(\eta|\boldsymbol{d}) = \mathbb{P}(\eta_i \leq \eta | D_i^W = \boldsymbol{d}), \quad \boldsymbol{d} \in \{0,1\}^{n_m}$$

$$G_{\eta,\epsilon|D,E}^W(\eta, \epsilon|\boldsymbol{d}, \boldsymbol{e}) = \mathbb{P}(\eta_i \leq \eta, \epsilon_j \leq \epsilon | D_{i,-j}^W = \boldsymbol{d}, E_{-i,j}^W = \boldsymbol{e}), \quad \boldsymbol{d} \in \{0,1\}^{n_m - 1}, \boldsymbol{e} \in \{0,1\}^{n_w - 1}$$

with analogous definitions for the school-optimal stable match and associated p.d.f.s $g_{\eta|D}^W$ and $g_{\eta,\epsilon|D,E}^W$. The main result is the following:

Lemma 3 *Under Assumption 1 and 2, we have:*

(i). $g_{\eta|D}^W$ *and* $g_{\eta|D}^M$ *satisfy:*

$$\lim_n \left| \frac{g_{\eta|D}^W(\eta|D_i^W)}{g_\eta(\eta)} - 1 \right| = \lim_n \left| \frac{g_{\eta|D}^M(\eta|D_i^M)}{g_\eta(\eta)} - 1 \right| = 1$$

(ii). $g_{\eta,\epsilon|D,E}^W$ *and* $g_{\eta,\epsilon|D,E}^M$ *satisfy:*

$$\lim_n \left| \frac{g_{\eta|D}^W(\eta, \epsilon|D_{i,-j}^W, E_{-i,j}^W)}{g_{\eta,\epsilon}(\eta, \epsilon)} - 1 \right| = \lim_n \left| \frac{g_{\eta|D}^M(\eta, \epsilon|D_{i,-j}^M, E_{-i,j}^M)}{g_{\eta,\epsilon}(\eta, \epsilon)} - 1 \right| = 1$$

The same results holds for the school side of the market.

PROOF: Let $g_{\eta,D}^{W}$ be the joint p.d.f. of taste shocks and availability indicators under the teacher optimal stable match. We can rewrite, by definition of a conditional density:

$$\frac{g_{\eta|D}^{W}(\eta|D_i^W)}{g_\eta(\eta)} = \frac{g_{\eta,D}^{W}(\eta, D_i^W)}{g_\eta(\eta)P(D_i^W)} = \frac{P(D_i^W|\eta_i = \eta)g_\eta(\eta)}{g_\eta(\eta)P(D_i^W)} = \frac{P(D_i^W|\eta_i = \eta)}{P(D_i^W)}$$

I then follow similar steps as in Menzel (2015) to show that:

$$\Big|\frac{P(D_i^W|\eta_i = \eta)}{P(D_i^W)} - 1\Big| \leq \sup_{\eta_1,\eta_2}\Big|\frac{P(D_i^W|\eta_i = \eta_1)}{P(D_i^W|\eta_i = \eta_2)} - 1\Big|$$

such that I only need to bound the probability that shifting η_i from η_1 to η_2 changes teacher i's opportunity set. We know from Lemma 2 that changing an availability indicator will trigger a chain of rematches that could change teacher i's opportunity set with probability less than $\frac{n^{-1/2}\overline{a}}{1-\lambda}$ as n approaches infinity. Here, we can show that shifting agent i's taste shocks would trigger at most two chains of rematches. Indeed, if the shift in taste shocks makes agent i prefers school l with $D_{il} = 1$ instead of her current employer school j, this changes both E_{ij} from 1 to 0 and E_{il} from 0 to 1. Thus, this would trigger two chains of rematches where both school j and the teacher which was displaced from school l by teacher i would need to find a new match. We can thus conclude that:

$$\frac{P(D_i^W|\eta_i = \eta_1)}{P(D_i^W|\eta_i = \eta_2)} - 1 \leq 2\frac{n^{-1/2}\overline{a}}{1-\lambda}$$

as $n \to \infty$ which can be shown to hold also in absolute value. As the right hand side converges to 0 as $n \to \infty$, this proves the first part of claim (i). The same result holds symetrically for the school side.

For part (ii), note that the argument can be extended in a similar way. If you change both school j and teacher i's taste shocks this can trigger at most 4 chains of rematches such that we can bound the probability of a shift in opportunity sets by $n^{-1/2}\frac{4\overline{a}}{1-\lambda}$ which can be made arbitrarily close to 0 as n approaches infinity.

D.4.3 Bounds for Inclusive Values

Since I have established exogeneity of opportunity sets under the school-optimal and teacher-optimal stable matches, the rest of the analysis focuses on characterizing the limit of inclusive values that arise under these extremal matchings. As in Menzel (2015), I show that both converge to a unique limit, implying that inclusive values arising from any stable matching also converge toward this limit.

I define $I_{wit}^{W} = I_{wit}(\mu_t^W)$ and $I_{mjt}^{W} = I_{mjt}(\mu_t^W)$ the inclusive values that arise from the sequence of teacher-optimal stable matches $\boldsymbol{\mu}^W$ in period t. Similarly, I define I_{wit}^M and I_{mjt}^M as the inclusive values that arise from the sequence of school-optimal stable matches $\boldsymbol{\mu}^M$ such that for any stable match μ_t^*, we have $I_{wit}^W \geq I_{wit}(\mu_t^*) \geq I_{wit}^M$ and $I_{mjt}^W \leq I_{wit}(\mu_t^*) \leq I_{mjt}^M$ for all t. I state the following result:

Lemma 4 *Under Assumption 1-3:*
(i). For all $i = 1, ..., n_w$ and $j = 1, ..., n_m$:

$$I_{wit}^M \geq \hat{\Gamma}_{wt}^M(x_{it}) + o_p(1) \quad \textit{and} \quad I_{mjt}^M \leq \hat{\Gamma}_{mt}^M(z_{jt}) + o_p(1)$$

where the analogous result holds for the teacher-optimal stable match with the side of inequalities reversed.
(ii). If the weight functions $\omega(x, z) \geq 0$ are bounded and form a Glivenko-Cantelli class in x, then

$$\sup_{x \in \mathcal{X}} \frac{1}{n} \sum_{j=1}^{n_m} \omega(x, z_{jt})(I_{mjt}^M - \hat{\Gamma}_{mt}^M(z_{jt})) \leq o_p(1)$$

and

$$\inf_{z \in \mathcal{Z}} \frac{1}{n} \sum_{i=1}^{n_w} \omega(x_{it}, z)(I_{wit}^M - \hat{\Gamma}_{wt}^M(x_{it})) \geq o_p(1)$$

The analogous conclusion holds for the teacher-optimal stable match where the sign of the inequalities is reversed and if $\omega(x, z) \geq 0$ are bounded and form a Glivenko-Cantelli class in z.

Proof: I first show that we can bound conditional choice probabilities given an opportunity set arising from a stable match using the extremal matchings. I first define the conditional probability that teacher i chooses school j given the realization of opportunity set M^M arising from the school-optimal stable match:

$$\Lambda^M_{wt}(x, z, M^M) = \mathbb{P}(U_{ijt} \geq \max_{k \in M^M_{it} \cup \{0\}} U_{ikt} | (M^M_{i\tau})^T_{\tau=t} = M^M, x_{it} = x, z_{jt} = z)$$

and the expectations about future match payoffs given future opportunity sets as:

$$\overline{U}^M_{t+1}(x, M^M) = \mathbb{E}\left[\max_{k \in M^M_{it+1} \cup \{0\}} U_{ikt+1} | (M^M_{i\tau})^T_{\tau=t+1} = M^M, x_{it+1} = x\right]$$

I also define the conditional choice probabilities and expectations about future payoffs in period t under exogenous opportunity sets as:

$$\Lambda_{wt}(x, z, M) = \mathbb{P}(U_{ijt} \geq \max_{k \in M \cup \{0\}} U_{ikt} | x_{it} = x, z_{jt} = z)$$

$$\overline{U}_{t+1}(x, M) = \mathbb{E}\left[\max_{k \in M \cup \{0\}} U_{ikt+1} | x_{it+1} = x\right]$$

As there are several stable matches such that $M^*_i = M^M_i$ and $W^*_j = W^M_j$ we can show that:

$$J\Lambda^M_{wt}(x, z, (M^M_{i\tau})^T_{\tau=t}) \leq J\Lambda_{wt}(x, z, (M^M_{i\tau})^T_{\tau=t}) + o_p(1)$$

$$\overline{U}^M_{t+1}(x, (M^M_{i\tau})^T_{\tau=t+1}) \geq \overline{U}_{t+1}(x, (M^M_{i\tau})^T_{\tau=t+1}) + o_p(1)$$

Using Proposition 2, we can then show that for $i = 1, ..., n_w$, $l_1 = 1, ..., n_m$ and $l_2 \neq l_1$:

$$\mathbb{E}\Bigg[J\Bigg(D^M_{il_1t} \exp\left\{\beta \int \overline{U}^M_{it+1}(s) w(s|x_{it}, z_{l_1t}) ds\right\}$$
$$- \Lambda^M_{mt}(x_{it}, z_{l_1t}, (I^M_{ml_1\tau})^T_{\tau=t}) \exp\left\{\beta \int \overline{U}^M_{t+1}(s, (M^M_{i\tau})^T_{\tau=t+1}) w(s|x_{it}, z_{l_1t}) ds\right\}\Bigg)\Bigg|(I^M_{ml_1\tau})^T_{\tau=t}, x_{it}, z_{l_1t}\Bigg] \to 0$$

and

$$\mathbb{E}\Bigg[J^2\Bigg(D^M_{il_1t} \exp\left\{\beta \int \overline{U}^M_{it+1}(s) w(s|x_{it}, z_{jt}) ds\right\}$$
$$- \Lambda^M_{mt}(x_{it}, z_{l_1t}, (I^M_{ml_1\tau})^T_{\tau=t}) \exp\left\{\beta \int \overline{U}^M_{t+1}(s, (I^M_{wi\tau})^T_{\tau=t+1}) w(s|x_{it}, z_{l_1t}) ds\right\}\Bigg)$$

$$
\begin{aligned}
&\times D^M_{il_2t}\exp\left\{\beta\int \overline{U}^M_{it+1}(s)w(s|x_{it},z_{l_1t})ds\right\}\\
&-\Lambda^M_{mt}(x_{it},z_{l_2t},(I^M_{ml_2\tau})^T_{\tau=t})\exp\left\{\beta\int \overline{U}^M_{t+1}(s,(I^M_{wi\tau})^T_{\tau=t+1})w(s|x_{it},z_{l_2t})ds\right\}\\
&\quad\left|(I^M_{ml_1\tau})^T_{\tau=t},(I^M_{ml_2\tau})^T_{\tau=t},(I^M_{wi\tau})^T_{\tau=t+1},x_{it},z_{l_1t},z_{l_2t}\right]\to 0
\end{aligned}
$$

Therefore, since under Assumption 1, we know that $\exp(U_t(x_{it},z_{jt}))$ is bounded, we can thus conclude that:

$$
\operatorname{Var}\left(\frac{1}{n}\sum_{k=1}^{n_m}\exp\left\{U_t(x_{it},z_{kt})+\beta\int \overline{U}^M_{t+1}(s,(I^M_{wi\tau})^T_{\tau=t+1})w(s|x_{it},z_{kt})ds\right\}J(D^M_{ik}-\Lambda^M_{mt}(x_{it},z_{kt},(I^M_{mk\tau})^T_{\tau=t}))\right)\to 0
$$

which implies that:

$$
\frac{1}{n}\sum_{k=1}^{n_m}\exp\left\{U_t(x_{it},z_{kt})+\beta\int \overline{U}^M_{t+1}(s,(I^M_{wi\tau})^T_{\tau=t+1})w(s|x_{it},z_{kt})ds\right\}J(D^M_{ikt}-\Lambda^M_{mt}(x_{it},z_{kt},(I^M_{mk\tau})^T_{\tau=t}))=o_p(1)
$$

Given that from Proposition 2:

$$
J\Lambda^M_{mt}(x,z,(W^M_{j\tau})^T_{\tau=t})\geq\frac{\exp\left\{V_t(x,z)+\beta_m\int\overline{V}^M_{t+1}(s)m(s|\boldsymbol{x},\boldsymbol{z})ds\right\}}{\exp\left\{\beta_m\int\overline{V}^M_{t+1}(s)m_0(s|\boldsymbol{x})ds\right\}+I^M_{mjt}}+o_p(1)
$$

This implies that:

$$
\frac{1}{n}\sum_{k=1}^{n_m}\exp\{U_t(x_{it},z_{kt})\}\left(JD^M_{ikt}-\frac{\exp\left\{V_t(x_{it},z_{kt})+\beta_m\int\overline{V}^M_{t+1}(s)m(s|x_{it},z_{kt})ds\right\}}{\exp\left\{\beta_m\int\overline{V}^M_{t+1}(s)m_0(s|z_{kt})ds\right\}+I^M_{mjt}}\right)\geq o_p(1)
$$

which proves the first claim of part (i) of Lemma 4. Similar steps can be used to bound inclusive values on the school side and for the teacher optimal sequence of stable matches.

Part (ii) follows from part (i) of the Lemma and the boundedness condition on ω which implies pointwise convergence. The Glivenko-Cantelli condition on ω then implies uniform convergence. This concludes the proof of Lemma 4.

The next step consists in establishing uniform convergence with respect to $\Gamma_{wt} \in \mathcal{T}_{wt}$ and $\Gamma_{mt} \in \mathcal{T}_{mt}$ of the fixed point mappings $\hat{\Psi}_{wt}$ and $\hat{\Psi}_{mt}$ to their population counterparts. I define:

$$\hat{\Psi}_{wt}[\mathbf{\Gamma}](x) = \frac{1}{n}\sum_{j=1}^{n_m} \psi_{wt}(z_{jt}, x; \mathbf{\Gamma})$$

where ψ_{wt} is defined as:

$$\psi_{wt}(z_{jt}, x; \mathbf{\Gamma}) = \frac{\exp\left\{U_t(x, z_{jt}) + V_t(x, z_{jt}) + \beta_w \int \overline{U}_{t+1}(s)w(s|x, z_{jt})ds + \beta_m \int \overline{V}_{t+1}(s)m(s|x, z_{jt})ds\right\}}{\exp\left\{\beta_m \int \overline{V}_{t+1}(s)m_0(s|z_{jt})ds\right\} + \Gamma_{mt}(z_{jt})}$$

Similarly, I define:

$$\hat{\Psi}_{mt}[\mathbf{\Gamma}](z) = \frac{1}{n}\sum_{i=1}^{n_w} \psi_{mt}(z, x_{it}; \mathbf{\Gamma})$$

where ψ_{mt} is defined as:

$$\psi_{mt}(z, x_{it}; \mathbf{\Gamma}) = \frac{\exp\left\{U_t(x_{it}, z) + V_t(x_{it}, z) + \beta_w \int \overline{U}_{t+1}(s)w(s|x_{it}, z)ds + \beta_m \int \overline{V}_{t+1}(s)m(s|x_{it}, z)ds\right\}}{\exp\left\{\beta_w \int \overline{U}_{t+1}(s)w_0(s|x_{it})ds\right\} + \Gamma_{wt}(x_{it})}$$

I define the class of functions $\mathcal{F}_w : \{\psi_w(., x; \mathbf{\Gamma}) : x \in \mathcal{X}, \mathbf{\Gamma} \in \mathcal{T}\}$ and $\mathcal{F}_m : \{\psi_m(z, ; \mathbf{\Gamma}) : z \in \mathcal{Z}, \mathbf{\Gamma} \in \mathcal{T}\}$.

Lemma 5 *Under Assumption 1:*
(i). The classes of functions $\mathcal{F}_w$ and $\mathcal{F}_w$ are Glivenko-Cantelli.
(ii). As $n \to \infty$ and for all t:

$$(\hat{\Psi}_{wt}[\mathbf{\Gamma}](x), \hat{\Psi}_{mt}[\mathbf{\Gamma}](z)) \to (\Psi_{wt}[\mathbf{\Gamma}](x), \Psi_{mt}[\mathbf{\Gamma}](z))$$

uniformly in $\mathbf{\Gamma} \in \mathcal{T}$, and $(x, z) \in \mathcal{X} \times \mathcal{Z}$.

Proof: Under Assumption 1, $\exp\{U(x, z) + V(x, z)\}$ is Lipschitz in x and z such that this class of functions is Glivenko-Cantelli. Γ_{mt} and Γ_{wt} are bounded and have bounded $p \geq 1$ derivatives for all t which makes the class of functions $\mathcal{T}$ Glivenko-Cantelli. Finally, as m and

w are continuous densities and that $\overline{U}$ and $\overline{V}$ are continuous note that the transformation $\psi_m(g,h) = \frac{g}{1+h}$ is bounded and continuous since h and g are bounded and continuous and $h \geq 0$. Theorem 3 in van der Vaart and Wellner (2000) implies claim (i) of Lemma 5. Part (ii) of Lemma 5 is a direct implication of part (i).

D.4.4 Proof of Theorem 3.1 (iii)

I finally turn to the proof of part (iii) of Theorem 1. I first apply Lemma 4 to show that for any $q \geq 1$:

$$\begin{aligned}\hat{\Gamma}^M_{wt}(x_{it}) &= \frac{1}{n}\sum_{k=1}^{n_m} \frac{\exp\left\{U_t(\boldsymbol{x}_{it},\boldsymbol{z}_{kt}) + V_t(\boldsymbol{x}_{it},\boldsymbol{z}_{kt}) + \beta\int \overline{U}^M_{it+1}(s)w(s|\boldsymbol{x}_{it},\boldsymbol{z}_{kt})ds + \beta\int \overline{V}^M_{kt+1}(s)m(s|\boldsymbol{x}_{it},\boldsymbol{z}_{kt})ds\right\}}{\exp\left\{\beta\int \overline{V}^M_{kt+1}(s)m_0(s|\boldsymbol{z}_{kt})ds\right\} + I^M_{mkt}} \\
&\geq \frac{1}{n}\sum_{k=1}^{n_m} \frac{\exp\left\{U_t(\boldsymbol{x}_{it},\boldsymbol{z}_{kt}) + V_t(\boldsymbol{x}_{it},\boldsymbol{z}_{kt}) + \beta\int \overline{U}^M_{t+1}(s)w(s|\boldsymbol{x}_{it},\boldsymbol{z}_{kt})ds + \beta\int \overline{V}^M_{t+1}(s)m(s|\boldsymbol{x}_{it},\boldsymbol{z}_{kt})ds\right\}}{\exp\left\{\beta\int \overline{V}^M_{t+1}(s)m_0(s|\boldsymbol{z}_{kt})ds\right\} + \hat{\Gamma}^M_{mt}(z_{kt})} + o_p(1)\end{aligned}$$

Analogous bounds can be formed for the inclusive value functions of the teacher-optimal stable match. We thus have that:

$$\hat{\Gamma}^M_{wt} \geq \hat{\Psi}^M_{wt}[\hat{\boldsymbol{\Gamma}}^M] + o_p(1) \quad \text{and} \quad \hat{\Gamma}^M_{mt} \leq \hat{\Psi}^M_{mt}[\hat{\boldsymbol{\Gamma}}^M] + o_p(1)$$

$$\hat{\Gamma}^W_{wt} \leq \hat{\Psi}^W_{wt}[\hat{\boldsymbol{\Gamma}}^W] + o_p(1) \quad \text{and} \quad \hat{\Gamma}^W_{mt} \geq \hat{\Psi}^W_{mt}[\hat{\boldsymbol{\Gamma}}^W] + o_p(1)$$

Given that $\hat{\Psi}_{wt}[\boldsymbol{\Gamma}]$ and $\hat{\Psi}_{mt}[\boldsymbol{\Gamma}]$ are nonincreasing and Lipschitz continuous in $\boldsymbol{\Gamma}$, we have:

$$\hat{\Gamma}^M_{wt} \geq \hat{\Psi}^M_{wt}[\hat{\boldsymbol{\Gamma}}^M] + o_p(1) \geq \hat{\Psi}^M_{wt}[\hat{\boldsymbol{\Psi}}^M[\hat{\boldsymbol{\Gamma}}^M]] + o_p(1)$$

Thus for any $\boldsymbol{\Gamma}^*$ solving the fixed point problem:

$$\Gamma^*_{wt} = \hat{\Psi}_{wt}[\boldsymbol{\Gamma}^*] + o_p(1) \quad \text{and} \quad \Gamma^*_{mt} = \hat{\Psi}_{mt}[\boldsymbol{\Gamma}^*] + o_p(1)$$

we thus have:

$$\hat{\Gamma}^M_{wt} \geq \Gamma^*_{wt} + o_p(1) \quad \text{and} \quad \hat{\Gamma}^M_{mt} \leq \Gamma^*_{mt} + o_p(1)$$

However, we know that the mapping $\hat{\Psi}$ is a contraction in logs, which means that it has a unique fixed point $\mathbf{\Gamma}^*$. In addition, the school-optimal stable match is unanimously preferred by schools while the teacher-optimal stable match is unanimously preferred by teachers (Roth and Sotomayor (1992)). This implies that $M_{it}(\mu^M) \subset M_{it}(\mu^*) \subset M_{it}(\mu^W)$ and $W_{it}(\mu^W) \subset W_{it}(\mu^*) \subset W_{it}(\mu^M)$ which means that for all i and j:

$$I^M_{wit} \leq I^*_{wit} \leq I^W_{wit} \quad \text{and} \quad I^W_{mjt} \leq I^*_{mjt} \leq I^M_{mjt}$$

This in turn implies that for all (x, z):

$$\hat{\Gamma}^M_{wt}(x) \leq \hat{\Gamma}^*_{wt}(x) \leq \hat{\Gamma}^W_{wt}(x) \quad \text{and} \quad \hat{\Gamma}^W_{mt}(z) \leq \hat{\Gamma}^*_{mt}(z) \leq \hat{\Gamma}^M_{mt}(z)$$

which implies that:

$$\Gamma^*_{wt} + o_p(1) \geq \hat{\Gamma}^W_{wt} \geq \hat{\Gamma}^M_{wt} \geq \Gamma^*_{wt} + o_p(1)$$

$$\Gamma^*_{mt} + o_p(1) \leq \hat{\Gamma}^W_{mt} \leq \hat{\Gamma}^M_{mt} \leq \Gamma^*_{mt} + o_p(1)$$

which in turn implies that:

$$\hat{\Gamma}^M_{wt} = \Gamma^*_{wt} + o_p(1) \quad \text{and} \quad \hat{\Gamma}^M_{mt} = \Gamma^*_{mt} + o_p(1)$$

$$\hat{\Gamma}^W_{wt} = \Gamma^*_{wt} + o_p(1) \quad \text{and} \quad \hat{\Gamma}^W_{mt} = \Gamma^*_{mt} + o_p(1)$$

Combining this with Lemma 3, this gives us for all $i = 1, ..., n_w$ and all $j = 1, ..., n_m$:

$$I^M_{wit} = \Gamma^*_{wt} + o_p(1) \quad \text{and} \quad I^M_{mjt} = \Gamma^*_{mt} + o_p(1)$$

$$I^W_{wit} = \Gamma^*_{wt} + o_p(1) \quad \text{and} \quad I^M_{mjt} = \Gamma^*_{mt} + o_p(1)$$

Note that given that inclusive value functions that would arise under any stable match μ^*_t defined as I^*_{wit} and I^*_{mjt} are such that $I^M_{wit} \leq I^*_{wit} \leq I^W_{wit}$ and $I^M_{mjt} \geq I^*_{mjt} \geq I^W_{mjt}$ the equality written above holds also for any I^*_{wit} and I^*_{mjt}.

I have shown that inclusive values can be approximated by the solution of the finite sample fixed point problem. Lemma 5 finally implies that the solution of the finite sample fixed point problem converges toward the solution of its population equivalent. This proves Theorem 1.(iii).

E Monte Carlo Simulations

To gain confidence in the validity of the theoretical results described in Section 4 and 5, I perform two Monte Carlo exercises. First, I simulate data from a market with different numbers of participating agents to verify whether empirical matching frequencies converge to their theoretical limit. Then, I then evaluate the performance of the Maximum Likelihood Estimator proposed in 5. I consider a market with three periods $T = 3$, normalize $\gamma_w = \gamma_m = 0$, and set $\beta_w = \beta_m = 0.9$. I then specify the flow payoffs as $U(x, z; \boldsymbol{\theta}) = \theta_1 + \theta_2 z$ and $V(x, z; \boldsymbol{\theta}) = \theta_1 + \theta_3 x$ for all t and set $\boldsymbol{\theta} = (1, 1, 1)$. I assume that $x_{i1} \sim \mathcal{N}(0, 1)$ and $z_{j1} \sim \mathcal{N}(0, 1)$. I assume the following laws of motion for x and z:

$$x_{it+1} = \begin{cases} x_{i1} + 1 & \text{if} \quad \mu_{wt}(i) \neq 0 \\ x_{i1} & \text{if} \quad \mu_{wt}(i) = 0 \end{cases} \quad , \quad z_{jt+1} = \begin{cases} z_{j1} + 1 & \text{if} \quad \mu_{mt}(j) \neq 0 \\ z_{j1} & \text{if} \quad \mu_{mt}(j) = 0 \end{cases}$$

This simulates a setting where teachers and schools become less attractive when they stay unmatched.

E.1 Convergence of Matching Frequencies

In this Monte Carlo exercise, I simulate data from the DGP described above for different market sizes indexed by n. In order to simulate the equilibrium, I first solve the fixed point problem described in Equation 4 to recover Γ^*_{wt} and Γ^*_{mt} and solve recursively for $\overline{U}_{t+1}$ and $\overline{V}_{t+1}$ for $t = \{1, 2\}$. I then draw a set of taste shocks ϵ_{ijt} and η_{ijt} for each period and each teacher-school pair and construct the lifetime utilities U_{ijt} and V_{ijt}. I then use the Deferred Acceptance algorithm to recover the teacher-optimal stable match in each period. The goal of this exercise is to evaluate whether the observed matching frequencies converge to their limit. More specifically I will look at whether the share of unmatched teachers in each period converges to its limit. Table E.1 shows the results of this exercise. We can clearly see that as the size of the market increases, the share of unmatched teachers observed in the simulated data converges to its limit, which is displayed in the bottom line. This shows that, even with moderate sample sizes, the limit economy seems to be a relatively good approximation for the finite economy.

E.2 Estimation

In this second experiment, I simulate data by following the same procedure for different values of n. I then estimate $\boldsymbol{\theta}$ using the procedure described in Section 5. Table E.2 shows that the estimator is unbiased even with small sample sizes. It is also consistent given that the standard deviation of the estimator decreases as the sample sizes increases.

Table E.1: Monte Carlo: Share of Unmatched teachers

n	$t = 1$	$t = 2$	$t = 3$
20	0.2600	0.1870	0.2675
50	0.2439	0.1734	0.2447
100	0.2389	0.1640	0.2339
200	0.2314	0.1565	0.2237
500	0.2263	0.1509	0.2147
1000	0.2228	0.1469	0.2095
2000	0.2206	0.1432	0.2053
Model	0.2076	0.1384	0.1965

NOTES. This table reports the average share of unmatched schools and teachers in each period taken over 200 sample draws for different sample sizes n.

Table E.2: Monte Carlo: MLE

n	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{\theta}_3$
20	0.952	0.986	1.023
	(0.475)	(0.352)	(0.334)
50	0.962	0.988	1.010
	(0.292)	(0.223)	(0.204)
100	0.969	0.994	1.007
	(0.192)	(0.156)	(0.140)
200	0.977	0.991	1.003
	(0.133)	(0.104)	(0.105)
500	0.984	0.994	1.003
	(0.088)	(0.067)	(0.063)
1000	0.992	0.995	1.002
	(0.060)	(0.047)	(0.046)
True value	1	1	1

NOTES. This table reports the average and standard deviation of the ML estimator of $\boldsymbol{\theta}$ over 500 sample draws for different sample sizes n.

F Alternative Model: Irreversible Matches

In this section, I present an alternative to the model discussed in Section 4. I consider a setting where matches are irreversible and the match is stable in each period given agents' continuation value of staying unmatched, as in Doval (2022), and show that all the results derived in this paper extend.

F.1 Model

In this model, in each period t, agents can either decide to form a match with an agent from the other side or decide to stay unmatched and wait to get better opportunities in period $t+1$. The timing works as follows:

Period 1: The set of teachers $\mathcal{I}_1$ and schools $\mathcal{J}_1$ arrive in the market. A matching μ_1 occurs and all teachers $i \in \mathcal{I}_1$ that stay unmatched such that $\mu_{w1}(i) = 0$ move on to the second period. Similarly, all schools $j \in \mathcal{J}_1$ which choose to leave their slot empty such that $\mu_{m1}(j) = 0$ move on to the second period. I define the set of teachers that choose to stay unmatched in period 1 as $\mathcal{I}_1^0(\mu)$. Similarly I define the set of schools that choose to leave their vacancy empty as $\mathcal{J}_1^0(\mu)$.

Period t: The set of teachers $\mathcal{I}_t$ and schools $\mathcal{J}_t$ arrive in the market along with the teachers that chose to stay unmatched in the previous period $\mathcal{I}_{t-1}^0(\mu)$ and the schools that chose to keep their slots empty in the previous period $\mathcal{J}_{t-1}^0(\mu)$. We define the set of teachers available in period t as $\mathcal{I}_t(\mu) = \mathcal{I}_t \cup \mathcal{I}_{t-1}^0(\mu)$ and the set of school available in period t as $\mathcal{J}_t(\mu) = \mathcal{J}_t \cup \mathcal{J}_{t-1}^0(\mu)$. A matching μ_t occurs and all teachers $i \in \mathcal{I}_t(\mu)$ such that $\mu_{wt}(i) = 0$ and schools $j \in \mathcal{J}_t(\mu)$ such that $\mu_{mt}(j) = 0$ participate in the next period. I define the set of teachers that choose to stay unmatched in period t as $\mathcal{I}_t^0(\mu)$. Similarly, I define the set of schools that choose to leave their vacancy empty as $\mathcal{J}_t^0(\mu)$.

Period T: The set of teachers $\mathcal{I}_T$ and schools $\mathcal{J}_T$ arrive in the market along with the teachers in $\mathcal{I}_{T-1}^0$ and the schools in $\mathcal{J}_{T-1}^0$. We define the set of teachers available in period T as

$\mathcal{I}_T(\mu) = \mathcal{I}_T \cup \mathcal{I}^0_{T-1}(\mu)$ and the set of schools available in period T as $\mathcal{J}_T(\mu) = \mathcal{J}_T \cup \mathcal{J}^0_{T-1}(\mu)$. From there a matching μ_T occurs and all teachers and schools choosing the outside option at this stage stay unmatched forever. The resulting matching is defined by $\mu = (\mu_t)_{t=1}^T$.

Firms and teachers are characterized by their observed attributes which collapse into two vectors of random variables $\boldsymbol{x}_{it}$ and $\boldsymbol{z}_{jt}$. I assume that the observed state variables of the new entrants in period t are drawn from the probability distribution functions m_t° and w_t°. I then assume that state variables evolve exogenously according to the Markov transition distribution functions m and w. This implies that aggregate states according to the following rule:

$$w_{t+1}(\boldsymbol{x}, \boldsymbol{\mu}) = \int_{\mathcal{X}_t} w(\boldsymbol{x}|s) f_t(s, *) ds + w^\circ_{t+1}(\boldsymbol{x})$$

$$m_{t+1}(\boldsymbol{z}, \boldsymbol{\mu}) = \int_{\mathcal{Z}_t} m(\boldsymbol{z}|s) f_t(*, h) dh + w^\circ_{t+1}(\boldsymbol{x})$$

I define the lifetime utility that teacher i gets from being matched with school j in period t as:

$$U_{ijt} = U_t(\boldsymbol{x}_{it}, \boldsymbol{z}_{jt}) + \sigma \eta_{ijt}$$

whereas the lifetime utility that school j gets from being matched with teacher i in period t is defined as:

$$V_{ijt} = V_t(\boldsymbol{x}_{it}, \boldsymbol{z}_{jt}) + \sigma \epsilon_{ijt}$$

I then define the lifetime utility that teacher i gets from staying unmatched and that school j gets from leaving its slot empty in period t as U_{i0t} and V_{0jt}:

$$U_{i0t} = \sigma \max_{k=1,\ldots,J} \eta_{i0,k} + \beta_w \int \overline{U}_{it+1}(\boldsymbol{x}_{it+1}) w(\boldsymbol{x}_{it+1}|\boldsymbol{x}_{it}) d\boldsymbol{x}_{it+1} - \beta_w \log(J)$$

$$V_{0jt} = \sigma \max_{k=1,\ldots,J} \epsilon_{0j,k} + \beta_m \int \overline{V}_{jt+1}(\boldsymbol{z}_{jt+1}) m(\boldsymbol{z}_{jt+1}|\boldsymbol{z}_{jt}) d\boldsymbol{z}_{jt+1} - \beta_m \log(J)$$

I then assume that Assumption 1-4 (ii) hold. I simply adjust 4 (iii) as the law of motion for aggregate states is defined as above. I also slightly modify Assumption 3 (i) such that $|\mathcal{I}_t| = [\exp(\gamma_{wt})n]$, $|\mathcal{J}_t| = [\exp(\gamma_{mt})n]$.

F.2 Linking Primitives to Equilibrium Sorting

I follow the same steps as in Section 4.2. I define F for a given random matching μ_t from a finite economy indexed by n as follows:

$$F_{nt}(x_{it}, z_{jt}|\mu_t) = \frac{1}{n} \sum_{i \in \mathcal{I}_t(\mu)} \sum_{j \in \mathcal{J}_t(\mu)} \mathbb{P}(x_{it} \leq x, z_{jt} \leq z, \mu_{wt}(i) = j)$$

I then denote F_t the limit of the distribution function F_{nt} as the size of the market n grows to infinity. I also define the joint density of matched characteristics as f_t.

As in the standard setting, I define the opportunity set faced by a given teacher $i \in \mathcal{I}_t$ in period t given a match μ as:

$$M_{it}(\mu) = \{j \in \mathcal{J}_t : V_{ijt} \geq V_{\mu_{mt}(j)jt}\}$$

Similarly, I define the opportunity set of school $j \in \mathcal{J}_t$ as:

$$W_{jt}(\mu) = \{i \in \mathcal{I}_t : U_{ijt} \geq U_{i\mu_{mt}(i)t}\}$$

The analogous of Proposition 1 follows directly from Assumption 4:

Proposition F.1 *Consider a match μ^* satisfying Assumption 4, for all $i \in \mathcal{I}_t$ and $j = \mathcal{J}_t$:*
(i) For all $t = 1, ..., T$:

$$U_{i\mu^*_{wt}(i)t} = \max_{k \in M_{it}(\mu^*) \cup \{0\}} U_{ikt} \quad \textit{and} \quad V_{\mu^*_{mt}(j)jt} = \max_{l \in W_{jt}(\mu^*) \cup \{0\}} V_{ljt}$$

(ii) Under Assumption 2, for all $t < T$:

$$\overline{U}_{it+1}(x) = \mathbb{E}_{\mathcal{S}_t} \left[\max_{k \in M_{it+1}(\mu^*) \cup \{0\}} U_{ikt+1} | x_{it+1} = x \right]$$

$$\overline{V}_{jt+1}(z) = \mathbb{E}_{\mathcal{S}_t} \left[\max_{l \in W_{jt+1}(\mu^*) \cup \{0\}} V_{ljt+1} | z_{jt+1} = z \right]$$

The proof is identical to the proof of Proposition 1. This result implies that an equilibrium match μ^* can be rewritten as the outcome of two dynamic discrete choice models where each agent's choice set is its opportunity set. However, each alternative, except the option of staying unmatched, is a terminating action.

I characterize the limit of conditional choice probabilities (CCPs) and expected future payoffs under arbitrary exogenous choice sets and by fixing the aggregate states distributions. I assume that $M_{it} = \{1, ..., J\}$ and $W_{jt} = \{1, ..., J\}$ for all t and I fix m_t and w_t for all t.

Proposition F.2 *Consider a given teacher $i \in \mathcal{I}_t$. Under Assumption 1-3 we have:*
(i) For all t, as $J \to \infty$:

$$J\mathbb{P}(U_{ijt} \geq U_{ikt}, k = \{0, 1, ..., J\} | \boldsymbol{x}_{it}, \boldsymbol{z}_{jt}) \longrightarrow \frac{\exp\left\{U_t(\boldsymbol{x}_{it}, \boldsymbol{z}_{jt})\right\}}{\exp\left\{\beta_w \int \overline{U}_{t+1}(s) w(s|\boldsymbol{x}_{it}) ds\right\} + \int \exp\left\{U_t(\boldsymbol{x}_{it}, h)\right\} m_t(h) dh}$$

$$\mathbb{P}(U_{i0t} \geq U_{ikt}, k = \{0, 1, ..., J\} | \boldsymbol{x}_{it}) \longrightarrow \frac{\exp\left\{\beta_w \int \overline{U}_{t+1}(s) w(s|\boldsymbol{x}_{it}) ds\right\}}{\exp\left\{\beta_w \int \overline{U}_{t+1}(s) w(s|\boldsymbol{x}_{it}) ds\right\} + \int \exp\left\{U_t(x_{it}, h)\right\} m_t(h) dh}$$

(ii) For all t:

$$\overline{U}_{t+1}(x) = \log\left(\exp\left\{\beta_w \int \overline{U}_{t+2}(s) w_0(s|x) ds\right\} + \int \exp\left\{U_{t+1}(x, h)\right\} m_{t+1}(h) dh\right) + \gamma + o(1)$$

where $\gamma \approx 0.5772$ is Euler's constant. Again, the proof is identical to the proof of Proposition 2. The same result holds symmetrically for the school side.

I now introduce that opportunity sets are unobserved and endogenous and show that the implications of Proposition F.2 allow us to tackle both of these issues. Using the same argument as in the standard case, Proposition F.2 implies that: (i) the probability that school j rematches with a specific teacher i vanishes to zero as the size of opportunity sets increases to infinity and (ii) the probability of choosing the outside option instead is nondegenerate in the limit. This implies that the dependence between taste shocks and opportunity sets vanishes in the limit.

I now consider a sequence of school-optimal stable matches μ^M. As opportunity sets' endogeneity vanishes in the limit for extremal matchings, we can then use Proposition F.2 (i) to bound teachers' CCPs in period t, assuming that we would observe the corresponding opportunity set $M_{it}(\mu_t^M)$ and future expected payoff function $\overline{U}_{it+1}^M$:

$$
\begin{aligned}
&n^{1/2}\mathbb{P}(U_{ijt} \geq \max_{k\in M_{it}(\mu_t^M)\cup\{0\}} U_{ikt}|x_{it}, z_{jt}, (z_{kt})_{k\in M_{it}(\mu_t^M)}, M_{i\tau}(\mu_t^M), \overline{U}_{it+1}^M) \\
&\qquad \leq \frac{\exp\left\{U_t(\boldsymbol{x}_{it}, \boldsymbol{z}_{jt})\right\}}{\exp\left\{\beta_w \int \overline{U}_{it+1}^M(s)w(s|\boldsymbol{x}_{it})ds\right\} + n^{-1/2}\sum_{k\in M_{it}(\mu_t^M)} \exp\left\{U_t(\boldsymbol{x}_{it}, \boldsymbol{z}_{kt})\right\}} + o(1)
\end{aligned} \tag{8}
$$

Similar bounds can be computed for a sequence of teacher-optimal stable match μ^W where the direction of the inequality is reversed. The same result also holds for the school side with the direction of the inequality reversed. Using Proposition F.2 (ii), we can also bound agents' expectations about their match payoff under a sequence of school-optimal stable matches μ^M as follows:

$$
\overline{U}_{it}^M(x) \geq \log\left(\exp\left\{\beta \int \overline{U}_{it+1}^M(s)w(s|x)ds\right\} + n^{-1/2}\sum_{k\in M_{it}(\mu_t^M)} \exp\left\{U_t(x, \boldsymbol{z}_{kt})\right\}\right) + \gamma + o(1) \tag{9}
$$

where again similar bounds can be computed for the teacher-optimal stable match and for the school side with the direction of the inequality reversed.

In Equations (8) and (9), $n^{-1/2}\sum_{k\in M_{it}(\mu_t^M)} \exp\left\{U_t(\boldsymbol{x}_{it}, \boldsymbol{z}_{kt})\right\}$ serves as a sufficient statistic that collapses all the information contained in opportunity sets which is needed to approximate CCPs and expectations about future payoffs.

I define teacher i's inclusive value given a sequence of realized matches μ^* as:

$$
I_{wit}^* = n^{-1/2}\sum_{k\in M_{it}(\mu_t^*)} \exp\left\{U_t(\boldsymbol{x}_{it}, \boldsymbol{z}_{kt})\right\}
$$

Similarly, I define school j's inclusive value given μ^* as:

$$I^*_{mjt} = n^{-1/2} \sum_{l \in W_{jt}(\mu^*_t)} \exp\left\{V_t(\boldsymbol{x}_{lt}, \boldsymbol{z}_{jt})\right\}$$

I also define I^M_{wit} and I^M_{mjt} as the inclusive values that would arise under a sequence of school-optimal stable matches μ^M in period t and I^W_{wit} and I^W_{mjt} as the inclusive values that would arise under a sequence of teacher-optimal stable matches μ^W in period t.

Inclusive values arising from a sequence of school-optimal and teacher-optimal stable matches in a given period t can be approximated by *expected inclusive value functions.* I rewrite I^M_{wit} as:

$$\begin{aligned} I^M_{wit} &= \frac{1}{n} \sum_{k \in \mathcal{J}_t(\mu)} \exp\left\{U(\boldsymbol{x}_{it}, \boldsymbol{z}_{kt})\right\} \times \sqrt{n}\mathbb{1}\{k \in M_{it}(\mu^M_t)\} \\ &= \frac{1}{n} \sum_{k \in \mathcal{J}_t(\mu)} \exp\left\{U(\boldsymbol{x}_{it}, \boldsymbol{z}_{kt})\right\} \sqrt{n}\mathbb{1}\{V_{ikt} \geq \max_{l \in W_{kt}(\mu^M_t) \cup \{0\}} V_{lkt}\} \end{aligned}$$

The inclusive value of a given teacher is determined by the set of schools that would accept her, which in turn depends on the preferences of all schools as well as their opportunity sets. Using the school analogous of Equation (1), I thus show that:

$$I^M_{wit} \geq \hat{\Gamma}^M_{wt}(x_{it}) + o_p(1) \quad \text{and} \quad I^M_{mjt} \leq \hat{\Gamma}^M_{mt}(z_{jt}) + o_p(1)$$

where $\hat{\Gamma}^M_{wt}$ and $\hat{\Gamma}^M_{mt}$ are the school-optimal expected inclusive value function of teachers and schools in period t which are defined as:

$$\hat{\Gamma}^M_{wt}(x_{it}) = \frac{1}{n} \sum_{k \in \mathcal{J}_t(\mu)} \frac{\exp\left\{U_t(\boldsymbol{x}_{it}, \boldsymbol{z}_{kt}) + V_t(\boldsymbol{x}_{it}, \boldsymbol{z}_{kt})\right\}}{\exp\left\{\beta \int \overline{V}^M_{kt+1}(s) m(s|\boldsymbol{z}_{kt}) ds\right\} + I^M_{mkt}}$$

$$\hat{\Gamma}^M_{mt}(z_{jt}) = \frac{1}{n} \sum_{l \in \mathcal{I}_t(\mu)} \frac{\exp\left\{U_t(\boldsymbol{x}_{lt}, \boldsymbol{z}_{jt}) + V_t(\boldsymbol{x}_{lt}, \boldsymbol{z}_{jt})\right\}}{\exp\left\{\beta \int \overline{U}^M_{lt+1}(s) w(s|\boldsymbol{x}_{lt}) ds\right\} + I^M_{wlt}}$$

where I define $\overline{U}^M_{it+1}$ and $\overline{V}^M_{jt+1}$ as follows:

$$\overline{U}^M_{it+1}(x) = \log\left(\exp\left\{\beta \int \overline{U}^M_{it+2}(s) w_0(s|x) ds\right\} + I^M_{wit+1}\right)$$

$$\overline{V}^M_{jt+1}(z) = \log\left(\exp\left\{\beta \int \overline{V}^M_{jt+2}(s) m_0(s|z) ds\right\} + I^M_{mjt+1}\right)$$

Note that similar bounds can be established for the inclusive values that would arise under the teacher-optimal stable match:

$$I^W_{wit} \leq \hat{\Gamma}^W_{wt}(x_{it}) + o_p(1) \quad \text{and} \quad I^W_{mjt} \geq \hat{\Gamma}^W_{mt}(z_{jt}) + o_p(1)$$

The proof follows the same steps as the proof of Lemma 4 in Appendix D.4.

The rest of the proof entails characterizing the fixed point problem and showing that inclusive values arising from an equilibrium match μ^* can be approximated by its solution. I define the fixed point mappings as follows:

$$\hat{\Psi}_{wt}[\mathbf{\Gamma}](x) = \frac{1}{n} \sum_{k \in \mathcal{J}_t(\mu)} \frac{\exp\left\{U_t(x, \boldsymbol{z}_{kt}) + V_t(x, \boldsymbol{z}_{kt})\right\}}{\exp\left\{\beta \int \overline{V}_{t+1}[\mathbf{\Gamma}](s) m(s|\boldsymbol{z}_{kt}) ds\right\} + \Gamma_{mt}(z_{kt})}$$

$$\hat{\Psi}_{mt}[\mathbf{\Gamma}](z) = \frac{1}{n} \sum_{l \in \mathcal{I}_t(\mu)} \frac{\exp\left\{U_t(\boldsymbol{x}_{lt}, z) + V_t(\boldsymbol{x}_{lt}, z)\right\}}{\exp\left\{\beta \int \overline{U}_{t+1}[\mathbf{\Gamma}](s) w(s|\boldsymbol{x}_{lt}) ds\right\} + \Gamma_{wt}(x_{lt})}$$

$$\overline{U}_{t+1}[\mathbf{\Gamma}](x) = \log\left(\exp\left\{\beta \int \overline{U}_{t+2}[\mathbf{\Gamma}](s) w_0(s|x) ds\right\} + \Gamma_{wt+1}(x)\right)$$

$$\overline{V}_{t+1}[\mathbf{\Gamma}](z) = \log\left(\exp\left\{\beta \int \overline{V}_{t+2}[\mathbf{\Gamma}](s) m_0(s|z) ds\right\} + \Gamma_{mt+1}(z)\right)$$

For a given equilibrium match μ^*, for any $x \in \mathcal{X}$ and $z \in \mathcal{Z}$ in each period t:

$$\hat{\Gamma}^*_{wt}(x) = \hat{\Psi}_{wt}[\hat{\mathbf{\Gamma}}^*](x) + o_p(1) \quad \text{and} \quad \hat{\Gamma}^*_{mt}(z) = \hat{\Psi}_{mt}[\hat{\mathbf{\Gamma}}^*](z) + o_p(1) \tag{10}$$

meaning that inclusive values in period t arising from an equilibrium match μ^* can be approximated by fixed points of the mappings $\hat{\Psi}_{wt}, \hat{\Psi}_{mt}$. To characterize the limit of inclusive values, I then consider the limit version of this fixed point problem:

$$\Gamma_{wt} = \Psi_{wt}[\mathbf{\Gamma}] \quad \text{and} \quad \Gamma_{mt} = \Psi_{mt}[\mathbf{\Gamma}] \quad \forall t \tag{11}$$

where

$$\Psi_{wt}[\mathbf{\Gamma}](x) = \int \frac{\exp\left\{U_t(x,h) + V_t(x,h)\right\}}{\exp\left\{\beta \int \overline{V}_{t+1}[\mathbf{\Gamma}](s)m(s|h)ds\right\} + \Gamma_{mt}(h)} m_t[\mathbf{\Gamma}](h)dh$$

$$\Psi_{mt}[\mathbf{\Gamma}](z) = \int \frac{\exp\left\{U_t(h,z) + V_t(h,z)\right\}}{\exp\left\{\beta \int \overline{U}_{t+1}[\mathbf{\Gamma}](s)w(s|h)ds\right\} + \Gamma_{wt}(h)} w_t[\mathbf{\Gamma}](h)dh$$

$$\overline{U}_{t+1}[\mathbf{\Gamma}](x) = \log\left(\exp\left\{\beta \int \overline{U}_{t+2}[\mathbf{\Gamma}](s)w_0(s|x)ds\right\} + \Gamma_{wt+1}(x)\right)$$

$$\overline{V}_{t+1}[\mathbf{\Gamma}](z) = \log\left(\exp\left\{\beta \int \overline{V}_{t+2}[\mathbf{\Gamma}](s)m_0(s|z)ds\right\} + \Gamma_{mt+1}(z)\right)$$

$$w_t[\mathbf{\Gamma}](x) = \int_{\mathcal{X}_t} w(x|s)f_{t-1}[\mathbf{\Gamma}](s,*)ds + w_t^\circ(x)$$

$$m_t[\mathbf{\Gamma}](z) = \int_{\mathcal{Z}_t} m(z|s)f_{t-1}[\mathbf{\Gamma}](*,h)dh + m_t^\circ(z)$$

The final step of the proof shows that this population fixed point problem has a unique solution and that the approximate solution of the finite sample fixed point problem converges to it. This is stated in the following result:

Theorem F.1 *Under Assumption 1-4:*
(i) The mapping $(\log \mathbf{\Gamma}_{\boldsymbol{w}}, \log \mathbf{\Gamma}_{\boldsymbol{m}}) \mapsto (\log \mathbf{\Psi}_{\boldsymbol{m}}[\mathbf{\Gamma}], \log \mathbf{\Psi}_{\boldsymbol{w}}[\mathbf{\Gamma}])$ *is a contraction.*
(ii) The fixed point problem described in Equation (11) always has a unique solution $\mathbf{\Gamma}^*_{\boldsymbol{m}}, \mathbf{\Gamma}^*_{\boldsymbol{w}}$.
(iii) For any equilibrium μ^*, $I^*_{wit} \longrightarrow \Gamma^*_{wt}(x_{it})$ *and* $I^*_{mjt} \longrightarrow \Gamma^*_{mt}(z_{jt})$ *for all* i, j *and* t.

The complete proof of this result can be found in Appendix F.4. Finally, from Theorem F.1 and Proposition F.2, we can fully characterize analytically the equilibrium of the model as a function of teachers' and schools' payoff functions. The limit joint density of matched characteristics f_t can be derived from the limit of conditional choice probabilities and has the following expression:

$$\frac{f_t(x,z)}{w_t(x)m_t(z)} = \frac{\exp\left\{U_t(x,z) + V_t(x,z) + \gamma_{wt} + \gamma_{mt}\right\}}{\left(\exp\left\{\beta \int \overline{U}^*_{t+1}(s)w(s|x)ds\right\} + \Gamma^*_{wt}(x)\right)\left(\exp\left\{\beta \int \overline{V}^*_{t+1}(s)m(s|z)ds\right\} + \Gamma^*_{mt}(z)\right)}$$

$$\frac{f_t(x,*)}{w_t(x)} = \frac{\exp\left\{\beta \int \overline{U}^*_{t+1}(s)w_0(s|x)ds + \gamma_{wt}\right\}}{\left(\exp\left\{\beta \int \overline{U}^*_{t+1}(s)w(s|x)ds\right\} + \Gamma^*_{wt}(x)\right)}$$

$$\frac{f_t(*,z)}{m_t(z)} = \frac{\exp\left\{\beta \int \overline{V}^*_{t+1}(s)m_0(s|z)ds + \gamma_{mt}\right\}}{\left(\exp\left\{\beta \int \overline{V}^*_{t+1}(s)m(s|z)ds\right\} + \Gamma^*_{mt}(z)\right)}$$

where $f_t(x,*)$ and $f_t(*,z)$ are, respectively, the density of the characteristics of unmatched teachers and unmatched schools. I define the equilibrium expected future payoff functions $\overline{U}^*_{t+1}$ and $\overline{V}^*_{t+1}$ recursively as:

$$\overline{U}^*_{t+1}(x) = \log\left(\exp\left\{\beta \int \overline{U}^*_{t+2}(s)w(s|x)ds\right\} + \Gamma^*_{wt+1}(x)\right)$$

$$\overline{V}^*_{t+1}(z) = \log\left(\exp\left\{\beta \int \overline{V}^*_{t+2}(s)m(s|z)ds\right\} + \Gamma^*_{mt+1}(z)\right)$$

and the equilibrium aggregate states distribution w^*_t and m^*_t as:

$$w^*_t(x) = \int_{\mathcal{X}_t} w(x|s)f_{t-1}[\mathbf{\Gamma}](s,*)ds + w^\circ_t(x)$$

$$m^*_t(z) = \int_{\mathcal{Z}_t} m(z|s)f_{t-1}[\mathbf{\Gamma}](*,h)dh + m^\circ_t(z)$$

F.3 Identification

The identification strategy follows the same steps as Section 5. I thus fix the value of the discount factors and consider two polar cases: (i) $T < \infty$ and nonstationarity and (ii) $T = \infty$ and stationarity.

F.3.1 Finite horizon

The identification argument in the finite horizon case can be done by backward induction. Starting from the last period T, we can identify the joint surplus as follows:

$$U_T(x,z) + V_T(x,z) = \log\left(\frac{f_T(x,z)}{f_T(x,*)f_T(*,z)}\right)$$

We can also identify Γ^*_{wT} and Γ^*_{mT} from the distribution of unmatched teachers and schools:

$$\Gamma^*_{wT}(x) = \frac{w_T(x)\exp(\gamma_{wT})}{f_T(x,*)} - 1$$

$$\Gamma^*_{mT}(z) = \frac{m_T(z)\exp(\gamma_{mT})}{f_T(*,z)} - 1$$

$\overline{U}_T$ and $\overline{V}_T$ can then be computed by backward induction:

$$\overline{U}_T(x) = \log(1 + \Gamma^*_{wT}(x)) + \gamma$$

$$\overline{V}_T(z) = \log(1 + \Gamma^*_{mT}(z)) + \gamma$$

From there, we can then repeat the same steps to identify the inclusive value functions and the joint surplus in period $T-1$. Finally, we iterate the procedure to identify the joint surplus and the inclusive value functions in all periods t. This results in the following proposition.

Proposition F.3 *Under Assumption 1-4 and for $T < \infty$:*

(i) The joint surplus function $U_t + V_t$ and the inclusive value functions Γ^*_{wt} and Γ^*_{mt} are identified for all t from f_t, the limiting joint distribution of matched characteristics in period t.

(ii) Without further restrictions, we cannot separately identify U_t and V_t for all t.

F.3.2 Infinite horizon

To allow for $T = \infty$, I impose Assumption 5 which implies $\Gamma_{mt} = \Gamma_m$ and $\Gamma_{wt} = \Gamma_w$ for all t, $\overline{U}_t = \overline{U}$ and $\overline{V}_t = \overline{V}$. This implies that we can write:

$$
\begin{aligned}
\frac{f(x,*)}{w(x)} &= \frac{\exp\left\{\beta \int \overline{U}^*(s) w_0(s|x) ds\right\}}{\left(\exp\left\{\beta \int \overline{U}^*(s) w_0(s|x) ds\right\} + \Gamma_w^*(x)\right)} \\
&= \frac{\exp\left\{\beta \int \overline{U}^*(s) w_0(s|x) ds\right\}}{\exp\left\{\overline{U}^*(x) - \gamma\right\}} = \exp\left\{\beta \int \overline{U}^*(s) w_0(s|x) ds - \overline{U}^*(x) + \gamma\right\}
\end{aligned}
$$

From there, we can invert this mapping to recover $\overline{U}^*$. We can follow the same steps to recover $\overline{V}$ from $f(*, z)$. It is then immediate to see that we can identify $U + V$ from $f(x, z)$.

Proposition F.4 *Under Assumption 1-5 and for $T = \infty$:*

(i). The joint surplus function $U + V$ and the inclusive value functions Γ_w^* and Γ_m^* are identified from the limiting joint distribution of matched characteristics in each period f.

(ii). Without further restrictions, we cannot separately identify U and V.

F.4 Proof Theorem F.1

I will start by proving part (i) of Theorem F.1. A first step is to restrict the space of functions in which the solutions to the fixed point problem described in Equation 11 can belong to. Namely, I will start by showing to we can restrict ourselves to a Banach space of continuous functions.

We start by constructing bounds for the solutions of this fixed point problem. Note that for all t, we can see that $\Psi_{wt}[\boldsymbol{\Gamma_m}](x) \geq 0$ and $\Psi_{mt}[\boldsymbol{\Gamma_w}](x) \geq 0$ for all (x, z) which implies that the solutions of this fixed point problem must be bounded from below by 0. To construct an upper bound we first need to construct a lower bound on $\overline{U}_t$ and $\overline{V}_t$. We proceed by backward induction. We know that $\overline{U}_T(x) = \log(1 + \Gamma_{wT}(x)) + \gamma$ which implies that $\overline{U}_T(x) \geq \gamma$ for all x. Iterating this procedure, we can then show that $\overline{U}_{T-1}(x) \geq \gamma(1 + \beta_w)$ and more generally that $\overline{U}_{t+1}(x) \geq \gamma \sum_{\tau=0}^{T-t} \beta_w^\tau$ and $\overline{V}_{t+1}(z) \geq \gamma \sum_{\tau=0}^{T-t} \beta_m^\tau$ for all (x, z). We also know from

Assumption 1, that U_t and V_t are bounded from above. We can thus show that

$$\Psi_{wt}[\boldsymbol{\Gamma_m}](x) \leq \frac{\exp\{\overline{U_t} + \overline{V_t}\}}{\gamma \sum_{\tau=1}^{T-t} \beta_m^{\tau-1}} \quad \forall x \in \mathcal{X}$$

$$\Psi_{mt}[\boldsymbol{\Gamma_w}](z) \leq \frac{\exp\{\overline{U_t} + \overline{V_t}\}}{\gamma \sum_{\tau=1}^{T-t} \beta_w^{\tau-1}} \quad \forall z \in \mathcal{Z}$$

To prove continuity of the mappings $\Psi_{wt}[\boldsymbol{\Gamma_m}]$ and $\Psi_{mt}[\boldsymbol{\Gamma_w}]$ we proceed by backward induction. Starting from $t = T$, we can rewrite $\Psi_{wT}[\boldsymbol{\Gamma_m}]$ as:

$$\Psi_{wT}[\boldsymbol{\Psi_m}[\boldsymbol{\Gamma_w}]](x) = \int \frac{\exp\{U_T(x,s) + V_T(x,s)\}}{1 + \int \frac{\exp\{U_T(t,s)+V_T(t,s)\}}{1+\Gamma_{wT}(t)} w_T[\boldsymbol{\Gamma_w}](t)dt} m_T[\boldsymbol{\Psi_m}[\boldsymbol{\Gamma_w}]](s)ds$$

which shows that continuity of the solution of $\Gamma_{wT} = \Psi_{wT}[\boldsymbol{\Psi_m}[\boldsymbol{\Gamma_w}]]$ follows directly from continuity of U_T and V_T as stated in Assumption 1. From there we can infer that $\overline{U}_T(x)$ is also continuous and we know that it is a non negative function which implies that $\Psi_{wT-1}[\boldsymbol{\Gamma_m}]$ will also be continuous. We can then iterate this argument to prove that the solutions of the fixed point problem described in Equation 11 must be continuous and bounded functions.

We now turn to the proof that the mapping $(\log \boldsymbol{\Gamma_w}, \log \boldsymbol{\Gamma_m}) \mapsto (\log \boldsymbol{\Psi_m}[\boldsymbol{\Gamma_w}], \log \boldsymbol{\Psi_w}[\boldsymbol{\Gamma_m}])$ is a contraction. We will start by showing that for alternative sets of functions $\boldsymbol{\Gamma_m} = (\Gamma_{mt})_{t=1}^T$ and $\tilde{\boldsymbol{\Gamma}}_{\boldsymbol{m}} = (\tilde{\Gamma}_{mt})_{t=1}^T$, there always exist a constant $\lambda < 1$ such that:

$$\Big|\Big| \log \boldsymbol{\Psi_w}[\boldsymbol{\Gamma_m}] - \log \boldsymbol{\Psi_w}[\tilde{\boldsymbol{\Gamma}}_{\boldsymbol{m}}] \Big|\Big|_\infty \leq \lambda \Big|\Big| \log \boldsymbol{\Gamma_m} - \log \tilde{\boldsymbol{\Gamma}}_{\boldsymbol{m}} \Big|\Big|_\infty$$

The mean value inequality for vector valued functions defined on Banach spaces implies that:

$$\Big|\Big| \log \boldsymbol{\Psi_w}[\boldsymbol{\Gamma_m}](x) - \log \boldsymbol{\Psi_w}[\tilde{\boldsymbol{\Gamma}}_{\boldsymbol{m}}](x) \Big|\Big|_\infty \leq \sup_{a\in[0,1]} \Big|\Big| D \log \boldsymbol{\Psi_w}[a \log \boldsymbol{\Gamma_m} + (1-a) \log \tilde{\boldsymbol{\Gamma}}_{\boldsymbol{m}}](x) \Big|\Big|_\infty \Big|\Big| \log \boldsymbol{\Gamma_m}(x) - \log \tilde{\boldsymbol{\Gamma}}_{\boldsymbol{m}}(x) \Big|\Big|_\infty$$

where $D \log \boldsymbol{\Psi_w}$ are the Gateaux derivatives of $\log \boldsymbol{\Psi_w}$. I will thus characterize and bound the following object for any $t \in [0,1]$ and any $x \in \mathcal{X}$:

$$D \log \boldsymbol{\Psi_w}[a \log \boldsymbol{\Gamma_m} + (1-a) \log \tilde{\boldsymbol{\Gamma}}_{\boldsymbol{m}}](x)$$

Note first that we can rewrite $\log \Psi_{wt}[\log \boldsymbol{\Gamma_m}](x)$ as:

$$\log \int \frac{\exp\left\{U_t(x,h) + V_t(x,h)\right\}}{\exp\left\{\beta \int \overline{V}_{t+1}[\log \boldsymbol{\Gamma_m}](s) m(s|h) ds\right\} + \exp\{\log \Gamma_{mt}(h)\}} m_t[\log \boldsymbol{\Gamma_m}](h) dh$$

where

$$m_t[\log \boldsymbol{\Gamma_m}](z) = \int_{\mathcal{Z}_t} m(z|s) f_{t-1}[\log \boldsymbol{\Gamma_m}](*, h) dh + m_t^{\circ}(z)$$

$$\overline{V}_{t+1}[\log \boldsymbol{\Gamma_m}](z) = \log \left(\exp\left\{\beta \int \overline{V}_{t+2}[\log \boldsymbol{\Gamma_m}](s) m(s|z) ds\right\} + \Gamma_{mt+1}(z) \right)$$

$$f_{t-1}[\log \boldsymbol{\Gamma_m}](*, z) = \frac{\exp\left\{\beta \int \overline{V}_t^*(s) m(s|z) ds\right\}}{\left(\exp\left\{\beta \int \overline{V}_t^*(s) m(s|z) ds\right\} + \Gamma_{mt-1}^*(z)\right)} m_{t-1}[\log \boldsymbol{\Gamma_m}](z)$$

Using the same steps as in the proof of Theorem 1 (i), we can show that:

$$\sup_{a \in [0,1]} \left|\left| D \log \boldsymbol{\Psi_w}[a \log \boldsymbol{\Gamma_m} + (1-a) \log \tilde{\boldsymbol{\Gamma}}_{\boldsymbol{m}}](x) \right|\right|_{\infty} < 1$$

which implies that for any alternative sets of functions $\boldsymbol{\Gamma_m} = (\Gamma_{mt})_{t=1}^T$ and $\tilde{\boldsymbol{\Gamma}}_{\boldsymbol{m}} = (\tilde{\Gamma}_{mt})_{t=1}^T$ there always exist a constant $\lambda < 1$ such that:

$$\left|\left| \log \boldsymbol{\Psi_w}[\boldsymbol{\Gamma_m}] - \log \boldsymbol{\Psi_w}[\tilde{\boldsymbol{\Gamma}}_{\boldsymbol{m}}] \right|\right|_{\infty} \leq \lambda \left|\left| \log \boldsymbol{\Gamma_m} - \log \tilde{\boldsymbol{\Gamma}}_{\boldsymbol{m}} \right|\right|_{\infty}$$

Symmetrical arguments can be applied to find that there for any alternative sets of functions $\boldsymbol{\Gamma_w} = (\Gamma_{wt})_{t=1}^T$ and $\tilde{\boldsymbol{\Gamma}}_{\boldsymbol{w}} = (\tilde{\Gamma}_{wt})_{t=1}^T$ always exist a constant $\lambda < 1$ such that:

$$\left|\left| \log \boldsymbol{\Psi_m}[\boldsymbol{\Gamma_w}] - \log \boldsymbol{\Psi_m}[\tilde{\boldsymbol{\Gamma}}_{\boldsymbol{w}}] \right|\right|_{\infty} \leq \lambda \left|\left| \log \boldsymbol{\Gamma_w} - \log \tilde{\boldsymbol{\Gamma}}_{\boldsymbol{w}} \right|\right|_{\infty}$$

This concludes the proof of part (i) of Theorem F.1 and shows that the mapping $(\log \boldsymbol{\Gamma_w}, \log \boldsymbol{\Gamma_m}) \mapsto (\log \boldsymbol{\Psi_m}[\boldsymbol{\Gamma_w}], \log \boldsymbol{\Psi_w}[\boldsymbol{\Gamma_m}])$ is a contraction.

Part (ii) of Theorem F.1 directly follows from part (i) and from the Banach fixed point theorem. Part (iii) follows from the same steps as the proof of Theorem 1 (iii).