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Equilibrium uniqueness in entry games with private information

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We study equilibria in static entry games with single-dimensional private information. Our framework embeds many models commonly used in applied work, allowing for firm heterogeneity and selective entry. We introduce the notion of strength, which summarizes a firm's ability to endure competition. In environments of applied interest, an equilibrium in which entry strategies are ordered according to the firms' strengths always exists. We call this equilibrium herculean. We derive simple and testable sufficient conditions guaranteeing equilibrium uniqueness and, consequently, a unique counterfactual prediction.

1. Introduction

■ Understanding firms' market entry decisions is a key element of economic policy and regulation. Predicting whether there will be timely entry after a merger or regulatory change requires a framework that determines the number and types of competitors. More broadly, a model with endogenous entry, prices, product characteristics, and welfare outcomes can be used to evaluate policies prospectively. When performing such analyses, researchers use the counterfactual equilibrium of an estimated model to assess the impact of the policy under consideration. A common

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challenge is the existence of multiple equilibria. Under multiplicity, the model may not yield a unique prediction to the applied question, hindering policy analysis (Berry and Tamer, 2006; Borkovsky et al., 2015). Computing multiple equilibria may also prove challenging when using numerical methods, which may limit the researcher's ability to gain a complete understanding of the impacts of a policy of interest (Iskhakov et al., 2016).

We study equilibrium uniqueness in static binary-action entry games with single-dimensional private information. Our framework accommodates a large variety of entry games, allowing for rich forms of firm *heterogeneity* and *selective* entry. Our main contribution is to provide a sufficient condition that guarantees equilibrium uniqueness. The condition is solely based on the model's fundamentals and verifying it does not require equilibrium computation. In many common applications, we can check the condition by performing a simple calculation. For example, Roberts and Sweeting (2013) and Grieco (2014) use numerical methods to show that their fitted models have a unique equilibrium. Using their estimates and our sufficient condition, we can confirm equilibrium uniqueness in their fitted models, highlighting the usefulness of our results. Our findings provide new tools for applied researchers studying entry.

We characterize firms' equilibrium behavior using a simple index, called *strength*, summarizing a firm's ability to endure competition. The strength of a firm is the unique *symmetric* cutoff strategy that makes the firm indifferent between entering and not entering the market. Facing equal competition, a stronger firm is more willing to enter the market than a weaker one. For the class of models studied, we show that there always exists an equilibrium in which entry strategies are ordered according to strength. We call this a *herculean* equilibrium. When our sufficient condition for equilibrium uniqueness holds, only one herculean equilibrium exists, and no non-herculean equilibrium is possible.

Our proposed framework encompasses static entry models commonly used in applied work. The approach accommodates a large variety of post-entry models, including auctions and competitions in price or quantity; it also allows for rich forms of firm heterogeneity, as firms are allowed to differ in their payoff functions or their distribution of types, capturing that firms might be heterogeneous in their public characteristics (e.g., firms might vary in their product characteristics, geographic locations, or levels of vertical integration). Payoffs might depend on the entry decisions and realized types of competitors, allowing a level of strategic interaction often ignored by the entry literature (auctions being an exception). For example, if firms are privately informed about their marginal costs of production, facing a competitor with a lower marginal cost will lead to a lower post-entry profit. The magnitude of this decrease depends on the firms' realized marginal costs, their degree of product substitutability, and the number of entrants. We enrich the set of models available to applied researchers by including these environments.

In the theoretical literature on market entry, Mankiw and Whinston (1986) study welfare in a symmetric model under complete information. Brock and Durlauf (2001) examine a symmetric coordination game with privately informed agents. Our modelling shares the idea that both the action and type of an agent affects the payoffs of other agents but differs in that entry decisions are strategic *substitutes* and in that we allow for asymmetric agents. Our article generalizes the existing literature on costly entry into second-price auctions (SPAs). Samuelson (1985) studies *ex ante* symmetric bidders. Tan and Yilankaya (2006) study two groups of asymmetric bidders ordered by first-order stochastic dominance (FOSD), whereas in Cao and Tian (2013) the two groups are ordered by entry costs. In Ye (2007), bidders are partially informed at entry and fully learn their valuations after entry occurs. Our framework allows for more general forms of bidder heterogeneity and, at the same time, embeds both informational environments. A firm's private information might correspond to its type or a signal about its type.

In the empirical literature, Bresnahan and Reiss (1990, 1991) and Berry (1992) developed the first empirical models of market entry that explicitly account for the strategic interaction

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between post-entry market competition and firms' entry decisions.[1] Under complete information, the entry game often contains multiple equilibria. Tamer (2003) shows that, without further assumptions, multiple equilibria can lead to set, rather than point, identification.[2] Using numerical methods, Seim (2006) shows that firms having private information may solve the problem of equilibrium multiplicity. Berry and Tamer (2006), however, construct examples of multiple equilibria under private information, raising the question of when uniqueness can be achieved. Glaeser and Scheinkman (2003) show that games of strategic complements in which competitors' types do not directly affect payoffs have a unique equilibrium when they satisfy a moderate social influence (MSI) condition.[3] We contribute to this discussion by identifying a testable condition guaranteeing equilibrium uniqueness in the context of games of strategic substitutes and general payoffs structures.

The importance of allowing for private information in entry models lies beyond the possibility of solving the multiple equilibria problem. Using complementary methodologies, Grieco (2014) and Magnolfi and Roncoroni (2023) test and reject the hypothesis that firms possess complete information at the moment of entry. Furthermore, compared to models that allow for private information, they show that assuming complete information delivers estimates that can lead to qualitatively different predictions. Roberts and Sweeting (2013, 2016) provide evidence of selection at entry, which cannot be accounted for by complete information models.

The article is organized as follows. For illustrative purposes, Section 2 presents our results in the context of a SPA. The section introduces and discusses the notions of strength and herculean equilibrium, developing key intuitions. Section 3 introduces the general model and extends the results showing that the existence of a herculean equilibrium is guaranteed and provides a sufficient condition for when the herculean equilibrium is the unique equilibrium of the game. Finally, Section 4 concludes. All the proofs are relegated to the Appendix.

2. An illustrative example

■ We begin by illustrating our results in the context of entry into an asymmetric SPA with independent private values. We generalize our results to a richer set of entry models in Section 3.

□ Second-price auction with entry costs.

Set up. Consider an SPA with reservation price $r \geq 0$. The auction consists of one seller, n potential bidders, and one indivisible good. Before making any entry decision, each bidder $i \in \{1, 2, \dots, n\}$ observes her valuation for the object, v_i, which is drawn from an atomless distribution function F_i with full support on $[0, \infty)$.[4] Each F_i is continuously differentiable and has a finite expectation. After observing v_i, each bidder, independently and simultaneously, decides whether to enter the auction. If bidder i decides to enter, she incurs in an entry cost $K_i > 0$. The tuple $(F_i, K_i)_{i=1}^n$, which includes the number of potential bidders n, is commonly known to all the bidders. Observe that bidders may differ in their distribution of valuations and entry costs. After bidders make their entry decisions, a participating bidder bids their valuation (i.e., its weakly dominant strategy).

Strategies, payoffs, and equilibrium. An entry strategy for bidder i is called *cutoff* if there is a threshold x_i such that bidder i enters the auction when its valuation is higher than x_i ($v_i \geq x_i$) and

[1] See also Ciliberto et al. (2020) in the context of entry and Bresnahan (1987), Berry (1994), and Berry et al. (1995) when the number of competitors is exogenous

[2] Sweeting (2009) shows that multiplicity can help with the model's identification in the context of coordination games. De Paula and Tang (2012) show that multiplicity can be used to infer the signs of strategic interactions. Marcoux (forthcoming) provides a statistical test for whether firms play the same equilibrium across a sample of entry decisions.

[3] The MSI condition has been used to establish uniqueness in the context of linear-payoffs models by Lee et al. (2014), Lin and Xu (2017), Xu (2018), and Lin et al. (2021).

[4] Our results still apply if the support of F_i were an interval $[0, b]$ with $b > 0$. We chose the current formulation to avoid the existence of corner solutions in which a bidder never enters.

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stays out otherwise. Online Appendix C shows that focusing on cutoff strategies is without loss of generality.

To ease the notation, we order bidders' identities according to their cutoffs, with x_1 being the bidder with the lowest cutoff and x_n the highest. For a given vector of cutoff strategies $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ define: (i) $A_i^k = \prod_{j>i}^k F_j(x_j)$, the probability that bidders playing cutoffs greater than (or *above*) bidder i, up to bidder k, do not enter the auction; and, (ii) $B_i(v) = \prod_{j<i} F_j(v)$, the probability that bidders playing cutoffs lower than (*below*) bidder i obtain valuations lower than v. Let $\mathbf{x}_i = (x_1, x_2, \ldots, x_i)$ be a vector containing the cutoff strategies up to bidder i. Bidder i's expected *revenue* of entering with a valuation $v_i = x_i$, when there are only i potential bidders, and the other $i-1$ bidders play cutoffs lower than x_i is: [5]

$$R_i(x_i; \mathbf{x}_{i-1}) = x_i B_i(x_i) - rA_0^{i-1} - \sum_{j=1}^{i-1} \left(A_j^{i-1} \int_{x_j}^{x_{j+1}} \max\{r, s\} dB_{j+1}(s) \right).$$

This revenue consists of bidder i's value, x_i, times the probability of being the highest valuation bidder, $B_i(x_i)$, minus the expected price paid. The expected price consists of the reserve price, r, when no competitor enters, which occurs with probability A_0^{i-1}, and the maximum between the reserve price and the highest competitors' bid when entry occurs. A price in the interval $[x_j, x_{j+1})$ is observed only if opponents playing cutoffs higher or equal to x_{j+1} stay out of the auction, which occurs with probability A_j^{i-1}. Thus, the price in such interval distributes according to B_{j+1}.

Given the opponents' entry cutoff $\mathbf{x}_{-i}$, bidder i's expected *profit* of entering the auction with a valuation x_i is equal to

$$\Pi_i(x_i; \mathbf{x}_{-i}) = A_i^n R_i(x_i; \mathbf{x}_{i-1}) - K_i. \tag{1}$$

The expected profit consists of the expected revenue minus the entry costs K_i. Bidder i loses the auction whenever an opponent with a higher valuation than x_i enters the auction. Thus, bidder i obtains a positive payoff only when opponents playing cutoffs larger than x_i stay out, which happens with probability A_i^n. In this event, bidder i competes in an auction with i potential bidders, all of which play cutoffs lower than x_i, thus receiving the expected revenue $R_i(x_i; \mathbf{x}_{i-1})$.

The function $\Pi_i(x_i; \mathbf{x}_{-i})$ is strictly increasing in each argument. A bidder's expected profit increases in its valuation x_i and in the opponents' entry cutoff x_j (i.e., when opponents enter less often). Because of this monotonicity, we can define bidder i's *best response* to $\mathbf{x}_{-i}$, a cutoff strategy, to be the unique valuation $\chi_i(\mathbf{x}_{-i})$ that solves $\Pi_i(\chi_i(\mathbf{x}_{-i}); \mathbf{x}_{-i}) = 0$. The best response function $\chi_i(\mathbf{x}_{-i})$ is continuous and satisfies $\chi_i(\mathbf{x}_{-i}) \geq r + K_i$, that is, bidders do not enter the auction if their valuation cannot cover the reservation price plus their entry cost, regardless of what their opponents are playing. Using implicit differentiation, we can show that bidder i's best response is monotonically decreasing in the opponents' cutoffs, $\partial \chi_i(\mathbf{x}_{-i})/\partial x_j < 0$; that is, when an opponent enters less often (higher x_j) a bidder is more willing to enter the auction (lower x_i).

A Bayesian equilibrium is a vector of cutoff strategies $\mathbf{x}$ such that every bidder i is indifferent to enter the auction when it draws a valuation equal to its cutoff strategy. That is, $\mathbf{x}$ is an equilibrium vector if and only if $\Pi_i(\mathbf{x}) \equiv \Pi_i(x_i; \mathbf{x}_{-i}) = 0$ or, equivalently, $\chi_i(\mathbf{x}_{-i}) = x_i$, for every bidder i. Online Appendix C shows that a Bayesian equilibrium always exists.

□ **Strength and herculean equilibrium.** We now introduce our two main definitions: bidder *strength* and *herculean* equilibrium. Strength uses the game fundamentals, $(F_i, K_i)_{i=1}^n$, to rank bidders' relative competitiveness. We use strength to identify an equilibrium that exists in every entry game—the herculean equilibrium. This equilibrium is the starting point to find conditions for equilibrium uniqueness.

[5] The following notation is being used throughout the article: $\sum_{\emptyset} = 0$ and $\prod_{\emptyset} = 1$.

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Definition 1 (Strength). The strength of bidder i, is the unique number s_i that solves $\Pi_i(s_i; s_i, \dots, s_i) = 0$; that is, the unique s_i satisfying:

$$(s_i - r)\prod_{j \neq i} F_j(s_i) = K_i. \tag{2}$$

We say that bidder i is stronger than bidder j if $s_i < s_j$.

Strength is well defined. It assigns a unique scalar s_i to each bidder i, delivering a complete ranking of the bidders.[6] The strength of bidder i is the unique cutoff s_i that is a best response to the other bidders playing the same cutoff strategy s_i, that is, the unique value s_i satisfying $\chi_i(s_i, \dots, s_i) = s_i$. Strength ranks bidders by using the unique symmetric strategy that makes a given bidder indifferent to entering the auction. When bidders are asymmetric, the strategy s_i might differ across bidders. The importance and usefulness of strength relies on summarizing the multidimensional characteristics of bidders, $(F_i, K_i)_{i=1}^n$, into a single scalar.[7]

In intuitive terms, strength ranks firms according to their ability to endure competition. The strength of bidder i encompasses information about a bidder's willingness to enter the auction, relative to that of its competitors. A lower cutoff strategy for bidder i means that bidder i is more willing to enter the auction, as it enters for lower valuations. A lower entry cutoff by competitors, on the other hand, implies that bidder i faces more competition, as competitors are entering more often. Thus, bidder i being stronger than j ($s_i < s_j$) indicates that i, despite facing more competition than j, is more willing to enter the auction. The next lemma shows that strength generalizes common notions of relative competitiveness used in the entry literature.

Lemma 1.

1) If bidders have identical entry costs but the bidders' values are ordered by FOSD, the dominating bidder is stronger.
2) If bidders have identical distributions of valuations, but different entry costs, the bidder with the lower entry costs is stronger.

The ranking provided by strength coincides with that provided by common heuristics used to determine the relative competitiveness of bidders, such as FOSD or entry-cost order. Strength, however, extends the ranking to scenarios in which relative competitiveness is not self-evident. Take, for example, a bidder whose distribution of valuations first-order stochastically dominates the other bidder but has a higher entry cost. This scenario may arise when "smaller" firms have subsidized entry (Marion, 2007). In this case, the former bidder might be stronger, as it is likely to draw a higher valuation, but it might also be weaker given its higher entry cost. Strength not only ranks bidders in this (or any other) scenario but also, as shown below, provides meaningful information about equilibrium behavior.

Definition 2 (Herculean Equilibrium). An equilibrium is called herculean if the equilibrium cutoffs are ordered by strength, with the stronger bidder playing the lower cutoff. That is, $x_i < x_j$ if and only if $s_i < s_j$.

Because stronger bidders are able to endure competition more, they should be more inclined to enter the auction. In terms of equilibrium behavior, the previous intuition translates to stronger bidders playing lower entry cutoffs. In symmetric games, on the other hand, every bidder is equally strong. The herculean equilibrium consists of symmetric strategies in which each bidder

[6] The function $(s_i - r)\prod_{j \neq i} F_j(s_i)$ is increasing, unbounded, and equal to 0 when $s_i = r$.

[7] Strength has advantages over other candidates to rank firms, such as expected payoff or entry probability. Online Appendix D presents examples illustrating the advantages of strength over these measures in obtaining information about the bidders' equilibrium behavior.

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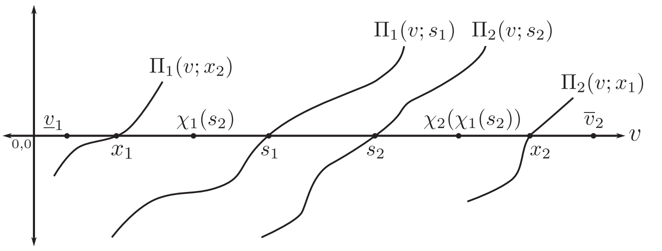
FIGURE 1

CONSTRUCTION OF A HERCULEAN EQUILIBRIUM FROM ITERATED BEST RESPONSES



Note: Starting from firm 2's strength, s_2, firm 1's best response, $\chi_1(s_2)$, is lower than its strength, $s_1 = \chi_1(s_1)$. Similarly, firm 2's best response to $\chi_1(s_2)$ is higher than $s_2 = \chi_2(s_2)$. Iterating these mutual best responses, create bounded monotonic sequences that converge to a herculean equilibrium.

plays a cutoff equal to its strength. Thus, in symmetric games, the herculean and symmetric equilibriums coincide.

Herculean equilibrium and strength are incomplete information analogs to risk-dominant equilibrium and risk factor in complete information games (Harsanyi and Selten, 1988). Both scalars, risk factor and strength, are found by computing an "indifferent entry" condition. In the context of complete information, a bidder's risk factor is the opponent's highest entry probability for which the bidder is willing to enter. On the other hand, a bidder's strength is the opponent's highest entry probability (lowest entry cutoff) for which the bidder enters if it obtains a valuation equal to said cutoff. Whereas in a herculean equilibrium, a stronger bidder is more likely to enter, the bidder with the lower risk factor enters a risk-dominant equilibrium.

□ **Auctions with two potential bidders.** We now illustrate our main results in the context of two potential bidders: a herculean equilibrium always exists and, under a weak cumulative distribution function (CDF)-concavity condition, it is the only equilibrium of the game. From now on, unless otherwise noted, we order bidders' identities by their strength, with bidder 1 being the strongest bidder.

Proposition 1. There always exists a herculean equilibrium. Moreover, the entry game has a unique equilibrium if, for each bidder i, the following condition holds:[8]

$$\frac{v f_i(v)}{F_i(v)} < 1 \quad \text{for all } v \in [\underline{v}_i, \overline{v}_i], \tag{3}$$

where $\underline{v}_i = K_i + r$ is bidder i's smallest entry cutoff that may lead to positive profits and $\overline{v}_i = \chi_i(\underline{v}_j)$ is bidder i's best response to $\underline{v}_j$ (i.e., bidder i's highest entry cutoff that she may play in an equilibrium.)

Proposition 1 provides two results. First, it establishes the existence of a herculean equilibrium, confirming the intuition that an equilibrium in which the strong bidder plays a lower entry cutoff should exist. To see the intuition, consider bidder 1's best response to the opponent's strength relative to its own strength. That is, $\chi_1(s_2)$ relative to $\chi_1(s_1) = s_1$, see Figure 1. Because bidder 1 is stronger, $s_1 < s_2$, bidder 1 faces less competition when bidder 2 plays s_2 instead of s_1. Consequently, bidder 1 enters more often, $\chi_1(s_2) < s_1 = \chi_1(s_1)$. Similarly, relative to s_2, bidder 2 faces more competition when bidder 1 plays $\chi_1(s_2) < s_2$. Thus, bidder 2 needs a higher valuation than its own strength to enter the auction, best responding with an entry cutoff that is higher than s_2, $\chi_2(\chi_1(s_2)) > \chi_2(s_2) = s_2$.

[8] Condition (3) can hold with equality if one firm is strictly stronger than the other.

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These incentives reinforce each other. Iterating mutual best responses starting from the bidders' strength generate two monotonic sequences of cutoffs that are, in each iteration, further apart. Because bidder 1's best response is bounded below by $\underline{v}_1$ and bidder 2's best response is bounded above by $\overline{v}_2$, this process converges to cutoffs $x_1 < x_2$ that are mutual best responses, that is, a herculean equilibrium. This iteration process can be used in applied research to find a herculean equilibrium when multiple equilibria exist.

Perhaps more importantly, Proposition 1 provides a sufficient condition on the CDFs' shape for the game to have a unique equilibrium. The uniqueness result is significant for applied work, as it provides a testable condition that guarantees that counterfactual equilibria will also be unique.[9] In intuitive terms, condition (3) is an equilibrium-stability condition. It guarantees that bidders do not overreact to a small change in the opponent's cutoff. We show that this lack of overreaction implies that a bidder's expected profit is monotonically increasing in its entry cutoff, even after considering the opponent's best response. In turn, this monotonicity implies that only one valuation makes a bidder indifferent to enter the auction, leading to a unique equilibrium.

To show that condition (3) implies equilibrium stability, let $x_i < x_j$. Using equation (1) when $n = 2$, bidder i's best response to x_j is $\chi_i(x_j) = r + K_i/F_j(x_j)$. Differentiating $\chi_i(x_j)$ with respect x_j, substituting for K_i, and using $x_i = \chi_i(x_j)$, we find

$$-\chi_i'(x_j) = (x_i - r)\frac{f_j(x_j)}{F_j(x_j)} < x_j\frac{f_j(x_j)}{F_j(x_j)} \leq 1,$$

where the first inequality follows from $x_i < x_j$ (and $r \geq 0$), and the last inequality from sufficient condition (3). That is, when bidder j increases its cutoff, bidder i best responds by decreasing its cutoff less than proportionally. Similarly, using implicit differentiation and analogous arguments, we can also show $-\chi_j'(x_i) = (x_i - r)f_i(x_i)/F_i(x_j) < 1$. Sufficient condition (3), then, guarantees that every pair of cutoff strategies satisfies the local stability condition $\chi_1'(x_2)\chi_2'(x_1) < 1$ (see Fudenberg and Tirole, 1991, p. 24). This implies equilibrium uniqueness. Graphically, an equilibrium is defined by a point at which best response functions cross. Best response functions are continuous and monotone. Thus, stable and unstable equilibria must alternate along the best response function. By condition (3), however, every equilibrium must be locally stable and, consequently, at most one equilibrium exists. As an equilibrium always exists, the game has a unique equilibrium.

Although the stability argument provides good intuition, we prove uniqueness directly by showing that a bidder's payoff is monotone in its own strategy even after considering the opponent's best response. Later, we scale this method to prove uniqueness under a larger set of players. For any strength order among bidders, define $\hat{\Pi}_i(x) = \Pi_i(x; \chi_j(x))$ to be bidder i's expected profit when their valuation is x, and the opponent best responds to the cutoff strategy x, $\chi_j(x)$. By definition, x is an equilibrium strategy when $\hat{\Pi}_i(x) = 0$.[10] We show that $\partial\hat{\Pi}_i(x)/\partial x > 0$ for every x, implying that $\hat{\Pi}_i(x)$ crosses zero only once, that is, a unique equilibrium exists. Let $\underline{x} = \min\{x, \chi_j(x)\}$ and $\overline{x} = \max\{x, \chi_j(x)\}$, differentiating $\hat{\Pi}_i(x)$ with respect to x we obtain:

$$\frac{\partial\hat{\Pi}_i(x)}{\partial x} = F_j(\overline{x}) + (\underline{x} - r)f_j(\chi_j(x))\chi_j'(x) = F_j(\overline{x})\big(1 - \chi_i'(\chi_j(x))\chi_j'(x)\big) > 0,$$

[9] In applied work, the distribution of values might have the structure $F_i(v) = F(v|X_i'\beta)$ where $X_i'\beta$ is a vector of bidder and auction characteristics (which may include auctions fixed effects). If $X_i'\beta$ is observed by an econometrician, condition (3) needs to apply conditional on $X_i'\beta$. If some of the elements in $X_i'\beta$ are unobserved, condition (3) delivers a set of unobserved values for which the game would have a unique equilibrium.

[10] A concise, albeit less intuitive, proof of existence of herculean equilibrium can be constructed using the function $\hat{\Pi}_i(x)$. Observe: (i) $\hat{\Pi}_1(x) < 0$ for $x \leq \underline{v}_1$; and (ii) because $s_1 < s_2$ and $\chi_2(x)$ is decreasing, $0 = \Pi_1(s_1; s_1) < \Pi_1(s_1; \chi_2(s_1)) = \hat{\Pi}_1(s_1)$. By the intermediate value theorem, there exists $x_1 \in (\underline{v}_1, s_1)$ such that $\hat{\Pi}_1(x_1) = 0$, a herculean equilibrium.

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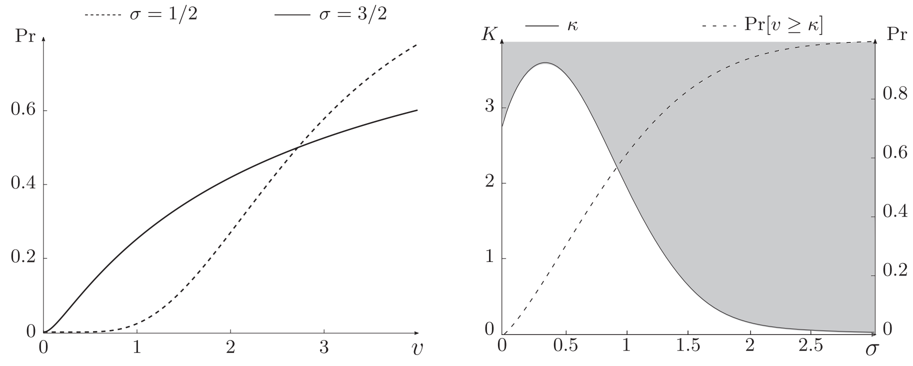
FIGURE 2

SUFFICIENCY WITH LOG-NORMAL VALUATIONS



(a) Log-normal CDFs ($\mu = 1$) **(b)** κ as a function of σ

Note: Panel (a) shows that log-normal CDFs are not concave. Panel (b) depicts the minimal threshold κ under which uniqueness condition (3) is guaranteed to hold, as a function of σ. The shaded area represents the set of entry costs under which the entry game has a unique equilibrium. $\Pr[v \geq \kappa]$ represents the proportion of valuations above κ.

where we used the expressions for $\chi_k'(x)$ derived above. The inequality follows from noting (as shown above) that condition (3) implies $-\chi_k'(x_{-k}) \in (0, 1)$, proving the result.

Equation (3), however, is not a necessary condition. It captures one of the possible mechanisms inducing a unique equilibrium. In particular, the condition ensures that the *shape* of best responses is such that they only cross once. Example 1c, below, illustrates another mechanism, not captured by (3), that will generate uniqueness: the degree of bidder asymmetry. Asymmetric bidders might have best responses at a different *scale*, ensuring that best responses cross once. For instance, if a bidder has a significantly lower expected valuation (or high entry cost), it will require an extreme (unlikely high) valuation for entry, and the game might have a unique equilibrium despite violating condition (3).

Lemma 2. (1) If (F_1, F_2) are concave, then (3) is satisfied and the equilibrium is unique. (2) If the distributions (F_1, F_2) become concave for high valuations, there exists a pair (κ_1, κ_2) such that, whenever $\underline{v}_i = r + K_i \geq \kappa_i$ for both bidders, the game has a unique equilibrium.[11]

Lemma 2 further characterizes sufficient condition (3). Lemma 2.1 shows that condition (3) is a weak form of CDF concavity. In particular, auctions with concave distributions of valuations (e.g., exponential or generalized Pareto) always have a unique equilibrium. Other distributions, such as beta, gamma, or Weibull, are concave for certain parameters, making condition (3) testable. Many distributions used in applications (such as the log-normal distribution) are concave for sufficiently high valuations. Lemma 2.2 shows that, for these eventually concave distributions, there exist sufficiently high entry costs, or reservation price, guaranteeing equilibrium uniqueness. This last result stands in contrast with traditional complete information intuitions, where large entry costs make entry by both firms unprofitable, leading to coordinated entry and equilibrium multiplicity. With private values, high entry costs (or reservation price) shifts the domain of feasible strategies $[\underline{v}_i, \bar{v}_i]$ to the concave segment of the CDFs, inducing equilibrium uniqueness.

Example 1 (Log-normal valuations). To illustrate the intuition behind strength, herculean equilibrium, and sufficient condition for uniqueness (3), consider a scenario with no reservation price,

[11] The proof of the lemma shows how to find the value κ_i for a given F_i.

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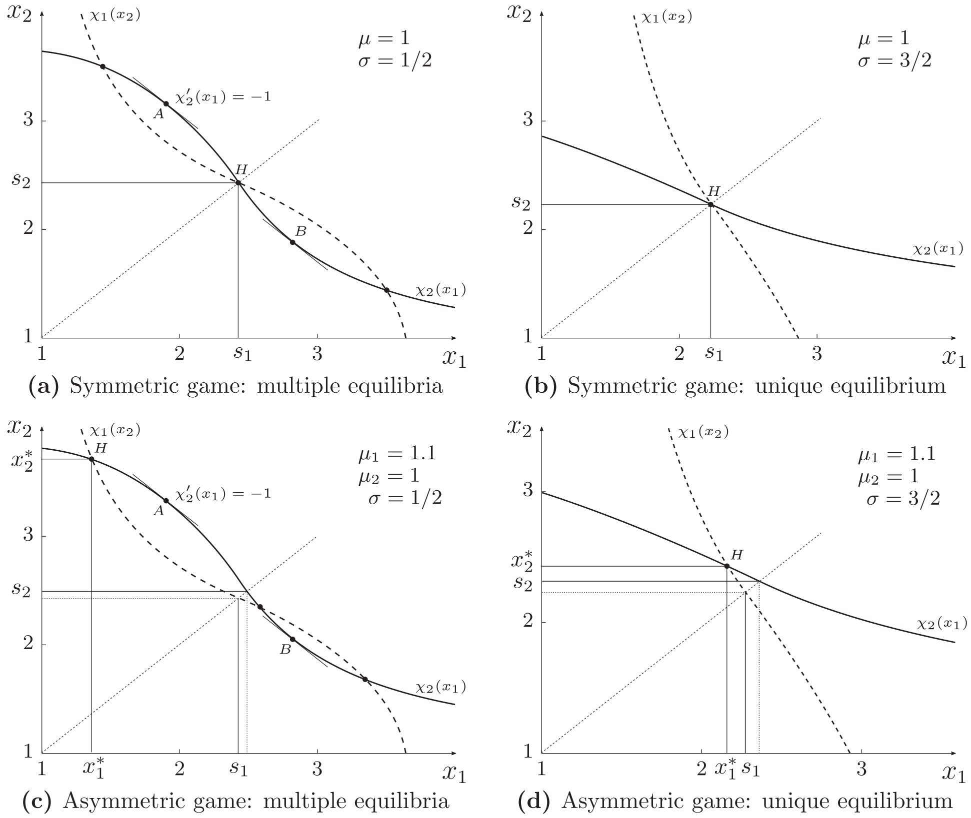
FIGURE 3

STRENGTH AND HERCULEAN EQUILIBRIUM UNDER LOG-NORMAL VALUATIONS



(a) Symmetric game: multiple equilibria **(b)** Symmetric game: unique equilibrium

(c) Asymmetric game: multiple equilibria **(d)** Asymmetric game: unique equilibrium

Note: The figure depicts bidders' best response function $\chi_i(x_j)$, their strength s_i, and the herculean equilibrium H, when valuations distribute log-normal in four different scenarios. Panels (a) and (b) depict symmetric auctions, whereas (c) and (d) asymmetric. Scenarios (a) and (c) have multiple equilibria. Sufficient condition for equilibrium uniqueness (3) does not hold between points A and B. In scenarios (b) and (d), condition (3) does hold and the game has a unique, the herculean, equilibrium.

$r = 0$, two entrants with identical entry cost, K, and valuations that are distributed log-normal with parameters (μ_i, σ). As illustrated by Figure 2a, this distribution family is not concave. Depending on its parameters, the entry game might have multiple or a unique equilibrium.

(a) **Uniqueness under sufficiently high entry costs:** Suppose symmetric bidders with $\mu_i = 1$. Because the log-normal distribution becomes concave for high values, by Lemma 2.2, for each value of σ we can find a threshold κ such that, for every $K \geq \kappa$, sufficient condition (3) holds. Figure 2b depicts the threshold κ and the mass of valuations above κ, as a function of σ. The shaded area represents the set of entry costs K under which the sufficient condition for uniqueness (3) holds. The relation between κ and σ is nonmonotonic, with κ converging to zero when σ is high enough. The proportion of valuations above κ, $\Pr[v \geq \kappa]$, monotonically increases in σ. That is, the larger the dispersion of the distribution, the less demanding the condition for uniqueness becomes. When $\sigma \to 0$, the mass of valuations above κ converges to zero. That is, as the game converges to a complete information game—where equilibrium multiplicity is known to exist—the sufficient condition for uniqueness is never met.

(b) **Multiplicity and uniqueness under symmetry:** We now illustrate the differences between multiple equilibria versus a unique equilibrium. Assume symmetric bidders, with $K = 1$

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and $\mu_i = 1$. Figure 3a,b illustrates bidders' best response functions and equilibria when $\sigma \in \{1/2, 3/2\}$ (see Figure 2a for the CDFs). When $\sigma = 1/2$ (Figure 3a), the auction has three equilibria, as the best responses cross at three different points. The segment between the points A and B highlights bidder 2's violation of the sufficient condition for uniqueness (3), as $-\chi_2'(x_1) > 1$. Because bidders are symmetric, the herculean equilibrium, denoted by H, is symmetric and equal to the bidders' strength.

In contrast, when $\sigma = 3/2$ (Figure 3b), sufficient condition (3) holds—the tuple (K, σ) is in the shaded area of Figure 2b. Best responses are flatter, satisfying $\chi_i'(x_j) < 1$ throughout. The game has a unique equilibrium (the herculean), which is also stable.

(c) **Asymmetric auctions:** We now illustrate strength and the herculean equilibrium in an asymmetric context. We repeat the previous analysis but now allow bidders to differ in μ. Bidder 1 is stronger, as it has higher expected valuations ($\mu_1 = 1.1 > 1 = \mu_2$). Figure 3c,d depicts the bidders' best response functions and the strength of each bidder. Strength is computed where a bidder's best response crosses the 45° line; that is, when $\chi_i(s_i) = s_i$. Because bidder 1 is stronger, a herculean equilibrium must lie above the 45° line. Figure 3c shows that when $\sigma = 1/2$, only one equilibrium is herculean, which is stable. The middle equilibrium is non-herculean and unstable. The other non-herculean equilibrium is stable. Figure 2a shows that as σ increases, the CDF becomes more concave. This flattens best responses and the sufficient condition for equilibrium uniqueness holds (see Figure 3d).

To conclude this example, it is interesting to observe what happens when μ_1 increases. Comparing Figure 3a,c, we can see that increasing the mean of bidder 1's distribution shifts bidder 2's best response upward (same shift can be observed comparing Figure 3b,d). This shift implies that the non-herculean equilibria get closer to each other. When μ_1 is sufficiently high, the upward shift of bidder 2's best response leads best responses to no longer cross to the right of the 45° line, inducing a unique equilibrium. As explained above, sufficient condition (3) fails to capture this mechanism for equilibrium uniqueness. Condition (3) is about the *shape* of best responses, whereas μ_i affect their *scale*.

□ **Auctions with *n* potential bidders.** We now extend Proposition 1 to auctions with n potential bidders. First, we illustrate that the result generalizes to environments in which bidders can be divided into two asymmetric groups. We then explain why our methods do not generally extend to an arbitrary number of groups. We extend the result to environments with an arbitrary number of groups by imposing further structure to the model.

Lemma 3. In an auction with n potential entrants, if two symmetric firms meet sufficient condition (3), they must play the same cutoff strategy in any equilibrium.

Two firms are called symmetric if they have identical entry cost K and distribution of valuations F. In a herculean equilibrium, symmetric firms must play symmetric strategies. Lemma 3 says that, under condition (3), restricting the uniqueness analysis to strategies in which symmetric firms play symmetric strategies is without loss of generality. A corollary of Lemma 3 is that condition (3) guarantees uniqueness in symmetric games with an arbitrary number of bidders.

A sketch of the proof of the Lemma is as follows. Let $\mathbf{x}$ be an equilibrium vector of cutoffs in which symmetric bidders i and j play $x_i < x_j$. Because of symmetry, bidders i and j have identical best response functions, $\chi(\mathbf{x}_{-k})$. Fix the competitors' strategies $\mathbf{x}_{-i,j}$ and, using symmetry, define $\hat{\Pi}(x) = \Pi_k(x; \chi(x, \mathbf{x}_{-i,j}), \mathbf{x}_{-i,j})$ for $k \in \{i, j\}$. A necessary condition for a cutoff x to be an equilibrium is $\hat{\Pi}(x) = 0$. Using the same steps as in the proof of Proposition 1, we can show that condition (3) implies $\partial\hat{\Pi}(x)/\partial x > 0$ for all x. If $\hat{\Pi}(x_i) = 0$, then $\hat{\Pi}(x_j) > 0$, contradicting that $\mathbf{x}$ is an equilibrium.

Two groups of bidders. Consider n bidders divided into two groups $g \in \{1, 2\}$. Each group g consists of n_g bidders, $n_1 + n_2 = n$, characterized by two tuples (F_g, K_g). Although bidders

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are symmetric *within* groups, bidders can be asymmetric *across* groups. The two-group model has been used in applied work when bidders can be divided by exogenous factors. Examples include mills and loggers in the timberwood industry (e.g., Athey et al., 2011), and favored and nonfavored bidders in highway procurement auctions (Krasnokutskaya and Seim, 2011).

Proposition 2. In the two-group model, there always exists a herculean equilibrium. If sufficient condition (3) holds for each group of bidders g, the herculean equilibrium is the unique equilibrium of the game.

Proposition 2 extends Proposition 1 to the two-group scenario. Because a herculean equilibrium prescribes symmetric firms to play symmetric strategies (i.e., group-symmetric strategies) the proof of existence mimics the two-bidder scenario. We define the group-symmetric best response as the best response of a bidder when every bidder in its group plays the same best-response strategy.[12] This definition generates one best response function per group. As before, iterating mutual (group-symmetric) best responses, starting from the bidders' strengths, pulls cutoffs further apart, converging to a herculean equilibrium.

By Lemma 3, restricting the analysis of uniqueness to group-symmetric strategies is without loss of generality. Following the uniqueness proof when $n = 2$, we use sufficient condition (3) to show that the expected profit of a bidder is strictly increasing in its group-symmetric strategy, even after taking into account the opponents' group-symmetric best response. Consequently, the expected profit of a bidder can cross zero once, inducing a unique equilibrium.

Asymmetric bidders. A herculean equilibrium might not exist in environments with $n \geq 3$ asymmetric bidders. We provide an example of nonexistence in Online Appendix D. This lack of existence precludes us from obtaining a general result about equilibrium uniqueness.

Our method of showing that iterated best responses are further apart than the bidders' strength does not extend to environments with $n \geq 3$ asymmetric bidders. The strength order between two bidders might reverse with the *behavior* of a third bidder. This reversal implies that, when iterating best responses of bidders, the best responses are no longer getting further apart, and the process might converge to a non-herculean equilibrium or not converge at all.

Consider an auction with no reservation price, $r = 0$, and three asymmetric bidders satisfying $s_1 < s_2 < s_3$. The bidders differ in their distribution of valuations but have identical entry costs, K. Using equation (2), bidder $i \in \{1, 2\}$ strength is determined by the solution to $s_i F_j(s_i) = K/F_3(s_i)$, see Figure 4. Iterating best responses between bidder 1 and 2 fixing $x_3 = s_3$, will produce best responses that are further apart as in the previous scenarios. Bidder 1's best response decreases in each iteration, and bidder 2's best response increases (as in Figure 1).

Consider now starting the iteration with bidder 3. Its best response to (s_1, s_2) is $\chi_3(s_1, s_2) > s_3$. Recompute the strength of bidders 1 and 2, but assuming that bidder 3 plays $\chi_3(s_1, s_2)$, that is, $\bar{s}_i F_j(\bar{s}_i) = K/F_3(\chi_3(s_1, s_2))$, see Figure 4 . Iterating best responses between bidders 1 and 2, fixing the behavior of bidder 3 at $\chi_3(s_1, s_2)$, might have different outcomes depending on the shape of $F_i(v)$. Panel (a) depicts a situation in which bidders are ordered by FOSD. In this example, the relative strength of bidders 1 and 2 remains invariant. As before, the iteration will mimic the process depicted in Figure 1. Panel (b) shows a scenario where the CDFs of bidders 1 and 2 cross. In contrast to the previous situation, iterating best responses will lead to a sequence of best responses in which bidder 2 will decrease in each iteration and bidder 1 will increase. That is, the process of iterating mutual best responses might not converge or converge to a non-herculean equilibrium. To reestablish our results, we need to impose further structure to the model.

[12] Formally, let $\tilde{\mathbf{x}} = (x_1, \ldots, x_1, x_2, \ldots, x_2)$ be a vector of group-symmetric strategies. Pick any bidder in group $g \in \{1, 2\}$ and let $\tilde{\Pi}_g(x_1, x_2) = \Pi_g(\tilde{\mathbf{x}})$ represent the expected profit of a bidder in group g, entering under valuation x_g, when group-symmetric strategies x_1 and x_2 are played. Define group g's group-symmetric best response $\chi_g(x)$ to be the function that solves $\tilde{\Pi}_g(\chi_g(x), x) = 0$.

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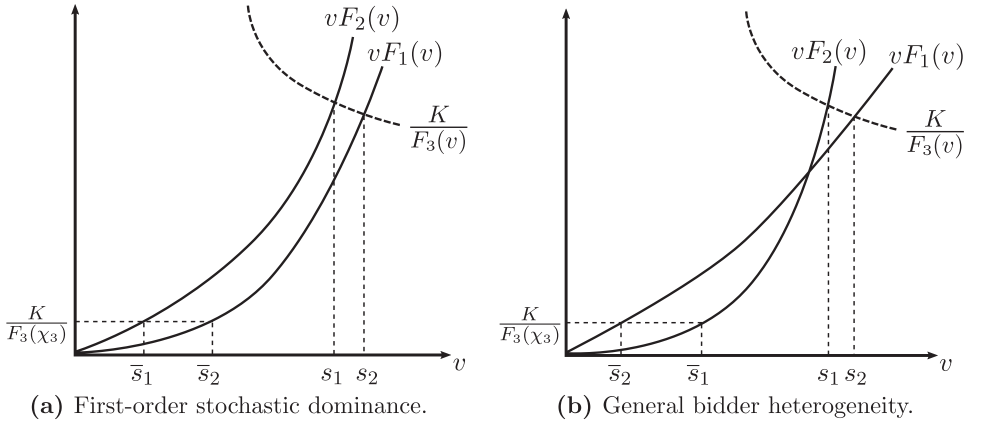
FIGURE 4

STRENGTH AND COMPETITION



Note: Bidder $i \in \{1, 2\}$ strength versus strength when bidder 3's behavior is fixed at $\chi_3(s_1, s_2)$. Panel (a) shows that when the bidders' CDFs are ordered by FOSD, the strength order between bidders 1 and 2 is robust to bidder 3's behavior. Panel (b) shows that when the CDFs cross, the strength order can change with the behavior of the third bidder.

Ordered bidders. We now show that a herculean equilibrium exists in scenarios in which the ranking provided by strength is robust to the opponents' behavior. We call these environments *ordered*.

Definition 3 (Ordered Auction). Let $\underline{v} = \min\{\underline{v}_i\}_{i=1}^n$ and $\overline{v} = \max\{\overline{v}_i\}_{i=1}^n$, where $\underline{v}_i = K_i + r$ and $\overline{v}_i = \chi_i(\underline{v}_{-i})$. An auction is ordered if for any two bidders i and j, with $i < j$, the following condition holds:

$$F_i(v)K_i \leq F_j(v)K_j \text{ for all } v \in [\underline{v}, \overline{v}]. \tag{4}$$

Lemma 4. If condition (4) holds, bidders are ordered by strength with bidder 1 being the strongest bidder.

Ordered environments include, as particular cases, situations in which bidders have: (i) identical entry costs, but distributions of valuations that are ordered by FOSD; or, (ii) identical distributions of valuations and different entry cost. It also allows, with certain restrictions, for bidders that stochastically dominate others but have higher entry costs, as illustrated in the next example.

Example 2. Consider a scenario in which the bidders' distribution of valuations belong to the exponentiated family (see Gupta et al., 1998); that is, $F_i(x) = F(x)^{\theta_i}$ for any distribution F and $\theta_i > 0$. Observe that bidder i first-order stochastically dominates j if and only if $\theta_i > \theta_j$. Let $\theta_i > \theta_j$, using $\overline{v}$ we find that every entry cost $K_i \leq K_j F(\overline{v})^{\theta_j - \theta_i}$ satisfies condition (4) and bidder i is stronger than j, that is, $s_i < s_j$. In particular, when $K_i \in (K_j, K_j F(\overline{v})^{\theta_j - \theta_i}]$, firm i first-order stochastically dominates j and has a higher entry cost.

Proposition 3. In an auction with n asymmetric bidders. If condition (4) holds, a herculean equilibrium exists. In addition, if sufficient condition (3) holds for each bidder i, the herculean equilibrium is the unique equilibrium of the entry game.

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Proposition 3 is neither a particular case nor a generalization of Proposition 2. Although the proposition extends the existence and uniqueness results to the case with n potential bidders, it also requires condition (4).[13]

We prove existence constructively using induction. To sketch the proof, order bidders by strength, with bidder 1 being the strongest. In each step i we show that, taking the best response functions of bidders $\{1, \ldots, i-1\}$ and cutoffs $\mathbf{x}^{i+1} \equiv (x_{i+1}, \ldots, x_n)$ as given, bidder i has a best response $\chi_i(\mathbf{x}^{i+1})$ satisfying $\chi_i(\mathbf{x}^{i+1}) > \chi_{i-1}(\chi_i(\mathbf{x}^{i+1}), \mathbf{x}^{i+1})$ for every $\mathbf{x}^{i+1}$. Thus, regardless of the cutoffs $\mathbf{x}^{i+1}$ chosen by weaker bidders in subsequent steps, the order between bidder's $i-1$ and i cutoffs will remain. This construction uses condition (4) to show the order among best-response functions is robust, delivering a herculean equilibrium at the last step.

We use sufficient condition (3) in the previous iteration to show uniqueness. In each induction step, we show that bidder i's expected payoff is strictly increasing in its cutoff valuation, even after considering best responses of stronger bidders. This monotonicity delivers a unique best-response function $\chi_i(\mathbf{x}^{i+1})$ in each iteration step. We use this property to show that no other herculean equilibrium exists, and that no non-herculean equilibrium is possible.

Sufficient condition (3) has to be checked for each potential bidder, translating into n conditions that need to be satisfied. In ordered environments, however, there are cases in which condition (3) only needs to be checked for a single bidder. First, consider a scenario in which bidders are ordered by entry costs, that is, the distribution of valuations, F, is symmetric among bidders. In this situation, because entry costs do not directly enter condition (3), if the condition holds on $[\underline{v}, \overline{v}]$, the condition would hold for every bidder in the entry game. Consider, also, the scenario in which firms are ordered by FOSD and belong to the exponentiated family $F_i(v) = F(v)^{\theta_i}$ (see Example 2). In this scenario, sufficient condition (3) for bidder i becomes

$$\frac{vf_i(v)}{F_i(v)} = \theta_i \frac{vf(v)}{F(v)} < 1.$$

The condition, thus, only needs to hold for the strongest bidder (highest θ_i).

3. A model of market entry

■ We now generalize the previous framework to include entry games used in the applied literature. We extend the previous results to environments with two groups of firms, with no restrictions on the degree of asymmetry across groups. In Online Appendix G, we extend the results to environments with an arbitrary number of ordered groups.

□ The baseline model.

Set up. Consider n firms simultaneously deciding on whether to enter a market. Firms are privately informed about their *type* v_i (a scalar), summarizing the firm's information about its profitability upon entering the market. Firm i's *post-entry* profit depends on: (i) the entry decision of every firm; (ii) firm i's type; and (iii) the types of other entrants. We assume that the type of firms not entering the market is payoff irrelevant. The type v_i is drawn according to a cumulative distribution function F_i, a continuously differentiable atomless distribution, with full support on $[a, b]$ where $a, b \in \overline{\mathbb{R}}$ (the extended reals). The distributions of types, F_i, are independent across firms but not (necessarily) identically distributed.

Let $E = \{1, 2, \ldots, n\}$ be the set of all potential entrants and $\mathcal{E}$ its power set. The set $\mathcal{E}$ contains every potential market structure that we can observe after entry decisions are made. We denote a (realized) market structure by $e \in \mathcal{E}$. The set e lists all the firms participating in a given market structure, whereas the set $e^c = E \backslash e$ lists all the firms that stay out. Similarly, for any firm $j \in e$, we use $e \backslash j$ to denote the market structure without firm j. Let $\mathcal{E}_i = \{e \in \mathcal{E} : i \in e\}$ be the

[13] Proposition 3 is a generalization of the existence of an ordered equilibrium result in Miralles (2008), which studied a scenario with n bidders ordered by FOSD and symmetric entry costs.

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set of market structures in which firm i enters. Denote by $v_e = (v_j)_{j\in e}$ the vector of realized types for every firm participating in market structure e. For example, $v_E = (v_1, v_2, \ldots, v_n)$ denotes the vector with the realized types of every firm. As a shortcut, we denote by v_{-i} the realized types of every firm except firm i and we write v_i instead of $v_{\{i\}}$ when i is the sole entrant.

Let $\pi_i(v_e)$ be a real valued function representing firm i's *post-entry* profit when the realized market structure is e and the realized types of the participating firms are v_e. To illustrate the workings of the notation, observe that $\pi_i(v_i)$ represents firm i's post-entry profit when i is the sole entrant and its type is v_i. Similarly, $\pi_i(v_E) = \pi_i(v_i, v_{-i})$ represents firm i's profit when every firm enters the market and the vector of realized types is given by v_E. In the SPA example, $\pi_i(v_e) = \max\{0, v_i - \max\{r, v_{e\setminus i}\}\} - K_i$. We normalize the payoff of a nonentrant to zero. Finally, we assume that $\pi_i(v_e)$ is continuous, integrable (with finite expectation) in each dimension of v_e, and differentiable almost everywhere with respect to its first argument v_i. We denote such a derivative by $\pi_i'(v_e)$.

The timing of the game is as follows. Before making any entry decision, each firm privately observes v_i. After observing v_i, each firm independently and simultaneously decides whether to enter the market. After entry decisions are made, market structure e is realized and each firm entering the market gets a payoff $\pi_i(v_e)$. The tuple $(F_i, \pi_i)_{i=1}^n$—which includes the number of potential entrants n—is commonly known to every potential entrant.

Main assumptions. For a given market structure e in which firm i enters the market ($e \in \mathcal{E}_i$), firm i's profit function satisfies the following three properties.

A 1 (Monotonicity). The profit function $\pi_i(v_e)$ is (i) weakly increasing in v_i; and (ii) strictly increasing in v_i when firm i is the sole entrant.

Assumption A1 gives economic meaning to the firms' type. Upon entering the market, firm i's profit (weakly) increases in v_i in any market structure e. A higher v_i can represent a lower marginal cost of production, a lower entry cost, a higher product quality, a better managerial ability, or a higher valuation for a good in an auction. In the SPA example, payoffs are monotone; they increase in v_i when bidder i is the entrant with the highest valuation and are constant in v_i otherwise.

For any market structure $e \in \mathcal{E}_i$, types v_e, and competitor $j \in e$, define firm i's *profit gain* induced by j's exit to be

$$\delta_{i,j}(v_e) = \pi_i(v_{e\setminus j}) - \pi_i(v_e). \tag{5}$$

The function $\delta_{i,j}(v_e)$ represents the increase in profit that firm i attains if firm j exits market structure e under types v_e. In two-player games, $\delta_{i,j}(v_e)$ represents the difference between monopoly and duopoly profits. In an SPA, with two potential bidders and valuations over the reserve price, $\delta_{i,j}(v_i, v_j) = \min\{v_i, v_j\} - r$.

A 2 (Substitutes). For each market structure e and competitor $j \in e$:

(i) $\pi_i(v_e)$ is weakly decreasing in v_j.
(ii) $\delta_{i,j}(v_e) \geq 0$.
(iii) There exists $\hat{v}_j$ such that $v_j \geq \hat{v}_j$ implies $\delta_{i,j}(v_e) > 0$.

Assumption A2 concerns the impact of competition on profits. It states that firms' entry actions are strategic substitutes, as competition decreases profits. In particular, the assumption states that $\pi_i(v_e)$ decreases when bidder i faces: (i) a more productive competitor (higher type v_j), or (ii) entry ($\delta_{i,j}(v_e) \geq 0$). Condition (iii) is a strengthening of (ii). It indicates that, for every

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competitor j and market structure e in which j participates, when j exits with a sufficiently high type, $v_j \geq \hat{v}_j$, firm i's payoffs are strictly larger.[14] An SPA satisfies (i), (ii), and (iii).

Let $\phi(v_e) = \prod_{j\in e} f_j(v_j)$ be the joint density of types of every firm participating in market structure e.

A 3 (Costly and Interior Entry). There exist values $\underline{v}_i < \overline{v}_i$ in the interior of the support of $F_i(v_i)$—that is, $\underline{v}_i, \overline{v}_i \in (a, b)$—such that:

(i) $\pi_i(\underline{v}_i) = 0$.
(ii)

$$\int_{(\underline{v}_j)_{j\in E\setminus i}}^{b} \pi_i(\overline{v}_i, v_{-i})\phi(v_{-i})d^{n-1}v_{-i} = 0,$$

where the multiple integral is over each of the $n-1$ dimensions of v_{-i}.

Assumption A3 concerns the nature of the entry problem. Condition (i) simply states that entry is costly. Firms need a sufficiently good type, $\underline{v}_i > a$, to enter the market as the sole entrant. In an SPA, $\underline{v}_i = r + K_i$, the reserve price plus the bidder's entry cost. Jointly with assumption A2, A3 implies that, when $v_i < \underline{v}_i$, firm i would never choose to enter the market under any market structure. That is, the value $\underline{v}_i$ represents the minimal type required to enter the market.

Condition (ii) states that any firm will enter the market if its type is sufficiently high. There exists a value $\overline{v}_i < b$ such that drawing $v_i > \overline{v}_i$ ensures entry, even if every potential competitor enters the market whenever $v_j \geq \underline{v}_j$. The assumption that $[\underline{v}_i, \overline{v}_i] \subset (a, b)$ guarantees that every equilibrium is interior; that is, no firm chooses to either never enter or always enter the market.

Partial revelation of information. Reinterpreting v_i as a signal and adding an affiliation assumption between the signal and the firms' type, the previous framework also accommodates models in which, before entry, private information is partially revealed to firms. The partial information framework allows for outcomes commonly observed in applied research but precluded in a full information model, such as *ex post* regret.[15]

Let $F_i(v_i, t_i)$ be firm i's joint cumulative distribution of *signals* v_i and *types* t_i with support on $[a, b] \times [c, d]$ with $c, d \in \overline{\mathbb{R}}$. The distributions F_i are independent across firms. Before making their costly entry decisions, a firm privately observes its signal v_i, allowing the firm to make inferences about its true type, t_i. Firms learn their type after entering the market. Let $F_i(v_i) = \int_c^d F_i(v_i, s)ds$ and let $F_i(t_i|v_i) = F_i(v_i, t_i)/F_i(v_i)$ be the CDF of t_i conditional on v_i.

A 4 (Affiliated Signals). For $v_i' > v_i$, $F_i(t_i|v_i') < F_i(t_i|v_i)$ for all t_i.

Assumption A4 states that higher signals lead to a higher expected type in terms of FOSD (cf. Gentry and Li, 2014; Marmer et al., 2013). Let $\tilde{\pi}_i(t_e)$ be firm i's profit under market structure e and vector of types for participating firms $t_e = (t_j)_{j\in e}$. Let n_e be the number of entrants in market structure e. Then, we reinterpret $\pi_i(v_e)$ as

$$\pi_i(v_e) = \int_c^d \tilde{\pi}_i(t_e)\prod_{k\in e} f_k(t_k|v_k)d^{n_e}t_e,$$

where the multidimensional integral is across each of the n_e dimensions of t_e. Given the properties of FOSD, if the profit function $\tilde{\pi}_i(t_e)$ satisfies analogous conditions to A1–A3, then $\pi_i(v_e)$ satisfies A1–A3 and the results below will hold.

[14] We could dispense of A2(iii) for our results, but we adopt it for brevity in the proofs.

[15] An example of *ex post* regret is bidders that pay the entry cost and submit bids below the reserve price (or do not submit bids), after updating their beliefs downward upon learning their true type.

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Example 3. To illustrate the breadth of models captured by assumptions A1–A4, we show it embeds two frameworks commonly used in applied work.

(a) **Linear model:** We say that the profit function is linear when

$$\pi_i(v_e) = \eta_i - \delta_i \cdot (n_e - 1) + v_i,$$

where η_i is a scalar summarizing both market and firm characteristics, and $\delta_i > 0$ is a parameter capturing how entry affects firm i's profits. [16] In this model, only firm j's entry decision, not its type, affects firm i's payoff. A common interpretation of the private information in the linear model is that $-v_i$ represents firm i's entry cost.[17]

(b) **SPA with partial information:** Consider an SPA in which bidders are partially informed about their valuations before entry.[18] Bidder i's valuation (or type) is given by $t_i = v_i\varepsilon_i$, where $v_i \sim F_i$ is the *signal* observed before the entry decision is made and ε_i is the *noise* observed after entry but before submitting a bid. We assume $\varepsilon_i \sim G$ with support in $[0, \infty)$ and is independent from v_i.
The expected payoff of a bidder entering with a signal v_i, when participating competitors observe signals $v_{e\setminus i}$ is:

$$\pi_i(v_e) = \int_{r/v_i}^{\infty} \left(\int_0^{v_i\varepsilon_i} (v_i\varepsilon_i - \max\{r, s\}) d\Psi_i(s, v_e) \right) dG(\varepsilon_i) - K_i.$$

Given the signal v_i, bidder i values the good in $t_i = v_i\varepsilon_i$, where ε_i has a cumulative distribution function of $G(\varepsilon_i)$. Entrants submit a bid equal to their valuation. Bidder i obtains the good when it is the highest valuation firm and pays the maximum between the opponents' valuation, s, and the reserve price, r. Conditional on $v_{e\setminus i}$, the maximal valuation among i's opponents has a CDF equal to $\Psi_i(s, v_e) = \prod_{j\in e\setminus i} G(s/v_j)$.

□ Strategies, payoffs, and equilibrium.

Payoffs and strategies. A *cutoff* strategy for firm i is a threshold x_i such that firm i enters the market whenever $v_i \geq x_i$ and stays out otherwise. Firm i's expected profit of entering the market with type v_i when facing opponents playing cutoffs $\mathbf{x}_{-i}$ is

$$\begin{aligned}\Pi_i(v_i; \mathbf{x}_{-i}) &= \mathbb{E}_{\varepsilon_i}\left[\mathbb{E}_{v_{-i}}[\pi_i(v_e) | v_{-i} \geq \mathbf{x}_{-i}] | \mathbf{x}_{-i}\right] \\ &= \sum_{e\in\mathcal{E}_i} \left\{ \left(\prod_{j\in e^c} F_j(x_j) \right) \int_{x_{e\setminus i}}^{b} \pi_i\big(v_i, v_{e\setminus i}\big)\phi(v_{e\setminus i}) d^{n_e-1} v_{e\setminus i} \right\}, \end{aligned} \tag{6}$$

where n_e is the number of entrants in market structure e.

Firm i's expected profit consists of an iterated expectation. First, given the opponents' strategy $\mathbf{x}_{-i}$, the outer expectation is over each market structure in which firm i participates, $e \in \mathcal{E}_i$. Then, for a given market structure e, the inner expectation is over the realization of types for every competitor v_{-i}, conditional on their type being above their entry cutoff. Expression (6) is the general analog of equation (1). Appendix B shows that (6) is strictly increasing in firm i's type v_i and in an opponent's cutoff, x_j. A higher entry cutoff x_j lowers the competitor's probability of entry, inducing firm i to face less competition.

[16] Although the term η_i is commonly known by the firms', an econometrician may not observe some elements in η_i. Typically, $\eta_i = X_i\beta_i + \zeta_i$, where X_i is a vector of observed firm and market characteristics and ζ_i is unobserved by the econometrician.

[17] Examples linear entry models with private information include: Seim (2006); Aguirregabiria and Mira (2007); Bajari et al. (2007); Pakes et al. (2007); Pesendorfer and Schmidt-Dengler (2008); Sweeting (2009); Aradillas-Lopez (2010); Bajari et al. (2010); Krasnokutskaya and Seim (2011); De Paula and Tang (2012); Vitorino (2012); Mazzeo et al. (2016).

[18] In the context of auctions, the partial information model been studied by Roberts and Sweeting (2013, 2016), Gentry and Li (2014), and Sweeting and Bhattacharya (2015).

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Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be a vector with cutoff strategies for every firm. A Bayesian *equilibrium* is a vector $\mathbf{x}$ such that $\Pi_i(\mathbf{x}) \equiv \Pi_i(x_i; \mathbf{x}_{-i}) = 0$ for every firm i. Online Appendix C shows that an equilibrium always exists and that *every* equilibrium is in cutoff strategies; that is, focusing on cutoff strategies is without loss of generality. We denote the partial derivative of $\Pi_i(\mathbf{x})$ with respect to x_i by $\Pi_i'(\mathbf{x})$.

Strength and herculean equilibrium. We now extend the notion of *strength* to the general framework. Strength uses the game fundamentals, $(F_i, \pi_i)_{i=1}^n$, to rank firms according to their ability to endure competition. As before, we use the firms' strength to identify the equilibrium that remains when the game has a unique equilibrium, the herculean equilibrium.

Definition 4 (Strength). The *strength* of firm i is the unique number $s_i \in \mathbb{R}$ that solves $\Pi_i(s_i; s_i, \ldots, s_i) = 0$, where $\Pi_i(\mathbf{x})$ is given by (6). We say that firm i is *stronger* than firm j if $s_i < s_j$.

Lemma 5. $\Pi_i(s_i; s_i, \ldots, s_i)$ is strictly increasing in s_i, crossing zero once.

The strength of firm i is the unique cutoff s_i that best responds to every competitor playing the same cutoff strategy s_i. A lower value of strength for firm i ($s_i < s_j$) indicates that firm i, despite facing more competition than j (i faces competitors with lower entry cutoffs), is more likely than j to enter the market (i plays a lower entry cutoff). Lemma 5 shows that strength is well defined, as it assigns a unique scalar s_i to each firm i, delivering a complete ranking of the firms. We call an equilibrium *herculean* if equilibrium cutoffs are ordered by strength, with stronger firms playing *lower* cutoffs.

The next definition is instrumental to characterize the sufficient conditions for equilibrium uniqueness.

Definition 5 (Expected Profit Gain). For any vector of cutoff strategies $\mathbf{x}$ define firm i's *expected profit gain* induced by firm j's exit to be

$$\hat{\Delta}_{i,j}(\mathbf{x}) = \sum_{e \in \mathcal{E}_i \setminus \mathcal{E}_j} \left\{ \left(\prod_{k \in e^c \setminus j} F_k(x_k) \right) \int_{(x_k)_{k \in e \setminus i}}^{b} \delta_{i,j}\left(x_i, x_j, v_{e\setminus i}\right) \phi(v_{e\setminus i}) d^{n_e - 1} v_{e\setminus i} \right\}, \tag{7}$$

where $\delta_{i,j}(v_e) \geq 0$ is firm i's *profit gain* induced by firm j's exit in market structure e with realized types v_e, as defined in (5).

Given a vector of cutoff strategies $\mathbf{x}$, firm i's expected profit gain induced by firm j's exit, $\hat{\Delta}_{i,j}(\mathbf{x})$, is the probability weighted sum over market structures that firm j can exit of firm i's profit gains due to j's exit, $\delta_{i,j}(v_e)$, integrated over every feasible realization of the competitors' type. The expected profit gain relates to the increase in profit that a firm experiences when a competitor (firm j) marginally increases its entry cutoff.[19] Expression (7) will help us characterize the shape of best responses and, consequently, when the entry game has a unique equilibrium. In an environment with two potential entrants, the expected profit gain equals the profit gain due to firm j not participating. In the context of an SPA, $\hat{\Delta}_{i,j}(\mathbf{x}) = \delta_{i,j}(x_i, x_j) = \min\{x_i, x_j\} - r$. Although assumption A2(ii) only implies that $\delta_{i,j}(v_e) \geq 0$, together with assumption A2(iii) we have that $\hat{\Delta}_{i,j}(\mathbf{x}) > 0$.

□ **Uniqueness with two groups of firms.** We now generalize our results to games where entrants can be divided into two groups according to their public characteristics. Firms are symmetric within their group. Across groups, firms can differ in their distribution of types and profit

[19] In particular, Lemma B2 shows that $\partial \Pi_i(\mathbf{x})/\partial x_j = f_j(x_j)\hat{\Delta}_{i,j}(\mathbf{x})$. An increase in x_j leads firm j to change its entry behavior with probability $f_j(x_j)$, inducing firm i to gain $\hat{\Delta}_{i,j}(\mathbf{x})$.

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functions. The two-group structure may arise naturally in applications where firms can be divided into incumbents and entrants, high- and low-quality firms, local and international producers, discount and traditional retailers, or legacy and low-cost airlines, among other examples.

Firms belong to one of two groups $g \in \{1, 2\}$. Group g consists of n_g potential entrants ($n_1 + n_2 = n$) described by the pair (π_g, F_g). For any firm i, let $g(i)$ represent firm i's group. We assume that profits are *symmetric* and *anonymous* within a group. That is, for every firm i, its profit under market structure e and realized types v_e is equal to $\pi_i(v_e) = \pi_{g(i)}(v_i, \mathbf{v}_r, \mathbf{v}_k)$, where r and k are the number of entrants in e, other than i, from group $g(i)$ and $-g(i)$, respectively. The vectors $\mathbf{v}_r$ and $\mathbf{v}_k$ represent the types of such entrants. A strategy is called group symmetric if for each firm i, $x_i = x_{g(i)}$. Without loss of generality, let group 1 be the strongest group.

Proposition 4. Let $\Delta_{i,j}(\mathbf{x}) = F_j(x_j)\hat{\Delta}_{i,j}(\mathbf{x})$. A herculean equilibrium always exists. The herculean equilibrium satisfies $x_1 < s_1 < s_2 < x_2$, where s_g and x_g are the strength and the equilibrium cutoff of group g. Furthermore, the game has a unique equilibrium if, for every firm i and each opponent j, conditions[20]

$$\frac{f_i(x_i)}{F_i(x_i)}\frac{\Delta_{i,j}(\mathbf{x})}{\Pi_i'(\mathbf{x})} < 1 \qquad \text{if } g(j) = g(i), \tag{8}$$

$$n_{-g(i)}\frac{f_i(x_i)}{F_i(x_i)}\frac{\Delta_{i,j}(\mathbf{x})}{\Pi_i'(\mathbf{x})} < 1 \qquad \text{if } g(j) \neq g(i), \tag{9}$$

hold for every vector $\mathbf{x}$ such that each dimension k satisfies $x_k \in [\underline{v}_{g(k)}, \overline{v}_{g(k)}]$.

Proposition 4 extends the existence of a herculean equilibrium result to the general framework in the two-group model. The proposition also provides bounds on the herculean equilibrium cutoffs, $x_1 \in (\underline{v}_1, s_1)$ and $x_2 \in (s_2, \overline{v}_2)$. As in the auction example, iterating mutual best responses, starting from the firms' strengths, will lead to a herculean equilibrium.

Proposition 4 also provides four conditions that need to be satisfied for equilibrium uniqueness—two conditions per group. The within-group condition (8) guarantees that, in equilibrium, firms only play group-symmetric strategies. The cross-group condition (9), on the other hand, guarantees that the herculean equilibrium is the only group-symmetric equilibrium of the game. Condition (9) bounds firm i's best response due to a group-symmetric deviation from the opposing group, $-g(i)$. Observe that the left-hand side of condition (9) is multiplied by the number of firms in group $-g(i)$. In group-symmetric strategies, there are $n_{-g(i)}$ opponents deviating simultaneously; thus, the condition needs to bound $n_{-g(i)}$ deviations at the same time. Comparing conditions (8) and (9), we can see that the former condition does not directly depend on $n_{g(i)}$. We can exploit the within-group symmetry among firms to obtain a bound that does not depend on the number of participants in the firm's group.

The following corollary is an immediate implication of Proposition 4 in the context of symmetric entry games.

Corollary 1. If sufficient condition (8) holds, a symmetric entry game has a unique equilibrium.

Example 4. Below we illustrate how to apply Proposition 4 in the context of the models introduced in Example 3.

(a) **Linear model:** When studying entry of supercenters into rural grocery markets, Grieco (2014) estimates a symmetric linear model with incomplete information and two potential entrants and $v_i \sim N(0, 1)$ (see Example 3a). In the smallest market, where coordination among entrants is more relevant and equilibrium multiplicity is more likely to emerge, the

[20] Conditions (8) and (9) can hold with equality if one group is strictly stronger than the other.

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model estimates are given by $\eta = -3.838$ and $\delta = 0.851$.[21] In this context, conditions (8) and (9) collapse into one, becoming

$$\delta F(x_j)\frac{f(x_i)}{F(x_i)} < 1 \text{ for } x_i, x_j \in [\underline{v}, \overline{v}].$$

Because the normal distribution is log-concave, $f(x_i)/F(x_i)$ decreases in x_i.[22] Consequently, it is sufficient to check the condition at x_i's lower bound and x_j's upper bound. Using the model estimates, we find $\delta F(\overline{v})f(\underline{v})/F(\underline{v}) = 10^{-4} < 1$. The condition is satisfied, and the equilibrium is unique.

(b) **SPA with partial information:** Roberts and Sweeting (2013, 2016) use an SPA with partial revelation of information model to study the United States Forest Service (USFS) timber auctions (see Example 3b). The auction consists of two groups of potential entrants, millers and loggers (groups 1 and 2, respectively). Before entry, each firm observes a signal v_i. For the representative (mean) auction, they estimate $\ln v_i \sim N(\mu_{g(i)}, 1.19)$, with $\mu_1 = 3.9607$ and $\mu_2 = 3.5824$. The estimated (symmetric) entry cost is \$2.0543/mfb (dollars per thousand board foot) and the auction's reserve price is \$27.77/mfb.[23] Searching numerically, they found a single equilibrium. We prove that the representative auction indeed has a unique equilibrium when $n_1 = n_2 = 1$. In Online Appendix F, we show that condition (3) implies conditions (8) and (9) in this context. Under log normality $v_i f_i(v_i)/F_i(v_i)$ is decreasing in v_i. Thus, condition (3) only needs to hold at $\underline{v}_i$. Then, $\underline{v}_1 f_1(\underline{v}_1)/F_1(\underline{v}_1) = 0.9436 < 1$ and $\underline{v}_2 f_2(\underline{v}_2)/F_2(\underline{v}_2) = 0.7568 < 1$, and the game has a unique equilibrium.

Extensions. The Online Appendix presents two extensions of the previous result.

A weaker sufficient condition. Observe that sufficient condition (9) becomes more demanding with an increase in the number of potential entrants—that is, ignoring the effects that the number of entrants has on the expected profit gain, $\hat{\Delta}_{i,j}(\mathbf{x})$. In Online Appendix E, we show that if the expected profit gain satisfies a condition analogous to supermodularity, we can relax Proposition 4 to only require sufficient condition (8) for every competitor, regardless of the group they belong to. In the Appendix, we also show that the supermodularity condition is satisfied in SPAs (Section 2) and in the linear model introduced in Example 3a. Consequently, in those environments the sufficient condition for uniqueness does not become more demanding as the number of potential entrants increase.

N groups of ordered entrants. Online Appendix G also extends our existence of herculean equilibrium and uniqueness results to an arbitrary number of entrants if, similar to the analysis at the end of Section 2, the environments are *ordered*. The Appendix also discusses scenarios in which the set of sufficient conditions can be reduced into a single condition, providing examples.

4. Concluding remarks

■ This article studies equilibrium uniqueness in static entry games with single-dimensional private information. To this end, we introduce the notions of strength and herculean equilibrium. We show that a herculean equilibrium always exists and develop sufficient conditions guaranteeing equilibrium uniqueness. The proposed framework embeds many models studied in the applied entry literature, accommodating firm heterogeneity and selection. With the aid of strength, we identify the herculean equilibrium. Strength can reduce the computational burden of calculating equilibria with heterogeneous firms, as it provides bounds for the herculean equilibrium.

[21] See Table 7, page 329: $\eta = \mu_0 - \mu_4 = -1.222 - 2.158 = -3.838$. The sufficient also holds for every other specification in the article.

[22] If $G(x) = \ln(F(x))$ is concave, then $G''(x) = \partial(f(x)/F(x))/\partial x < 0$.

[23] In their model, $v_i = \theta_i \varepsilon_i$ where $\ln \theta_i \sim N(\mu_{g(i)}, 0.3321)$ and $\ln \varepsilon_i \sim N(0, 0.8579)$. See Tables 3 or 4 in Roberts and Sweeting (2013, 2016), respectively.

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We use our sufficient conditions jointly with the estimates in empirical studies on the literature to illustrate the application of these conditions. We show that their empirical models have a unique equilibrium.

This article focuses on entry games when firms' entry decisions are strategic substitutes. In games of strategic complements (i.e., when entry becomes more desirable when other firms enter), the restriction to cutoff strategies remains without loss of generality. However, a symmetric entry game under strategic complementarity might have multiple symmetric equilibria (see Brock and Durlauf, 2001; Sweeting, 2009). Consequently, because strength coincides with symmetric equilibrium cutoffs in symmetric games, strength might not be uniquely defined in these types of games. Strategic complementarity also hinders the existence of a unique equilibrium. We can show that a strategic complement analogous to sufficient condition (8) delivers a unique equilibrium in the context of two firms. However, our methods do not directly extend to those cases when more than two competitors are present.

The focus of this article is on static entry games with private information. We emphasize developing a framework that embeds most of the applied work on endogenous market formation. Beyond the presented results, we see these new developments as the starting point for studying equilibrium uniqueness in dynamic entry games with incomplete information. We hope the tools developed here enable further research in dynamic environments.

Appendix A: Omitted Proofs

Proof of Lemma 1. The result follows from Lemma 4, as both scenarios satisfy condition (4). □

Proof of Proposition 1. It follows from Proposition 2 when $n_1 = n_2 = 1$. □

Proof of Lemma 2. The proof of both statements make use that a concave differentiable function is bounded above by its first-order Taylor approximation; that is, for every x and y such that $x > y$

$$F(x) - F(y) \geq (x - y)f(x). \tag{A1}$$

The first claim follows from taking $y = 0$ and using $F(0) = 0$.

For the second statement, let y in equation (A1) be the inflection point under which $F_i(v)$ becomes concave. Because of concavity, $F_i''(x) \leq 0$, and $f_i(x)$ is nonincreasing for every $x \geq y$. Because $F_i(v)$ is bounded above by 1, $f_i(x)$ converges to zero as x goes to infinity. If $F_i(y) \leq f_i(y)y$, let $\kappa_i \geq y$ be the valuation that satisfies $F_i(y) = f_i(\kappa_i)y$. Then, for every $x \geq \kappa_i \geq y$ we have:

$$F_i(x) \geq xf_i(x) + F_i(y) - yf_i(x) \geq xf_i(x) + F_i(y) - yf_i(\kappa_i) = xf_i(x),$$

and the inequality hods. If $F_i(y) > f_i(y)y$, let $\kappa_i = y$ and for every $x \geq \kappa_i$ we have: $F_i(x) \geq xf_i(x) + F_i(y) - yf_i(x) \geq xf_i(x)$, proving the result. □

Proof of Lemma 3. It follows from Lemma B3 in Appendix B. □

Proof of Proposition 2. Using equation (1) and the definition of equilibrium we begin by observing that every herculean equilibrium is characterized by cutoffs $x_1 \leq x_2$ that jointly solve

$$(x_1 - r)F_1(x_1)^{n_1-1}F_2(x_2)^{n_2} = K_1$$

$$F_2(x_2)^{n_2-1}\left[x_2F_1(x_2)^{n_1} - rF_1(x_1)^{n_1} - \int_{x_1}^{x_2} v d(F_1(v)^{n_1})\right] = K_2.$$

Existence. By construction. If $s_1 = s_2 = s$ there is a herculean equilibrium with cutoffs $x_1 = x_2 = s$. Assume $s_1 < s_2$. By Lemma 3, focusing on group-symmetric strategies is without loss of generality. Let $\chi_1(x)$ be group 1's group-symmetric best response to group 2 playing the group-symmetric strategy x. By definition of strength, $\chi_1(s_1) = s_1$. Using implicit differentiation it can be checked that $\chi_1'(x) < 0$ (see uniqueness proof below). Define $\hat{\Pi}_2(x) = \Pi_2(\chi_1(x); x)$ to be the expected profit of a firm in group 2 when it draws valuation x, every other firm in group 2 plays the group-symmetric strategy x, and group 1 best responds with the group-symmetric cutoff $\chi_1(x)$, that is

$$\hat{\Pi}_2(x) = F_2(x)^{n_2-1}\left[xF_1(\bar{x})^{n_1} - rF_1(\chi_1(x))^{n_1} - \int_{\chi_1(x)}^{\bar{x}} v d(F_1(v)^{n_1})\right] - K_2,$$

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where $\bar{x} = \max\{\chi_1(x), x\}$. An equilibrium (x_1, x_2) is given when $\hat{\Pi}_2(x_2) = 0$ and $x_1 = \chi_1(x_2)$. Observe that $x_2 \in (s_1, \infty)$ is necessary and sufficient for a herculean equilibrium (i.e., for $x_1 < x_2$). This follows from $\chi_1(x)$ being decreasing in x and $\chi_1(s_1) = s_1$. Then, $x_1 = \chi_1(x_2) < x_2$ if and only if $x_2 \in (s_1, \infty)$.

We prove existence by the intermediate value theorem. By the bounded expectation assumption, $\hat{\Pi}_2(x)$ is unbounded above. Hence, because $\hat{\Pi}_2(x)$ is continuous, it is sufficient to show that $\hat{\Pi}_2(s_1) < 0$. This follows from observing

$$\hat{\Pi}_2(s_1) = \Pi_2(s_1; s_1) < \Pi_2(s_2; s_2) = 0,$$

where the inequality follows from $\Pi_2(s; s)$ being increasing in s (by Lemma B2) and the definition of strength, s_2.

Uniqueness. We show that the function $\hat{\Pi}_2(x)$ is strictly increasing, thus it can cross zero only once. We derive the proof in two parts: (i) There exists a unique herculean equilibrium, that is, $\hat{\Pi}_2(x)$ is strictly increasing in x when $x > s_1$; and (ii) There is no equilibrium in which $x_2 < x_1$, that is, $\hat{\Pi}_2(x)$ is strictly increasing in x when $x < s_1$.[24]

To prove part (i) we start differentiating $\hat{\Pi}_2(x)$ in the scenario when $x > s_1$

$$\hat{\Pi}_2'(x) = F_2(x)^{n_2-1}\Bigg\{F_1(x)^{n_1} + n_1\chi_1'(x)(\chi_1(x) - r)f_1(\chi_1(x))F_1(\chi_1(x))^{n_1-1}$$
$$+(n_2 - 1)\frac{f_2(x)}{F_2(x)}\left[xF_1(x)^{n_1} - rF_1(\chi_1(x))^{n_1} - \int_{\chi_1(x)}^{x} yd(F_1(y)^{n_1})\right]\Bigg\}.$$

Because $F_2(x)^{n_2-1} > 0$, it is sufficient to show that the term in braces is nonnegative for all $x \geq s_1$. Implicitly differentiating $\chi_1(x)$ using $\Pi_1(\chi_1(x); x) = 0$

$$\chi_1'(x) = -\frac{n_2(\chi_1(x) - r)F_1(\chi_1(x))}{F_1(\chi_1(x)) + (n_1 - 1)(\chi_1(x) - r)f_1(\chi_1(x))}\frac{f_2(x)}{F_2(x)} < 0$$

replacing into the expression in braces delivers

$$(n_2 - 1)\frac{f_2(x)}{F_2(x)}\left[xF_1(x)^{n_1} - rF_1(\chi_1(x))^{n_1} - \int_{\chi_1(x)}^{x} yd(F_1(y)^{n_1})\right]$$
$$+\left[F_1(x)^{n_1} - \frac{n_1n_2(\chi_1(x) - r)^2f_1(\chi_1(x))F_1(\chi_1(x))^{n_1}}{F_1(\chi_1(x)) + (n_1 - 1)(\chi_1(x) - r)f_1(\chi_1(x))}\frac{f_2(x)}{F_2(x)}\right]. \tag{A2}$$

We show that a lower bound for the expression above is always positive. Maximize the subtracting integral term in the first square brackets by taking the upper bound $x\int_{\chi_1(x)}^{x} dF_1(y)^{n_1}$ in the integral to obtain

$$xF_1(x)^{n_1} - rF_1(\chi_1(x))^{n_1} - x(F_1(x)^{n_1} - F_1(\chi_1(x))^{n_1}) = (x - r)F_1(\chi_1(x))^{n_1} > 0.$$

Because $r \geq 0$, sufficient condition (3) implies

$$(x - r)f_i(x) \leq xf_i(x) \leq F_i(x). \tag{A3}$$

Using this observation, we maximize the subtracting term in the second square brackets by substituting $F_1(\chi_1(x))$ for $(\chi_1(x) - r)f_1(\chi_1(x))$ in the denominator. Then, equation (A2) becomes

$$F_1(\chi_1(x))^{n_1}\left[\left(\frac{F_1(x)^{n_1}}{F_1(\chi_1(x))^{n_1}} - 1\right) + (n_2 - 1)(x - \chi_1(x))\frac{f_2(x)}{F_2(x)}\right] > 0,$$

where $x > \chi_1(x)$ for $x > s_1$ was used to obtain the inequality. Hence the lower bound of (A2) is positive and $\hat{\Pi}_2(x)$ is increasing in x.

To prove part (ii) we differentiate $\hat{\Pi}_2(x)$ when $x < s_1$ (i.e., $x < \chi_1(x)$)

$$\hat{\Pi}_2'(x) = F_1(\chi_1)^{n_1}F_2(x)^{n_2-1}\left[1 + (x - r)\left((n_2 - 1)\frac{f_2(x)}{F_2(x)} + n_1\chi_1'\frac{f_1(\chi_1)}{F_1(\chi_1)}\right)\right],$$

where we used χ_1 instead of $\chi_1(x)$ to ease the notation. We show that a lower bound of $\hat{\Pi}_2'(x)$ is positive. We start by deriving a lower bound for $\chi_1'(x)$. Implicitly differentiating $\chi_1(x)$ using $\Pi_1(\chi_1(x); x) = 0$ we get that $\chi_1'(x)$ is equal to:

$$\frac{-n_2(x - r)F_2(x)^{n_2-1}f_2(x)}{F_2(\chi_1)^{n_2} + (n_1 - 1)\big(\chi_1F_2(\chi_1)^{n_2} - rF_2(x)^{n_2} - \int_x^{\chi_1} vd(F_2(v)^{n_2})\big)\frac{f_1(\chi_1)}{F_1(\chi_1)}} < 0.$$

First, maximize the subtracting integral term in the denominator by taking the upper bound $\chi_1(x)\int_x^{\chi_1(x)} dF_2(y)^{n_2} = \chi_1(x)(F_2(\chi_1(x))^{n_2} - F_2(x)^{n_2})$. Then, rearranging, the denominator becomes

$$F_2(\chi_1)^{n_2}\left(1 + (n_1 - 1)(\chi_1 - r)\frac{f_1(\chi_1)}{F_1(\chi_1)}\right) \geq n_1(\chi_1 - r)\frac{f_1(\chi_1)}{F_1(\chi_1)}F_2(\chi_1)^{n_2},$$

[24] By $s_1 < s_2$, we already know that $x_2 = s_1$ is not an equilibrium.

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where in the inequality we used property (A3), which is implied by sufficient condition (3). Then, substituting in the denominator

$$0 > \chi_1'(x) \geq -\frac{n_2}{n_1}\frac{(x-r)}{(\chi_1(x)-r)}\frac{F_1(\chi_1(x))}{f_1(\chi_1(x))}\frac{f_2(x)}{F_2(x)} > -\frac{n_2}{n_1}\frac{F_1(\chi_1(x))}{f_1(\chi_1(x))}\frac{f_2(x)}{F_2(x)},$$

where in the last inequality we used that, by assumption, $x < \chi_1(x)$. Replacing $\chi_1'(x)$ into $\hat{\Pi}_2'(x)$, we obtain

$$\hat{\Pi}_2'(x) > F_1(\chi_1)^{n_1}F_2(x)^{n_2-1}\left[1-(x-r)\frac{f_2(x)}{F_2(x)}\right] \geq 0,$$

where in the last inequality we used that sufficient condition (3) implies $1 \geq (x-r)f_2(x)/F_2(x)$, proving the result. □

Proof of Lemma 4. By definition of i's strength $(s_i - r)\prod_{j\neq i}F_j(s_i) = K_i$. Equation (4) implies $K_{i+1}F_{i+1}(s_i)/F_i(s_i) \geq K_i$. Substituting for K_i on the RHS of i's strength and rearranging: $(s_i - r)\prod_{j\neq i+1}F_j(s_i) \leq K_{i+1}$. Because the LHS is increasing in s_i, $s_{i+1} \geq s_i$. □

Proof of Proposition 3. *Existence*: For a given vector $\mathbf{v} = (v_1, \ldots, v_n)$, let $\mathbf{v}_i = (v_1, \ldots, v_i)$ represent the elements of $\mathbf{v}$ from the 1st to the ith dimension and $\mathbf{v}^i = (v_i, \ldots, v_n)$ the elements from the ith to the nth. Start by ordering bidders by strength, with bidder 1 being the strongest and n the weakest. Recall equation (1), $\Pi_i(x_i; \mathbf{x}_{-i}) = A_i^n R_i(x_i; \mathbf{x}_{i-1}) - K_i$. An equilibrium $\mathbf{x} = (x_1, \ldots, x_n)$ exists if and only if $\Pi_i(\mathbf{x}) \equiv \Pi_i(x_i; \mathbf{x}_{-i}) = 0$ holds for every i.

We construct a herculean equilibrium $\mathbf{x}$ recursively. We start by constructing x_1 as a function of $\mathbf{x}^2$. For any vector $\mathbf{x}^2$, define $\chi_1(\mathbf{x}^2)$ to be the value of x_1 that solves $\Pi_1(x_1; \mathbf{x}^2) = 0$; that is, $x_1 = r + K_1/A_1^n$.

Construct x_2 using $\chi_1(\mathbf{x}^2)$. By substituting $\chi_1(\mathbf{x}^2)$ in for the value of x_1 in $\Pi_2(\mathbf{x})$, we can write $\hat{\Pi}_2(\mathbf{x}^2) = \Pi_2(x_2; \chi_1(\mathbf{x}^2), \mathbf{x}^3)$ which is a function of $\mathbf{x}^2$ only. That is, with a slight abuse of notation, $\hat{\Pi}_2(\mathbf{x}^2) = A_2^n R_2(\mathbf{x}^2) - K_2$ where

$$R_2(\mathbf{x}^2) = R_2(\chi_1(\mathbf{x}^2); x_2) = x_2F_1(x_2) - rF_1(\chi_1(\mathbf{x}^2)) - \int_{\chi_1(\mathbf{x}^2)}^{x_2} v\,dF_1(v)$$

is the revenue function $R_2(\mathbf{x}_2)$ after replacing the function $\chi_1(\mathbf{x}^2)$ for the value of x_1. The finite expectation assumption implies that $\hat{\Pi}_2(x_2, \mathbf{x}^3)$ is unbounded above in x_2. Fix any $\mathbf{x}^3$. Define $\hat{x}_2$ to be the *largest* value of x_2 that satisfies $\hat{x}_2 = \chi_1(\hat{x}_2, \mathbf{x}^3)$. Observe that $\hat{x}_2$ always exists, as $x_2 \in \mathbb{R}_+$ and $\chi_1(x_2, \mathbf{x}^3)$ is a continuous function of x_2 with range in $(r + K_1, \overline{v}_1)$. Also, for every $x_2 > \hat{x}_2$, $x_2 > \chi_1(x_2, \mathbf{x}^3)$. Otherwise, x_2 and $\chi_1(x_2, \mathbf{x}^3)$ would cross again and $\hat{x}_2$ would not be the largest crossing point.

Using $\hat{x}_2 = \chi_1(\hat{x}_2, \mathbf{x}^3) = r + K_1/(F_2(\hat{x}_2)A_2^n)$, we find

$$\hat{\Pi}_2(\hat{x}_2, \mathbf{x}^3) = (\hat{x}_2 - r)A_2^nF_2(\hat{x}_2) = K_1F_1(\hat{x}_2)/F_2(\hat{x}_2) - K_2.$$

If the bidders are equally strong, that is, condition (4) holds with equality, $\hat{\Pi}_2(\hat{x}_2, \mathbf{x}^3) = 0$. Then, we can define $\chi_2(\mathbf{x}^3) = \hat{x}_2$. If bidder 2 is strictly weaker, condition (4) implies $\hat{\Pi}_2(\hat{x}_2, \mathbf{x}^3) < 0$. Thus, by the intermediate value theorem, there exists $\chi_2(\mathbf{x}^3) > \hat{x}_2$ such that $\hat{\Pi}_2(\chi_2(\mathbf{x}^3), \mathbf{x}^3) = 0$. Because $\chi_2(\mathbf{x}^3) > \hat{x}_2$, we have $\chi_2(\mathbf{x}^3) > \chi_1(\chi_2(\mathbf{x}^3), \mathbf{x}^3)$ for any $\mathbf{x}^3$, implying that the order will not reverse when constructing cutoffs for weaker firms (though, the actual values of χ_1 and χ_2 do change with $\mathbf{x}^3$). Observe that, by replacing $x_2 = \chi_2(\mathbf{x}^3)$ into $\chi_1(\mathbf{x}^2)$, we can write χ_1 and χ_2 as a function of $\mathbf{x}^3$. That is, $\chi_1(\mathbf{x}^3) = \chi_1(\chi_2(\mathbf{x}^3), \mathbf{x}^3)$.

Suppose we have shown that, for any vector $\mathbf{x}^i$, $\chi_1(\mathbf{x}^i) \leq \chi_2(\mathbf{x}^i) \leq \cdots \leq \chi_{i-1}(\mathbf{x}^i)$ (strict whenever $s_{k-1} < s_k$). For each step $k \leq i-1$, $\chi_k(\mathbf{x}^{k+1})$ has been recursively constructed by: (i) replacing the previous-step solution $\chi_{k-1}(\mathbf{x}^k)$ into $\chi_j(\mathbf{x}^{k-1})$ for $j \leq k-2$, so that $\chi_j(\mathbf{x}^k) = \chi_j(\chi_{k-1}(\mathbf{x}^k), \mathbf{x}^k)$; (ii) defining

$$\hat{\Pi}_k(\mathbf{x}^k) = \Pi_k(x_k; \chi_1(\mathbf{x}^k), \ldots, \chi_{k-1}(\mathbf{x}^k), \mathbf{x}^{k+1});$$

and, (iii) defining $\chi_k(\mathbf{x}^{k+1})$ to be the highest value x_k that solves $\hat{\Pi}_k(x_k, \mathbf{x}^{k+1}) = 0$. We show that there exists $\chi_i(\mathbf{x}^{i+1}) \geq \chi_{i-1}(\chi_i(\mathbf{x}^{i+1}), \mathbf{x}^{i+1})$ (strict if $s_i > s_{i-1}$) solving $\hat{\Pi}_i(\chi_i(\mathbf{x}^{i+1}), \mathbf{x}^{i+1}) = 0$. By equation (1), $\hat{\Pi}_{i-1}(\mathbf{x}^{i-1}) = 0$ implies $R_{i-1}(\mathbf{x}^{i-1}) = K_{i-1}/A_{i-1}^n$. Substituting the vector of solutions $(\chi_j(\mathbf{x}^i))_{j=1}^{i-1}$ we can write $\Pi_i(\mathbf{x})$ as $\hat{\Pi}_i(\mathbf{x}^i) = A_i^nR_i(\mathbf{x}^i) - K_i$. Because of the finite expectation assumption, $\hat{\Pi}_i(\mathbf{x}^i)$ is unbounded above in x_i. Fix any vector $\mathbf{x}^{i+1}$. Take $\hat{x}_i$ to be the largest value of x_i that satisfies $\hat{x}_i = \chi_{i-1}(\hat{x}_i, \mathbf{x}^{i+1})$. This value exists by the same argument given to find $\hat{x}_2$ and it also satisfies $x_i > \chi_{i-1}(x_i, \mathbf{x}^{i+1})$ for $x_i > \hat{x}_i$. Using $\hat{x}_i = \chi_{i-1}(\hat{x}_i, \mathbf{x}^{i+1})$ and Lemma B.1.2 (see the Auxiliary Result section) we know[25]

$$R_i(\hat{x}_i; \mathbf{x}^{i+1}) = F_{i-1}(\chi_{i-1}(\hat{x}_i, \mathbf{x}^{i+1}))R_{i-1}(\chi_{i-1}(\hat{x}_i, \mathbf{x}^{i+1}); \hat{x}_i, \mathbf{x}^{i+1}).$$

Then, using the property $R_{i-1}(\chi_{i-1}(\mathbf{x}^i); \mathbf{x}^i) = K_{i-1}/A_{i-1}^n$ and $\hat{x}_i = \chi_{i-1}(\hat{x}_i, \mathbf{x}^{i+1})$, we can write $\hat{\Pi}_i(\hat{x}_i, \mathbf{x}^{i+1}) = K_{i-1}F_{i-1}(\hat{x}_i)/F_i(\hat{x}_i) - K_i$. If bidders $i-1$ and i are equally strong, $\hat{\Pi}_i(\hat{x}_i, \mathbf{x}^{i+1}) = 0$ by condition (4) and we define

[25] The equation above uses the recursion notation. The formulation from the lemma is

$$R_i(\hat{x}_i; \mathbf{x}_{i-2}, \chi_{i-1}(\hat{x}_i, \mathbf{x}^{i+1})) = F_{i-1}(\chi_{i-1}(\hat{x}_i, \mathbf{x}^{i+1}))R_{i-1}(\chi_{i-1}(\hat{x}_i, \mathbf{x}^{i+1}); \mathbf{x}_{i-2}).$$

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$\chi_i(\mathbf{x}^{i+1}) = \hat{x}_i$. If bidder i is strictly weaker than $i-1$, condition (4) implies $\hat{\Pi}_i(\hat{x}_i, \mathbf{x}^{i+1}) < 0$. Then, by the intermediate value theorem, there exists $\chi_i(\mathbf{x}^{i+1}) > \hat{x}_i$ such that $\hat{\Pi}_i(\chi_i(\mathbf{x}^{i+1}), \mathbf{x}^{i+1}) = 0$. Finally, because $\chi_i(\mathbf{x}^{i+1}) > \hat{x}_i$, we have $\chi_i(\mathbf{x}^{i+1}) > \chi_{i-1}(\chi_i(\mathbf{x}^{i+1}), \mathbf{x}^{i+1})$ for any $\mathbf{x}^{i+1}$, and the order between the cutoffs will be robust to the construction of the equilibrium cutoffs for weaker firms. □

Uniqueness: We begin by outlining the induction argument. We order bidders from strongest to weakest. We first show that the strongest bidder has a unique best response to any vector of cutoffs by weaker opponents, $\mathbf{x^2}$. Then, we show that bidder 2 has a unique best response to weaker opponents' cutoffs, $\mathbf{x^3}$, taking bidder 1's unique best response function as given. We also show that these best responses are ordered: bidder 2 always play a higher entry cutoff. Finally, assuming that we have shown that the $k-1$ strongest bidders have a unique best response and that these best responses are ordered, we show that bidder k has a unique best response to any cutoff by weaker bidders, $\mathbf{x}^{k+1}$, and that bidder k always play a higher cutoff than bidder $k-1$. This shows that there is a unique herculean equilibrium. We, then, use the previous argument to show that it also implies that no non-herculean equilibrium exists. Conditions (3) and (4) are used throughout the proof.

Preliminaries. Define $\mathbf{\Pi}_k : \mathbb{R}^n \to \mathbb{R}^k$ to be a function equal to $\Pi_i(\mathbf{x})$ (see equation (1)) in the $i \leq k$ dimension.[26] Fix k, by the existence proof we know that, for every $j \leq k$, there exists recursively defined functions $\chi_j(\mathbf{x}^{k+1})$ satisfying $\mathbf{\Pi}_k(\chi_1(\mathbf{x}^{k+1}), \ldots, \chi_k(\mathbf{x}^{k+1}), \mathbf{x}^{k+1}) = 0$. For any $i \leq k$, the total differential of $\Pi_i(\chi_1(\mathbf{x}^{k+1}), \ldots, \chi_k(\mathbf{x}^{k+1}), \mathbf{x}^{k+1})$ with respect to x_j, $j > k$, is:

$$A_i^n\left[\sum_{s=1}^{i-1} A_s^{i-1} R_s(\mathbf{x}_s) f_s(x_s)\frac{d\chi_s}{dx_j} + B_i(x_i)\frac{d\chi_i}{dx_j} + R_i(\mathbf{x}_i)\left(\sum_{s>i}^{k} h_s(x_s)\frac{d\chi_s}{dx_j} + h_j(x_j)\right)\right], \tag{A4}$$

where $h_i(v) = f_i(v)/F_i(v)$ is the reversed hazard rate of F_i (see Online Appendix D.3 for a step-by-step derivation of A4). Using (A4) and the implicit differentiation of $\mathbf{\Pi}_k$, we can write the vector of derivatives $\mathbf{d}_k = (d\chi_1/dx_{k+1}, \ldots, d\chi_k/dx_{k+1})^T$ (T denotes transpose), as the solution to the following system of linear equations:

$$A_i^n[M_k\mathbf{d}_k + \mathbf{R}_k h_{k+1}(x_{k+1})] = 0, \tag{A5}$$

where $\mathbf{R}_k = (R_1(x_1), R_2(\mathbf{x}_2), \ldots, R_k(\mathbf{x}_k))^T$ and M_k is equal to

$$\begin{pmatrix} B_1(x_1) & R_1(x_1)h_2(x_2) & R_1(x_1)h_3(x_3) & \cdots & R_1(x_1)h_k(x_k) \\ A_1^1R_1(x_1)f_1(x_1) & B_2(x_2) & R_2(\mathbf{x}_2)h_3(x_3) & \cdots & R_2(\mathbf{x}_2)h_k(x_k) \\ A_1^2R_1(x_1)f_1(x_1) & A_2^2R_2(\mathbf{x}_2)f_2(x_2) & B_3(x_3) & \cdots & R_3(\mathbf{x}_3)h_k(x_k) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_1^{k-1}R_1(x_1)f_1(x_1) & A_2^{k-1}R_2(\mathbf{x}_2)f_2(x_2) & A_3^{k-1}R_3(\mathbf{x}_3)f_3(x_2) & \cdots & B_k(x_k) \end{pmatrix}.$$

If M_k is invertible, the solution to (A5) is given by:

$$\mathbf{d}_k = -M_k^{-1}\mathbf{R}_k h_{k+1}(x_{k+1}). \tag{A6}$$

We will show that M_k is invertible. Then, using the derivatives in (A6), we show that $d\hat{\Pi}_{k+1}(\mathbf{x}^{k+1})/dx_{k+1} > 0$. This implies that $\hat{\Pi}_{k+1}(\mathbf{x}^{k+1})$ single-crosses zero in the x_{k+1} dimension, and $\chi_{k+1}(\mathbf{x}^{k+2})$ is uniquely defined. By the existence proof, we also know that $\chi_k(\chi_{k+1}(\mathbf{x}^{k+2}), \mathbf{x}^{k+2}) < \chi_{k+1}(\mathbf{x}^{k+2})$. Then, by the induction argument, each step of the construction $\chi_{k+1}(\mathbf{x}^{k+2})$ is ordered and uniquely defined. Thus, the herculean equilibrium is unique.

Claim A1. There exists a unique herculean equilibrium.

Proof. Fix a step k and let $(\chi_j(\mathbf{x}^{k+1}))_{j=1}^k$ be the vector of functions constructed until step k in the recursion in the existence proof above. For ease in notation, for any $\mathbf{x}^{k+1}$ we write $(\chi_j(\mathbf{x}^{k+1}))_{j=1}^k = \mathbf{x}_k$. We need to show that there is a unique value of x_{k+1} that solves $\hat{\Pi}_{k+1}(\mathbf{x}^{k+1}) = 0$ for any vector $\mathbf{x}^{k+2}$. In particular, we show $d\hat{\Pi}_{k+1}(\mathbf{x}^{k+1})/dx_{k+1} > 0$, so that $\hat{\Pi}_{k+1}(\mathbf{x}^{k+1})$ single crosses zero from below.

Using (A4),

$$\frac{d\hat{\Pi}_{k+1}(\mathbf{x}^{k+1})}{dx_{k+1}} = A_{k+1}^n(\mathbf{m}_k\mathbf{d}_k + B_{k+1}(x_{k+1}))$$

where $\mathbf{m}_k = (A_1^kR_1(x_1)f_1(x_1), A_2^kR_2(\mathbf{x}_2)f_2(x_2), \ldots, A_k^kR_k(\mathbf{x}_k)f_k(x_k))$. Using (A6), if M_k is invertible we can write $\mathbf{d}_k = -M_k^{-1}\mathbf{R}_k h_{k+1}(x_{k+1})$ and

$$\frac{d\hat{\Pi}_{k+1}(\mathbf{x}_{k+1})}{dx_{k+1}} = A_{k+1}^n(B_{k+1}(x_{k+1}) - q_k h_{k+1}(x_{k+1}))$$

[26] For ease in notation, we use $\Pi_i(\mathbf{x})$ and $R_i(\mathbf{x}_i)$ to refer to $\Pi_i(x_i; \mathbf{x}_{-i})$ and $R_i(x_i; \mathbf{x}_{i-1})$.

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where $q_k = \mathbf{m}_k M_k^{-1} \mathbf{R}_k$ is a scalar. Because $A_{k+1}^n > 0$, it is sufficient to show that the term inside the parenthesis is positive for all relevant values of x_{k+1}. We prove the previous statement and the invertibility of M_k by induction.

Observe $\Pi_1(\mathbf{x}) = A_1^n(x_1 - r) - K_1$, thus $d\Pi_1(\mathbf{x})/dx_1 > 0$ and bidder 1 has a unique best response (given by $\chi_1(\mathbf{x}^2) = r + K_1/A_1^n$). For bidder 2, observe $M_1 = B_1(x_1) = 1$ is invertible and $q_1 = (x_1 - r)^2 f_1(x_1)$ is well defined. Then, $B_2(x_2) - q_1 h_2(x_2) = F_1(x_2) - (x_1 - r)^2 f_1(x_1) h_2(x_2)$. Using condition (3) twice, $x_1 F_1(x_1)/x_2$ is an upper bound for the subtracting term. Because, by construction, we are interested in $x_2 \geq x_1$, $B_2(x_2) - q_1 h_2(x_2) > 0$. Then, $d\hat{\Pi}_2(\mathbf{x}^2)/dx_2 > 0$ and $\chi_2(\mathbf{x}^3)$ is uniquely defined.

Suppose we have shown that M_{j-1} is invertible and $B_j(x_j) - q_{j-1} h_j(x_j) > 0$ for all $j \leq k$. Let $l_k = (B_k(x_k) - q_{k-1} h_k(x_k))^{-1}$ and observe that $l_k > 0$ by induction hypothesis; then, by the definition of M_k

$$M_k = \begin{pmatrix} M_{k-1} & \mathbf{R}_{k-1} h_k(x_k) \\ \mathbf{m}_{k-1} & B_k(x_k) \end{pmatrix}.$$

Using blockwise inversion,

$$M_k^{-1} = \begin{pmatrix} M_{k-1}^{-1} + h_k(x_k) l_k (M_{k-1}^{-1} \mathbf{R}_{k-1} \mathbf{m}_{k-1} M_{k-1}^{-1}) & -h_k(x_k) l_k (M_{k-1}^{-1} \mathbf{R}_{k-1}) \\ -l_k(\mathbf{m}_{k-1} M_{k-1}^{-1}) & l_k \end{pmatrix}$$

and the inverse of M_k is well defined. We need to show $B_{k+1}(x_{k+1}) - q_k h_{k+1}(x_{k+1}) > 0$. Observing that $\mathbf{R}_k = (\mathbf{R}_{k-1}, R_k(\mathbf{x}_k))^T$, $\mathbf{m}_k = (\mathbf{m}_{k-1} F_k(x_k), R_k(\mathbf{x}_k) f_k(x_k))$, and using the definition of M_k^{-1} and l_k we can write:

$$q_k = F_k(x_k) q_{k-1} + f_k(x_k)(R_k(\mathbf{x}_k) - q_{k-1})^2 \big/ (B_k(x_k) - q_{k-1} h_k(x_k)). \tag{A7}$$

Thus, $B_{k+1}(x_{k+1}) - q_k h_{k+1}(x_{k+1}) > 0$ is equivalent to show:

$$\left(\frac{B_k(x_{k+1}) F_k(x_{k+1})}{f_k(x_k) h_{k+1}(x_{k+1})} - \frac{q_{k-1}}{h_k(x_k)} \right) (B_k(x_k) - q_{k-1} h_k(x_k)) > (R_k(\mathbf{x}_k) - q_{k-1})^2$$

where $B_{k+1}(x_{k+1}) = B_k(x_{k+1}) F_k(x_{k+1})$ was used. By the existence proof we are only interested in $x_{k+1} \geq x_k$; using this condition, that $B_k(v)$ is decreasing in v, and condition (3) we find that $(B_k(x_k) x_k - q_{k-1})^2$ is a lower bound for the LHS of the expression above. Lemma B.1.1 shows $B_i(x_k) x_k \geq R_k(\mathbf{x}_k)$. Thus we just need to show that $B_k(x_k) x_k - q_{k-1} \geq 0$, which is done by proving $R_k(\mathbf{x}_k) - q_{k-1} \geq 0$. We do this by induction. Because q_0 is not defined, we begin with $i = 2$. Integrating by parts $R_2(\mathbf{x}^2)$, $R_2(\mathbf{x}^2) - q_1$ is equal to

$$(x_1 - r) F_1(x_1) + \int_{x_1}^{x_2} F_1(v) dv - (x_1 - r)^2 f_1(x_1) > \int_{x_1}^{x_2} F_1(v) dv \geq 0$$

where $x_1 \geq x_1 - r$ and condition (3) was used in the last step. Suppose we have shown $R_j(\mathbf{x}_j) \geq q_{j-1}$ for $j \leq i$. We show $R_{i+1}(\mathbf{x}_{i+1}) \geq q_i$. Using equation (A7), this is equivalent to:

$$R_{i+1}(\mathbf{x}_{i+1})/F_i(x_i) - q_{i-1} - (R_i(\mathbf{x}_i) - q_{i-1})^2 \Big/ \left(\frac{B_i(x_i)}{h_i(x_i)} - q_{i-1} \right) \geq 0.$$

Lemma B.1.2 shows $R_{i+1}(\mathbf{x}_{i+1})/F_i(x_i) \geq R_i(\mathbf{x}_i)$. By the induction hypothesis $R_i(\mathbf{x}_i) \geq q_{i-1}$ and we can rewrite the condition as

$$1 \geq (R_i(\mathbf{x}_i) - q_{i-1}) \Big/ \left(\frac{B_i(x_i)}{h_i(x_i)} - q_{i-1} \right).$$

The result follows from condition (3) and Lemma B.1.1. Thus $R_{i+1}(\mathbf{x}_{i+1}) \geq q_i$, which proves $d\hat{\Pi}_{k+1}(\mathbf{x}^{k+1})/dx_{k+1} > 0$ for all $x_{k+1} \geq x_k$ and a unique herculean equilibrium exists.

Claim A2. There is no non-herculean equilibria.

Proof. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be an ordered vector of equilibrium cutoffs. beginning from the lower cutoff, let i be the first bidder to play a smaller cutoff than a stronger bidder $i+1$; that is, $x_i < x_{i+1}$ but $s_i > s_{i+1}$. In other words, every bidder $k \leq i$ have their cutoffs in the same order as their strength. Because of this, we can use our recursive construction in the existence proof and our induction argument in the uniqueness proof up to bidder i, so that best responses are uniquely defined for any vector $\mathbf{x}^{i+1}$ that bidders may play.

Let us analyze $\hat{\Pi}_{i+1}(\mathbf{x}^{i+1})$. Because $\hat{\Pi}_i(\mathbf{x}^i) = 0$ we know $R_i(\mathbf{x}^i) = K_i/A_i^n$. Writing $\hat{\Pi}_{i+1}(\mathbf{x}^{i+1}) = A_{i+1}^n R_{i+1}(\mathbf{x}^{i+1}) - K_{i+1}$. Take x_{i+1} to be the value that satisfies $x_{i+1} = \chi_i(x_{i+1}, \mathbf{x}^{i+2}) = x_i$ and notice that Lemma B.1.2 implies $R_{i+1}(x_i, \mathbf{x}^{i+2}) = F_i(x_i) R_i(x_i, x_i, \mathbf{x}^{i+2})$. Then, using $R_i(\mathbf{x}^i) = K_i/A_i^n$, we can write $\hat{\Pi}_{i+1}(x_i, \mathbf{x}^{i+2}) = K_i F_i(x_i)/F_{i+1}(x_i) - K_{i+1} > 0$; which is *positive* under (4) and the condition that bidder $i+1$ is stronger than bidder i. We need to show that there is no $x_{i+1}^* > x_i$ such that $\hat{\Pi}_{i+1}(x_{i+1}^*, \mathbf{x}^{i+2}) = 0$. This follows from the proof of uniqueness, as condition (3) implies $d\hat{\Pi}_{i+1}(\mathbf{x}^{i+1})/dx_{i+1} > 0$ for $x_{i+1}^* > x_i$, implying the result. □

Proof of Lemma 5. We show that s_i exists and that $\sigma_i(s) \equiv \Pi_i(s; s, \ldots, s)$ single crosses zero.

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Existence: Observe that assumptions A3 and A2 jointly imply $\sigma_i(\underline{v}_i) < 0$. Similarly, assumption A3 and Lemma B2 (see Appendix B) imply, $\sigma_i(\overline{v}_i) \geq \Pi_i(\overline{v}_i; a_{-i}) > 0$ (where a is the lower bound of the support of F_i). Then, by the intermediate value theorem, there exist $\hat{s}$ such that $\sigma_i(\hat{s}) = 0$.

Uniqueness: By Lemma B2 and the chain rule, we have that $\sigma_i'(s) > 0$. Thus, $\sigma_i(s)$ single crosses zero; that is, there is a unique value s_i satisfying $\sigma_i(s_i) = 0$. □

Proof of Proposition 4. *Proof preliminaries*: If $s_1 = s_2$ the herculean equilibrium corresponds to the strength of the firms. Assume, without loss of generality, that $s_1 < s_2$. Let $\hat{\mathbf{x}} = (x_1, x_1, \ldots, x_1, x_2, x_2, \ldots, x_2)$ be a vector of group-symmetric cutoff strategies. Pick any firm in group $i \in \{1, 2\}$ and let $\Pi_i^{gs}(x_1, x_2) = \Pi_i(\hat{\mathbf{x}})$ —where gs stands for group symmetric— represent the expected profit of a firm belonging to group i entering with a valuation x_i, when opponents play group-symmetric strategies x_1 and x_2. Observe that the function $\Pi_i^{gs}(x_1, x_2)$ has a two-dimensional domain, taking as input the group-symmetric strategy of each group.

Lemma B3 in the Auxiliary Results section implies that, under condition (8), restricting to group-symmetric strategies is without loss. If there is another type of equilibrium, it must be that symmetric firms play asymmetric cutoffs, contradicting the Lemma. Define $\chi_1(x)$ to be the function that solves $\Pi_1^{gs}(\chi_1(x), x) = 0$. Thus, $\chi_1(x)$ corresponds to group 1's symmetric best response to group 2 playing the group-symmetric cutoff x. By Lemma B2, $\Pi_1^{gs}(x_1, x_2)$ is increasing in each argument, and the value $\chi_1(x)$ exists and is unique; that is, $\chi_1(x)$ is well defined. □

Lemma A1. $\chi_1(s_1) = s_1$ and, under condition (9), $0 > \chi_1'(x) > -\frac{f_2(x)}{F_2(x)}\frac{F_1(\chi_1(x))}{f_1(\chi_1(x))}$.

Proof. By definition of strength we know $\Pi_1^{gs}(s_1, s_1) = 0$, therefore $\chi_1(s_1) = s_1$. Let G_i be the set of firms belonging to group i. Using implicit differentiation, the chain rule, that groups members are symmetric, and Lemma B2

$$\chi_1'(x) = -\frac{\frac{\partial \Pi_1^{gs}(\chi_1(x),x)}{\partial x_2}}{\frac{\partial \Pi_1^{gs}(\chi_1(x),x)}{\partial x_1}} = -\frac{\sum_{j\in G_2}\frac{\partial \Pi_1(\hat{\mathbf{x}})}{\partial x_j}}{\sum_{j\in G_1}\frac{\partial \Pi_1(\hat{\mathbf{x}})}{\partial x_j}} = \frac{-n_2\frac{f_2(x)}{F_2(x)}\Delta_{1,2}(\hat{\mathbf{x}})}{\Pi_1'(\hat{\mathbf{x}}) + (n_1 - 1)\frac{f_1(\chi_1(x))}{F_1(\chi_1(x))}\Delta_{1,1}(\hat{\mathbf{x}})},$$

where $\Delta_{i,j}(\hat{\mathbf{x}}) = F_j(x_j)\hat{\Delta}_{i,j}(\hat{\mathbf{x}})$ is defined by equation (7). Because numerator and denominator are positive, the equation above proves $\chi_1'(x) < 0$ for all x. For the lower bound of $\chi_1(x)$ observe that $\Delta_{1,1}(\hat{\mathbf{x}}) > 0$. Take a lower bound for $\chi_1'(x)$ by making $\Delta_{1,1}(\hat{\mathbf{x}})$ zero. The lower bound $\chi_1'(x) > -\frac{f_2(x)}{F_2(x)}\frac{F_1(\chi_1(x))}{f_1(\chi_1(x))}$ follows by using sufficient condition (9). □

Existence of a herculean equilibrium: Define $\hat{\Pi}_2(x) = \Pi_2^{gs}(\chi_1(x), x)$. This function is continuous and corresponds to the expected profit of a firm in group 2 when it enters the market under valuation x, group 2 plays the group-symmetric cutoff x, and group 1 plays their group-symmetric best response $\chi_1(x)$. Define x_2 to be the value satisfying $\hat{\Pi}_2(x_2) = 0$ and let $x_1 = \chi_1(x_2)$. Observe that $x_2 \in (s_1, \infty)$ is necessary and sufficient for $x_1 < x_2$. This is so, as $\chi_1(x)$ is decreasing in x and $\chi_1(s_1) = s_1$. The next claim proves that a herculean equilibrium ($x_1 < x_2$) exists, $x_1 < s_1$ and $x_2 > s_2$.

Claim A3. $\hat{\Pi}_2(s_2) < 0$ and there exists $\tilde{x} > s_2$ such that $\hat{\Pi}_2(\tilde{x}) > 0$. Thus, by the intermediate value theorem, the herculean equilibrium cutoff $x_2 \in (s_2, \tilde{x})$ exists.

Proof. Because group two is weak, and $\chi_1(x)$ is decreasing in x, we know that $\chi_1(s_2) < \chi_1(s_1) = s_1 < s_2$ (where Lemma A1 was used in the equality). Lemma B2 and the definition of strength implies $\hat{\Pi}_2(s_2) = \Pi_2^{gs}(\chi_1(s_2), s_2) < \Pi_2^{gs}(s_2, s_2) = 0$, proving $\hat{\Pi}_2(s_2) < 0$. For the second part of the claim, observe that, by Lemma B2, $\Pi_2^{gs}(x_1, x_2)$ is increasing in x_1; then, $\Pi_2^{gs}(\chi_1(x), x) \geq \Pi_2^{gs}(a, x)$ for all x. Take $\tilde{x} = \overline{v}_2$ and observe that, by assumption A3, $\Pi_2^{gs}(a, \tilde{x}) > 0$, proving the result. □

Uniqueness of equilibrium: Observing that, under condition (8), Lemma B3 applies. Therefore, it is without loss to restrict the analysis to group-symmetric strategies. To prove uniqueness, then, we need to show that no other herculean equilibrium exists and that we can not have an equilibrium where $x_2 < x_1$.

Claim A4. There exists a unique herculean equilibrium.

Proof. To prove uniqueness within the herculean class, we show $\hat{\Pi}_2'(x) > 0$ so that $\hat{\Pi}_2(x)$ single crosses zero from below. Recall $\hat{\mathbf{x}} = (\chi_1(x), \ldots, \chi_1(x), x, \ldots, x)$. Differentiating $\hat{\Pi}_2(x)$, using the chain rule, and that firms play group-symmetric strategies, we obtain

$$\hat{\Pi}_2'(x) = \sum_{j\in G_2}\frac{\partial \Pi_2(\hat{\mathbf{x}})}{\partial x_j} + \chi'(x)\sum_{j\in G_1}\frac{\partial \Pi_2(\hat{\mathbf{x}})}{\partial x_j}$$

$$> \Pi_2'(\hat{\mathbf{x}}) + (n_2 - 1)\frac{f_2(x)}{F_2(x)}\Delta_{2,2}(\hat{\mathbf{x}}) - n_1\frac{f_2(x)}{F_2(x)}\Delta_{2,1}(\hat{\mathbf{x}})$$

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$$>(n_2 - 1)\frac{f_2(x)}{F_2(x)}\Delta_{2,2}(\hat{\mathbf{x}}) > 0.$$

The first inequality follows from using equation (B2) and the bound for $\chi_1'(x)$ in Lemma A1. The second inequality follows from sufficient condition (9). Proving that the derivative is positive and uniqueness within the herculean class. □

Claim A5. There is no group-symmetric equilibrium in which the strong group plays a higher cutoff than the weak group.

Proof. We show that no non-herculean equilibrium-that is, $x_1 > x_2$ but $s_1 < s_2$—can exist. Define $\chi_2(x)$ to be the function that satisfies $\Pi_2^{gs}(x, \chi_2(x)) = 0$; $\chi_2(x)$ corresponds to group two's best response to the cutoff of group one when $x_1 = x$. Using the same arguments as for $\chi_1(x)$, $\chi_2(x)$ is also well defined. Similarly, following the steps of Lemma A1, it can be shown: $\chi_2(s_2) = s_2$, $\chi_2'(x) < 0$, and under condition (9), $\chi_2'(x)$ is bounded below by $-\frac{f_1(x)F_2(\chi_2(x))}{F_1(x)f_2(\chi_2(x))}$.

Define the continuous function $\hat{\Pi}_1(x) = \Pi_1^{gs}(x, \chi_2(x))$ which corresponds to the expected profit of a firm in group 1 when entering the market under valuation x and its opponents play the pair of group-symmetric strategies $(x, \chi_2(x))$. We show that there is no x satisfying $x_1 = x > \chi_2(x) = x_2$ and $\hat{\Pi}_1(x) = 0$; that is, no non-herculean equilibrium exists. Start by observing that $x > \chi_2(x)$ if and only if $x \in (s_2, \infty)$. In Lemma 5 we showed the function $\sigma_1(s) = \Pi_1^{gs}(s, s)$ is strictly increasing in s. By the definition of strength and by firm 2 being weak ($s_1 < s_2$),

$$\sigma_1(s_1) = \Pi_1^{gs}(s_1, s_1) = 0 < \sigma_1(s_2) = \Pi_1^{gs}(s_2, s_2) = \Pi_1^{gs}(s_2, \chi_2(s_2)) = \hat{\Pi}_1(s_2),$$

showing that $\hat{\Pi}_1(s_2) > 0$. Following analogous steps to those in Claim A4, which requires the using lower bound for $\chi_2'(x)$ and sufficient condition (9), we can show that $\hat{\Pi}_1'(x) > 0$. Then, because $\hat{\Pi}_1(s_2) > 0$ and $\hat{\Pi}_1'(x) > 0$ for all x, $\hat{\Pi}_1(x)$ never crosses zero when $x > s_2$ and the result follows. □

Appendix B: Auxiliary Results

Lemma B1. In a second-price auction, let $(x_1, x_2, \ldots, x_n)$ be an ordered vector of cutoff strategies (i.e., $x_1 \leq x_2 \leq \cdots x_n$). Then, the following properties hold:

1. $x_i B_i(x_i) \geq R_i(\mathbf{x}_i)$ and strict if $r > 0$ or if exists $j < i$ such that $x_j < x_i$.
2. $R_i(\mathbf{x}_i) > F_{i-1}(x_{i-1})R_{i-1}(\mathbf{x}_{i-1})$ and with equality if $x_i = x_{i-1}$.

Proof. Recall the definition of $R_i(\mathbf{x}_i)$ in equation (1). For the first claim simply observe,

$$x_i B_i(x_i) - R_i(\mathbf{x}_i) = rA_0^{i-1} + \sum_{k=1}^{i-1}\left(A_k^{i-1}\int_{x_k}^{x_{k+1}} s dB_{k+1}(s)\right)$$

which is strictly positive if $r > 0$ or if there exists a bidder $j < i$ such that $x_j < x_i$. For the second claim we show that $R_i(\mathbf{x}_i) = F_{i-1}(x_{i-1})R_{i-1}(\mathbf{x}_{i-1}) + \int_{x_{i-1}}^{x_i} B_i(s)ds$, which proves the claim. Rewriting $R_i(\mathbf{x}_i)$ using definition in (1):

$$R_i(\mathbf{x}_i) = x_i B_i(x_i) - F_{i-1}(x_{i-1})\left[rA_0^{i-2} - \sum_{k=1}^{i-2}\left(A_k^{i-2}\int_{x_k}^{x_{k+1}} s dB_{k+1}(s)\right)\right] - \int_{x_{i-1}}^{x_i} s dB_i(s).$$

Integrating by parts the last term, $R_i(\mathbf{x}_i)$ becomes:

$$x_{i-1}B_i(x_{i-1}) - F_{i-1}(x_{i-1})\left[rA_0^{i-2} - \sum_{k=1}^{i-2}\left(A_k^{i-2}\int_{x_k}^{x_{k+1}} s dB_{k+1}(s)\right)\right] + \int_{x_{i-1}}^{x_i} B_i(s)ds.$$

Because, by definition, $B_i(x_{i-1}) = B_{i-1}(x_{i-1})F_{i-1}(x_{i-1})$, the result follows. □

Lemma B2. $\Pi_i(\mathbf{x})$ is strictly increasing in every dimension of $\mathbf{x}$.

Proof of Lemma B2. Start with the derivative of Π_i with respect to x_i, then

$$\frac{\partial \Pi_i}{\partial x_i} \equiv \Pi_i'(\mathbf{x}) = \sum_{e\in\mathcal{E}_i}\left\{\left(\prod_{j\in e^c} F_j(x_j)\right)\int_{\{x_j\}_{j\in e\setminus i}}^{b} \pi_i'\big(x_i, v_{e\setminus i}\big)\phi(v_{e\setminus i})d^{n_e-1}v_{e\setminus i}\right\} > 0, \tag{B1}$$

which is positive as, by assumption A3, there is a positive probability that firm i is the sole entrant. To compute $\partial\Pi_i/\partial x_j$, pick a market structure e in which j stays out ($j \in e^c$). Conditional on e, the derivative of Π_i with respect to x_j is equal to

$$f_j(x_j)\left(\prod_{k\in e^c\setminus j} F_k(x_k)\right)\int_{\{x_k\}_{k\in e\setminus i}}^{b} \pi_i\big(x_i, v_{e\setminus i}\big)\phi(v_{e\setminus i})d^{n_e-1}v_{e\setminus i}.$$

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Now take market structure e, from above, and use Leibnitz differentiation, to compute $\partial \Pi_i/\partial x_j$ conditional on market structure $e \cup j$; that is, entry decisions by every firm remain the same as in e except firm j, which now participates

$$-f_j(x_j)\left(\prod_{k\in e^c\setminus j} F_k(x_k)\right)\int_{\{x_k\}_{k\in e\setminus i}}^{b} \pi_i\big(x_i, x_j, v_{e\setminus i}\big)\phi(v_{e\setminus i})d^{n_e-1}v_{e\setminus i}.$$

Observe that both expressions from above only differ in sign and in the profit function that is integrated over. Summing both equations delivers

$$f_j(x_j)\left(\prod_{k\in e^c\setminus j} F_k(x_k)\right)\int_{\{x_k\}_{k\in e\setminus i}}^{b} \delta_{i,j}\big(x_i, x_j, v_{e\setminus i}\big)\phi(v_{e\setminus i})d^{n_e-1}v_{e\setminus i},$$

where $\delta_{i,j}(v_e) \geq 0$ is defined in equation (5). Summing across every market structure in which j stays out and using equation (7) we obtain

$$\frac{\partial \Pi_i}{\partial x_j} = f_j(x_j)\hat{\Delta}_{i,j}(\mathbf{x}) = \frac{f_j(x_j)}{F_j(x_j)}\Delta_{i,j}(\mathbf{x}) > 0. \tag{B2}$$

Thus, the derivative is positive. □

Lemma B3. Under condition (8), two symmetric firms that best respond to each other must play the same cutoff strategy.

Proof. Consider two symmetric firms, p and q, and fix *any* profile of cutoffs strategies $\mathbf{x}_{E\setminus\{p,q\}}$ for the rest of the firms. The equilibrium condition for firm p holds whenever there exists x_p and x_q such that $\Pi_p(x_p; x_q, \mathbf{x}_{E\setminus\{p,q\}}) = 0$. Define $\chi(x_p)$ to be firm q's best response to x_p (and to $\mathbf{x}_{E\setminus\{p,q\}}$, which is fixed throughout the proof). By Lemma B2, $\Pi_p(x_p; x_q, \mathbf{x}_{E\setminus\{p,q\}})$ is strictly increasing in both x_p and x_q, which implies that $\chi(x_p)$ exists and is uniquely defined for each x_p. To prove the Lemma, we need to prove three claims. □

Claim B1. There exists a unique pair of symmetric cutoffs, $x_p = x_q = z$, such that $\Pi_p(z; z, \mathbf{x}_{E\setminus\{p,q\}}) = 0$.

Proof. Suppose firms p and q play a symmetric cutoff, $x_p = x_q = z$. Define $\hat{\sigma}(z) = \Pi_p(z; z, \mathbf{x}_{E\setminus\{p,q\}}) = \Pi_q(z; z, \mathbf{x}_{E\setminus\{p,q\}})$, where the last equality follows from symmetry among firms. Thus, if the equilibrium condition is satisfied by firm p, it is also satisfied by firm q. We want to show there exists a unique value $\hat{z}$ such that $\hat{\sigma}(\hat{z}) = 0$. Following analogous steps to those in Lemma 5, it is easy to show $\hat{\sigma}(\underline{v}_p) < 0$ and $\hat{\sigma}(\overline{v}_p) > 0$; so that, there exists $\hat{z}$ such that $\hat{\sigma}(\hat{z}) = 0$. Similarly, using Lemma B2 and the chain rule, we can show that $\hat{\sigma}'(z) > 0$. Hence, the value $\hat{z}$ is unique. □

Claim B2. Under condition (9):[27] $0 > \chi'(x_p) > -\frac{f(x_p)}{F(x_p)}\frac{F(\chi(x_p))}{f(\chi(x_p))}$.

Proof. Let $\mathbf{x} = (x_p, \chi(x_p), \mathbf{x}_{E\setminus\{p,q\}})$. Using implicit differentiation and equations (B1) and (B2) from Lemma B2, we obtain

$$\chi'(x_p) = -\frac{\frac{\partial \Pi_q(\chi(x_p); x_p, \mathbf{x}_{E\setminus\{p,q\}})}{\partial x_p}}{\frac{\partial \Pi_q(\chi(x_p); x_p, \mathbf{x}_{E\setminus\{p,q\}})}{\partial x_q}} = -\frac{f(x_p)}{F(x_p)}\frac{\Delta_{q,p}(\mathbf{x})}{\Pi_q'(\mathbf{x})} < 0,$$

which is negative as the denominator and numerator are positive. To obtain the lower bound for $\chi'(x_p)$ simply use condition (8). □

Claim B3. An increase in x_p, when firm q best responds by playing $\chi(x_p)$, leads firm p to strictly increase its profit; that is, $\Pi_p(x_p; \chi(x_p), \mathbf{x}_{E\setminus\{p,q\}})$ is increasing in x_p.

Proof. Differentiating $\Pi_p(x_p; \chi(x_p), \mathbf{x}_{E\setminus\{p,q\}})$ with respect to x_p, using the chain rule, and equations (B1) and (B2) we obtain

$$\begin{aligned}\frac{d\Pi_p(\mathbf{x})}{dx_p} &= \frac{\partial \Pi_p(\mathbf{x})}{\partial x_p} + \frac{\partial \chi(x_p)}{\partial x_p}\frac{\partial \Pi_p(\mathbf{x})}{\partial x_q}\\ &= \Pi_p'(\mathbf{x}) + \frac{\partial \chi(x_p)}{\partial x_p}\frac{f(\chi(x_p))}{F(\chi(x_p))}\Delta_{p,q}(\mathbf{x}) > \Pi_p'(\mathbf{x}) - \frac{f(x_p)}{F(x_p)}\Delta_{p,q}(\mathbf{x}) > 0,\end{aligned}$$

where $\mathbf{x} = (x, \chi(x), \mathbf{x}_{E\setminus\{p,q\}})$. The first inequality follows from Claim B2, whereas the second from condition (8); which proves the claim. □

[27] For ease in notation, we use symmetry, and drop the subindexes from F when referring to firms p and q.

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We prove Lemma B3 by contradiction. Recall that $\mathbf{x}_{E\setminus\{p,q\}}$ is fixed throughout the proof. Suppose, without loss of generality, that there exists $x_q < x_p$ constituting an equilibrium. By Claim B1 there exists a unique value $\hat{z}$ such that $\hat{\sigma}(\hat{z}) = 0$.

Suppose first $x_q < \hat{z} < x_p$. Because

$$\hat{\sigma}(\hat{z}) = \Pi_p(\hat{z}; \hat{z}, \mathbf{x}_{E\setminus\{p,q\}}) = \Pi_p(\hat{z}; \chi(\hat{z}), \mathbf{x}_{E\setminus\{p,q\}}) = 0,$$

and $x_p > \hat{z}$, Claim B3 implies that we must have $\Pi_p(x_p; \chi(x_p) = x_q, \mathbf{x}_{E\setminus\{p,q\}}) > 0$; which contradicts (x_p, x_q) being an equilibrium.

Suppose now $x_q < x_p < \hat{z}$. Lemma B2 and Claim B1 imply

$$0 = \hat{\sigma}(\hat{z}) > \hat{\sigma}(x_p) = \Pi_p(x_p; x_p, \mathbf{x}_{E\setminus\{p,q\}}) > \Pi_p(x_p; \chi(x_p) = x_q, \mathbf{x}_{E\setminus\{p,q\}})$$

which contradicts (x_p, x_q) being an equilibrium. Analogous arguments can be constructed for the case $\hat{z} < x_q < x_p$, proving the Lemma.

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