# Recursive Hurwicz Expected Utility and Neo-Additive Sources*

* preliminary and incomplete

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**Abstract**

In a dynamic extension of Gul and Pesendorfer's (2015) (static) Hurwicz Expected Utility (HEU), preferences are defined over information decision problems in which a decision-maker can make his choice from a menu of options contingent on the realization of a signal. Both his static preferences as well as his interim preferences that guide his choice after receipt of the signal admit an HEU representation. Furthermore, the interim preferences are invariant to any choice that the DM could have made had the realization of the signal been different. This allows us to recover his preferences over information decision problems recursively, and implies that any plan of action that was ex ante optimal remains so after receipt of the signal.

In addition, Gul and Pesendorfer's concept of a source is extended to any *probabilitistically sophisticated* source that need only be continuous with respect to non extreme outcomes. For an information decision problems in which both the signal and the menu of options are measurable with respect to a particular non-extreme-outcome (that is, "neo-")continuous source, it turns out only in the case of a neo-*additive* source are interim preferences *source consistent* in the sense they can be formulated entirely in terms of the source and the parameters that characterize the restriction of the decision-maker's static preferences to acts that are adapted to that source.

**JEL Classification:** D80, D81

**Key words:** ambiguity, model consistency, consequentialism, dynamic consistency, neo-additive capacity

# 1 Introduction

In order to develop a dynamic extension of Gul and Pesendorfer's (2015) (static) model of Hurwicz Expected Utility (HEU), consider the preferences of a decision-maker (hereafter, DM) defined over information decision problems (IDPs). The objects of choice are acts that associate a monetary prize to every state of nature. Each IDP comprises a signal, which I take to be a function from the state space to a space of (conceivable) realizations, and a menu of acts, from which the DM makes a choice after learning the realization of the signal.[1]

First notice, the DM's static preferences over acts are naturally embedded in his preferences over IDPs. Simply attribute a preference for one act over another, should the DM prefer an IDP with an uninformative signal that has the former act as its only available option to another IDP also involving the same uninformative signal that has the latter as its only available option.

Now assume the restriction of his preferences to such IDPs conforms to HEU. This entails the existence of a prior $\mu$, a Bernoulli utility $v$, and an ambiguity aversion (parameter) $\alpha$ that resides in the unit interval. The prior is a probability measure defined on a (rich) $\sigma$-algebra of events that Gul and Pesendorfer refer to as *ideal* since they are ones for which the associated uncertainty lends itself to precise quantification by the prior. Let $\mathcal{E}$ denote the domain of $\mu$. Gul and Pesendorfer also refer to any act adapted to $\mathcal{E}$ as *ideal*.

For any pair of ideal acts $g$ and $g'$, the DM's static preferences rank $g$ over $g'$ if and only if

$$\int v \, dF_g^\mu \geqslant \int v \, dF_{g'}^\mu ,$$

where $F_g^\mu$ (respectively $F_{g'}^\mu$) denotes the cumulative distribution function (CDF) induced from $g$ (respectively, $g'$) by his prior $\mu$.[2] That is, the restriction of his static preferences to ideal acts conforms to subjective expected utility.

Furthermore, the restriction of his preferences to *ideal* IDPs, by which I mean those problems in which each of the signal's information cells (that is, preimages of its realizations) is an ideal event and every option in the menu is an ideal act, may be represented by the function that assigns to each ideal IDP comprising a signal $\sigma$

[1] Assume each menu is finite as is the range of each signal.

[2] More precisely, for each outcome $x$, $F_g^\mu(x)$ is the quantity $\mu$ assigns to the (ideal) event the act $g$ yields an outcome less than or equal to $x$.

adapted to $\mathcal{E}$ and a menu $M$ of ideal acts, the utility

$$\sum_{s\in S:\mu(\sigma^{-1}(s))>0} \mu\left(\sigma^{-1}(s)\right)\left(\max_{g\in M}\int v\, dF_g^{\mu_s^\sigma}\right),$$

where: *(i)* $S$ is the space of conceivable signal realizations with generic element $s$; *(ii)* $\mu\left(\sigma^{-1}(s)\right)$ is the ex ante probability the realization of the signal $\sigma$ is $s$; and, *(iii)* for each outcome $x$, $F_g^{\mu_s^\sigma}(x)$ denotes the probability the (ideal) act $g$ yields an outcome no greater than $x$ conditional on the realization of the signal being $s$.[3]

Notice that, after learning the realization of the signal, the DM's choice from the menu of ideal acts is governed by an interim preference relation whose restriction to ideal acts admits an expected utility representation characterized by the same Bernoulli utility and by a posterior that corresponds to the Bayesian update of his prior conditional on the information cell corresponding to that realization having obtained. Moreover, these interim preferences are *consequentialist* as they do not depend on what (ideal) act would have determined the outcome had the realization of the signal been different. As a consequence, any plan of action in the ideal IDP that is ex ante optimal is also *dynamically consistent*: after learning the realization of the signal, the DM never has a strict incentive not to follow through with that plan of action.

An ideal IDP $(\sigma\,, M)$ may also be evaluated *recursively.* Working backwards, for each of the possible (and non-null) realizations of the signal, first find the certainty equivalent of the most preferred act from the menu conditional on the information cell corresponding to that realization obtaining. Next, construct an act by assigning each of these conditional certainty equivalents to the corresponding information cell. The DM's valuation of $(\sigma\,, M)$ may be expressed as the (static) subjective expected utility of that act. More precisely, first associate with $(\sigma\,, M)$ an auxiliary act $a^{(\sigma\,,M)}$ obtained by setting

$$a^{(\sigma\,,M)}(\omega) := v^{-1}\left(\max_{g\in M}\int v\, dF_g^{\mu_s^\sigma}\right)$$

whenever $\mu\left(\sigma^{-1}(\sigma(\omega))\right) > 0$.[4] Applying the expected utility representation of the

[3] More precisely, $F_g^{\mu_s^\sigma}(x) := \mu\big(\{\omega\in\Omega\colon f(\omega)\leqslant x\}\cap\sigma^{-1}(s)\big)\,/\mu\left(\sigma^{-1}(s)\right)$, where $\Omega$ denotes the state space with generic element $\omega$.

[4] Should $\mu\left(\sigma^{-1}(\sigma(\omega))\right) = 0$ (that is, the event $\sigma^{-1}(s)$ in which $\omega$ resides is null), we are free to set $a^d(\omega)$ equal to any (arbitrary) outcome.

DM's static preferences, yields

$$\int v\, dF^{\mu}_{a^d} = \sum_{s \in S : \mu(\sigma^{-1}(s))>0} \mu\left(\sigma^{-1}(s)\right) \left( \max_{g \in M} \int v\, dF^{\mu^\sigma_s}_{g} \right) .$$

The model developed in section 2 is one that permits such a recursive evaluation of *any* IDP. Moreover, the interim preferences that guide the DM's choice after receipt of a signal are:

1. *model consistent* – they admit an HEU representation with the *same* risk attitudes and the *same* aversion toward ambiguity as the DM's static preferences;

2. *consequentialist* – they are invariant to what act would have determined the outcome had the realization of the signal been different; and,

3. *dynamically consistent* – any plan of action that is ex ante optimal will be followed through.

I shall refer to members of this class as *Recursive Hurwicz Expected Utility (R-HEU)* maximizers. I also show an R-HEU maximizer never places a negative value on information if and only if either *(i)* every signal is measurable with respect to his prior, or *(ii)* he is maximally averse to ambiguity.

Next, I establish in section 3, there exist many other "priors" each of which has the property that the restriction of an R-HEU maximizer's static preferences to acts that are measurable with respect to such a prior admits a probabilistically sophisticated rank dependent utility (RDU) representation. Not only do these include all the sources characterized by Gul and Pesendorfer (2015) but, in addition, it also allows for ones in which the RDU representation of an act adapted to that source need not be continuous with respect to extreme (that is, best and worst) outcomes. Hence I refer to this class of priors as *non extreme outcome (neo-)continuous* sources. One special subclass corresponds to the neo-additive (capacity) model introduced and analyzed in Chateauneuf et al. (2007). For IDPs involving signals and acts that are all measurable with respect to a neo-additive source, both the static and interim preferences admit neo-EU representations. That is, model consistency holds not just with respect to HEU but also with respect to the neo-additive model. Furthermore, I establish it is only the restriction of his preferences to neo-additive source IDPs that exhibit this more stringent form of model consistency.

Section 4 contains an axiomatization of the R-HEU model. Underpinning it is the idea that the DM, when evaluating an IDP, anticipates after receipt of the signal's realization, he will *know* the information cell corresponding to that realization has obtained while its complement has not. In other words, he knows he will be assigning that information cell a *precise* conditional probability of one (and correspondingly, assigning its complement a conditional probability of zero). So the dynamic restrictions imposed jointly on the DM's ex ante preferences with respect to the signal and his interim preferences associated with that signal entail the set of *effectively ideal* events for his interim preferences consisting of the smallest $\sigma$-algebra of events that includes all the ideal events gleaned from his static preferences as well as the partition of information cells corresponding to the signal's possible realizations.

I discuss in Section 5 our approach in the context of some of the extant literature that includes a detailed comparison with Gul and Pesendorfer's (2021) recursive and consequentialist theory of belief revision. I conclude in Section 6. All proofs appear in Appendix A.

# 2 The Model

Adopting Gul and Pesendorfer's (2014) setting of purely subjective uncertainty, $\Omega$ (with generic element $\omega$) denotes the state space, subsets of which are referred to as events. The non-degenerate interval $X = [\underline{x}\,, \overline{x}]$ comprises the set of prizes and $\mathcal{F}$, the set of mappings from $\Omega$ to $X$, is the set of (Savage) acts, the objects of choice. Each outcome $x \in X$ is also identified with the (constant) act $f$ in which $f(\omega) = x$ for all $\omega$. And with further (albeit fairly standard) abuse of notation, $X$ will also refer to the set of constant acts.

For any pair of events $B, E \subseteq \Omega$, $B \backslash E$ shall denote the set of elements that are in $B$ but not in $E$. For any pair of acts $f$ and $g$ in $\mathcal{F}$ and any event $B \subset \Omega$, $f_B g$ is identified with the act that agrees with $f$ on $B$ and with $g$ on $\Omega \backslash B$.

Let $S$ denote the universe of conceivable realizations for any possible signal. It includes the *null* realization $\varnothing$.

The first component of an *information decision problem (IDP)* is a signal (structure) described by a function $\sigma\colon \Omega \to S$ with finite range, with the interpretation that the DM learns the realization of the signal is $s$ should the event $\sigma^{-1}(s)$ obtain. Let $\Sigma$ denote the set of signals available to the DM which I take to be a non-empty subset of all functions from $\Omega$ to $S$. In particular, $\Sigma$ contains the *null* (that is, *un-*

*informative*) signal $\mathbf{0}$, the constant mapping in which $\mathbf{0}(\omega) = \varnothing$ for all $\omega$ in $\Omega$. The second component is a finite menu of acts $M \subset \mathcal{F}$, $|M| < \infty$. Let $\mathcal{M}$ denote the set of menus. As any signal $\sigma \in \Sigma$ may be paired with any menu $M \in \mathcal{M}$ to form an IDP $(\sigma\,, M)$, the set of IDPs thus comprises the product set $\Sigma \times \mathcal{M}$.

A *plan (of action)* $\phi$ assigns to each $s \in S$ an act $\phi_s \in \mathcal{F}$. A plan of action $\phi$ is *feasible* in the IDP $(\sigma\,, M)$, if $\phi_s \in M$ for each $s \in \sigma(\Omega)$. With a slight abuse of the notation, each feasible plan in an IDP is *identified* with the act it generates. That is, a feasible plan $\phi$ for the IDP $(\sigma\,, M)$ is identified with the act defined by setting $\phi(\omega) := \phi_s(\omega)$ if $\omega \in \sigma^{-1}(s)$.

For each IDP $(\sigma\,, M)$ in $\Sigma \times \mathcal{M}$, let $\Phi(\sigma, M) \subset \mathcal{F}$ denote the set of acts generated by feasible plans in that IDP. I refer to any pair $(\sigma, f)$ in $\Sigma \times \mathcal{F}$ as a *signal-act* and note it may be viewed as a the subclass of IDPs in which the menu contains just a single option. Thus I view and treat $\Sigma \times \mathcal{F}$ (the set of signal-acts) as a subset of $\Sigma \times \mathcal{M}$ (the set of IDPs).

The DM is characterized in part by a binary relation $\succsim$ defined over $\Sigma \times \mathcal{M}$ that guides his selection, made ex ante, among IDPs. I refer to the restriction of $\succsim$ to $\Sigma \times \mathcal{F}$ as the DM's *signal-act preferences*. For each particular signal $\sigma$ in $\Sigma$, I refer to the restriction of $\succsim$ to $\{\sigma\} \times \mathcal{F}$ as his *ex ante preferences* with respect to that signal. Finally, for those signal-acts involving the null signal $\mathbf{0}$, I refer to the restriction of $\succsim$ to $\{\mathbf{0}\} \times \mathcal{F}$ as his *static preferences*.

In addition, there is associated with the DM a set of *interim preferences*

$$\{\succsim_s^\sigma \subseteq \mathcal{F} \times \mathcal{F} \colon \sigma \in \Sigma\,, s \in \sigma(\Omega)\,\}\,.$$

That is, there is an interim preference relation defined over acts for each signal paired with each of its possible realizations.

A *prior* is a countably-additive, complete and non-atomic probability measure defined on a $\sigma$-algebra of subsets of $\Omega$. Let $\Pi$ denote the set of all priors with generic element $\pi$.

Like his counterpart in Gul and Pesendorfer (2015), our DM's static preferences are characterized by a triple $\langle \mu\,, v\,, \alpha \rangle$, where $\mu \in \Pi$ with domain $\mathcal{E}$ is his prior, $v \colon X \to \mathbb{R}$ is his Bernoulli utility, a continuous and strictly increasing function; and, $\alpha \in [0\,,1]$ is his degree of aversion to any ambiguity he perceives present.

For any $g$ measurable with respect to $\mu$, its static subjective expected utility is given by

$$V_\emptyset^{\mathbf{0}}(f) := \int v\, dF_g^\mu\,,$$

where $F_g^\mu$ is the cumulative distribution function (CDF) over outcomes associated with $g$ induced by his prior $\mu$.

More generally, for any prior $\pi \in \Pi$ and any act $f$ measurable with respect to $\pi$, $F_f^\pi$ shall denote the right-continuous non-decreasing function obtained by setting $F_f^\pi(x) := \pi\left(\{\omega \in \Omega \colon f(\omega) \leqslant x\}\right)$.

Consider now how the DM evaluates a signal-act $(\mathbf{0}\,, f)$, in which the act $f$ need *not* necessarily be measurable with respect to $\mu$. Although he may not be able to associate with $f$ a precise CDF, he can always compute least-upper- and greatest-lower-bounds.

Since in the sequel I employ this construction for priors (as well as posteriors) other than $\mu$, I define cumulative probability bounds with respect to a generic prior $\pi$ in $\Pi$.

**Definition 1 (Least-upper- and greatest-lower- cumulative-probabilities)** *Fix a prior $\pi \in \Pi$ with domain $\mathcal{E}^\pi$. For each act $f \in \mathcal{F}$ and each outcome $x \in X$, set*

$$\overline{F}_f^\pi(x) := \inf_{B \in \mathcal{E}^\pi,\, B \supseteq \{\omega \in \Omega \colon f(\omega) \leqslant x\}} \pi(B)\ ,$$

$$\underline{F}_f^\pi(x) := \sup_{B \in \mathcal{E}^\pi,\, B \subseteq \{\omega \in \Omega \colon f(\omega) \leqslant x\}} \pi(B)\ .$$

By construction both $\overline{F}_f^\pi$ and $\underline{F}_f^\pi$ are right-continuous non-decreasing functions over $X$ with $\overline{F}_f^\pi \geqslant \underline{F}_f^\pi$. Moreover, $\overline{F}_f^\pi = \underline{F}_f^\pi \left(= F_f^\pi\right)$ whenever $f$ is measurable with respect to $\pi$.

The static Hurwicz expected utility of the signal act $(\mathbf{0}, f)$ may now be expressed as:

$$V_\emptyset^{\mathbf{0}}(f) = \int v\, d\left[\alpha \overline{F}_f^\mu + (1-\alpha)\, \underline{F}_f^\mu\right]\ . \tag{1}$$

Although this differs from the $\alpha$-MEU formulation presented in Gul and Pesendorfer (2015), since

$$\int v\, d\left[\alpha \overline{F}_f^\mu + (1-\alpha)\, \underline{F}_f^\mu\right] = \alpha \int v\, d\overline{F}_f^\mu + (1-\alpha) \int v\, d\underline{F}_f^\mu\,,$$

their equivalence on the set simple acts, the domain considered by Gul and Pesendorfer, follows from the following proposition.

**Proposition 1** *For each act $f \in \mathcal{F}$ in which $|f(\Omega)| < \infty$:*

$$\int v\, d\overline{F}^{\mu}_{f} = \min_{\pi \in \Pi^{\mu}} \sum_{x \in X} v(x)\pi\left(f^{-1}(x)\right) \text{ and } \int v\, d\underline{F}^{\mu}_{f} = \max_{\pi \in \Pi^{\mu}} \sum_{x \in X} v(x)\pi\left(f^{-1}(x)\right),$$

*where $\Pi^{\mu} \subset \Pi$ is the set of* extensions *of $\mu$, that is, $\pi \in \Pi^{\mu}$, if $\mathcal{E}^{\pi} \supset \mathcal{E}$ and $\pi(E) = \mu(E)$ for every $E$ in $\mathcal{E}$.*

To describe how the DM evaluates an arbitrary signal-act and, more generally, an arbitrary IDP, entails the use of the *measurable split* generated by the preimage of a signal. In order to define this partition of the state space I require some additional notation and definitions, starting with the inner-measure of $\mu$, denoted by $\mu_*$, which is defined by setting

$$\mu_*(B) := \sup_{E \subseteq B : E \in \mathcal{E}} \mu(E) \text{ for each } B \subseteq \Omega .$$

Since $\mu$ is countably additive, the infimum is attained. I shall refer to the measurable event $[B]_*$ in $\mathcal{E}$, as the *inner-sleeve* of $B$, if $[B]_* \subseteq B$ and $\mu([B]_*) = \mu_*(B)$. The inner-sleeve of $B$ may be viewed as the largest meaurable subset of $B$. Correspondingly, the *outer-sleeve* of $B$, denoted by $[B]^*$, is defined by setting $[B]^* := \Omega \backslash [\Omega \backslash B]_*$ and hence may be viewed as the *smallest* measurable *superset* of the event $B$. I refer to $\mu^*(B) := \mu([B]^*)$ as the outer-measure of $B$.[5]

**Definition 2 (A Signal's Measurable Split)** *The* measurable split *generated by the preimage of a signal $\sigma \in \Sigma$, denoted by $\{E^{\sigma}_{Q} \in \mathcal{E} : Q \subseteq \sigma(\Omega), Q \neq \emptyset\}$, is a partition of the state space that is inductively defined as follows:*

1. *For each realization $s \in \sigma(\Omega)$, set $E^{\sigma}_{\{s\}} := [\sigma^{-1}(s)]_*$.*
2. *For each $Q \subseteq \sigma(\Omega)$ such that $|Q| > 1$, set*

$$E^{\sigma}_{Q} := \left[\sigma^{-1}(Q)\right]_* \backslash \left( \bigcup_{\widehat{Q} \subset Q} E^{\sigma}_{\widehat{Q}} \right) .$$

*I refer to $E^{\sigma}_{Q}$ as the $\sigma$-*marginal inner-sleeve *of the set of signal realizations $Q$.*

[5] Notice that the inner-and outer-sleeves are unique up to a set of $\mu$-measure 0.

As an illustration, consider the measurable split generated by the preimage image of a binary signal $\widehat{\sigma}$ that maps each state to one of two possible realizations, $s'$ or $s''$. The measurable split is the three element partition of the state-space

$$\Big\{ \underset{\big[\widehat{\sigma}^{-1}(s')\big]_*}{\underset{\|}{E^{\widehat{\sigma}}_{\{s'\}}}} \;,\; \underset{\big[\widehat{\sigma}^{-1}(s'')\big]_*}{\underset{\|}{E^{\widehat{\sigma}}_{\{s''\}}}} \;,\; \underset{\Omega\setminus\left(\big[\widehat{\sigma}^{-1}(s')\big]_*\cup\big[\widehat{\sigma}^{-1}(s'')\big]_*\right)}{\underset{\|}{E^{\widehat{\sigma}}_{\{s',s''\}}}} \Big\}.$$

The first (respectively, second) element corresponds to the largest measurable subset in which the signal's realization is $s'$ (respectively, $s''$). For the third element, all the DM can discern is that the realization will be either $s'$ or $s''$. However, he is unable to attribute any fraction of the probability his prior assigns to this element of the split to either $s'$ or $s''$ alone.

More generally for an arbitrary signal $\sigma$ and for each non-empty set of its realizations $Q \subseteq \sigma(\Omega)$, recalling the approach of Dempster (1967) and Shafer (1976), I interpret $\mu\big(E^{\sigma}_{Q}\big)$ (the probability assigned by the prior to the $\sigma$-marginal inner-sleeve of $Q$) as measuring the weight the DM places on the evidence that directly supports the realization of the signal residing in $Q$ that cannot be further refined *ex ante* in terms of any of its strict subsets.

Analogous to Gul and Pesendorfer's (2021) two-stage updating procedure for compound random variables, in order to construct his recursive evaluation of an arbitrary signal-act $(\sigma\,,f)$, the DM first forms a *proxy* of his prior $\mu$ that renders all the information cells of the signal unambiguous. This in turn entails the set of *effectively ideal* events of the interim preferences in IDPs involving this signal to comprise the smallest $\sigma$-algebra of events that includes all ideal events of his static preferences as well as all the information cells of the signal.

So as to avoid any double counting, the DM employs a type of "principle of indifference" and attributes an equal fraction $|Q|^{-1}$ of the prior probability $\mu\big(E^{\sigma}_{Q}\big)$ to the weight the proxy assigns to the conjunction the signal realization is $s$ (an element of $Q$) *and* the event $E^{\sigma}_{Q}$ obtains. This implies that the proxy is an extension of the prior.

**Definition 3 (Signal Proxy)** *Fix a signal $\sigma \in \Sigma$. Let $\mathcal{E}^{\sigma}$ denote the smallest $\sigma$-algebra containing $\mathcal{E}$ as well as the pre-image of each realization of $\sigma$. The* proxy *$\mu^{\sigma} \in \Pi^{\mu}$ with domain $\mathcal{E}^{\sigma}$, that the DM associates with the signal $\sigma$ is the extension* *of his prior obtained by setting for each $B \in \mathcal{E}^\sigma$:*

$$\mu^\sigma(B) := \sum_{s \in S} \left( \sum_{Q \subseteq \sigma(\Omega):\, s \in Q} \frac{\mu\left( [B \cap \sigma^{-1}(s)]^* \cap E_Q^\sigma \right)}{|Q|} \right)$$

As a check to see that the proxy is indeed an extension of the DM's prior, notice that for any ideal event $E$ in $\mathcal{E}$, $[E \cap \sigma^{-1}(s)]^* \cap E_Q^\sigma = E \cap [\sigma^{-1}(s)]^* \cap E_Q^\sigma = E \cap E_Q^\sigma$. Hence

$$\begin{aligned} \mu^\sigma(E) &= \sum_{s \in S} \left( \sum_{Q \subseteq \sigma(\Omega):\, s \in Q} \frac{\mu\big(E \cap E_Q^\sigma\big)}{|Q|} \right) \\ &= \sum_{Q \subseteq \sigma(\Omega)} \left( \sum_{s \in Q} \frac{\mu\big(E \cap E_Q^\sigma\big)}{|Q|} \right) = \sum_{Q \subseteq \sigma(\Omega)} \mu\big(E \cap E_Q^\sigma\big) = \mu(E) \ . \end{aligned}$$

The DM next updates this proxy according to Bayes's rule and uses these updated beliefs to form the conditional HEU preferences that determine the conditional certainty equivalents of $f$ for each of the signal's information cells. For each realization $s \in S$ such that $\mu^\sigma(\sigma^{-1}(s)) > 0$, let $\mu_s^\sigma$ denote the Bayesian update of $\mu^\sigma$ conditional on $\sigma^{-1}(s)$ obtaining. That is, for each $B \in \mathcal{E}^\sigma$ set

$$\mu_s^\sigma(B) := \frac{\mu^\sigma(B \cap \sigma^{-1}(s))}{\mu^\sigma(\sigma^{-1}(s))} \ .$$

Notice by definition, $\mu_\emptyset^{\mathbf{0}} \equiv \mu$. Furthermore, since

$$\mu^\sigma(\cdot) = \sum_{s \in S:\, \mu^\sigma(\sigma^{-1}(s)) > 0} \mu^\sigma\big(\sigma^{-1}(s)\big)\, \mu_s^\sigma(\cdot) \ ,$$

for each act $f$ in $\mathcal{F}$ wehave:

$$\begin{aligned} &\int v \, d\left[ \alpha \overline{F}_f^{\mu^\sigma} + (1-\alpha)\, \underline{F}_f^{\mu^\sigma} \right] \\ &= \sum_{s \in S:\, \mu^\sigma(\sigma^{-1}(s)) > 0} \mu^\sigma\big(\sigma^{-1}(s)\big) \left( \int v \, d\left[ \alpha \overline{F}_f^{\mu_s^\sigma} + (1-\alpha)\, \underline{F}_f^{\mu_s^\sigma} \right] \right) \end{aligned}$$

All the elements required to define the class of Recursive HEU maximizers have now been assembled.

**Definition 4 (The Class of Recursive HEU Maximizers)** *An ex ante preference relation* $\succsim$ *over IDPs that has associated with it the set of interim preferences* $\{\succsim_s^\sigma \subset \mathcal{F} \times \mathcal{F} \colon \sigma \in \Sigma\, , s \in \sigma(\Omega)\,\}$ *is a member of the class of* Recursive Hurwicz Expected Utility Maximizers *if there exists a prior* $\mu$ *(with domain* $\mathcal{E}$*), a Bernoulli utility* $v$*, and an ambiguity aversion* $\alpha$*, such that the ex ante and the interim preferences admit representations* $V \colon \Sigma \times \mathcal{M} \to \mathbb{R}$*, and* $V_s^\sigma : \mathcal{F} \to \mathbb{R}$*, respectively, where*

(*i*) *for each signal* $\sigma \in \Sigma$*, each of its realizations* $s \in \sigma(\Omega)$ *in which* $\mu^\sigma(\sigma^{-1}(s)) > 0$*, and each act* $f \in \mathcal{F}$*:*

$$V_s^\sigma(f) = \int v \, d\left[\alpha \overline{F}_f^{\mu_s^\sigma} + (1-\alpha)\, \underline{F}_f^{\mu_s^\sigma}\right];$$

(*ii*) *and, for each IDP* $(\sigma\, , M)$ *in* $\Sigma \times \mathcal{M}$ *:*

$$V(\sigma\, , M) = V_\emptyset^{\mathbf{0}}\left(a^{(\sigma\, , M)}\right)$$

*where* $a^{(\sigma\, , M)}$ *is the auxiliary act associated with* $(\sigma, M)$ *obtained by setting*

$$a^{(\sigma,M)}(\omega) := \begin{cases} v^{-1}(\max_{f \in M} V_s^\sigma(f)) & \text{if } \omega \in \sigma^{-1}(s) \text{ and } \mu^\sigma(\sigma^{-1}(s)) > 0 \\ \underline{x} & \text{otherwise} \end{cases}$$

By construction, for any act $f$ that is adapted to a signal $\sigma$ (that is, $f(\omega) = f(\widehat{\omega})$ whenever $\sigma(\omega) = \sigma(\widehat{\omega})$), we have $V(\sigma\, , f) = V(\mathbf{0}\, , f)$. To interpret this property, notice that $(\sigma\, , f)$ (respectively, $(\mathbf{0}\, , f)$) is a signal-act in which all uncertainty resolves *early* (respectively, *late*). Hence we view the DM assigning both signal acts the same utility as embodying a *time-neutrality* property as we are requiring the DM be indifferent between any pair of signal-acts in which the outcome is determined by the same act, and where all uncertainty resolves either early or late.[6] Hence, unlike the compound risk models of Kreps and Porteus (1978), Epstein and Zin (1989) and Grant et al. (1998), an R-HEU maximizer cannot exhibit an *intrinsic* love of or aversion to information.

[6] This is also a feature of Gul and Pesendorfer's (2021) belief revision model under ambiguity.

## 2.1 When is Information Always (Weakly) Valuable?

Turning now to the question of when an R-HEU maximizer values information, for each signal-act $(\sigma\,, f)$ in $\Sigma \times \mathcal{F}$, I show in Lemma 1 in Appendix A that

$$\overline{F}_f^{\mu} \geqslant \overline{F}_f^{\mu^\sigma} \geqslant \underline{F}_f^{\mu^\sigma} \geqslant \underline{F}_f^{\mu}\,.$$

That is, these CDFs are ranked in terms of the partial ordering of first-order stochastic dominance.

I interpret a strict preference $(\mathbf{0}\,, f) \succ (\sigma\,, f)$ as the DM exhibiting, in the context of the act $f$, an *intrinsic aversion* to the information embodied in the signal $\sigma$ (compared to the baseline of no information). If, however, $\sigma$ is measurable with respect to $\mu$, then since $\mu^\sigma = \mu$, we have $\overline{F}_f^{\mu} = \overline{F}_f^{\mu^\sigma}$ and $\underline{F}_f^{\mu^\sigma} = \underline{F}_f^{\mu}$, which in turn implies $V(\mathbf{0}\,, f) = V(\sigma\,, f)$, or equivalently, $(\mathbf{0}\,, f) \sim (\sigma\,, f)$. That is, if $\sigma$ is measurable with respect to the DM's prior, then he cannot exhibit, in the context of *any* act, an intrinsic aversion to the information embodied in that signal.

Alternatively, if an R-HEU maximizer exhibits maximal aversion to ambiguity, that is, $\alpha = 1$, then from the well-known result that all expected utility maximizers agree with the partial ordering of first-order stochastic dominance we have $V(\mathbf{0}\,, f) = \int v\, d\overline{F}_f^{\mu} \leqslant \int v\, d\overline{F}_f^{\mu^\sigma} \leqslant \int v \overline{F}_{a(\sigma,f)}^{\mu} = V(\sigma\,, f)$. That is, for *no* signal can the DM exhibit, in the context of *any* act, an intrinsic aversion to the information embodied in that signal.

Conversely, in order for an R-HEU maximizer to exhibit, in the context of *some* act, an intrinsic aversion to the information embodied in *some* signal, he cannot be maximally ambiguity averse and there must exist at least one signal that is not measurable with respect to his prior.

Adding the standard instrumental value of information from the DM being able to make his choice contingent on the signal's realization, leads to the following characterization as to when information is always (weakly) valuable.

**Proposition 2** *Consider a DM who is an R-HEU maximizer characterized by the triple $\langle \mu\,, v\,, \alpha \rangle$. The following are equivalent.*

1. *The DM's evaluation of information is never negative, that is,*

$$(\sigma\,, M) \succsim (\mathbf{0}\,, M) \text{ for all } (\sigma\,, M) \in \Sigma \times \mathcal{M}\,.$$

2. *Either every $\sigma \in \Sigma$ is measurable with respect to $\mu$ or $\alpha = 1$.*

## 2.2 The Equal Distribution of the "Harsanyi Dividend".

I conclude this section with the following observation. Fix a signal $\sigma$ in $\Sigma$. We can associate with $\sigma$ a cooperative game in which each realization $s$ in $\sigma(\Omega)$ is deemed a "player". The value of the coalition $Q \subseteq \sigma(\Omega)$ of players is set equal to the inner-measure of that coalition's preimage, that is, $\mu_*(\sigma^{-1}(Q))$. Notice that $\mu(E_Q^\sigma)$, the probability the prior assigns to the $\sigma$-marginal inner-sleeve of $Q$, corresponds to what Harsanyi (1982) dubbed the *"dividend"* generated by the coalition $Q$. Take the "value" of player $s$ to be the weight the proxy $\mu^\sigma$ assigns $\sigma^{-1}(s)$, the information cell associated with the realization $s$. Applying Definition 3 this value may be expressed as

$$\sum_{Q \subseteq \sigma(\Omega)} \left( \sum_{s \in Q} \frac{\mu\big(E_Q^\sigma\big)}{|Q|} \right) .$$

That is, the assignment entails sharing the Harsanyi dividend of each coalition *equally* among the players in that coalition.[7] We shall see in section 4 below that this equal sharing of the dividend generated by a set of realizations among its members underpins the derivation of the DM's proxy that renders each information cell unambiguous.

# 3 Neo-continuous sources

As well as accommodating Ellsberg-style evidence of non-neutral attitudes toward ambiguity, a key feature of the HEU model is its ability to address source-preference and Allais-style violations of expected utility theory. This is because in addition to his prior $\mu$, there are other (indeed many other) "priors" for which the restriction of the DM's static preferences to acts measurable with respect to such a prior admit a particular *probabilistically sophisticated generalized expected utiity* representation, namely, the Rank Dependent Utility (RDU) model first introduced and axiomatized by Quiggin (1982).[8]

For each prior $\pi \in \Pi$ with domain $\mathcal{E}^\pi$, let $\mathcal{F}^\pi$ denote the set of acts that are measurable with respect to $\pi$. The restriction of the DM's static preferences to $\mathcal{F}^\pi$ admits an RDU representation if there exists a *probability transformation* $\tau\colon [0\,,1] \to$

[7] Harsanyi shows this equal distribution rule results in each player being assigned her *Shapley value*.

[8] We are using the term probabilistically sophistication in the sense of Machina and Schmeidler (1992).

$[0\,,1]$ which is strictly increasing and normalized with $\tau(0) := 0$ and $\tau(1) := 1$,[9] such that for every pair of acts $f$ and $f'$ in $\mathcal{F}^\pi$:

$$\int v\, d\left(\tau \circ F^\pi_f\right) \geqslant \int v\, d\left(\tau \circ F^\pi_{f'}\right) \iff V\left(\mathbf{0}\,, f\right) \geqslant V\left(\mathbf{0}\,, f'\right)$$

In line with the motivation that underpins Gul and Pesendorfer's (2015) notion of a source, the focus here is on those priors that can rationalize the restriction of the static preferences between pairs of binary bets for given stakes involving events that reside in the domain of that prior.

**Definition 5 (Non-extreme-outcome (neo)-continuous sources)** *A prior* $\pi \in \Pi$ *is a* neo-continuous source *if for every pair of events* $A$ *and* $B$ *in* $\mathcal{E}^\pi$ *and every pair of outcomes* $x > y$:

$$(\mathbf{0}\,, x_A y) \succsim (\mathbf{0}\,, x_B y) \iff \pi(A) \geqslant \pi(B)\,.$$

To provide a rationale for the terminology employed here as well as to see the connection between my notion of a (neo-continuous) source and and that of Gul and Pesendorfer, let's begin by recalling Gul and Pesendorfer refer to a function $\gamma\colon [0, 1] \to [0, 1]$ as a power series if there is a sequence $a = (a_1, a_2, \ldots)$ such that $a_i \in [0, 1]$, $\sum_i a_i = 1$ and $\gamma(t) = \sum_i a_i \cdot t^i$. Let $\Gamma$ denote the set of all power series. Extending Gul and Pesendorfer's notion, a prior $\pi \in \Pi$ *conforms conditionally on the ideal event* $E$ (in $\mathcal{E}$) to the power series $\gamma \in \Gamma$ if

$$\mu_*(A) = \begin{cases} \mu(E)\gamma(\pi(A))\,, & \text{if } \pi(A) < 1 \\ 1 & \text{if } \pi(A) = 1 \end{cases} \qquad \text{for all } A \in \mathcal{E}^\pi\,. \tag{2}$$

Gul and Pesendorfer consider the (special) case of a prior conforming to the power series on the entire state space, and so require $\mu_*(A) = \gamma(\pi(A))$.

Just as Gul and Pesendorfer established HEU permits infinitely many distinct sources and that a power series characterizes each one, the following generalizes their Proposition 2.

**Proposition 3** (i) *A prior is a neo-continuous source if and only if it conforms conditionally on some non-null ideal event to some power series;* (ii) *for every non-*

[9] Gul and Pesendorfer (2015) also require $\tau$ to be continuous but I include probability transformations that allow for jumps at 0 and at 1.

*null ideal event and every power series, there exists a neo-continuous source that conforms conditionally on that event to that power series.*

Notice, by fixing a neo-continuous source $\pi$ that conforms conditionally on the ideal event $E$ to the power series $\gamma$, it readily follows for any *non-null* and *non-universal* event $A$ from that source (that is, the event is in $\mathcal{E}^{\pi}$ and the probability $\pi$ assigns it is strictly positive but less than one), its outer-measure is given by

$$\begin{aligned}\mu^{*}(A) &= 1-\mu_{*}(\Omega\backslash A) = 1-\mu(E)\gamma\left(\pi\left(\Omega\backslash A\right)\right)\\ &= 1-\mu(E)+\mu(E)-\mu(E)\gamma\left(1-\pi\left(A\right)\right)\\ &= 1-\mu(E)+\mu(E)\,\widehat{\gamma}\left(\pi\left(A\right)\right)\end{aligned}$$

where $\widehat{\gamma}(p) := 1-\gamma\left(1-p\right)$ is the *conjugate dual* of $\gamma$. Hence for any bet based on $A$ involving the stakes $x > y$, we have

$$\begin{aligned}&V^{\mathbf{0}}_{\emptyset}(x_A y) = \left(\alpha\mu_{*}\left(A\right)+\left(1-\alpha\right)\mu^{*}\left(A\right)\right)v\left(x\right)+\left(1-\alpha\mu_{*}\left(A\right)-\left(1-\alpha\right)\mu^{*}\left(A\right)\right)v(y)\\ &= \left(\left(1-\lambda\right)\left[\alpha\gamma\left(p\right)+\left(1-\alpha\right)\widehat{\gamma}\left(p\right)\right]+\left(1-\alpha\right)\lambda\right)v\left(x\right)\\ &+\left(\left(1-\lambda\right)\left[\alpha\widehat{\gamma}\left(1-p\right)+\left(1-\alpha\right)\gamma\left(1-p\right)\right]+\alpha\lambda\right)v\left(y\right)\\ &= \left(1-\lambda\right)\left(\left[\alpha\gamma\left(p\right)+\left(1-\alpha\right)\widehat{\gamma}\left(p\right)\right]v\left(x\right)+\left[\alpha\widehat{\gamma}\left(1-p\right)+\left(1-\alpha\right)\gamma\left(1-p\right)\right]v\left(y\right)\right)\\ &+\lambda\left(\left(1-\alpha\right)v\left(x\right)+\alpha v\left(y\right)\right)\end{aligned}$$

where $\lambda = 1-\mu(E)$ and $p = \pi(A)$. That is, the utility of the bet may be expressed as a $(1-\lambda, \lambda)$-convex combination of

$$\left[\alpha\gamma\left(p\right)+\left(1-\alpha\right)\widehat{\gamma}\left(p\right)\right]v\left(x\right)+\left[\alpha\widehat{\gamma}\left(1-p\right)+\left(1-\alpha\right)\gamma\left(1-p\right)\right]v\left(y\right) \tag{3}$$

and

$$\left(1-\alpha\right)v\left(x\right)+\alpha v\left(y\right) \tag{4}$$

Expression (3) corresponds to the RDU of the bet $x_A y$ (for which $F^{\pi}_{x_A y}(y) = 1-p$) with respect to the (continuous) probability transformation $\tau = \alpha\widehat{\gamma}+(1-\alpha)\gamma$. Expression (4), on the other hand, is the convex combination of the utilities of the worst and best outcomes in that bet in which the weight on the utility of the worst outcome equals the DM's amibiguity aversion.

The next proposition states for any act adapted to a neo-continuous source, its static HEU representation can be expressed in terms of such a convex combination. To state it, we introduce the following notation. For any CDF $F$, let $\underline{x}_F$ (respectively, $\overline{x}_F$) denote the greatest lower (respectively, least upper) bound of the support of $F$. That is, $\underline{x}_F := \sup\{y : F(y) = 0\}$ and $\overline{x}_F := \inf\{y : F(y) = 1\}$.

**Proposition 4** *Suppose a prior $\pi \in \Pi$ is a neo-continuous source that conforms conditionally on the ideal event $E \in \mathcal{E}$ to the power series $\gamma$. Set $\lambda := 1-\mu(E)$ and $\tau := \alpha\widehat{\gamma} + (1-\alpha)\gamma$. Then for each act $f \in \mathcal{F}^\pi$:*

$$V_\emptyset^{\mathbf{0}}(f) = (1-\lambda)\int v\, d\left[\tau \circ F_f^\pi\right] + \lambda\left(\alpha v(\underline{x}_{F_f^\pi}) + (1-\alpha)\, v(\overline{x}_{F_f^\pi})\right) . \tag{5}$$

Thus we see the restriction of the static preferences to acts in $\mathcal{F}^\pi$ admits a representation defined over the CDFs of the acts induced by the source, characterized by the quartet, comprising (i) the Bernoulli utility $v$, (ii) the ambiguity aversion $\alpha$, (iii) the parameter $\lambda$, and (iv) the power series $\gamma$.

For the special case in which $\gamma$ is the identity, the righthand side of expression (5) collapses to the neo-expected utility of an act which is how a decision maker from the neo-additive capacity model introduced by Chateauneuf et al. (2007) evaluates acts. Chateauneuf et al. interpret the parameter $\lambda$ as a measure of the amount of ambiguity the DM perceives he is facing with the complementary weight $1-\lambda$ measuring the confidence he places on the probability $\pi$. We shall refer to $\lambda$ as the DM's lack of confidence in the *source*. In evaluating $F_f^\pi$ (the CDF of the act $f$ induced by the source $\pi$), the DM places weight $1-\lambda$ on the RDU computed with respect to the probability transformation $\tau = \alpha\widehat{\gamma} + (1-\alpha)\,\gamma$. He then accounts for his lack of confidence in the source by assigning a fraction $\alpha$ of the remaining weight $\lambda$ to the utility of the worst outcome in the support of $F_f^\pi$, with the residual placed on the utility of the best outcome in the support of $F_f^\pi$.

## 3.1 Updating neo-additive sources

Fix a prior $\pi \in \Pi$ that is a neo-continuous source with respect to some non-null ideal event $E$. Analogous to the class of ideal IDPs discussed in the introduction, an IDP $(\sigma, M)$ is a $\pi$-IDP if: (i) $\sigma^{-1}(s) \in \mathcal{E}^\pi$ with $\pi(\sigma^{-1}(s)) > 0$ for every realization $s$ in $\sigma(\Omega)$; and, (ii) $M \subset \mathcal{F}^\pi$.[10]

[10] We observe that the class of ideal IDPs is the class of $\mu$-IDPs.

To see how we may express the representation of the restriction of the DM's preferences to $\pi$-source IDPs in which $\pi$ is a neo-*additive* source, consider first a signal-act $(\sigma, f)$ that is a $\pi$-source IDP. Set $n^\sigma := |\{s \in S\colon \pi(\sigma^{-1}(s)) > 0\}|$ and for each realization $s$ such that $\pi(\sigma^{-1}(s)) > 0$, let $\pi_s^\sigma(\cdot)$ denote the update of $\pi$ conditional on the realization of $\sigma$ is $s$. It is defined by setting for each $A$ in $\mathcal{E}^\pi$,

$$\pi_s^\sigma(A) := \frac{\mu(A \cap \sigma^{-1}(s) \cap E)/\mu(E)}{\mu(\sigma^{-1}(s) \cap E)/\mu(E)} = \frac{\pi(A \cap \sigma^{-1}(s))}{\pi(\sigma^{-1}(s))}.$$

Applying the definitions and results from above yields

$$\begin{aligned}
&\int v\, d\left[\alpha \overline{F}_f^{\mu^\sigma} + (1-\alpha)\, \underline{F}_f^{\mu^\sigma}\right] \\
&= \sum_{s \in \sigma(\Omega)} \left[\underbrace{\mu\left(\sigma^{-1}(s) \cap E\right)}_{(1-\lambda)\pi(\sigma^{-1}(s))} \int v\, dF_f^{\pi_s^\sigma} + \frac{\lambda}{n^\sigma}\left(\alpha v\left(\underline{x}_{F_f^{\pi_s^\sigma}}\right) + (1-\alpha)\, v\left(\overline{x}_{F_f^{\pi_s^\sigma}}\right)\right)\right] \\
&= \sum_{s \in \sigma(\Omega)} \underbrace{\left((1-\lambda)\,\pi\left(\sigma^{-1}(s)\right) + \lambda \frac{1}{n^\sigma}\right)}_{p_s^\sigma := \mu^\sigma(\sigma^{-1}(s))} \left[\underbrace{\frac{(1-\lambda)\,\pi\left(\sigma^{-1}(s)\right)}{(1-\lambda)\,\pi\left(\sigma^{-1}(s)\right) + \lambda/n^\sigma}}_{1-\lambda_s^\sigma} \int v\, dF_f^{\pi_s^\sigma} \right.\\
&\qquad \left. + \underbrace{\frac{\lambda/n^\sigma}{(1-\lambda)\,\pi\left(\sigma^{-1}(s)\right) + \lambda/n^\sigma}}_{\lambda_s^\sigma}\left(\alpha v\left(\underline{x}_{F_f^{\pi_s^\sigma}}\right) + (1-\alpha)\, v\left(\overline{x}_{F_f^{\pi_s^\sigma}}\right)\right)\right] \\
&= \sum_{s \in \sigma(\Omega)} p_s^\sigma \left[(1-\lambda_s^\sigma) \int v\, dF_f^{\pi_s^\sigma} + \lambda_s^\sigma \left(\alpha v\left(\underline{x}_{F_f^{\pi_s^\sigma}}\right) + (1-\alpha)\, v\left(\overline{x}_{F_f^{\pi_s^\sigma}}\right)\right)\right]
\end{aligned}$$

where

$$\begin{aligned}
p_s^\sigma &= (1-\lambda)\,\pi\left(\sigma^{-1}(s)\right) + \lambda \frac{1}{n_\sigma}\,, \\
\lambda_s^\sigma &= \frac{\lambda/n_\sigma}{p_s^\sigma} = \frac{\lambda}{(1-\lambda)\left(\frac{\pi(\sigma^{-1}(s))}{1/n_\sigma}\right) + \lambda}\,,
\end{aligned}$$

Hence, if for each realization $s \in S$ for which $\pi\left(\sigma^{-1}(s)\right) > 0$ we set

$$V_s^{\sigma}(f) := (1-\lambda_s^{\sigma}) \int v \, dF_f^{\pi_s^{\sigma}} + \lambda_s^{\sigma} \left( \alpha v(\underline{x}_{F_f^{\pi_s^{\sigma}}}) + (1-\alpha)\, v(\overline{x}_{F_f^{\pi_s^{\sigma}}}) \right) .$$

then for the $\pi$-IDP $(\sigma, M)$, we have

$$V(\sigma\,, M) = (1-\lambda) \int v \, dF^{\pi}_{a^{(\sigma,M)}} + \lambda \left( \alpha v(\underline{x}_{F^{\pi}_{a^{(\sigma,M)}}}) + (1-\alpha)\, v(\overline{x}_{F^{\pi}_{a^{(\sigma,M)}}}) \right)$$

where $a^{(\sigma,M)}$ is the auxiliary act associated with $(\sigma, M)$ constructed by setting

$$a^{(\sigma,M)}(\omega) := \begin{cases} v^{-1}(\max_{f\in M} V_s^{\sigma}(f)) & \text{if } \omega \in \sigma^{-1}(s) \text{ and } \pi\left(\sigma^{-1}(s)\right) > 0 \\ \underline{x} & \text{otherwise} \end{cases} .$$

The signifcance of the above algebraic manipulations is that they demonstrate for any IDP adapted to a given neo-additive source $\pi$, the signal's realization-contingent parameters of the neo-EU representation of the interim preferences, may be formulated entirely in terms of the quartet $\langle \pi\,, v\,, \alpha\,, \lambda \rangle$, comprising the source, the Bernoulli utility, the aversion to ambiguity, and the lack of confidence in the source. That is, not only are the interim preferences model-consistent in terms of remaining within the HEU model, but they remain within the neo-EU model and admit formulations that eshew any (explicit) reference to the DM's prior $\mu$.

An implication of the next proposition is that if for each source there is at least one non-degenerate binary signal adapted to that source, then it is only the neo-additive source IDPs that exhibit this more stringent *source* model consistency property.

**Proposition 5** *Without loss of generality, set $v(\bar{x}) := 1$ and $v(\underline{x}) := 0$. Take $\pi \in \Pi$ (with domain $\mathcal{E}^{\pi}$) to be a source that conforms to the power series $\gamma(p) = p^n$.*[11] *Suppose there exists an event $A \in \mathcal{E}^{\pi}$ with $\pi(A) \in (0,1)$ and a signal $\sigma \in \Sigma$, for which $\sigma(\omega) = s'$ if $\omega \in A$ and $\sigma(\omega) = s''$, otherwise. Then for any $\widehat{A} \in \mathcal{E}^{\pi}$ such that $\widehat{A} \subset A$ and $\pi(\widehat{A}) \in (0, \pi(A))$:*

$$V\left(\sigma, \bar{x}_{\widehat{A}}\underline{x}\right) = \left(\alpha\gamma\left(\pi(A)\right) + (1-\alpha)\widehat{\gamma}\left(\pi(A)\right)\right) V_{s'}^{\sigma}\left(\overline{x}_{\widehat{A}}\underline{x}\right)$$

*where*

$$V_{s'}^{\sigma}\left(\bar{x}_{\widehat{A}}\underline{x}\right) = \alpha\gamma\left(\frac{\pi\left(\widehat{A}\right)}{\pi\left(A\right)}\right) + (1-\alpha)\,\widehat{\gamma}\left(\frac{\pi\left(\widehat{A}\right)}{\pi\left(A\right)}\right) ,$$

[11] That is, $\pi$ conforms conditionally on $\Omega$ to $\gamma(p) = p^n$.

*if and only if* $\gamma(p) \equiv p$, *that is,* $n = 1$.

# 4 Characterization

To state the axioms for the characterization requires the following definitions, many taken from Gul and Pesendorfer (2015) albeit suitably adapted for a dynamic setting. The event $E$ is *ideal* if both the event and its complement $\Omega \backslash E$ exhibit a form of Savage's property **P2** with respect to the DM's static preferences. Formally, the event $E$ is deemed ideal if for every quartet of acts $f$, $f'$, $\widehat{f}$ and $\widetilde{f}$ in $\mathcal{F}$:

$$\Big[(\mathbf{0}, f_E \widehat{f}) \succsim (\mathbf{0}, f'_E \widehat{f}) \text{ and } (\mathbf{0}, \widehat{f}_E f) \succsim (\mathbf{0}, \widehat{f}_E f')\Big] \\ \Longrightarrow \\ \Big[(\mathbf{0}, f_E \widetilde{f}) \succsim (\mathbf{0}, f'_E \widetilde{f}) \text{ and } (\sigma, \widetilde{f}_E f) \succsim (\mathbf{0}, \widetilde{f}_E f')\Big] \,.$$

As Gul and Pesendorfer observe for an HEU maximizer, ideal events represent those events for which the DM associates the least uncertainty (corresponding to the events in the domain of the prior $\mu$ in Definition 4). The DM uses ideal events to quantify the uncertainty of other, non-ideal events. Let $\mathcal{E}$ denote the set of all ideal events. We refer to an act as *ideal* if it is adapted to $\mathcal{E}$. Formally, an act $g$ is ideal if $g^{-1}(Y) \in \mathcal{E}$ for all $Y \in \mathcal{B}(X)$ where $\mathcal{B}(X)$ is the smallest $\sigma$-algebra of subsets of $X$ containing all open subsets of $X$.[12] Let $\mathcal{G} \subset \mathcal{F}$ denote the set of ideal acts.

An event $B$ is *null* if $(\mathbf{0}\,, \widehat{f}_B f) \sim (\mathbf{0}\,, f)$ for all $f\,, \widehat{f} \in \mathcal{F}$. Let $\mathcal{N}$ denote the set of null events. For each non-null event $A \notin \mathcal{N}$ and each act $f \in \mathcal{F}$, define $f^+(A) := \inf\{Y \in \mathcal{B}(X)\colon A \backslash f^{-1}(Y) \in \mathcal{N}\}$. We interpret $f^+(A)$ as the "support" of the act $f$ on the non-null event $A$.

An event $D$ is *diffuse* if for every non-null ideal event $E$ in $\mathcal{E} \backslash \mathcal{N}$:

$$E \cap D \neq \emptyset \neq E \cap \Omega \backslash D\,.$$

From the perspective of the DM's static preferences, diffuse events are the most uncertain, since they (and their complements) contain no non-null ideal events. Let $\mathcal{D}$ denote the set of all diffuse events. We say an act $h$ is *diffuse* if for every $Y \in \mathcal{B}(X)$, $h^{-1}(Y) \in \mathcal{D}$ whenever $h^{-1}(Y) \notin \mathcal{N}$ and $h^{-1}(X \backslash Y) \notin \mathcal{N}$. That is, whenever it is neither impossible nor certain an element from the set of outcomes $Y$ will result, the DM is unable to use ideal events to quantity with any more precision the likelihood

[12] Equivalently, $g$ is ideal if $g^{-1}(\{y\colon y \leqslant x\}) \in \mathcal{E}$ for all $x \in X$.

an outcome from that set $Y$ will occur. Let $\mathcal{H} \subset \mathcal{F}$ denote the set of diffuse acts.

With these preliminaries in hand I begin by assuming the DM's static preferences admit an HEU representation.[13]

**Assumption H (Static HEU Maximization)** *There exists a prior $\mu$ with domain $\mathcal{E}$, a Bernoulli utility $v$, and an ambiguity aversion $\alpha$, such that for all $f$ and $f'$ in $\mathcal{F}$, $(\mathbf{0}\,, f) \succsim (\mathbf{0}\,, f')$ if and only if*

$$\int v\, d\left[\alpha \overline{F}^{\mu}_{f} + (1-\alpha)\, \underline{F}^{\mu}_{f}\right] \geqslant \int v\, d\left[\alpha \overline{F}^{\mu}_{f'} + (1-\alpha)\, \underline{F}^{\mu}_{f'}\right]$$

Next consider a signal-act $(\sigma\,, f)$ in which the act is adapted to the signal, that is, $f(\omega) = f(\widehat{\omega})$ whenever $\sigma(\omega) = \sigma(\widehat{\omega})$. This as a signal-act in which all the uncertainty is resolved *early*, since once the DM learns the realization of the signal he knows what outcome will result. Alternatively, by replacing $\sigma$ with $\mathbf{0}$, the signal act that results, $(\mathbf{0}\,, f)$, also has all its uncertainty resolved at one point in time, albeit now *late*. The first axiom states the DM cannnot express an intrinsic strict preference between all early versus all late resolution of uncertainty.

**Axiom 1 (Timing Neutrality)** *For each signal $\sigma \in \Sigma$: $(\mathbf{0}\,, f) \sim (\sigma\,, f)$ for every act in which $\sigma(\omega) = \sigma(\widehat{\omega}) \implies f(\omega) = f(\widehat{\omega})$.*

The following two axioms are adapted from Ghirardato (2002). The first requires that the interim preferences associated with a given signal and any of its non-null realizations should not depend on how the acts behave on states outside the information cell corresponding to that realization. The second imposes a dynamic restriction on the DM's ex ante preferences with respect to a given signal and the interim preferences associated with each of the signal's non-null realizations.

**Axiom 2 (Consequentialism)** *Fix a signal $\sigma \in \Sigma$. For every realization $s \in \sigma(\Omega)$ such that $\sigma^{-1}(s) \notin \mathcal{N}$ and every pair of acts $f\,, \widetilde{f}$ in $\mathcal{F}$: $\quad f \sim^{\sigma}_{s} f_{\sigma^{-1}(s)}\widetilde{f}$.*

**Axiom 3 (Dynamic Consistency)** *Fix a signal $\sigma \in \Sigma$. For every realization $s \in \sigma(\Omega)$ such that $\sigma^{-1}(s) \notin \mathcal{N}$ and every pair of acts $f\,, \widetilde{f}$ in $\mathcal{F}$:*

[13] In Appendix B we provide an axiomatic characterization of Assumption H that draws heavily on the representation result in Gul and Pesendorfer (2015). Gul and Pesendorfer's result in turn relies on their characterization of the EUU model in Gul and Pesendorfer (2014). As Grant et al. (2023) have recently shown, however, those axioms are not sufficient to ensure the preferences admit an EUU representation. Hence our axiomatization also utilizes the modifications suggested by Grant et al. (2023).

$$f \succsim_s^\sigma \widetilde{f} \implies (\sigma\,, f_{\sigma^{-1}(s)}\widetilde{f}) \succsim (\sigma\,, \widetilde{f})$$

Echoing the intuition offered by Ghirardato for requiring dynamic consistency take this form, I interpret Axiom 3 as saying that if the DM anticipates he would (weakly) prefer the act $f$ over the act $\widetilde{f}$ were he told the realization of the signal $\sigma$ was $s$, then given a (default) plan of action that is identified with the act $\widetilde{f}$, the possibility to postpone his choice until actually observing the realization $s$ does not make him any worse off.

I next require every event identified as ideal with respect to the static preferences remains so for each of the interim preferences. Formally, the axiom states for any interim preference relation each ideal event is both "left-ideal" (it satisfies Savage's postulate P2) and "right-ideal" (its complement satisfies P2). From Lemma B0 in Gul and Pesendorfer (2014, p25) it then follows that the event is ideal for these interim preferences.

**Axiom 4 (Preservation of Ideal Events)** *For every signal $\sigma \in \Sigma$, every realization $s \in \sigma(\Omega)$, every ideal event $E \in \mathcal{E}$ and every pair of acts $f$ and $f'$ in $\mathcal{F}$:*

$$f_E f' \succsim_s^\sigma f' \implies f \succsim_s^\sigma f'_E f \text{ and } f'_E f \succsim_s^\sigma f' \implies f \succsim_s^\sigma f_E f' \,.$$

To provide some intuition for the next axiom, consider a signal and its associated measurable split as specified in definition 2. Recall in the discussion immediately following that definition, the probability assigned by the prior to an element of the measurable split associated with a set of possible realizations of the signal was interpreted as measuring the weight the DM places on evidence that directly supports the signal's realization coming from that set that cannot be further refined in terms of any of its strict subsets. So, in line with his inability to refine that probability any further, should the DM be informed the signal's realization is a *particular* element in that set, then the probability he assigns to an ideal event conditional on that element of the measurable split obtaining should remain unchanged. More generally, the axiom requires the DM's betting preferences be invariant over such conditional bets.

**Axiom 5 (Invariant Interim Risk Preferences)** *Fix a signal $\sigma \in \Sigma$ with associated measurable split $\{E_Q^\sigma \in \mathcal{E} : Q \subseteq \sigma(\Omega)\,, Q \neq \emptyset\}$. For each $Q \subseteq \sigma(\Omega)$ such that $E_Q^\sigma \notin \mathcal{N}$, and each $s \in Q$:*

$$\Big(\mathbf{0}\,, (x_E y)_{E_Q^\sigma} \underline{x}\Big) \succsim \Big(\mathbf{0}\,, z_{E_Q^\sigma} \underline{x}\Big) \iff (x_E y)_{E_Q^\sigma} \underline{x} \succsim_s^\sigma z_{E_Q^\sigma} \underline{x}\,,$$

*for all ideal $E \in \mathcal{E}$ and all outcomes $x$ , $y$ , and $z$ in $X$.*

Motivation for the next axiom comes from the following property of the static preferences that stems from them admitting an HEU representation.

**Proposition 6** *Suppose $\succsim$ satisfies Assumption H. Fix a signal $\sigma \in \Sigma$ with associated measurable split $\{E^\sigma_Q \in \mathcal{E} : Q \subseteq \sigma(\Omega)\, , Q \neq \emptyset\}$. Then for each $Q \subseteq \sigma(\Omega)$ such that $E^\sigma_Q \notin \mathcal{N}$,*

$$(\mathbf{0}, \big(x_{\sigma^{-1}(s)}y\big)_{E^\sigma_Q} f) \sim (\mathbf{0}, \big(x_{\sigma^{-1}(s')}y\big)_{E^\sigma_Q} f)\,,$$

*for any pair of realizations $s$ , $s'$ in $Q$, any pair of outcomes $x > y$, and any act $f$.*

In terms of the DM's static preferences, we see from Proposition 6 that conditional on $E^\sigma_Q$ obtaining, the DM is indifferent between a bet on $\sigma^{-1}(s)$ and a bet on $\sigma^{-1}(s')$. In other words, from the perspective of his static preferences the pair of events $E^\sigma_Q \cap \sigma^{-1}(s)$ and $E^\sigma_Q \cap \sigma^{-1}(s')$ are exchangeable. This exchangeability property exhibited by his static preferences, suggests that when constructing the proxy that renders each information cell of the signal unambiguous, the DM should distribute the "Harsanyi dividend" associated with each "coalition of realizations" $Q$ (that is, the weight his prior attaches to the event $E^\sigma_Q$) *uniformly* among its members. The following definition along with the next axiom embody this line of reasoning.

**Definition 6 (Equally-Distributed Harsanyi-Dividend Ideal Partition)** *Fix a signal $\sigma \in \Sigma$, with associated measurable split $\{E^\sigma_Q \in \mathcal{E} : Q \subseteq \sigma(\Omega)\, , Q \neq \emptyset\}$. For each non-empty $Q \subseteq \sigma(\Omega)$, let $\{E^\sigma_{Q,s} \colon s \in Q\}$ denote an ideal partition of $E^\sigma_Q$ that comprises* $|Q|$ uniformly weighted *elements. That is, $\mu(E^\sigma_{Q,s}) = \mu(E^\sigma_Q)/|Q|$ for each $s \in Q$.*

*An* equally-distributed Harsanyi-dividend ideal partition*, $\{E^\sigma_s \colon s \in \sigma(\Omega\}$, that can be associated with the signal $\sigma$ is then formed by setting for each $s$ in $\sigma(\Omega)$:*

$$E^\sigma_s := \bigcup_{Q \subseteq \sigma(\Omega) \colon s \in Q} E^\sigma_{Q,s}\,.$$

*Notice that,*

$$[\sigma^{-1}(s)]_* \;\left( = E^\sigma_{\{s\}} \right) \subseteq E^\sigma_s \subseteq \left( \bigcup_{Q \subseteq \sigma(\Omega) \colon s \in Q} E^\sigma_Q \;= \right) [\sigma^{-1}(s)]^*$$

*for each $s$ in $\sigma(\Omega)$.*

The next axiom requires after the DM learns the signal's realization, his interim preferences over bets on elements of the measurable split conform to his static preferences over *conditional* versions of such bets where the conditioning event is the element of an associated equally-distributed Harsanyi-dividend ideal partition corresponding to that realization.

**Axiom 6 (Equal distribution of Harsanyi Dividend)** *Fix a signal $\sigma \in \Sigma$ with associated measurable split $\{E^\sigma_Q \in \mathcal{E} : Q \subseteq \sigma(\Omega)\,, Q \neq \emptyset\}$. Let $\{E^\sigma_s : s \in \sigma(\Omega\}$ denote an equally-distributed Harsanyi-dividend ideal partition that can be associated with $\sigma$.*

*For each $Q \subseteq \sigma(\Omega)$ such that $E^\sigma_Q \notin \mathcal{N}$, and each $s \in Q$:*

$$\Big(\mathbf{0}\,, (x_{E^\sigma_Q} y)_{E^\sigma_s} \underline{x}\Big) \succsim \big(\mathbf{0}\,, z_{E^\sigma_s} \underline{x}\big) \iff x_{E^\sigma_Q} y \succsim^\sigma_s z\,,$$

*for all outcomes $x\,, y\,,$ and $z$ in $X$.*

I also require the DM's attitude toward any (remaining) ambiguity he perceives present remains unchanged. In particular, the unconditional certainty equivalent of a diffuse act equates with the interim conditional certainty equivalent of any diffuse act with the same (conditional) support.

**Axiom 7 (Invariant Uncertain Outcome Preferences)** *Fix a signal $\sigma \in \Sigma$. For any realization $s \in \sigma(\Omega)$ with $\sigma^{-1}(s) \notin \mathcal{N}$, any pair of diffuse acts $h$ and $\widehat{h}$ in $\mathcal{H}$, any outcome $z$ in $X$, and any event $E \in \mathcal{E}^\sigma$ such that $E \subseteq \sigma^{-1}(s)$ and $E \notin \mathcal{N}$,*

$$\text{if } h^+(\Omega) = \widehat{h}^+(E) \text{ then } (\mathbf{0}\,, h) \succsim (\mathbf{0}\,, z) \iff \widehat{h} \succsim^\sigma_s z_E \widehat{h}\,.$$

Finally, the DM may only express a preference for one IDP over another when there exists an available plan of action in the former that dominates according to her signal-act preferences all available plans of action in the latter.

**Axiom 8 (Ex ante optimal planning)** *For any pair of IDPs $(\sigma, M)$ and $(\sigma', M')$ in $\Sigma \times \mathcal{M}$, $(\sigma, M) \succsim (\sigma', M')$ if and only if there exists $\phi \in \Phi(\sigma, M)$ for which $(\sigma\,, \phi) \succsim (\sigma'\,, \phi')$ for all $\phi' \in \Phi(\sigma, M')$.*

The representation theorem follows.

**Theorem 1** *Suppose Assumption H holds. Then the relation $\succsim$ with associated interim preferences $\{\succsim^\sigma_s \subset \mathcal{F} \times \mathcal{F} : \sigma \in \Sigma\,, s \in \sigma(\Omega)\,\}$ are those of an R-HEU maximizer if and only if Axioms 1 – 8 hold.*

# 5 Related Literature

In the standard model of subjective expected utility, where beliefs are represented by a probability measure, the rule applied almost universally to incorporate information is Bayesian updating. And indeed, as noted in the introduction, for the class of ideal IDPs, an R-HEU maximizer's updated beliefs are simply the Bayesian update of his prior conditional on the information cell corresponding to the signal's realization having obtained.

For non-additive beliefs there have been two major approaches in the literature. The first is a statistical approach that considers for different updating rules the statistical properties of the conditional expectations derived from such rules. Examples include Denneberg (1994, 2002), Jaffray (1992), Lapied et al. (2012), Lehrer (2005), Shafer (1976), Walley (1991), and, more recently and most relevant here, Gul and Pesendorfer (2021).

The other approach and the one taken in this paper, is decision-theoretic. The updating rule arises from axioms imposed on the preferences both ex ante and interim: a nonexhaustive list of examples includes Epstein and Schneider (2003), Gilboa and Schmeidler (1993), Hanany and Klibanoff (2007), Pires (2002), Siniscalchi (2011), Sarin and Wakker (1998), and Wang (2003).

In contrast to all of the above, however, without any need to impose any a priori restriction on the domain of acts and/or signals, our dynamic extension of HEU allows for a DM with ex ante and interim preferences that not only both conform to HEU but are also consequentialist and dynamically consistent.[14]

At first blush this might seem at odds with Ghirardato (2002) who established, even without requiring a priori the ex ante preferences be SEU: imposing both of his notions of dynamic consistency and consequentialism *entailed* the ex ante preferences conforming to SEU. However, his result can be reconciled with the current framework by noting a key difference is that the set of interim preferences we consider is richer. Recall, the DM's interim preferences are allowed to depend not just on the conditioning event (that is, the information cell corresponding to the realization of the signal) but also on the entire partition of information cells induced by all the signal's possible realizations.

[14]Hanany and Klibanoff (2006), Siniscalchi (2006), and Sarin and Wakker (1998) drop consequentialism and retain dynamic consistency. Epstein and Schneider (2003) retain both in a multiple prior model for a fixed signal and then require the set of priors to be *rectangular* with respect to the information cells of the signal.

Gul and Pesendorfer's (2021) recursive and consequentialist theory of belief revision also allows updated beliefs and hence conditional evaluations to (potentially) depend on the entire signal and not just the information cell corresponding to a particular realization. To compare their approach with the one here, first notice a compound random variable in their setting may naturally be identified with a signal-act $(\sigma, f)$ in ours. They encode the DM's *prior* perception of uncertainty with a totally monotone capacity, $\nu$. In their model the DM evaluates a signal-act $(\mathbf{0}, f)$ (that is, one involving the null signal $\mathbf{0}$) by setting

$$V^{\text{GP}}(\mathbf{0}, f) := \int v \circ f \, d\nu \,,$$

the Choquet integral of the state-contingent utility $v \circ f$ with respect to $\nu$.[15] Notice if we take $\nu$ to be $\mu_*$, the inner-measure associated with an R-HEU maximizer's prior $\mu$, then since $\int v \circ f \, d\mu_* = \int v \circ f \, d\overline{F}^{\mu}_f$, these static preferences correspond to those of an HEU maximizer with an extreme aversion to any ambiguity he perceives present, that is, $\alpha = 1$.

For signal-acts involving a non-null signal, analogous to what an R-HEU maximizer does with his prior probability, their DM first forms a *"proxy"* capacity, which we shall denote by $\nu^\sigma$, that is derived from $\nu$ in a manner rendering each of the information cells of the signal *unambiguous*, just as the proxy prior does for an R-HEU maximizer.

The DM next updates the proxy for each realization $s$, for which $\nu^\sigma(\sigma^{-1}(s)) > 0$, according to Bayes' rule,

$$\nu^\sigma_s(A) := \frac{\nu^\sigma(A \cap \sigma^{-1}(s))}{\nu^\sigma(\sigma^{-1}(s))} \,,$$

and computes $\int v \circ f \, d\nu^\sigma_s$ (the Choquet integral of $v \circ f$ with respect to the conditional capacity $\nu^\sigma_s$). Finally he assigns the signal-act $(\sigma \, , f)$ utility

$$V^{\text{GP}}(\sigma, f) := \int v^\sigma_f \, d\nu \,,$$

where $v^\sigma_f$ is any state-contingent utility in which $v^\sigma_f(\omega) = \int v \circ f \, d\nu^\sigma_s$ whenever $\omega \in$

[15] If, without loss of generality, we set $v(\underline{x}) := 0$ and $v(\overline{x}) := 1$, then

$$\int v \circ f \, d\nu = \int \nu \left( \{ \omega \in \Omega \colon v \left( f \left( \omega \right) \right) \geqslant u \} \right) \, du \,.$$

where the righthand side of the equality is a (standard) Lebesgue integral.

$\sigma^{-1}(s)$ and $\nu^\sigma(\sigma^{-1}(s)) > 0$.

In Gul and Pesendorfer's setting the state space is (essentially) finite.[16] This allows them to define the proxy capacity via a modification of the prior capacity's Möbius transform. In the current setting, however, in which the initial beliefs are characterized by a prior that is a *convex-ranged probability*, it seems more appropriate to follow the approach we took in Definition 3 above, and derive the proxy capacity via a modification of the prior probability. Furthermore, since in their apportionment of the weights of the Möbius transform of the prior capacity *"no event that intersects multiple elements of the information partition is given weight by the Möbius transform of the proxy capacity"* (Gul and Pesendorfer, 2021, p8) this suggests the following modification be made to the prior probability to derive an analog of their proxy capacity in our setting. Let $\mu^\sigma_{\text{GP}}$ denote a "Gul and Pesendorfer (2021) inspired" proxy prior probability defined over $\mathcal{E}^\sigma$, that is obtained by setting for each event $B \in \mathcal{E}^\sigma$:

$$\mu^\sigma_{\text{GP}}(B) := \frac{\sum_{s\in S} \mu\left(\left[B \cap \sigma^{-1}(s)\right]_*\right)}{\sum_{s\in S} \mu\left(\left[\sigma^{-1}(s)\right]_*\right)}$$

And take the proxy capacity to be the inner-measure of $\mu^\sigma_{\text{GP}}$. That is, set

$$\nu^\sigma(A) := \sup_{B\subseteq A : B \in \mathcal{E}^\sigma} \mu^\sigma_{\text{GP}}(B) \text{ for each } A \subseteq \Omega\,.$$

Notice that $\nu^\sigma$ does indeed render each information cell unambiguous in the sense that, for each $s \in S$:

$$\nu^\sigma\big(\sigma^{-1}(s)\big) + \nu^\sigma\big(\Omega\backslash\sigma^{-1}(s)\big) = \frac{\mu\left(\left[\sigma^{-1}(s)\right]_*\right)}{\sum_{s'\in S} \mu\left(\left[\sigma^{-1}(s')\right]_*\right)} + \frac{\sum_{s'\neq s} \mu\left(\left[\sigma^{-1}(s')\right]_*\right)}{\sum_{s'\in S} \mu\left(\left[\sigma^{-1}(s')\right]_*\right)} = 1\,.$$

[16] Formally, their state space is homeomorphic to the set of natural numbers but they assume the prior capacity has a finite support.

Updating $\nu^\sigma$ via Bayes rule then yields

$$\begin{aligned}
\nu_s^\sigma(A) &:= \frac{\widehat{\mu}_*^\sigma\left(A\cap\sigma^{-1}(s)\right)}{\widehat{\mu}_*^\sigma\left(\sigma^{-1}(s)\right)} = \frac{\mu([A\cap\sigma^{-1}(s)]_*)}{\sum_{s'\in S}\mu([\sigma^{-1}(s')]_*)}\times\frac{\sum_{s'\in S}\mu([\sigma^{-1}(s')]_*)}{\mu([\sigma^{-1}(s)]_*)}\\
&= \frac{\mu([A\cap\sigma^{-1}(s)]_*)}{\mu\left([\sigma^{-1}(s)]_*\right)} = \frac{\mu_*(A\cap\sigma^{-1}(s))}{\mu_*(\sigma^{-1}(s))}\\
&= \frac{\nu(A\cap\sigma^{-1}(s))}{\nu(\sigma^{-1}(s))}\,.
\end{aligned}$$

The simplicity of this expression and particularly the property that, for any pair of signals $\sigma$ and $\sigma'$ and pair of realizations $s$ and $\widehat{s}$, the conditional capacity $\nu_s^\sigma$ equals $\nu_{\widehat{s}}^{\widehat{\sigma}}$ whenever $\sigma^{-1}(s)=\widehat{\sigma}^{-1}(\widehat{s})$ (that is, the two realizations correspond to the same conditioning event obtaining), might make this seem an appealing and tractable approach to follow. However, except for the case of a signal adapted to ideal events, $\mu_{\text{GP}}^\sigma$ is *not* an extension of $\mu$. In particular, for any subset of signals $Q\subseteq\sigma(\Omega)$ with more than one element, *none* of the weight $\mu\big(E_Q^\sigma\big)$ (the probability assigned by the prior to the $\sigma$- marginal inner-sleeve of $Q$) is apportioned to the proxy $\nu^\sigma$ and hence to the updated capacities. So recalling again the approach of Dempster (1967) and Shafer (1976), we interpret this as the DM *only* weighting evidence that directly supports a particular realization obtaining. Correspondingly this means his interim preferences after the learning the realization of the signal effectively ignore *all* of the evidence that only *indirectly* supports that realization occurring.

In light of the above, I contend the approach taken in the current setting rests on more solid evidential foundations.

# 6 Concluding Remarks

*yet to be written*

# A Proofs

## Proof of Proposition 1

Fix $\pi \in \Pi^{\mu}$. Consider $f \in \mathcal{F}$ with $f(\Omega) = \{x_1, \cdots, x_n\}$ where $x_1 < \cdots < x_n$ and $f$ is measurable with respect to $\pi$. Since $\pi$ is an extension of $\mu$, for all $\pi$-measurable events $B \subset \Omega$,

$$\mu_*(B) \leqslant \pi(B) \leqslant 1 - \mu_*(\Omega \backslash B) \ .$$

Set $F_f^{\pi}(x) := \pi(\{\omega \in \Omega : f(\omega) \leqslant x\})$. Hence $F_f^{\pi}(x) \leqslant \overline{F}_f^{\mu}(x)$ for all $x \in X$ by definition of $\overline{F}_f^{\mu}$, that is, $F_f^{\pi}$ first-order stochastically dominates $\overline{F}_f^{\mu}$. Thus,

$$\int v \circ f \, d\pi = \int v \, dF_f^{\pi} \geqslant \int v \, d\overline{F}_f^{\mu}$$

Since $\pi$ is arbitrary,

$$\min_{\pi \in \Pi_{\mu}} \int v \circ f \, d\pi \geqslant \int v \, d\overline{F}_f^{\mu}$$

Let $R^f$ be the $\sigma$ algebra generated by $\{f^{-1}(x) : x \in f(\Omega)\}$ and define a probability measure $\pi'$ on $R^f \cup R$ by setting

$$\pi'(E) = \sum_{x_i \in f(E), i \neq 1} (\overline{F}_f^{\mu}(x_i) - \overline{F}_f^{\mu}(x_{i-1})) + \overline{F}_f^{\mu}(x_1) \mathbf{1}_{\{1 \in f(E)\}}$$

if $E \in R^f$ and $\pi'(E) = \mu(E)$ otherwise. By construction $\pi' \in \Pi_{\mu}$ and so

$$\min_{\pi \in \Pi_{\mu}} \int v \circ f \, d\pi \leqslant \int v \circ f \, d\pi' = \int v \, d\overline{F}_f^{\mu}$$

Therefore, we have $\min_{\pi \in \Pi_{\mu}} \int v \circ f \, d\pi = \int v \, d\overline{F}_f^{\mu}$. We can construct an analogous argument to show the second equality in Proposition 1 holds.

## Proof of Proposition 2

Let $\mathcal{E} \subset \mathcal{E}'$ with corresponding probability measure $\mu$ and $\mu'$. For all $E \in \mathcal{E}$, $\mu(E) = \mu'(E)$. We can define the least upper CDF and greatest lower CDF w.r.t. $\mu$ and $\mu'$, which we denote by $\overline{F}^{\mu}$, $\overline{F}^{\mu'}$, $\underline{F}^{\mu}$ and $\underline{F}^{\mu'}$, respectively.

**Lemma 1** *For any $f \in \mathcal{F}$, the following inequalities hold:*

$$\overline{F}^{\mu}_{f} \geqslant \overline{F}^{\mu'}_{f} \geqslant \underline{F}^{\mu'}_{f} \geqslant \underline{F}^{\mu}_{f}$$

**Proof.** By the definition of least upper CDF, given $x \in X$, we have

$$\overline{F}^{\mu}_{f}(x) = \inf_{E \in \mathcal{E},\, E \supseteq \{\omega \in \Omega :\, f(\omega) \leqslant x\}} \mu(E) \geqslant \inf_{E \in \mathcal{E}',\, E \supseteq \{\omega \in \Omega :\, f(\omega) \leqslant x\}} \mu'(E) = \overline{F}^{\mu'}_{f}(x)$$

since the infimum is taken over the larger set on the RHS of the above equality. Similarly, we have $\underline{F}^{\mu'}_{f}(x) \geqslant \underline{F}^{\mu}_{f}(x)$ for all $x \in X$. ■

We are now ready to prove the proposition. Suppose to the contrary that $0 \leqslant \alpha < 1$. Let $\mathcal{E} \subset \mathcal{E}^{\sigma}$ and $\mathcal{E} \neq \mathcal{E}^{\sigma}$, that is, there is $B \in \mathcal{E}^{\sigma}$ but $B \notin \mathcal{E}$. We can simply let $\sigma(\omega) = s_1$, if $\omega \in B$ and $\sigma(\omega) = s_2$ otherwise, and consider the act $\overline{x}_B\underline{x}$. Statement 1 of Proposition 2 implies

$$\alpha v(\underline{x}) + (1-\alpha)v(\overline{x}) \leqslant \mu^{\sigma}(B)v(\underline{x}) + (1-\mu^{\sigma}(B))v(\overline{x})$$

and so $\alpha \geqslant \mu^{\sigma}(E)$. Since $\mu^{\sigma}(B)$ is arbitrary, $\alpha \geqslant 1$, which leads to a contradiction. ■

## Proof of Proposition 3

(i) Fix a prior $\pi$. If $\pi$ conforms conditionally on conditionally on the non-null ideal event $E \in \mathcal{E} \backslash \mathcal{N}$ to the power series $\gamma$ in $\Gamma$, then for any pair of events $A$ and $B$ in $\mathcal{E}^{\pi}$ and any pair of outcomes $x > y$, it follows from expression (3) that $(\mathbf{0}, x_A y) \succsim (\mathbf{0}, x_B y)$ if and only if $\pi(A) \geqslant \pi(B)$. Hence $\pi$ is a neo-continuous source.

(ii) Fix a non-null ideal event $E \in \mathcal{E} \backslash \mathcal{N}$ and a power series $\gamma \in \Gamma$. Adapting the constructions employed by Gul and Pesendorfer (2015) in their proof of their Proposition 2(ii), it follows there exists a $\sigma$-algebra of events, $\mathcal{E}_E$, and a probability $\pi$ with support $E$ such that for each $A \in \mathcal{E}_E$ $\mu_*(A) = \mu(E)\gamma\left(\pi(A)\right)$.

As $\pi$ is convex-valued we can construct the following sequence of $\pi$-measurable partitions of $E$, $(\mathbb{E}^1, \ldots, \mathbb{E}^n, \ldots)$.

1. $\mathbb{E}^1 := \{E^1_0, E^1_1\}$ where $\pi(E^1_0) = \pi(E^1_1) = 1/2$

2. $\mathbb{E}^2 := \{E^2_{00}, E^2_{00}, E^2_{10}, E^2_{11},\}$ where $E^2_0 = E^2_{00} \cup E^2_{01}$, $E^1_1 = E^2_{10} \cup E^2_{11}$ and $\pi(B) = 1/4$ for all $B \in \mathbb{E}^2$.

$\vdots$

$n$. $\mathbb{E}^n := \{\ldots, E^n_{b0}, E^n_{b1}\ldots\}$, where $b$ is an $n-1$ bit binary number, and where for each $b$, $E^{n-1}_b = E^n_{b0} \cup E^1_{b1}$ and $\pi(E^n_{b0}) = \pi(E^1_{b1}) = 2^{-n}$

$\vdots$

Gul and Pesendorfer (2014) show if the continuum hypothesis holds, then there exists a collection of pairwise disjoint diffuse events $\{D_1, D_2, \ldots\}$ such that $\bigcup_i D_i = \Omega$. Consider the following sequence of partitions of $\Omega$.

1. $\mathbb{P}^1 := \{E^1_0 \cup \bigcup_i (D_{2i-1} \cap (\Omega \backslash E)), E^1_1 \cup \bigcup_i (D_{2i} \cap (\Omega \backslash E))\}$

2. $$\mathbb{P}^2 := \left\{E^2_{00} \cup \bigcup_i (D_{4i-3} \cap (\Omega \backslash E)), E^2_{01} \cup \bigcup_i (D_{4i-2} \cap (\Omega \backslash E)), E^2_{10} \cup (D_{4i-1} \cap (\Omega \backslash E)), E^2_{11} \cup \bigcup_i (D_{4i} \cap (\Omega \backslash E))\right\}$$

$\vdots$

Notice each partition in this sequence is a refinement of its predecessor and for each $n$, each element in the partition $\mathbb{P}^n$ has inner measure $(1-\lambda)\gamma(2^{-n})$, where $\lambda = 1-\mu(E)$. So consider the sequence of $\sigma$-algebras $(\mathcal{E}^n)^\infty_{n=1}$ in which $\mathcal{E}^n$ is the $\sigma$- algebra generated by the partition $\mathbb{P}^n$. For each $n$ and each $A \in \mathcal{E}^n$, set $\pi^n(A) := \pi(A)$.

By construction, we have for each $A \in \mathcal{E}^n$,

$$\mu_*(A) := \begin{cases} (1-\lambda)\,\gamma\,(\pi^n(A)) & \text{if } \pi^n\,(A) < 1 \\ 1 & \text{otherwise} \end{cases}$$

Hence the restriction of $\succsim$ to binary bets $x_A y$, with $x > y$ and $A \in \mathcal{E}^n$ admits a biseparable representation of the form $\nu^n(A)v(x) + (1-\nu^n(A))v(y)$ where $\nu$ is the capacity given by:

$$\nu^n(A) = \begin{cases} (1-\lambda)\,\gamma\,(\pi^n(A)) + \lambda\,(1-\alpha) & \text{if } \pi^n(A) \in [2^{-n}, 1) \\ 1 & \text{if } \pi^n(A) = 1 \end{cases}$$

Finally, take $(\bar{\pi}, \mathcal{E}^{\bar{\pi}})$ to be the completion of $(\pi^\infty, \bigcup_n \mathcal{E}^n)$, where $\pi^\infty$ is the measure on $\bigcup_n \mathcal{E}^n$ obtained by setting $\pi^\infty(A) := \pi(A)$ for every $A \in \bigcup_n \mathcal{E}^n$. . ■

## Proof of Proposition 4

## Proof of Proposition 5

Without any essential loss of generality, we can model the state space as an $n$-dimensional unit hyper-cube:

$$\Omega = \underbrace{[0,1] \times [0,1] \times \ldots \times [0,1]}_{n \text{ times}}$$

and take $\mu$ to be the Lebesgue product measure. As we noted above in the proof of proposition 3, Gul and Pesendorfer (2014) show if the continuum hypothesis holds, then there exists a collection of pairwise disjoint diffuse events $\{D_1, D_2, \ldots,\}$ such that $\bigcup_i D_i = \Omega$. For each Borel subset $B$ of the unit interval $[0,1]$, set

$$\begin{aligned} A^B := {} & (B \times [0,1] \times [0,1] \times \ldots \times [0,1] \times [0,1] \cap D_1) \\ & \cup ([0,1] \times B \times [0,1] \times \ldots \times [0,1] \times [0,1] \cap D_2) \\ & \cup ([0,1] \times [0,1] \times B \times \ldots \times [0,1] \times [0,1] \cap D_3) \\ & \vdots \\ & \cup ([0,1] \times [0,1] \times [0,1] \times \ldots \times B \times [0,1] \cap D_{n-1}) \\ & \cup \left([0,1] \times [0,1] \times [0,1] \times \ldots \times [0,1] \times B \cap \bigcup_{i=n}^{\infty} D_i\right) \end{aligned}$$

Notice

$$\left[A^B\right]_* = \underbrace{B \times B \times \ldots \times B}_{n \text{ times}} = B^n \text{ and } \left[A^B\right]^* = \Omega \backslash ([0,1] \backslash B)^n$$

For each Borel subset of $[0,1]$, $B$, set $\widehat{\pi}(A^B) := \nu(B)$, where $\nu$ is the Lebesgue measure defined over the Borel subsets of the unit interval $[0,1]$.

Take $\pi$ to be the (unique) probability measure that extends $\widehat{\pi}$ to $\mathcal{E}^\pi$, the smallest $\sigma$-algebra containing all the elements in

$$\left\{A^B : B \text{ is a Borel subset of } [0,1]\right\} .$$

For ease of exposition and without any loss of generality we take our R-HEU maximizer to be one that exhibits extreme aversion to ambiguity, that is, one for which $\alpha = 1$.

Consider the binary signal

$$\sigma(\omega) = \begin{cases} s' & \text{if } \omega \in A^B \\ s'' & \text{if } \omega \notin A^B \end{cases}$$

where $B$ is some Borel subset of the unit interval $[0,1]$. Its ideal split is

$$\begin{aligned}\left\{E^\sigma_{\{s'\}}, E^\sigma_{\{s',s''\}}, E^\sigma_{\{s''\}}\right\} &= \left\{\left[A^B\right]_*, \left[\Omega\backslash A^B\right]_*, \Omega\backslash\left(\left[A^B\right]_* \cup \left[\Omega\backslash A^B\right]_*\right)\right\} \\ &= \left\{B^n, ([0,1]\backslash B)^n, \Omega\backslash\left(B^n \cup ([0,1]\backslash B)^n\right)\right\}\end{aligned}$$

Now consider a bet on $A^{\widehat{B}}$, with stakes $\overline{x} > \underline{x}$, where $\widehat{B}$ is a subset of $B$, thus making $A^{\widehat{B}}$ a subset of $A^B$. We have $\pi\left(A^{\widehat{B}}\right) = \nu(\widehat{B}) < \nu(B) = \pi(A^B)$. Moreover,

$$\begin{aligned}\mu^\sigma(A^B) &= \mu\left(\left[A^B\right]^* \cap E^\sigma_{\{s'\}}\right) + \frac{1}{2}\mu\left(\left[A^B\right]^* \cap E^\sigma_{\{s',s''\}}\right) \\ &= \nu(B^n) + \frac{1}{2}\nu\left(\Omega\backslash\left(B^n \cup ([0,1]\backslash B)^n\right)\right) \\ &= \nu(B)^n + \frac{1}{2}\left[1 - \nu(B)^n - (1-\nu([B]))^n\right] \\ &= \frac{1 + \nu(B)^n - (1-\nu([B]))^n}{2}\end{aligned}$$

Hence,

$$v\left(V^{\sigma}_{s'}\left(\bar{x}_{\widehat{A}}\underline{x}\right)\right)=\mu^{\sigma}_{s'}\left(A^{\widehat{B}}\right)=\frac{\mu^{\sigma}\left(A^{\widehat{B}}\right)}{\mu^{\sigma}(A^{B})}$$
$$=\frac{2\nu\left(\widehat{B}\right)^{n}}{1+\nu\left(B\right)^{n}-\left(1-\nu\left(B\right)\right)^{n}}\neq\frac{\nu\left(\widehat{B}\right)^{n}}{\nu\left(B\right)^{n}}=\left(\frac{\pi\left(A^{\widehat{B}}\right)}{\pi(A^{B})}\right)^{n}$$

whenever $n\neq 1$.

■

## Proof of Theorem 1

### A.1 Outline of sufficiency proof

Fix a signal $\sigma\in\Sigma$ with associated measurable split $\{E^{\sigma}_{Q}\in\mathcal{E}:Q\subseteq\sigma(\Omega)\,,Q\neq\emptyset\}$. Let $\{E^{\sigma}_{s}\colon s\in\sigma(\Omega\}$ denote an equally-distributed Harsanyi-dividend ideal partition that can be associated with $\sigma$.

Part I. Construction of proxy and representation of conditional risk preferences.

1. Fix $s\in\sigma(\Omega)$. By Axiom 2 (consequentialism) $\Omega\backslash\sigma^{-1}(s)$ is a null event as are all of its subsets. In conjunction with Axiom 4 (preservation of ideal events), set of ideal events $\mathcal{E}^{\sigma}$ for $\succsim^{\sigma}_{s}$ is smallest sigma-algebra containing $\mathcal{E}$ and the information cells corresponding to the signal's realizations. So restriction of $\succsim^{\sigma}_{s}$ to acts adapted to $\mathcal{E}^{\sigma}$ admits an SEU representation $(v',\mu^{\sigma}_{s})$.

2. For each $s\in\sigma(\Omega)$, set $\mu^{\sigma}(\sigma^{-1}(s)):=\mu(E^{\sigma}_{s})$.

3. For each $E\in\mathcal{E}$, set

$$\mu^{\sigma}(E):=\sum_{s\in S\colon\,\mu^{\sigma}(\sigma^{-1}(s))>0}\mu^{\sigma}_{s}(E)\mu^{\sigma}\big(\sigma^{-1}(s)\big)\ .$$

By Axiom 2 (consequentialism) it follows

$$\mu^{\sigma}\big(E\cap E^{\sigma}_{Q}\cap\sigma^{-1}(s)\big)=\mu^{\sigma}_{s}(E\cap E^{\sigma}_{Q})\mu^{\sigma}\big(\sigma^{-1}(s)\big)$$

4. Applying Axiom 5 we have $v' = v$ and for all $Q \subset \sigma(\Omega)$ and all ideal $E$

$$\frac{\mu_s^\sigma\big(E \cap E_Q^\sigma\big)}{\mu_s^\sigma\big(E_Q^\sigma\big)} = \frac{\mu\big(E \cap E_Q^\sigma\big)}{\mu\big(E_Q^\sigma\big)}\,. \tag{6}$$

Hence, if $\mu^\sigma(\sigma^{-1}(s)) > 0$ then from (6), we obtain

$$\frac{\mu^\sigma\big(E \cap E_Q^\sigma \cap \sigma^{-1}(s)\big)}{\mu^\sigma\big(E_Q^\sigma \cap \sigma^{-1}(s)\big)} = \frac{\mu\big(E \cap E_Q^\sigma\big)}{\mu\big(E_Q^\sigma\big)}\,. \tag{7}$$

5. Invoking Axiom 6, we have by applying the HEU representation of static preferences that LHS preference statement corresponds to the following inequality

$$\frac{\mu\left(E_Q^\sigma \cap E_s^\sigma\right)}{\mu\left(E_s^\sigma\right)} v(x) + \left[1 - \frac{\mu\left(E_Q^\sigma \cap E_s^\sigma\right)}{\mu\left(E_s^\sigma\right)}\right] v(y) \geqslant v(z)$$

But notice that by the construction of $E_s^\sigma$,

$$\mu\big(E_Q^\sigma \cap E_s^\sigma\big) = \frac{\mu\big(E_Q^\sigma\big)}{|Q|}$$

So applying the SEU representation of the interim preferences, the RHS preference statement corresponds to the following inequality

$$\mu_s^\sigma(E_Q^\sigma) v(x) + \left[1 - \mu_s^\sigma(E_Q^\sigma)\right] v(y) \geqslant v(z)$$

But notice that

$$\mu_s^\sigma(E_Q^\sigma) = \frac{\mu^\sigma\big(E_Q^\sigma \cap \sigma^{-1}(s)\big)}{\mu^\sigma(\sigma^{-1}(s))}$$

When the two inequalities hold with equality we have

$$\frac{\mu(E_Q^\sigma)/|Q|}{\mu(E_s^\sigma)} = \frac{\mu^\sigma\left(E_Q^\sigma \cap \sigma^{-1}(s)\right)}{\mu^\sigma(\sigma^{-1}(s))}$$

And since we have

$$\mu^{\sigma}\left(E^{\sigma}_{Q}\cap\sigma^{-1}(s)\right)=\frac{\mu\left(E^{\sigma}_{Q}\right)}{|Q|}$$

substituting this into equation (7) yields:

$$\mu^{\sigma}\left(E\cap E^{\sigma}_{Q}\cap\sigma^{-1}(s)\right)=\frac{\mu\left(E\cap E^{\sigma}_{Q}\right)}{|Q|}\,.$$

Hence for all $E\in\mathcal{E}$ we have

$$\mu^{\sigma}\left(E\cap\sigma^{-1}(s)\right)=\left(\sum_{Q\subseteq\sigma(\Omega):\,s\in Q}\frac{\mu\big(E\cap E^{\sigma}_{Q}\big)}{|Q|}\right)\,.$$

Moreover, for all $B\in\mathcal{E}^{\sigma}$:

$$\mu^{\sigma}(B\cap\sigma^{-1}(s)):=\left(\sum_{Q\subseteq\sigma(\Omega):\,s\in Q}\frac{\mu\left(\left[B\cap\sigma^{-1}(s)\right]^{*}\cap E^{\sigma}_{Q}\right)}{|Q|}\right)\,.$$

Part II Deriving certainty equivalent of arbitrary signal-act.

1. By Axiom 7, we have $\alpha'=\alpha$
2. Assigning utility to arbitrary signal-act $(\sigma\,,f)\in\Sigma\times\mathcal{F}$.
   (a) Repeated application of Axiom 3 (Dynamic Consistency) yields

   $$(\sigma\,,f)\sim(\sigma\,,\phi^{f})$$

   where $\phi^{f}:S\to X$ is a plan of action in which for each (non-null) realization $s\in\sigma(\Omega)$, $\phi^{f}(s)\sim^{\sigma}_{s}f$.
   (b) Applying Axiom 1 (Timing Neutrality) yields $(\sigma\,,\phi^{f})\sim(\mathbf{0}\,,\phi^{f})$.
   (c) Applying HEU representation of static preferences set $V(\sigma\,,f):=V(\mathbf{0}\,,\phi^{f})$.
3. Assigning utility to arbitrary IDP. Applying Axiom 8 set

$$V(\sigma,M):=\max_{\phi\in\Phi(\sigma,M)}V(\sigma\,,\phi)\,.$$

# B An Axiomatization of Assumption H

Four of the following axioms are taken directly from Gul and Pesendorfer (2015) but adapted so that they apply to signal-acts involving the null signal. We refer the reader to the discussion in Gul and Pesendorfer (2014) for the motivation and explanation of the role played by these axioms in their characterization of (static) HEU Maximizers. And following Grant et al. (2023), we strengthen the analog to Gul and Pesendorfer's (2015) third axiom and make a slight modification to the analog of the first part of their sixth (continuity) axiom. As Grant et al. (2023) show, the former rules out state-dependence of the interval utility (which in the context of Axiom H.4 ensures the state independence of the ambiguity aversion $\alpha$) while the latter is required to establish the set of ideal events is countably additive.

**Axiom H.1 (Ordering)** *For any three acts $f, f', f''$ in $\mathcal{F}$:*

(*i*) *either $(\mathbf{0}, f) \succsim (\mathbf{0}, f')$ or $(\mathbf{0}, f') \succsim (\mathbf{0}, f)$, or both; and,*

(*ii*) *if $(\mathbf{0}, f) \succsim (\mathbf{0}, f')$ and $(\mathbf{0}, f') \succsim (\mathbf{0}, f'')$ then $(\mathbf{0}, f) \succsim (\mathbf{0}, f'')$.*

**Axiom H.2 (Statewise Monotonicity)** *For any pair of acts $f, f'$ in $\mathcal{F}$: if $f(\omega) > f'(\omega)$ for all $\omega \in \Omega$ then $(\mathbf{0}, f) \succ (\mathbf{0}, f')$.*

**Axiom H.3 (Invariant Conditional Certainty Equivalents for Diffuse Bets)** *For any pair of outcomes $x, y$ in $X$, any ideal event $E$ in $\mathcal{E}$, and any pair of diffuse events $D, D'$ in $\mathcal{D}$: $(\mathbf{0}, y_D x) \succsim z \implies (\mathbf{0}, (y'_D x)_E x) \succsim z_E x$.*

**Axiom H.4 (Comparative Probability)** *For any pair of events $A$ and $B$, and any pair of outcomes $x > y$, if $(\mathbf{0}, x_A y) \succsim (\mathbf{0}, x_B y)$ then $(\mathbf{0}, w_A z) \succsim (\mathbf{0}, w_B z)$, for all pairs of outcomes $w > z$.*

**Axiom H.5 (Small ideal-event continuity)** *For any pair of ideal acts $g$ and $g'$ in $\mathcal{G}$, if $(\mathbf{0}, g) \succ (\mathbf{0}, g')$ then there exists a finite partition of $\Omega$, $\{E_1, \ldots, E_n\}$ in which each $E_i \in \mathcal{E}$ for all $i = 1, \ldots, n$, such that $(\mathbf{0}, \underline{x}_{E_i} g) \succ (\mathbf{0}, \overline{x}_{E_i} g')$ for all $i = 1, \ldots, n$.*

**Axiom H.6 (Pointwise and uniform continuity)** *For any acts $f$, $f'$, $g$ and $h$, any sequence of acts $f^n$ and any (decreasing) sequence of ideal events $E^n$ with $E_{n+1} \subset E_n$:*

(*i*) *Suppose $(\mathbf{0}, g \succsim f E^n f') \succsim (\mathbf{0}, h)$ for all $n$. Then $(\mathbf{0}, g) \succsim (\mathbf{0}, f \cap E^n f') \succsim (\mathbf{0}, h)$.*

*(ii)* *Suppose* $(\mathbf{0}, g) \succsim (\mathbf{0}, f_n) \succsim (\mathbf{0}, h)$ *for all* $n$*. Then* $f_n \in \mathcal{F}$ *converges uniformly to* $f$ *implies* $(\mathbf{0}, g) \succsim (\mathbf{0}, f) \succsim (\mathbf{0}, h)$*.*

As an immediate corollary of Gul and Pesendorfer's (2015) Proposition 1 (p. 470) we have the following.

**Lemma 2** *Fix a preference relation* $\succsim$ *over* $\Sigma \times \mathcal{F}$*. The restriction to* $\{\mathbf{0}\} \times \mathcal{F}$ *satisfies Axioms H.1 – H.6 if an only if Assumption H holds.*

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