Search Theory of Imperfect Competition with Decreasing Returns to Scale*

GUIDO MENZIO
NYU and NBER

October 2023

Abstract

I study a version of the search-theoretic model of imperfect competition by Burdett and Judd (1983) in which sellers operate a production technology with decreasing rather than constant returns to scale. Equilibrium exists and is unique, and its structure depends on the extent of search frictions. If search frictions are large enough, the price distribution is non-degenerate and atomless. If search frictions are neither too large nor too small, the price distribution is non-degenerate with an atom at the lowest price. If search frictions are small enough, the price distribution is degenerate. Equilibrium is efficient if and only if the price distribution is degenerate. Generically, neither the structure nor the welfare properties of equilibrium are the same as in Burdett and Judd (1983). As in Burdett and Judd (1983), however, equilibrium outcomes span the spectrum from pure monopoly to perfect competition as search frictions decline.

JEL Codes: D43, D83, J31 .

Keywords: Search frictions, Imperfect competition, Price dispersion.

*Guido Menzio: Department of Economics, New York University, 19 West 4th Street, New York, NY 10012 (email: gm1310@nyu.edu). I am grateful to Jaroslav Borovicka, Lukas Nord, and Guillaume Rocheteau for useful comments. Declarations of interest: none.

1 Introduction

Search frictions provide a natural theory of imperfect competition in product markets. The theory is simple and compelling. Due to search frictions, buyers cannot purchase a product from just any seller, but only from a finite sample of sellers. As the number of non-captive buyers—i.e. buyers who can purchase the product from multiple sellers—relative to the number of captive buyers—i.e. buyers who can purchase the product from only one seller—goes from zero to infinity, the equilibrium spans the spectrum going from monopoly to perfect competition.

The search-theoretic framework of imperfect competition was developed by Butters (1977), Varian (1980), and Burdett and Judd (1983). Traditionally, the framework has been used to study the dispersion of the price of a particular good (Sorensen 2000, Hong and Shum 2006, Galenianos, Pacula and Persico 2012, Kaplan and Menzio 2015, Bethune, Choi and Wright 2020), the dispersion of the relative price of different goods (Kaplan, Menzio, Rudanko and Trachter 2019), and the fluctuations of the price of a particular good at a particular store (Varian 1980, Menzio and Trachter 2018). More recently, the framework has been used to study a broader set of questions, such as price stickiness and monetary policy (Head, Liu, Menzio and Wright 2012, Burdett, Trejos and Wright 2017, Burdett and Menzio 2018), consumption inequality (Pytka 2018, Nord 2022), product design and market concentration (Menzio 2023, Albrecht, Vroman and Menzio 2023), and business cycles (Kaplan and Menzio 2015). The framework, adapted to the labor market, has been used to study residual wage inequality (Burdett and Mortensen 1998, Bontemps, Robin and van den Berg 2000, Mortensen 2003).

In this paper, I contribute to the development of the search-theoretic framework of imperfect competition by characterizing the properties of equilibrium under the assumption that sellers face an increasing marginal cost of production. That is, I characterize the properties of equilibrium when sellers operate a decreasing returns to scale production function. Butters (1977), Varian (1980), and Burdett and Judd (1983) assume that all sellers face the same, constant marginal cost of production. Existing variations of the search-theoretic framework of imperfect competition either maintain the assumption that all sellers face the same, constant marginal cost, or assume that sellers face different but constant marginal costs. That is, the existing literature assumes that sellers operate a constant return to scale production function. For some applications, the assumption of constant returns to scale may be appropriate (e.g. production with only variable factors that can be purchased or produced at constant prices). For other applications, the assumption of decreasing returns to scale may be more realistic (e.g. production with either fixed factors or with variable factors that are purchased or produced at increasing price). For example, in a version of Lagos and Wright (2005) where the decentralized market operates as in Burdett and Judd (1983), it would be natural to assume that individual sellers face a strictly increasing disutility from producing additional units of output. In the labor market context of Burdett and Mortensen (1998), it may be sometimes more realistic to assume that individual firms have a strictly decreasing marginal product of labor because they produce differentiated goods.

I consider the market for some consumer good. On one side of the market, there is a continuum of identical sellers. Each seller posts a price p for the good, and produces the good according to some cost function $c(q)$ that is strictly increasing and strictly convex in the quantity q. On the other side of the market, there is a double continuum of identical buyers. Each buyer demands one unit of the good and enjoys a utility u from consuming it. Due to informational or physical frictions, a buyer cannot purchase from just any seller, but only from those with which he comes into contact. The process through which buyers contact sellers is such that a buyer contacts one randomly selected seller with probability $1-\alpha$, in which case the buyer is captive, and two randomly selected sellers with probability α, in which case the buyer is non-captive. The parameter α is an inverse measure of search frictions, since the ratio of non-captive to captive buyers goes from 0 to infinity, as α goes from 0 to 1. The structure of the market is the same as in Burdett and Judd (1983), except that the sellers' marginal cost of production is strictly increasing rather than constant.

This seemingly small modification to the environment of Burdett and Judd (1983) changes the structure of equilibrium. It is easy to understand why this is the case. In Burdett and Judd (1983), the equilibrium is such that sellers post prices according to a distribution F whose support is some interval $[p_\ell, p_h]$, where $p_\ell > c$, $p_h = u$ and c denotes the sellers' constant marginal cost. The distribution F is atomless. Any mass point at a price p_0 would create a downward discontinuity in the demand curve and, in turn, it would induce an individual seller to deviate from p_0 to $p_0 - \epsilon$ so as to undercut the mass of competitors at p_0. The shape of the distribution F is such that sellers enjoy the same profit by posting any price on the support $[p_\ell, p_h]$. A seller that posts the highest price on the distribution enjoys a high profit margin, $p_h - c$, but only trades with the captive buyers that it meets. A seller that posts the lowest price on the distribution enjoys a smaller profit margin, $p_\ell - c$, but trades with all the buyers that it meets. The price p_ℓ equates the profits of the two sellers. More generally, the price function $p(x)$, which maps a seller's quantile x in the distribution F into a seller's price, is such that profits are constant for all $x \in [0,1]$. As search frictions decline, in the sense that α converges to 1, the fraction of captive buyers falls to 0 and so does the profit of the seller posting the highest price on the distribution. Therefore, as search frictions decline, the profit for a seller posting the lowest price on the distribution must also fall to 0, which requires the price p_ℓ to converge to the marginal cost c. If sellers face an increasing marginal cost, the price charged by the seller with the lowest price would have to fall below the marginal cost in order to drive the seller's profit to 0. Such a price, however, would not be consistent with the seller's profit maximizing behavior.

What is then the structure of equilibrium when sellers face a strictly increasing marginal cost of production? The answer, I find, depends on the extent of search frictions. If search frictions are sufficiently large, in the sense that α is smaller than some cutoff α_1, the equilibrium has the same structure as in Burdett and Judd (1983). That is, the equilibrium is such that the price distribution F is atomless and its support is some interval $[p_\ell, p_h]$, with $p_h = u$. A seller enjoys the same profit by posting any price on the support of F. If a seller posts a lower price, it sells more units of the good but enjoys a lower profit per unit, both because of the lower price and the higher average cost. If a seller posts a higher price, it sells fewer units of the good but enjoys a higher profit per unit, both because of the higher price and the lower average cost. Intuitively, when search frictions are large enough, equilibrium profits are large and, for this reason, the price function $p(x)$ that equalizes the profit of sellers at different quantiles of the distribution F is everywhere higher than the sellers' marginal cost.

If search frictions are neither too large, in the sense that α is greater than α_1, nor too small, in the sense that α is smaller than some other cutoff α_2, the equilibrium is such that the distribution F has a mass point at some price p_0 and a density over some interval $[p_\ell, p_h]$, with $p_\ell > p_0$ and $p_h = u$. The structure of equilibrium is different than in Burdett and Judd (1983). There is a mass of sellers posting the lowest price p_0. These sellers do not wish to lower their price to $p_0 - \epsilon$ because the price p_0 is equal to their marginal cost. There is a density of sellers posting prices in the interval $[p_\ell, p_h]$. These sellers trade less output but make that up with a higher profit margin. Since any seller posting a price greater than p_0 sells discretely fewer units that a seller posting p_0, the distribution F has a gap between p_0 and p_ℓ. Intuitively, as α increases from 0 to α_1, the equilibrium profit declines and the lowest price in the atomless distribution F falls towards the seller's marginal cost which, in turn, is increasing in α. If α were to increases beyond α_1, the lowest price in an atomless distribution F would become lower than the seller's marginal cost and, for this reason, it would not be consistent with profit maximization. Instead, as α increases beyond α_1, sellers start bunching at a lowest price p_0 that is equal to their marginal cost. The bunching of sellers at the lowest price leads to a decline in the quantity that these sellers trade, a decline in their marginal cost, a decline in their price (since the price has to be equal to the marginal cost) and, in turn, a decline in their profit.

If search frictions are sufficiently small, in the sense that α is greater than α_2, the equilibrium is such that the distribution F is degenerate at a price p_0 equal to the seller's marginal cost. The equilibrium is the same as in a frictionless and perfectly competitive model, in the sense that every seller trades the same quantity of the good at a price that is equal to the marginal cost of production. Sellers do not wish to lower their price because the price p_0 is equal to their marginal cost. Sellers do not wish to increase their price because, if they did, they would only trade with captive buyers and there are not enough captive buyers in the market. Intuitively, as α increases from α_1 to α_2, the profit of a seller posting the lowest price p_0 must decline because it must be equal to the profit of a seller posting the highest price $p_h = u$. The decline in the profit for a seller posting p_0 is attained by an increase in the fraction of sellers posting p_0. As α reaches α_2, all the sellers post the price p_0 and the right tail of the distribution disappears. Further increases in α do not change the equilibrium outcomes.

Even though replacing the assumption of constant returns to scale with the assumption of decreasing returns to scale affects the structure of equilibrium, it does not change the relationship between search frictions and the extent of competition. Just as in Burdett and Judd (1983), the equilibrium limits to the monopoly outcome as the ratio of non-captive to captive buyers converges to zero (i.e., $\alpha \to 0$), and it limits to the competitive outcome as the ratio of non-captive to captive buyers diverges to infinity (i.e., $\alpha \to 1$). Just as in Burdett and Judd (1983), sellers' profits decline monotonically as the ratio of non-captive to captive buyers increases. The structure of equilibrium is different than in Burdett and Judd (1983) because the competitive outcomes are different. When sellers have a constant marginal cost of production, the competitive outcome is such that every seller posts a price equal to the marginal cost. The competitive outcome does not pin down how much of the good each seller produces. When sellers have an increasing marginal cost of production, the competitive outcome is such that every seller posts a price equal to the marginal cost and every seller produces the same quantity of output. An atomless price distribution generates dispersion in the quantity of output produced by different sellers, even as the extent of price dispersion becomes arbitrarily small. For this reason, an atomless price distribution can approach the competitive outcome when sellers have a constant marginal cost, but cannot approach the competitive outcome when sellers have an increasing marginal cost. When sellers have an increasing marginal cost, the competitive outcome can only be approached if the price distribution becomes degenerate.

The assumption of decreasing returns to scale does affect the welfare properties of equilibrium. In Burdett and Judd (1983), the equilibrium is always efficient. The equilibrium features price dispersion and, in turn, dispersion in the quantity of output produced by different sellers. But, since sellers have the same constant marginal cost of production, dispersion in the quantity of output produced by different seller does not lead to any inefficiency. In contrast, when sellers have a strictly increasing marginal cost of production, the equilibrium is efficient if and only if there is no dispersion in the quantity of output produced by different sellers and, in turn, if and only if there is no dispersion in prices. Hence, when sellers have a strictly increasing marginal cost of production, the equilibrium is efficient if and only if search frictions are sufficiently small, in the sense that α is greater than α_2. If search frictions are not small enough, in the sense that α is not greater than α_2, the equilibrium features price dispersion, quantity dispersion, and it is inefficient.

2 Environment

I consider the market for a homogeneous good. On one side of the market, there is a continuum of ex-ante identical sellers with measure 1. On the other side of the market there is a double continuum of ex-ante identical buyers with measure $b > 0$ per seller.

Each seller chooses a price p and produces the good according the cost function $c(q)$, where $q \geq 0$ is the quantity of output and $c(q)$ is a twice-differentiable, strictly increasing, strictly convex function such that $c(0) = 0$, $c'(0) = 0$, $c'(\infty) = \infty$. Each buyer enjoys a utility $u - p$ if he purchases a unit of the good at the price p, where $u > c'(b)$ is the buyer's valuation of the good.[1] Each buyer enjoys a utility of 0 if he does not purchase the good.

The market is frictional, in the sense that a buyer cannot purchase from any seller, but only from those sellers with which he comes into contact. The assumption is meant to capture either informational frictions, i.e. the buyer does not know the location of all the sellers, or physical frictions, i.e. the buyer cannot reach all of the sellers. In the spirit of Burdett and Judd (1983), I assume that a buyer contacts one randomly-selected seller with probability $1 - \alpha$, in which case the buyer is said to be *captive*, and two randomly-selected sellers with probability α, in which case the buyer is said to be *non-captive*, where $\alpha \in (0, 1)$. The buyer observes the price charged by the contacted sellers and, based on those prices, he decides from which seller to demand a unit of the good. A seller can refuse to trade with some buyers, in which case the buyer can demand the good from his other contact.

I am now in the position to define an equilibrium.

Definition 1. (Equilibrium) *An equilibrium is a cumulative distribution $F(p)$ of prices such that:* (i) *A buyer demands the good from the contacted seller with the lowest price, as long as the lowest price is non-greater than u;* (ii) *If a buyer contacts two sellers with the same price non-greater than u, he demands the good from either seller with probability $1/2$;* (iii) *For every price p on the support of F, the profit of a seller attains its maximum;* (iv) *For every price p on the support of F, a seller trades with every buyer who demands a unit of the good.*

The environment is the same as in Burdett and Judd (1983), except that sellers face a strictly increasing rather than a constant marginal cost of production. The definition of equilibrium is the same as in Burdett and Judd (1983), except for condition (iv). Condition (i) is the optimality condition for the buyer's problem. Condition (ii) is a tie-breaking condition for the buyer's problem. Condition (iii) is the optimality condition for the seller's problem. Condition (iv) is a restriction on equilibrium. The condition does not restrict the seller's strategy, since the seller is allowed to turn down some of the buyers' demand. The condition instead requires that, in equilibrium, the seller finds it optimal to meet the buyers' demand. Condition (iv) simplifies the characterization of equilibrium, even though I suspect that it does not have any bite.

[1] In Section 5, I discuss the case in which $u \leq c'(b)$.

3 General properties of equilibrium

In this section, I derive some general properties of equilibrium. These properties imply that there are only three possible types of equilibria: (i) Equilibria in which the distribution F is degenerate at a mass point p_0, with p_0, where $p_0 \in (0, u]$; (ii) Equilibria in which the distribution F is non-degenerate, it has a mass point at p_0 and a density over an interval $[p_\ell, p_h]$, where $p_0 > 0$, $p_\ell > p_0$ and $p_h = u$; (iii) Equilibria in which the distribution F is non-degenerate, it does not have any mass points, and it has a density over an interval $[p_\ell, p_h]$, where $p_\ell > 0$ and $p_h = u$.

It is useful to start by deriving an expression for the profit $V(p)$ enjoyed by a seller posting an arbitrary price p. For $p \in [0, u]$, $V(p)$ is given by

$$\begin{aligned} V(p) = \quad & \max_q pq - c(q) \\ & \text{s.t. } q \in [0, d(p)]. \end{aligned} \tag{3.1}$$

where $d(p)$ denotes the buyers' demand and is given by

$$d(p) = b \left\{1 - \alpha + 2\alpha \left[1 - F(p) + \mu(p)/2\right]\right\}. \tag{3.2}$$

The expression for the buyers' demand $d(p)$ in (3.2) is easy to understand, given that $F(p)$ denotes the measure of sellers with a price non-greater than p and $\mu(p)$ denotes the measure of sellers with a price equal to p. The seller meets a measure $b(1-\alpha)$ of captive buyers, i.e. buyers who do not have any other contact. A captive buyer demands the good from the seller with probability 1. The seller meets a measure $b2\alpha$ of non-captive buyers, i.e. buyers who have a second contact. A non-captive buyer demands the good from the seller with probability $1 - F(p) + \mu(p)/2$, where $1 - F(p)$ is the probability that the buyer's second contact has a price strictly greater than p, and $\mu(p)/2$ is the probability that the buyer's second contact has a price equal to p and the buyer chooses to purchase the good from the seller. The expression for the seller's profit $V(p)$ in (3.1) is also easy to understand. The seller chooses how much output q to produce so as to maximize revenues, pq, net of costs, $c(q)$, taking as given the buyers' demand $d(p)$. For $p > u$, $V(p) = 0$ since a buyer does not demand the good from a seller that posts a price p that exceeds his valuation u of the good.

As mentioned in the definition of equilibrium, I restrict attention to equilibria in which sellers find it optimal to meet buyers' demand. Formally, I restrict attention to equilibria in which, for every price p on the support of the distribution F, the solution to the seller's problem in (3.1) is to produce a quantity of output q equal to the buyers' demand $d(p)$. Since I restrict attention to equilibria in which sellers meet buyers' demand, the expression for $d(p)$ in (3.2) does not include any buyers whose second contact posts a price lower than p but turn out to be rationed.

I am now in the position to establish some lemmas which provide a general characterization of equilibrium. In the first lemma, I prove that, in any equilibrium, the maximized seller's profit V^*, with $V^* = \max_p V(p)$, must be strictly positive. The intuition behind this lemma is the same as in Butters (1977), Varian (1980), and Burdett and Judd (1983). Namely, a seller can achieve a strictly positive profit by posting a price p equal to the buyer's valuation u and by selling only to captive buyers. The lemma implies that the distribution F cannot contain on its support any price p strictly greater than u, since $V(p) = 0$ for any $p > u$.

Lemma 1: *In any equilibrium, the seller's profit V^* is strictly positive.*

Proof: The profit for a seller that posts the price u is

$$\begin{aligned} V(u) = \quad & \max_q pq - c(q), \\ & \text{s.t. } q \in [0, b\{1 - \alpha + 2\alpha(1 - F(u)) + \alpha\mu(u)\}]. \end{aligned} \tag{3.3}$$

For any F, the seller can choose q equal to $b(1-\alpha)$. Since $u \geq c'(b)$ and $c'(b) > c'(b(1-\alpha))$, it follows that $u > c'(b(1-\alpha))$. Since c is strictly convex and $c(0) = 0$, $qc'(q) > c(q)$ for any $q > 0$. Combining these observations yields $ub(1-\alpha) > c'(b(1-\alpha))b(1-\alpha) > c(b(1-\alpha))$ and, hence, $V(u)$ is strictly positive. Since $V^* \geq V(u)$, it follows that V^* is strictly positive. ■

In the second lemma, I show that a seller posting a price p finds it optimal to meet the buyers' demand $d(p)$ if p is greater or equal to $c'(d(p))$, while it finds it optimal to ration the buyers if p is strictly smaller than $c'(d(p))$. The intuition for the result is straightforward. Namely, a seller is going to meet the buyers' demand $d(p)$ if and only if the marginal cost of producing $d(p)$ units of output is non-greater than the price p. The result has an important implication. Since equilibrium requires that sellers meet the buyers' demand, it follows that the support of the distribution F cannot contain any price p such that $p < c'(d(p))$.

Lemma 2: (i) *A seller posting a price p such that $p \geq c'(d(p))$ finds it optimal to meet the buyers' demand $d(p)$. A seller posting a price p such that $p < c'(d(p))$ finds it optimal to ration buyers.* (ii) *In any equilibrium, the support of the price distribution F does not include any price p such that $p < c'(d(p))$.*

Proof: (i) Consider a seller posting a price p. The problem of the seller is

$$\begin{aligned} V(p) = \quad & \max_q pq - c(q) \\ & \text{s.t. } q \in [0, d(p)]. \end{aligned} \tag{3.4}$$

The objective function in (3.4) is strictly concave in q. Therefore, the solution to (3.4) is $d(p)$ if the derivative of the objective function with respect to q is non-negative when evaluated at $d(p)$, while the solution to (3.4) is strictly smaller than $d(p)$ if the derivative of the objective function with respect to q is strictly negative when evaluated at $d(p)$. That is, if $p - c'(d(p)) \geq 0$, the solution to (3.4) is $d(p)$. If $p - c'(d(p)) < 0$, the solution to (3.4) is some q strictly smaller than $d(p)$.

(ii) On the way to a contradiction, suppose that there exists an equilibrium in which the support of F includes a price p such that $p_0 < c'(d(p))$. From part (i), it follows that a seller posting the price p finds it optimal to produce a quantity $q < d(p)$. This contradicts condition (iv) in the definition of equilibrium. ■

In the third lemma, I prove that, in any equilibrium, the distribution F cannot have a mass point at a price p_0 such that p_0 is strictly greater than the seller's marginal cost $c'(d(p_0))$. The intuition behind this lemma is essentially the same as in Butters (1977), Varian (1980) and Burdett and Judd (1983). Namely, a mass point at some price p_0 creates a discontinuity in the demand faced by the seller, since any price strictly smaller than p_0 gives the seller the option to trade with all, rather than half, of the non-captive buyers whose second contact has a price of p_0. Therefore, if p_0 exceeds the marginal cost of production $c'(d(p_0))$, an individual seller can strictly increase its profit by posting the price $p_0 - \epsilon$ rather than the price p_0, for some ϵ small enough. Notice that, unlike in Butters (1977), Varian (1980) or Burdett and Judd (1983), the deviation from p_0 to $p_0 - \epsilon$ may require the seller to ration some buyers in order to be profitable.

Lemma 3: *In any equilibrium, the price distribution F does not have a mass point at any price p_0 such that $p_0 > c'(d(p_0))$.*

Proof: On the way to a contradiction, suppose that there is an equilibrium in which F has a mass point at some price p_0 such that $p_0 > c'(d(p_0))$. The equilibrium profit for a seller posting the price p_0 is given by

$$V(p_0) = p_0 d(p_0) - c(d(p_0)), \tag{3.5}$$

where

$$d(p_0) = b\left\{1 - \alpha + 2\alpha\left[1 - F(p_0) + \mu(p_0)/2\right]\right\}. \tag{3.6}$$

For any $\epsilon > 0$, the profit for a seller posting the price $p_0 - \epsilon$ is given by

$$\begin{aligned} V(p_0 - \epsilon) = \quad & \max_q (p_0 - \epsilon)q - c(q) \\ & \text{s.t. } q \in [0, d(p_0 - \epsilon)]. \end{aligned} \tag{3.7}$$

where

$$d(p_0 - \epsilon) \geq b\left\{1 - \alpha + 2\alpha\left[1 - F(p_0) + \mu(p_0)\right]\right\}. \tag{3.8}$$

The expression in (3.8) states that the demand $d(p_0 - \epsilon)$ is at least equal to the sum of measure of captive buyers, $b(1 - \alpha)$, the measure of non-captive buyers whose second contact posts a price strictly smaller than p_0, $b2\alpha(1 - F(p_0))$, and the measure of non-captive buyers whose second contact posts a price equal to p_0, $b2\alpha\mu(p_0)$. Comparing (3.8) and (3.7), it follows that $d(p_0 - \epsilon)$ exceeds $d(p_0)$ by at least $b\alpha\mu(p_0)$.

Let q^* be such that $c'(q^*)$ equals $c'(d(p_0)) + [p_0 - c'(d(p_0))]/2$. That is q^* is such that the seller's marginal cost is equal to $c'(d(p_0))$ plus half of the strictly positive difference between p_0 and $c'(d(p_0))$. Let δ denote $q^* - d(p_0)$. Notice that δ is strictly positive and does not depend on ϵ. First, consider the case in which q^* is smaller than the lower bound on $d(p_0 - \epsilon)$ on the right-hand side of (3.8). In this case, the difference between $V(p_0 - \epsilon)$

and $V(p_0)$ is such that

$$\begin{aligned}
& V(p_0-\epsilon)-V(p_0) \\
\geq \quad & -d(p_0)\epsilon+\delta\,(p_0-\epsilon)-[c(q^*)-c(d(p_0))] \\
> \quad & -d(p_0)\epsilon+\delta\,(p_0-\epsilon)-\delta c'(q^*) \\
= \quad & -d(p_0)\epsilon+\delta\,(p_0-\epsilon)-\delta\left\{c'(d(p_0))+[p_0-c'(d(p_0))]/2\right\} \\
= \quad & -d(p_0)\epsilon-\delta\epsilon+\delta\,[p_0-c'(d(p_0))]/2,
\end{aligned} \tag{3.9}$$

where the second line makes use of the fact that q^* is a feasible quantity for a seller posting the price $p_0-\epsilon$, the third line makes use of the fact that c is strictly convex, the fourth line makes use of the definition of q^*, and the last line is an algebraic manipulation of the previous one. Since (3.9) holds for any $\epsilon>0$, $p_0-c'(d(p_0))>0$ and $\delta>0$, there exists an ϵ small enough such that $V(p_0-\epsilon)-V(p_0)>0$. Therefore, $V(p_0)<V(p_0-\epsilon)\leq V^*$, which implies that p_0 cannot be on the support of F and, hence, we reached a contradiction.

Now, consider the case in which q^* is greater than the lower bound on $d(p_0-\epsilon)$ on the right-hand side of (3.8). In this case, let $\hat{q}$ be equal to $b(1-\alpha)+b2\alpha\,[1-F(p_0)+\mu(p_0)]$. Let $\hat{\delta}$ denote $\hat{q}-d(p_0)$ and notice that $\hat{\delta}$ is strictly positive and does not depend on ϵ. The difference between $V(p_0-\epsilon)$ and $V(p_0)$ is such that

$$\begin{aligned}
& V(p_0-\epsilon)-V(p_0) \\
\geq \quad & -d(p_0)\epsilon+\hat{\delta}(p_0-\epsilon)-[c(\hat{q})-c(d(p_0))] \\
> \quad & -d(p_0)\epsilon+\hat{\delta}(p_0-\epsilon)-\hat{\delta}c'(\hat{q}) \\
> \quad & -d(p_0)\epsilon+\hat{\delta}(p_0-\epsilon)-\hat{\delta}\left\{c'(d(p_0))+[p_0-c'(d(p_0))]/2\right\} \\
= \quad & -d(p_0)\epsilon-\hat{\delta}\epsilon+\hat{\delta}\,[p_0-c'(d(p_0))]/2,
\end{aligned} \tag{3.10}$$

where the second line makes use of the fact that $\hat{q}$ is a feasible quantity for a seller posting the price $p_0-\epsilon$, the third line makes use of the fact that c is strictly convex, the fourth line makes use of the fact that $\hat{q}<q^*$ and, hence, $c'(\hat{q})<c'(q^*)$ and $c'(q^*)$ equals $c'(d(p_0))+[p_0-c'(d(p_0))]/2$. Since (3.10) holds for any $\epsilon>0$, $p_0-c'(d(p_0))>0$ and $\hat{\delta}>0$, there exists an ϵ small enough such that $V(p_0-\epsilon)-V(p_0)>0$. Therefore, $V(p_0)<V(p_0-\epsilon)\leq V^*$, which implies that p_0 cannot be on the support of F and, hence, we reached a contradiction. ■

In the fourth lemma, I show that the distribution F can have at most one mass point. Moreover, I show that, if the distribution F does have a mass point at some price p_0, then p_0 must be equal to the seller's marginal cost $c'(q(p_0))$ and p_0 must be the lowest price on the support of F. These findings are easy to understand. The fact that any mass point must be at a price equal to the seller's marginal cost follows from Lemma 2 and Lemma 3. The fact that there can be at most one mass point follows from the fact that there is at most one price that is equal to the seller's marginal cost, since a seller's output is strictly decreasing in the price and, hence, the seller's marginal cost is strictly decreasing in the price. The fact that the mass point must be the lowest price on the support of F follows immediately from Lemma 2.

Lemma 4: *In any equilibrium in which the distribution F has mass points*: (i) *There is a unique mass point p_0 and p_0 is such that $p_0 = c'(d(p_0))$*; (ii) *The lowest price on the support of F is p_0*; (iii) *The buyer's demand at p_0 is $d(p_0) = b\{1 + \alpha(1 - F(p_0))\}$*.

Proof: (i) Consider an equilibrium in which the distribution F has a mass point at some p_0. From Lemma 2, it follows that p_0 is such that $p_0 \geq c'(d(p_0))$. From Lemma 3, it follows that p_0 is such that $p_0 \leq c'(d(p_0))$. Combining these observations yields $p_0 = c'(d(p_0))$.

I now want to establish that the mass point is unique. On the way to a contradiction, suppose that the distribution F has another mass point at some price $p_1 \neq p_0$. First, consider the case of $p_1 > p_0$. In this case, the buyers' demand for a seller posting the price p_1 is given by

$$\begin{aligned} d(p_1) &= b\left\{1 - \alpha + 2\alpha\left[1 - F(p_1) + \mu(p_1)/2\right]\right\} \\ &< b\left\{1 - \alpha + 2\alpha\left[1 - F(p_0) + \mu(p_0)/2\right]\right\} = d(p_0), \end{aligned} \tag{3.11}$$

where the second line makes use of the fact that $F(p_0) \leq F(p_1) - \mu(p_1)$. Since $d(p_1) < d(p_0)$, it follows that $c'(d(p_1)) < c'(d(p_0))$. The previous observation together with $p_0 < p_1$ and $p_0 = c'(d(p_0))$ yields $c'(d(p_1)) < p_1$. Since p_1 is a mass point, however, it must be the case that $c'(d(p_1)) = p_1$. I reached a contradiction.

Next, consider the case of $p_1 < p_0$. In this case, the buyers' demand for a seller posting the price p_1 is given by

$$\begin{aligned} d(p_1) &= b\left\{1 - \alpha + 2\alpha\left[1 - F(p_1) + \mu(p_1)/2\right]\right\} \\ &> b\left\{1 - \alpha + 2\alpha\left[1 - F(p_0) + \mu(p_0)/2\right]\right\} = d(p_0), \end{aligned} \tag{3.12}$$

where the second line makes use of the fact that $F(p_1) \leq F(p_0) - \mu(p_0)$. Since $d(p_1) > d(p_0)$, it follows that $c'(d(p_1)) > c'(d(p_0))$. The previous observation together with $p_0 > p_1$ and $p_0 = c'(d(p_0))$ yields $c'(d(p_1)) > p_1$. Since p_1 is a mass point, however, it must be the case that $c'(d(p_1)) = p_1$. Again, I reached a contradiction.

(ii) On the way to a contradiction, suppose that the distribution F has a mass point at some p_0 and that there exists a $p_1 < p_0$ on the support of F. The buyers' demand for a seller posting the price p_1 is given by

$$\begin{aligned} d(p_1) &= b\left\{1 - \alpha + 2\alpha\left[1 - F(p_1)\right]\right\} \\ &> b\left\{1 - \alpha + 2\alpha\left[1 - F(p_0) + \mu(p_0)/2\right]\right\} = d(p_0), \end{aligned} \tag{3.13}$$

Since $d(p_1) > d(p_0)$, it follows that $c'(d(p_1)) > c'(d(p_0))$. The previous observation together with $p_1 < p_0$ and $p_0 = c'(d(p_0))$ yields $c'(d(p_1)) > p_1$. Since p_1 is on the support of F, Lemma 2 implies that $c'(d(p_1)) \leq p_1$. I reached the desired contradiction.

(iii) Suppose that the distribution F has a mass point at some p_0. The buyers' demand at p_0 is given by

$$\begin{aligned} d(p_0) &= b\{1-\alpha+2\alpha[1-F(p_1)+\mu(p_0)/2]\} \\ &= b\{1-\alpha+2\alpha[1-F(p_0)+F(p_0)/2]\} \\ &= b\{1+\alpha[1-F(p_0)]\}, \end{aligned} \tag{3.14}$$

where the second line makes use of the fact that the mass point p_0 is the lowest price on the distribution F and, hence, $\mu(p_0)=F(p_0)$. ■

Lemma 4 states that the price distribution F has at most one mass point. Therefore, there are three possible types of equilibria: (i) Equilibria in which the distribution F is degenerate at the mass point; (ii) Equilibria in which the distribution F is non-degenerate but has a mass point; (iii) Equilibria in which the distribution F is non-degenerate and does not have a mass point. I now characterize some properties of the equilibria of type (ii) and (iii).

Lemma 5 deals with equilibria in which the distribution F does not have a mass point. It shows that the support of F is an interval $[p_\ell, p_h]$ such that p_h is equal to the buyers' valuation u. The logic behind this lemma is exactly the same as in Butters (1977), Varian (1980) and Burdett and Judd (1983). Namely, if the distribution F has a gap between p_1 and p_2, the buyers' demand $d(p)$ is constant for all p in the interval $[p_1, p_2]$. Hence, an individual seller obtains a strictly higher profit by posting the price p_2 rather than the price p_1, which implies that p_1 is not on the support of F. Similarly, if the highest price p_h on the distribution F is strictly smaller than u, the buyers' demand $d(p)$ is constant for any p in the interval $[p_h, u]$. Hence, an individual seller obtains a strictly higher profit by posting the price u rather than the price p_h, which implies that p_h cannot be on the support of F.

Lemma 5: *In any equilibrium in which the distribution F does not have a mass point, the support of F is an interval $[p_\ell, p_h]$, with $p_h = u$.*

Proof: Consider an equilibrium in which the distribution F is non-degenerate and does not have a mass point. In such an equilibrium, the buyers' demand $d(p)$ is given by $b\{1-\alpha+2\alpha(1-F(p))\}$.

I first show that there cannot be a gap on the support of F. On the way to a contradiction, suppose that there is a gap on the support of F between p_1 and p_2, with p_1 and p_2 on the support of F and $p_1 < p_2$. The profit for a seller positing the price p_1 is given by

$$\begin{aligned} V(p_1) &= b\{1-\alpha+2\alpha[1-F(p_1)]\}\,p_1 - c\left(b\{1-\alpha+2\alpha[1-F(p_1)]\}\right) \\ &< b\{1-\alpha+2\alpha[1-F(p_2)]\}\,p_2 - c\left(b\{1-\alpha+2\alpha[1-F(p_2)]\}\right) \\ &= V(p_2). \end{aligned} \tag{3.15}$$

The first and the third lines in (3.15) make use of the fact that, for any p on the support of F, the seller finds it optimal to meet the buyers' demand $d(p)$, and $d(p)$ is given by

$b\{1-\alpha+2\alpha(1-F(p))\}$. The second line makes use of the fact that $F(p_2)=F(p_1)$ and $p_2>p_1$. Since $V(p_1)<V(p_2)$ and $V(p_2)\leq V^*$, it follows that p_1 cannot be on the support of F. This is the desired contradiction.

Since the support of F cannot have any gaps, the support of F must be an interval $[p_\ell,p_h]$. I now show that p_h is equal to u. Suppose that p_h is strictly greater than u. In this case, the profit $V(p_h)$ for a seller posting the price p_h is equal to 0. Since Lemma 1 shows that $V^*>0$, it follows that $p_h>u$ cannot be on the support of F. Suppose that p_h is strictly smaller than u. In this case, the profit $V(p_h)$ for a seller posting the price p_h is equal to $b(1-\alpha)p_h-c(b(1-\alpha))$. The profit $V(u)$ for a seller posting the price u is equal to $b(1-\alpha)p_h-c(b(1-\alpha))$. Since $p_h<u$, it follows that $V(p_h)<V(u)\leq V^*$ and, hence, $p_h<u$ cannot be on the support of F. Therefore, p_h must be equal to u. ■

Lemma 6 deals with equilibria in which the distribution F is non-degenerate and has a mass point. The lemma shows that the support of F is the mass point p_0 and an interval $[p_\ell,p_h]$, with $p_\ell>p_0$ and $p_h=u$. Lemma 4 implies that the mass point p_0 must be the lowest price on the support of F. The same argument used in Lemma 5 implies that, away from the mass point, the support of F must be an interval $[p_\ell,p_h]$ with $p_h=u$. The support of F must have a gap between p_0 and p_ℓ because the quantity of output sold by a seller that posts the price p_0 is strictly greater than the quantity of output sold by a seller that posts the price p_ℓ. Since the profit of the two sellers has to be identical, p_ℓ must be strictly smaller than p_0.

Lemma 6: *In any equilibrium in which the distribution F is non-degenerate and has a mass point p_0, the support of F is $p_0\cup[p_\ell,p_h]$, with $p_\ell>p_0$ and $p_h=u$.*

Proof: Consider an equilibrium in which the distribution F is non-degenerate and has a mass point. From Lemma 4, it follows that the mass point p_0 is the lowest price on the support of F and such that p_0 equals $c'(d(p_0))$, with $d(p_0)=b\{1+\alpha[1-F(p_0)]\}$. From Lemma 4, it also follows that any price $p_1\neq p_0$ on the support of F is strictly greater than p_0. For any such price $d(p_1)=b\{1-\alpha+2\alpha(1-F(p_1))\}$ since there is only one mass point.

I first show that the support of F does not include any prices in the neighborhood $(p_0,p_0+\epsilon]$, with $\epsilon>0$. On the way to a contradiction, suppose that the price p_1 is on the support of F. The difference between the profit for a seller posting the price p_0 and a seller posting the price p_1 is given by

$$\begin{aligned}V(p_0)-V(p_1)\quad &= d(p_0)p_0-d(p_1)p_1-[c(d(p_0))-c(d(p_1))]\\ &=[d(p_0)-d(p_1)]\,p_0-[p_1-p_0]\,d(p_1)-\int_{d(p_1)}^{d(p_0)}c'(x)dx\\ &=\int_{d(p_1)}^{d(p_0)}[c'(d(p_0))-c'(x)]\,dx-[p_1-p_0]\,d(p_1)\end{aligned}\tag{3.16}$$

where the last line in (3.16) makes use of the fact that $p_0=c'(d(p_0))$. The last line shows that $V(p_0) - V(p_1)$ can be expressed as the difference of two terms. Since $d(p_0)$ is equal to $b\,[1+\alpha(1-F(p_0))]$ and $d(p_1)$ is non-greater than $b[1-\alpha+2\alpha(1-F(p_0))]$, the first term is such that

$$\begin{aligned} & \int_{d(p_1)}^{d(p_0)} [c'(d(p_0)) - c'(x)]\, dx \\ \geq \; & \int_{b[1-\alpha+2\alpha(1-F(p_0))]}^{b[1+\alpha(1-F(p_0))]} [c'(b\,[1+\alpha(1-F(p_0))]) - c'(x)] dx \end{aligned} \tag{3.17}$$

For the same reason, the second term in the last line of (3.16) is such that

$$\begin{aligned} & [p_1 - p_0]\, d(p_1) \\ \leq \; & [p_1 - p_0]\, b[1-\alpha+2\alpha(1-F(p_0))]. \end{aligned} \tag{3.18}$$

The lower bound on the right-hand side of (3.17) is strictly positive, since $b\,[1+\alpha(1-F(p_0))]$ is strictly greater than $b\,[1-\alpha+2\alpha(1-F(p_0))]$ and c is strictly convex. Moreover, the lower bound is independent of p_1. The upper bound on the right-hand side of (3.18) is also strictly positive but becomes arbitrarily small for p_1 close to p_0. These observations imply that, for all $p_1 \in (p_0, p_0+\epsilon]$, $V(p_0) - V(p_1) > 0$ and, hence, p_1 cannot be on the support of F. A contradiction.

Next, I show that there cannot be a gap on the support of F between any two prices p_1 and p_2 where p_1 and p_2 are on the support of F and $p_0 < p_1 < p_2$. On the way to a contradiction, suppose there is a gap in the support of F between p_1 and p_2. The profit for a seller positing the price p_1 is

$$\begin{aligned} V(p_1) \; & = b\,\{1-\alpha+2\alpha[1-F(p_1)]\}\, p_1 - c\,(b\,\{1-\alpha+2\alpha[1-F(p_1)]\}) \\ & < b\,\{1-\alpha+2\alpha[1-F(p_2)]\}\, p_2 - c\,(b\,\{1-\alpha+2\alpha[1-F(p_2)]\}) \\ & = V(p_2). \end{aligned} \tag{3.19}$$

where the second line makes use of $F(p_2) = F(p_1)$ and $p_2 > p_1$. Since $V(p_1) < V(p_2)$, the price p_1 cannot be on the support of F.

I have shown that the support of F cannot include any price $p_1 < p_0$, any price $p_1 \in (p_0, p_0+\epsilon)$, and it cannot have any gaps between any two prices p_1 and p_2 that are strictly greater than p_0. Therefore, the support of F is given by p_0 and some interval $[p_\ell, p_h]$ with $p_\ell > p_0 + \epsilon$. Using the same argument as in Lemma 5, I can show that p_h must be equal to u. ■

4 Existence of different types of equilibria

In the previous section, I established that there are only three possible types of equilibria: (i) Equilibria in which F is degenerate; (ii) Equilibria in which F non-degenerate and has a mass point; (iii) Equilibria in which F is non-degenerate and does not have any mass points. In this section, I derive necessary and sufficient conditions on the fundamentals under which each type of equilibrium exists and characterize its properties.

4.1 Degenerate distribution

In the first type of equilibrium, the distribution F is a mass point at some price p_0. It is easy to characterize this type of equilibrium. Part (i) of Lemma 4 implies that the price p_0 equals the seller's marginal cost $c'(d(p_0))$. Part (iii) of Lemma 4 implies that the buyers' demand $d(p_0)$ is equal to b. Therefore, the profit for a seller posting the price p_0 is

$$V(p_0) = bc'(b) - c(b). \tag{4.1}$$

Since p_0 is on the support of F, it follows that $V(p_0) = V^*$.

For the equilibrium to exist, it must be the case that a seller cannot attain a profit strictly greater than V^* by posting a price different from p_0. If a seller posts a price $p < p_0$, its profit is given by

$$\begin{aligned} V(p) \quad &= \max_q pq - c(q), \text{ s.t. } q \in [0, b(1+\alpha)] \\ &< \max_q c'(b)q - c(q), \text{ s.t. } q \in [0, b(1+\alpha)] \\ &= c'(b)b - c(b) = V^*. \end{aligned} \tag{4.2}$$

The first line in (4.2) makes use of the fact that $d(p) = b(1+\alpha)$ for all $p < p_0$. The second line makes use of the fact that $p < p_0$ and $p_0 = c'(b)$. The third line follows from the observation that the quantity q that solves the maximization problem in the second line is b. The inequalities in (4.2) imply that a seller cannot attain a profit strictly greater than V^* by posting a price $p < p_0$.

If a seller posts a price p equal to the buyers' valuation u, its profit is given by

$$\begin{aligned} V(u) \quad &= \max_q uq - c(q), \text{ s.t. } q \in [0, b(1-\alpha)] \\ &= ub(1-\alpha) - c(b(1-\alpha)). \end{aligned} \tag{4.3}$$

The first line in (4.3) makes use of the fact that $d(p)$ equals $b(1-\alpha)$ for all $p > p_0$, and $u > c'(b) = p_0$. The second line follows from the observation that the solution to the problem in the first line is a quantity q equal to $b(1-\alpha)$, since the derivative of the objective function evaluated at $b(1-\alpha)$ is $u - c'(b(1-\alpha)) > u - c'(b) > 0$. Overall, (4.3) implies a seller posting a price p equal to u cannot attain a profit strictly greater than V^* if and only if

$$ub(1-\alpha) - c(b(1-\alpha)) \leq c'(b)b - c(b). \tag{4.4}$$

If a seller posts a price p in the interval (p_0, u), its profit is given by

$$\begin{aligned} V(p) &= \max_q pq - c(q), \text{ s.t. } q \in [0, b(1-\alpha)] \\ &< \max_q uq - c(q), \text{ s.t. } q \in [0, b(1-\alpha)] \\ &\leq V^*, \end{aligned} \tag{4.5}$$

The first line in (4.5) makes use of the fact that $d(p)$ equals $b(1-\alpha)$ for all $p \in (p_0, u)$. The second line makes use of the fact that $p < u$. The third line makes use of (4.3) and (4.4). Overall, the inequalities in (4.5) imply that a seller posting a price $p \in (p_0, u)$ cannot attain a profit strictly greater than V^* as long as condition (4.4) is satisfied. Finally, note that a seller cannot attain a profit strictly greater than V^* by posting a price strictly greater than the buyer's valuation u since $V^* > 0$ and $V(p) = 0$ for all $p > u$.

For the equilibrium to exist, it must also to be the case that a seller chooses to meet the buyers' demand whenever it posts a price on the support of the distribution F. The only price on the support of F is p_0, which is equal to $c'(d(p_0))$. Since the price p_0 is non-smaller than the marginal cost $c'(d(p_0))$, part (i) of Lemma 2 implies that a seller finds it optimal to meet the buyers' demand.

The following proposition summarizes the conditions for existence and the properties of an equilibrium in which the price distribution is degenerate.

Proposition 1: (Degenerate distribution)

(i) *An equilibrium with a degenerate distribution F exists if and only if*

$$ub(1-\alpha) - c(b(1-\alpha)) \leq c'(b)b - c(b). \tag{4.6}$$

(ii) *If an equilibrium with a degenerate distribution F exists, it is unique. The equilibrium is such that every seller posts the price $p_0 = c'(b)$ and trades $d(p_0) = b$ units of the good.*

4.2 Non-degenerate distribution with a mass point

The second type of equilibrium is such that the price distribution F is non-degenerate and has a mass point. I now proceed to characterize this type of equilibrium. From part (i) of Lemma 4, it follows that the mass point is at a price p_0 equal to the seller's marginal cost $c'(d(p_0))$. From part (iii) of Lemma 4, it follows that p_0 is the lowest price on the distribution F and $d(p_0)$ is equal to $b[1 + \alpha(1 - F(p_0))]$. Therefore, the profit for a seller posting p_0 is given by

$$V(p_0) = c'(b[1 + \alpha(1 - x_0)])b[1 + \alpha(1 - x_0)] - c(b[1 + \alpha(1 - x_0)]), \tag{4.7}$$

where $x_0 = F(p_0)$ denotes the measure of sellers that post a price equal to p_0. Since p_0 is on the support of F, $V(p_0)$ must be equal to V^*.

From Lemma 6, it follows that the support of F includes an interval $[p_\ell, p_h]$, with $p_\ell > p_0$ and $p_h = u$, and $d(p)$ is equal to $b[1 - \alpha + 2\alpha(1 - F(p))]$ for all $p \in [p_\ell, p_h]$. Therefore, the profit for a seller positing the price p equal to the buyers' valuation u is

$$\begin{aligned} V(u) &= ub\,[1 - \alpha + 2\alpha(1 - F(u))] - c\,(b\,[1 - \alpha + 2\alpha(1 - F(u))]) \\ &= ub(1 - \alpha) - c(b(1 - \alpha)), \end{aligned} \tag{4.8}$$

where the second line makes use of the fact that $F(u) = 1$. Since u is on the support of F, $V(u)$ must be equal to V^*.

Equating the profit of a seller posting the price p_0 and the profit of a seller posting the price u yields

$$c'(b[1 + \alpha(1 - x_0)])b[1 + \alpha(1 - x_0)] - c(b[1 + \alpha(1 - x_0)]) = ub(1 - \alpha) - c(b(1 - \alpha)). \tag{4.9}$$

The equation above pins down x_0, the fraction of sellers that post a price equal to p_0. The fraction x_0 must be strictly greater than 0—since F has a mass point at p_0—and strictly smaller than 1—since F is non-degenerate. The left-hand side of (4.9) is a strictly decreasing function of x_0, takes the value $c'(b(1 + \alpha))b(1 + \alpha) - c(b(1 + \alpha))$ for $x_0 = 0$, and the value $c'(b)b - c(b)$ for $x_0 = 1$. The right-hand side of (4.9) is independent of x_0. From these observations, it follows that x_0 is strictly greater than 0 if and only if

$$c'(1 + \alpha)b(1 + \alpha) - c(1 + \alpha) > ub(1 - \alpha) - c(b(1 - \alpha)). \tag{4.10}$$

and it is strictly smaller than 1 if and only if

$$c'(b)b - c(b) < ub(1 - \alpha) - c(b(1 - \alpha)). \tag{4.11}$$

Assume that conditions (4.10) and (4.11) are satisfied.

The profit for a seller posting a price $p \in [p_\ell, p_h]$ is such that

$$V(p) = pb\,[1 - \alpha + 2\alpha(1 - x(p))] - c(b\,[1 - \alpha + 2\alpha(1 - x(p))]), \tag{4.12}$$

where $x(p) = F(p)$ denotes the fraction of sellers posting a price smaller than p. Since any $p \in [p_\ell, p_h]$ is on the support of F, $V(p)$ must be equal to V^*.

Equating the profit of a seller posting the price $p \in [p_\ell, p_h]$ and the profit of a seller posting the price u yields

$$pb\,[1 - \alpha + 2\alpha(1 - x(p))] - c(b\,[1 - \alpha + 2\alpha(1 - x(p))]) = ub(1 - \alpha) - c(b(1 - \alpha)). \tag{4.13}$$

The above equation pins down $x(p)$, the fraction of sellers that post a price smaller than p. Since $F(p_\ell) = F(p_0)$ and $p_\ell > p_0$, the fraction $x(p)$ must be such that $x(p_\ell) = x_0$ for some $p_\ell > p_0$. Since $F(p_h) = 1$ and $p_h = u$, the fraction $x(p)$ must be such that $x(u) = 1$. Since $F(p)$ is strictly increasing over the interval (p_ℓ, p_h), the fraction $x(p)$ must be strictly increasing in p.

Rather than characterizing the function $x(p)$, it is easier to characterize its inverse $p(x)$, i.e. the price posted by a seller that is at the x-th quantile of the price distribution. To characterize $p(x)$ notice that the equal profit condition (4.13) is equivalent to

$$p'(x) = \frac{2\alpha}{1-\alpha+2\alpha x}\{p(x) - c'(b[1-\alpha+2\alpha(1-x)]\} \tag{4.14}$$

and

$$p(1) = u. \tag{4.15}$$

The expression in (4.14) is obtained by totally differentiating (4.13) and guarantees that the seller's profit $V(p)$ is constant over the interval $[p_\ell, p_h]$. The expression in (4.15) is obtained from (4.13) and $x(u) = 1$, and it guarantees that the seller's profit $V(p)$ is equal to the right-hand side of (4.13) at $p_h = u$.

The expression in (4.14) is a differential equation for $p(x)$. If and only if the price $p(x)$ equals the marginal cost $c'(b[1-\alpha+2\alpha(1-x)])$, $p'(x) = 0$. That is, $c'(b[1-\alpha+2\alpha(1-x)])$ is the null-cline of the differential equation (4.14). If the price $p(x)$ exceeds the marginal cost $c'(b[1-\alpha+2\alpha(1-x)])$, $p'(x) > 0$. That is, everywhere above the null-cline, $p(x)$ is strictly increasing. If the price $p(x)$ is lower than the marginal cost $c'(b[1-\alpha+2\alpha(1-x)])$, $p'(x) < 0$. That is, everywhere below the null-cline, $p(x)$ is strictly decreasing. Since the null-cline is strictly decreasing in x, $p(x)$ can only cross the null-cline from below. The properties of the differential equation (4.14) are illustrated in Figure 1. The relevant solution to the differential equation (4.14) is the one that satisfies the boundary condition (4.15). Since $p(1) = u$ and $u > c'(b(1-\alpha))$, this solution to the differential equation is above the null-cline at $x = 1$. Since $p(x)$ can cross the null-cline only from below, the relevant solution to the differential equation is such that $p'(x) > 0$ for all $x \in (x_0, 1)$ if and only if the relevant solution is above the null-cline at $p(x_0)$. That is, if and only if $p(x_0) \geq c'(b[1-\alpha+2\alpha(1-x_0)])$.

In order to find $p(x_0)$, one could solve the differential equation (4.14) subject to the boundary condition (4.15). It is however easier to use that fact that (4.14) and (4.15) guarantee that the seller's profit from posting the price $p(x_0)$ is equal to the right-hand side of (4.13), which, in turn, is equal to the seller's profit from posting the price p_0. The equal profit condition between $p(x_0)$ and p_0 is

$$\begin{aligned} & p(x_0)b[1-\alpha+2\alpha(1-x_0)] - c(b[1-\alpha+2\alpha(1-x_0)]) \\ = \quad & c'(b[1+\alpha(1-x_0)])b[1+\alpha(1-x_0)] - c(b[1+\alpha(1-x_0)]). \end{aligned} \tag{4.16}$$

where the right-hand side makes use of the fact that $p_0 = c'(b[1+\alpha(1-x_0)])$. The solution to (4.16) is a $p(x_0)$ such that $p(x_0)$ is strictly greater than $c'(b[1-\alpha+2\alpha(1-x_0)])$. For $p(x_0)$ equal to $c'(b[1-\alpha+2\alpha(1-x_0)])$, the left-hand side of (4.16) is strictly smaller than the right-hand side because $c'(q)q - c(q)$ is a strictly increasing function of q and $1-\alpha+2\alpha(1-x_0)$ is strictly smaller than $1+\alpha(1-x_0)$. A fortiori, if $p(x_0)$ is strictly smaller than $c'(b[1-\alpha+2\alpha(1-x_0)])$, the left-hand side of (4.16) is strictly smaller than

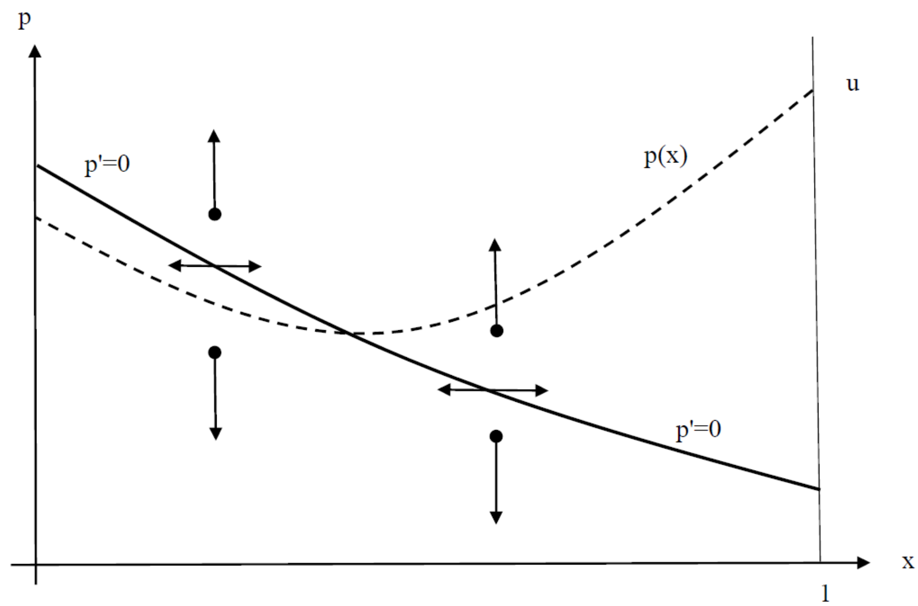

Figure 1: Differential equation (4.14)

the right-hand side. Hence, the solution to (4.16) must be a price $p(x_0)$ strictly greater than $c'(b[1-\alpha+2\alpha(1-x_0)])$. This implies that $p'(x) > 0$ for all $x \in (x_0, 1)$.

Let me rewrite (4.16) as

$$\begin{aligned} & p(x_0)b[1-\alpha+2\alpha(1-x_0)] - c(b[1-\alpha+2\alpha(1-x_0)]) \\ = \; & \max_q c'(b[1+\alpha(1-x_0)])q - c(q). \end{aligned} \tag{4.17}$$

where the right-hand side of (4.17) makes use of the fact that the quantity q that maximizes $p_0 q - c(q)$ is $b[1+\alpha(1-x_0)]$ and p_0 is $c'(b[1+\alpha(1-x_0)])$. The solution to (4.17) is a $p(x_0)$ strictly greater than p_0. For $p(x_0) \leq p_0$, the left-hand side of (4.17) is strictly smaller than the right-hand side, since $p(x_0) \leq p_0$ and $b[1-\alpha+2\alpha(1-x_0)]$ is different than $b[1+\alpha(1-x_0)]$. Hence, the solution to (4.17) must be a price $p(x_0)$ strictly greater than p_0.

From the above observations, it follows that the solution to the differential equation (4.14)-(4.15) is such that $p(x_0) > p_0$, $p'(x) > 0$ for all $x \in (x_0, 1)$, and $p(1) = u$. In turn, since $p'(x) > 0$, it follows that $p(x)$ is greater than $c'(b[1-\alpha+2\alpha(1-x)])$ for all $x \in [x_0, 1]$. These properties of $p(x)$ imply that $x(p)$, the inverse of $p(x)$, is such that $x(p_\ell) = x_0$ for some $p_\ell > p_0$, $x'(p) > 0$ for all $p \in (p_\ell, p_h)$, and $x(p_h) = 1$ with $p_h = u$. That is, $x(p)$ satisfies all the equilibrium requirements. Moreover, $x(p)$ is such that p is greater than $c'(b[1-\alpha+2\alpha(1-x(p))])$ for all $p \in [p_\ell, p_h]$.

For the equilibrium to exist, I still need to show that a seller cannot make a profit strictly greater that V^* by posting a price p that is not on the support of F. Consider a seller posting a price p strictly smaller than p_0. The seller's profit is given by

$$\begin{aligned} V(p) &= \max_q pq - c(q), \text{ s.t. } q \in [0, b(1+\alpha)] \\ &< \max_q p_0 q - c(q), \text{ s.t. } q \in [0, b(1+\alpha)] \\ &= p_0 b[1+\alpha(1-x_0)] - c(b[1+\alpha(1-x_0)]) = V^*, \end{aligned} \tag{4.18}$$

where the first line follows from the fact that $d(p) = b(1+\alpha)$ for all $p < p_0$, the second line follows from the fact that $p < p_0$, and the third line follows from the fact that the solution to the maximization problem in the second line is $b[1+\alpha(1-x_0)]$, since p_0 is equal to $c'(b[1+\alpha(1-x_0)])$. The inequalities in (4.18) imply that a seller cannot attain a profit greater than V^* by posting a price p strictly smaller than p_0.

Consider a seller posting a price p in the interval (p_0, p_ℓ). The seller's profit is

$$\begin{aligned} V(p) &= \max_q pq - c(q), \text{ s.t. } q \in [0, b[1-\alpha+2\alpha(1-x_0)]] \\ &< \max_q p_\ell q - c(q), \text{ s.t. } q \in [0, b[1-\alpha+2\alpha(1-x_0)]] \\ &= p_\ell b[1-\alpha+2\alpha(1-x_0)] - c(b[1-\alpha+2\alpha(1-x_0)]) = V^*, \end{aligned} \tag{4.19}$$

where the first line makes use of the fact $d(p) = b[1-\alpha+2\alpha(1-x_0)]$ for any $p \in (p_0, p_\ell)$, the second line makes use of the fact that $p < p_\ell$, and the third line makes use of the fact that the solution to the maximization problem in the second line is $b[1-\alpha+2\alpha(1-x_0)]$ since the price p_ℓ is greater than the marginal cost $c'(b[1-\alpha+2\alpha(1-x_0)])$. The inequalities in (4.19) imply that a seller cannot attain a profit greater than V^* by posting a price p in the interval (p_0, p_ℓ). Finally, notice that a seller cannot make a profit greater than V^* by posting a price p greater than the buyer's valuation u, since $V(p) = 0$ for all $p > u$ and $V^* > 0$.

Lastly, for the equilibrium to exist, it must also to be the case that a seller chooses to meet the buyers' demand whenever it posts a price on the support of the distribution F. At the price p_0, the buyers' demand $d(p_0)$ is equal to $b[1+\alpha(1-x_0)]$. The price p_0 is equal to the marginal cost $c'(d(p_0))$ and, by Lemma 2, the seller finds it optimal to meet the buyers' demand. At any price $p \in [p_\ell, p_h]$, the buyer's demand $d(p)$ is equal to $b[1-\alpha+2\alpha(1-F(p))]$. The solution to the differential equation (4.14)-(4.15) implies that p is greater than the marginal cost $c'(d(p))$ and, by Lemma 2, the seller finds it optimal to meet the buyers' demand.

The following proposition summarizes the conditions for the existence and the properties of an equilibrium in which the distribution is non-degenerate and has a mass point.

Proposition 2: (Non-degenerate distribution with a mass point)

(i) *An equilibrium in which F is non-degenerate and has a mass point exists if and only if*

$$c'(1+\alpha)b(1+\alpha) - c(1+\alpha) > ub(1-\alpha) - c(b(1-\alpha)) \tag{4.20}$$

and

$$c'(b)b - c(b) < ub(1-\alpha) - c(b(1-\alpha)). \tag{4.21}$$

(ii) *If an equilibrium in which F is non-degenerate and has a mass point exists, it is unique. The equilibrium is such that:*

(a) *The support of F is $p_0 \cup [p_\ell, p_h]$, where $p_0 = c'(b[1+\alpha(1-x_0)])$, $p_\ell > p_0$ is given by (4.16), and p_h is equal to u.*

(b) *A measure x_0 of sellers post the price p_0 and trade $b[1+\alpha(1-x_0)]$ units of output, where $x_0 \in (0,1)$ is the solution to (4.9).*

(c) *A measure $1-x_0$ of sellers post prices in the interval $[p_\ell, p_h]$. For any $x \in [x_0, 1]$, a seller at the x-th quantile of the distribution F posts the price $p(x)$ and trades $b[1-\alpha+2\alpha(1-x)]$ units of output, where $p(x)$ is the solution to (4.14)-(4.15).*

4.3 Non-degenerate distribution without mass points

The third type of equilibrium is such that the price distribution F is non-degenerate and does not have any mass points. I now proceed to characterize this type of equilibrium.

From Lemma 5, it follows that the support of F is some interval $[p_\ell, p_h]$ and $d(p)$ is equal to $b[1-\alpha+2\alpha(1-F(p))]$ for all $p \in [p_\ell, p_h]$, with $p_h = u$. Therefore, the profit for a seller posting the price u is

$$V(u) = ub(1-\alpha) - c(b(1-\alpha)). \tag{4.22}$$

The profit for a seller posting any price $p \in [p_\ell, p_h]$ is such that

$$V(p) = pb[1-\alpha+2\alpha(1-x(p))] - c(b[1-\alpha+2\alpha(1-x(p))]), \tag{4.23}$$

where $x(p) = F(p)$ denotes the fraction of sellers posting a price smaller than p. Since u is on the support of F, $V(u)$ must be equal to V^*. Since any $p \in [p_\ell, p_h]$ is on the support of F, $V(p)$ must be also equal to V^*.

Equating (4.22) and (4.23) yields

$$pb[1-\alpha+2\alpha(1-x(p))] - c(b[1-\alpha+2\alpha(1-x(p))]) = ub(1-\alpha) - c(b(1-\alpha)). \tag{4.24}$$

The expression above pins down $x(p)$, the fraction of sellers with a price smaller than p. Since $F(p_\ell) = 0$, the fraction $x(p)$ must be such that $x(p_\ell) = 0$. Since $F(p_h) = 1$ and $p_h = u$, the fraction $x(p)$ must be such that $x(u) = 1$. Since $F(p)$ is strictly increasing over the interval (p_ℓ, p_h), the fraction $x(p)$ must be strictly increasing in p.

Rather than characterizing $x(p)$, it is easier to characterize its inverse $p(x)$. To this aim, notice that the equal profit condition (4.24) is equivalent to

$$p'(x) = \frac{2\alpha}{1-\alpha+2\alpha x}\{p(x) - c'(b[1-\alpha+2\alpha(1-x)]\} \tag{4.25}$$

and

$$p(1) = u. \tag{4.26}$$

The expressions (4.25)-(4.26) describe exactly the same differential equation as (4.14)-(4.16). Therefore, as shown in the previous section, the solution to the differential equation is such that $p'(x) > 0$ for all $x \in (0,1)$ if and only if $p(0) \geq c'(b(1+\alpha))$. Moreover, as shown in the previous section, if $p'(x) > 0$ for all $x \in (0,1)$, the solution to the differential equation is such that $p(x)$ is greater than $c'(b[1-\alpha+2\alpha(1-x)])$.

From (4.23), it follows that $p(0)$ is given by

$$p(0)b(1+\alpha) - c(b(1+\alpha)) = ub(1-\alpha) - c(b(1-\alpha)). \tag{4.27}$$

Clearly, $p(0) > c'(b(1+\alpha))$ if and only if

$$c'(b(1+\alpha))b(1+\alpha) - c(b(1+\alpha)) \leq ub(1-\alpha) - c(b(1-\alpha)). \tag{4.28}$$

If and only if condition (4.28) is satisfied, the solution to the differential equation (4.25)-(4.26) is a function $p(x)$ such that $p(0) = p_\ell$, $p(1) = u$, and $p'(x) > 0$ for all $x \in (0,1)$. Therefore, if and only if (4.28) holds, there exists a solution to the equal profit condition (4.24) such that $x(p_\ell) = 0$, $x(p_h) = 1$ with $p_h = u$, and $x'(p) > 0$ for all $p \in (p_\ell, p_h)$. Moreover, if (4.28) holds, the price p is greater than the marginal cost $c'(b[1-\alpha+2\alpha(1-F(p))])$ for all $p \in [p_\ell, p_h]$. Assume that condition (4.28) is satisfied.

I now need to show that a seller cannot make a profit strictly greater that V^* by posting a price p that is not on the support of F. Consider a seller posting a price p strictly smaller than p_ℓ. The seller's profit is given by

$$\begin{aligned} V(p) \quad &= \max_q pq - c(q), \text{ s.t. } q \in [0, b(1+\alpha)] \\ &< \max_q p_\ell q - c(q), \text{ s.t. } q \in [0, b(1+\alpha)] \\ &= p_\ell b(1+\alpha) - c(b(1+\alpha)) = V^*, \end{aligned} \tag{4.29}$$

where the first line follows from the fact that $d(p) = b(1+\alpha)$ for all $p < p_\ell$, the second line follows from the fact that $p < p_\ell$, and the third line follows from the fact that the solution to the maximization problem in the second line is $b(1+\alpha)$, since p_ℓ is greater than $c'(b(1+\alpha))$. The inequalities in (4.29) imply that a seller cannot attain a profit greater than V^* by posting a price p strictly smaller than p_ℓ. Similarly, a seller cannot attain a profit strictly greater than V^* by posting a price p strictly greater than u, since $V(p) = 0$ for all $p > u$ and $V^* > 0$.

Finally, I need to show that a seller posting the price p finds it optimal to meet the buyers' demand $d(p)$ for every $p \in [p_\ell, p_h]$. At any price $p \in [p_\ell, p_h]$, the buyer's demand $d(p)$ is equal to $b[1-\alpha+2\alpha(1-F(p))]$. The solution to the differential equation (4.25)-(4.26) implies that p is greater than the marginal cost $c'(d(p))$ and, by Lemma 2, the seller finds it optimal to meet the buyers' demand.

The following proposition summarizes the conditions for the existence and the properties of an equilibrium in which the distribution is non-degenerate and does not have any mass points.

Proposition 3: (Non-degenerate distribution without mass points)

(i) *An equilibrium in which F is non-degenerate distribution and has no mass points exists if and only if*

$$c'(b(1+\alpha))b(1+\alpha) - c(b(1+\alpha)) \leq ub(1-\alpha) - c(b(1-\alpha)). \tag{4.30}$$

(ii) *If an equilibrium in which F is non-degenerate and has no mass points exists, it is unique. The equilibrium is such that:*

(a) *The support of F is the interval $[p_\ell, p_h]$, where p_ℓ is given by (4.27) and p_h is equal to u.*

(b) *For any $x \in [0,1]$, a seller at the x-th quantile of the price distribution F posts the price $p(x)$ and trades $b[1-\alpha+2\alpha(1-x)]$ units of output, where $p(x)$ is the solution of (4.25)-(4.26).*

5 Existence, uniqueness and nature of equilibrium

In the previous section, I derived necessary and sufficient conditions for the existence of each of the three types of equilibria. I also proved that, if an equilibrium of a particular type exists, it is unique. In this section, I establish that an equilibrium always exists and is always unique but the equilibrium is of a different type depending on the value of the fundamentals.

The conditions for the existence of different types of equilibria depend on the value taken by three expressions:

$$c'(b)b - c(b), \tag{5.1}$$

$$ub(1-\alpha) - c(b(1-\alpha)), \tag{5.2}$$

$$c'(b(1+\alpha))b(1+\alpha) - c(b(1+\alpha)). \tag{5.3}$$

Figure 2 plots the three expressions above as a function of α. Condition (5.1) is the solid line, condition (5.2) is the dotted line, condition (5.3) is the dashed line. First, consider the expression in (5.1). The expression is strictly positive because $c'(b) > c(b)/b$. The expression is independent of α. Second, consider the expression in (5.2). For $\alpha = 0$, the

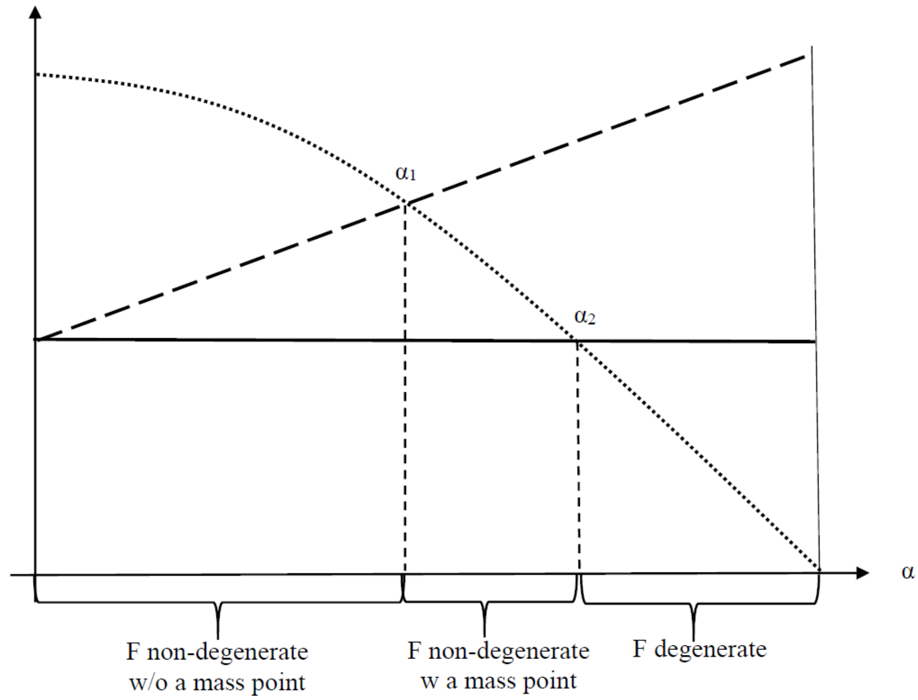

Figure 2: Conditions (5.1), (5.2) and (5.3)

expression takes the value $ub - c(b)$, which is strictly greater than $c'(b)b - c(b)$ because $u > c'(b)$ and, hence, strictly positive. For $\alpha = 1$, the expression takes the value 0. The expression is strictly decreasing in α because its derivative, $-ub + bc'(b(1-\alpha))$, is strictly negative. Lastly, consider the expression in (5.3). For $\alpha = 0$, the expression takes the value $c'(b)b - c(b)$. The expression is strictly increasing in α because $c'(q)q - c(q)$ is strictly increasing in q and $b(1+\alpha)$ is strictly increasing in α.

Let α_1 denote the value of α for which the expression in (5.2) is equal to the expression in (5.3). Clearly, α_1 exists and belongs to the interval $(0, 1)$. Let α_2 denote the value for which the expression in (5.2) is equal to the expression in (5.1). Clearly, α_2 exists and belongs to the interval $(\alpha_1, 1)$. For any $\alpha \in (0, \alpha_1]$, (5.2) is greater than (5.3), and (5.3) is strictly greater than (5.1). For any $\alpha \in (\alpha_1, \alpha_2)$, (5.3) is strictly greater than (5.2), and (5.2) is strictly greater than (5.1). For any $\alpha \in [\alpha_2, 1)$, (5.3) is strictly greater than (5.1), and (5.1) is greater than (5.2).

Proposition 3 states that an equilibrium where F is non-degenerate and does not have any mass points exists if and only if (5.2) is greater than (5.3). This is the case if and only if α belongs to the interval $(0, \alpha_1]$. Proposition 2 states that an equilibrium where F is non-degenerate and has a mass point exists if and only if (5.3) is strictly greater than (5.2) and (5.2) is strictly greater than (5.1). This is the case if and only if α belongs to the interval (α_1, α_2). Proposition 1 states that an equilibrium where F is degenerate exists if and only if (5.1) is greater than (5.2). This is the case if and only if α belongs to the interval $[\alpha_2, 1)$. Moreover, Propositions 1, 2, 3 state that each type of equilibrium is unique if it exists. From these observations, it follows that, for any value of $\alpha \in (0,1)$, the equilibrium exists and is unique but its nature depends on the value of α.

Theorem 1 below contains a summary of my findings.

Theorem 1: (Existence, uniqueness and nature of equilibrium)

(i) *For any $\alpha \in (0,1)$, the equilibrium exists and is unique.*

(ii) *There exist two cutoffs α_1 and α_2, with $0 < \alpha_1 < \alpha_2 < 1$, such that:*

(a) *For any $\alpha \in (0, \alpha_1]$, the equilibrium is such that F is non-degenerate and does not have a mass point. The equilibrium has the properties described in Proposition 3.*

(a) *For any $\alpha \in (\alpha_1, \alpha_2)$, the equilibrium is such that F is non-degenerate and does have a mass point. The equilibrium has the properties described in Proposition 2.*

(b) *For any $\alpha \in [\alpha_2, 1)$, the equilibrium is such that F is degenerate. The equilibrium has the properties described in Proposition 1.*

Let me comment on Theorem 1.[2] To this aim, recall that α is an inverse measure of search frictions, in the sense that the ratio of non-captive to captive buyers increases from zero to infinity as α goes from 0 to 1. When search frictions are large, in the sense that α is smaller than α_1, the equilibrium is such that the price distribution F is atomless and its support is the interval $[p_\ell, p_h]$, with $p_h = u$. The equilibrium price distribution has the same properties as in Burdett and Judd (1983) and for the same reasons. In particular, the equilibrium price distribution is atomless because any mass point p_0 would generate a downward discontinuity in the demand curve and, for this reason, an individual seller would strictly prefer posting a price below p_0 than the price p_0. The support of the equilibrium price distribution is an interval because any gaps in the support would generate a constant demand curve and, for this reason, an individual seller would strictly prefer posting a price to the right of the gap than the price to the left of the gap. The highest price on the support of the equilibrium price distribution is u because the seller with the highest price trades only with captive buyers.

When search frictions are intermediate, in the sense that α is between α_1 and α_2, the equilibrium is such that the price distribution F has support $p_0 \cup [p_\ell, p_h]$, with $p_\ell > p_0$

[2] Theorem 1 is derived under the assumption $u > c'(b)$. What is the equilibrium if $u \leq c'(b)$? It is easy to show that there exists no equilibrium in which all buyers enter the market. Indeed, if all buyers enter the market, any potential equilibrium is such that some sellers either post a price p that is strictly smaller than $c'(d(p))$, which is not consistent with the requirement that sellers meet the buyers' demand, or a price p that is strictly greater than u, which is not consistent with the sellers' profit maximization. Any equilibrium must be such that some measure $\hat{b}$ of buyers enter the market and some measure $b - \hat{b}$ of buyers stay out of the market. Hence, in any equilibrium, a buyer must be indifferent between entering the market and staying out of the market. The buyers' indifferent condition holds if and only if the price distribution is degenerate at $p_0 = u$. In turn, the price distribution is degenerate at $p_0 = u$ if and only if the seller's marginal cost $c'(\hat{b})$ is equal to u. Therefore, if $u \leq c'(b)$, an equilibrium exists and is unique.

and $p_h = u$. At p_0, the equilibrium price distribution has a mass point. Over the interval $[p_\ell, p_h]$, the equilibrium price distribution is atomless. The mass point p_0 creates a discontinuity in the demand curve. A seller posting the price p_0 has no incentive to deviate to a lower price and increase quantity because its marginal cost is equal to p_0. Not every seller, however, can post the price p_0. If every seller posted the price p_0, an individual seller would be better off posting a price equal to u and trading only with captive buyers. Since for any $p < p_0$, the price p exceeds the marginal cost, sellers that do not post p_0 must distribute themselves in an atomless fashion over some interval $[p_\ell, p_h]$. For the same reasons as in Burdett and Judd (1983), p_h must be equal to u. Since the mass point at p_0 creates a downward discontinuity in the demand curve, p_ℓ must be strictly greater than p_0.

When search frictions are small, in the sense that α is greater than α_2, the equilibrium is such that the price distribution F is degenerate at some price p_0. A seller has no incentive to deviate to a price lower than p_0 because its marginal cost is equal to p_0. A seller has no incentive to deviate to a price greater than p_0 because there are not enough captive buyers to make the deviation profitable.

It is easy to understand the transformation of the equilibrium structure as search frictions decline. When search frictions are large, the difference in the quantity of output produced by the seller with the lowest price (i.e., the seller that trades with every contacted buyer) and the quantity of output produced by the seller with the highest price (i.e., the seller that trades only with the contacted buyers who are captive) is small and so is the difference in their marginal costs. For this reason, the equilibrium has the same structure as in Burdett and Judd (1983), a model in which sellers have a constant marginal cost.

As search frictions become smaller, the quantity of output produced by the seller with the lowest price grows, while the quantity of output produced by the seller with the highest price shrinks. As a result, the marginal cost of the seller with the highest price increases, while the marginal cost of the seller with the lowest price decreases. Moreover, as search frictions become smaller, the lowest price must fall to keep the profit of a seller with the lowest price equal to the profit of a seller with the highest price u. Eventually, the seller with the lowest price faces a marginal cost equal to its price. At this point, further reductions in search frictions break the Burdett and Judd (1983) equilibrium because the seller with the lowest price is unwilling to further increase its quantity and to further reduce its price. Quite naturally, the equilibrium becomes such that sellers start bunching at the lowest price, where price and marginal cost are equated. The measure of sellers at the lowest price is such that the profit of these sellers must be the same as the profit of a seller with the highest price u. The measure of sellers posting the lower price reduces the quantity traded by each of these sellers, their marginal cost, their price and, in turn, their profit. Since the profit of a seller with the highest price decreases as search frictions become smaller, the measure of sellers at the lowest price must grow. Eventually, search frictions become sufficiently small that all the sellers post the lowest price. At this point, further reductions in search frictions have no effect on equilibrium outcomes.

It is useful to examine the behavior of equilibrium in the limit for $\alpha \to 0$ and for $\alpha \to 1$. For $\alpha \to 0$, almost every buyer is in contact with a single seller, the price distribution F converges to the monopoly price u, and every seller trades the same quantity of output. For $\alpha \to 1$, almost every buyer is in contact with multiple sellers, the price distribution F is a mass point at the competitive price $c'(b)$, and every seller trades the same quantity of output. As α goes from 0 to 1, equilibrium profits decline monotonically. In this sense, as α increases from 0 to 1, the equilibrium spans the spectrum from pure monopoly to perfect competition.

Let me contrast the behavior of equilibrium with Burdett and Judd (1983). In Burdett and Judd (1983), the unique equilibrium always features an atomless price distribution. Intuitively, any mass point at a price p_0 generates a downward discontinuity at p_0. Since the seller's marginal cost is constant and prices are greater than marginal cost (because equilibrium profits are strictly positive), the downward discontinuity can always be exploited by an individual seller to increase its profit. Since the equilibrium price distribution is atomless, sellers are strictly ranked by buyers. Hence, the seller with the lowest price trades a quantity of output that keeps increasing as search frictions decline, while the seller with the highest price trades a quantity of output that declines towards zero. For $\alpha \to 0$, the equilibrium is such that every sellers posts a price approximately equal to u and every seller trades the same quantity of output. For $\alpha \to 1$, the equilibrium is such that every seller posts a price approximately equal to the marginal cost c but, since sellers are strictly ranked, the concentration of trade is the highest. As α goes from 0 to 1, equilibrium profits decline monotonically. In this sense, also in Burdett and Judd (1983), the equilibrium spans the spectrum from pure monopoly to perfect competition. In Burdett and Judd (1983), however, the competitive outcome is approached in an asymmetric fashion—in the sense that the seller with the lowest price produces much more than the seller with the highest price. An asymmetric approach to perfect competition is possible when sellers have a constant marginal cost of production, i.e. when production features constant returns to scale, and hence the size of each seller is indeterminate. An asymmetric approach to the competitive outcome is not possible, however, when sellers have a strictly increasing marginal cost of production, i.e. when production features decreasing returns to scale, and, hence, perfect competition requires every seller to have the same size.

6 Welfare properties of equilibrium

In this section, I examine the welfare properties of equilibrium. I define welfare as the sum of the buyers' and sellers' payoffs. For any $\alpha \in (0, \alpha_1]$, welfare is

$$\begin{aligned} W &= \int_0^1 \{b[1-\alpha+2\alpha(1-x)]u - c(b[1-\alpha+2\alpha(1-x)])\}\, dx \\ &= bu - \int_0^1 c(b[1-\alpha+2\alpha(1-x)])dx \end{aligned} \tag{6.1}$$

The expression in (6.1) is easy to understand. For $\alpha \in (0, \alpha_1]$, the equilibrium distribution F is non-degenerate and does not have any mass points. Consider a seller at the x-th quantile of F. The payoff to the seller is given by the difference between revenues, $b[1-\alpha+2\alpha(1-x)]p(x)$, and production costs, $c(b[1-\alpha+2\alpha(1-x)])$. The payoff to the customers of the seller is given by $b[1-\alpha+2\alpha(1-x)](u-p(x))$. Therefore, the payoffs to the seller and its customers is the difference between $b[1-\alpha+2\alpha(1-x)]u$ and $c(b[1-\alpha+2\alpha(1-x)])$. Welfare is given by the integral with respect to x of the payoffs to a seller at the x-th quantile of F and to its customers.

For any $\alpha \in (\alpha_1, \alpha_2)$, welfare is

$$\begin{aligned} W &= \int_0^{x_0} \{b[1+2\alpha(1-x_0)]u - c(b[1+2\alpha(1-x_0)])\}\, dx \\ &\quad + \int_{x_0}^1 \{b[1-\alpha+2\alpha(1-x)]u - c(b[1-\alpha+2\alpha(1-x)])\}\, dx \\ &= bu - x_0 c(b[1+2\alpha(1-x_0)]) - \int_{x_0}^1 c(b[1-\alpha+2\alpha(1-x)])dx. \end{aligned} \tag{6.2}$$

For $\alpha \in (\alpha_1, \alpha_2)$, the equilibrium distribution F is such that a measure x_0 of sellers posts the price p_0 and a measure $1-x_0$ of sellers posts prices distributed over the interval $[p_\ell, p_h]$ with $p_\ell > p_0$. Consider a seller posting the price p_0. The payoff to the seller is given by the difference between revenues, $b[1+2\alpha(1-x_0)]p_0$, and production costs, $c(b[1+2\alpha(1-x_0)])$. The payoff to the seller's customers is given by $b[1+2\alpha(1-x_0)](u-p_0)$. Therefore, the sum of the payoffs to the seller and its customers is the difference between $b[1+2\alpha(1-x_0)]u$ and $c(b[1+2\alpha(1-x_0)])$. Consider a seller at the x-th quantile of the price distribution, with $x \in [x_0, 1]$. The sum of the payoffs to the seller and its customers is given by the difference between $b[1-\alpha+2\alpha(1-x)]u$ and $c(b[1-\alpha+2\alpha(1-x)])$.

For any $\alpha \in [\alpha_2, 1)$, welfare is

$$W = \int_0^1 (bu - c(b))dx = bu - c(b). \tag{6.3}$$

For $\alpha \in [\alpha_2, 1)$, the equilibrium distribution F is degenerate at p_0. Consider a seller posting the price p_0. The payoff to the seller is given by the difference between revenues,

bp_0, and production costs, $c(b)$. The payoff to the seller's customers is given by $b(u-p_0)$. Therefore, the sum of the payoffs to the seller and its customers is given by $bu - c(b)$.

Now consider the problem of a social planner

$$\begin{aligned} W^* = \quad & \max_{g_1,g_2} \int_0^1 \{q(i)u - c\,(q(i))\}di, \text{ s.t.} \\ & q(i) = b\left[(1-\alpha)g_1(i) + 2\alpha\left(\int_0^1 g_2(i,j)dj\right)\right], \\ & g_1(i) \geq 0,\ g_1(i) \leq 1, \\ & g_2(i,j) \geq 0,\ g_2(i,j) + g_2(i,j) \leq 1. \end{aligned} \tag{6.4}$$

Let me explain the optimization problem above. The objective of a utilitarian planner is to maximize welfare. The planner chooses the probability $g_1(i)$ with which a buyer who is in contact only with seller $i \in [0,1]$ purchases the good, and the probability $g_2(i,j)$ with which a buyer who is in contact with both seller i and seller j purchases the good from seller i. The probability that a buyer who is in contact with seller i purchases the good must be greater than 0 and smaller than 1. The probability that a buyer in contact with seller i and seller j purchases the good from seller i must be greater than 0. The probability that a buyer in contact with seller i and seller j purchases the good from either seller must be smaller than 1.

Consider a relaxed version of the planner's problem in which the non-negativity constraints $g_1(i) \geq 0$ and $g_2(i,j) \geq 0$ are omitted. For such a relaxed problem, the necessary conditions for optimality are

$$b(1-\alpha)\left(u - c'(q(i))\right) = \lambda_1(i), \tag{6.5}$$

$$2b\alpha\left(u - c'(q(i))\right) = \lambda_2(i,j), \tag{6.6}$$

$$g_1(i) \leq 1,\ \lambda_1(i) \geq 0, \tag{6.7}$$

and

$$g_2(i,j) + g_2(j,i) \leq 1,\ \lambda_2(i,j) \geq 0, \tag{6.8}$$

where $\lambda_1(i)$ denotes the multiplier on the constraint $g_1(i) \leq 1$, $\lambda_2(i,j)$ denotes the multiplier on the constraint $g_2(i,j) + g_2(j,i)$, and the inequalities in (6.7) and (6.8) hold with complementary slackness.

Take an arbitrary seller i. It follows from (6.6) that $\lambda_2(i,j)$ equals $2b\alpha\left(u - c'(q(i))\right)$ for all j. Since $\lambda_2(i,j)$ is equal to $\lambda_2(j,i)$ and $\lambda_2(j,i)$ is equal to $2b\alpha\left(u - c'(q(j))\right)$, $2b\alpha\left(u - c'(q(j))\right)$ is equal to $2b\alpha\left(u - c'(q(i))\right)$ and, hence, $q(j) = q(i) = q$ for all i and j. That is, the solution to the relaxed planner's problem is such that every seller produces the same quantity of output q. Since there are b buyers per seller and every seller produces the same quantity, it follows that $q \leq b$. Since $u > c'(b)$, it follows from (6.5) and (6.6) that $\lambda_1(i) > 0$ and $\lambda_2(i,j) > 0$ and, in turn, $g_1(i) = 1$ and $g_2(i,j) + g_2(j,i) = 1$ for all i and j. That is, the solution to the relaxed social planner's problem is such that a buyer always purchases the good and, hence, $q = b$. Clearly, the solution to the relaxed planner's problem is feasible in the original planner's problem, as it can be achieved by setting $g_1(i) = 1$ and $g_2(i,j) = g_2(j,i) = 1/2$. Therefore, the solution to the relaxed planner's problem is the solution to (6.4) and

$$W^* = bu - c(b). \tag{6.9}$$

Comparing welfare in equilibrium and in the solution to the planner's problem leads immediately to the following theorem.

Theorem 2 (Efficiency). *The equilibrium is efficient if* α *belongs to the interval* $[\alpha_2, 1)$. *The equilibrium is inefficient if* α *belongs to the interval* $(0, \alpha_2)$.

Theorem 2 states that the equilibrium is efficient if and only if search frictions are low enough. Irrespective of search frictions, the equilibrium price distribution F is such that every seller posts a price non-greater than the buyer's valuation for the good. Therefore, every buyer purchases a unit of the good and the total value of buyers' consumption is maximized. If search frictions are sufficiently low, the price distribution F is degenerate. Therefore, every seller produces the same quantity of output and the total cost of sellers' production is minimized. Since the total value of buyers' consumption is maximized and the total cost of sellers' production is minimized, the equilibrium is efficient. If search frictions are not low enough, the equilibrium price distribution F is non-degenerate. Therefore, not all sellers produce the same quantity of output and the total cost of sellers' production is not minimized. Hence, the equilibrium cannot efficient.

Theorem 2 identifies another difference between the model in which sellers have an increasing marginal cost of production and the model in which sellers have a constant marginal cost of production (i.e., Burdett and Judd 1983). In Burdett and Judd (1983), the equilibrium is always efficient. Irrespective of search frictions, the equilibrium price distribution F is such that every seller posts a price non-greater than the buyer's valuation for the good. Therefore, every buyer purchases a unit of the good and the total value of buyers' consumption is maximized. Irrespective of search frictions, the equilibrium price distribution is not degenerate. Even though sellers produce different quantities of output, the total cost of sellers' production is minimized because sellers have a constant, common marginal cost. Hence, the equilibrium is always efficient.

7 Conclusions

I studied a search-theoretic model of imperfect competition in the spirit of Burdett and Judd (1983), in which the sellers' production function has decreasing rather than constant returns to scale. I proved that the equilibrium exists, it is unique, and its structure depends on the extent of search frictions. When search frictions are large enough, the equilibrium is such that the price distribution F is non-degenerate, atomless, and its support is some interval $[p_\ell, p_h]$, with $p_h = u$. When search frictions are neither too large nor too small, the equilibrium is such that the price distribution F is non-degenerate, it has an atom at some p_0, where p_0 is equal to the seller's marginal cost, and a strictly positive density over some interval $[p_\ell, p_h]$, with $p_\ell > p_0$ and $p_h = u$. When search frictions are small enough, the equilibrium is such that the price distribution F is degenerate at some p_0, where p_0 is equal to the seller's marginal cost. Thus, unless search frictions are sufficiently large, the structure of equilibrium is different than in Burdett and Judd (1983). I also proved that the equilibrium is efficient if and only if the price distribution F is degenerate. Hence, unless search frictions are sufficiently small, the equilibrium is inefficient, in contrast with Burdett and Judd (1983). However, as in Burdett and Judd (1983), the equilibrium becomes more competitive as search frictions become smaller and it approaches the competitive outcome as search frictions vanish.

The characterization of equilibrium in this paper could be fruitfully extended in a couple of ways. First, it would be interesting to extend the analysis to the case in which sellers operate heterogeneous rather than homogeneous production technologies with decreasing returns to scale. As in the case of homogeneous technologies, it is clear that the equilibrium price distribution would have to have a mass point at some p_0 when search frictions are sufficiently small. Unlike in the case of homogeneous technologies, the quantity of output produced by sellers at p_0 would have to be different, since it would have to equalize to p_0 the marginal cost of production for sellers with different technologies. Therefore, the equilibrium would have to require buyers to randomize in a non-uniform fashion when meeting two sellers that post the price p_0. Second, it would be interesting to extend the analysis to the labor market model of Burdett and Mortensen (1988), which is a labor market version of Burdett and Judd (1983). This extension would involve more than relabeling sellers to firms and buyers to workers because search, in Burdett and Mortensen (1998), is sequential rather than simultaneous and, for this reason, the workers' reservation wage (the analogue of the buyers' valuation) is an equilibrium object rather than a parameter.

References

[1] Albrecht, J., G. Menzio, and S. Vroman. 2023. "Vertical Differentiation in Frictional Product Markets." *Journal of Political Economy: Macro*, 1: 586-632.

[2] Bethune, Z., M. Choi, and R. Wright. 2020. "Frictional Goods Markets: Theory and Applications." *Review of Economic Studies*, 87: 691-720.

[3] Bontemps, C., J. Robin, and G. van den Berg. 2000. "Equilibrium Search with Continuous Productivity Dispersion" Theory and Non-Parametric Estimation." *International Economic Review*, 41: 305-358.

[4] Burdett, K., and K. Judd. 1983. "Equilibrium Price Dispersion." *Econometrica*, 51: 955-970.

[5] Burdett, K., and G. Menzio. 2018. "The (Q,S,s) Pricing Rule." *Review of Economic Studies*, 85: 892-928.

[6] Burdett, K., and D. Mortensen. 1998. "Wage Differentials, Employer Size, and Unemployment." *International Economic Review*, 39: 955-970.

[7] Burdett, K., A. Trejos and R. Wright. 2017. "A New Suggestion for Simplifying the Theory of Money." *Journal of Economic Theory*, 172: 423-450.

[8] Butters, G. 1977. "Equilibrium Distributions of Sales and Advertising Prices." *Review of Economic Studies*, 44: 465-491.

[9] Galenianos, M., R. Pacula, and N. Persico. 2012. "A Search-Theoretic Model of the Retail Market for Illicit Drugs." *Review of Economic Studies*, 79: 1239-1269.

[10] Head, A., L. Liu, G. Menzio, and R. Wright. 2012. "Sticky Prices: A New Monetarist Approach." *Journal of the European Economic Association*, 10: 939-973.

[11] Hong, H., and M. Shum. 2006. "Using Price Distributions to Estimate Search Costs." *Rand Journal of Economics*, 37: 257-275.

[12] Kaplan, G., and G. Menzio. 2015. "The Morphology of Price Dispersion." *International Economic Review*, 56: 892-928.

[13] Kaplan, G., and G. Menzio. 2016. "Shopping Externalities and Self-Fulfilling Unemployment Fluctuations." *Journal of Political Economy*, 124: 771-825.

[14] Kaplan, G., G. Menzio, L. Rudanko, and N. Trachter. 2019. "Relative Price Dispersion: Evidence and Theory." *American Economic Journal: Micro*, 11: 68-124.

[15] Lagos, R., and R. Wright. 2005. "A Unified Framework for Monetary and Policy Analysis." *Journal of Political Economy*, 113: 463-484.

[16] Menzio, G. 2023. "Optimal Product Design: Implications for Competition and Growth of Declining Search Frictions." *Econometrica*, 91: 605-639.

[17] Menzio, G., and N. Trachter. 2018. "Equilibrium Price Dispersion Within and Across Stores." *Review of Economic Dynamics*, 28: 205-220.

[18] Mortensen, D.. 2003. *Wage Dispersion: Why Are Similar Workers Paid Differently?* MIT Press, Cambridge, MA.

[19] Nord, L. 2022. "Shopping, Demand Composition, and Equilibrium Prices." Manuscript, European University Institute.

[20] Pytka, K. 2018. "Shopping Effort in Self-Insurance Economies." Manuscript, University of Mannheim.

[21] Sorensen, A. 2000. "Equilibrium Price Dispersion in Retail Markets for Prescription Drugs." *Journal of Political Economy*, 108: 833-862.

[22] Varian, H. 1980. "A Model of Sales." *American Economic Review*, 61: 37-70.