Managing Public Portfolios*

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Abstract

We develop a unified framework for optimally managing public portfolios for a class of macro-finance models that include widely-used specifications for households' risk and liquidity preferences, market structures for financial assets, and trading frictions. An optimal portfolio hedges fluctuations in interest rates, primary surpluses, liquidities and inequalities. It recognizes liquidity benefits that government debts provide and internalizes equilibrium effects of public policies on financial asset prices. We express an optimal portfolio in terms of statistics that are functions only of macro and financial market data. An application to the U.S. shows that hedging interest rate risk plays a dominant role in shaping an optimal maturity structure of government debt.

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1 Introduction

This paper isolates and quantifies motives that shape optimal government portfolios of financial assets for a class of heterogeneous household general equilibrium models. This class of models includes popular specifications of households' risk and liquidity preferences, sets of tradable securities, as well as restrictions that limit access to markets. We characterize the main forces that shape an optimal portfolio with a small number of statistics that are functions only of observables. We apply our approach to study the optimal maturity structure of the U.S. government debt. For the U.S. we find that an optimal portfolio has a simple shape that is well-approximated by portfolio shares that decline exponentially with maturity and that has a longer overall duration than the current U.S. maturity structure.

Our framework includes domestic households, foreign investors, and a benevolent government. Households can be heterogeneous and derive utility from consumption and leisure; in addition, they can also derive indirect utility from holdings financial assets. This indirect utility summarizes shadow benefits and costs from holding assets that provide liquidity services, or affect borrowing constraints or trading frictions. A benevolent government planner uses distortionary taxes to finance exogenous public expenditures. Households, government and foreign investors trade an arbitrary set of financial assets. Our specification of household preferences and demand of foreign investors is flexible enough to represent asset pricing models including ones with recursive utilities, discount factor shocks, ambiguity aversion, preferred habitats. Both closed and open economies are included.

We isolate forces that determine an optimal government portfolio by studying consequences of perturbing the government portfolio at any history along a competitive equilibrium and then applying small-noise expansions. This allows us to express the optimal portfolio as a function of some statistics that are functions of macro and financial market data. We show that these statistics let us characterize an optimal portfolio, and that there is no need to take a stance on a particular model as long as that model is consistent with these statistics. This ability to sidestep specifying such details is important because there remain disagreements within the asset pricing literature about the sources of asset price fluctuations.

The key notion that emerges in our analysis is *a target portfolio* that a benevolent government would optimally choose in the absence of any rebalancing costs that may arise when a big government has pricing power in financial markets. The target portfolio captures a trade-off between hedging future risks that the government will face against providing liquidity services now. The future risks come from fluctuations in interest rates, primary surpluses, measures of liquidity, and inequalities across households. These risks are summarized by covariances of returns on assets in the government portfolio with various financial and macroeconomic variables. The value of the liquidity services in the present is summarized by a particular measure of liquidity premia on various assets.

If rebalancing the government portfolio does not affect asset prices, as in a small open economy, then it is optimal for the government to set its portfolio to the target portfolio. More generally, the formula for the optimal portfolio includes costs of rebalancing. We show that those costs are proportional to the distance between the target portfolio and the portfolio with which the government enters a period, and the price elasticity of various assets to changes in their supply induced by government trades.

Our framework can be used to study any set of securities. We apply it to a particular structure in which the government portfolio contains only public debts of different maturities. We use data on the returns of U.S. government and corporate bonds, taxes, and primary surpluses to estimate each component of the target portfolio. We find that a single force – interest rate risk – contributes most to the shape of the target portfolio. That means that the target portfolio takes a very simple form—portfolio shares of debts decline roughly geometrically in their maturities, with the rate of decline given by households' discount factor. Moreover, maintaining this portfolio requires minimal rebalancing, which implies that the optimal portfolio is roughly equal to the target portfolio for virtually any price elasticities.

This finding is driven by several statistics. U.S. government bonds are a poor hedge against primary surplus, liquidity risks, and inequality risk. Their returns are much more volatile than, and not very correlated with, either future primary surpluses or various measures of future liquidity premia on government bonds. Furthermore, primary surpluses are procyclical, while liquidity premia are countercyclical, which means that these two risks have offsetting effects on the target portfolio. Compared to primary surpluses and liquidity risks, measures of inequality are even less correlated with returns and desire to hedge them contributes very little to the optimal portfolio. Liquidity premia also seem to be similar across different maturities of government bonds, which leaves interest rate risk as the only quantitatively meaningful term in the target portfolio.

Unlike the situation with primary surplus risks, liquidity risks, and inequality risks, there exists a simple portfolio that can hedge interest rate risks well. Interest rate risks affect the government only when it needs to roll over its existing debt. By choosing a maturity structure that matches the duration of debts to expected primary surpluses, the government can eliminate anticipated debt rollovers and thereby hedge most of the interest rate risk. We show that in a stationary environment such a portfolio can be replicated by issuing a growth-rate-adjusted consol. A portfolio structured to minimize interest rate risk also minimizes required rebalancings. This, in turn, implies that costs of portfolio adjustments have small quantitative impacts on the optimal portfolio.

Related Literature Our paper is related to an extensive Ramsey literature on the optimal composition of government debt, such as Lucas and Stokey (1983), Zhu (1992), Chari et al. (1994), Angeletos (2002), Buera and Nicolini (2004), Farhi (2010); Faraglia et al. (2018); Lustig et al. (2008), Bhandari et al. (2017a). Those authors used simple versions of the closed economy neoclassical growth models to characterize optimal public portfolios. However, those models fail to approximate empirical relationships among asset prices, asset supplies, and macroeconomic variables, key objects that determine how well alternative securities hedge risks. We overcome that deficiency by considering a much more general specification of preferences and asset demands that includes *multiple* mechanisms that can account for the observed asset pricing behavior.

Realistic asset pricing dynamics dramatically change many insights about optimal public portfolios that emerged from that earlier literature. For example, in their quantitative model calibrated to the U.S. economy, Buera and Nicolini (2004) find that the government should issue long-term debt valued at tens or even hundreds times GDP while simultaneously taking offsetting short (i.e., negative) positions in short-term debt of similar magnitudes. They also find that government holdings of debts of similar maturities may differ by hundreds percent of GDP; that the composition of the optimal portfolio is very sensitive to the menu of traded maturities; and that relatively small aggregate shocks caused very significant portfolio rebalancing. In contrast, our optimal portfolio is very stable over time and has simple declining maturity weights qualitatively like those observed in US data. We show that the dramatic differences in these findings are driven by counterfactual asset pricing implications of the standard neoclassical growth model.

Our paper builds on a large literature in finance that focuses on understanding asset price determination, such as Ai and Bansal (2018), Bansal and Yaron (2004), Albuquerque et al. (2016), Krishnamurthy and Vissing-Jorgensen (2012), Greenwood and Vayanos (2014). Those authors proposed a variety of modifications to the standard neoclassical environment so that it is consistent with the observed behavior of asset prices. By setting up a framework that incorporates all of these mechanisms and obtaining expressions for the optimal portfolios that depend only on a small number of statistics that are functions of aggregates and asset returns, we sidestep taking a stand on their relative importance.

Work by Bohn (1990) is probably the closest in spirit to ours. He studied a representative agent model with distortionary taxes and computed an optimal government portfolio as a function of covariances that he estimated for U.S. data. Unlike in our setting, in Bohn's model, consumers are risk-neutral, tax distortions are ad-hoc, financial securities provide no liquidity services, the set of those assets is restrictive, and all asset prices are exogenous.

Our findings are also related to some recent work by Debortoli et al. (2017, 2022). For a deterministic version of Lucas and Stokey (1983), they find that issuing a consol aligns incentives across successive governments and eliminates time inconsistency. We study a government with commitment but still find that in a stationary world the optimal portfolio is well approximated by a (growth-adjusted) consol—a security that hedges the empirically dominant interest rate risk and eliminates needs to rollover or rebalance the portfolio.[1]

We obtain for an optimal government portfolio formulas that are related to the formulas for private portfolios that appear in classic portfolio theory contributions of Samuelson (1970), Merton (1969, 1971), Campbell and Viceira (1999, 2001), and Viceira (2001). While individual investors in the classical portfolio theory and the government in our model both choose portfolios to hedge their risks, there are substantial differences in the forces that determine portfolio composition. Neither liquidity services nor price impacts feature in the classical portfolio theory in which investors are small relative to the market. The trade-offs between risks and returns of various assets captured by Sharpe ratios and risk-aversions that play the central role in the classical portfolio theory are entirely absent in our government's portfolio problem. This is because the government is benevolent and shares agents preferences. This implies that it cannot improve welfare by simply replicating any trade that households can do themselves. Instead, the government portfolio depends on a statistic that captures additional costs (such as trading frictions) or benefits (such as liquidity services) that assets provide to agents beyond pure transfers of resources across time. We refer to this measure as a excess liquidity premia and provide a way to measure it for all securities. Finally, our formulas capture additional motives such hedging fluctuations in inequalities across households that are relevant for public portfolios but not present in discussions of private portfolios.

In recent papers, Jiang et al. (2019, 2020) document a number of puzzling facts about market values of total debt and primary surpluses in the U.S. These facts are puzzling when debt valuation is viewed from a lens of an arbitrage-free and frictionless asset pricing framework.

[1] Our work is also related to a recent paper by Bigio et al. (2019) that studies the optimal composition of government portfolios of bonds of different maturities. They largely abstract from the interest rate risk, primary surplus, and liquidity channels that we emphasize and focus on understanding how price impacts from debt issuance affect portfolio composition. Because they impose an exogenous cap on the maturities that the government can issue, that the government wants to rebalance its portfolio even in the absence of all risks.

Our setting departs from such a framework by incorporating market segmentation as well as a broad notion of liquidity services that U.S debts provide. However, our focus in this paper on how the market value of government debt is optimally allocated across various securities and not much on the determinants of the level itself.

Methodologically, we are related to two strands of literature. We borrow our approach of using a small number of statistics to characterize an optimal government portfolio from a recent applied public finance literature, notably Saez (2001) and Chetty (2009). That literature generally focuses on settings where a government faces no risk. When applied to our problem directly, this approach yields no clear and transparent insights. We make progress by augmenting it with some small-noise approximations. Small noise approximations have been used frequently both in finance (e.g., Samuelson (1970), Devereux and Sutherland (2011)) and computational economics (e.g., Guu and Judd (2001), Schmitt-Grohe and Uribe (2004), Bhandari et al. (2021)). The particular class of expansions that we use does not require us to assume stationarity or to ignore heteroskedasticity. That makes it particularly suitable to study portfolio problems in dynamic stochastic economies.

Outline The rest of the paper is organized as follows. In Section 2, we describe the class of economic settings. In Section 3, we use a special case of our general economy to describe a variational approach that characterize an optimal portfolio and the economic forces that pin it down. In Section 4, we apply our theory to infer an optimal portfolio for the U.S. and compare it to the observed portfolio. In Section 5, we consider several extensions that relax the restrictions imposed in Section 3 special case. We show that the qualitative insights from Section 3 and the many of the quantitative insights derived in Section 4 continue to hold more generally. Section 6 concludes. Proofs of all statements in the main text are relegated to the online appendix.

2 General environment

We consider a discrete time, infinite period economy populated by three groups of agents: a government, households, and foreign investors. All exogenous disturbances in period t are summarized by $s_t \subset \mathbb{R}^N$, where $N \leq \infty$. The initial history s_0 is predetermined. A history of shocks is $s^t = (s_0, ..., s_t)$. We use $\Pr\left(s^{T+t}\right)$ and $\Pr\left(s^{T+t}|s^T\right)$ to denote probabilities of s_{T+t} conditional on information in period 0 and s^T respectively. Any variable x_t appearing below is a function of s^t. Most of the time, we omit explicit reference to a history and simple write x_t rather than $x_t\left(s^t\right)$. Whenever a specific history s^T is clear from the context, we use $\mathbb{E}_T x_{T+t}$

to denote the expectation of x_{T+t} conditional on history s^T. We use $\mathbf{x}$ to denote $\left\{x_t\left(s^t\right)\right\}_{t,s^t}$.

All agents trade a countable set of securities. Security i is characterized by an exogenous stream of dividends $\mathbf{D}^i$ and net supply $\boldsymbol{S}^i$. The price of security i in period t is denoted by Q_t^i. The set of securities is exogenous but essentially arbitrary, and may include, as special cases, the full set of history-contingent Arrow securities or history-nonncontingent bonds of various maturities. For now, all securities are assumed to be real but we discuss nominal securities in Section 3.4. The government, households, and foreign investors may face addition restrictions on their ability to trade securities. We discuss such restrictions below. We let $\mathcal{G}_t(s^t)$ or $\mathcal{G}_t$ to denote the subset of securities that the government can trade in history s^t.

We now describe each group of agents in more detail.

Government. We start with a government budget *identity*

$$X_t + \sum_{i\in\mathcal{G}_t} Q_t^i B_t^i = \sum_{i\in\mathcal{G}_{t-1}} \left(Q_t^i + D_t^i\right) B_{t-1}^i, \tag{1}$$

where X_t is the primary surplus, and $\left\{B_t^i\right\}_{i\in\mathcal{G}_t}$ are government holdings of various securities. We adopt the convention that positive values of B_t^i denote the government's liabilities. Accounting identity (1) states that the market value of liabilities at the beginning of a period equals the sum of the primary budget surplus and the market value of government liabilities at the end of it. The primary surplus is the difference between the government's revenues and its expenditures. We use $\mathcal{T}_t$ to denote tax revenues and G_t to denote expenditures, so that $X_t \equiv \mathcal{T}_t - G_t$.

We assume that in every set $\mathcal{G}_t$ there is a *one-period government bond*, a security that is available in zero net supply, that can be issued (i.e., held in positive quantity) by the government, and that returns one unit of the consumption good as payout in in period $t+1$. Other securities in $\mathcal{G}_t$ can be arbitrary. We refer to a set $\left\{Q_t^i B_t^i\right\}_{i\in\mathcal{G}_t}$ as a *portfolio of government securities*. We use a convention that B_t^0 refers to the one-period government bond issued in period t.[2]

Let $B_t \equiv \sum_{i\in\mathcal{G}_t} Q_t^i B_t^i$ be the market value of the government portfolio and $\omega_t^i \equiv Q_t^i B_t^i / B_t$ be the portfolio share of security $i \in \mathcal{G}_t$. Let $\overrightarrow{\omega}_t$ be a column vector that has as its elements $\left\{\omega_t^i\right\}_{i\in\mathcal{G}_t\setminus\{0\}}$, that is, the portfolio shares of all securities other than the one-period bond that the government can trade at t. The elements in vector $\overrightarrow{\omega}_t$ always sum to $1-\omega_t^0$.

[2] This convention involves a slight abuse of notation. To keep notation simple, it is convenient to enumerate at time 0 all securities that are ever traded in the future by $i = 1, 2, \ldots$ and use $\mathcal{G}_t\left(s^t\right)$ to denote the subset of those securities that can be traded in s^t. This convention implies that there are infinitely many one-period government bonds, one for each s^t, but only one such bond is in each $\mathcal{G}_t\left(s^t\right)$. This is the bond that we call B_t^0 or $B_t^0\left(s^t\right)$.

Households. Households are heterogeneous. Household h has recursive preferences

$$V_{h,t} = U_{h,t}\left(c_{h,t} - v_{h,t}\left(y_{h,t}\right), \left\{Q_t^i b_{h,t}^i\right\}_{i \in \mathcal{G}_t}, G_t\right) + \beta \mathbb{W}_{h,t}\left(V_{h,t+1}\right), \tag{2}$$

where $\mathbb{W}_{h,t}$ is a functional that maps $t+1$ measurable random variables to t measurable random variables. Here $c_{h,t}$ and $y_{h,t}$ are consumption and earnings of household h and $b_{h,t}^i$ are holdings of security i by household h. We assume that functions $U_{h,t}$ and $v_{h,t}$ are twice differentiable, strictly increasing in their first arguments and concave, and that $\mathbb{W}_{h,t}$ is twice continuously differentiable, strictly increasing, and increasing in first- and second-order stochastic dominance,[3] with a property that for any time-t measurable random variable x_{t+1} we have $\mathbb{W}_{h,t}\left(x_{t+1}\right) = x_{t+1}$.

Household h solves

$$\max_{\mathbf{c}_h, \mathbf{y}_h, \left\{\mathbf{b}_h^i\right\}_i} V_{h,0} \tag{3}$$

subject to initial portfolio $\left\{b_{h,-1}^i\right\}_i$ and

$$c_{h,t} + \sum_i Q_t^i b_{h,t}^i = y_{h,t} - \mathcal{T}_t\left(y_{h,t}\right) + \sum_i \left(Q_t^i + D_t^i\right) b_{h,t-1}^i, \tag{4}$$

and

$$\varphi_{h,t}\left(\left\{Q_t^i b_{h,t}^i\right\}_i\right) \geq \mathbf{0}. \tag{5}$$

Our specification of the household problem in (3)–(5) includes a broad class of models that are used in applied work in macro and finance. That breadth allows us to collect insights that transcend structural details that differ across a variety of models.

In preference specification (2), we let the period utility $U_{h,t}$ depend explicitly on $\left\{Q_t^i b_{h,t}^i\right\}_{i \in \mathcal{G}_t}$ and on government spending G_t. This allows us to be agnostic as to whether or not households get utility directly simply from holding some subset of government-traded securities (e.g., as with the convenience benefits or liquidity services analyzed by Krishnamurthy and Vissing-Jorgensen (2012)) or whether government expenditures G_t directly enhance utilities of some households or whether government expenditures are just dropped in the ocean. Furthermore, the fact that period utility function $U_{h,t}$ can vary by history and that the functional $\mathbb{W}_{h,t}$ has very few restrictions means that our framework includes many models of households attitude towards risk and discounting.[4] One substantive restriction on preferences that we impose in

[3] In other words, $\mathbb{W}_{h,t}\left(x_{t+1}^1\right) \geq \mathbb{W}_{h,t}\left(x_{t+1}^2\right)$ whenever random variable x_{t+1}^1 first- or second-order stochastically dominates x_{t+1}^2.

[4] Ai and Bansal (2018) showed that this specification of functional $\mathbb{W}_{h,t}$ includes, as special cases, standard time-separable preferences, recursive preferences of Epstein and Zin (1989), the variational preferences of Maccheroni et al. (2006a,b), the multiplier preferences of Hansen and Sargent (2008) and Strzalecki (2011), the

specification (2) is that there is no income effects in labor supply. This assumption substantially simplifies our approach. We explore how well our approach works in models with income effects in Section 5.4.

The functional $\varphi_{h,t}$ that appears in equation (5) captures a wide variety of asset trading restrictions faced by households. For instance, the restriction that household h cannot trade some security j in period t can be represented by two inequalities $Q_t^j b_{h,t}^j \geq 0$ and $Q_t^j b_{h,t}^j \leq 0$ as a part of functional $\varphi_{h,t}$. In parallel with one-period government bond, we assume that there exists in each period t a security to which we refer as a *one-period private bond*. This security matures in period $t+1$ and pays one unit of consumption good as dividend, and there is some subset of households for whom it does not appear in the constraint set $\varphi_{h,t}$ or the utility function $U_{h,t}$. We use $Q_t^{0,pvt}$ to denote the price of this security. Other than that, $\varphi_{h,t}$ is arbitrary. Thus, specification (5) takes no stance on whether households can trade the same securities as the government, or whether all or only some households can borrow and lend from each other. Later in this paper we explore how various assumptions on asset trading among households affect the government portfolio problem. We write summation $\sum_i$ in the budget constraint (4) over all securities, since $\varphi_{h,t}$ can restrict holdings of any subset of those securities to zeros.

Foreign investors. For now, we simply assume that foreign investors are a set of time-t measurable, twice continuously differentiable demand functions $\left\{\mathcal{B}_t^i\left(\left\{\mathbf{Q}^i\right\}_i\right)\right\}_{i,t}$, where $\mathcal{B}_t^i$ may be subject to exogenous shocks. Later on, we explore several cases of this general specification — a small open economy ($\mathcal{B}_t^i$ is perfectly elastic), a closed economy ($\mathcal{B}_t^i$ is perfectly inelastic with $\mathcal{B}_t^i = 0$ for all i, t), and preferred habitat models that give rise to downward sloping demand curves for government debt in the spirit of Greenwood and Vayanos (2014) and Koijen and Yogo (2019).

Definition 1. For given initial conditions $\left\{b_{h,-1}^i, B_{h,-1}^i\right\}_{i,h}$, and government policy $\left(\overrightarrow{\boldsymbol{\omega}}, \mathbf{B}, \boldsymbol{\mathcal{T}}, \mathbf{G}\right)$, a *competitive equilibrium* is a collection $\left(\left\{\mathbf{c}_h, \mathbf{y}_h, \mathbf{b}_h^i, \mathbf{B}^i, \mathbf{Q}^i, \mathbf{Y}\right\}_{i,h}\right)$ such that (i) $\left(\mathbf{c}_h, \mathbf{y}_h, \left\{\mathbf{b}_h^i\right\}_i\right)$ solves (3), (ii) $\left(\boldsymbol{\mathcal{T}}, \mathbf{Y}, \mathbf{G}, \left\{\mathbf{Q}^i, \mathbf{B}^i\right\}_i\right)$ satisfies (1), (iii) $\sum_h \mathbf{y}_h = \mathbf{Y}$ and $\sum_h \mathbf{b}_h^i + \boldsymbol{\mathcal{B}}^i = \boldsymbol{\mathcal{S}}^i + \mathbf{B}^i$ for all i, (iv) $\omega_t^i = Q_t^i B_t^i / \sum_{i \in \mathcal{G}_t} Q_t^i B_t^i$.

The focus of our analysis will be on optimal portfolios.

second-order expected utility of Ergin and Gul (2009), the smooth ambiguity preferences of Klibanoff et al. (2005), Klibanoff et al. (2009), the disappointment aversion preference of Gul (1991), and the recursive smooth ambiguity preference of Hayashi and Miao (2011). Moreover, by relaxing the differentiability assumption on $\mathbb{W}_{h,t}$, one can extend them to the maxmin expected utility of Gilboa and Schmeidler (1989), and Epstein and Schneider (2003). The stochastic function $U_{h,t}$ can express the discount shock formulation used in Albuquerque et al. (2016).

Definition 2. For given initial conditions and a set of Pareto weights $\{\varpi_h\}_h$, a competitive equilibrium associated with $\left(\overrightarrow{\boldsymbol{\omega}}, \mathbf{B}, \boldsymbol{\mathcal{T}}, \mathbf{G}\right)$ is optimal if there is no competitive equilibrium associated with $\left(\overrightarrow{\boldsymbol{\omega}}', \mathbf{B}', \boldsymbol{\mathcal{T}}', \mathbf{G}\right)$ that delivers strictly higher social welfare $\sum_h \varpi_h V_{h,0}$. We call $\overrightarrow{\boldsymbol{\omega}}$ *the optimal public portfolio.*

Government policies must satisfy the government budget constraint in all histories. If the government changes its portfolio at any date, then future paths of debts, taxes, or expenditures must adjust. We focus on the optimal portfolio choice for any stochastic process $\mathbf{G}$. This allows us to be agnostic about whether government expenditures are exogenous or endogenous, and, when they are endogenous, whether or not they are chosen optimally. We discuss this further in Section 5.5.

3 The benchmark economy

We start with a special case of the Section 2 environment that we refer to as our *benchmark economy.* This allows us to explain our methodology and highlight key insights transparently. As we show in Section 5, relaxing restrictions imposed in the benchmark economy only strengthens results that prevail in the benchmark economy.

Definition 3. The *benchmark economy*

1. Is small and open;
2. Has identical households, so that we can drop subscript h;
3. Has *linear* taxes $\mathcal{T}_t(y_t) = \tau_t y_t$ for some random process $\{\tau_t\}$;
4. Has a constant *elasticity of earnings* $v_t(y_t) = \theta_t^{-1/\gamma} \frac{y_t^{1+1/\gamma}}{1+1/\gamma}$ for some $\gamma > 0$ and positive random process $\{\theta_t\}$;
5. Has government bonds that are *perfect substitutes*, meaning that they appear $U_t\left(\cdot, \sum_{i \in \mathcal{G}_t} Q_t^i b_t^i, \cdot\right)$ and $\varphi_t\left(\cdot, \sum_{i \in \mathcal{G}_t} Q_t^i b_t^i\right)$ in the utility function and constraint set, respectively.
6. Has time-varying *multiplicative discount factor shocks* δ_t so that $U_t() = \delta_t U()$ for some function U and positive random process $\{\delta_t\}$;
7. Has a constant *intertemporal elasticity of substitution* so that there exists a scalar $IES \geq 0$ such that $U_c(\Gamma x, B', \Gamma G)/U_c(x, B, G) = \Gamma^{-1/IES}$ for any positive scalars (x, G, Γ) and any (B, B'), where U_c is the derivative of U with respect to its first argument.

Condition 1 allows us to ignore responses of asset prices to government actions. This lets us check our findings with results from studies of portfolio problems in settings with atomistic private investors. We relax this condition in Section 5.8. Condition 2 is a useful starting point that abstracts from heterogeneity and we relax it in Section 5.7. Conditions 3 and 4 yield simple algebraic expressions for deadweight losses from taxation. They require few substantive restrictions; we go on to drop even those conditions in Sections 5.7 and 5.4. Condition 5 sets up an important benchmark in which all non-pecuniary benefits of government securities – including their direct utility consequences as well as their effects on trading frictions – depend only on their total market value. It sets a natural starting point for proceeding to consider effects of non-pecuniary forces on a government portfolio. We drop this condition in Section 5.3. Finally, conditions 6 and 7 are used only to illustrate how our model applies in a stationary setting in which expected growth rates of all real variables are equal.

3.1 Key notions

We define summary measures of tax distortions, discounting for time and risk, and trading frictions that arise in any competitive equilibrium. We shall use these measures to characterize optimal portfolios.

In our benchmark economy, our measure of tax distortions is

$$\xi_t \equiv \frac{\partial \mathcal{T}_t / \partial \tau_t}{Y_t} = \frac{\partial \ln (\tau_t Y_t)}{\partial \ln \tau_t} = 1 - \gamma \frac{\tau_t}{1-\tau_t}. \tag{6}$$

The numerator, $\partial \mathcal{T}_t / \partial \tau_t$, is the actual response of tax revenues to a marginal increase in tax rates; the denominator, Y_t, is the statutory response to this increase, i.e., the increase in tax revenues if household pre-tax earnings were held fixed. The ratio measures the deadweight loss from taxation. If $\xi_t = 1$, taxes are not distortionary. Equation (6) shows that the ratio ξ_t can also be interpreted as the tax revenue elasticity $\partial \ln \mathcal{T}_t / \partial \ln \tau_t$. The third equality in (6) follows from the household intratemporal optimality condition

$$y_t = \theta_t (1-\tau_t)^{\gamma}. \tag{7}$$

The *return* on holding a security i from time t to $t+1$ is $R^i_{t+1} \equiv \left(Q^i_{t+1} + D^i_{t+1}\right)/Q^i_t$, with $R^0_{t+1} = 1/Q^0_t$ and $R^{0,pvt}_{t+1} = 1/Q^{0,pvt}_t$ being returns on one-period government and private bonds. Returns R^i_{t+1} on all risky securities are time-$(t+1)$ measurable, but both R^0_{t+1} and $R^{0,pvt}_{t+1}$ are known at time t and, therefore, time-t measurable. The *excess return* of security i

is $r^i_{t+1} \equiv R^i_{t+1} - R^0_{t+1}$. Three stochastic discount rates between periods t and $t+k$ are

$$
\begin{aligned}
Q_{t,k} &\equiv Q^0_t \times ... \times Q^0_{t+k-1}, \\
Q^{pvt}_{t,k} &\equiv Q^{0,pvt}_t \times ... \times Q^{0,pvt}_{t+k-1}, \\
\mathcal{Q}_{t,k} &\equiv \frac{1}{\sum_{i\geq 1} r^i_{t+1}\omega^i_t + R^0_{t+1}} \times ... \times \frac{1}{\sum_{i\geq 1} r^i_{t+k}\omega^i_{t+k-1} + R^0_{t+k}},
\end{aligned} \tag{8}
$$

where $\sum_{i\geq 1}$ denotes the sum over all assets $i \in \mathcal{G}\setminus\{0\}$. The first two stochastic discount rates use cumulative returns on one-period government-issued and privately-issued bonds, respectively, while $\mathcal{Q}_{t,k}$ uses cumulative returns on government portfolios. We use a convention that $Q_{t,0} = Q^{pvt}_{t,0} = \mathcal{Q}_{t,0} = 1$ so that the government budget constraint in some period $T+1$ can be written in the present value form as

$$
\mathbb{E}_{T+1} \sum_{t=1}^{\infty} \mathcal{Q}_{T+1,t-1} X_{T+t} = B_T \left[R^0_{T+1} + \sum_{i\geq 1} \omega^i_T r^i_{T+1} \right]. \tag{9}
$$

Households' intertemporal optimality conditions are complicated, a consequence of the large number of possible specifications of preferences and non-pecuniary costs and benefits of securities, either due to direct utility benefits or trading frictions. For most of our analysis, these complexities can be side-stepped. Let $\beta^t \Pr\left(s^t\right) M_t\left(s^t\right)$ be the Lagrange multiplier on the household budget constraint (4). For each security i we define a wedge A^i_t by

$$
\frac{1}{A^i_t} \equiv \mathbb{E}_t \frac{\beta M_{t+1}}{M_t} R^i_{t+1}. \tag{10}
$$

This wedge equals one whenever security i brings with it no non-pecuniary benefits because it enters neither utility function U_t nor asset trading constraint φ_t so that $A^{0,pvt}_t = 1$ for all t. Note that $\ln A^0_t = \ln Q^0_t - \ln Q^{0,pvt}_t$ is the difference between prices of government and private one period bonds so we refer to $\ln A^0_t$ as a *liquidity premium*.[5] We define $A_{t,0} = 1$ and $A_{t,k}$ as

$$
A_{t,k} \equiv A^0_t \times ... \times A^0_{t+k-1}
$$

for $k\geq 1$. Thus, $\ln A_{t,k} = \ln Q_{t,k} - \ln Q^{pvt}_{t,k}$ corresponds to the accumulated liquidity premium between periods t and $t+k$ for any $k \geq 0$.

In our benchmark economy, government securities are perfect substitutes with one another, which implies that liquidity wedges for all government securities government securities are equal.

[5] The empirical literature has found that government debts are often traded at higher prices (i.e., offer lower returns) than virtually equally riskless debts issued by highly rated private corporations. There exist various explanations for this. We refer to this price difference as a "liquidity premium" without necessarily taking a stand on whether it is driven by liquidity services, tax benefits, or other types of convenience benefits that government debt provides.

Lemma 1. *If government securities are perfect substitutes then in any competitive equilibrium* $A_t^i = A_t^0$ *for all* t, $i \in \mathcal{G}_t$.

3.2 Analysis

We first use a variational approach to analyze determinants of an optimal public portfolio and then proceed to apply a class of small-noise approximations that considerably simplifies isolating key forces.

Start from a competitive equilibrium and consider a family of perturbations. Suppose that at history s^T the government (i) increases holdings of security $j \in \mathcal{G}_T$ by ϵ while simultaneously reducing holdings of the one-period government bond by the same amount; (ii) unwinds this transaction in period $T+1$, thereby realizing excess returns $r_{T+1}^j \epsilon$; (iii) rolls over these returns for another $t-1$ periods by putting them into one-period government bonds; and (iv) adjusts taxes to distribute these returns back to households in period $T+t$.

By construction, this perturbation increases government revenues by $\epsilon r_{T+1}^j / Q_{T+1,t-1}$ in period $T+t$. As $\epsilon \to 0$, the marginal effect from this perturbation on taxes is $\partial_{j,T,t,\epsilon} \tau_{T+t} = 1/\left(\xi_{T+t} Y_{T+t}\right)$. Applying the envelope theorem to maximization problem (3), the welfare impact of this perturbation is

$$\partial_{j,T,t,\epsilon} V_0 = \beta^{T+t} \Pr\left(s^T\right) \mathbb{E}_T M_{T+t} \frac{r_{T+1}^j}{Q_{T+1,t-1}} \frac{1}{\xi_{T+t}} sign(\epsilon). \tag{11}$$

A necessary condition for optimality is that there are no welfare improving perturbations. If both positive and negative ϵ are feasible, we obtain[6]

$$\mathbb{E}_T M_{T+t} \frac{r_{T+1}^j}{Q_{T+1,t-1}} \frac{1}{\xi_{T+t}} = 0 \text{ for all } T, t \geq 1, j \in \mathcal{G}_T. \tag{12}$$

To bring out the economics under equation (12), it is useful to compare it to household intertemporal optimality conditions

$$\begin{aligned} \mathbb{E}_T M_{T+t} \frac{r_{T+1}^j}{Q_{T+1,t-1}^{pvt}} &= \frac{1}{A_T^j} - \frac{1}{A_T^0} \\ &= 0 \text{ for all } T, t \geq 1, j \in \mathcal{G}_T. \end{aligned} \tag{13}$$

The first line follows from households' Euler equations and holds for any security j. The second line follows from the fact that in the bechmark economy government securities are

[6] Our empirical application focuses on optimal government holdings of debts of different maturities. We find that it is optimal for the government to issue positive quantities debts of all available maturities. This means that both positive and negative ϵ are feasible, so we focus on this situation in our theoretical analysis. Using our approach, it is possible but algebraically cumbersome to incorporate corner solutions.

perfect substitutes so that according to Lemma 1 their liquidity wedges are equal. There are two main differences between (12) and (13). A first is in how returns are valued on the margin. The shadow value of a unit of resources in household's hands in period $T+t$ is the Lagrange multiplier on the household budget constraint, M_{T+t}. The shadow value of a unit of resources in government's hands is M_{T+t}/ξ_{T+t}. The tax revenue elasticity ξ_{T+t} appears here because of the deadweight cost of transferring resources between households and the government. A second difference between (12) and (13) arises because households and the government transfer resources intertermporally at different prices, $Q_{T+1,t-1}$ and $Q^{pvt}_{T+1,t-1}$ respectively.

While equations (12) and (13) are similar, there is an important conceptual difference. The household optimality condition (13) holds in *any* competitive equilibrium, whether or not the government sets its policies optimally. Equation (12) holds only when government policies are optimal. Implications of equation (12) can be better exhibited if we "net out" the household optimality condition (13) from it.

We can gather more insights by using small-noise expansions to approximate our optimality conditions. We consider the following class of second-order approximations. We can write the underlying state process s_{T+t} for $t \geq 0$ as

$$s_{T+t} = \mathbb{E}_T s_{T+t} + \varepsilon_{T+t},$$

where $\mathbb{E}_T \varepsilon_{T+t} = \mathbf{0}$. Let $\overline{s}_{T+t} \equiv \mathbb{E}_T s_{T+t}$ and consider a family of stochastic processes parameterized by scalar $\sigma \geq 0$ where the process for the underlying states is given by $s_{T+t}(\sigma) = \overline{s}_{T+t} + \sigma \varepsilon_{T+t}$. The case $\sigma = 1$ corresponds to our economy, while the case $\sigma = 0$ corresponds to an economy in which all uncertainty vanishes after history s^T. Our approximation is based on second-order Taylor expansions of equilibrium conditions with respect to σ around $\sigma = 0$. A related approach is often used in portfolio theory.[7] We use signs "$\simeq$" to denote relationships that hold up to third order of approximation and "$\approx$" to denote a relationship in a deterministic limit that emerges as $\sigma \to 0$.

Subtracting (13) from (12) and applying our small noise approximation, we obtain

$$cov_T\left(\ln \xi_{T+t}, r^j_{T+1}\right) \simeq -cov_T\left(\ln A_{T+1,t-1}, r^j_{T+1}\right) \text{ for all } T, t \geq 1, j \in \mathcal{G}_T. \tag{14}$$

This equation highlights that an optimal policy strives to equate fluctuations in the tax revenue elasticity ξ_{T+t} to fluctuations in liquidity premia $\ln A_{T+1,t-1}$ at all time horizons t. Other things

[7] Samuelson (1970) might be the first one to use it in portfolio applications. Schmitt-Grohe and Uribe (2004) provide a classic exposition of this approach to study macroeconomic models. Devereux and Sutherland (2011) apply such perturbations to study portfolio problems in open economy models. Our small noise expansion is slightly different from theirs as we use small noise expansion after a specific s^T rather than around a steady-state at period 0.

being equal, fluctuations in deadweight loses are costly. If the government and households can borrow at the same rate of interest, so that $\ln A_{T+1,t-1} = 0$ for all T, t, then the government should use its risky securities to minimize fluctuations in $\ln \xi_{T+t}$ by setting the covariance on the left-hand side of (14) to zero. This tax smoothing prescription is similar to Bohn (1990) (see his equation (8)) but it holds in a much more general environment than he studied. When the interest rates that households and the government face are not the same, deviations from tax smoothing are called for. While there are contemporaneous welfare gains from issuing more public debt in states with high liquidity premia, servicing that additional debt requires levying higher taxes in the future. Equation (14) captures the optimal way to balance tax smoothing against liquidity provision.

Because tax revenues must be sufficient to finance primary deficits and debt service, the government budget constraint establishes a tight link between tax optimality conditions (14) and an optimal public portfolio. To investigate ramifications of this link, we start with the following second-order approximation of the budget constraint (9)

$$\begin{aligned}&\sum_{t=1}^{\infty} \mathbb{E}_T Q_{T+1,t-1} cov_T \left(X_{T+t}, r^j_{T+1} \right) + \sum_{t=2}^{\infty} \mathbb{E}_T X_{T+t} cov_T \left(Q_{T+1,t-1}, r^j_{T+1} \right) \\ \simeq & B_T \sum_{i \geq 1} \omega^i_T cov_T \left(r^i_{T+1}, r^j_{T+1} \right) \text{ for all } T, j \in \mathcal{G}_T.\end{aligned} \tag{15}$$

This is simply a second-order approximation of an identity that states that fluctuations in returns on government portfolio (the right-hand side of (9)) should be consistent with fluctuations in primary surpluses and interest rates (the left-hand side (9)). But because it is framed in terms of covariances, it is easy to relate it to the optimality conditions (14). By itself, equation (15) imposes few restrictions as it holds in equilibrium at both optimal and suboptimal government policies. To obtain a prescription for an optimal portfolio $\overrightarrow{\omega}_T$, we shall combine equation (15) with (14).

To prepare to combine equations (14) and (15), first note that fluctuations in the primary surplus X_{T+t} can emanate from two sources: fluctuations in taxes τ_{T+t} (and hence in the tax revenue elasticity ξ_{T+t}) and fluctuations in (θ_{T+t}, G_{T+t}). We want to distinguish these two sources of fluctuations. Define

$$\ln Y^{\perp}_t \equiv \ln Y_t - \gamma \ln (1 - \tau_t) .$$

Since γ is the elasticity of earnings with respect to the retention rate $(1 - \tau_t)$, the variable $\ln Y^{\perp}_t$ removes fluctuations in output arising from fluctuations in tax rates. If we define $X^{\perp}_{T+t}$ as

$$X^{\perp}_{T+t} \equiv \mathbb{E}_T \mathcal{T}_{T+t} \times \ln Y^{\perp}_{T+t} - \mathbb{E}_T G_{T+t} \times \ln G_{T+t},$$

we can obtain a decomposition

$$cov_T\left(X_{T+t}, r^j_{T+1}\right) \simeq cov_T\left(X^{\perp}_{T+t}, r^j_{T+1}\right) - \mathbb{E}_T\zeta_{T+t}\mathbb{E}_T Y_{T+t} cov_T\left(\ln \xi_{T+t}, r^j_{T+1}\right), \tag{16}$$

where $\zeta_{T+t} \equiv \gamma^{-1}\left(1-(1+\gamma)\tau_{T+t}\right)^2$.

Equation (16) decomposes fluctuations in the primary surplus into two components: fluctuations driven by changes in tax rates (the second term on the right-hand side of (16)) and fluctuations driven by other shocks (the first term on the right-hand side of (16)). If we combine (14), (15), and (16), we obtain the main result of this section.

Theorem 1. *An optimal public portfolio in the benchmark economy satisfies*

$$\begin{aligned}&\sum_{t=1}^{\infty}\mathbb{E}_T Q_{T+1,t-1} cov_T\left(X^{\perp}_{T+t}, r^j_{T+1}\right) + \sum_{t=2}^{\infty}\mathbb{E}_T X_{T+t} cov_T\left(Q_{T+1,t-1}, r^j_{T+1}\right)\\&+\sum_{t=2}^{\infty}\mathbb{E}_T Q_{T+1,t-1}\mathbb{E}_T\zeta_{T+t}\mathbb{E}_T Y_{T+t} cov_T\left(\ln A_{T+1,t-1}, r^j_{T+1}\right) \simeq B_T\sum_{i\geq 1}\omega^i_T cov_T\left(r^i_{T+1}, r^j_{T+1}\right)\end{aligned} \tag{17}$$

for all $T, j \in \mathcal{G}_T$.

The right-hand side of equation (17) contains optimal portfolio weights $\overrightarrow{\omega}_T$ that are chosen so that fluctuations in the return on the public portfolio hedge three distinct risks that the government faces and that are summarized by the covariances on the left-hand side of (17). We call these three the *primary surplus*, the *interest rate*, and the *liquidity* risk, respectively. Equation (17) shows that, other things being equal, securities have higher weight in the optimal portfolio if their returns increase when primary surpluses are high,[8] when interest rates are low, and when liquidity premia are high. States with lower than expected present values of primary surpluses (either directly through changes in expected revenues and spending or via higher interest rates) or with lower than expected liquidity premia, require a costly increase in tax rates unless the government holds liabilities whose value fall in those same states.[9]

There is no reason to expect one security to be equally good at hedging all risks at all time horizons. Thus, the coefficients that multiply these covariances can be interpreted as quasi-weights that scale risks across different time periods as well as across different types of risks for a given time period. To understand these quasi-weights, it is useful to focus on a special case of our benchmark economy that we call a stationary benchmark economy.

[8] Recall our convention that B_T denotes government obligations, i.e. debts. If returns on debt positively covary with the primary surplus, debt obligations become lower in states in which the primary surplus decreases.

[9] The benchmark economy assumes $\gamma > 0$ and formula (17) continues to apply as $\gamma \to 0$. However, in the limiting case when $\gamma = 0$, taxes are non-distortionary, Ricardian equivalence holds, and *any* government portfolio is optimal.

Definition 4. An optimal competitive equilibrium is *stationary* (at s^T) if there are some constants Γ and R such that for all $t \geq 1$ (i) $\mathbb{E}_T \frac{G_{T+t+1}}{G_{T+t}} \approx \mathbb{E}_T \frac{\theta_{T+t+1}}{\theta_{T+t}} \approx \Gamma$, (ii) $\mathbb{E}_T \delta_{T+t} \approx \delta_T$, (iii) $\mathbb{E}_T R^i_{T+t} \approx R$ for all i, and (iv) $\mathbb{E}_T \frac{c_{T+t+1}}{c_{T+t}} \approx \Gamma$.

Stationarity is a convenient benchmark under which all real variables grow at a constant rate Γ in the deterministic limit. Conditions (i) and (ii) state that expenditures G_{T+t} and productivity θ_{T+t} grow at rate Γ, and that there no predictable trend in the rate of discount. Condition (iii) ensures that in the deterministic economy all securities earn the same holding period returns, so that in a stochastic economy all excess returns are ultimately driven by risk. These conditions imply that in the optimal equilibrium of a deterministic economy, tax rates are constant and that output Y_{T+t} and the primary surplus X_{T+t} both grow at rate Γ. Condition (iv) is simply a balanced growth requirement that ensures that consumption grows at the same rate as output. It can be dispensed with (see the appendix) but our discussion is more streamlined with it.

This stationary economy allows us simplify the weights that appear in the public portfolio. To state our results succinctly, we define four covariance matrices Σ^Q_T, Σ^X_T, Σ^A_T, Σ_T as follows

$$\Sigma^Q_T[j,t] = cov_T\left(\frac{\ln Q_{T+1,t}}{Q^0_T}, r^j_{T+1}\right), \qquad \Sigma^A_T[j,t] = cov_T\left(A_{T+1,t}, r^j_{T+1}\right),$$
$$\Sigma^X_T[j,t] = cov_T\left(\frac{X^{\perp}_{T+t}}{\mathbb{E}_T Y_{T+t}}, r^j_{T+1}\right), \qquad \Sigma_T[j,i] = cov_T\left(r^i_{T+1}, r^j_{T+1}\right).$$

Corollary 1. *In a stationary benchmark economy, an optimal public portfolio satisfies*

$$\Sigma_T \overrightarrow{\omega}_T \simeq \left[\pi^Q \Sigma^Q_T + \pi^X_T \Sigma^X_T + \pi^A_T \Sigma^A_T\right] \overrightarrow{\hat{\beta}}, \tag{18}$$

where $\hat{\beta} = \beta \Gamma^{1-1/IES}$, $\pi^Q = 1 - \hat{\beta}$, $\pi^X_T = \hat{\beta}^{-1} \Gamma Y_T / B_T$, $\pi^A_T = \Gamma \zeta_T Y_T / B_T$, *and* $\overrightarrow{\hat{\beta}}$ *is a column vector with coefficients* $\overrightarrow{\hat{\beta}}[t] = \hat{\beta}^t$.

Stationarity allows us to obtain simple and interpretable formulas for optimal quasi-weightings of different risks in the public portfolio. Intertemporally, all three risks in period $T+t$ are weighted by $\hat{\beta}^t$, which depends on the discount factor β, the growth rate of the economy Γ, and the coefficient of the intertemporal substitution IES. The common assumption that $IES = 1$ implies that $\hat{\beta} = \beta$. Intratemporally, the three risks are weighted with quasi-weights π^Q, π^X_T and π^A_T. These weights imply that the relative importance of hedging interest rate risks is higher when the debt-to-GDP ratio is larger (since π^X_T and π^A_T decrease in B_T/Y_T), and that the relative importance of hedging the liquidity risk is lower when taxes τ_T are higher (since π^A_T decreases in τ_T). The economic logic driving the first insight is that interest rate risk matters to the government when it rolls over its debts; thus, the more debt there is that needs to be rolled over, the larger is welfare cost of managing interest rate risk, and the bigger is the role of hedging that risks in forming an optimal portfolio. The economic logic driving the second insight is that to manage the liquidity risk the government needs to vary its outstanding debt, and consequently change future taxes. Since deadweight losses are convex in tax rates, the cost of uncertainty from actively managing liquidity risk is larger when the current tax burden is already high.

If matrix Σ_T is invertible then an optimal portfolio is unique and given by

$$\overrightarrow{\omega}_T^* \equiv \left[\pi^Q \Sigma_T^{-1} \Sigma_T^Q + \pi_T^X \Sigma_T^{-1} \Sigma_T^X + \pi_T^A \Sigma_T^{-1} \Sigma_T^A\right] \overrightarrow{\hat{\beta}}. \tag{19}$$

We refer to $\overrightarrow{\omega}_T^*$ as a *target portfolio.* It will play an important role in our extension to economies that are neither small nor open.

Observe that although the optimal portfolio in equation (19) depends on various measures of risk, it does not include a term that captures either expected excess returns or risk aversion, objects that plays central roles in standard portfolio theory (e.g., Samuelson (1970), Merton (1971), Campbell and Viceira (1999), Viceira (2001)). This finding could have been anticipated from our earlier discussion of government and household optimality conditions (12) and (13). Since the government is benevolent, it has the same attitude towards risks and returns as households. So long as government securities are perfect substitutes, there is no reason for the government to chase higher excess returns on securities – households can get those same excess returns for themselves without bearing deadweight losses from taxation. Only if government securities are *imperfect* substitutes for private securities should the government depart from focusing exclusively on hedging risks. We discuss this case in Section 5.3.

3.3 Optimal bond portfolio

So far, we considered an optimal public portfolio consisting of an arbitrary set of securities. But bonds are the most common securities that governments have used to smooth aggregate fluctuations. In this section, we show how focusing on a portfolio of bonds brings several additional insights.

For simplicity, we assume that all bonds are zero coupon discount bonds. Let Q_T^t be the period T price of a pure discount bond that matures in period $T+t$. The long t-period discount rate Q_T^t is equal to the expectation of the product of one-period discount rates over the next t periods, $Q_{T,t}^0$ plus a term that reflects liquidity and risk premia. It can be shown that the risk premium term is of the second order. As an implication of covariances themselves being of the second order, $cov_T\left(Q^t_{T+1}, r^j_{T+1}\right) \simeq cov_T\left(Q_{T+1,t}, r^j_{T+1}\right)$. Furthermore, fluctuations in holding period returns of a t-period pure discount bond, $r^t_{T+1} = Q^t_{T+1}/Q^{t+1}_T$, are closely related to fluctuations in a t-period-ahead interest rate, $1/Q^t_{T+1}$, since both are driven by fluctuations in the same price Q^t_{T+1}. The most direct way to see the implication of these two observations is by deriving a counterpart of expression (19) for a market structure with a full set of pure discount bonds.

Corollary 2. *When the set of government securities consists of the full set of pure discount bonds, then $\Sigma_T \simeq \Sigma^Q_T$ in the optimal competitive equilibrium. If Σ_T is invertible, then the target portfolio of bonds in the benchmark stationary economy is*

$$\overrightarrow{\omega}^*_T = \left(1-\hat{\beta}\right)\overrightarrow{\hat{\beta}} + \left[\pi^X_T\Sigma^{-1}_T\Sigma^X_T + \pi^A_T\Sigma^{-1}_T\Sigma^A_T\right]\overrightarrow{\hat{\beta}}. \tag{20}$$

The central insight from this corollary is that bonds are a very good instrument for hedging interest rate risk. By matching the duration of debts to the duration of liabilities, the government can eliminate *expected* debt roll-overs. The remaining unexpected roll-over risks has only third-order effect on welfare. The portfolio of government bonds that hedges interest rate risk equals $\left(1-\hat{\beta}\right)\overrightarrow{\hat{\beta}}$ and can be replicated by a consol paying coupons that grow at rate Γ.

3.4 Nominal economy

So far, we have focused on real economies but with only minimal changes our analysis extends to nominal economies. In the appendix, we formally define a nominal version of our benchmark economy in which the government can trade nominal rather than real securities. Let P_t be the nominal price level and $\Pi_t \equiv P_t/P_{t-1}$ be the inflation rate. We extend the definition of stationarity to include a condition that $\mathbb{E}_T\Pi_{T+t}\approx\Pi_T$ for all T, t so that inflation is approximately a random walk, consistent some models that approximate U.S. data (see, for example, Atkeson and Ohanian (2001) or Stock and Watson (2007)). The parameter Γ still denotes the growth rate of real variables. In the appendix we verify

Corollary 3. *In the nominal economy, equations (18), (19) and (20) hold, except that now all variables in Σ_T, Σ^X_T, Σ^A_T, Σ^Q_T are measured in nominal terms, and the coefficients that multiply Σ^X_T and Σ^A_T become $\Pi_T\pi^X_T$ and $\Pi_T\pi^A_T$ respectively.*

4 The target portfolio in the U.S. data

Formulas for the optimal portfolios derived in Section 3 have the convenient property that key objects have straightforward empirical counterparts. In addition to population covariance matrices that can be approximated by sample covariance matrices, the target portfolio depends on only three preference parameters: an elasticity of earnings γ, an intertemporal elasticity of substitution IES, and a time discount factor β. These three parameters are routinely calibrated in applied work and there is widespread consensus about their plausible magnitudes.

Our optimal portfolio formulas are closely related to the "sufficient statistics" approach to characterizing optimal policies that has become popular in public finance and macroeconomics (see Chetty (2009) for an overview). The small-noise expansions that we used are new to this literature. Prior to our work, the sufficient statistics approach has been used either to study optimal policies in deterministic models (e.g., the optimal tax problems in Saez (2001) or Golosov et al. (2014)) or models with very simple stochastic structures (e.g., the optimal unemployment insurance analysis of Chetty (2006)). Small-noise expansions allow us to extend this approach to much richer stochastic environments that can be used to study questions such as portfolio management in realistic settings.

In this section, we use U.S. data to evaluate the target portfolio formulas. We focus on a portfolio of bonds of different maturities, as bonds are the securities that are most commonly used by governments to respond to business cycle frequency shocks (we discuss other securities in Section 5.6). As with all formulas from the "sufficient statistics" literature, it is important to keep in mind, when bringing data to the theory, that theoretical objects are measured under optimal policies, while their empirical counterparts are measured under existing policies. Although we ignore this distinction for now, we return to it in Section 5.2.

4.1 Data

We use U.S. national income and product accounts for data on GDP, primary surplus, tax revenues and expenditures. We use data on average marginal tax rates from Barro and Redlick (2011) that we extend to 2017. To measure returns on government debts of different maturities, we use the Fama Maturity Portfolios published by CRSP. There are 11 such portfolios, of which ten portfolios correspond to maturities of 6 to 60 months in 6 months intervals, and a final portfolio for maturities between 60 and 120 months. We add a twelfth portfolio that consists of the nominal 3-Month Treasury Bill, published by the Federal Reserve Board of Governors. All data are quarterly, nominal, and extend from 1952 to 2017. Finally, we use data on the yield curve of High Quality Market (HQM) corporate bonds provided by the U.S. Treasury to infer the short-maturity return on privately-traded bonds. The shortest maturity that Treasury reports is one year, and we use the yield curve to impute the 3 month yield. The data for HQM bonds are available from 1984. More details about data sources and construction are in

Appendix B.1.1.

In Table 1, we present summary statistics of contemporaneous covariances, means, and autocorrelations. For convenience, all variables are multiplied by 100 and reported in quarterly percentage points. Several patterns that emerge from this table will play an important role in shaping an optimal portfolio. Covariances of excess returns of government bonds of different maturities are several orders of magnitude larger than covariances of excess returns with primary surpluses, tax rates, or liquidity premia. Furthermore, covariances of excess returns with primary surpluses have opposite signs from covariances of excess returns with liquidity premia. This reflects that the primary government surplus is procyclical, but that risk and liquidity premia are countercyclical.

Table 1: COVARIANCE MATRIX

	Excess returns r_t^j for various maturities j											Surplus to GDP	Tax rate	Liquidity premium
	6m	12m	18m	24m	30m	36m	42m	48m	54m	60m	120m	X_t/Y_t	τ_t	$\ln A_t^0$
6m	0.092	0.2	0.29	0.36	0.43	0.48	0.5	0.53	0.56	0.61	0.69	-0.01	0.017	0.003
12m		0.49	0.73	0.91	1.1	1.2	1.3	1.4	1.5	1.6	1.8	-0.10	-0.021	0.007
18m			1.1	1.4	1.7	1.9	2.1	2.2	2.4	2.6	3	-0.17	-0.027	0.011
24m				1.8	2.2	2.5	2.7	3	3.1	3.5	3.9	-0.26	-0.068	0.013
30m					2.8	3.2	3.5	3.7	3.9	4.4	5	-0.31	-0.091	0.016
36m						3.6	4	4.3	4.5	5.1	5.8	-0.40	-0.081	0.018
42m							4.4	4.8	5.1	5.6	6.5	-0.45	-0.140	0.020
48m								5.4	5.6	6.2	7.2	-0.50	-0.180	0.021
54m									6.1	6.7	7.7	-0.56	-0.190	0.023
60m										7.8	8.6	-0.62	-0.170	0.023
120m											10	-0.75	-0.290	0.027
X_t/Y_t												4.30	0.940	-0.000
τ_t													1.900	-0.014
$\ln A_t^0$														0.002
Mean	0.076	0.14	0.2	0.23	0.26	0.3	0.33	0.33	0.36	0.29	0.44	2.5	30	0.043
Autocorr	-0.11	-0.08	-0.09	-0.08	-0.09	-0.07	-0.05	-0.03	-0.04	-0.07	-0.03	0.96	0.92	0.84

Notes: Excess returns 6m, 12m, ... are the nominal excess returns in Fama maturity portfolios corresponding to 6-12 months, 12-18 months, ... maturity bins, respectively. Surplus is measured as federal tax receipts (including contributions to social insurance) less federal government consumption expenditure (including transfer payments to persons) from the BEA. The tax rates series is an average marginal tax rate on income computed by Barro and Redlick (2011) and extended to 2017. The liquidity premium on the short bond, $\ln A_t^0$, is inferred from prices of government-issued and high quality privately issued bonds. All data are quarterly and in percentage points. All series are for 1952-2017 with the exception of the short liquidity premium that is for 1984-2017.

4.2 Target portfolios from U.S. data

Since most U.S. public debt is in the form nominal bonds, we use the nominal versions of equations (19) and (20) to evaluate target portfolios. We set the three preference parameters that appear in those formulas to $\gamma = \frac{1}{2}$, $IES = 1$, and $\beta = 0.99$, which are commonly used values. We set $\Gamma = 1.005$ to target an annual two percent growth rate of real variables, roughly in line with U.S. data. We set $\tau_T = \frac{1}{3}$ and $\frac{B_T}{Y_T} = 4$ so that taxes and debt to (quarterly) GDP are similar to current U.S. levels, and $\Pi = 1.005$ to be in line with the U.S. Federal Reserve target of a two percent annual inflation rate.

We now discuss how population covariance matrices Σ_T^{-1}, Σ_T^Q, Σ_T^A, Σ_T^X can be approximated. There are several challenges. First, Table 1 reports sample counterparts of ergodic covariances, while our theory is about covariances conditional on a period-T information set. Second, our formulas require an inverse of the covariance matrix of returns, Σ_T^{-1}, and it is known[10] that simply calculating an in-sample covariance matrix and then taking its inverse can lead to large sampling errors. Finally, we need to measure not only covariances of returns with contemporary realizations of various macroeconomic variables but also their realizations at all future horizons.

We overcome these challenges by adopting a parsimonious dynamic factor structure representation.[11] Let z_t be a stacked vector that consists of excess returns $\left\{r_t^j\right\}_j$ for the 11 portfolios of different maturities j, the liquidity premium $\ln A_t^0$, and de-trended nominal $\ln Y_t^{\perp,\$}$ (constructed from nominal GDP and tax rates and the Section 3.2 definition of $\ln Y_t^{\perp,\$}$) and nominal expenditures $\ln G_t^{\$}$. We use z_t^k to denote the k^{th} element of this vector. We posit the following stochastic process

$$\begin{aligned} z_t^k &= \alpha_k + \rho_k z_{t-1}^k + \kappa_k f_t + \varepsilon_t^k \text{ for all } k, \\ f_t &= \alpha_f + \rho_f f_{t-1} + \varepsilon_t^f, \end{aligned} \tag{21}$$

where f_t is a factor and $\left\{\varepsilon_t^k, \varepsilon_t^f\right\}_{k,t}$ are residuals. We set f_t to be the first principal component extracted from observed returns, the government surplus, output, and the risk-free rate, and denote the variances of the residuals by $\left\{\sigma_k^2, \sigma_f^2\right\}_{k,f}$. We use the subscripts $k \in \{Y, G, A\}$ to denote the variables $\ln Y_t^{\perp,\$}$, $\ln G_t^{\$}$, and $\ln A_t^0$, and $k = j$ to denote returns on bonds of maturity j. We report estimates in Table 2.

[10] See, for example, early work by Jobson and Korkie (1980), Merton (1980), Michaud (1989) and more recent work by Jagannathan and Ma (2003) and DeMiguel et al. (2007).

[11] Factor representations are popular in finance for estimating Σ_T^{-1} (see, e.g., MacKinlay and Pastor (2000), Chan et al. (1999), Senneret et al. (2016)). We superimpose a VAR structure on the factor model to obtain covariance estimates at all leads and lags. This extension is similar in spirit to the Factor Augmented Vector Auto Regressions (FAVAR) literature (see, e.g. Bernanke et al. (2005) and Bai et al. (2016)).

Table 2: FACTOR MODEL ESTIMATION (BASELINE)

	Excess returns r_t^j for various maturities j														
	6m	12m	18m	24m	30m	36m	42m	48m	54m	60m	120m	$\ln G_t^{\$}$	$\ln Y_t^{\perp,\$}$	$\ln A_t^0$	f_t
α_k	0.086 (0.014)	0.155 (0.025)	0.220 (0.033)	0.245 (0.035)	0.284 (0.039)	0.315 (0.039)	0.346 (0.038)	0.344 (0.037)	0.372 (0.037)	0.304 (0.043)	0.444 (0.030)	-0.177 (0.016)	-0.153 (0.008)	0.006 (0.003)	0.024 (0.501)
ρ_k	-0.107 (0.043)	-0.057 (0.035)	-0.041 (0.030)	-0.043 (0.025)	-0.042 (0.023)	-0.025 (0.020)	-0.022 (0.018)	-0.008 (0.016)	-0.022 (0.015)	-0.027 (0.015)	0.003 (0.009)	1.000 (nan)	1.000 (nan)	0.828 (0.047)	0.000 (nan)
κ_k	0.028 (0.002)	0.074 (0.003)	0.118 (0.004)	0.157 (0.004)	0.199 (0.005)	0.230 (0.005)	0.257 (0.005)	0.285 (0.005)	0.306 (0.005)	0.345 (0.005)	0.404 (0.004)	-0.032 (0.016)	-0.047 (0.008)	0.001 (0.000)	0.000 (nan)
σ_k^2	0.044 (0.004)	0.154 (0.014)	0.267 (0.024)	0.300 (0.027)	0.378 (0.034)	0.384 (0.034)	0.356 (0.031)	0.345 (0.031)	0.341 (0.030)	0.460 (0.041)	0.222 (0.020)	4.231 (0.375)	1.147 (0.102)	0.000 (0.000)	63.753 (5.637)
R2	0.536	0.698	0.771	0.840	0.870	0.898	0.922	0.938	0.946	0.943	0.979	0.015	0.109	0.727	0.000

Notes: This table records the OLS estimates of the factor model (21). Standards errors are in parenthesis.The row titled "R2" are values of R-squared for each equation in the system (21). The sample for excess returns and primary surpluses normalized by outputs is 1952-2017, and the sample for the one-period liquidity premium is 1984-2017. The time period is a quarter.

We consider several variants of this factor model. For concreteness, in the body of the paper we report results for the special case that satisfies the stationary conditions given in Definition 4. This requires additional restrictions to (21) that set ρ_Y and ρ_G to be equal to one, and ρ_f to zero. We also estimate our factor model without imposing these restrictions and find that our results are virtually unchanged. We also allow for heteroskedastic shocks by estimating $\left\{\sigma_k^2, \sigma_f^2\right\}_{k,f}$ for each date and show that that variations in the target portfolio from time-varying covariances are fairly small.

We are interested in constructing optimal portfolios of bonds for arbitrary sets of maturities—for instance all bonds of maturities $1 \dots N \leq \infty$ quarters—but CRSP bond return data are available for only a subset of maturities. To implement our formulas, we extrapolate our estimates $\left\{\kappa_j, \sigma_j^2\right\}_j$ from the 11 maturities j for which we have data to any $j \geq 1$ using a convenient functional form $\kappa_j = e^0 - e^0 \exp\left(-e^1 j\right)$ and similarly for σ_j^2 . This functional form allows for a parsimonious parametrization how the loadings vary with maturities. The coefficient e^1 captures the slope while the coefficient e^0 bounds the range of values between $\left[0, e^0\right]$.[12]

The factor model (21) allows us to construct the target portfolio of bonds for any subset of maturities $\mathcal{G}$ as follows. Pick any collection of maturities $\mathcal{G}$. The factor structure (21) implies that Σ_T^{-1}, Σ_T^Q, Σ_T^A, Σ_T^X for that $\mathcal{G}$ can be easily constructed, and we can compute portfolios

[12] Fits are reported in the appendix. There we also show that our main results change little with an alternative extrapolation in which we use linear extrapolations for intermediate maturities, and by assuming that $\left(\kappa_j, \sigma_j^2\right)$ for all maturities greater j greater than 40 quarters are equal to $\left(\kappa_{40}, \sigma_{40}^2\right)$.

that hedge primary surplus, liquidity and interest rate risks in closed form. Define a constant $\chi^{-2} = \sigma_f^{-2} + \sum_{i \in \mathcal{G}} \kappa_i^2 \sigma_i^{-2}$, the components of the optimal portfolio satisfy

$$\begin{aligned} \Sigma_T^{-1}\Sigma_T^X \overrightarrow{\beta}\,[j] &= \underbrace{\frac{\hat{\beta}}{1-\hat{\beta}}\left(\kappa_Y \frac{\mathcal{T}_T^{\$}}{Y_T^{\$}} - \kappa_G \frac{G_T^{\$}}{Y_T^{\$}}\right)}_{\equiv K_T^X} \left(\frac{\kappa_j}{\sigma_j^2}\chi^2\right), \\ \Sigma_T^{-1}\Sigma_T^A \overrightarrow{\beta}\,[j] &= \underbrace{\left(\frac{\hat{\beta}}{1-\rho_A}\right)\left[\left(\frac{1}{1-\hat{\beta}}\right) - \left(\frac{\rho_A}{1-\hat{\beta}\rho_A}\right)\right]\kappa_A}_{\equiv K_T^A} \left(\frac{\kappa_j}{\sigma_j^2}\chi^2\right), \end{aligned} \tag{22}$$

$$\Sigma_T^{-1}\Sigma_T^Q \overrightarrow{\beta}\,[j] = \hat{\beta}^j + \underbrace{\sum_{t \notin \mathcal{G}} \hat{\beta}^t \kappa_t}_{\equiv K_T^Q} \left(\frac{\kappa_j}{\sigma_j^2}\chi^2\right). \tag{23}$$

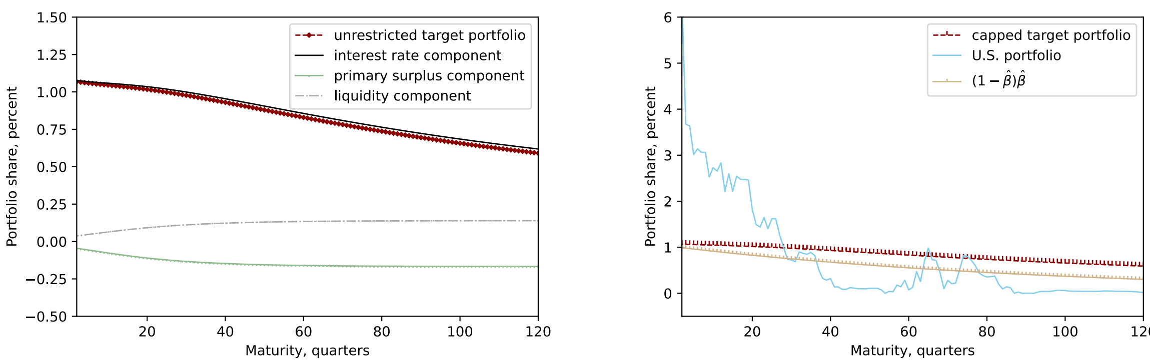
These expressions highlight several points about forming optimal portfolios. Consider the first equality in (22). Fluctuations in $\ln X_t^{\perp,\$}$ are driven both by the *common* component, proportional to f_t, and by the *idiosyncratic* $\varepsilon_{x,t}$ component orthogonal to the factor. A common factor shock affects the present value of $\ln X_t^{\perp,\$}$ proportionally to K_T^X. Common and idiosyncratic shocks also make returns fluctuate. The common component proportional to σ_f^2 helps hedge primary surplus risk, while the idiosyncratic component that is proportional to σ_j^2 does not. The ratio χ^2/σ_j^2 summarizes this trade-off. Equation (22) shows a simple rule for hedging surpluses, with bonds that have higher κ_j/σ_j^2 getting higher weights. The second equation in (22) shows that a portfolio that hedges liquidity risk takes similar form.

Hedging interest rate risk is different. While returns on bond of maturity j can hedge only a common component of fluctuations in the primary surplus and liquidity, it can hedge both common and idiosyncratic components of the j-period ahead interest rate. This is the first term on the right hand side of (23) in which $\hat{\beta}^j$ appears because of discounting, since hedging short- horizon fluctuations in interest rates is more important than fluctuations in very long term interest rates. For interest rates of duration $i \notin \mathcal{G}$, only the common component can be hedged. This component is captured by the second term on the right hand side of (23) and has the same structure as (22).

While factor structure (21) is particularly simple, it conveys broad principles for forming hedging portfolios that prevail more generally. This point is useful to keep that in mind as we describe features of the U.S. data that drive our main quantitative results.

We now turn to discussing quantitative aspects of the target portfolio. Using estimates from Table 2, we construct target portfolios for two collections of maturities $\mathcal{G}$. First, we allow $\mathcal{G}$ to consist of all maturities up to some finite number N periods. We set $N = 120$ so that the longest maturity is 120 quarters, consistent with long-standing practices by the U.S. government. We call this a *capped* target portfolio. Second, we report an optimal portfolio for when the government can issue bonds of any maturity, $N = \infty$. We call it the *unrestricted* target portfolio. We show these portfolios in Figure 1.

Figure 1(a) shows the capped target portfolio and its interest rate component $\pi^Q \Sigma_T^{-1} \Sigma_T^Q$, its primary surplus component $\pi_T^X \Sigma_T^{-1} \Sigma_T^X$, and its liquidity component $\pi_T^A \Sigma_T^{-1} \Sigma_T^A$. Evidently, the target portfolio almost exactly coincides with its interest rate component, with the other two components contributing little. This makes sense in light of our Section 4.1 analysis. As we documented in Table 1, although observed returns on bonds are fairly volatile, co-movement with macroeconomic variables are small. This makes bonds a poor hedge of these risks. Consequently, the target portfolio aims mostly to hedge interest rate risk.[13]



(a) (b)

Figure 1: Portfolio shares of securities with maturities from 2 quarters to 120 quarters. In panel (a) we plot the target (capped) portfolio and its 3 components that hedge interest rate risk, primary surplus risk, and liquidity risk, respectively. In panel (b) we plot the capped target portfolio and compared it to the 2017 U.S. federal debt portfolio (See Appendix B.1.1 for data sources and construction).

Several additional inferences can be made from Figure 1(a). First, the role of primary surplus and liquidity hedging components increases with increases in a bond's duration. This is driven by the fact that returns on longer bonds are more correlated with macroeconomic variables than are returns on shorter bonds. This can be seen both from Table 1 and from the fact that coefficients κ_j/σ_j^2 in Table 2 are increasing in j. This is also consistent with

[13]That bond prices exhibit little systematic co-variation with macroeconomic variables has been documented by a number of authors (e.g., see references in the handbook chapter of Duffee (2013)). A related literature on predictability of bond returns (see Cochrane and Piazzesi (2005), Ludvigson and Ng (2009)) also finds that a significant portion of the predictability of bond excess returns comes from a few linear combinations of contemporaneous bond prices rather than macroeconomic variables.

findings of Campbell and Shiller (1991) and Cochrane and Piazzesi (2005) who document that the predictability in bond returns increases with increases in maturity. While this makes long bonds a better hedge against fluctuations in macroeconomic variables than shorter bonds, the magnitudes are small.

Figure 1(a) also shows that that primary surplus and liquidity components have opposite signs and so offset each other. This is driven by primary surpluses being pro-cyclical while the liquidity premium is counter-cyclical,[14] so hedging the former risk calls for having less government debt in recessions while hedging the latter risk calls for more government debt.

We now discuss an unrestricted portfolio. First note that $\lim_{N\to\infty}\chi^2 \to 0$. From equation (22) and (23), it is easy to see that this implies that maturity by maturity

$$\Sigma_T^{-1}\Sigma_T^X\overrightarrow{\beta}\,[j] \to 0, \quad \Sigma_T^{-1}\Sigma_T^A\overrightarrow{\beta}\,[j] \to 0, \quad \Sigma_T^{-1}\Sigma_T^Q\overrightarrow{\beta}\,[j] \to \beta^j \text{ for all } j > 0. \tag{24}$$

But sums (across maturities j) of the three portfolio components $1^{\mathrm{T}}\Sigma_T^{-1}\Sigma_T^X\overrightarrow{\beta}$, $1^{\mathrm{T}}\Sigma_T^{-1}\Sigma_T^A\overrightarrow{\beta}$, and $1^{\mathrm{T}}\Sigma_T^{-1}\Sigma_T^Q\overrightarrow{\beta}$ are finite and satisfy

$$\frac{\pi_T^X 1^{\mathrm{T}}\Sigma_T^{-1}\Sigma_T^X\overrightarrow{\beta}}{\pi^Q 1^{\mathrm{T}}\Sigma_T^{-1}\Sigma_T^Q\overrightarrow{\beta}} \to \pi_T^X\frac{K_T^X}{\kappa_\infty}, \qquad \frac{\pi_T^A 1^{\mathrm{T}}\Sigma_T^{-1}\Sigma_T^A\overrightarrow{\beta}}{\pi^Q 1^{\mathrm{T}}\Sigma_T^{-1}\Sigma_T^Q\overrightarrow{\beta}} \to \pi_T^A\frac{K_T^A}{\kappa_\infty} \tag{25}$$

as $N \to \infty$, where $\kappa_\infty = \lim_{j\to\infty}\kappa_j$.[15]

Equations (24) and (25) have the following interpretation. As the number of maturities available grows, the government can reduce the adverse hedging effect of idiosyncratic volatility from issuing any particular bond by spreading portfolio weights across maturities. Thus, the contribution of any single maturity to hedging common risk approaches zero roughly at a rate $1/N$. This explains equation (24). While the importance of any particular maturity to hedging the common component of risk diminishes as N increases, the total contribution of the portfolio to hedging the three risks remains finite. Equation (25) shows that the relative contributions of the three risks to the target portfolio can be summarized by only three numbers, namely, K_T^X, K_T^A and κ_∞ as well by the quasi-weights π_T^X and π_T^A.

It is easy to use equations (25) and our estimates reported in Table 2 to infer that the target portfolio mostly focuses on hedging interest rate risks. The two limits in (25), that is, the sums of the portfolio shares that hedge the primary surplus risk and the liquidity risk are -0.17 and 0.14, respectively, (see the appendix for the source of these calculations). Thus, the importance of hedging primary surpluses and liquidity risks is much smaller than hedging of

[14] This also implies that that covariances of returns with the primary surplus take the opposite sign from their covariances with the liquidity premium in Table 1.

[15] Given our functional form for extrapolation, $\kappa_\infty = e^0$.

interest rate risk. In addition. primary surplus and liquidity risks mostly offset each other. As a result, hedging interest rate risk contributes most to the target portfolio.

The blue line in Figure 1(b) plots the actual U.S. portfolio of government bonds in 2017. Relative to the target portfolio, the U.S. government overweights short maturities and underweights long maturities in its portfolio. The Macaulay duration, computed as $\sum_{t\in\mathcal{G}} t\omega[t]$, for the observed U.S. portfolio is about 5 years, while that for the optimal target portfolio with capped maturities is about 14 years.

5 Qualifications and extensions

Here we discuss the relationship of this paper to existing work on optimal public portfolios, dependence of the key formulas (18) and (20) on observed versus optimal allocations, and implications of relaxing assumptions under the benchmark economy.

5.1 Debt portfolios in neoclassical models

A large literature in macroeconomics starting with, Lucas and Stokey (1983), Zhu (1992) and Chari et al. (1994), studies optimal public portfolios in "neoclassical" models with complete markets and a representative agent who has time separable expected utility preferences over consumption and leisure. Angeletos (2002) showed that it is both feasible and optimal for a government with access to the full set of pure discount bonds to implement a complete market allocation. He derives explicit expressions for the required portfolio. Buera and Nicolini (2004) and Farhi (2010)) found that plausible calibrations of the neoclassical model requires an optimal portfolio with huge long and short positions.[16] Those portfolios differ markedly from the simple portfolio that we obtained in Section 4.

In this section, we want to understand sources of those differences. We also want to see how well our simple statistical rules for forming an optimal portfolio perform in environments where some of the assumptions used to derive our rules are violated, e.g., absence of income and price effects.[17] We follow the model of Buera and Nicolini (2004) closely. We assume that households are identical, that they maximize

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-1/IES}}{1-1/IES} - \frac{y_t^{1+1/\gamma}}{1+1/\gamma} \right]$$

[16]Lustig et al. (2008) study a nominal version of the neoclassical model and impose short-selling as well as maximum maturity restrictions on the government portfolio. They find that these restrictions are binding and that an optimal portfolio issues debt almost exclusively in the maximal maturity bond.

[17]In online Appendix C, we extend our methods to study the target portfolio in a closed economy.

given their initial debt holdings and subject to a sequence of budget constraints

$$c_t + \sum_i Q_t^i b_t^i = (1-\tau_t)\, y_t + \sum_i \left(Q_t^i + D_t^i\right) b_{t-1}^i,$$

where the set of securities is assumed to be a set of pure discount bonds of all maturities. The economy is closed, bonds are in the zero net supply, and the government chooses bonds and taxes $\boldsymbol{\tau}$ to finance an exogenous stochastic government expenditure process $\mathbf{G}$. This economy satisfies all of the conditions that underly our benchmark economy except that it is closed and that income effects are present.

We first construct an optimal bond portfolio using standard numerical methods. We call this the *theoretical* optimal portfolio. We follow Buera and Nicolini (2004) and set $IES = 1/2$ and $\gamma = 1$. We assume that $\ln G_t$ follows an AR(1) process and calibrate the mean, variance, and first-order autocorrelation of this process to U.S. data. We discretize this process AR(1) process by confining possible realizations to be on a grid with 50 points. We set the initial level of debt to be four times (quarterly) output in a corresponding complete market economy.[18]

Since the Markov state can take 50 possible values, results of Angeletos (2002) imply that an optimal allocation can be achieved using only the bonds with the first 50 maturities. We use formulas that Angeletos derived in his Corollary to his Theorem 1 to compute that optimal portfolio and report it in the green line in Figure 2.[19] By construction, the ratio of the total market value debt to annual GDP is close to 1, but this conceals large variations in market values of positions at specific maturities. Consistent with findings of Buera and Nicolini, our optimal Angeletos portfolio exhibits huge long-short positions and variations in them across Markov states. Market values of bonds of a given maturity can range from $+1,500$ to $-1,000$ *times* annual GDP.

What would our statistical summary approach to approximating an optimal portfolio tell us for this economy? Returns on different bonds are highly correlated in the neoclassical economy, which makes the matrix of returns Σ_T nearly singular. For that reason, we focus on formula (18), which does not require inverting Σ_T. To make formula (18) operational, we fix a tolerance level $\epsilon > 0$ and study portfolios $\overrightarrow{\omega}_T$ that satisfy

$$\left\| \Sigma_T \overrightarrow{\omega}_T - \left[\pi^Q \Sigma_T^Q + \pi_T^X \Sigma_T^X + \pi_T^A \Sigma_T^A\right] \overrightarrow{\hat{\beta}} \right\| \leq \epsilon, \tag{26}$$

where $\|\cdot\|$ is the L^1 norm and the matrix Σ_T^A of liquidity premia is identically zero. For all tolerance levels that we have studied, we found that a portfolio that satisfies (26) is very close

[18] See online appendix B.2 for more details.

[19] Actually, there are 50 different portfolios, one for each possible value of G. Here we plot portfolio for one of the middle values ($s = 24$) of realizations of G for concreteness, but it is representative of the portfolio shapes in all other states.

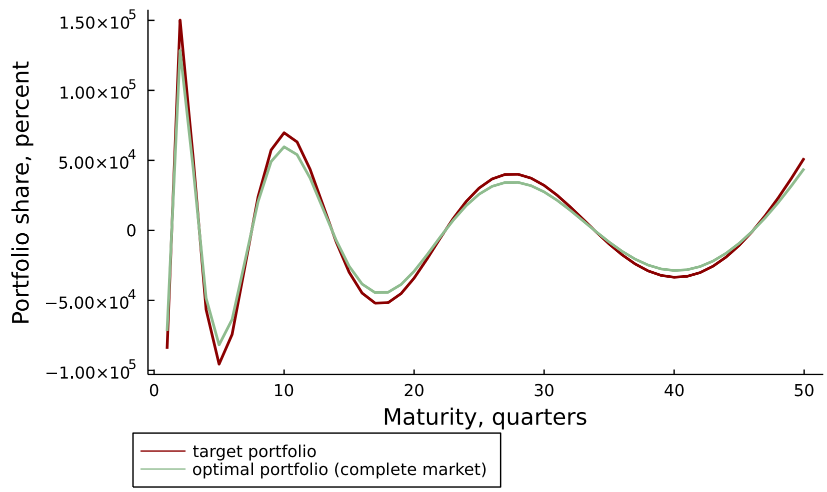


Figure 2: Government portfolio shares ω^i of 50 pure discount bonds of maturities $i \in \{1, \dots, 50\}$ quarters. The dark line is the portfolio implementing the complete market allocation following Angeletos (2002). The light line is the target portfolio defined by equation (18) taken for the average state 24 in the ergodic distribution of G. The tolerance is $\epsilon = 10^{-5}$.

to the theoretical optimal portfolio computed above. The red line in Figure 2 presents this portfolio. Thus, in the Angeletos environment, having ignored income and price effects in deriving equation (18) seems not to have impaired its ability quantitatively to approximate an optimal portfolio well.

Since the matrix Σ_T is nearly singular, other portfolios also approximately satisfy equation (18). With highly correlated returns there are multiple portfolios that can attain levels of welfare that are close to welfare attainable by trading a complete set of Arrow securities. We find that all such portfolios take large long-short positions. For example, following a suggestion of Angeletos, we can consider an optimal portfolio that consists of only a one-period bond and a consol that pays one unit of consumption in perpetuity. It is typically possible to find such a portfolio that satisfies (26). In this portfolio, it is optimal for the government to issue debt of 7.46 times annual GDP in the consol and to save -6.46 times annual GDP in the risk-free bond. This finding is consistent with findings from a similar exercise in Angeletos (2002) and confirms that in the neoclassical growth model, long maturity debt is an excellent hedge against primary surplus risk.

Since our statistical formulas are reliable guides for constructing an optimal portfolios in the neoclassical model, we can use them to understand what drives differences between our prescribed optimal government portfolio and the one that emerges from the standard growth model. In the appendix, we produce versions of Tables 1 and 2 but now estimated from simulations of a neoclassical growth model instead of U.S. data. We find that simulations of the neoclassical model generate counterfactual statistics for volatilities of bond prices and also for their co-movements with macroeconomic aggregates. For instance, for long maturities the variance of returns is between 0.025 and 0.035, which is 300 times smaller than their counterparts in U.S. data. The covariances of returns with primary government surpluses are only 10-20 times smaller, indicating much higher correlations. Furthermore, returns and surpluses are positively correlated and of opposite sign from those in U.S. data. According to formula (25), $\frac{K^X}{\kappa_\infty}$ is a key determinant of an optimal portfolio. Estimating the factor model (21) using data simulated from the neoclassical economy gave us a $\frac{K^X}{\kappa_\infty}$ that is about 20 times larger and has an opposite sign to that found from U.S. data.

Thus, a standard neoclassical model misrepresents the asset return movements that shape an optimal portfolio. It is an inappropriate tool for studying optimal public portfolios, whose composition depend critically on the properties of co-movements between returns and macroeconomic variables.[20] Bhandari et al. (2017b) described extensions of a neoclassical growth model, such as discount factor shocks in the spirit of Albuquerque et al. (2016), that can help realign theoretical results with statistics summarized in Tables 1 and thereby imply an optimal public portfolio closer to those prescribed in Section 4.2.

5.2 Current vs optimal policies

Although we have computed statistics called for by formulas (19) and (20) using U.S. data generated under then-prevailing U.S. policies, the theory that motivates these formulas states that the statistics are to be computed under outcomes produced by *optimal* policies. While we can detect differences between prescribed optimal and observed U.S. portfolios in Figure 1, we argue here that measuring our key statistics at current policies is unlikely to have substantially affected our findings.

Our assertion that interest rate risk is the dominant factor shaping an optimal portfolio would change only if increasing the duration of the U.S. portfolio would materially change the covariances Σ_T, Σ_T^X and Σ_T^A that govern the optimal portfolio. That seems unlikely for several reasons.

First, in a large class of macroeconomic models, covariances of returns with other variables

[20] Researchers have explored optimal portfolios in a neoclassical model when governments face additional frictions. For example, Faraglia et al. (2018) studied the effect of transaction costs, while Debortoli et al. (2017) studied the effects of government commitment on the formation of the optimal portfolio. Our results present reservations about this approach. Government frictions have little effect on asset pricing implications of the neoclassical model and hence those versions would still have unrealistic predictions about hedging properties of various bonds and other securities.

are primarily determined by exogenous shocks;[21] changes in government policies have only modest effects on their values. We have confirmed this assertion both in the neoclassical model we discussed in Section 5.1 and in an extension of Albuquerque et al. (2016) that uses discount factor shocks calibrated to match returns on debts in U.S. data.[22] In both cases, we calibrated primitives using a competitive allocation under policies that reflect observed U.S. policies and also under optimal fiscal policies. We found no substantial differences in magnitudes of our key statistics.

An alternative approach is to see whether these covariances are affected by government policies directly in the data. Establishing such causal relationship is a very challenging empirical exercise. We proceed as follows. First, we extended our factor model (21) to allow for time-varying volatilities in all variables and estimated that GARCH-like specification using the method of maximum likelihood. That allowed us to construct conditional volatilities at different dates in the U.S. data. While our estimated conditional volatilities are not constant and exhibit spikes during recessions, they show little relationship with the duration of U.S. government portfolio.[23]

Since the duration of the government portfolio is an endogenous object, we also explored the following quasi-experiment. Figure 3, taken from Garbade (2007), shows two clear breaks in the maturity structure of U.S. government debt, one around 1965 and the other around 1975. As Garbade (2007) explains, changes in regulations drove both breaks. For many years, the U.S. Treasury issued debt mainly in three categories: bills (maturity below 1 year), notes (below 5 years), and bonds (above 5 years). After 1918 there were a statutory ceiling on bond coupons of 4.25% and a restriction that bonds had to be issued at par; bills and notes were not subject to such restrictions. A gradual increase in safe-corporate yields after the war made those restrictions start to bind around 1965, prompting the Treasury to switch from issuing bonds to issuing bills and notes. This resulted in a sharp decrease in the average maturity of U.S. debt over the next decade. In the mid-1970s, Congress enacted several laws that allowed the Treasury to issue bonds with coupon payments exceeding 4.25% and to issue notes with maturities up to 10 years. These measures allowed the Treasury to increase the maturity of its portfolio substantially after 1975.

We investigated whether there were structural breaks in our factor model around 1965

[21]See for instance Bansal and Yaron (2004), or Albuquerque et al. (2016).

[22]For details about that exercise, see our earlier version of the paper, Bhandari et al. (2017b).

[23]While we see systematic spike in volatilities, we do not detect meaningful effects of those spikes on an optimal public portfolio. In particular, a spike in the liquidity component largely offset the spike in the primary surplus component of the target portfolio formula. The optimal portfolio seems to adjust somewhat in response to changes in stochastic volatility, but the associated variation is quantitatively fairly modest. See Figure 8 of the appendix.

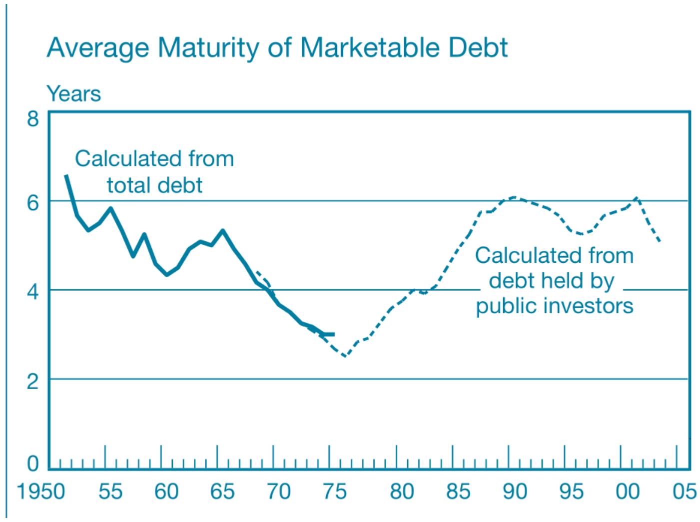


Figure 3: The duration of outstanding U.S. government debt. Source: Garbade (2007).

and 1975. We did not find evidence that the primary surplus component $\mathbf{1} \cdot \pi_T^X \Sigma_t^{-1} \Sigma_t^X$ was affected by those dates. A Chow test did not reject the null that factor loadings summarized by $\left(\kappa_Y \frac{\mathcal{T}_T^\$}{Y_T^\$} - \kappa_G \frac{G_T^\$}{Y_T^\$}\right)$ were same pre and post 1965 and 1975.

5.3 Government and private bonds are imperfect substitutes

In our baseline economy, we assumed that government bonds are perfect substitutes. We now extend our theory by relaxing this assumption to derive a formula for the optimal portfolio when bonds are imperfect substitutes. Quantifying the extra term using U.S. data, we find small quantitative departures for the optimal portfolio.

Recall, that when government bonds are perfect substitutes, their liquidity wedges are equal according to Lemma 1. This need not be true in general. We define an excess liquidity premium a_t^i for $i \in \mathcal{G}_t$ as

$$a_t^i \equiv \frac{1}{A_t^0} - \frac{1}{A_t^i},$$

and use $\overrightarrow{a}_t$ to denote a vector of excess liquidity premia for all $i \in \mathcal{G}_t \setminus \{0\}$.

When bonds are imperfect substitutes, optimality condition (12) becomes

$$\mathbb{E}_T \frac{\beta^t M_{T+t}}{M_T} \left(R_{T+2}^0 \times ... \times R_{T+t}^0\right) \frac{r_{T+1}^j}{\xi_{T+t}} = -\frac{1}{R_T^{pvt}} a_T^j \text{ for all } T, t \geq 1, j \in \mathcal{G}_T.$$

Now the optimality condition has an additional term that reflects that securities can differ not only in their hedging benefits (the term on the left side of this equation) but also in their liquidity benefits (the term on the right side of this equation). Like equation (12), this equation shows how the government optimally confronts the trade-off between providing hedging and providing liquidity. It is straightforward to follow the steps taken in Section 3.2 and thereby extend both Theorem 1 and its corollaries to cover situations when government securities are imperfect substitutes. In particular, equation (19) would become

$$\Sigma_T \boldsymbol{\omega}_T \simeq \left[\pi^Q \Sigma_T^Q + \pi_T^X \Sigma_T^X + \pi_T^A \Sigma_T^A\right] \overrightarrow{\beta} + \pi_T^a \overrightarrow{a}_T,$$

where $\pi_T^a = \pi_T^A / \left(1 - \hat{\beta}\right)$. It shows that assets with higher excess liquidity premia should have higher weights in public portfolios.

It is enlightening to extract implications of this formula for an optimal portfolio of bonds. As before, we use R_{T+1}^i to denote a return on a government issued pure discount bond that matures in period $T+1+i$. We assume that households can also issue a pure discount bond that matures in $T+1+i$ but that brings no non-pecuniary benefits. We use $R_{T+1}^{i,pvt}$ to denote its return. Let $\alpha_{T+1}^i = R_{T+1}^{i,pvt} - R_{T+1}^i$.

From the household optimality condition, we must have

$$1 = \mathbb{E}_T \frac{\beta M_{T+1}}{M_T} R_{T+1}^{i,pvt}. \tag{27}$$

Therefore, we can show that the excess liquidity wedge a_T^i satisfies

$$a_T^i = \underbrace{\frac{1}{R_{T+1}^{0,pvt}} \mathbb{E}_T \left(\alpha_{T+1}^i - \alpha_{T+1}^0\right)}_{\text{relative yield slope}} + \underbrace{cov_T \left(\frac{\beta M_{T+1}}{M_T}, \alpha_{T+1}^i - \alpha_{T+1}^0\right)}_{\text{risk correction}}. \tag{28}$$

Equation (28) lets us isolate statistics that determine how excess liquidity premia vary with maturity i. The "relative yield slope" term captures expected excess returns of privately- vs publicly-issued bonds. In U.S. data, it is increasing in maturity i. Think about yield curves for government and private bonds. Yield curves for both government and high quality public debt are generally upward sloping; but the private yield curve is typically *steeper* than the public one. Thus, the relative yield slope statistic implies that longer maturities are, on the margin, more desirable. The "risk correction" term also depends on i. Depending on the sign of the correlation of α_{T+1}^i with the household's stochastic discount factor (SDF) $\frac{\beta M_{T+1}}{M_T}$, the risk correction statistic can either reinforce or offset the first relative yield curve statistic.

In the appendix, we use a formal factor structure along lines of Koijen et al. (2017) to estimate a household SDF that prices relatively safe corporate bonds of different maturities using (27) and estimated a_T^i. We find that the risk correction term is generally of similar magnitude to but of the opposite sign than the relative yield slope term, implying that coefficients a_T^i are small and cannot be statistically distinguished from zero. Thus, we conclude that our benchmark portfolios summarized in Figure 1 provide a good approximation to an optimal maturity structure of public debt even if government bonds are imperfect substitutes.

5.4 Variable elasticity and nonlinear taxation

Our benchmark specification assumes that the elasticity of earnings is a constant γ and that the tax function is linear. This simplified our derivation of the tax revenue elasticity ξ_t. More general specifications of tax functions and preferences about supplying labor will modify the formula for ξ_t. Modulo this change, the envelope condition (11), the optimality conditions (14), construction of $X_{T+t}^{\perp}$ and Theorem 1 all remain unchanged.

We begin by extending our analysis to cover general preferences

$$U_t = U_t\left(c_t - v_t\left(y_t\right), \left\{Q_t^i b_t^i\right\}_{i \in \mathcal{G}_t}, G_t\right),$$

where $v_t\left(\cdot\right)$ is a twice differentiable, strictly concave function that varies with histories s^t. With such preferences, the elasticity of earnings γ now satisfies $\gamma_t = v_t''\left(y_t\right) y_t / v_t'\left(y_t\right)$ and a tax revenue elasticity becomes $\xi_t = 1 - \gamma_t \tau_t / \left(1 - \tau_t\right)$.

Next, we consider general non-linear tax functions. Suppose that in period t, the government uses a twice differential tax schedule $\mathcal{T}_t\left(\cdot\right)$, so that households who earn $\hat{y}$ receive after-tax earnings of $\hat{y} - \mathcal{T}_t\left(\hat{y}\right)$. We need to generalize the notion of deadweight losses from perturbing such a tax schedule. Consider changing the tax function in a direction $\mathcal{H}\left(\cdot\right)$ so that households face an earnings tax schedule $\mathcal{T}_t\left(\cdot\right) + \delta \mathcal{H}\left(\cdot\right)$, where δ is a scalar. Following our Section 3.1 analysis, we define a tax revenue elasticity as a ratio of the actual change in tax revenues $\partial \mathcal{T}_t / \partial \delta$ to the statutory change in tax revenues $\mathcal{H}\left(Y_t\right)$, i.e., $\xi_t \equiv \frac{\partial \mathcal{T}_t / \partial \delta}{\mathcal{H}\left(Y_t\right)}$. A formula for ξ_t in the general non-linear case is

$$\xi_t = 1 - \gamma \frac{\mathcal{T}_t'\left(y_t\right)}{1 - \mathcal{T}_t'\left(y_t\right)} \frac{1}{1 + \gamma \frac{y_t \mathcal{T}_t''\left(y_t\right)}{1 - \mathcal{T}_t'\left(y_t\right)}} \frac{y_t \mathcal{H}'\left(y_t\right)}{\mathcal{H}\left(y_t\right)}. \tag{29}$$

Formula (29) might suggest that ξ_t should depend on the particular perturbation $\mathcal{H}$ and thus take a form $\xi_t^{\mathcal{H}}$. But this is not the case. Let $\mathbb{H}$ be a collection of feasible perturbations. An immediate consequence of optimality of the tax system is that $\xi_t^{\mathcal{H}}$ should be equal for all $\mathcal{H} \in \mathbb{H}$. In particular, if linear perturbations are included in $\mathbb{H}$ then to construct an optimal portfolio we can focus only on them, as we did in Section 3.2.[24]

[24] If lump sum taxes are included in $\mathbb{H}$ then ξ_t is always 1 as only lump sum taxes are used to collect revenues. Our comment still stands, since small perturbations of linear taxes around $\tau_t = 0$ are non-distortionary.

5.5 Endogenous spending and inflation policies

We derived our formulas for an optimal portfolio while holding government spending and inflation policies fixed. In principle, the government can hedge risks by adjusting its expenditures, which would certainly affect at the very least the covariance matrix Σ_T^X. Whether the current path of expenditures is optimal or not requires one to model costs and benefits of a stochastic process $\mathbf{G}$, which is something outside of the scope of this paper. But regardless of whether a process $\mathbf{G}$ has been optimally chosen, an optimal public portfolio would still satisfy formulas (19) and (20).

An analogous assertion applies to an extension of our model that includes nominal securities. Inflation can alter returns on nominal securities and allow a government to put additional state-contingencies into returns. If a government has full control of the nominal price level, in principle it can replicate complete markets by altering properties of Σ_T, Σ_T^X, and Σ_T^A. In practice, the government's ability to fine-tune inflation appears to be limited: since 2012, when the U.S. Federal Reserve bank officially announced the 2% inflation target, inflation (measured using the consumer price index) has been outside a $2\% \pm 0.5\%$ band for 23 out of 42 quarters.

5.6 What are government debts?

Our calculations treated U.S. government bonds as comprehensive measures of U.S. debt. Auerbach et al. (1994), Lucas and Zeldes (2009), and Lucas (2016) argue that U.S. government debt is actually much higher as it includes implicit promises embedded in the Social Security system and guarantees for household mortgages and students loans. But the U.S. government also owns many assets including public lands and waterways. How should those additional debts and assets affect our analysis?

It is useful to start by observing that the government's budget equation (1) is an accounting identity. Whether we count a promise in period T of a \$1 of Social Security payment in period $T + t$ as part of future expenditures G_{T+t} or as part of current debt of a t period maturity is arbitrary. What matters for an optimal portfolio problem is which securities can be adjusted over the frequencies under analysis. In practice, the U.S. holdings of Social Security obligations, mortgage and debt guarantees, and rivers and parks adjust quite infrequently. The U.S. Treasury and the Federal Reserve exert substantial day-to-day control over the composition of government debts of different maturities, but they exercise no control over those assets. For this reason, we treat all those implicit securities and asset returns as a part of $\mathbf{X}$ and focus instead on an optimal composition of government debts. Having said that, with appropriate redefinitions of B and X, formula (19) can be used to study how the U.S. government can better hedge its risks by adjusting its holdings of public lands and waterways.

5.7 Household heterogeneity

We now extend our framework to include heterogeneous households who differ in their skills and their access to markets. Heterogeneity adds two motives that affect the optimal portfolio: hedging fluctuations in inequality and overcoming trading frictions that affect only a subset of agents. In this section, we show that both motives lengthen the duration of an optimal portfolio.

Suppose that household h has household-specific productivity $\theta_{h,t}$. Also suppose that households can be partitioned into two sets: $\mathbb{T}$, a set of households who can trade securities, and $\mathbb{N}$, a set of households who cannot trade securities. We maintain all other assumptions from our benchmark economy. We then consider our Section 3.2 perturbation. The welfare effect $\partial_{j,T,\epsilon} V_{0,}$ of this perturbation is

$$\partial_{j,Tt,\epsilon} V_0 = \beta^{T+t} \Pr\left(s^T\right) \sum_h \left[\varpi_h \mathbb{E}_T \beta M_{h,T+t} \frac{r^j_{T+1}}{Q_{T+1,t-1}} \xi^{-1}_{T+t} \frac{y_{h,T+t}}{Y_{T+t}} \right] \times sign(\epsilon), \tag{30}$$

where $M_{h,T+t}$ is the Lagrange multiplier on a type h household's budget constraint. Comparing equation (30) to its representative agent counterpart equation (11), there are two new terms highlighting the new forces that are present in heterogeneous agent settings.

The first is that the inverse tax revenue elasticity is weighted by $\frac{y_{h,T+t}}{Y_{T+t}}$, which is the share of household type h's income. To the extent these shares fluctuate, there is a motive for the government to use the returns on its portfolio to hedge those fluctuations. The second is the presence of the Lagrange multipliers $\{M_h\}$ on budget constraints for all households. In the representative agent counterpart, we used household optimality in security markets, that is, equation (13) to "net out" the implications on government optimality. With heterogeneous agents, the counterpart of equation (13) holds only for $h \in \mathbb{T}$. Thus, fluctuations in the wedge between the Lagrange multipliers of the traders and non-traders (a measure of deviation from perfect risk-sharing) capture a planners' desire to trade on behalf of agents who have trouble trading.

These two forces are summarized by two new simple statistics. Movements in inequality are summarized by a measure $\sum_h \mu_{h,t} \ln\left(s_{h,T+t}\right)$, where $s_{h,t} = y_{h,t}/Y_t$ and $\left\{\mu_{h,t}\right\}_{h,t}$ is a deterministic sequence of weights (see the appendix for formulas) that add up to one for all t and depend on both relative productivities and Pareto weights. It is easy to check that this measure is decreasing in the dispersion of incomes. Next, define $\ln\left(M_{\mathbb{T},T+t}\right)$ and $\ln\left(M_{\mathbb{N},T+t}\right)$ as an average of the Lagrange multipliers on budget constraints of traders and non traders, respectively, e.g., $\ln\left(M_{\mathbb{T},T+t}\right) \equiv \sum_{h\in\mathbb{T}} \mu_{h,t} \ln\left(M_{h,T+t}\right) \Big/ \sum_{h\in\mathbb{T}} \mu_{h,t}$. The imperfect risk sharing force is captured by $\ln\left(M_{\mathbb{T},T+t}\right) - \ln\left(M_{\mathbb{N},T+t}\right)$.[25] Following steps resembling those in our derivation equation (18), we obtain the following result.

Corollary 4. *In a stationary benchmark economy with heterogeneity, an optimal public portfolio satisfies*

$$\Sigma_T \boldsymbol{\omega}_T \simeq \left[\pi^Q \Sigma_T^Q + \pi_T^X \Sigma_T^X + \pi_T^A \Sigma_T^A + \pi_T^A \Sigma_T^{ineq} + \pi_T^A \Sigma_T^M\right] \overrightarrow{\hat{\beta}}, \tag{31}$$

where $\Sigma_T^{ineq}[j,t] = cov_T\left(\sum_h \mu_{h,t} \ln\left(\frac{1}{s_{h,T+t}}\right), r_{T+1}^j\right)$, $\Sigma_T^M[t,j] = cov_T\left(\mu_{\mathbb{N},t}\left[\ln\left(M_{\mathbb{T},T+t}\right) - \ln\left(M_{\mathbb{N},T+t}\right)\right], r_{T+1}^j\right)$ *and constants* $\hat{\beta}, \pi^Q, \pi_T^X, \pi_T^A$ *are the same as in Corollary 1.*

We now discuss the implications of the extra terms in the optimal portfolio relative to expression (18) that we derived in the benchmark economy. The concerns for inequality fluctuations manifest in the sign and the magnitude of Σ_T^{ineq}. A literature in macro and labor (see Storesletten et al. (2004), Guvenen et al. (2014)) documents that income inequality is countercyclical. Our Section 4.1 description of bond excess returns emphasized they too are countercyclical with larger predictable components for longer duration bonds. That makes us expect Σ_T^{ineq} to be positive and larger in magnitude for longer bonds. Equation (31) then implies that concerns for fluctuating income shares should push the government to issue additional debts at longer maturities.

We get a sense of the magnitude of the inequality-hedging portfolio from the following back of the envelope calculation. Assume that a household type $h = L$ represents a group of individuals who are in the left-tail (or bottom L percentile) of the income distribution, and that the planner sets $\mu_{L,t} = 1$. Then $\Sigma_T^{ineq}[j,t]$ depends on how the income share of the bottom L percentile covaries with returns. We can use our factor model in equation (21) with an additional equation to parameterize $\pi_T^A \Sigma^{-1} \Sigma_T^{ineq} \overrightarrow{\beta}[j] = \pi_T^A \underbrace{\frac{\hat{\beta}\kappa^{ineq}}{1-\rho^{ineq}\hat{\beta}}}_{\equiv K_T^{ineq}} \left(\frac{\kappa_j}{\sigma_j^2}\chi^2\right)$ with two new objects: κ^{ineq}, a loading of inequality on the common factor, and ρ^{ineq}, the first-order autoccorelation in a measure of inequality. We set $L = 25\%$ and use income share data from Guvenen et al. (2014) to obtain $\kappa^{ineq} = 0.002$ and $\rho^{ineq} = 0.92$.[26] Our estimate of K_T^{ineq}

[25] The formulation of government optimality using aggregated Lagrange multipliers of various groups is closely related to "multiplier approach" of Chien et al. (2011) who show that equilibria of a large class of heterogeneous agent, incomplete markets environments can be characterized and efficiently computed using a multipliers representation.

[26] Guvenen et al. (2014) use SSA data and provide means as well as quantiles of labor earnings at an annual frequency from 1978-2011. We first detrend the raw measure of inequality and then project it onto the unemployment rate to obtain a quarterly inequality series. We estimated κ^{ineq} and ρ^{ineq} by applying OLS to the regression equation $\ln\frac{Y_t}{y_{L,t}} = \alpha^{ineq} + \rho^{ineq}\ln\frac{Y_{t-1}}{y_{L,t-1}} + \kappa^{ineq} f_t + \sigma^{ineq}\epsilon^{ineq}$.

is about 10 times smaller in magnitude than K_T^X and K_T^A, which capture the fiscal hedging and liquidity hedging components, respectively, and that the parts of the portfolio that hedge inequality accounts for less than 1% of total debt. This finding reflects the weak correlation of bond returns with macro factors, especially with movements in income inequality.

Besides fluctuations in income inequality, equation (31) shows that heterogeneity adds a term that depends on how ratios of the average Lagrange multipliers across agents vary across time. Movements in this ratio reflect differences in trading frictions across households. When non-traders have more volatile consumption (presumably because they have fewer avenues to smooth) than the traders, the government can use its debt portfolio to shift some risk from non-traders to traders and improve average welfare.

To get a sense of what heterogeneous trading frictions mean for the duration of an optimal portfolio, we capture the differences in consumption risk using a parsimonious formulation that sets $\ln(M_{\mathbb{N},T+t}) = (1+\psi)\ln(M_{\mathbb{T},T+t})$; the scalar parameter ψ is intended to measure strength of trading frictions. When non-traders face more risk, so that multiplier $\ln(M_{\mathbb{N},T+t})$ is more volatile than $\ln(M_{\mathbb{T},T+t})$, the parameter $\psi > 0$. Substituting into the definition of Σ_T^M we get

$$\Sigma_T^M[t,j] = -\psi\mu_{\mathbb{N},t} cov_T\left(\ln(M_{\mathbb{T},T+t}), r_{T+1}^j\right) \tag{32}$$

Equation (32) has several insights. It says that when $\psi > 0$, the government should borrow more using securities that have larger negative values of the covariance $cov_T\left(\ln(M_{\mathbb{T},T+t}), r_{T+1}^j\right)$. A security whose returns are low when marginal values of wealth are high are more "risky" from an investor's perspective. A strategy in which the government borrows more in such risky securities and invests more in (or lowers issuance of) the risk-free asset makes the overall public portfolio less risky. On the margin, it generates a welfare gain because it allows the government to lower the volatility of the non-traders after-tax incomes. When such risky securities are of longer duration (which is generally the case with long duration bonds), such a strategy would increase the duration of the public portfolio.

Although equation (32) is stated in terms of $M_{\mathbb{T},T+t}$, we can use the counterpart of equation (32) for the traders and rewrite it as

$$-\mathrm{cov}_T\left(\ln(M_{\mathbb{T},T+t}), r_{T+1}^j\right) \simeq \mathbb{E}_T r_{T+1}^j - \mathrm{cov}_T\left(\ln Q_{T+1,t-1}, r_{T+1}^j\right) + \mathrm{cov}_T\left(\ln A_{T+1,t-1}, r_{T+1}^j\right),$$

where all the terms on the right-hand side can be measured from return data that we used in Section (4). In the appendix, we use estimates from our Section 4.2 factor model to quantify those terms for a special case in which the government trades a risk-free and a growth-adjusted consol.

5.8 Responses of prices to government policies

In this section, we investigate the implications of relaxing the assumption that government trades have no effect on asset prices.

Two broad classes of price determination models are commonly used in the literature that studies government portfolio: closed economy models (Lucas and Stokey (1983); Angeletos (2002); Debortoli et al. (2017); Faraglia et al. (2018)) in which a representative household prices all securities, and various models of segmented markets or preferred habitats (Greenwood and Vayanos (2014); Koijen and Yogo (2019); Bigio et al. (2019)) in which a group of outside investors prices assets. We focus here on preferred habitat models and utilize their simplicity and flexibility in matching data. We leave analysis of closed economy models to the online appendix.[27]

We build on Greenwood and Vayanos (2014) (GV) in which all marginal changes in a government portfolio are absorbed by outside investors that they call "arbitrageurs" and who are short-lived, risk-averse, and optimally choose their holdings of government debts of different maturities. Following GV, assume that (a) foreign demand for the risk-free bond is perfectly elastic, and (b) prices of all securities and supplies of government bonds satisfy

$$\ln Q_t^i = \lambda\left[i\right] - \Lambda\left[i,j\right] B_t^j, \tag{33}$$

where i refers to all securities, $j \in \mathcal{G}_t \setminus \{0\}$ are government bonds, and (c) price impacts operate by changing compensation for duration risk, which means that $\overline{\Lambda} = \partial_\sigma \Lambda = \mathbf{0}$.

Our perturbation remains the same as in Section 3.2. The envelope theorem implies that our perturbation affects welfare because it changes taxes and asset prices. In the benchmark economy without price effects, those welfare effects were summarized by $\left(\frac{M_{T+1}}{\xi_{T+1}}\right) r_{T+1}^j$. The counterpart of government optimality in preferred habitat models is

$$0 = \vartheta_T^j + \mathbb{E}_T \frac{\beta M_{T+1}}{M_T} r_{T+1}^j \left(\frac{1}{\xi_{T+1}}\right), \tag{34}$$

where

$$\begin{aligned}\vartheta_T^j \equiv \frac{1}{\xi_T\left(s^T\right)} \sum_i \partial_{j,T,t,\epsilon} Q_T^i \left(B_T^i - B_{T-1}^{i+1}\right) + \sum_{i\geq 1} \partial_{j,T,t,\epsilon} Q_T^i \left(b_{T-1}^i - b_T^i\right) \\ + \sum_i \left(1 - \frac{1}{A_T^i}\right) b_T^i\left(s^T\right) \partial_{j,T,t,\epsilon} Q_T^i\left(s^T\right).\end{aligned}$$

[27] In a closed economy, a perturbation of portfolio at some history s^T affects prices at all past and future histories. In online appendix C, we show how to adapt the variational approach of Section 3.2 to such settings. There we derive a formula for the target portfolio and also show that price responses in the closed economy model are inconsistent with their empirical counterparts.

In preferred habitat models there are two additional effects. The first is an effect on government revenues from changing bond prices and the consequent change in taxes. This effect is given by $\left(\frac{M_T}{\xi_T}\right)\left(\sum_i \partial_{j,T,t,\epsilon} Q_T^i \left(B_T^i - B_{T-1}^{i+1}\right)\right)$, where $\partial_{j,T,t,\epsilon} Q_T^i$ tells how much the price of security i is affected by the perturbation. A second effect instigated by a price response comes through private sector decisions and is given by a sum of two terms: (i) $M_T \sum_i \partial_{j,T,t,\epsilon} Q_T^i \left(b_{T-1}^i - b_T^i\right)$, which is a household analog of the effect on income coming from changes in asset prices, and (ii) $M_T \sum_i \left(1 - \frac{1}{A_T^i}\right) b_T^i \left(s^T\right) \partial_{j,T,t,\epsilon} Q_T^i \left(s^T\right)$, which captures effects on direct utilities provided by asset holdings and trading frictions.

Implications of optimality condition (34) largely parallel those in our Section 3 analysis of equation (12). To illustrate the additional insights concisely, we focus on the stationary economy and further assume that domestic households' portfolio of government debts are described by a rule $\phi_T B_{i,T} \approx b_{i,T}$ so that ϕ_T is the fraction of the public debt held by domestic households.

Given our timing assumptions, we refer to $\overrightarrow{\omega}_T$ as *the end of t period portfolio* to distinguish it from what we shall define as a *beginning of period portfolio*. A *beginning of period portfolio* was chosen in the previous period but is evaluated at current prices; we denote it using a vector $\overrightarrow{\omega}_T^+$ with elements $\omega_T^+[i] = \frac{Q_T^i B_{T-1}^{i+1}}{B_T}$. Let $\Lambda_T^{QE}[i,j] \equiv Y_T \partial_{j,T,t,\epsilon} \ln Q_T^i \approx Y_T \Lambda[i,j] \left(\frac{\Gamma}{\hat{\beta}}\right)^j$ be a transformation of $\Lambda[i,j]$ that allows us to express it as semi-elasticities of bond prices with respect to a change in the ratio of the value of government debts to GDP. We have

Corollary 5. *Let $\overrightarrow{\omega}_T^*$ be the target portfolio when price effects are zero. In our preferred habitat benchmark economy with $\phi_T B_{i,T} \approx b_{i,T}$, the optimal portfolio of government bonds satisfies*

$$\overrightarrow{\omega}_T \simeq \overrightarrow{\omega}_T^* - (1 - \phi_T \xi_T) \pi_T^{QE} \Sigma_T^{-1} \Lambda_T^{QE} \left(\overrightarrow{\omega}_T - \overrightarrow{\omega}_{T-1}^+\right) \tag{35}$$

with

$$\pi_T^{QE} = \left(\frac{\Gamma \zeta_T}{1 - \hat{\beta}}\right) \left(\frac{\Gamma}{\hat{\beta}}\right)^2.$$

Corollary 5 indicates that with price impacts an optimal portfolio consists of the Section 3 target portfolio $\overrightarrow{\omega}_T^*$ plus an additional term that comes from costs of portfolio rebalancing and is proportional to $\Lambda_T^{QE}\left(\overrightarrow{\omega}_T - \overrightarrow{\omega}_{T-1}^+\right)$. The gap $\overrightarrow{\omega}_T - \overrightarrow{\omega}_{T-1}^+$ captures the magnitude of portfolio rebalancing in period T; we scale that gap with the matrix of price elasticities Λ_T^{QE} to get the costs of rebalancing. Formula (35) shows that how price responses imply deviations from the target portfolio.

When price effects are large, target and optimal portfolios can have complicated dynamics. However, if $\overrightarrow{\omega}_T^*$ consists mainly of the interest rate risk, which Section 4 shows to be the empirically relevant case for the U.S., then Corollary 5 has the following sharp implication about an optimal public portfolio.

Corollary 6. *Suppose that* $\overrightarrow{\omega}^*_T = (1-\beta)\overrightarrow{\hat{\beta}}$. *Then the optimal portfolio of bonds in a stationary preferred habitat economy satisfies*

$$\overrightarrow{\omega}_T \approx \overrightarrow{\omega}^*_T.$$

A remarkable aspect of Corollary 6 is that an optimal portfolio does not depend on magnitudes of price responses to government portfolio adjustments. To interpret the economic force behind this outcome, recall from Section 4 that interest rate risk is best hedged by constructing a portfolio that minimizes rebalancing. In stationary environments, such a portfolio sets $\overrightarrow{\omega}_T - \overrightarrow{\omega}^+_{T-1} = \mathbf{0}$; so the last term in equation (35) disappears for any value of Λ^{QE}.

Magnitudes of price effects. The Corollary 6 condition holds only approximately in our Section 4 estimates. To reassess our conclusions about the target portfolio, we next use GV's estimates of price impacts of QE type policies.

In the GV model, price impact on government debts of different maturities are functions of only one statistic, the duration defined as $D_t = B^0_T + \sum_{i=1}^{\infty}(i+1)B^i_T$, and take an affine form

$$\ln Q^i_t = \lambda_0 + \lambda^i D_t, \tag{36}$$

where λ_0 and $\left\{\lambda^i\right\}_{i=1}^{\infty}$ are parameters. GV use instrumental variables to estimate

$$\ln \text{yield}^i_t = b^0 + b^i \frac{D_t}{\text{GDP}} + \text{controls}_t + \text{noise}^i_t, \tag{37}$$

where $\ln \text{yield}^i_t \equiv -\left(\frac{1}{i+1}\right)\ln Q^i_t$. GV infer point estimates $b^i \approx [\frac{.2}{100}, \frac{.4}{100}]$ that imply that a one GDP decrease in maturity-weighted debt[28] would bring a 30 to 40 basis point decrease in yields. That finding is consistent with the view that QE actions lower duration risk borne by bond investors, with that lower duration risk being accompanied by lower term spreads.[29]

It is easy to recover a matrix Λ^{QE}_T from $\left\{b^i\right\}_i$. Using the definition of yields and the fact that GV assume that government bond prices are functions only of the duration of the government portfolio (36), it turns out that in a stationary economy

$$\Lambda^{QE}_T[i,j] = (i+1)\times b^i \left[\frac{j+1}{[Q^0]^j} - \frac{1}{Q^0_T}\right]. \tag{38}$$

[28] For example, in a typical QE the government simultaneously buys long-maturity debt and issues short maturity debt.

[29] Although GV don't explicitly test $\beta^1 = 0$, their theory implies that prices of risk-free debt are unaffected by duration risk; they include risk-free debt prices among their controls in (37)

We use the GV point estimates of b^i for their reported maturities and extrapolate to other maturities by fitting the functional form for factor loadings that we used in our benchmark model.[30] In the appendix we report the fit and a heatmap of Λ^{QE} (all normalized by its mean value) computed using equation (38) and the extrapolated $\{b^n\}$ sequence. Price impacts are large when securities involved are both of longer maturities. We set $\phi_T = 0.7$ to obtain a domestic share of US debt at 70%, an estimate that we obtain from FRED.

We can now describe the implied optimal portfolio of public debts using equations (35). We use Section 4.2 parameter estimates that imply that $\omega^*_{T,}$, π^{QE}_{T+t}, and Σ_{T+t} are all independent of t. Still, formula (35) prescribes a non-trivial dependence of portfolio $\overrightarrow{\omega}_T$ on portfolio $\overrightarrow{\omega}_{T-1}$. We focus on the following stationary point of equation (35),

$$\overrightarrow{\boldsymbol{\omega}} = \left(I + (1 - \phi\xi) \left(\frac{\Gamma\zeta}{1 - \overline{Q}^0 \Gamma} \right) \left(\frac{1}{\overline{Q}^0} \right)^2 \Sigma^{-1} \Lambda^{QE} \left(I - L^+ \right) \right)^{-1} \overrightarrow{\boldsymbol{\omega}}^*,$$

where matrix L^+ is such that $\overrightarrow{\omega}^+ = L^+ \overrightarrow{\omega}$.

Figure 4 reports the optimal capped portfolio $\overrightarrow{\boldsymbol{\omega}}$ in our preferred habitat model and compares it to an optimal capped portfolio in the Section 4 small open economy. Evidently, incorporating price impacts increases holdings of short maturities and decreases holding of long maturities. This maturity tilting reflects that the GV calibration provides larger price impacts at longer maturities. The government now faces a tradeoff with respect to longer maturities. Issuing long maturities can help to hedge interest rate risk but requires keeping the share of debt in those roughly constant. If the number of available maturities is capped, frequent balancing is required. Since costs of rebalancing are larger for longer maturities, the planner tilts the portfolio toward shorter maturities. However, the associated decline in the Macaulay duration is only about 6 months. So the two portfolios appear to be quite similar. The economics underlying this outcome flows from Corollary 6: empirically $\overrightarrow{\omega}^*_T$ is close to $\left(1 - \hat{\beta}\right) \overrightarrow{\hat{\beta}}$, so price responses have small effects on an optimal public portfolio.

[30] In particular, Table 2 of GV reports estimates for bonds of maturities $2, 3, 4, 5, 10$ years. We assume that $b^n = \overline{b}_0 + \overline{b}_0 \exp\left(-\overline{b}_1 \times n\right)$ and find coefficients $\left\{\overline{b}_0, \overline{b}_1\right\}$ that minimize squared errors. Our results are robust to other extrapolation schemes.

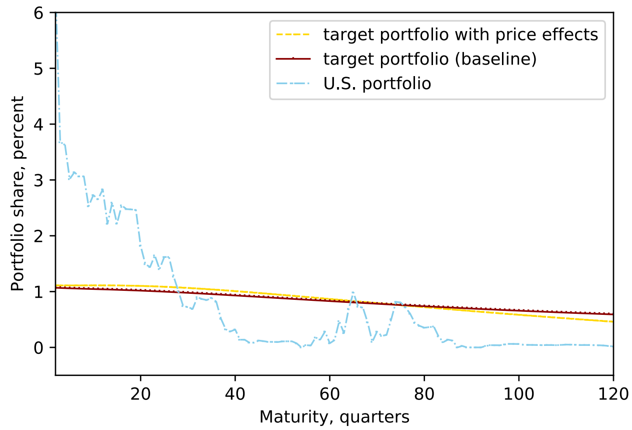


Figure 4: Portfolio shares of securities with maturities from 2 quarters to 120 quarters. The red line is the optimal portfolio without price effects, and the orange line is the optimal portfolio with price effects. The blue line plots the 2017 U.S. federal debt portfolio (see Appendix B.1.1 for data sources and construction).

6 Conclusion

We have proposed an analytical framework that includes a broad class of dynamic stochastic equilibrium models containing various heterogeneities among households, limits on market participations, sources of liquidity and stochastic discount factors. We show how to characterize policies that transcend details of particular models in this class with a small number of statistics that are functions only of asset prices and macroeconomic variables. We have used small-noise expansions to characterize and approximate optimal public portfolios in terms of those statistics. For U.S. data, we find that an optimal portfolio is simple and stable over time, and that it approximately replicates a growth-adjusted consol. We show that the source of differences between our findings and those provided by earlier computations of optimal public portfolios in neoclassical models comes from features of those earlier models that cause them to misrepresent covariances of asset returns with macroeconomic aggregates.

This paper focuses exclusively on economies in which a government commits and does not default. A natural next step is to alter those assumptions by proceeding along lines advocated by Arellano and Ramanarayanan (2012), Aguiar et al. (2019), Bocola and Dovis (2019), and others. We hope to take steps in those directions next.

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Online Appendix

A Appendix: Theoretical analysis

A.1 Proof of Lemma 1

Proof. Let $\beta^t \Pr\left(s^t\right) M\left(s^t\right) \mathfrak{N}\left(s^t\right)$ be the vector of Lagrange multipliers on constraints (5) and let $\mathbb{W}_{t,s_k}\left(V_{t+1}\right)$ be the derivative of $\mathbb{W}_t$ with respect to $V_{t+1}\left(s^t, s_k\right)$ for any s_k. The optimality condition for $b_t^i\left(s^t\right)$ in any competitive equilibrium can be written as

$$\left[\left(\frac{\mathbb{W}_{0,s_1} \times ... \times \mathbb{W}_{t-1,s_t}}{\Pr\left(s^t\right) M_t\left(s^t\right)}\right) \frac{\partial U_t\left(s^t\right)}{\partial\left(Q_t^i b_t^i\right)} + \mathfrak{N}\left(s^t\right) \cdot \frac{\partial \varphi_t\left(s^t\right)}{\partial\left(Q_t^i b_t^i\right)} - 1\right] \tag{39}$$
$$+ \beta \sum_{s^{t+1}|s^t} \Pr\left(s^{t+1}|s^t\right) \frac{M_{t+1}\left(s^{t+1}\right)}{M_t\left(s^t\right)} R_{t+1}^i\left(s^{t+1}\right) = 0,$$

where $s^t = (s_1, .., s_t)$. When government securities are perfect substitutes, $\frac{\partial U_t\left(s^t\right)}{\partial\left(Q_t^i b_t^i\right)}$ and $\frac{\partial \varphi_t\left(s^t\right)}{\partial\left(Q_t^i b_t^i\right)}$ are the same for all $i \in \mathcal{G}_t$. Therefore, this equation implies that $A_t^i = A_t^0$ and

$$\mathbb{E}_t \frac{\beta M_{t+1}}{M_t} r_{t+1}^j = 0 \, \text{for all } t, j \in \mathcal{G}_t \tag{40}$$

in any competitive equilibrium. □

A.2 Proofs for Section 3.2

Let $x_t\left(\sigma\right)$ be any equilibrium variable in the σ-economy. We use second order Taylor expansions of the equilibrium conditions with respect to σ around $\sigma = 0$. Let $\overline{x}_t, \partial_\sigma x_t, \partial_{\sigma\sigma} x_t$ be the zeroth-, first- and second-order terms in these expansions. In this notation,

$$x_t\left(\sigma\right) \simeq \overline{x}_t + \sigma \partial_\sigma x_t + \frac{\sigma^2}{2} \partial_{\sigma\sigma} x_t, \quad x_t\left(\sigma\right) \approx \overline{x}_t.$$

Note that the statement that $x_t \approx 0$ is equivalent to $\overline{x}_t = 0$. We first show several preliminary results that will be used throughout this section.

Lemma 2. *In the optimal equilibrium in the benchmark economy,* $\overline{r}_{T+1}^j = \mathbb{E}_T \partial_\sigma r_{T+1}^j = 0$ *for all* $T, j \in \mathcal{G}_T$*, which also implies that*

$$\overline{\mathcal{Q}}_{T+1,t} = \overline{Q}_{T+1,t}, \quad \mathbb{E}_{T+1} \partial_\sigma \mathcal{Q}_{T+1,t} = \mathbb{E}_{T+1} \partial_\sigma Q_{T+1,t} \text{ for } T, t. \tag{41}$$

Proof. We first show that $\overline{r}^j_{T+1} = \mathbb{E}_T \partial_\sigma r^j_{T+1} = 0$ for all $T, j \in \mathcal{G}_T$ under conditions of the lemma. The zeroth order expansion of equation (12) for $t = 1$ is

$$\overline{M}_{T+1} \frac{\overline{r}^j_{T+1}}{\overline{\xi}_{T+1}} = 0 \text{ for all } T, j \in \mathcal{G}_T. \tag{42}$$

Neither $\overline{M}_{T+1}$ nor $\overline{\xi}_{T+1}$ can be zero, which implies that $\overline{r}^j_{T+1} = 0$. Using this result, the first-order approximation of equation (12) for $t = 1$ is

$$\overline{M}_{T+1} \frac{\overline{r}^j_{T+1}}{\overline{\xi}_{T+1}} \mathbb{E}_T \partial_\sigma r^j_{T+1} = 0 \, \text{for all } T, j \in \mathcal{G}_T,$$

which implies that $\mathbb{E}_T \partial_\sigma r^j_{T+1} = 0$. Applying this result to first-order expansions of $Q_{t,k}$ and $\mathcal{Q}_{t,k}$ in equation (8) gives (41). □

The previous lemma also implies the following useful corollary.

Corollary 7. *For any equilibrium variables x_t, z'_t, z''_t in the optimal equilibrium in the benchmark economy, the following relationship holds for any T, $j \in \mathcal{G}_T$.*

$$\mathbb{E}_T \left[z'_{T+1} z''_{T+1} \right] cov_T \left(x_{T+1}, r^j_{T+1} \right) \simeq \mathbb{E}_T \left[z'_{T+1} \right] \mathbb{E}_T \left[z''_{T+1} \right] cov_T \left(x_{T+1}, r^j_{T+1} \right) \simeq \overline{z}'_{T+1} \overline{z}''_{T+1} \mathbb{E}_T \partial_\sigma x_{T+1} \partial_\sigma r^j_{T+1}.$$

Proof. Using Lemma 2, we have

$$\mathbb{E}_T \left[x_{T+1} r^j_{T+1} \right] \simeq \mathbb{E}_T \left[\partial_\sigma x_{T+1} \partial_\sigma r^j_{T+1} + \frac{1}{2} \overline{x}_{T+1} \partial_{\sigma\sigma} r^j_{T+1} \right], \; \left(\mathbb{E}_T x_{T+1} \right) \left(\mathbb{E}_T r^j_{T+1} \right) \simeq \mathbb{E}_T \left[\frac{1}{2} \overline{x}_{T+1} \partial_{\sigma\sigma} r^j_{T+1} \right],$$

and, therefore,

$$cov_T \left(x_{T+1}, r^j_{T+1} \right) = \left[\mathbb{E}_T x_{T+1} r^j_{T+1} - \mathbb{E}_T x_{T+1} \mathbb{E}_T r^j_{T+1} \right] \simeq \mathbb{E}_T \partial_\sigma x_{T+1} \partial_\sigma r^j_{T+1}.$$

Since $\overline{cov_T \left(x_{T+1}, r^j_{T+1} \right)} = \partial_\sigma cov_T \left(x_{T+1}, r^j_{T+1} \right) = 0$, we obtain

$$\begin{aligned} \mathbb{E}_T \left[z'_{T+1} z''_{T+1} \right] cov_T \left(x_{T+1}, r^j_{T+1} \right) &\simeq \overline{z}'_{T+1} \overline{z}''_{T+1} \mathbb{E}_T \partial_\sigma x_{T+1} \partial_\sigma r^j_{T+1}, \\ \mathbb{E}_T \left[z'_{T+1} \right] \mathbb{E}_T \left[z''_{T+1} \right] cov_T \left(x_{T+1}, r^j_{T+1} \right) &\simeq \overline{z}'_{T+1} \overline{z}''_{T+1} \mathbb{E}_T \partial_\sigma x_{T+1} \partial_\sigma r^j_{T+1}. \end{aligned}$$

□

Lemma 3. *Equation (14) holds in the optimal equilibrium of the benchmark economy.*

Proof. The second-order expansion of (13), invoking results of Lemma 2, gives

$$\begin{aligned} 0 &= \frac{1}{2} \left[\overline{\frac{M_{T+t}}{Q^{pvt}_{T+1,t-1}}} \right] \mathbb{E}_T \partial_{\sigma\sigma} r^j_{T+1} + \left[\overline{\frac{M_{T+t}}{Q^{pvt}_{T+1,t-1}}} \right] \mathbb{E}_T \partial_\sigma \ln M_{T+t} \partial_\sigma r^j_{T+1} - \left[\overline{\frac{M_{T+t}}{Q^{pvt}_{T+1,t-1}}} \right] \mathbb{E}_T \partial_\sigma \ln Q^{pvt}_{T+1,t-1} \partial_\sigma r^j_{T+1} \\ &= \frac{1}{2} \mathbb{E}_T \partial_{\sigma\sigma} r^j_{T+1} + \mathbb{E}_T \partial_\sigma \ln M_{T+t} \partial_\sigma r^j_{T+1} - \mathbb{E}_T \partial_\sigma \ln Q^{pvt}_{T+1,t-1} \partial_\sigma r^j_{T+1}. \end{aligned}$$

Similarly, the second-order expansion of (12) gives

$$0=\frac{1}{2}\mathbb{E}_T\partial_{\sigma\sigma}r^j_{T+1}+\mathbb{E}_T\partial_\sigma\ln M_{T+t}\partial_\sigma r^j_{T+1}-\mathbb{E}_T\partial_\sigma\ln Q_{T+1,t-1}\partial_\sigma r^j_{T+1}-\mathbb{E}_T\partial_\sigma\ln\xi_{T+1}\partial_\sigma r^j_{T+1}.$$

Combine these two equations and use the fact that $\ln A_{T+1,t-1}=\ln Q_{T+1,t-1}-\ln Q^{pvt}_{T+1,t-1}$ to get

$$\mathbb{E}_T\partial_\sigma\ln\xi_{T+1}\partial_\sigma r^j_{T+1}=-\mathbb{E}_T\partial_\sigma\ln A_{T+1,t-1}\partial_\sigma r^j_{T+1}.$$

This equation implies (14) by Corollary 7. □

Lemma 4. *Equation (15) holds in the optimal equilibrium of the benchmark economy.*

Proof. Lemma 2 implies that the zeroth order approximation of equation (9) is

$$\sum_{t=1}^{\infty}\overline{Q}_{T+1,t}\overline{X}_{T+t-1}=\overline{B}_T\overline{R}^0_{T+1}. \tag{43}$$

Multiply equation (9) by r^j_{T+1} and take expectations at time T. The Law of Iterated Expectations implies

$$\mathbb{E}_T\sum_{t=1}^{\infty}\mathcal{Q}_{T+1,t-1}X_{T+t}r^j_{T+1}=\mathbb{E}_TB_T\left[R^0_{T+1}+\sum_{i\geq1}\omega^i_Tr^i_{T+1}\right]r^j_{T+1}. \tag{44}$$

Take the second-order expansion of equation (44), note that the terms multiplying $\partial_{\sigma\sigma}r^j_{T+1}$ cancel out due to (43) and that $\mathbb{E}_T\partial_\sigma\mathcal{Q}_{T+1,t-1}\partial_\sigma r^j_{T+1}=\mathbb{E}_T\left(\mathbb{E}_{T+1}\partial_\sigma\mathcal{Q}_{T+1,t-1}\right)\partial_\sigma r^j_{T+1}=\mathbb{E}_T\partial_\sigma Q_{T+1,t-1}\partial_\sigma r^j_{T+1}$ by Lemma 2 to obtain

$$\mathbb{E}_T\sum_{t=1}^{\infty}\overline{X}_{T+t}\partial_\sigma Q_{T+1,t-1}\partial_\sigma r^j_{T+1}+\mathbb{E}_T\sum_{t=1}^{\infty}\overline{Q}_{T+1,t-1}\partial_\sigma X_{T+t}\partial_\sigma r^j_{T+1}=\overline{B}_T\mathbb{E}_T\left[\sum_{i\geq1}\overline{\omega}^i_T\partial_\sigma r^i_{T+1}\partial_\sigma r^j_{T+1}\right]. \tag{45}$$

Finally, note that $Q_{T+1,0}=1$ and, therefore, $\partial_\sigma Q_{T+1,0}=0$. Together with Corollary 7 this establishes equation (15). □

Lemma 5. *Equation (16) holds in the optimal equilibrium of the benchmark economy.*

Proof. From (7), we have $\ln Y_t=\ln\theta_t+\gamma\ln(1-\tau_t)$, therefore $\ln Y^\perp_t=\ln\theta_t$. Thus, using Corollary 7, we have

$$cov_T\left(X^\perp_{T+t},r^j_{T+1}\right)\simeq\overline{\tau_{T+t}Y_{T+t}}\mathbb{E}_T\partial_\sigma\ln Y^\perp_{T+t}\partial_\sigma r^j_{T+1}-\overline{G}_{T+t}\mathbb{E}_T\partial_\sigma\ln G_{T+t}\partial_\sigma r^j_{T+1}.$$

Similarly, $X_{T+t} = \tau_{T+t}Y_{T+t} - G_{T+t}$ and thus

$$\begin{aligned} cov_T\left(X_{T+t}, r^j_{T+1}\right) &\simeq \overline{\tau_{T+t}Y_{T+t}}\Bigg[\mathbb{E}_T\partial_\sigma \ln Y^\perp_{T+t}\partial_\sigma r^j_{T+1} - \frac{\gamma}{1-\overline{\tau}_{T+t}}\mathbb{E}_T\partial_\sigma\tau_{T+t}\partial_\sigma r^j_{T+1} \\ &+\frac{1}{\overline{\tau}_{T+t}}\mathbb{E}_T\partial_\sigma\tau_{T+t}\partial_\sigma r^j_{T+1}\Bigg] - \overline{G}_{T+t}\mathbb{E}_T\partial_\sigma \ln G_{T+t}\partial_\sigma r^j_{T+1} \\ &\simeq cov_T\left(X^\perp_{T+t}, r^j_{T+1}\right) + \overline{Y}_{T+t}\left(\frac{1-\overline{\tau}_{T+t}-\gamma\overline{\tau}_{T+t}}{1-\overline{\tau}_{T+t}}\right)\mathbb{E}_T\partial_\sigma\tau_{T+t}\partial_\sigma r^j_{T+1}. \end{aligned}$$

Direct calculations show that $\partial_\sigma \ln \xi_{T+t} = -\frac{\gamma}{(1-\overline{\tau}_{T+t}-\gamma\overline{\tau}_{T+t})(1-\overline{\tau}_{T+t})}\partial_\sigma\tau_{T+t}$. Therefore, we have

$$\frac{1-\overline{\tau}_{T+t}-\gamma\overline{\tau}_{T+t}}{1-\overline{\tau}_{T+t}}\mathbb{E}_T\partial_\sigma\tau_{T+t}\partial_\sigma r^j_{T+1} = -\frac{\left(1-\overline{\tau}_{T+t}-\gamma\overline{\tau}_{T+t}\right)^2}{\gamma}\mathbb{E}_T\partial_\sigma \ln \xi_{T+t}\partial_\sigma r^j_{T+1}.$$

Combine with the previous equation, use definition of ζ_{T+t} and Corollary 7 to obtain (16). $\square$

We need to prove the following result before proceeding to study stationary economy.

Lemma 6. *In any equilibrium $\overline{\mathbb{W}_{t,s_k}\left(V_{t+1}\right)} = \Pr\left(s_k|s^t\right)$ and, therefore, in any equilibrium the benchmark economy* $\left(\overline{\frac{1}{R^{j,pvt}_{t+1}}}\right) = \frac{\overline{\delta}_{t+1}}{\overline{\delta}_t}\frac{U_c\left(\overline{c}_{t+1}-\frac{1}{\overline{\theta}^{1/\gamma}_{t+1}}\frac{\overline{y}^{1+1/\gamma}_{t+1}}{1+1/\gamma},\left\{\overline{Q}^i_{t+1}\overline{b}^i_{t+1}\right\}_{i\in\mathcal{G}_{t+1}},\overline{G}_{t+1}\right)}{U_c\left(\overline{c}_t-\frac{1}{\overline{\theta}^{1/\gamma}_t}\frac{\overline{y}^{1+1/\gamma}_t}{1+1/\gamma},\left\{\overline{Q}^i_t\overline{b}^i_t\right\}_{i\in\mathcal{G}_t},\overline{G}_t\right)}$ *for all $t \geq T$.*

Proof. Consider any random variable $\epsilon = \{\epsilon(s_l)\}_l$ with $\sum_l \Pr\left(s_l|s^t\right)\epsilon(s_l) = 0$. Let $F_t(\sigma) \equiv \mathbb{W}_t\left(\{\overline{x}+\sigma\epsilon_l\}_l\right)$. Its derivatives is $F'_t(0) = \sum_{s_k}\mathbb{W}_{t,s_k}\left(\{\overline{x}\}_l\right)\epsilon(s_k)$. Since $\mathbb{W}$ is increasing in the second-order stochastic dominance, $F'_t(0) \leq 0$. Together with $\sum_l \Pr\left(s_l|s^t\right)\epsilon(s_l) = 0$ then the first condition can be written as

$$\sum_{s_k}\left[\mathbb{W}_{t,s_k}\left(\{\overline{x}\}_l\right) - \Pr\left(s_k|s^t\right)\right]\epsilon(s_k) \leq 0.$$

Since ϵ is arbitrary, $\mathbb{W}_{t,s_k}\left(\{\overline{x}\}_l\right) = \Pr\left(s_k|s^t\right)$.

The consumption optimality condition for household is

$$\mathbb{W}_{0,s_1} \times ... \times \mathbb{W}_{t-1,s_t} \times U_{c,t}\left(s^t\right) = \Pr\left(s^t\right)M_t\left(s^t\right),$$

where $s^t = (s_1, .., s_t)$, which implies that

$$\overline{\delta}_{T+t}U_c\left(\overline{c}_{T+t} - \frac{1}{\overline{\theta}^{1/\gamma}_{T+t}}\frac{\overline{y}^{1+1/\gamma}_{T+t}}{1+1/\gamma}, \left\{\overline{Q}^i_{T+t}\overline{b}^i_{T+t}\right\}_{i\in\mathcal{G}_{T+t}}, \overline{G}_{T+t}\right) = \overline{M}_{T+t}.$$

Since $R^{j,pvt}_{T+t}$ satisfies $\left(\overline{\frac{1}{R^{j,pvt}_{T+t+1}}}\right) = \frac{\beta\overline{M}_{T+t+1}}{\overline{M}_{T+t}}$, for all j,we obtain the result of the lemma. $\square$

We are now ready to show the properties of stationary economy.

Lemma 7. *In a stationary optimal equilibrium of the benchmark economy, for all* $t \geq 1$, $\tau_{T+t} \approx \tau_T$, $Q_{T+t} \approx \beta \Gamma^{-1/IES}$,

$$\mathbb{E}_T \frac{X_{T+t+1}}{X_{T+t}} \approx \mathbb{E}_T \frac{Y_{T+t+1}}{Y_{T+t}} \approx \mathbb{E}_T \frac{B_{T+t+1}}{B_{T+t}} \approx \Gamma,$$

and $\mathbb{E}_T \frac{X_{T+t}}{Y_{T+t}} \approx \frac{(1-\hat{\beta})}{\hat{\beta}} \frac{B_T}{Y_T}$ *where* $\hat{\beta} = \beta \Gamma^{1-1/IES}$.

Proof. Since $A_t^{j,pvt} = 1$ for all j, t,

$$1 = \overline{\left(\frac{\beta M_{T+t+1}}{M_{T+t}}\right)} \overline{R}^{j,pvt}_{T+t+1} = \overline{\left(\frac{\beta M_{T+t+1}}{M_{T+t}}\right)} R \text{ for all } j, t$$

by condition (iii) of the definition of stationarity. Therefore, equations (11) and (12) imply

$$\frac{1}{\overline{\xi}_{T+t}} = \overline{\left(\frac{\beta M_{T+t+1}}{M_{T+t}}\right)} R \frac{1}{\overline{\xi}_{T+t+1}} \text{ for all } t,$$

which, in turn, implies $\overline{\tau}_{T+t} = \overline{\tau}_T$ for all t. The optimality condition of households (7) and condition (i) then implies that $\overline{\left(\frac{Y_{T+t+1}}{Y_{T+t}}\right)} = \Gamma$ and, therefore, $\overline{\left(\frac{X_{T+t+1}}{X_{T+t}}\right)} = \Gamma$ for all t.

Let $Q \equiv 1/R$. The government budget constraint (9) is $\overline{B}_T = \sum_{t=1}^{\infty} Q^t \overline{X}_{T+t} = \frac{Q\Gamma}{1-Q\Gamma} \overline{X}_T$, which implies that $\overline{\left(\frac{B_{T+t+1}}{B_{T+t}}\right)} = \Gamma$ and

$$\overline{\left(\frac{X_{T+t}}{Y_{T+t}}\right)} = \left(\frac{1-Q\Gamma}{Q\Gamma}\right) \overline{\left(\frac{B_{T+t}}{Y_{T+t}}\right)} = \left(\frac{1-Q\Gamma}{Q\Gamma}\right) \overline{\left(\frac{B_T}{Y_T}\right)}. \tag{46}$$

The household optimality condition for c_t in the benchmark economy is

$$\mathbb{W}_{0,s_1} \times ... \times \mathbb{W}_{t-1,s_t} \times \delta_t \left(s^t\right) U_c \left(s^t\right) = \Pr \left(s^t\right) M_t \left(s^t\right).$$

Lemma 6 implies that the zeroths order approximation of this equation is

$$\overline{\delta}_{T+t} U_c \left(\overline{c}_{T+t} - \frac{1}{\overline{\theta}_{T+t}^{1/\gamma}} \frac{\overline{y}_{T+t}^{1+1/\gamma}}{1+1/\gamma}, \left\{\overline{b}^i_{T+t}\right\}_{i \in \mathcal{G}_t}, \overline{G}_{T+t}\right) = \overline{M}_{T+t}$$

and, therefore, by the properties (iii) and (iv) and the definition of IES we have

$$Q = \overline{\left(\frac{\beta M_{T+t+1}}{M_{T+t}}\right)} = \beta \Gamma^{-1/IES}.$$

Thus, $Q\Gamma = \beta \Gamma^{1-1/IES}$. This, together with definition of $\hat{\beta}$ and (46), implies

$$\overline{\left(\frac{X_{T+t}}{Y_{T+t}}\right)} = \frac{1-\hat{\beta}}{\hat{\beta}} \overline{\left(\frac{B_T}{Y_T}\right)}.$$

The statement of the lemma then follows from applying Corollary 7. □

Note that the only place where we used condition (iv) in the proof is in deriving expression Q in terms of growth rates of real variables and obtaining weights $\hat{\beta}$ in terms of growth rates of real variables on balanced growth path. Condition (iv) is necessary for a balanced growth path but not for our results. For concreteness, suppose that U is separable in the first argument and let $u_t \equiv c_t - \frac{1}{\theta_t^{1/\gamma}} \frac{y_t^{1+1/\gamma}}{1+1/\gamma}$ and $\Gamma_t^u \equiv \left(\frac{u_t}{u_{t-1}}\right)$. Conditions (i)-(iii) imply that Γ_t^u is independent of t and so that $Q = \beta \left(\Gamma^u\right)^{-1/IES}$. Thus, the only thing that changes in the analysis is that we replace $\Gamma^{1-1/IES}$ with $\Gamma \times \left(\Gamma^u\right)^{-1/IES}$ and adjust the definition of $\hat{\beta}$ accordingly.

A.2.1 Proof of Corollary 1

Proof. Equation (17) can equivalently be written as

$$\sum_{t=1}^{\infty} \overline{Q}_T^0 \overline{Q}_{T+1,t-1} \left(\frac{\overline{Y}_{T+t}}{\overline{Y}_T}\right) \left(\frac{\overline{X}_{T+t}}{\overline{Y}_{T+t}}\right) \mathbb{E}_T \partial_\sigma \ln Q_{T+1,t-1} \partial_\sigma r_{T+1}^j + \sum_{t=1}^{\infty} \overline{Q}_T^0 \overline{Q}_{T+1,t-1} \left(\frac{\overline{Y}_{T+t}}{\overline{Y}_T}\right) \mathbb{E}_T \frac{\partial_\sigma X_{T+t}^\perp}{\overline{Y}_{T+t}} \partial_\sigma r_{T+1}^j$$
$$+ \sum_{t=1}^{\infty} \overline{Q}_T^0 \overline{Q}_{T+1,t-1} \left(\frac{\overline{Y}_{T+t}}{\overline{Y}_T}\right) \overline{\zeta}_{T+t} \mathbb{E}_T \partial_\sigma r_{T+1}^j \partial_\sigma \ln A_{T+1,t-1} = \overline{Q}_T^0 \frac{\overline{B}_T}{\overline{Y}_T} \mathbb{E}_T \left(\sum_{i \geq 1} \partial_\sigma r_{T+1}^j \partial_\sigma r_{T+1}^i \overline{\omega}_T^i\right).$$

By Lemma 7, in the stationary economy $\overline{Q}_T^0 \overline{Q}_{T+1,t-1} \left(\frac{\overline{Y}_{T+t}}{\overline{Y}_T}\right) = \hat{\beta}^t$, $\overline{\zeta}_{T+t} = \overline{\zeta}_T$ and $\frac{\overline{X}_{T+t}}{\overline{Y}_{T+t}} = \frac{1-\hat{\beta}}{\hat{\beta}} \frac{B_T}{Y_T}$. Substitute these expressions and re-arrange to obtain

$$\left(1 - \hat{\beta}\right) \sum_{t=1}^{\infty} \hat{\beta}^t \mathbb{E}_T \frac{\partial_\sigma \ln Q_{T+1,t}}{Q_T^0} \partial_\sigma r_{T+1}^j + \hat{\beta}^{-1} \Gamma \frac{\overline{Y}_T}{\overline{B}_T} \sum_{t=1}^{\infty} \hat{\beta}^t \mathbb{E}_T \frac{\partial_\sigma X_{T+t}^\perp}{\overline{Y}_{T+t}} \partial_\sigma r_{T+1}^j$$
$$+ \overline{\zeta}_T \Gamma \frac{\overline{Y}_T}{\overline{B}_T} \sum_{t=1}^{\infty} \hat{\beta}^t \mathbb{E}_T \partial_\sigma r_{T+1}^j \partial_\sigma \ln A_{T+1,t} = \mathbb{E}_T \left(\sum_{i \geq 1} \partial_\sigma r_{T+1}^j \partial_\sigma r_{T+1}^i \overline{\omega}_T^i\right).$$

Apply Corollary 7 and write in matrix form to get (18). □

A.2.2 Proof of Corollary 2

Corollary 2 follows from the following lemma.

Lemma 8. *Let Q_T^t, r_T^t be the period-T price and excess return of a pure discount bond that expires in period $T+t$. Then in the optimal equilibrium of the baseline economy*

$$cov_T\left(\ln Q_{T+1,t}^t, r_{T+1}^j\right) \simeq cov_T\left(\ln Q_{T+1,t}, r_{T+1}^j\right) \tag{47}$$

and

$$Q_T^0 cov_T\left(r_{T+1}^t, r_{T+1}^j\right) \simeq cov_T\left(\ln Q_{T+1,t}, r_{T+1}^j\right) \tag{48}$$

for all $j \in \mathcal{G}_T$. The latter implies $\Sigma_T \simeq \Sigma_T^Q$ when the government trades the full set of pure discount bonds.

Proof. From the definition of A_T^{t+1} and R_T^t, we have the following recursion: $Q_T^{t+1} = \mathbb{E}_T \frac{\beta M_{T+1}}{M_T} A_T^{t+1} Q_T^t$. This implies that price Q_T^{t+1} must satisfy

$$
\begin{aligned}
&= \mathbb{E}_{T+1}\left[\left(\frac{\beta^t M_{T+t}}{M_{T+1}}\right)\left(A_{T+1}^t \times A_{T+2}^{t-1} \times ... \times A_{T+t-1}^0\right)\right] \\
&= \mathbb{E}_{T+1}\left[\left(\frac{\beta^t M_{T+t}}{M_{T+1}}\right)\left(A_{T+1}^0 \times A_{T+2}^0 \times ... \times A_{T+t-1}^0\right)\right] \\
&= \mathbb{E}_{T+1}\left[\frac{\beta M_{T+2} A_{T+1}^0}{M_{T+1}} \times ... \times \frac{\beta M_{T+t} A_{T+t-1}^0}{M_{T+t-1}}\right],
\end{aligned}
$$

where the second equation follows form Lemma 1. Similarly, $Q_{T+1,t}$ is given by

$$
Q_{T+1,t} = \mathbb{E}_{T+1}\frac{\beta M_{T+2} A_{T+1}^0}{M_{T+1}} \times ... \times \mathbb{E}_{T+t-1}\frac{\beta M_{T+t} A_{T+t-1}^0}{M_{T+t-1}}.
$$

Using the Law of Iterated Expectations, we obtain

$$
\begin{aligned}
\mathbb{E}_T \partial_\sigma \ln Q_{T+1,t} \partial_\sigma r_{T+1}^j &= \mathbb{E}_T\left[\mathbb{E}_{T+1}\partial_\sigma \ln \frac{\beta M_{T+2} A_{T+1}^0}{M_{T+1}} \partial_\sigma r_{T+1}^j + ... + \mathbb{E}_{T+t-1}\partial_\sigma \ln \frac{\beta M_{T+t} A_{T+t-1}^0}{M_{T+1}} \partial_\sigma r_{T+1}^j\right] \\
&= \mathbb{E}_T\left[\partial_\sigma \ln \frac{\beta M_{T+2} A_{T+1}^0}{M_{T+1}} \partial_\sigma r_{T+1}^j + ... + \partial_\sigma \ln \frac{\beta M_{T+t} A_{T+t-1}^0}{M_{T+1}} \partial_\sigma r_{T+1}^j\right] \\
&= \mathbb{E}_T \partial_\sigma \ln Q_{T+1}^t \partial_\sigma r_{T+1}^j.
\end{aligned}
$$

This expression is equivalent to (47) by Corollary 7.

To show (48), first observe that $r_{T+1}^t = Q_{T+1}^t / Q_T^{t+1} - 1/Q_T^0$ and, therefore,

$$
\begin{aligned}
\mathbb{E}_T \partial_\sigma r_{T+1}^t \partial_\sigma r_{T+1}^j &= \left(\overline{\frac{Q_{T+1}^t}{Q_T^{t+1}}}\right)\mathbb{E}_T\left[\partial_\sigma \ln Q_{T+1}^t \partial_\sigma r_{T+1}^j - \partial_\sigma \ln Q_T^{t+1} \partial_\sigma r_{T+1}^j\right] - \left(\overline{\frac{1}{Q_T^0}}\right)\mathbb{E}_T \partial_\sigma \ln Q_T^0 \partial_\sigma r_{T+1}^j \\
&= \left(\overline{\frac{Q_{T+1}^t}{Q_T^{t+1}}}\right)\mathbb{E}_T\left[\partial_\sigma \ln Q_{T+1}^t \partial_\sigma r_{T+1}^j\right] = \left(\overline{\frac{1}{\beta M_{T+1} A_T^0 / M_t}}\right)\mathbb{E}_T\left[\partial_\sigma \ln Q_{T+1}^t \partial_\sigma r_{T+1}^j\right] \\
&= \left(\overline{\frac{1}{Q_T^0}}\right)\mathbb{E}_T\left[\partial_\sigma \ln Q_{T+1}^t \partial_\sigma r_{T+1}^j\right],
\end{aligned}
$$

where the second equation follows from the fact that $\partial_\sigma \ln Q_T^{t+1}$ and $\partial_\sigma \ln Q_T^0$ are measurable with respect to T and $\mathbb{E}_T \partial_\sigma r_{T+1}^j = 0$ by Lemma 2. This equation is equivalent to (48) by Corollary 7. $\square$

A.3 Nominal economy

We now describe a nominal version of the economy. Let P_t be the price level and suppose all securities are nominal. Security 0 now refers to a nominal one-period bond that pays one dollar next period. The household and government budget constraint in the nominal economy are

$$P_t c_t + \sum_i Q_t^i b_t^i = P_t y_t - \tau_t P_t y_t + \sum_i \left(Q_t^i + D_t^i\right) b_{t-1}^i$$

and

$$P_t X_t + \sum_{i \in \mathcal{G}_t} Q_t^i B_t^i = \sum_{i \in \mathcal{G}_{t-1}} \left(Q_t^i + D_t^i\right) B_{t-1}^i,$$

respectively. All returns and liquidity premia are now in nominal terms. It is irrelevant in the benchmark economy where U_t, φ_t, $\left\{\mathcal{B}_t^i\right\}_i$ are functions of nominal or real value of security holdings. The definition of competitive equilibrium and optimum competitive equilibrium remain unchanged except in nominal economy they are defined for $(\mathbf{G}, \mathbf{P})$ rather than $\mathbf{G}$.

Our analysis of the benchmark economy extends with minimal changes to nominal economy. For any real variable x_t we use notation $x_t^{\$}$ to denote its nominal value, $x_t^{\$} \equiv P_t x_t$. All variables are defined in the same way as in Section 3 and it is easy to see that Lemma 1 continues to hold in this settings.

It is easy to see that the perturbation we considered in Section 3 requires tax adjustments $r_{T+1}^i / \left(Q_{T+1,t-1} \xi_{T+t} Y_{T+t}^{\$}\right)$ so that equation (12) remain unchanged. The budget constraint now holds in nominal terms so that equation (15) in the nominal economy becomes

$$\sum_{t=2}^{\infty} \mathbb{E}_T X_{T+t}^{\$} cov_T \left(Q_{T+1,t-1}, r_{T+1}^j\right) + \sum_{t=1}^{\infty} \mathbb{E}_T Q_{T+1,t-1} cov_T \left(X_{T+t}^{\$}, r_{T+1}^j\right) \simeq B_T \sum_{i \geq 1} \omega_T^i cov_T \left(r_{T+1}^i, r_{T+1}^j\right). \tag{49}$$

If we define $X_{T+t}^{\perp,\$}$ as

$$X_{T+t}^{\perp,\$} \equiv \mathbb{E}_T \mathcal{T}_{T+t}^{\$} \times \left(\ln Y_t^{\$} - \gamma \ln (1 - \tau_t)\right) - \mathbb{E}_T G_{T+t}^{\$} \times \ln G_{T+t}^{\$}$$

and follow the steps of Lemma 5, we obtain

$$cov_T \left(X_{T+t}^{\$}, r_{T+1}^j\right) \simeq cov_T \left(X_{T+t}^{\perp,\$}, r_{T+1}^j\right) - \mathbb{E}_T \zeta_{T+t} \mathbb{E}_T Y_{T+t}^{\$} cov_T \left(\ln \xi_{T+t}, r_{T+1}^j\right). \tag{50}$$

Use equations (49) and (50) and follow the steps of proof of equation (17) to obtain

$$\begin{aligned}
&\mathbb{E}_T \left(\sum_{i \geq 1} \partial_\sigma r_{T+1}^j \partial_\sigma r_{T+1}^i \overline{\omega}_T^i\right) \overline{Q}_T^0 \frac{\overline{B}_T}{\overline{P}_T \overline{Y}_T} = \\
&- \sum_{t=1}^{\infty} \overline{Q}_T^0 \overline{Q}_{T+1,t-1} \left(\frac{\overline{P}_{T+t}}{\overline{P}_T} \frac{\overline{Y}_{T+t}}{\overline{Y}_T}\right) \overline{\zeta}_{T+t} \left(\mathbb{E}_t \partial_\sigma r_{T+1}^j \partial_\sigma \ln A_{T+1,t-1}\right) \\
&+ \sum_{t=1}^{\infty} \overline{Q}_T^0 \overline{Q}_{T+1,t-1} \left(\frac{\overline{P}_{T+t}}{\overline{P}_T} \frac{\overline{Y}_{T+t}}{\overline{Y}_T}\right) \mathbb{E}_T \frac{\partial_\sigma X_{T+t}^{\perp,\$}}{\overline{P}_{T+t} \overline{Y}_{T+t}} \partial_\sigma r_{T+1}^j \\
&+ \sum_{t=1}^{\infty} \overline{Q}_T^0 \overline{Q}_{T+1,t-1} \left(\frac{\overline{P}_{T+t}}{\overline{P}_T} \frac{\overline{Y}_{T+t}}{\overline{Y}_T}\right) \left(\frac{\overline{X}_{T+t}}{\overline{Y}_{T+t}}\right) \mathbb{E}_T \partial_\sigma \ln Q_{T+1,t-1} \partial_\sigma r_{T+1}^j.
\end{aligned} \tag{51}$$

In the stationary nominal economy, $\frac{\overline{P}_{T+t}}{\overline{P}_T}\frac{\overline{Y}_{T+t}}{\overline{Y}_T} = (\Pi_T \Gamma)^t$, the price of a nominal one-period government bond to the zeroth order satisfies

$$\overline{Q}^0_{T+t} = \beta \frac{U_{c,T+t+1}}{U_{c,T+t}} \frac{\overline{P}_T}{\overline{P}_{T+1}} = \frac{\beta \Gamma^{-1/IES}}{\Pi_T},$$

the nominal discount rate $Q_{T+1,t-1}$ satisfies $\overline{Q}_{T+1,t-1} = \left(\frac{\beta}{\Pi_T}\Gamma^{-1/IES}\right)^{t-1}$, and $\frac{\overline{B}_T}{\overline{P}_T \overline{X}_T} = \frac{\sum_{t=1}^{\infty}\left(\frac{\beta}{\Pi_T}\Gamma^{-1/IES}\Gamma\Pi_T\right)^t}{\frac{\beta}{\Pi_T}\Gamma^{-1/IES}}$, $\overrightarrow{\hat{\beta}}_T[t] = \overline{Q}^0_T \overline{Q}_{T+1,t-1}\left(\frac{\overline{P}_{T+t}}{\overline{P}_T}\frac{\overline{Y}_{T+t}}{\overline{Y}_T}\right) = \left(\beta \Gamma^{1-1/IES}\right)^t = \hat{\beta}^t$. Use these expressions in equation (51) and apply Corollary 7 to get

$$\mathbb{E}_T\left(\sum_{i\geq 1} \partial_\sigma r^j_{T+1} \partial_\sigma r^i_{T+1} \overline{\omega}^i_T\right) = \left(1-\hat{\beta}\right)\sum_{t=1}^{\infty} \hat{\beta}^t cov_T\left(\frac{\ln Q_{T+1,t}}{Q^0_T}, r^j_{T+1}\right)$$
$$+\left(\frac{\Gamma\Pi_T}{\hat{\beta}}\right)\left(\frac{Y^\$_T}{B_T}\right)\sum_{t=1}^{\infty}\hat{\beta}^t cov_T\left(\frac{X^{\perp,\$}_{T+t}}{\mathbb{E}_T Y^\$_{T+t}}, r^j_{T+1}\right) + (\Gamma\Pi_T\zeta_T)\left(\frac{Y^\$_T}{B_T}\right)\sum_{t=1}^{\infty}\hat{\beta}^t cov_T\left(\ln A_{T+1,t}, r^j_{T+1}\right)$$

In matrix form this equation becomes

$$\Sigma_T \overrightarrow{\boldsymbol{\omega}}_T \simeq \left[\pi^Q \Sigma^Q_T + \Pi_T \pi^X_T \Sigma^X_T + \Pi_T \pi^A_T \Sigma^A_T\right] \overrightarrow{\hat{\beta}},$$

which proves the nominal version of equation (18) given in Corollary 3. The returns with the full set of nominal bonds satisfy $\Sigma_T \simeq \Sigma^Q_T$, which implies the nominal version of (20) given in Corollary 3.

A.4 Household Heterogeneity

Suppose household h has household specific productivity $\theta_{h,t}$ and we partition the households into two groups: $\mathbb{T}$ represent the set of households who can trade bonds and $\mathbb{N}$ represent the set of households who cannot trade bonds. Other than that, we focus on the baseline economy, in which all government assets are perfect substitutes and economy is small and open. Individual optimality implies

$$y_{h,t} = \theta^{1+\gamma}_{h,t}\left(1-\tau_t\right)^\gamma$$

and we can define total output as $Y_t = \sum_h y_{h,t}$. Assuming a linear tax function, we have

$$\frac{\partial \Upsilon_t}{\partial \tau_t} = Y_t + \tau_t \sum_h \frac{\partial y_{h,t}}{\partial \tau_t} = Y_t - \gamma\frac{\tau_t}{1-\tau_t}\sum_h y_{h,t} = Y_t\left(1-\gamma\frac{\tau_t}{1-\tau_t}\right),$$

so tax revenue elasticity is the same as before.

$$\xi_t = 1-\gamma\frac{\tau_t}{1-\tau_t}.$$

Consider a perturbation in which the government swaps ϵ of security $j \in \mathcal{G}_T$ for a risk-free bond in period T, undoes this perturbation in period $T+1$ and realized excess return r^j_{T+1} and then rolls it over for t periods using one-period bond before returning it back to the household. The same analysis as with the representative agent yields a welfare gain for agent h as

$$\partial_{j,T,\epsilon} V_{h,0} \propto \mathbb{E}_T M_{h,T+t} \left(Q_{T+1,t-1}\right)^{-1} r^j_{T+1} y_{h,T+t} \frac{\partial \tau_{T+t}}{\partial \Upsilon_{T+t}} = \mathbb{E}_T \beta^t M_{h,T+t} \left(Q_{T+1,t-1}\right)^{-1} r^j_{T+1} \frac{y_{h,T+t}}{Y_{T+t}} \frac{1}{\xi_{T+t}}.$$

So an optimality condition will be

$$\mathbb{E}_T \sum_h \varpi_h M_{h,T+t} \left(Q_{T+1,t-1}\right)^{-1} r^j_{T+1} \frac{y_{h,T+t}}{Y_{T+t}} \frac{1}{\xi_{T+t}} = 0,$$

where ϖ_h are Pareto weights. A second-order expansion of this equation then yields

$$\begin{aligned}
0 = \mathbb{E}_T \Bigg\{ & \frac{1}{2} \sum_h \varpi_h \left[\overline{M_{h,T+t}}\right] \left(\overline{Q_{T+1,t-1}}\right)^{-1} \partial_{\sigma\sigma} r^j_{T+1} \left[\overline{\frac{y_{h,T+t}}{Y_{T+t}} \frac{1}{\xi_{T+t}}}\right] \\
&+ \sum_h \varpi_h \left[\overline{M_{h,T+t}}\right] \left(\overline{Q_{T+1,t-1}}\right)^{-1} \partial_\sigma \ln\left(M_{h,T+t}\right) \partial_\sigma r^j_{T+1} \left[\overline{\frac{y_{h,T+t}}{Y_{T+t}} \frac{1}{\xi_{T+t}}}\right] \\
&+ \sum_h \varpi_h \left[\overline{M_{h,T+t}}\right] \left(\overline{Q_{T+1,t-1}}\right)^{-1} \partial_\sigma \ln\left(\frac{y_{h,T+t}}{Y_{T+t}}\right) \partial_\sigma r^j_{T+1} \left[\overline{\frac{y_{h,T+t}}{Y_{T+t}} \frac{1}{\xi_{T+t}}}\right] \\
&- \sum_h \varpi_h \left[\overline{M_{h,T+t}}\right] \left(\overline{Q_{T+1,t-1}}\right)^{-1} \partial_\sigma \ln\left(\xi_{T+t}\right) \partial_\sigma r^j_{T+1} \left[\overline{\frac{y_{h,T+t}}{Y_{T+t}} \frac{1}{\xi_{T+t}}}\right] \\
&- \sum_h \varpi_h \left[\overline{M_{h,T+t}}\right] \left(\overline{Q_{T+1,t-1}}\right)^{-1} \partial_\sigma \ln\left(Q^{rf}_{T+1,t-1}\right) \partial_\sigma r^j_{T+1} \left[\overline{\frac{y_{h,T+t}}{Y_{T+t}} \frac{1}{\xi_{T+t}}}\right] \Bigg\}.
\end{aligned}$$

Canceling out the terms that do not depend on h and dividing out by the coefficient on $\mathbb{E}_T \partial_{\sigma\sigma} r^j_{T+1}$ yields the optimality condition

$$\begin{aligned}
0 = & \mathbb{E}_T \Bigg[\frac{1}{2} \partial_{\sigma\sigma} r^j_{T+1} + \sum_h \mu_{h,T+t} \partial_\sigma \ln\left(M_{h,T+t}\right) \partial_\sigma r^j_{T+1} + \partial_\sigma \ln\left(\xi_{T+t}\right) \partial_\sigma r^j_{T+1} + \partial_\sigma \ln\left(Q_{T+1,t-1}\right) \partial_\sigma r^j_{T+1} \\
& + \sum_h \mu_{h,T+t} \partial_\sigma \ln\left(\frac{y_{h,T+t}}{Y_{T+t}}\right) \partial_\sigma r^j_{T+1} \Bigg] \qquad (52)
\end{aligned}$$

where $\mu_{h,T+t} \equiv \varpi_h \left[\overline{M_{h,T+t}}\right] \overline{s_{h,T+t}} \Big/ \left(\sum_h \varpi_h \left[\overline{M_{h,T+t}}\right] \overline{s_{h,T+t}}\right)$ are a deterministic sequence of weights that sum to one with $s_{h,T+t} \equiv \frac{y_{h,T+t}}{Y_{T+t}}$.

As government bonds are perfect substitutes, for all $h \in \mathbb{T}$ we must have

$$\mathbb{E}_T M_{h,T+t} \left(Q^{pvt}_{T+1,t-1}\right)^{-1} r^j_{T+1} = 0.$$

Expanding this equation yields

$$
\begin{aligned}
0 = \frac{1}{2}\left[\overline{M_{h,T+t}}\right]\left(\overline{Q^{pvt}_{T+1,t-1}}\right)^{-1}\partial_{\sigma\sigma} r^{j}_{T+1} - \left[\overline{M_{h,T+t}}\right]\left(\overline{Q^{pvt}_{T+1,t-1}}\right)^{-1}\partial_{\sigma}\ln\left(Q^{pvt}_{T+1,t-1}\right)\partial_{\sigma} r^{j}_{T+1} \\
+ \left[\overline{M_{h,T+t}}\right]\left(\overline{Q^{pvt}_{T+1,t-1}}\right)^{-1}\partial_{\sigma}\ln\left(M_{h,T+t}\right)\partial_{\sigma} r^{j}_{T+1}
\end{aligned}
$$

for all $h \in \mathbb{T}$. This simplifies to

$$
0 = \mathbb{E}_T\left[\frac{1}{2}\partial_{\sigma\sigma} r^{j}_{T+1} - \partial_{\sigma}\ln\left(Q^{pvt}_{T+1,t-1}\right)\partial_{\sigma} r^{j}_{T+1} + \partial_{\sigma}\ln\left(M_{h,T+t}\right)\partial_{\sigma} r^{j}_{T+1}\right].
$$

Since $\ln Q^{pvt}_{T+1,t-1} = \ln Q_{T+1,t-1} - \ln A_{T+1,t-1}$, we have.

$$
\frac{1}{2}\mathbb{E}_T\partial_{\sigma\sigma} r^{j}_{T+1} = \mathbb{E}_T\left[-\partial_{\sigma}\ln A_{T+1,t-1}\partial_{\sigma} r^{j}_{T+1} + \partial_{\sigma}\ln\left(Q_{T+1,t-1}\right)\partial_{\sigma} r^{j}_{T+1} - \partial_{\sigma}\ln\left(M_{h,T+t}\right)\partial_{\sigma} r^{j}_{T+1}\right] \tag{53}
$$

As this holds for all $h \in \mathbb{T}$ we can average over all traders, using weights $\mu_{h,T+t}$, to obtain

$$
\frac{1}{2}\mathbb{E}_T\partial_{\sigma\sigma} r^{j}_{T+1} = \mathbb{E}_T\left[-\partial_{\sigma}\ln A_{T+1,t-1}\partial_{\sigma} r^{j}_{T+1} + \partial_{\sigma}\ln\left(Q_{T+1,t-1}\right)\partial_{\sigma} r^{j}_{T+1} - \partial_{\sigma}\ln\left(M_{\mathcal{T},T+t}\right)\partial_{\sigma} r^{j}_{T+1}\right] \tag{54}
$$

where $\ln\left(M_{\mathbb{T},T+t}\right)$ is the average SDF of all traders:

$$
\ln\left(M_{\mathbb{T},T+t}\right) \equiv \sum_{h\in\mathbb{T}}\mu_{h,T+t}\ln\left(M_{h,T+t}\right) \Big/ \sum_{h\in\mathbb{T}}\mu_{h,T+t}.
$$

The same equation does not hold for the non-traders but we do have that for all $h \in \mathbb{N}$

$$
\begin{aligned}
\frac{1}{2}\mathbb{E}_T\partial_{\sigma\sigma} r^{j}_{T+1} =& \mathbb{E}_T\Bigg[-\partial_{\sigma}\ln A_{T+1,t-1}\partial_{\sigma} r^{j}_{T+1} + \partial_{\sigma}\ln\left(Q_{T+1,t-1}\right)\partial_{\sigma} r^{j}_{T+1} - \partial_{\sigma}\ln\left(M_{h,T+t}\right)\partial_{\sigma} r^{j}_{T+1} \\
&+ \left(\partial_{\sigma}\ln\left(M_{h,T+t}\right) - \partial_{\sigma}\ln\left(M_{\mathcal{T},T+t}\right)\right)\partial_{\sigma} r^{j}_{T+1}\Bigg].
\end{aligned} \tag{55}
$$

We can now use equations (53) and (55) substitute for $\frac{1}{2}\partial_{\sigma\sigma} r^{j}_{T+1}$ in (52) to get

$$
\begin{aligned}
-\mathbb{E}_T\partial_{\sigma}\ln\xi_{T+t}\partial_{\sigma} r^{j}_{T+1} =& \mathbb{E}_T\Bigg[\partial_{\sigma}\ln A_{T+1,t-1}\partial_{\sigma} r^{j}_{T+1} + \partial_{\sigma}\left\{\sum_{h}\mu_{h,T+t}\ln\left(\frac{1}{s_{h,T+t}}\right)\right\}\partial_{\sigma} r^{j}_{T+1} \\
&+ \partial_{\sigma}\left\{\sum_{h\in\mathbb{N}}\mu_{h,T+t}\left(\ln\left(M_{\mathbb{T},T+t}\right) - \ln\left(M_{h,T+t}\right)\right)\right\}\partial_{\sigma} r^{j}_{T+1}\Bigg].
\end{aligned}
$$

We can further simplify this expression by defining

$$
\ln\left(M_{\mathbb{N},T+t}\right) \equiv \sum_{h\in\mathbb{N}}\mu_{h,T+t}\ln\left(M_{h,T+t}\right) \Big/ \left(\sum_{h\in\mathbb{N}}\mu_{h,T+t}\right)
$$

as the "average" SDF of the non-traders, then

$$-\text{cov}_T\left(\ln \xi_{T+t}, r^j_{T+1}\right) \simeq \text{cov}_T\left(\partial_\sigma \ln A_{T+1,t-1}\partial_\sigma r^j_{T+1}\right) + \text{cov}_T\left(\sum_h \mu_{h,T+t}\ln\left(\frac{1}{s_{h,T+t}}\right), r^j_{T+1}\right)$$
$$+ \mu_{\mathbb{N},T+t}\text{cov}_T\left(\ln\left(M_{\mathbb{T},T+t}\right) - \ln\left(M_{\mathbb{N},T+t}\right), \partial_\sigma r^j_{T+1}\right) \quad (56)$$

where $\mu_{\mathbb{N},T+t} \equiv \left(\sum_{h\in\mathbb{N}}\mu_{h,T+t}\right)$ is the "share" of non-traders. Equation (56) adds two additional terms to equation (14) in main text that capture the effect of heterogeneity on the planners desire to smooth taxes. The first term, $\text{cov}_T\left(\sum_h \mu_{h,T+t}\ln\left(\frac{1}{s_{h,T+t}}\right), r^j_{T+1}\right)$, captures the planners desire to raise taxes in states of the world where inequality is high. The second term, $\mu_{\mathbb{N},T+t}\text{cov}_T\left(\ln\left(M_{\mathbb{T},T+t}\right) - \ln\left(M_{\mathbb{N},T+t}\right), \partial_\sigma r^j_{T+1}\right)$, captures the fact that the planner is trading on behalf of agents without access to asset markets and therefore will want to raise taxes in states of which the non-traders place less weight on relative to those agents with access to asset markets. This effect is scaled by the relative size of the non-traders. Following the steps of Theorem 1 and Corollary 1 we get

$$\Sigma_T\boldsymbol{\omega}_T \simeq \left[\pi^Q\Sigma^Q_T + \pi^X_T\Sigma^X_T + \pi^A_T\Sigma^A_T + \pi^A_T\Sigma^{ineq}_T + \pi^A_T\Sigma^M_T\right]\overrightarrow{\hat{\beta}}$$

where $\Sigma^{ineq}_T[t,j] = cov_T\left(\sum_h \mu_{h,T+t}\ln\left(\frac{1}{s_{h,T+t}}\right), r^j_{T+1}\right)$ is covariance matrix of returns with inequality and

$$\Sigma^M_T[j,t] = \mu_{\mathbb{N},T+t}\text{cov}_T\left(\ln\left(M_{\mathbb{T},T+t}\right) - \ln\left(M_{\mathbb{N},T+t}\right), r^j_{T+1}\right),$$

is the covariance of returns with the relative stochastic discount factors of traders and non-traders.

We can get a feel for how trading frictions affect the optimal portfolio by studying a special case. We further specialize to a simpler market structure in which the government trades only a risk-free security and a growth-adjusted consol. Let excess return on the consol be denoted by r^∞_t. Finally, we impose that the stochastic discount factor of the non-traders is scaled version of the stochastic discount factor of the traders: $\ln\left(M_{\mathbb{N},T+t}\right) = (1+\psi)\ln\left(M_{\mathbb{T},T+t}\right)$. This introduces a new parameter, ψ, that captures the severity of trading frictions as $\psi > 0$ implies that the SDF of the non-traders is more volatile of those of the traders.

Under this last assumption the covariance of the relative stochastic discount factors simplifies to

$$\text{cov}_T\left(\ln\left(M_{\mathbb{T},T+t}\right) - \ln\left(M_{\mathbb{N},T+t}\right), r^j_{T+1}\right) = -\psi\text{cov}_T\left(\ln\left(M_{\mathbb{T},T+t}\right), r^j_{T+1}\right).$$

As the traders trade the consol, we can use the traders' Euler equation, equation (54), to substitute out for this covariance and obtain

$$-\text{cov}_T\left(\ln\left(M_{\mathbb{T},T+t}\right), r^j_{T+1}\right) \simeq \mathbb{E}_T r^j_{T+1} - \text{cov}_T\left(\ln Q_{T+1,t-1}, r^j_{T+1}\right) + \text{cov}_T\left(\ln A_{T+1,t-1}, r^j_{T+1}\right).$$

Under our stationarity assumptions we have $\mu_{\mathcal{N},T+t} = \mu_{\mathcal{N},T}$ and can therefore express Σ_T^M as the sum of three terms

$$\Sigma_T^M = \mu_{\mathbb{N},T}\psi\left(\mathcal{R}_T - \overline{Q}_T^0\Sigma_T^Q + \Sigma_T^A\right)$$

where $\mathcal{R}_T[j,t] = \hat{\beta}^{-1}\mathbb{E}_T r_{T+1}^j$.

The effect of non-traders on the optimal portfolio is given by $\pi_T^A\Sigma_T^{-1}\Sigma_T^M\overrightarrow{\hat{\beta}}$. This simplifies under this market structure of a growth adjusted consol and a risk free bond as Σ_T is now a single number representing the conditional covariance of the growth adjusted consol. We can also make progress on the components of $\Sigma_T^M\overrightarrow{\hat{\beta}}$, starting with $\mathcal{R}_T\overrightarrow{\hat{\beta}} = \frac{\mathbb{E}_T r_{T+1}^j}{1-\hat{\beta}}$. Next we note that

$$\begin{aligned}
\Sigma_T^Q\overrightarrow{\hat{\beta}} = \sum_{t=1}^{\infty}\hat{\beta}^t\mathrm{cov}_T\left(\frac{1}{Q_T^0}\ln Q_{T+1,t}, r_{T+1}^{\infty}\right) &\simeq \frac{\Gamma}{\hat{\beta}}\sum_{t=1}^{\infty}\hat{\beta}^t\mathbb{E}_T\partial_\sigma\ln Q_{T+1,t}\partial_\sigma r_{T+1}^{\infty} \\
&\simeq \frac{\Gamma}{\hat{\beta}}\mathbb{E}_T\sum_{t=1}^{\infty}\Gamma^t\partial_\sigma Q_{T+1,t}\partial_\sigma r_{T+1}^{\infty} \\
&\simeq \frac{\Gamma}{\hat{\beta}}\mathbb{E}_T\partial_\sigma Q_{T+1,t}^{\infty}\partial_\sigma r_{T+1}^{\infty} \\
&\simeq \frac{\Gamma}{1-\hat{\beta}}\mathrm{cov}_T(r_{T+1}^{\infty}, r_{T+1}^{\infty}) = \frac{\Gamma}{1-\hat{\beta}}\Sigma_T.
\end{aligned}$$

Finally, we have that $\Sigma_T^A\overrightarrow{\hat{\beta}} = \sum_{t\geq 1}\hat{\beta}^t cov_T\left(\ln A_{T+1,t}^{rf}, r_{T+1}^{\infty}\right)$. All put together we have that

$$\Sigma_T^{-1}\Sigma_T^M\overrightarrow{\hat{\beta}} \approx \frac{\mu_{\mathbb{N},T}\psi}{1-\hat{\beta}}\left(\frac{\mathbb{E}_T r_{T+1}^{\infty}}{\mathrm{var}_T(r_{T+1}^{\infty})} - \hat{\beta} + (1-\hat{\beta})\frac{\sum_{t\geq 1}\hat{\beta}^t\mathrm{cov}_T\left(\ln A_{T+1,t}^{rf}, r_{T+1}^{\infty}\right)}{\mathrm{var}_T(r_{T+1}^{\infty})}\right).$$

Our empirical estimates have found that holding period returns on government debts of all maturities co-vary positively with the liquidity premium so we can assume that $\sum_{t\geq 1}\hat{\beta}^t cov_T\left(\ln A_{T+1,t}^{rf}, r_{T+1}^{\infty}\right)$. This implies that

$$\Sigma_T^{-1}\Sigma_T^M\overrightarrow{\hat{\beta}} > \frac{\mu_{\mathbb{N},T}\psi}{1-\hat{\beta}}\left(\frac{\mathbb{E}_T r_{T+1}^{\infty}}{\mathrm{var}_T(r_{T+1}^{\infty})} - \hat{\beta}\right). \tag{57}$$

So the presence of non-traders will lengthen the maturity as long as $\frac{\mathbb{E}_T r_{T+1}^{\infty}}{\mathrm{var}_T(r_{T+1}^{\infty})} > \hat{\beta}$. We can construct estimates for both $\mathbb{E}_T r_{T+1}^{\infty}$ and $\mathrm{var}_T(r_{T+1}^{\infty})$ using the fact that the growth adjusted consol is the infinite sum of zero coupon bonds of all maturities weighted by $\hat{\beta}^j$. To check equation (57), we use the estimates of the factor model and find that the left hand side is 137 which is significantly larger that $\hat{\beta} < 1$.

A.5 Price Effects

In this section we fill in the steps to derive formula (35) and the proof of Corollary 6.

Lemma 9. *In the preferred habitat benchmark economy with $\phi_T B_{i,T} \approx b_{i,T}$, the government optimality is given by*

$$cov_T\left(\ln \xi_{T+1+k}, r^j_{T+1}\right) + cov_T\left(\ln A^0_{T+1,k}, r^j_{T+1}\right) \simeq \overline{\xi_T} Q^{-1} \vartheta^j_T. \tag{58}$$

and equation (35) is satisfied.

Proof. Assumption $\phi_T B_{i,T} \approx b_{i,T}$ implies

$$-\sum_i \left[\partial_{j,T,t,\epsilon} Q^i_T\left(s^T\right)\left(b^i_T\left(s^T\right) - b^i_{T-1}\left(s^{T-1}\right)\right)\right] \approx \phi_T \sum_i \left[\partial_{j,T,t,\epsilon} Q^i_T\left(s^T\right)\left(B^i_T\left(s^T\right) - B^i_{T-1}\left(s^{T-1}\right)\right)\right].$$

From assumption $\overline{\Lambda} = \partial_\sigma \Lambda = \mathbf{0}$ and equation (34) it follows that $\overline{r}^j_{T+1} = \mathbb{E}_T \partial_\sigma r^j_{T+1} = 0$ and as before, stationarity implies $A^i_T \approx 1$. Now, take second-order expansion of equation (34) to get

$$\begin{aligned} 0 = &(1 - \phi_T \xi_T)\left(\frac{\sum_{i\geq 1} \partial_{\sigma\sigma}\partial_{j,T,t,\epsilon} Q\left(\overline{B}^i_T - \overline{B}^{i+1}_{T-1}\right)}{\xi_T}\right) \\ &+ \mathbb{E}_T \frac{\beta M_{T+1}}{M_T} \frac{r^j_{T+1}}{\xi_{T+1+k}}\left[\left(\frac{\beta M_{T+2}}{M_{T+1}} R^0_{T+2}\right) \times ... \left(\frac{\beta M_{T+1+k}}{M_{T+k}} R^0_{T+1+k}\right)\right] \end{aligned} \tag{59}$$

Under our perturbation and exploiting the form (33), we get

$$\partial_{j,T,t,\epsilon} \ln Q^i_T = \frac{\Lambda_T[i,j]}{Q^j_T} - \frac{\Lambda_T[0,j]}{Q^0_T}$$

which combined with the GV assumption $\Lambda_T[0,j] = 0$ and $\overline{\Lambda} = \partial_\sigma \Lambda = \mathbf{0}$ imply that $\partial_{\sigma\sigma}\partial_{j,T,t,\epsilon} Q^i_T = \frac{\overline{Q}^i_T \Lambda_T[i,j]}{\overline{Q}^j_T}$. Using this, it is easy to see that

$$\begin{aligned} \sum_{i\geq 1} \partial_{\sigma\sigma}\partial^j_\epsilon Q^i_T\left(B^i_T - B^{i+1}_{T-1}\right) &= \left(\frac{B_T}{Y_T}\right)\sum_{i\geq 1}\left(\frac{Y_T \Lambda_T[i,j]}{\overline{Q}^j_T}\right)\left(\overline{\omega}_T[i] - \overline{\omega}_{T-1}[i+1]\frac{\overline{Q}^i_T}{\overline{Q}^{i+1}_{T-1}}\frac{\overline{B}_{T-1}}{\overline{B}_T}\right) \\ &= \left(\frac{B_T}{Y_T}\right)\sum_{i\geq 1}\Lambda^{QE}_T[i,j]\left(\overline{\omega}_T[i] - \overline{\omega}^+_{T-1}[i]\right) \end{aligned} \tag{60}$$

Following the same steps as in Lemma (3), we get

$$\begin{aligned} &\mathbb{E}_T \frac{\beta M_{T+1}}{M_T}\frac{r^j_{T+1}}{\xi_{T+1+k}}\left[\left(\frac{\beta M_{T+2}}{M_{T+1}} R^0_{T+2}\right) \times ... \left(\frac{\beta M_{T+1+k}}{M_{T+k}} R^0_{T+1+k}\right)\right] \\ &= -\overline{\xi^{-1}_T Q}\,\mathbb{E}_T\left[\mathbb{E}_T \partial_\sigma r^j_{t+1} \partial_\sigma \ln A^0_{T+1,k}\right] - \overline{\xi^{-1}_T Q}\,\mathbb{E}_T \partial_\sigma \log \xi_{T+1+k} \partial_\sigma r^j_{t+1}. \end{aligned} \tag{61}$$

Substitute (60) and (61) in (59) and applying Lemma (7) to get equation (58), where the extra term ϑ^j_T simplifies to

$$\vartheta^j_T = (1 - \phi_T \xi_T)\left(\frac{B_T}{Y_T}\right)\left(\frac{1}{\overline{\xi}_T}\right)\sum_{i\geq 1}\Lambda^{QE}_T[i,j]\left(\omega_T[i] - \omega^+_{T-1}[i]\right). \tag{62}$$

where $\Lambda_T^{QE}[i,j] \equiv Y_T \Lambda[i,j] \left(\frac{\Gamma}{\hat{\beta}}\right)^j$ and $\omega_{T-1}^{+}[i] \equiv \hat{\beta}^{-1} \omega_{T-1}[i+1]$. [31]

The final expression (35) follows from substituting the budget constraint (50), and equations (58) and (62) for all j. □

Lemma 10. *Corollary 6 holds*

In a stationary economy

$$Q_T^i \approx Q = \beta \Gamma^{1-1/IES} \quad \frac{B_{T-1}^{i+1}}{B_T} \approx \Gamma^{-1} \omega_{T-1}[i+1].$$

From the definition of $\omega_T^{+}[i] = \frac{Q_T^i B_{T-1}^{i+1}}{B_T}$, we get that

$$\omega_T^{+}[i] = \beta \Gamma^{-1/IES} \omega_{T-1}[i+1]. \tag{63}$$

When $\Sigma^X = \Sigma^A = 0$, from Lemma 2 we get $\omega_T^* = \left(1 - \hat{\beta}\right) \overrightarrow{\hat{\beta}}$ and substituting for $\hat{\beta} = \beta \Gamma^{1-1/IES}$, we obtain

$$\omega_{T-1}^*[i+1] = \left(1 - \beta \Gamma^{1-1/IES}\right) \left(\beta \Gamma^{1-1/IES}\right)^{i+1}. \tag{64}$$

Substitute in (64) in (63) to get

$$\omega_T^{+}[i] = \left(1 - \beta \Gamma^{1-1/IES}\right) \left(\beta \Gamma^{1-1/IES}\right)^{i} = \omega_T^*[i].$$

The statement of the corollary follows from noticing that the second term in equation (35) drops out when $\omega_T^{+} = \omega_T^*$.

B Appendix: Empirical analysis

B.1 Results reported in Section 4

B.1.1 Data

Output, expenditures, tax revenues

We use the U.S. national income and product accounts to measure output, tax revenues. For our measure of output $Y_t^{\$}$ we use U.S. GDP. We measure nominal tax revenues $\mathcal{T}_t^{\$}$ as Federal Total Current Tax Receipts + Federal Contribution To Social Insurance and public expenditures $G_t^{\$}$ as Federal Consumption Expenditures + Federal Transfer Payments To Persons from BEA. All series are nominal and de-trended with constant time trends.

[31] Lemma 7 continues to hold in the stationary preferred habitat economy because $\bar{r}_{T+1}^j = \mathbb{E}_T \partial_\sigma r_{T+1}^j = 0$.

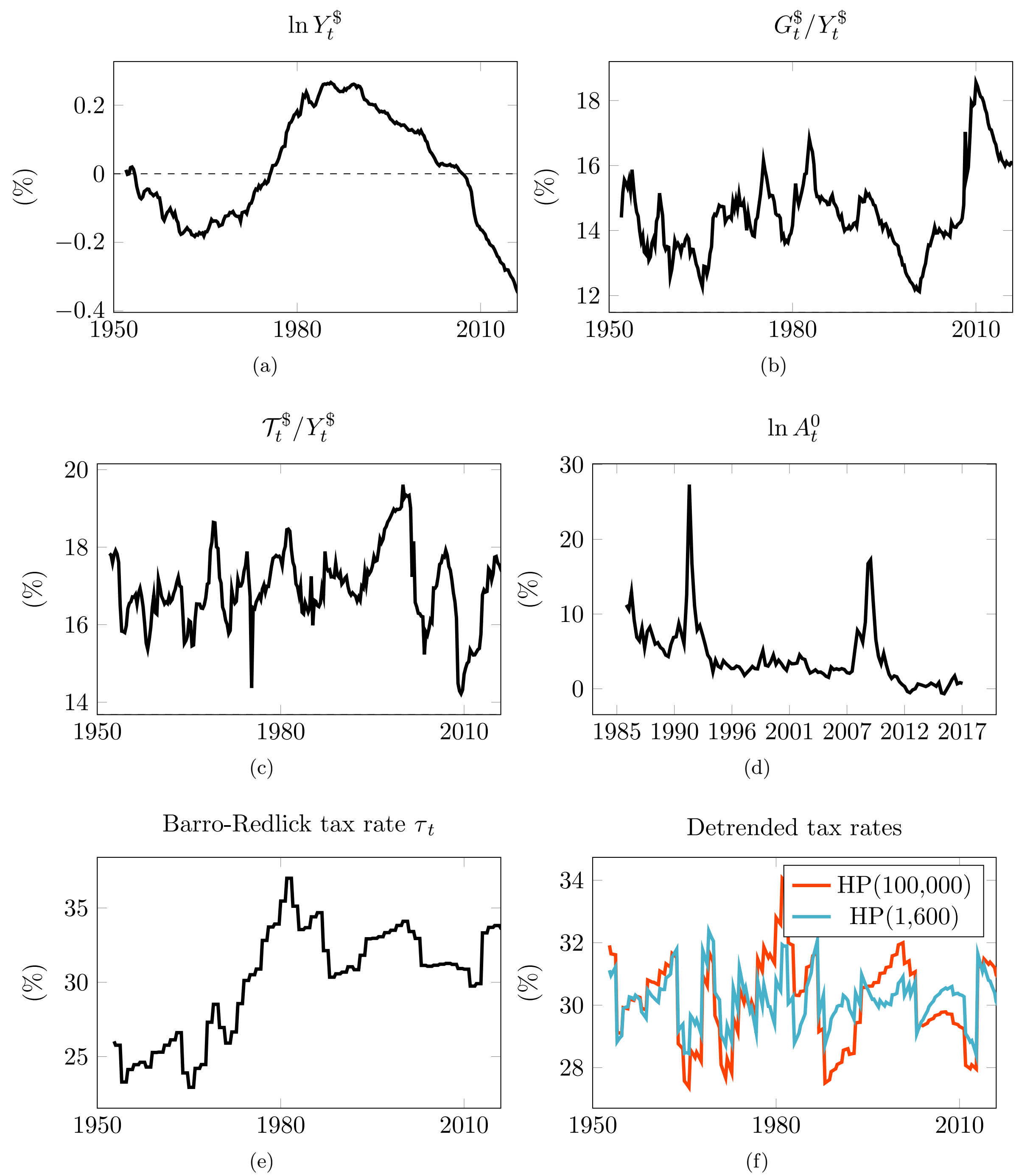


Figure 5: Summary of macroeconomic time series. Panel (a) plots detrended log nominal GDP, panel (b) plots the nominal government expenditure measured as Federal Consumption Expenditures + Federal Transfer Payments To Persons divided by nominal GDP, panel (c) plots nominal revenues divided by nominal GDP, panel (d) plots the imputed 3 month return on privately-issued debt, panel (e) plots the average marginal tax rate on income, panel (f) plots two ways of detrending the series in panel (e).

Tax rates

As a measure of tax rates τ_t we use the measure of the average marginal federal tax rate from Barro and Redlick (2011). Their series end in 2012 but we follow their steps and extrapolate this series for the years 2013-2017 using the Statistics of Income publicly available data from the Taxstats website. The series for the raw tax rates are plotted in Figure 5(e). It is clear from the series that there is a structural break in taxes around 1975. In our analysis we use want to focus on movements in taxes around business cycle frequencies and therefore we want to remove this break. We pursue two ways of doing that. First, we follow the business cycle literature and apply a Hodrick-Prescott (HP) filter with the penalty parameter set to 1,600. The resulting series is shown as the teal-blue line in the right panel Figure 5(f). While this procedure eliminates the low frequency movements in taxes, it also makes the resulting series "too smooth" post 1975. As an alternative, we adjust the penalty parameter until we achieve both goals: remove low frequency movements around 1975 and preserve the volatility of tax rates after and before 1975. The resulting series is show in the red line (at a penalty parameter of 100,000) in the right panel. We use the red line as a baseline measure of tax rates, but all our results are virtually unchanged if we use the teal line instead (see sub-section B.1.4).

Asset returns and government portfolio of bonds

We use the Fama Maturity Portfolios published by CRSP. There are 11 such portfolios, out of which ten portfolios correspond to maturities of 2 to 20 quarter in 2 quarter intervals, and a final portfolio for maturities between 30 and 40 quarters. We use the convention that the upper cut-off for each maturity corresponds to j in the mapping of data to the theory. That is, we use returns on portfolio of bonds of maturities between 2 to 4 quarter to measure r_t^j to $j = 4$, between 4 to 6 quarters to measure r_t^j for $j = 6$, etc. With this convention $j = 40$ is the largest maturity. We aggregate monthly log-returns by summing them across months within each quarter.

To measure returns on private bonds we use the yield curve of High Quality Market (HQM) Corporate Bonds computed by the U.S. Treasury.[32] The yields are available for select maturities with the shortest one being one year, while our quarterly model requires imputing returns on 3-months private bonds. For our baseline dataset, we followed McCulloch (1975) and interpolated the nominal bond yields using cubic splines and then used that interpolation to obtain the 3-month returns. We experimented with alternative extrapolation procedures, such

[32] The data can be accessed at https://www.treasury.gov/resource-center/economic-policy/corp-bond-yield/pages/corp-yield-bond-curve-papers.aspx

as using quadratic splines, and did not find any meaningful effect on our results. We use these returns to construct the liquidity premium $\ln A_t^0$.

Maturity structure of the U.S. government debt

We use the CRSP Treasuries Monthly Series to get the amount outstanding B_t^i for all (including TIPS and other inflation-protected bonds) federally issued (marketable) debt between 1952 and 2017, normalized by its face value. Each bond is uniquely identified by its cusips number n. CRSP also supplies us the Macaulay duration i for the outstanding amount, and the nominal market price $Q_t^{n,i}$ of each bond outstanding. For a few bonds where duration is absent, we set the duration equal to maturity date $-$ current date.

We follow Jiang et al. (2019), and construct at each date t, the market value $Q_t^i B_t^i$ held by the US government in bonds of Macaulay duration i, by summing across cusips n, such that $Q_t^i B_t^i = \sum_n Q_t^{n,i} B_t^{n,i}$. We then sum across all Macaulay duration i to get the market value of the government debt portfolio $B_t \equiv \sum_{i \in \mathcal{G}_t} Q_t^i B_t^i$ at each date t. We finally compute the portfolio weight in the US government debt portfolio for each maturity i using that $\omega_t^i = \frac{Q_t^i B_t^i}{B_t}$.

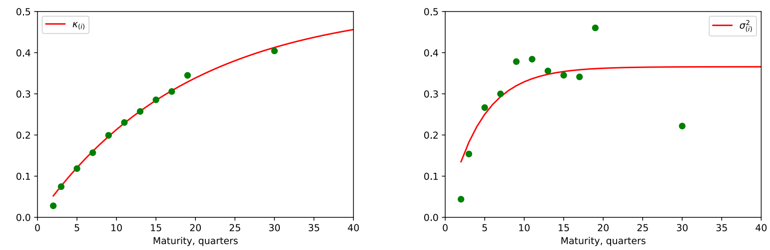
B.1.2 Estimations and extrapolations

We estimate model our factor model (21) using OLS. In the main text (Table 2), we report the estimates for the baseline specification in which we restricted $\rho_G = \rho_Y = 1$ and $\rho_f = 0$. This estimation procedure produces estimates of $\left(\alpha_j, \rho_j, \kappa_j, \sigma_j^2\right)$ for eleven j, with the highest being $j = 40$. For constructing our target portfolios, we need to extrapolate $\left(\rho_j, \kappa_j\right)$ for all $j > 1$. In the baseline extrapolation, we estimate δ_j and σ_j^2 by fitting the closest exponential function: $f(j) = e^0 - e^0 \exp(-e^1 \times j)$ for $f(j) \in \left\{\delta_j, \sigma_j^2\right\}$. We fit the parameters e^0 and e^1 to minimize sum of squares between fitted and actual values of δ_j and σ_j^2. Alternatively, we also experimented to linearly extrapolate between any two adjacent j, and assume that $\left(\kappa_j, \sigma_j^2\right) = \left(\kappa_{40}, \sigma_{40}^2\right)$ for $j > 40$. The point estimates and this extrapolation is reported in Figure 6(a). We also experimented with alternative extrapolation, presented in Figure 6(b) that we report in Section B.1.4.

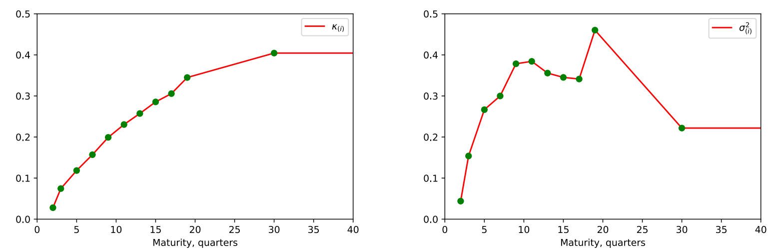
B.1.3 Deriving equations (22) and (23)

We start from the factor structure (21), which we rewrite in matrix form for each subscripts $k \in \{Y, G, A\}$ as:

$$\mathbb{A}^k \begin{bmatrix} z_{t+T}^k \\ f_{T+t} \end{bmatrix} = \boldsymbol{\alpha}^k + \mathbb{B}^k \begin{bmatrix} z_{t+T-1}^k \\ f_{T+t-1} \end{bmatrix} + \boldsymbol{\varepsilon}_{T+t}^k,$$



(a) Fit for extrapolation of the factor model estimates of $\left(\kappa_j, \sigma_j^2\right)$ using $f(j) = e^0 - e^0 \exp(-e^1 \times j)$(21)



(b) Fit for extrapolation of the factor model estimates of $\left(\kappa_j, \sigma_j^2\right)$ using linear splines(21)

Figure 6

where we have stacked the coefficients as follows:

$$\boldsymbol{\alpha}^k \equiv \begin{bmatrix} \alpha_y \\ \alpha_f \end{bmatrix}, \mathbb{A}^k \equiv \begin{bmatrix} 1 & -\kappa_k \\ 0 & 1 \end{bmatrix}, \mathbb{B}^k = \begin{bmatrix} \rho_k & 0 \\ 0 & \rho_f \end{bmatrix}, \boldsymbol{\varepsilon}_{T+t}^k = \begin{bmatrix} \varepsilon_{T+t}^k \\ \varepsilon_{f,T+t} \end{bmatrix}.$$

We then invert this VAR(1) representation to get to the vector MA(t):

$$\begin{aligned}
\begin{bmatrix} z_{T+t}^k \\ f_{T+t} \end{bmatrix} &= \left(\mathbb{A}^k\right)^{-1} \boldsymbol{\alpha}^k + \left(\mathbb{A}^k\right)^{-1} \mathbb{B}^k \begin{bmatrix} z_{T+t-1}^k \\ f_{T+t-1} \end{bmatrix} + \left(\mathbb{A}^k\right)^{-1} \boldsymbol{\varepsilon}_{T+t}^k \\
&= \left(\mathbb{A}^k\right)^{-1} \left[\boldsymbol{\alpha}^k + \mathbb{B}^k \begin{bmatrix} z_{T+t-2}^k \\ f_{T+t-2} \end{bmatrix}\right] + \left(\mathbb{A}^k\right)^{-1} \boldsymbol{\varepsilon}_{T+t}^k + \left(\mathbb{A}^k\right)^{-1} \mathbb{B}^k \left(\mathbb{A}^k\right)^{-1} \boldsymbol{\varepsilon}_{T+t-1}^k \\
&\vdots \\
&= \mathbb{E}_T \left(\begin{bmatrix} z_{T+t}^k \\ f_{T+t} \end{bmatrix}\right) + \sum_{\tau=0}^{t-1} \left(\left(\mathbb{A}^k\right)^{-1} \mathbb{B}^k\right)^{\tau} \left(\mathbb{A}^k\right)^{-1} \boldsymbol{\varepsilon}_{T+t-\tau}^k.
\end{aligned}$$

From the first row of this vector MA(t)representation, we can read MA(t) representation for each component z_{T+t}^k:

$$z_{T+t}^k = \mathbb{E}_T \left[z_{T+t}^k\right] + \sum_{\tau=0}^{t-1} \left[\rho_k^\tau \varepsilon_{T+t-\tau}^k + \frac{\kappa_k \left(\rho_f^{\tau+1} - \rho_k^{\tau+1}\right)}{\rho_f - \rho_k} \varepsilon_{f,T+t-\tau}\right].$$

We use the MA(t) representation to obtain formula for the matrices Σ_T^{-1}, Σ_T^Q, Σ_T^A, Σ_T^X. First, note that for any k and any $t \geq 1$:

$$
\begin{aligned}
cov_T\left(z_{T+t}^k, r_{T+1}^j\right) &= cov_T\left(\rho_k^{t-1}\varepsilon_{T+1}^k + \frac{\kappa_k\left(\rho_f^t - \rho_k^t\right)}{\rho_f - \rho_k}\varepsilon_{f,T+1}, \varepsilon_{T+1}^j + \kappa_j\varepsilon_{f,T+1}\right) \\
&= \frac{\left(\rho_f^t - \rho_k^t\right)}{\rho_f - \rho_k}\kappa_k\kappa_j\sigma_f^2 + \iota_{\{k=j\}}\sigma_j^2.
\end{aligned}
$$

Applying that formula for $t = 1$ and $k = j$, we get

$$
\Sigma_T\left[j,t\right] = cov_T\left(r_{T+1}^j, r_{T+1}^t\right) = \kappa_j\kappa_t\sigma_f^2 + \iota_{\{t=j\}}\sigma_j^2.
$$

Furthermore we can easily check that, using that $\chi^{-2} = \sigma_f^{-2} + \sum_{t\in\mathcal{G}}\kappa_t^2\sigma_t^{-2}$,

$$
\Sigma^{-1}\left[i,j\right] = \iota_{\{i=j\}}\sigma_i^{-2} - \chi^2\kappa_i\kappa_j\sigma_i^{-2}\sigma_j^{-2}.
$$

Using Corollary 2, when the set of government securities consists of the full set of pure discount bonds, we have $\Sigma_T^Q\left[j,t\right] \simeq \Sigma_T\left[j,t\right]$, and hence

$$
\Sigma_T^Q\left[j,t\right] \simeq \kappa_j\kappa_t\sigma_f^2 + \iota_{\{t=j\}}\sigma_j^2.
$$

We then use stationarity and the definition of $X_{T+t}^{\perp,\$} \equiv \Gamma\left[\mathcal{T}_T^\$ \times \ln Y_{T+t}^{\perp,\$} - G_T^\$ \times \ln G_{T+t}^\$\right]$ to get that

$$
\begin{aligned}
\Sigma_T^X\left[j,t\right] &= cov_T\left(\frac{X_{T+t}^{\perp,\$}}{\mathbb{E}_T Y_{T+t}^\$}, r_{T+1}^j\right) \\
&= \frac{\mathcal{T}_T^\$}{Y_T^\$}cov_T\left(\ln Y_{T+t}^{\perp,\$}, r_{T+1}^j\right) - \frac{G_T^\$}{Y_T^\$}cov_T\left(\ln G_{T+t}^\$, r_{T+1}^j\right) \\
&= \kappa_j\left(\frac{\mathcal{T}_T^\$}{Y_T^\$}\kappa_Y\frac{\rho_f^t - \rho_Y^t}{\rho_f - \rho_Y} - \frac{G_T^\$}{Y_T^\$}\kappa_G\frac{\rho_f^t - \rho_G^t}{\rho_f - \rho_G}\right)\sigma_f^2. \qquad (65)
\end{aligned}
$$

Finally, we use the definition of $A_{t,k} \equiv A_t^0 \times ... \times A_{t+k-1}^0$ and that $\overline{A_t^0} = 1$ to compute

$$
\begin{aligned}
\Sigma_T^A\left[j,t\right] &= cov_T\left(A_{T+1,t}, r_{T+1}^j\right) \\
&= cov_T\left(A_{T+1}^0 \times ... \times A_{t+T}^0, r_{T+1}^j\right) \\
&\simeq \sum_{\ell=0}^{t-1} cov_T\left(A_{T+1+\ell}, r_{T+1}^j\right) \\
&\simeq \sum_{\ell=0}^{t-1}\frac{\left(\rho_f^{\ell+1} - \rho_A^{\ell+1}\right)}{\rho_f - \rho_A}\kappa_A\kappa_j\sigma_f^2 \\
&\simeq \frac{\kappa_A\kappa_j\sigma_f^2}{\rho_f - \rho_A}\left[\rho_f\frac{1-\rho_f^t}{1-\rho_f} - \rho_A\frac{1-\rho_A^t}{1-\rho_A}\right]. \qquad (66)
\end{aligned}
$$

In the third line, we use the $\simeq$ sign because we take a first-order approximation of the product of A^0. Note that in our baseline case with $\rho_G = \rho_Y = 1$ and $\rho_f = 0$, those formula simplify to

$$\Sigma_T^X [j,t] = \kappa_j \left(\frac{\mathcal{T}_T^\$}{Y_T^\$} \kappa_Y - \frac{G_T^\$}{Y_T^\$} \kappa_G \right) \sigma_f^2, \Sigma_T^A [j,t] = \kappa_A \kappa_j \sigma_f^2 \frac{1-\rho_A^t}{1-\rho_A}.$$

We can now compute the three components determining the portfolio allocation. First note that:

$$\begin{aligned}
\sum_{\ell \in \mathcal{G}} \Sigma_T^{-1} [j,\ell] \kappa_\ell &= \sum_{\ell \in \mathcal{G}} \left(\iota_{\{j=\ell\}} \sigma_j^{-2} - \chi^2 \kappa_j \kappa_\ell \sigma_j^{-2} \sigma_\ell^{-2} \right) \kappa_\ell \\
&= \kappa_j \sigma_j^{-2} \left[1 - \frac{\sum_{\ell \in \mathcal{G}} \kappa_\ell^2 \sigma_\ell^{-2}}{\sigma_f^{-2} + \sum_{t \in \mathcal{G}} \kappa_t^2 \sigma_t^{-2}} \right] \\
&= \frac{\kappa_j \sigma_j^{-2}}{1 + \sum_{t \in \mathcal{G}} \kappa_t^2 \sigma_f^2 \sigma_t^{-2}}.
\end{aligned}$$

The primary surplus component $\pi_T^X \Sigma_T^{-1} \Sigma_T^X \overrightarrow{\beta}$ is given by

$$\begin{aligned}
\pi_T^X \Sigma_T^{-1} \Sigma_T^X \overrightarrow{\beta} [j] &= \pi_T^X \left(\sum_{\ell \geq 1} \Sigma_T^{-1} [j,\ell] \kappa_\ell \right) \left(\sum_{t \geq 1} \left(\frac{\mathcal{T}_T^\$}{Y_T^\$} \kappa_Y \frac{\rho_f^t - \rho_Y^t}{\rho_f - \rho_Y} - \frac{G_T^\$}{Y_T^\$} \kappa_G \frac{\rho_f^t - \rho_G^t}{\rho_f - \rho_G} \right) \sigma_f^2 \hat{\beta}^t \right) \\
&= K_{X,T} \frac{\kappa_j}{\sigma_j^2} \chi^2,
\end{aligned}$$

with $K_{X,T} = \pi_T^X \hat{\beta} \left(\frac{\mathcal{T}_T^\$}{Y_T^\$} \frac{\kappa_Y}{\rho_f - \rho_Y} \left[\frac{\rho_f}{1-\hat{\beta}\rho_f} - \frac{\rho_Y}{1-\hat{\beta}\rho_Y} \right] - \frac{G_T^\$}{Y_T^\$} \frac{\kappa_G}{\rho_f - \rho_G} \left[\frac{\rho_f}{1-\hat{\beta}\rho_f} - \frac{\rho_G}{1-\hat{\beta}\rho_G} \right] \right)$. Similarly, the liquidity component $\pi_T^A \Sigma_T^{-1} \Sigma_T^A \overrightarrow{\beta}$ is

$$\begin{aligned}
\pi_T^A \Sigma_T^{-1} \Sigma_T^A \overrightarrow{\beta} [j] &\simeq \pi_T^A \left(\sum_{\ell \geq 1} \Sigma_T^{-1} [j,\ell] \kappa_\ell \right) \left(\sum_{t \geq 1} \frac{\kappa_A \sigma_f^2}{\rho_f - \rho_A} \left[\rho_f \frac{1-\rho_f^t}{1-\rho_f} - \rho_A \frac{1-\rho_A^t}{1-\rho_A} \right] \hat{\beta}^t \right) \sigma_f^2 \\
&= K_{A,T} \frac{\kappa_j}{\sigma_j^2} \chi^2,
\end{aligned}$$

with $K_{A,T} = \pi_T^A \hat{\beta} \left(\frac{\kappa_A}{\rho_f - \rho_A} \left[\frac{\rho_f}{1-\rho_f} \left[\frac{1}{1-\hat{\beta}} - \frac{\rho_f}{(1-\hat{\beta}\rho_f)} \right] - \frac{\rho_A}{1-\rho_A} \left[\frac{1}{1-\hat{\beta}} - \frac{\rho_A}{(1-\hat{\beta}\rho_A)} \right] \right] \right)$. Finally, the interest rate risk component $\pi^Q \Sigma_T^{-1} \Sigma_T^Q \overrightarrow{\beta}$ is given by

$$\begin{aligned}
\pi^Q \Sigma_T^{-1} \Sigma_T^Q \overrightarrow{\beta}\,[j] &= \pi^Q \left(\sum_{\ell \in \mathcal{G}} \Sigma_T^{-1}[j,\ell]\,\kappa_\ell \right) \left(\sum_{t \geq 1} \left(\kappa_t \sigma_f^2 + \frac{\iota_{\{t=\ell\}} \sigma_\ell^2}{\kappa_\ell} \right) \hat{\beta}^t \right) \\
&= \pi^Q \kappa_j \sigma_j^{-2} \left[\frac{\sum_{t \geq 1} \kappa_t \sigma_f^2 \hat{\beta}^t}{\sigma_f^{-2} + \sum_{t \in \mathcal{G}} \kappa_t^2 \sigma_t^{-2}} \right] \\
&\cdots + \pi^Q \left(\sum_{\ell \in \mathcal{G}} \left[\iota_{\{j=\ell\}} \sigma_j^{-2} - \frac{\kappa_j \kappa_\ell \sigma_j^{-2} \sigma_\ell^{-2}}{\sigma_f^{-2} + \sum_{t \in \mathcal{G}} \kappa_t^2 \sigma_t^{-2}} \right] \sigma_\ell^2 \hat{\beta}^\ell \right) \\
&= \hat{\beta}^j + K_{Q,T} \frac{\kappa_j}{\sigma_j^2} \chi^2,
\end{aligned}$$

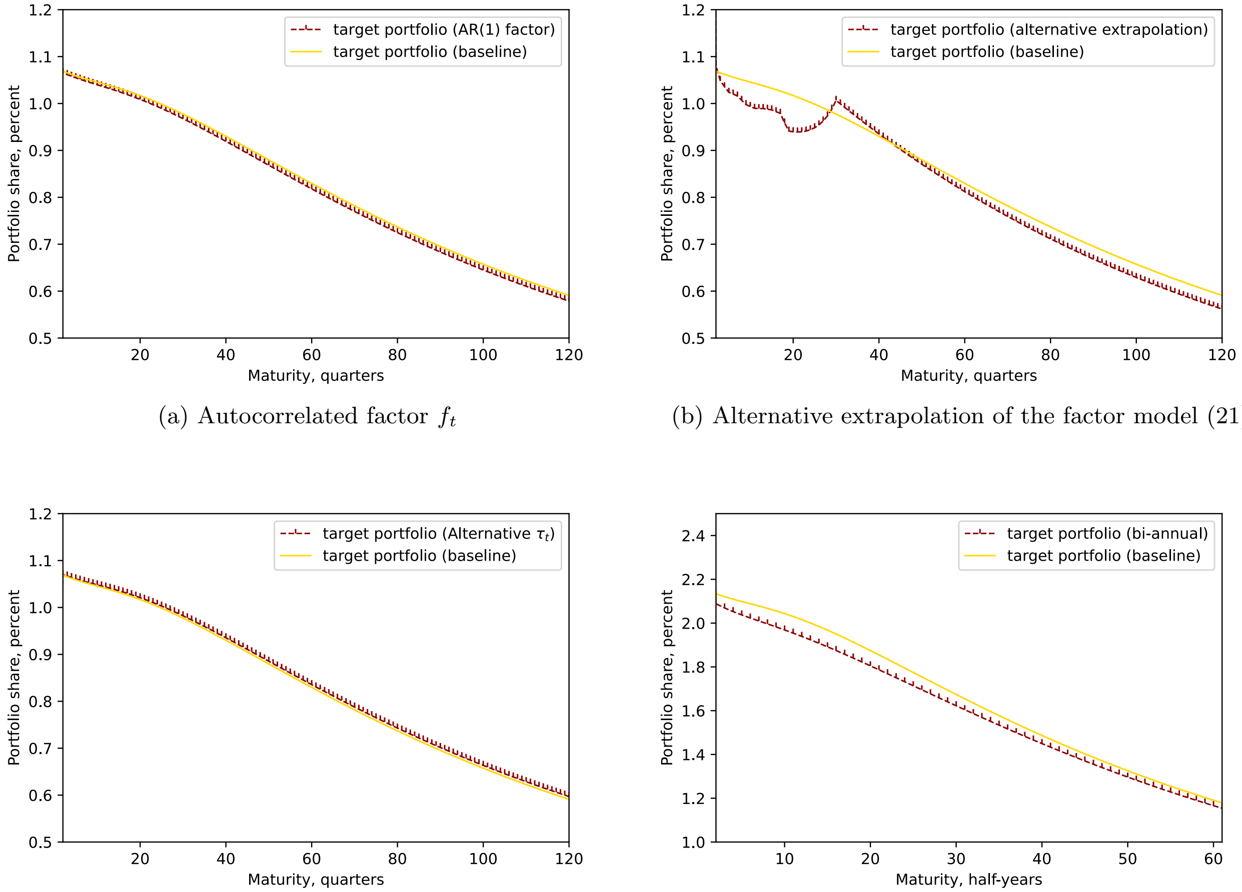
where $K_{Q,T} = \pi^Q \left[\sum_{t \geq 1} \kappa_t \hat{\beta}^t - \sum_{t \in \mathcal{G}} \kappa_t \hat{\beta}^t \right]$. From here it is easy to see that as the number of maturities go to infinity, $K_{Q,T} \to 0$.

In our baseline case with $\rho_G = \rho_Y = 1$ and $\rho_f = 0$, these formula simplify to expressions (22) and (23).

We quantify the coefficients K_X, K_A, K_Q for the capped portfolio using the estimates from our factor model. The constants $\chi^2 = \underbrace{\frac{1}{\underbrace{\sigma_f^{-2}}_{0.015} + \underbrace{\sum_{i \in \mathcal{G}} \kappa_i^2 \sigma_i^{-2}}_{74.76}}}_{0.013}$, $\pi_T^X = \underbrace{\underbrace{\hat{\beta}^{-1}}_{1.015} \Gamma \underbrace{Y_T/B_T}_{0.25}}_{0.2537}$, $\pi_T^A = \underbrace{\underbrace{\Gamma}_{1.005} \underbrace{\zeta_T}_{0.605} \underbrace{Y_T/B_T}_{0.25}}_{0.152}$, $\pi^Q = \underbrace{1 - \hat{\beta}}_{0.01}$ and

$$K_{X,T} = \underbrace{\underbrace{\left(\frac{\hat{\beta}}{1 - \hat{\beta}} \right)}_{98.99} \underbrace{\left(\underbrace{\kappa_Y}_{-0.47} \underbrace{\frac{\mathcal{T}_T^\$}{Y_T^\$}}_{0.17} - \underbrace{\kappa_G}_{-0.0317} \underbrace{\frac{G_T^\$}{Y_T^\$}}_{0.15} \right)}_{-0.00321}}_{-0.320},$$

$$K_{A,T} = \underbrace{\underbrace{\left(\frac{\underbrace{\kappa_A}_{0.00098} \underbrace{\hat{\beta}}_{0.99}}{1 - \underbrace{\rho_A}_{0.827}} \right)}_{0.0056} \underbrace{\left[\underbrace{\frac{1}{1 - \hat{\beta}}}_{99.99} - \underbrace{\frac{\overbrace{\rho_A}^{0.827}}{1 - \underbrace{\hat{\beta} \rho_A}_{0.819}}}_{4.59} \right]}_{95.4}}_{0.534},$$



(a) Autocorrelated factor f_t

(b) Alternative extrapolation of the factor model (21)

(c) Alternative tax series

(d) Bi-annual frequency

Figure 7: The capped target portfolio $\overrightarrow{\omega}^*_T$ for various alternatives (in yellow) versus the baseline capped target portfolio (in red)

$$K_{Q,T} = \underbrace{\left(\sum_{t \notin G}^{\infty} \hat{\beta}^t \kappa_t \right)}_{0.153}.$$

We see that the magnitude of $\pi_T^X K_{X,T}$ is roughly equal to $\pi_T^A K_{A,T}$ but they have opposite signs and is much smaller than $\left(1 - \hat{\beta}\right)$. This explains why the portfolio that hedges primary surplus offsets the portfolio that hedges liquidity risks but both are less important than hedging interest rate risks.

Table 3: FACTOR MODEL ESTIMATION RESULTS (AR(1) FACTOR STRUCTURE)

	Excess returns r_t^j for various maturities j														
	6m	12m	18m	24m	30m	36m	42m	48m	54m	60m	120m	$\ln G_t^{\$}$	$\ln Y_t^{\perp,\$}$	$\ln A_t^0$	f_t
α_k	0.086	0.155	0.220	0.245	0.284	0.315	0.346	0.344	0.372	0.304	0.444	-0.177	-0.319	0.006	0.026
	(0.014)	(0.025)	(0.033)	(0.035)	(0.039)	(0.039)	(0.038)	(0.037)	(0.037)	(0.043)	(0.030)	(0.016)	(0.008)	(0.003)	(0.502)
ρ_k	-0.107	-0.057	-0.041	-0.043	-0.042	-0.025	-0.022	-0.008	-0.022	-0.027	0.003	1.001	1.009	0.828	-0.035
	(0.043)	(0.035)	(0.030)	(0.025)	(0.023)	(0.020)	(0.018)	(0.016)	(0.015)	(0.015)	(0.009)	(0.008)	(0.004)	(0.047)	(0.063)
κ_k	0.028	0.074	0.118	0.157	0.199	0.230	0.257	0.285	0.306	0.345	0.404	-0.032	-0.048	0.001	0.000
	(0.002)	(0.003)	(0.004)	(0.004)	(0.005)	(0.005)	(0.005)	(0.005)	(0.005)	(0.005)	(0.004)	(0.016)	(0.008)	(0.000)	(nan)
σ_k^2	0.044	0.154	0.267	0.300	0.378	0.384	0.356	0.345	0.341	0.460	0.222	4.231	1.125	0.000	63.676
	(0.004)	(0.014)	(0.024)	(0.027)	(0.034)	(0.034)	(0.031)	(0.031)	(0.030)	(0.041)	(0.020)	(0.375)	(0.100)	(0.000)	(5.65)
R2	0.536	0.698	0.771	0.840	0.870	0.898	0.922	0.938	0.946	0.943	0.979	0.985	0.996	0.727	0.001

Notes: This table records the OLS estimates of the factor model (21) without imposing $\rho_f = 0, \rho_Y = \rho_G = 1$. Standards errors are in parenthesis. The sample for excess returns and primary surpluses normalized by outputs is 1952-2017, and the sample for the one-period liquidity premium is 1984-2017. The time period is a quarter.

B.1.4 Robustness

AR(1) factor structure In this section, we consider the general estimation of (21) without any a-priori restrictions on parameters. Table 3 presents estimation results.

The construction of matrices Σ_T^{-1} and Σ_T^Q remain unchanged while matrices Σ_T^A and Σ_T^X are now constructed using expressions (65) and (66). We construct the capped target portfolio implied by the general AR(1) structure and compare to our baseline portfolio in panel (a) of Figure 7.

Alternative extrapolation method, tax series, time aggregation, calculation of returns In Section B.1.2 we discussed an alternative extrapolation procedure for coefficients $\left(\kappa_j, \sigma_j^2\right)$, while in Section B.1.1 we presented an alternative procedure to de-trend tax series. None of these alternative approached affect our conclusions in any meaningful way. We reported the implied unrestricted target portfolios under this alternative procedures respectively in panel (b) and (c) of Figure 7.

We also experimented with alternative ways to calculate returns with different time frequencies. In the baseline, we used quarterly measures of returns, surpluses and taxes to ensure the largest sample such that we could measure asset prices and macro data in a consistent way. To verify if our results are driven by our choice of the frequency, we use returns and other macro variables at biannual frequencies. The shortest maturity available is now of 6 months, which we take as our measure of the one-period government bond R_t^0. As before, we construct the biannual holding period return by summing monthly returns for each portfolio which are separated by 6 month intervals. For other macro variables, we aggregate two consecutive quarters to obtain the biannual series. Using this data, we apply the same procedure as the baseline (extracting the factor, estimating the factor model, constructing the conditional covariances) and obtain the optimal portfolio. We show the implied unrestricted target portfolios in the panel (d) of Figure 7. In order to compare it to our baseline results which have portfolios by quarterly bins, we aggregate the baseline portfolio weights to biannual weights using $\omega_{biannual}[i] \equiv \omega[2i-1] + \omega[2i]$,where i indexes the 6 month intervals and the right hand size is the baseline target portfolio. We find that that the two biannual portfolios are very similar.

B.2 Results reported in Section 5

B.2.1 Additional discussion for Section 5.1

To simulate the neoclassical model, we solve a complete markets Ramsey allocation as in Lucas and Stokey (1983) by posing the following maximization problem. Given some $t = 0$ state $s_0 \in \mathcal{S}$ and household savings $b_0\left(s^0\right)$, the Ramsey problem can be expressed as

$$\max_{c_t(s^t),y_t(s^t)} \mathbb{E}_0 \sum_{t=0}^{\infty} U\left(c_t, \frac{y_t}{\theta_t}\right) \tag{67}$$

subject to

$$y_t\left(s^t\right) = c_t\left(s^t\right) + G\left(s_t\right), \tag{68}$$

$$b_0\left(s^0\right) u_c\left(s^0\right) = \sum_{t=0}^{\infty} \sum_{s^t} \beta^t \pi_t\left(s^t\right) \left[u_c\left(s^t\right) c_t\left(s^t\right) + u_y\left(s^t\right) y_t\left(s^t\right)\right], \tag{69}$$

where the *implementability constraints,* equation (69) is derived by taking the time-0 budget constraint and replacing after-tax wages as well as bond prices.

We assume that the state space $\mathcal{S}$ is discrete (described below) and non-linearly solve the optimal allocation using the first-order conditions of the Ramsey planning problem. The resulting optimal allocation is represented using two sets of vectors of dimension $2|\mathcal{S}|$, one set for consumption and labor choices at $t = 0$ and another set for all $s_t \in \mathcal{S}$ for $t \geq 1$. Using the Ramsey allocation $\{c_t, y_t\}$, we can back out other related objects such tax rates $\tau_t = 1 - \frac{\left(\frac{y_t}{\theta_t}\right)^{\gamma}}{c_t^{-\sigma}}$; primary surplus $X_t = \tau_t y_t - G_t$; and zero-coupon bond prices $Q_t^n = \mathbb{E}_t \frac{c_{t+n}^{-\sigma}}{c_t^{-\sigma}}$.

We follow Buera and Nicolini (2004) and assume that the preferences of households are isoelastic $U\left(c_t, \frac{y_t}{\theta_t}\right) = \frac{c_t^{1-\sigma}}{1-\sigma} - \frac{\left(\frac{y_t}{\theta_t}\right)^{1+\gamma}}{1+\gamma}$ with parameters $\sigma = 2$ and $\gamma = 1$. The economy is closed, so the demand of assets from foreign investors is zero and there are liquidity services provided by government bonds. The only source of uncertainty comes from the exogenous stochastic process of government expenditures G_t, which follows an AR(1) process

$$\ln G_t = \alpha_G + \rho_G \ln G_{t-1} + \sigma_G \epsilon_t$$

We set $(\alpha_G, \rho_G, \sigma_G)$ to obtain a mean G/Y of 15%, auto correlation of 0.95 and a standard deviation $\frac{1.2}{15}$ which are in line with the U.S. data that we use in Section 4.1. We discretize the $\ln G_t$ process with $|\mathcal{S}| = 50$ grid points. For our calculations, we set the level of initial debt B_0 so that the annualized initial level of government liabilities to GDP is 100%.

We use this parameterization to construct several versions of the optimal portfolio. First, for a given $s \in \mathcal{S}$, we apply Corollary to Theorem 1 in Angeletos (2002) and obtain the optimal portfolio $\omega_T^{CM,n}\left(s^T\right) = \omega^{CM,n}\left(s_T = s\right)$ for $n = 1, \ldots 50$ maturities that implements the complete markets allocation. We use the bond prices and present value of primary surpluses all of which can be backed out given the objects from the Ramsey allocation. In Figure 2, red color line, we plot $\left\{\omega^{CM,n}\right\}_{n=1}^{50}$ for $s_T = s_{24}$ which is the modal state.

Details for Figure 2 To obtain the target portfolio $\overrightarrow{\omega}_T$ given some history s^T, we need to solve for a vector of portfolio shares that satisfies

$$\Sigma_T \overrightarrow{\omega}_T = \left[\pi_T^Q \Sigma_T^Q + \pi_T^X \Sigma_T^X\right] \overrightarrow{\hat{\beta}}.$$

Before explaining how we get $\overrightarrow{\omega}_T$, we make two observations. First, given the properties of the Ramsey allocations, $\Sigma_T, \pi_T^Q, \pi_T^X, \Sigma_T^X$ only depend on state s_T, which we set to s_{24} and as before can be computed in closed form using the complete market allocation that we have already solved. Second, as mentioned in the main text the returns of different bonds are highly correlated in the neoclassical economy, which makes the matrix of returns Σ_T to be effectively non-invertible and there are a range of portfolios that satisfy inequality (26) for a given ϵ. To obtain the target portfolio that is plotted in Figure 2 blue color, we set $\epsilon = 1e - 3$ and pose the following minimization problem

$$\min_{\tilde{\boldsymbol{\omega}}} \left\|\tilde{\boldsymbol{\omega}} - \overrightarrow{\omega}_T^{ABN}\right\|$$

such that

$$\left\|\Sigma_T \tilde{\boldsymbol{\omega}}^n - \left[\pi^Q \Sigma_T^Q + \pi_T^X \Sigma_T^X + \pi_T^A \Sigma_T^A\right] \overrightarrow{\hat{\beta}}\right\| \leq \mathbf{1}^\intercal \epsilon.$$

where $||\cdot||$ we mean the sup norm . This formulation conveniently delivers an objective that is quadratic while the constraint set is linear and convex; and we use a standard methods (OSQP library) to solve the minimization problem.

From the outcomes in Figure 2, there are two clear observations. First, even though Corollary 1 abstracted from income and price effects, the resulting target portfolio provides an excellent fit to the optimal portfolio that replicated the complete market allocation in this neoclassical economy. Second, the portfolio in the neoclassical economy are very different than our target portfolio computed using U.S. data. It has large negative (695 times annual GDP in risk-free assets) positions in the risk-free bond and large and offsetting positions in risky bonds with flipping signs. As a point of reference, in the target portfolio that is calibrated to US data, positive debt is issued in all maturities with the maximum being around 1 percent, and the share in the risk-free debt is quite small 0.7 percent.

Risk-free bond and consol We chose the first 50 maturity zero coupon bonds to keep the discussion closer to the market structure analyzed in Section 4. The formula Theorem 1 of Angeletos (2002) can be applied to any $|\mathcal{S}|$ securities. Furthermore, if we replace inverse in equation (12) of Angeletos (2002) of with pseudo-inverse[33], we can also obtain the formula for a set of securities smaller than $|\mathcal{S}|$. Our online codes are flexible to implement any market structure but since the general patterns are not that different we discuss only one special case, in which the market structure has a risk-free bond and consol. Besides being a case analyzed in detail in Angeletos (2002), an attractive feature of risk-free bond and consol market structure is that Σ_T is a scalar, and the target portfolio (also a scalar) is uniquely pinned down. In this case, using the modified Angeletos (2002) formula, we find that the holdings in the consol, ω^{consol} is 746 percent of annual GDP and an offsetting position of in the risk free bond ω^0 of -646 percent of annual GDP. The counterpart $\overrightarrow{\omega}_T = \left[\frac{\pi_T^Q \Sigma_T^Q}{\Sigma_T} + \frac{\pi_T^X \Sigma_T^X}{\Sigma_T}\right] \overrightarrow{\hat{\beta}}$ for our target portfolio, we get 677 percent in the consol and an offsetting -577 percent in the risk-free bond. This fact both the portfolios are so close to each other reasserts the validity of our formula for the target portfolio.

Next comparing the share of holdings in the consol (741 percent of total debt) to the sum of shares of the portfolio that hedges primary surplus risk $\mathbf{1}^\intercal \Sigma_T^{-1} \pi_T^X \Sigma_T^X$ (-15% in the baseline with 120 maturities and -17% in the theoretical unrestricted target portfolio which we estimate from the US data in Section 4), we find that it is about 40-50 times larger with an opposite sign.

Understanding the sources of differences between the optimal portfolios We want to understand the sources that drive the differences between the optimal portfolios computed

[33] A pseudoinverse is minimum (Euclidean) norm solution to a system of linear equations with multiple solutions. See also discussion in Section V.B in Angeletos (2002).

using U.S. data and the neoclassical model. First, we construct the counterpart of Table 1 using data simulated for 265 quarters from neoclassical model economy. The results are reported in Table 4 below. Compared to Table 1 in the main text, we see that returns in the simulated economy are much less volatile. For instance, for long maturities the variance of returns is between 0.025 and 0.035 which is 300 times smaller than what we get for the U.S. counterparts. The covariances of returns with primary surplus are only 10-20 times smaller signaling a much higher correlation. Furthermore, the sign of the covariance with primary surplus is positive for long maturities while it is negative in for the U.S. data.

Table 4: COVARIANCE MATRIX FOR NEOCLASSICAL MODEL

	Excess returns r_t^j for various maturities j											Surplus to GDP	Tax rate	Liquidity premium
	6m	12m	18m	24m	30m	36m	42m	48m	54m	60m	120m	X_t/Y_t	τ_t	$\ln A_t^0$
6m	0.00015	0.00042	0.00067	0.00089	0.0011	0.0013	0.0014	0.0016	0.0017	0.0019	0.0023	0.0029	0	0
12m		0.0012	0.0019	0.0026	0.0031	0.0037	0.0041	0.0046	0.0049	0.0053	0.0066	0.0083	0	0
18m			0.003	0.0041	0.005	0.0058	0.0066	0.0072	0.0079	0.0084	0.01	0.013	0	0
24m				0.0054	0.0066	0.0078	0.0088	0.0097	0.01	0.011	0.014	0.018	0	0
30m					0.0082	0.0095	0.011	0.012	0.013	0.014	0.017	0.022	0	0
36m						0.011	0.013	0.014	0.015	0.016	0.02	0.025	0	0
42m							0.014	0.016	0.017	0.018	0.023	0.029	0	0
48m								0.017	0.019	0.02	0.025	0.032	0	0
54m									0.02	0.022	0.027	0.034	0	0
60m										0.023	0.029	0.037	0	0
120m											0.036	0.046	0	0
X_t/Y_t												0.58	0	0
τ_t													0	0
$\ln A_t^0$														0
Mean	0.0008	0.0023	0.0036	0.0047	0.0058	0.0067	0.0076	0.0083	0.009	0.0096	0.012	4.1	18	0
Autocorr	0.11	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.92	0.84	0

We simulate the neoclassical model for 265 quarters that correspond to the sample period 1952-2017. Excess returns 6m, 12m, ... are the nominal excess returns in Fama maturity portfolios corresponding to 6-12 months, 12-18 months, ... maturity bins, respectively. All data is quarterly and in percentage points.

Our factor structure suggests a parsimonious way to understand why the neoclassical portfolio has the features we highlighted, that is, large savings in risk-free bonds and offsetting positions in risky assets. To see that, consider a limiting case when the market structure has bonds of all maturities. In the main text equations (22) provides closed-form expressions for the share in the risk-free security and the risky portfolio as a function of of the factor loadings and highlights the role of the ratio $\frac{K^X}{\kappa_\infty}$ in driving the differences.

	Excess returns r_t^j for various maturities j														
	6m	12m	18m	24m	30m	36m	42m	48m	54m	60m	120m	$\ln G_t^{\$}$	$\ln Y_t^{\perp,\$}$	$\ln A_t^0$	f_t
α_k	0.001	0.003	0.004	0.006	0.007	0.008	0.009	0.010	0.010	0.011	0.014	-0.098	5.503	0.000	-0.003
	(0.001)	(0.002)	(0.003)	(0.004)	(0.005)	(0.006)	(0.007)	(0.008)	(0.008)	(0.009)	(0.011)	(0.170)	(0.017)	(0.000)	(0.025)
ρ_k	-0.034	-0.035	-0.036	-0.037	-0.037	-0.038	-0.038	-0.039	-0.039	-0.040	-0.041	0.455	0.452	0.000	0.861
	(0.061)	(0.061)	(0.061)	(0.061)	(0.061)	(0.061)	(0.061)	(0.061)	(0.061)	(0.061)	(0.060)	(0.025)	(0.025)	(0.000)	(0.032)
κ_k	-0.006	-0.019	-0.030	-0.039	-0.049	-0.057	-0.064	-0.071	-0.077	-0.082	-0.103	3.869	0.387	0.000	0.000
	(0.001)	(0.003)	(0.004)	(0.006)	(0.007)	(0.008)	(0.009)	(0.010)	(0.011)	(0.012)	(0.014)	(0.170)	(0.017)	(0.000)	(nan)
σ_k^2	0.000	0.001	0.003	0.004	0.007	0.009	0.012	0.014	0.017	0.019	0.030	1.908	0.019	0.000	0.163
	(0.000)	(0.000)	(0.000)	(0.000)	(0.001)	(0.001)	(0.001)	(0.001)	(0.001)	(0.002)	(0.003)	(0.169)	(0.002)	(0.000)	(0.014)
R2	0.173	0.174	0.174	0.175	0.175	0.175	0.176	0.176	0.176	0.176	0.177	0.936	0.936	nan	0.745

Table 5: Factor model estimation results Neoclassical model

Notes: This table records the OLS estimates of the factor model (21) without imposing $\rho_f = 0, \rho_Y = \rho_G = 1$. Standards errors are in parenthesis. All series are for a 265 quarters that correspond to the sample period 1952-2017.

We follow the same steps as in Section 4.2 and estimate the factor model but now using data simulated from the neoclassical economy. We make two changes relative to the baseline. We allow for ρ_G and ρ_Y to be smaller than 1 as the neoclassical model had AR(1) as the data generating process. Second, to estimate κ_∞ we extrapolate using $\kappa_j = a^0 - a^1 \exp\left(-a^2 \times j\right)$.[34] Plugging in for the values of K^X, κ_∞, we obtain $\frac{K^X}{\kappa_\infty}$ equals 9.72 in the neoclassical economy as compared to -0.6 using the U.S. data. Thus, from the lens of our expression (25), a big part of the differences between the optimal portfolios can be understood from the fact that $\frac{K^X}{\kappa_\infty}$ in the neoclassical model is much larger and of opposite sign from what we obtained in Section 4.2 using US data. Qualitatively it implies a larger and position with opposite opposite in the portfolio that hedges primary surplus risk. Quantitatively, the finding that $\frac{K^X}{\kappa_\infty}$ is about 20 times larger than in the neoclassical economy explains well the gaps in the portfolio that hedges the primary surplus risk. The reason for this can be traced back to observations we made about covariance of returns with each other and primary deficits in Table 4.

B.2.2 Additional discussion for Section 5.2

Heteroskedastic shocks In the main text, we assumed that the shocks ε_t were homoskedastic, that is, we imposed that $\{\sigma_k\}$ for $k \in \{j, Y, G, A, f\}$ are constant through time. We relax that assumption and augment the baseline factor model 21 with the following univariate GARH

[34] This is a slightly general version so that its fits the neoclassical data well. Our results are virtually the same if we use the linear splines extrapolation as in the Online Appendix B.1.2.

processes $\{\sigma_k\}$

$$\sigma_{k,t}^2 = \overline{\sigma}_k^2 + \sum_{j=1}^{p} \rho_{kp}^{GARCH} \varepsilon_{zt-p}^2 + \sum_{j=1}^{q} \varrho_{kq}^{GARCH} \sigma_{\varepsilon z,t-q}^2$$

and impose that all ε are standard Gaussian and independent of each other. We now estimate the system using maximum likelihood and assuming $p = 2$ and $q = 1$.

The consequence of heteroskedastic shocks is that structure of the expressions for Σ_T and Σ_T^{-1} as well as $\Sigma_T^k$for $k \in \{X, A, Q\}$ remains the same but they have time-varying parameters $\sigma_{f,t}$ and $\sigma_{j,t}$ for each return maturities j.[35] We use the same extrapolation scheme as the baseline to obtain (σ_j, κ_j) for other maturities. And finally, as an implication, the optimal target portfolio and its components also inherit that time-variation.[36]

Results In Figure 8, we plot the time-series for elements in $\{\sigma_{j,t}\}$ and $\sigma_{f,t}$. The volatilities for returns (including the factor) and macroaggregates are high in the early 80s and the great recession of 2008-2010 and quite stable in the intervening periods.

Keeping everything else the same, periods when the factor is more volatile increases the covariance of returns with each other as well as the covariance of returns with surpluses and liquidity risk. Thus, a priori the effect on the optimal portfolio is ambiguous. To gauge how much the portfolio moves overtime, we start by plotting in Figure 9), the 90-10 interval by maturity, that is, for each maturity we construct the 90th and 10th percentile across dates. We see that for lower maturities the portfolio shares varies by as much as 20-25 basis points and the fluctuations are much smaller for larger maturities.

To understand the sources of this variation, we separate out the primary surplus risk portfolio and the liquidity risk portfolios using expressions (22) and report the sum of the portfolio shares across maturities for every period. In Figure 10, we see that both these shares are quite stable through time, and more or less offset each other.

B.3 Additional details for Section 5.3

In this section we estimate excess liquidity premia a^i and statistically test $\mathbb{E}a_t^i = 0$. We first describe the estimation framework and then our findings.

[35]The time-variation in $\left\{\sigma_G^2, \sigma_Y^2, \sigma_A^2\right\}$ drops out because the covariances of hedging terms are driven by the common component captured in the factor $\left\{\sigma_{f,t}^2\right\}$.

[36]In principle, the fiscal risk and liquidity risk portfolio could vary because quasi-weights π_T^X and π_T^A or $\vec{\beta}$ vary with time. To focus on the impact of heteroskedastic shocks, we keep them constant and equal to the values that we used in the main text and only allow the target portfolio to vary due to time-varying covariances.

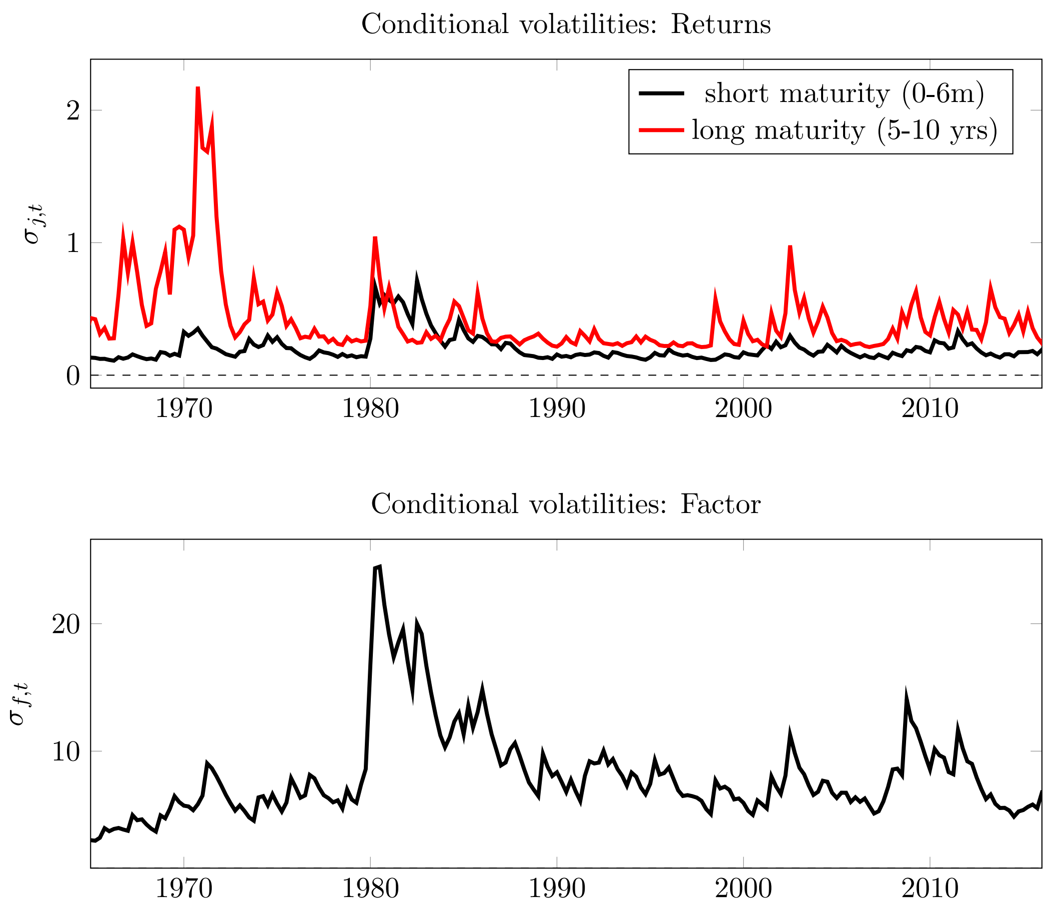


Figure 8: Conditional volatilities of returns, factor, using the estimated GARCH model

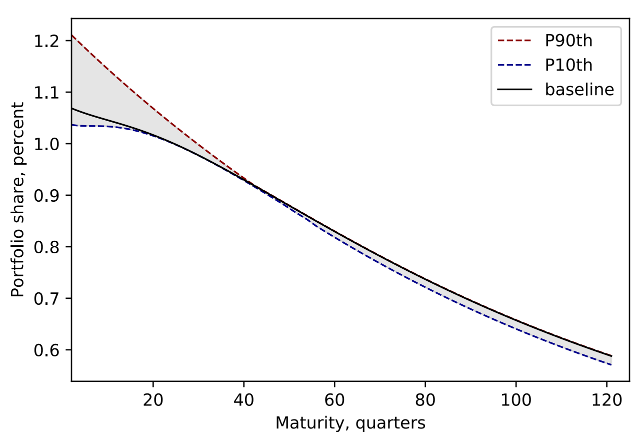


Figure 9: 90-10 interval of portfolio shares (maturities from 2 quarters to 120 quarters) with heteroskedastic shocks.

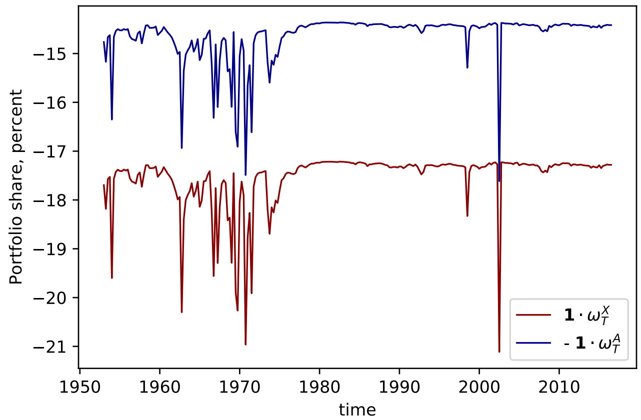


Figure 10: Components of target portfolio with heteroskedastic shocks. The blue line plots the sum of the shares of the portfolio that hedges the primary surplus risk, that is, $\mathbf{1} \cdot \pi_T^X \Sigma_t^{-1} \Sigma_t^X$ and the red line plots the negative of sum of the shares of the portfolio that hedges the liquidity risk, that is, $-\mathbf{1} \cdot \pi_T^A \Sigma_t^{-1} \Sigma_t^A$.

Framework From equation (10), we know that

$$a_t^i = -\mathbb{E}_t \frac{\beta M_{t+1}}{M_t} \left(R_{t+1}^i - R_t^0 \right) = -\mathbb{E}_t \frac{\beta M_{t+1}}{M_t} r_t^i.$$

To back out a, we need to estimate $\frac{\beta M_{t+1}}{M_t}$. We start by assuming that the SDF is affine in a vector of some demeaned factors $\mathbf{f}_t^{pvt}$:

$$\frac{\beta M_{t+1}}{M_t} = -c_0 - \mathbf{c_1} \cdot \mathbf{f}_t^{pvt},$$

and then use the fact that the there is no liquidity wedge for privately traded bonds. Thus we are looking for $(c_0, \mathbf{c_1})$ that minimize the error in $\mathbb{E}_t \frac{\beta M_{t+1}}{M_t} \boldsymbol{R}_t^{pvt} = \mathbf{1}$ for a given set of returns on private bonds $\boldsymbol{R}_t^{pvt}$. This yields a familiar expression for estimates of $(c_0, \mathbf{c_1})$

$$c_0 = -\left(\mathbb{E} R_t^{0,pvt} \right)^{-1},$$

$$\boldsymbol{c}_1^{'} = -c_0 \mathbb{E} \left(\boldsymbol{r}_{t+1}^{pvt} \right)^{\intercal} \left(\mathbb{E} \left[\boldsymbol{r}_{t+1}^{pvt} \right] \left(\boldsymbol{f}_{t+1}^{pvt} \right)^{\intercal} \right) \left(\left[\mathbb{E} \mathbf{f}_{t+1}^{pvt} \left(\left[\boldsymbol{r}_{t+1}^{pvt} \right] \right)^{\intercal} \right] \left[\mathbb{E} \left[\boldsymbol{r}_{t+1}^{pvt} \right] \left(\boldsymbol{f}_{t+1}^{pvt} \right)^{\intercal} \right] \right)^{-1}.$$

Fama and MacBeth (1973) show that $(c_0, \mathbf{c_1})$ can be estimated using a two step process in which we first run a return by return time-series regression to estimate security specific "betas" and then we run a cross section regression for each date to back out "lambdas" or factor risk premia, $\boldsymbol{\lambda}_t^{pvt}$

$$r_t^{pvt,j} = \gamma^j + \boldsymbol{\beta}^j \cdot \mathbf{f}_t^{pvt} + \epsilon_t^j,$$

$$\mathbf{r}_t^{pvt} = \boldsymbol{\alpha}_t + \boldsymbol{\beta}' \cdot \boldsymbol{\lambda}_t^{pvt},$$

and we can recover $(c_0, \mathbf{c_1})$ using

$$c_0 = -\mathbb{E}\left[R_t^{0,pvt}\right]^{-1},$$
$$c_1 = -c_0 \left(\Sigma_t^{\mathbf{f}}\right)^{-1} \mathbb{E}\left[\boldsymbol{\lambda}_t^{pvt}\right].$$

Using Fama McBeth procedure is useful because it immediately lends to an application for Delta method for computing the standard errors on the a. Let$\widehat{\ }$be the estimated counterparts of the theoretical objects. We can express $\widehat{\mathbb{E}\left[a_t^i\right]}$ as some function $s(\cdot)$ such that

$$\widehat{\mathbb{E}\left[a_t^i\right]} \equiv s\left(\left\{r_{t+1}^i\right\}_{t\geq 0}, \left\{R_t^{0,pvt}\right\}_{t\geq 1}, \left\{\mathbf{f}_t^{pvt}\right\}_{t\geq 1}, \left\{\boldsymbol{\lambda}_t^{pvt}\right\}_{t\geq 1}\right).$$

Applying the Delta method, we get that:

$$\sigma^2(\mathbb{E}\left[a_t^i\right]) = T \times \nabla s' \Sigma^a \nabla s,$$

where

$$\nabla s = -\frac{1}{T}\begin{bmatrix} \widehat{\mathbb{E}}\left[R_t^{0,pvt}\right]^{-1} \mathbf{1}_{T\times 1} \\ \left[\widehat{\mathbb{E}}\left[r_{t+1}^i\right] + \left(\widehat{\Sigma_t^{\mathbf{f}}}\right)^{-1} \widehat{\mathbb{E}}\left[\boldsymbol{\lambda}_t^{pvt}\right] \widehat{cov_t}(\mathbf{f}_t^{pvt}, r_{t+1}^i)\right] \widehat{\mathbb{E}}\left[R_t^{0,pvt}\right]^{-2} \mathbf{1}_{T\times 1} \\ 0 \\ \widehat{\mathbb{E}}\left[R_t^{0,pvt}\right]^{-1} \left(\widehat{\Sigma_t^{\mathbf{f}}}\right)^{-1} \widehat{cov_t}(\mathbf{f}_t^{pvt}, r_{t+1}^i) \mathbf{1}_{T\times 1} \end{bmatrix} \quad \Sigma^a = cov\begin{bmatrix} \left\{r_{t+1}^i\right\}_{t\geq 0} \\ \left\{R_t^{0,pvt}\right\}_{t\geq 1} \\ \left\{\mathbf{f}_t^{pvt}\right\}_{t\geq 1} \\ \left\{\boldsymbol{\lambda}_t^{pvt}\right\}_{t\geq 1} \end{bmatrix}.$$

Estimation To estimate the excess liquidity premia and its standard errors, we need three things: a set of factors Factors $\mathbf{f}_t^{pvt}$, a measure of private risk-free rate $R^{0,pvt}$, and a set of excess returns $\boldsymbol{r}_t^{pvt}$. We describe those choices and then our results.

For the SDF $\frac{\beta M_{t+1}}{M_t}$ estimation, we impose a 3 factors structure to the SDF as in Koijen et al. (2017). The first factor is Cochrane and Piazzesi (2005)'s "CP factor". The second factor is the level (LVL) factor, which is constructed as the first component of the forward rate covariance matrix, following Cochrane and Piazzesi (2008). The third factor is the value-weighted stock market excess return from CRSP. We then estimate the SDF with the returns on 5 portfolios of corporate bonds of credit ratings AAA, AA, and A, constructed from Bloomberg (formerly Barclays) indices,[37] and available from 1989 to 2015. These indices measure the investment grade, fixed-rate, taxable corporate bond market. They include USD-denominated securities publicly issued by US and non-US industrial, utility and financial issuers.[38] We use as a private

[37] We thank Alexandros Kontonikas for sharing with us the data used in and used in Guo, Kontonikas and Maio (2020)

[38] For more details, see https://www.bloomberg.com/professional/product/indices/bloomberg-fixed-income-indices-fact-sheets-publications/

risk free rate $R_t^{0,pvt}$ our previous estimates of A_t^0 such that $R_t^{0,pvt} = \frac{R_t^0}{1-A_t^0}$.[39]

We then apply our estimation framework. Our findings are reported in Table 6. We see that although the point estimates are negative relfecting the larger share of risk-adjustment, the main takeaway is that that all maturities the estimates are statistically not different from zero. Thus we cannot reject $\mathbb{E}a_t^i = 0$.

Table 6: Estimates of the time-averaged excess liquidity premium $\widehat{\mathbb{E}\left[a_t^i\right]}$

maturity i	$\hat{\mathbb{E}}\left[a_t^i\right]$	s.e	t-stat	p-value
6 months	-0.03	(0.12)	-0.24	0.81
12 months	-0.05	(0.33)	-0.16	0.87
18 months	-0.05	(0.63)	-0.07	0.94
24 months	-0.06	(0.83)	-0.07	0.94
30 months	-0.08	(1.04)	-0.07	0.94
36 months	-0.09	(1.23)	-0.07	0.94
42 months	-0.10	(1.47)	-0.07	0.95
48 months	-0.10	(1.65)	-0.06	0.95
54 months	-0.10	(1.85)	-0.05	0.96
60 months	-0.10	(1.92)	-0.05	0.96
120 months	-0.11	(2.28)	-0.05	0.96

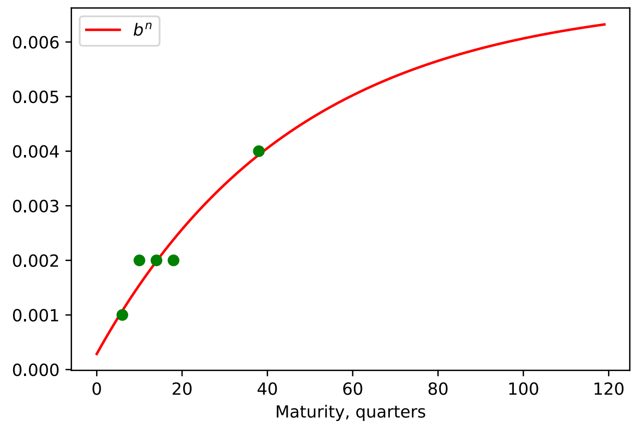
Notes: This table records the estimates of the average excess liquidity premium, its standard errors, and the associated t-statistics and p-value for the 11 Fama Maturity Portfolios. We take an average liquidity premium of 0 as our null hypothesis. The sample is 1989-2015. The units of the average excess liquidity premium is quarterly and in percentage points. We follow Fama and MacBeth (1973) and control for cross-sectional correlations but we assume that there is no serial correlations in the estimation of the SDF. We compute the standard errors using the Delta method.

B.4 Details for Section 5.8

Next we describe the how we build Λ_T^{QE} from the Greenwood and Vayanos (2014) estimates. We use the GV point estimates of b^i for their reported maturities and extrapolate for other maturities by fitting the same functional form that we used in the baseline for factor loadings.[40] The fit is reported in the left panel of Figure 11. In the right panel of Figure 11, we show the heatmap of Λ^{QE} (all normalized by its mean value) computed using expression (38) and the extrapolated $\{b^n\}$. The price impacts are larger around the south east region around the

[39] Alternatively, we also use the return on the portfolio of AAA corporate bonds of intermediate maturities (1 to 10 years) from Barclays as our (private) risk-free rate $R_t^{0,pvt}$. That doesn't affect much the results (the point estimates are modified but still very non-significant).

[40] In particular, Table 2 of GV reports estimates for bonds of maturities $2, 3, 4, 5, 10$ years. We assume that $b^n = \bar{b}_0 + \bar{b}_0 \exp\left(-\bar{b}_1 \times n\right)$ and find coefficients $\left\{\bar{b}_0, \bar{b}_1\right\}$ that minimize the least square errors. The results are robust to other extrapolation schemes.



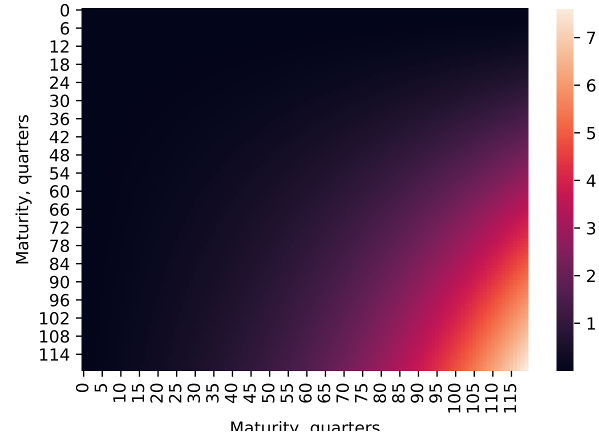


Figure 11: The left panel plots the fit for coefficients b^n. The right panel shows the normalized heatmap for the price impact matrix: Λ^{QE}

diagonal. Thus price impacts are large when securities involved are both of longer maturities.

C Closed Economy

In this appendix, we study a closed neoclassical version of our *benchmark economy*. Unlike the benchmark open economy specification in Section 3, a change in the governments portfolio will necessarily change the price of assets in economy; and, compared to the segmented markets version of the benchmark economy presented in Section 5.8, a change in the portfolio composition at date T will also affect the price of securities in all other periods.

In what follows, we show how to to adjust our variational approach to incorporate such effects on prices. Our main result is to characterize the price effects and using that we show that the closed economy neoclassical setting implies price responses that are counterfactual relative to the evidence reviewed in Section 5.8. Besides the different structure on price effects, the rest of the analysis of a closed economy including the steps to obtain the expression for the optimal portfolio are identical to Section 3.2. In Section C.1, we formally describe the neoclassical closed economy environment that we study, then introduce the perturbation and analyze the welfare effects and optimality of the government. The proofs of the main results are in Section C.2.

C.1 Analysis

In addition to the assumptions of the benchmark economy we assume that:

1. Household preferences are time separable

$$V_t = U_t \left(c_t - \frac{(y_t/\theta_t)^{1+1/\gamma}}{1+1/\gamma} \right) + \beta \mathbb{E}_t V_{t+1}.$$

   Following the neoclassical tradition, we abstract from trading frictions and non-pecuniary benefits of government debt by assuming that $\{Q_t^i b_t^i\}_{i \in \mathcal{G}_t}$ does not enter into the utility function and $\varphi_t\left(\{Q_t^i b_t^i\}\right) \equiv 0$.

2. Government expenditures $\mathbf{G}$ are exogenous.

3. Foreign investors are absent, $\boldsymbol{\mathcal{B}}^i = \mathbf{0}$, for all i and all assets are in zero net supply.[41]

4. The set of available securities can replicate a consol. We will let Q_t^∞ denote the price of the consol at date t.

Under these assumptions asset market clearing implies that

$$b_t^i = B_t^i$$

and

$$c_t + G_t = Y_t.$$

Absence of trading frictions and non-pecuniary benefits of government securities the household optimality conditions imply

$$\mathbb{E}_t M_{t+1} R_{t+1}^i = M_t \text{ or } M_t Q_t^i = \mathbb{E}_t \left[M_{t+1} \left(d_{t+1}^i + Q_{t+1}^i \right) \right] \tag{70}$$

Perturbation Following Section 3.2, we use a variational approach to isolate the optimal public portfolio. We consider any competitive equilibrium and introduce a perturbation at a particular history s^T by assuming that the government purchases $\frac{\epsilon}{Q_T^j(s^T)}$ units of security j which is funded by selling $\frac{\epsilon}{Q_T^{rf}(s^T)}$ of the risk free bond. This asset swap produces an additional $r_{T+1}^j(s^{T+1})\epsilon$ of excess returns at all histories s^{T+1} following s^T. We assume that the government uses those resources to purchase an additional $\frac{r_{T+1}^j(s^T)\epsilon}{1+Q_{T+1}^\infty(s^{T+1})}$ of the consol while keeping its holdings of all other assets constant. Due to its nature of swapping a longer security for a risk-free bond we will refer to this as a Quantitative Easing (or QE) perturbation and

[41] That all assets are in zero net supply is for notational simplicity. Assuming positive net supply simply adds another term to the resource constraint equivalent to changing exogenous government expenditures.

formally define it by

$$\partial_{j,T,\epsilon} B^i_t(s^t) = \begin{cases} \frac{\epsilon}{Q^0_T(s^T)} & \text{if } i = rf \text{ and } s^t = s^T, \\ -\frac{\epsilon}{Q^j_T(s^T)} & \text{if } i = j \text{ and } s^t = s^T, \\ -\frac{1}{1+Q^\infty_{T+1}(s^T)} \left(r^j_{T+1}\left(s^{T+1}\right)\right)\epsilon & \text{if } i = \infty \text{ and } s^t \succ s^T, t > T, \\ 0 & \text{otherwise.} \end{cases}$$

The change in portfolio composition necessarily requires a change in taxes to balance the governments budget constraint,

$$G_t + \sum_{i \geq 0}(Q^i_t + d^i_t)B^i_{t-1} = \tau_t Y_t + \sum_{i \geq 0} Q^i_t B^i_t.$$

Differentiating with respect to ϵ in the direction of the QE perturbation yields the following response of tax revenues

$$-\partial_{j,T,\epsilon}\left(\tau_t Y_t\right) = \frac{r^j_{T+1}\left(s^{T+1}\right)}{1 + Q^\infty_{T+1}\left(s^{T+1}\right)} \left(I_{\{s^t \succ s^T\}} \right) + \sum_{i \geq 0} \partial_{j,T,\epsilon} Q^i_t(s^t) \left(B^i_t(s^{t-1}) - B^i_{t-t}(s^t) \right) \tag{71}$$

where $I_{\{s^t \succ s^T\}}$ is an indicator returning 1 if history s^t follows from s^T and zero otherwise. Intuitively the effect of the perturbation on tax revenues is a combination of two effects. The first, $\frac{r^j_{T+1}\left(s^{T+1}\right)}{1+Q^\infty_{T+1}(s^{T+1})}\left(I_{\{s^t \succ s^T\}}\right)$, are the direct effects that are a result of the excess returns generated by the asset swap. The second,$\sum_{i \geq 0} \partial_{j,T,\epsilon} Q^i_t \left(B^i_t(s^t) - B^i_{t-1}(s^{t-1})\right)$, is the indirect effect that arises because the asset swap in period T changes prices not only in all future periods but also in all past periods starting from the initial date 0.

Assuming that the equilibrium manifold is sufficiently smooth, we can apply the envelope theorem to the household's maximization problem to obtain the welfare impact of this perturbation as $\epsilon \to 0$. The welfare effect of this perturbation comes from its effect on both tax rates and security prices and is given by

$$
\begin{aligned}
\partial_{j,T,\epsilon} V_0 &= \mathbb{E}_0 \sum_{t\geq 0} M_t \left(-\frac{\partial_{j,T,\epsilon}\left(\tau_t Y_t\right)}{\xi_t} + \sum_{i\geq 0} \partial_{j,T,\epsilon} Q_t^i \left(b_{t-1}^i - b_t^i\right) \right) \\
&= \mathbb{E}_0 \sum_{t\geq 0} M_t \left(-\frac{\partial_{j,T,\epsilon}\left(\tau_t Y_t\right)}{\xi_t} + \sum_{i\geq 0} \partial_{j,T,\epsilon} Q_t^i \left(B_{t-1}^i - B_t^i\right) \right) \\
&= \mathbb{E}_0 \left[\sum_{t\geq 0} M_t \sum_{i\geq 0} \partial_{j,T,\epsilon} Q_t^i \left(\frac{\xi_t B_{t-1}^i - B_{t-1}^i}{\xi_t} - \frac{\xi_t B_t^i - B_t^i}{\xi_t} \right) + \sum_{t\geq T+1} \left(\frac{M_t}{\xi_t}\right) \left(I_{\{\tilde{s}^t \succ s^T\}}\right) \left(\frac{r_{T+1}^j}{1+Q_{T+1}^{\infty}} \right) \right] \\
&= \mathbb{E}_0 \left[\sum_{t\geq 0} M_t \left(\frac{\xi_t - 1}{\xi_t}\right) \sum_{i\geq 0} \partial_{j,T,\epsilon} Q_t^i \left(B_{t-1}^i - B_t^i\right) + \sum_{t\geq T+1} \left(\frac{M_t}{\xi_t}\right) \left(I_{\{\tilde{s}^t \succ s^T\}}\right) \left(\frac{r_{T+1}^j}{1+Q_{T+1}^{\infty}} \right) \right] \\
&= \Pr_0\left(s^T\right) M_T\left(s^T\right) \left[PE_{j,T,\epsilon} + \mathbb{E}_T \sum_{k\geq 1} \left(\frac{M_{T+k}}{M_T}\right) \left(\frac{r_{T+1}^j}{1+Q_{T+1}^{\infty}}\right) \frac{1}{\xi_{T+k}} \right] \qquad (72)
\end{aligned}
$$

with

$$
PE_{j,T,\epsilon} = \frac{1}{\Pr_0\left(s^T\right) M_T\left(s^T\right)} \mathbb{E}_0 \left[\sum_{t\geq 0} M_t \left(\frac{\xi_t - 1}{\xi_t}\right) \sum_{i\geq 0} \partial_{j,T,\epsilon} Q_t^i \left(B_{t-1}^i - B_t^i\right) \right].
$$

The term $\mathbb{E}_T \sum_{k\geq 1} \left(\frac{M_{T+k}}{M_T}\right) \left(\frac{r_{T+1}^j}{1+Q_{T+1}^{\infty}}\right) \frac{1}{\xi_{T+k}}$ parallels the effect of the same perturbation in the open economy benchmark model, and can be analyzed in a similar manner. Now, in addition to that term, we also have $PE_{j,T,\epsilon}$ that captures the effect on asset prices for all histories starting from time 0 onward. In the next section we will show how our second order expansions can allow us express that term using covariances that can be measured in the data.

Characterizing the Price Effects The perturbation affects asset prices through its effect on the stochastic discount factor of the household. This can be seen by differentiating the household Euler equation (70) with respect to ϵ in the direction of the perturbation to get

$$
\left(\partial_{j,T,\epsilon} M_t\right) Q_t^i + M_t \left(\partial_{j,T,\epsilon} Q_t^i\right) = \mathbb{E}_t \left[\partial_{j,T,\epsilon} M_{t+1} \left(d_{t+1}^i + Q_{t+1}^i\right) + M_{t+1} \left(\partial_{j,T,\epsilon} Q_{t+1}^i\right)\right].
$$

As the perturbation affects the stochastic discount factor through changes in tax rates we define $\xi_{M,t} \equiv \frac{\partial \log M_t}{\partial(\tau_t y_t)}$ as the semi-elasticity of $\log M_t$ with respect to the tax revenues which implies $\partial_{j,T,\epsilon} M_t = M_t \xi_{M,t} \partial_{j,T,\epsilon}\left(\tau_t y_t\right)$. Under our assumptions, this semi-elasticity is given by

$$
\xi_{M,t} = -\psi_t \times \frac{1}{Y_t - G_t - \theta_t v\left(Y_t\right)} \times \left(\frac{\xi_t - 1}{\xi_t}\right)
$$

where $\psi_t \equiv \frac{-[c_t - v_t(Y_t)]U''(c_t - v_t(Y_t))}{U'(c_t - v_t(Y_t))}$ is the coefficient of relative risk aversion.

To get a better understanding of how these terms contribute the price effects in the closed economy we'll focus on a stationary version of the economy

Definition 5. An optimal competitive equilibrium is *stationary from time T* if there exists a constant R_T such that for all $t > T$ (i) $\mathbb{E}_T G_t \approx G_T$ (ii) $\mathbb{E}_T \delta_t \approx \delta_T$ (iii) $\mathbb{E}_T R_t^i \approx R_T$ for all i and (iv) $\mathbb{E}_T c_t \approx c_T$.

This definition of stationary differs from the stationarity of the main text in that we assume a growth rate of $\Gamma = 1$. All of our results extend to a positive growth rate assuming that the utility function is CRRA.[42] Our first set of results concern the asset pricing implications of the QE perturbation. We will leave the proof of both propositions to the end of the section.

Proposition 1. *For a neoclassical model which is stationary from time T*

1. The QE perturbation keeps asset prices zero to the first-order

$$\partial_\sigma \partial_{j,T,\epsilon} Q_t^i = 0 \quad \forall \quad i, t \geq 0$$

2. The QE perturbation only affects risk-premia at T

$$\mathbb{E}_t \partial_{\sigma\sigma} \partial_{j,T,\epsilon} r_{t+1}^i = 0 \quad \forall \quad t \neq T$$

   and at date T

$$\mathbb{E}_T \partial_{\sigma\sigma} \partial_{j,T,\epsilon} r_{T+1}^i = \frac{2\psi_T}{Y_T - G_T - \theta_T v\left(Y_T\right)} \times \left(\frac{1-\xi_T}{\xi_T}\right)\left(\frac{1}{1+\overline{Q}_{T+1}^\infty}\right) \mathbb{E}_t \partial_\sigma r_{t+1}^j \partial_\sigma r_{t+1}^i > 0,$$

   where ψ_T is coefficient of relative risk aversion.

This proposition states that the QE perturbation does not effect prices to zeroth or first order. This is inline with our modeling of price effects in Section 5.8 where we assume that the effect prices is at second order. Intuitively, to zeroth and first-order all assets have the same expected return so the QE perturbation only changes the risk profile of the household's stochastic discount factor which, in turn, will only effect prices to second order. Moreover, the proposition states that the effect on asset prices in the closed economy are counterfactual to what has been documented in the data. Estimates by Greenwood and Vayanos (2014) and others find that find that $\Lambda^{QE}[rf, j] \approx 0$ and $\Lambda^{QE}[i, j] > 0$ for $i > rf$ which implies that expected excess returns should decrease with the QE perturbation rather than increase:

$$\mathbb{E}_T \partial_{\sigma\sigma} \partial_{j,T,\epsilon} r_{T+1}^i = -\frac{\overline{Q}_{T+1}^i}{\overline{Q}_T^i} \frac{\partial_{\sigma\sigma} \partial_{j,T,\epsilon} Q_T^i}{\overline{Q}_T^i} < 0.$$

[42]The main difference is that we will require that the government smooth excess returns using a growth-adjusted consol rather than a pure consol.

When governments buy back long term debt by issuing short term debt, short term rates appear to be unchanged so expected excess returns are driven by the fall in the term premia as the increased demand drives up prices.

In contrast, in the closed economy, the government returns the excess returns from the QE swap via taxes which results in making states of the world where excess returns are high (low) better (worse) for the household by lowering (raising) tax rates in those states. As a result, the value of the asset decreases which raises the risk-premia. As noted, this is in inconsistent with the segmented market literature which finds that the excess returns on long maturity debt are lower after QE.

Finally, we are able to use our expansions to characterize the price effects

Proposition 2. *For a neoclassical economy which is stationary from time 0, if all initial debt* $\left\{B_{-1}^{i}\right\}_{i}$ *was risk-free then* $PE_{j,T,\epsilon} \simeq \left(\frac{\bar{\xi}}{\bar{\xi}-1}\right)^{-1} \Psi_T\left(s^T\right)$ *where*

$$\begin{aligned}\Psi_T\left(s^T\right) =& \frac{-2\overline{B}\overline{\xi}_M(\overline{Q}^{rf}-1)}{\left(1-\overline{B}(\overline{Q}^{rf}-1)\overline{\xi}_M\right)} \sum_{t=T+1}^{\infty}\left(\frac{\left(\overline{Q}_t^{rf}\right)^{t-T}}{1+\overline{Q}_{T+1}^{\infty}}\right) cov_T\left(\partial_\sigma \ln M_t, \partial_\sigma r_{T+1}^{j}\right) \\ & -\frac{2\overline{\xi}_M\overline{B}}{\left(1-\overline{B}(\overline{Q}^{rf}-1)\overline{\xi}_M\right)} \sum_{t=T+1}\left(\frac{\left(\overline{Q}_t^{rf}\right)^{-T}}{1+\overline{Q}_{T+1}^{\infty}}\right) cov_T\left(\partial_\sigma r_{T+1}^{j}, \partial_\sigma \ln Q_t^{rf}\right) \\ & -\frac{2\overline{\xi}_M}{\left(1-\overline{B}(\overline{Q}^{rf}-1)\overline{\xi}_M\right)} \sum_{j\geq 1}\frac{\overline{Q}_t^{rf}}{1+\overline{Q}_{T+1}^{\infty}\left(s^{T+1}\right)} cov_T\left(\partial_\sigma r_{T+1}^{j}, \partial_\sigma r_{T+1}^{j}\right) \\ & -\frac{2\overline{B}}{\left(1-\overline{B}(\overline{Q}^{rf}-1)\overline{\xi}_M\right)} \sum_{t=T}^{\infty}\left(\frac{\left(\overline{Q}_t^{rf}\right)^{t-T}}{1+\overline{Q}_{T+1}^{\infty}} cov_T\left(\partial_\sigma \xi_{M,t}-\partial_\sigma \xi_{M,t+1}, \partial_\sigma r_{T+1}^{j}\right)\right)\end{aligned}$$

As we have noted without any assumptions price effects are given by

$$PE_{j,T,\epsilon} = \frac{1}{\Pr_0\left(s^T\right) M_T\left(s^T\right)} \mathbb{E}_0\left[\sum_{t\geq 0} M_t\left(\frac{\xi_t-1}{\xi_t}\right)\sum_{i\geq 0}\partial_{j,T,\epsilon} Q_t^i\left(B_{t-1}^i - B_t^i\right)\right]$$

where a swap of securities at a particular history can affect asset prices at all other histories—past and future—due to general equilibrium effects on the stochastic discount factor that now directly depends on the tax rates. Proposition 2 allows us to characterize these price effects with a closed form expression using entirely time T covariances that are measurable in the data.

C.2 Proofs for Propositions 1 and 2

C.2.1 Proof of Proposition 1

We begin by noting that at the zeroth order, we get $\overline{\xi}^{j}_{M,t} = -\frac{\overline{\psi_t}}{\overline{Y}_t - \overline{G}_t - \overline{\theta}_t v(\overline{Y}_t)} \times \left(\frac{\overline{\xi}_t - 1}{\overline{\xi}_t}\right) = \overline{\xi}_{M,T}$, is independent of time and the details of the perturbation. We proceed by proving a series of lemmas documenting the results of Proposition 1

Lemma 11. *Expected excess returns are zero to the zeroth and the first order*

Proof. The zeroth of (70) gives us

$$\overline{r}^{i}_{t+1} = 0$$

Take first-order expansion to get

$$\mathbb{E}_t \partial_\sigma r^i_{t+1} \overline{M_{t+1}} + \mathbb{E}_t \overline{r^i_{t+1}} \partial_\sigma M_{t+1} = 0$$

and thus

$$\mathbb{E}_t \partial_\sigma r^i_{t+1} = 0.$$

$\square$

Lemma 12. *To the first-order, price effects are zero, that is, for all i, t: $\partial_\sigma \partial_{j,T,\epsilon} Q^i_t = 0$*

Proof. Start from the definition of Q^i_t

$$Q^i_t \left(s^t\right) = \mathbb{E}_{s^t} \sum_{k \geq 1} \frac{M_{t+k}}{M_t} D^i_{t+k}.$$

$$\partial_\sigma \partial_{j,T,\epsilon} Q^i_t = \mathbb{E}_t \sum_{k \geq 1} \left(\partial_{j,T,\epsilon} \partial_\sigma \log M_{t+k} - \partial_{j,T,\epsilon} \partial_\sigma \log M_t\right) \left(\frac{\overline{M_{t+k}}}{\overline{M}_t}\right) D^i_{T+k}.$$

A necessary and sufficient condition for price effects to be zero at the firs-order is that $k \geq 1$

$$\mathbb{E}_t \left(\partial_{j,T,\epsilon} \partial_\sigma \log M_{t+k} - \partial_{j,T,\epsilon} \partial_\sigma \log M_t\right) = 0 \tag{73}$$

Use the definition of $\xi_M \left(s^t\right)$ to get $\partial_{j,T,\epsilon} \log M_t \left(s^t\right) = \partial_{j,T,\epsilon} \left(\tau_t \left(s^t\right) Y_t \left(s^t\right)\right) \times \xi_M \left(s^t\right)$. To first-order

$$\partial_\sigma \partial_{j,T,\epsilon} \log M_t = \partial_\sigma \partial_{j,T,\epsilon} \left(\tau_t Y_t\right) \times \overline{\xi}_{M,t}$$

Then (73) is equivalently expressed as

$$\mathbb{E}_t \left(\partial_\sigma \partial_{j,T,\epsilon} \log M_{t+k} - \partial_\sigma \partial_{j,T,\epsilon} \log M_t\right) = \overline{\xi}_{M,t} \left(\mathbb{E}_t \partial_\sigma \partial_{j,T,\epsilon} \left(\tau_{t+k} Y_{t+k}\right) - \partial_\sigma \partial_{j,T,\epsilon} \left(\tau_t Y_t\right)\right)$$

We check condition (73) by guess and verify.

Suppose $\partial_\sigma\partial_{j,T,\epsilon}Q_t^i = 0$ for $t \geq 0$, then for all $t \geq 0$ and from equations (71)

$$-\partial_\sigma\partial_{j,T,\epsilon}\left(\tau_t Y_t\right) = \partial_\sigma\left(\frac{r_{T+1}^j\left(s^{T+1}\right)}{1+Q_{T+1}^\infty\left(s^{T+1}\right)}\right)I_{\{s^t\succ s^{T+1}\}} = \frac{\partial_\sigma r_{T+1}^j\left(s^{T+1}\right)}{1+\overline{Q}_{T+1}^\infty}I_{\{s^t\succ s^{T+1}\}}$$

When $t \geq T+1$

$$\mathbb{E}_t\left(\partial_\sigma\partial_{j,T,\epsilon}\log M_{T+1+k} - \partial_\sigma\partial_{j,T,\epsilon}\log M_{T+1}\right) = \overline{\xi}_{M,T+1}\left(\frac{\partial_\sigma r_{T+1}^j\left(s^{T+1}\right)}{1+\overline{Q}_{T+1}^\infty} - \frac{\partial_\sigma r_{T+1}^j\left(s^{T+1}\right)}{1+\overline{Q}_{T+1}^\infty}\right)I_{\{s^t\succ s^{T+1}\}} = 0$$

When $t \leq T$, we can use the fact that to the first order, expected excess returns are zero from Lemma (12) to establish that (73) holds. □

Lemma 13. *In the closed economy the effect of the perturbation on expected excess returns is*

$$\mathbb{E}_t\partial_{\sigma\sigma}\partial_{j,T,\epsilon}r_{t+1}^i = 0 \quad \forall \quad t \neq T$$

and at date T

$$\mathbb{E}_T\partial_{\sigma\sigma}\partial_{j,T,\epsilon}r_{T+1}^i = \frac{2\psi_T}{Y_T - G_T - \theta_T v\left(Y_T\right)} \times \left(\frac{1-\xi_T}{\xi_T}\right)\left(\frac{1}{1+\overline{Q}_{T+1}^\infty}\right)\mathbb{E}_t\partial_\sigma r_{t+1}^j\partial_\sigma r_{t+1}^i > 0$$

Proof. The first-order expansion $\partial_{j,T,\epsilon}M_t$ after using Lemma 12 gives us

$$\partial_\sigma\partial_{j,T,\epsilon}M_{t+1} = -\overline{\xi}_{M,t}\overline{M}_{t+1}\left\{\partial_\sigma\left(\frac{r_{T+1}^j}{1+Q_{T+1}^\infty}\right)I_{\{\boldsymbol{s}^t\succ\boldsymbol{s}^{T+1}\}}\right\}$$

Use this along with the second order expansion of households optimality condition (70) to obtain

$$0 = \mathbb{E}_t\partial_\sigma r_{t+1}^i\left(-\overline{\xi}_{M,t}\overline{M}_{t+1}\left\{\partial_\sigma\left(\frac{r_{T+1}^j}{1+q_{T+1}^\infty}\right)I_{\{\boldsymbol{s}^t\succ\boldsymbol{s}^{T+1}\}}\right\}\right) + \mathbb{E}_t\partial_{\sigma\sigma}\partial_{j,T,\epsilon}r_{t+1}^i\overline{M}_{t+1}$$

For $t < T$, $I_{\{\boldsymbol{s}^t\succ\boldsymbol{s}^{T+1}\}} = 0$ and thus $\mathbb{E}_t\partial_{\sigma\sigma}\partial_{j,T,\epsilon}r_{t+1}^i = 0$.

For $\boldsymbol{s}^t \succ \boldsymbol{s}^{T+1}$, use Law of iterated expectations to get

$$0 = \mathbb{E}_{T+1+k}\partial_\sigma r_{T+1+k}^i\left(-\overline{\xi}_{M,T+1+k}\overline{M}_{T+1+k}\underbrace{\mathbb{E}_{T+1}\left\{\partial_\sigma\left(\frac{r_{T+1}^j}{1+Q_{T+1}^\infty}\right)\right\}}_{=0}\right) + \mathbb{E}_{T+1+k}\partial_{\sigma\sigma}\partial_{j,T,\epsilon}r_{T+k+2}^i\overline{M}_{T+2+k}$$

and use Lemma (12) to get $\mathbb{E}_t\partial_{\sigma\sigma}\partial_{j,T,\epsilon}r_{t+1}^i = 0$ for $\boldsymbol{s}^t \succ \boldsymbol{s}^{T+1}$.

Finally for $t = T$

$$0 = \mathbb{E}_T\partial_\sigma r_{T+1}^i\left(-\overline{\xi}_{M,T}\overline{M}_{T+1}\left\{\partial_\sigma\left(\frac{r_{T+1}^j}{1+q_{T+1}^\infty}\right)\right\}\right) + \mathbb{E}_t\partial_{\sigma\sigma}\partial_{j,T,\epsilon}r_{t+1}^i\overline{M}_{T+1}.$$

Substitute for $\overline{\xi}_{M,T}$ and simplify to to get

$$\mathbb{E}_T\partial_{\sigma\sigma}\partial_{j,T,\epsilon}r^i_{T+1} \simeq \frac{2\psi_T}{Y_T - G_T - \theta_T v\left(Y_T\right)} \times \left(\frac{1-\xi_T}{\xi_T}\right)\left(\frac{1}{1+\overline{Q}^{\infty}_{T+1}}\right)\mathbb{E}_t\partial_\sigma r^j_{T+1}\partial_\sigma r^i_{t+1}.$$

Since $\xi_T = 1 - \gamma\frac{\tau_T}{1-\tau_T} < 1$, $Y_T - G_T - \theta_T v\left(Y_T\right) > 0$ from Inada conditions, and $\psi_T > 0$, we get that $\mathbb{E}_T\partial_{\sigma\sigma}\partial_{j,T,\epsilon}r^j_{T+1} > 0$. □

C.2.2 Proof of Proposition 2

The second order expansion of the price effects

$$\partial_{\sigma\sigma}\left(\Pr_0\left(s^T\right)M_T\left(s^T\right)PE_{j,T,\epsilon}\right) = \mathbb{E}_0\left[\sum_{t\geq 0}\left(\frac{\bar{\xi}_t - 1}{\bar{\xi}_t}\right)M_t\sum_{i\geq 0}\partial_{\sigma\sigma}\partial_{j,T,\epsilon}Q^i_t\left(\overline{B}^i_{t-1} - \overline{B}^i_t\right)\right] \tag{74}$$

which equals

$$\left(\frac{\bar{\xi}_0 - 1}{\bar{\xi}_0}\right)\overline{M}_0\sum_{i\geq 0}\partial_{\sigma\sigma}\partial_{j,T,\epsilon}Q^i_0\overline{B}^i_{-1} + \left(\frac{\bar{\xi}_0 - 1}{\bar{\xi}_0}\right)\mathbb{E}_0\left[\sum_{t\geq 0}\sum_{i\geq 0}\overline{B}^i_t\left(\overline{M}_{t+1}\partial_{\sigma\sigma}\partial_{j,T,\epsilon}Q^i_{t+1} - \overline{M}_t\partial_{\sigma\sigma}\partial_{j,T,\epsilon}Q^i_t\right)\right]. \tag{75}$$

Its easy to see that $\left(\frac{\bar{\xi}_0-1}{\bar{\xi}_0}\right)\overline{M}_0\sum_{i\geq 0}\partial_{\sigma\sigma}\partial_{j,T,\epsilon}Q^i_0\overline{B}^i_{-1} = \left(\frac{\bar{\xi}_0-1}{\bar{\xi}_0}\right)\overline{M}_0\sum_{i\neq rf}\partial_{\sigma\sigma}\partial_{j,T,\epsilon}Q^i_0\overline{B}^i_{-1} = 0$ under the assumption that initial debt was risk-free.

The household pricing equation implies

$$M_tQ^i_t = \mathbb{E}_t\left[M_{t+1}\left(Q^i_{t+1} + D^i_{t+1}\right)\right] \tag{76}$$

Differentiating by $\partial_{j,T,\epsilon}$ gives

$$\left(\partial_{j,T,\epsilon}M_t\right)Q^i_t + M_t\partial_{j,T,\epsilon}Q^i_t = \mathbb{E}_t\left[\left(\partial_{j,T,\epsilon}M_{t+1}\right)\left(Q^i_{t+1} + D^i_{t+1}\right) + M_{t+1}\partial_{j,T,\epsilon}Q^i_{t+1}\right]$$

Let's start by looking at $t < T$, We know that $\partial_\sigma\partial_{j,T,\epsilon}M_{t+1} = 0$ so taking the second derivative with respect to σ yields

$$\mathbb{E}_t\left[\overline{M}_{t+1}\partial_{\sigma\sigma}\partial_{j,T,\epsilon}Q^i_{t+1} - \overline{M}_t\partial_{\sigma\sigma}\partial_{j,T,\epsilon}Q^i_t\right] = \overline{Q}^i_t\mathbb{E}_t\left[\left(\partial_{\sigma\sigma}\partial_{j,T,\epsilon}M_t\right) - \left(\partial_{\sigma\sigma}\partial_{j,T,\epsilon}M_{t+1}\right)\overline{R}^{rf}_{t+1}\right].$$

For $t > T$ and $s^t \succ s^T$ we have $\frac{\partial_{j,T,\epsilon}M_t}{M_t} = \xi^j_{M,t}\partial_{j,T,\epsilon}\left(\tau_tY_t\right)$ and hence $\partial_\sigma\partial_{j,T,\epsilon}M_t = \overline{M}_t\bar{\xi}_{M,t}\frac{\partial_\sigma r^j_{T+1}}{1+\overline{Q}^{\infty}_{T+1}}$. The second-order expansion of equation (76) is

$$\begin{aligned}2\partial_\sigma\partial_{j,T,\epsilon}M_t\partial_\sigma Q^i_t + \left(\partial_{\sigma\sigma}\partial_{j,T,\epsilon}M_t\right)\overline{Q}^i_t + \overline{M}_t\partial_{\sigma\sigma}\partial_{j,T,\epsilon}Q^i_t &= \mathbb{E}_t\left[2\partial_\sigma\partial_{j,T,\epsilon}M_{t+1}\partial_\sigma\left(Q^i_{t+1} + D^i_{t+1}\right)\right]\\ &+ \left(\partial_{\sigma\sigma}\partial_{j,T,\epsilon}M_{t+1}\right)\left(\overline{Q}^i_{t+1} + D^i_{t+1}\right) + \overline{M}_{t+1}\partial_{\sigma\sigma}\partial_{j,T,\epsilon}Q^i_{t+1}\end{aligned}$$

We know that

$$\mathbb{E}_t\left[\partial_\sigma \partial_{j,T,\epsilon} M_{t+1} \partial_\sigma\left(Q_{t+1}^i + D_{t+1}^i\right)\right] = \bar{\xi}_{M,T+1} \frac{\partial_\sigma r_{T+1}^j}{1+\overline{Q}_{T+1}^\infty} \mathbb{E}_t\left[\overline{M}_{t+1} \partial_\sigma\left(Q_{t+1}^i + D_{t+1}^i\right)\right]$$

so we get

$$\begin{aligned}
&\mathbb{E}_t\left[\partial_\sigma \partial_{j,T,\epsilon} M_{t+1} \partial_\sigma\left(Q_{t+1}^i + D_{t+1}^i\right)\right] - \partial_\sigma \partial_{j,T,\epsilon} M_t \partial_\sigma Q_t^i \\
&= \frac{\partial_\sigma r_{T+1}^j}{1+\overline{Q}_{T+1}^\infty} \bar{\xi}_{M,T+1}\left(\mathbb{E}_t\left[\overline{M}_{t+1} \partial_\sigma\left(Q_{t+1}^i + D_{t+1}^i\right)\right] - \overline{M}_t \partial_\sigma Q_t^i\right) \\
&= \frac{\partial_\sigma r_{T+1}^j}{1+\overline{Q}_{T+1}^\infty} \bar{\xi}_{M,T+1} \overline{Q}_t^i\left(\partial_\sigma M_t - \partial_\sigma M_{t+1} \overline{R}_{t+1}^{rf}\right)
\end{aligned}$$

with the last equality coming from

$$\partial_\sigma M_t \overline{Q}_t^i + \overline{M}_t \partial_\sigma Q_t^i = \mathbb{E}_t\left[\partial_\sigma M_{t+1}\left(\overline{Q}_{t+1}^i + \overline{D}_{t+1}^i\right) + \overline{M}_{t+1} \partial_\sigma\left(Q_{t+1}^i + D_{t+1}^i\right)\right].$$

Note that this only depends on i through $\overline{Q}^i$ thus for $t > T$

$$\begin{aligned}
&\mathbb{E}_t\left[\overline{M}_{t+1} \partial_{\sigma\sigma} \partial_{j,T,\epsilon} Q_{t+1}^i - \overline{M}_t \partial_{\sigma\sigma} \partial_{j,T,\epsilon} Q_t^i\right] \\
&= \overline{Q}_t^i \mathbb{E}_t\left[\left(\partial_{\sigma\sigma} \partial_{j,T,\epsilon} M_t\right) - \left(\partial_{\sigma\sigma} \partial_{j,T,\epsilon} M_{t+1}\right) \overline{R}_{t+1}^{rf} - \bar{\xi}_{M,T+1} \overline{M}_t \frac{\partial_\sigma r_{T+1}^j}{1+\overline{Q}_{T+1}^\infty} \frac{\partial_\sigma Q_t^{rf}}{\overline{Q}_t^{rf}}\right]
\end{aligned}$$

where the last term is simplified by noting that $\overline{M}_t \frac{\partial_\sigma Q_t^{rf}}{\overline{Q}_t^{rf}} = \mathbb{E}_t\left[\frac{1}{\overline{Q}_t^{rf}} \partial_\sigma M_{t+1} - \partial_\sigma M_t\right]$.

Finally, we have the $t = T$ and $s^t = s^T$ term which gives

$$\begin{aligned}
&\mathbb{E}_T\left[\overline{M}_{T+1} \partial_{\sigma\sigma} \partial_{j,T,\epsilon} Q_{T+1}^i - \overline{M}_T \partial_{\sigma\sigma} \partial_{j,T,\epsilon} Q_T^i\right] \\
&= \overline{Q}_T^i \mathbb{E}_t\left[\left(\partial_{\sigma\sigma} \partial_{j,T,\epsilon} M_T\right) - \left(\partial_{\sigma\sigma} \partial_{j,T,\epsilon} M_{T+1}\right) \overline{R}_{T+1}^{rf} - \frac{\bar{\xi}_{M,T+1} \overline{M}_{T+1}}{1+\overline{Q}_{T+1}^\infty\left(s^{T+1}\right)} \partial_\sigma r_{T+1}^j \partial_\sigma r_{T+1}^i\right].
\end{aligned}$$

Now we note that all the terms $\overline{M}_{t+1} \partial_{\sigma\sigma} \partial_{j,T,\epsilon} Q_{t+1}^i - \overline{M}_t \partial_{\sigma\sigma} \partial_{j,T,\epsilon} Q_t^i$ in the price effect sum have a component $\overline{Q}_t^i\left(\left(\partial_{\sigma\sigma} \partial_{j,T,\epsilon} M_t\right) - \left(\partial_{\sigma\sigma} \partial_{j,T,\epsilon} M_{t+1}\right) \overline{R}_{t+1}^{rf}\right)$ in them. We gain some tractability by substituting $\partial_{\sigma\sigma} \partial_{j,T,\epsilon} M_t = \overline{M}_t \bar{\xi}_M \partial_{\sigma\sigma} \partial_{j,T,\epsilon}\left(\tau_t Y_t\right) + 2\overline{M}_t \partial_\sigma \xi_{M,t} \partial_\sigma \partial_{j,T,\epsilon}\left(\tau_t Y_t\right)$ and doing so makes

$$\begin{aligned}
&\mathbb{E}_0\left[\sum_{t\geq 0}\sum_{i\geq 0}\overline{B}_t^i\left(\overline{M}_{t+1}\partial_{\sigma\sigma}\partial_{j,T,\epsilon}Q_{t+1}^i-\overline{M}_t\partial_{\sigma\sigma}\partial_{j,T,\epsilon}Q_t^i\right)\right]\\
&=\mathbb{E}_0\left[\sum_{t=0}^{\infty}\overline{B}_t\overline{M}_t\bar{\xi}_{M,t}\left(\partial_{\sigma\sigma}\partial_{j,T,\epsilon}\left(\tau_tY_t\right)-\partial_{\sigma\sigma}\partial_{j,T,\epsilon}\left(\tau_{t+1}Y_{t+1}\right)\right)\right]\\
&-2\Pr(s^T)\bar{\xi}_{M,T+1}\mathbb{E}_T\left[\sum_{t=T+1}\overline{B}_t\overline{M}_t\frac{\partial_\sigma r_{T+1}^j}{1+\overline{Q}_{T+1}^{\infty}}\frac{\partial_\sigma Q_t^{rf}}{\overline{Q}_t^{rf}}\right] \qquad (77)\\
&-2\Pr(s^T)\bar{\xi}_{M,T+1}\mathbb{E}_T\left[\sum_{j\geq 1}\frac{\overline{M}_{T+1}}{1+\overline{Q}_{T+1}^{\infty}\left(s^{T+1}\right)}\partial_\sigma r_{T+1}^j\partial_\sigma r_{T+1}^i\right]\\
&+2\Pr(s^T)\mathbb{E}_T\left[\sum_{t=T}^{\infty}\overline{B}_t\overline{M_t}\left(\partial_\sigma\xi_{M,t}\partial_\sigma\partial_{j,T,\epsilon}\left(\tau_tY_t\right)-\partial_\sigma\xi_{M,t+1}\partial_\sigma\partial_{j,T,\epsilon}\left(\tau_{t+1}Y_{t+1}\right)\right)\right] \qquad (78)
\end{aligned}$$

Most of these objects we can easily put some structure on except for

$$\mathbb{E}_0\left[\sum_{t=0}^{\infty}\overline{B}_t\overline{M}_t\bar{\xi}_{M,t}\left(\partial_{\sigma\sigma}\partial_{j,T,\epsilon}\left(\tau_tY_t\right)-\partial_{\sigma\sigma}\partial_{j,T,\epsilon}\left(\tau_{t+1}Y_{t+1}\right)\right)\right],$$

there we have note that $\overline{B}_t=\overline{B}_0=\overline{B}$, $\overline{M}_t=\left(Q^{rf}\right)^t\overline{M_0}$ and $\bar{\xi}_{M,t}=\bar{\xi}_{M,0}=\bar{\xi}_M$. Put together we have

$$\begin{aligned}
&\mathbb{E}_0\left[\sum_{t=0}^{\infty}\overline{B}_t\overline{M}_t\bar{\xi}_{M,t}\left(\partial_{\sigma\sigma}\partial_{j,T,\epsilon}\left(\tau_tY_t\right)-\partial_{\sigma\sigma}\partial_{j,T,\epsilon}\left(\tau_{t+1}Y_{t+1}\right)\right)\right]\\
&=\overline{B\xi}_{M,}\mathbb{E}_0\left[\sum_{t=0}^{\infty}\left(Q^{rf}\right)^t\left(\partial_{\sigma\sigma}\partial_{j,T,\epsilon}\left(\tau_tY_t\right)-\partial_{\sigma\sigma}\partial_{j,T,\epsilon}\left(\tau_{t+1}Y_{t+1}\right)\right)\right]\overline{M}_0\\
&=\overline{B\xi}_{M,}\mathbb{E}_0\left[\sum_{t=0}^{\infty}\left(Q^{rf}\right)^t(Q^{rf}-1)\partial_{\sigma\sigma}\partial_{j,T,\epsilon}\left(\tau_tY_t\right)\right]\overline{M}_0
\end{aligned}$$

we can then plug into $\partial_{\sigma\sigma}\partial_{j,T,\epsilon}\left(\tau_t Y_t\right)$ to get

$$\begin{aligned}
&\mathbb{E}_0\left[\sum_{t=0}^{\infty}\overline{B}_t\overline{M}_t\overline{\xi}_{M,t}\left(\partial_{\sigma\sigma}\partial_{j,T,\epsilon}\left(\tau_t Y_t\right)-\partial_{\sigma\sigma}\partial_{j,T,\epsilon}\left(\tau_{t+1}Y_{t+1}\right)\right)\right]\\
&=\overline{B}(Q^{rf}-1)\overline{\xi}_M\overline{M}_0\mathbb{E}_0\left[\sum_{t=0}^{\infty}\left(Q^{rf}\right)^t\sum_{i\geq 0}\partial_{\sigma\sigma}\partial_{j,T,\epsilon}Q_t^i\left(\overline{B}_{t-1}^i-\overline{B}_t^i\right)\right]\\
&+\overline{B}\overline{\xi}_M\overline{M}_0(Q^{rf}-1)\Pr(s^T)\mathbb{E}_T\left[\sum_{t=t+1}^{\infty}\left(Q^{rf}\right)^t\partial_{\sigma\sigma}\left(\frac{r_{T+1}^j\left(s^{T+1}\right)}{1+\overline{Q}_{T+1}^{\infty}}\right)\right]\\
=&\overline{B}(Q^{rf}-1)\overline{\xi}_M\frac{\overline{\xi}}{\overline{\xi}-1}\partial_{\sigma\sigma}\left(\Pr_0\left(s^T\right)M_T\left(s^T\right)PE_0^j\left(s^T\right)\right) \qquad (79)\\
&+\overline{B}\overline{\xi}_M\overline{M}_0(Q^{rf}-1)\Pr(s^T)\mathbb{E}_T\left[\sum_{t=t+1}^{\infty}\left(Q^{rf}\right)^t\partial_{\sigma\sigma}\left(\frac{r_{T+1}^j\left(s^{T+1}\right)}{1+\overline{Q}_{T+1}^{\infty}}\right)\right]
\end{aligned}$$

Going back to the HH version of this perturbation we get

$$\mathbb{E}_T\left[\sum_{t=T+1}^{\infty}M_t\frac{r_{T+1}^j}{1+\overline{Q}_{T+1}^{\infty}}\right]=0$$

As second order expansion of this gives

$$\overline{M}_0\mathbb{E}_T\left[\sum_{t=T+1}^{\infty}\left(Q^{rf}\right)^t\partial_{\sigma\sigma}\left(\frac{r_{T+1}^j\left(s^{T+1}\right)}{1+\overline{Q}_{T+1}^{\infty}}\right)\right]=-2\mathbb{E}_T\left[\sum_{t=T+1}^{\infty}\partial_\sigma M_t\frac{\partial_\sigma r_{T+1}^j\left(s^{T+1}\right)}{1+\overline{Q}_{T+1}^{\infty}}\right] \qquad (80)$$

Putting all together we get (combining equations (74),(77),(79), and (80))

$$\begin{aligned}
\left(\frac{\overline{\xi}}{\overline{\xi}-1}\right)\partial_{\sigma\sigma}PE_{j,T,\epsilon}=&\frac{-2\overline{\xi}_M}{\left(1-\overline{B}_0(Q^{rf}-1)\right)\left(Q^{rf}\right)^T}\mathbb{E}_T\left[\sum_{t=T+1}\overline{B}_t\frac{\partial_\sigma r_{T+1}^j}{1+\overline{Q}_{T+1}^{\infty}}\frac{\partial_\sigma Q_t^{rf}}{\overline{Q}_t^{rf}}\right]\\
&-\frac{2\overline{B}\overline{\xi}_M(Q^{rf}-1)}{\left(1-\overline{B}(Q^{rf}-1)\right)}\mathbb{E}_T\left[\sum_{t=T+1}^{\infty}\left(Q^{rf}\right)^{t-T}\partial_\sigma\ln M_t\frac{\partial_\sigma r_{T+1}^j\left(s^{T+1}\right)}{1+\overline{Q}_{T+1}^{\infty}}\right]\\
&\frac{-2\overline{\xi}_M}{\left(1-\overline{B}(Q^{rf}-1)\right)\left(Q^{rf}\right)^T}\mathbb{E}_T\left[\sum_{j\geq 1}\frac{Q^{rf}}{1+\overline{Q}_{T+1}^{\infty}\left(s^{T+1}\right)}\partial_\sigma r_{T+1}^j\partial_\sigma r_{T+1}^i\right]\\
&-\frac{2\overline{B}}{\left(1-\overline{B}(Q^{rf}-1)\right)}\mathbb{E}_T\left[\sum_{t=T}^{\infty}\left(Q^{rf}\right)^{t-T}\frac{\partial_\sigma r_{T+1}^j\left(s^{T+1}\right)}{1+\overline{Q}_{T+1}^{\infty}}\left(\partial_\sigma\xi_{M,t}-\partial_\sigma\xi_{M,t+1}\right)\right]
\end{aligned}$$

as desired.