Bundling under Double-Crossing Preferences*

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A monopolist sells two goods to a buyer with additive, type-increasing valuations. Moreover, the valuation for one good increases more convexly than for the other, implying a double-crossing property in the preferences for bundles. We characterize the optimal mechanism by extending the canonical Myersonian approach. Above a critical type, the allocations are monotone and maximal separation arises whenever stochastic bundles are used, while below this threshold, there is at most one interval where the quantities move in opposite directions, with maximal bunching. The screening structure depends on non-local incentive compatibility across the threshold, which is represented as a majorization constraint.

Keywords: Bundling, multi-dimensional screening, double-crossing property, relative convexity, majorization constraint

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1 Introduction

Firms often sell multiple products in monopolistically competitive markets. However, the nature of the optimal selling mechanisms remains poorly understood, despite a long-standing literature dating back to Stigler (1963). Many commonly observed bundling practices still lack concrete explanations. Higher-priced bundles do not always offer more of every component. There are also seemingly random promotions that appear unpredictably at certain times or in specific locations.

The complexity of the multi-product screening problem (e.g. Rochet and Choné, 1998) departs sharply from the single-good case, where a posted price maximizes expected revenue (Riley and Zeckhauser, 1983). The reason is the failure of the "single-crossing property (SCP)". If there exist orderings over buyer types and over allocations such that higher types have increasingly stronger preferences for higher allocations, optimal mechanisms are characterized simply by a monotone quantity schedule and binding local incentive compatibility (IC) constraints (Myerson, 1981). This becomes an overly strong assumption with two or more goods.

This paper characterizes optimal bundling in a setting that instead satisfies a "double-crossing property (DCP)". In our two-good model, the buyer's valuations are additive and both increasing in type.[1] Moreover, the ratio of marginal gains from an increase in type is monotone increasing, or the valuation of one good is more *convexly increasing* than that of the other. This "relative convexity" property, which adapts a similar concept from Matthews and Moore (1987), implies *single-peaked* marginal rate of substitution (MRS), as well as DCP in the preferences for bundles.

Once we step away from a single-crossing environment, local IC no longer guarantees global IC. Under DCP, Matthews and Moore (1987) show that only *downward* IC (DIC) matters. However, there may be binding *non-local* DIC constraints associated with non-montonicity in allocations, and little is known about how to characterize them. We obtain this characterization and relax the optimization problem in the spirit of Myerson (1981). The essential non-local DIC constraints occur *across* a critical type, where the relative attractiveness of the more convex good reaches its minimum. Moreover, these constraints can be expressed as a *majorization constraint*, recently introduced to economic problems by Kleiner, Moldovanu, and Strack (2021).

[1] Early studies focused on the benefits of bundling with negatively correlated or independently distributed values (e.g. Stigler, 1963; Adams and Yellen, 1976; Armstrong, 1999).

Our main result establishes a specific form of the optimal mechanism that we call "Low-type Bunching and High-type Stochastic Separation (LBHSS)". Above the critical type, the quantities are type-increasing and maximal separation arises whenever stochastic bundles are used, while below this threshold, there is at most one interval where the two quantities move in opposite directions, with maximal bunching. Each type in the non-monotone interval is matched assortatively to a type above the threshold under the essential non-local DIC relations. Using this property, we develop ironing techniques that deliver the structure of optimal bundling. A full characterization is obtained with an additional regularity condition on the type distribution.

These results build on some recent studies that ask when relatively simple forms of deterministic bundling are optimal. With single-dimensional types and non-additive valuations, Haghpanah and Hartline (2021) find a ratio-monotonicity condition for "pure bundling", while Yang (2025) explores more general "nested bundling" (i.e. upgrade pricing). With additive valuations, Kwak (2025) shows that monotone MRS is necessary and sufficient for robust optimality of separate sales, while nested bundling is considered by Bergemann et al. (2021). We identify conditions for more complex bundling. Instead of reverse-engineering the conditions for a specific solution, the optimal mechanism is derived from a set of primitive assumptions. Relative convexity is itself a ratio-monotonicity condition, albeit of a higher order, and it leads to non-monotonicity in MRS.

Daskalakis, Deckelbaum, and Tzamos (2017) develop a duality approach to characterize the optimal mechanism in the general linear utility setting.[2] However, their methods are difficult to apply for obtaining the actual solution beyond pure bundling. We extend the conventional Myersonian approach, using transparent relaxations of constraints that reveal the underlying incentive structure. Our results show precisely how different bundling practices, from separate sales to stochastic bundling, can be explained within a single framework defined by a condition with a clear economic meaning.[3] In the existing literature, the need for non-monotonicity and randomization has been discussed, but only in an ad hoc manner and without a general account of when, or in what form, they should be expected (e.g. Thanassoulis, 2004; Manelli

[2]For other duality approaches to multidimensional screening, see, among others, Carroll (2017) and Cai, Devanur, and Weinberg (2019).

[3]LBHSS includes "mixed bundling", which is deterministic but non-monotone. It posts a price for each possible deterministic bundle in a non-additive manner (Adams and Yellen, 1976; McAfee, McMillan, and Whinston, 1989).

and Vincent, 2007; Hart and Reny, 2015; Hart and Nisan, 2019). This is true even with two goods.[4] LBHSS offers such a characterization and is derived from a natural and tractable extension of SCP, with scope for further generalization to many goods.

Matthews and Moore (1987) consider double-crossing preferences in an insurance model with two goods. They also assume a single-dimensional type space, but the utility function is non-linear. We extend their analysis to a linear setting and achieve a full derivation of optimal mechanism, including a characterization of binding non-local DIC constraints. More recently, Chen, Ishida, and Suen (2022) consider the implications of DCP for signaling. Instead of utility functions, they impose double-crossing on indifference curves. Their equilibrium analysis also relies on identifying non-adjacent types whose pairwise incentive constraints affect the outcome. The corresponding actions and allocations are pooled, contrasting our contractual solution.

The rest of the paper is organized as follows. Section 2 describes the model and introduces the concepts of relative convexity and DCP. Section 3 sets up the optimization problem. Section 4 obtains key properties of the optimal mechanism. Section 5 presents the main results on LBHSS. Section 6 offers economic interpretations of the results and discusses an extension to more than two goods. Section 7 concludes. All formal proofs appear in the Appendix and are supplemented by an Online Appendix.

2 Model

2.1 Basic Setting

A monopolist sells two goods, indexed by $i \in \{1,2\}$, to a single buyer, whose type is drawn from a finite set $[N] = \{1, 2, \ldots, N\}$. The buyer's valuation for each i, v_n^i, is positive and strictly increasing in n. While the buyer knows his own type, the seller only knows its prior distribution $f := (f_1, f_2, \ldots, f_N)$ with full support and the cumulative distribution denoted by $F_n = \sum_{i=1}^n f_i$.

A bundle is a pair of quantities $q = (q^1, q^2) \in [0,1]^2$. We say that q is deterministic if $q^i \in \{0,1\}$ for all i and stochastic otherwise. The production cost is zero.[5] Given a bundle q sold at price p, the utility of type n is given by $U_n(q,p) := v_n^1 q^1 + v_n^2 q^2 - p$.

[4] In a two-good model with additive and independent values, Pavlov (2011) offers an example of optimal random bundling, while Manelli and Vincent (2006) report cases in which deterministic bundling is dominated by random bundling.

[5] We may also interpret q as normalized quantities of goods whose marginal production costs are constant, less than the minimum valuation, and zero for the empty bundle.

The value of the buyer's outside option is also zero.

A (direct) mechanism is $M = \{(q_n, p_n)\}_{n \in [N]}$. The incentive-compatibility (IC) and individual rationality (IR) constraints are

$$U_n(q_n, p_n) \geq U_n(q_{n'}, p_{n'}) \quad \forall n, n' \in [N]; \tag{IC}$$

$$U_n(q_n, p_n) \geq 0 \quad \forall n \in [N]. \tag{IR}$$

The seller solves

$$\max_{M \in \mathcal{M}} \sum_{n=1}^{N} f_n p_n, \tag{OP}$$

where $\mathcal{M}$ refers to the set of mechanisms satisfying the IC and IR constraints. Our main results are independent of the type distribution f.

2.2 Relative Convexity and Double-Crossing Property

The key assumption is the following property of the buyer's preferences.

Definition 1 (Relative convexity) $\frac{v^2_{n+1} - v^2_n}{v^1_{n+1} - v^1_n}$ *is strictly increasing in* n.

In words, the ratio of marginal gains from an increase in type is monotone increasing. This implies that v^2_n is more *convexly increasing* than v^1_n, hence the name of the property. We refer to good 1 (good 2) as the less (more) convex good. Definition 1 adapts the non-increasing absolute risk aversion property in Matthews and Moore (1987) to a multi-product monopoly setting.

Relative convexity implies *single-peaked* marginal rate of substitution (MRS). There exists a type at which $\frac{v^2_n}{v^1_n}$ reaches its minimum, decreasing up to that point and increasing thereafter. We assume, for simplicity, that this minimum is attained at a unique type, denoted by $n^* := \arg\min_n \frac{v^2_n}{v^1_n}$.[6]

Lemma 1 *Under relative convexity,* $\frac{v^2_n}{v^1_n}$ *is strictly decreasing for* $n \leq n^*$ *and strictly increasing for* $n \geq n^*$.

Proof. See Appendix A.1. ■

Unless $n^* \in \{1, N\}$, relative convexity implies non-monotone MRS. Figure 1a depicts the convex nature of Definition 1.

[6] In fact, there can be at most two adjacent types with the minimum MRS.

(a) Non-monotone MRS

(b) Double-crossing property

Figure 1: Properties of relative convexity

Remark 1 (Example) *Suppose that type n's gross utility from consuming good $i \in \{1,2\}$ is given by $\tilde{v}_n^i = \alpha_i n$, where $\alpha_i > 0$. Consuming i also incurs cost c_n^i such that the net utility is $v_n^i := \tilde{v}_n^i - c_n^i$. If $c_n^2 = 0$ for all n, and if the marginal disutility of good 1, $\Delta c_n^1 := c_{n+1}^1 - c_n^1$, is strictly increasing, then this preference relation is consistent with relative convexity. Section 6.1 offers a further discussion of our key assumption.*

Another implication of relative convexity is the double-crossing property.

Definition 2 (Double-crossing property) *$U_n(q,p)$ satisfies the "double-crossing property (DCP)" if the following hold. Consider $n' < n^0 < n''$ such that for some $q = (q^1, q^2)$ and $\tilde{q} = (\tilde{q}^1, \tilde{q}^2)$ with $q \neq \tilde{q}$ and $p, \tilde{p} \in \mathbb{R}$,*

$$U_{n'}(q,p) \leq U_{n'}(\tilde{q},\tilde{p}); \; U_{n^0}(q,p) \geq U_{n^0}(\tilde{q},\tilde{p}); \; U_{n''}(q,p) \leq U_{n''}(\tilde{q},\tilde{p}). \tag{1}$$

Then, $q^1 > \tilde{q}^1$ and $q^2 < \tilde{q}^2$; furthermore,

$$U_n(q,p) < U_n(\tilde{q},\tilde{p}) \;\; \forall n > n''; \; U_n(q,p) < U_n(\tilde{q},\tilde{p}) \;\; \forall n < n'. \tag{2}$$

In words, $U_n(q,p)$ and $U_n(\tilde{q},\tilde{p})$ can cross twice at most. If they cross exactly twice, it must be that $q^1 > \tilde{q}^1$ and $q^2 < \tilde{q}^2$, and the lowest and highest types prefer $(\tilde{q},\tilde{p})$ over (q,p), while intermediate types prefer (q,p) over $(\tilde{q},\tilde{p})$, as depicted in Figure 1b.[7] If $q > \tilde{q}$ (under component-wise order), since v_n^1 and v_n^2 are strictly increasing,

[7] In Matthews and Moore (1987), (2) holds only when two of (1) hold strictly. Hence, our definition can be considered as a strict version of DCP.

the (strict) single-crossing property (SCP) holds:[8] for every n and any $p, \tilde{p}$,

$$U_n(q,p) \geq U_n(\tilde{q},\tilde{p}) \implies U_{n'}(q,p) > U_{n'}(\tilde{q},\tilde{p}) \ \forall n' > n;$$
$$U_n(q,p) \leq U_n(\tilde{q},\tilde{p}) \implies U_{n'}(q,p) < U_{n'}(\tilde{q},\tilde{p}) \ \forall n' < n.$$

Matthews and Moore (1987) show that with DCP, local *downward* incentive compatibility (DIC) constraints bind in an optimal mechanism. As a result, all the DIC constraints, together with the IR constraint for the lowest type, constitute a set of *essential* constraints. A solution to (OP) thus solves the following relaxed problem:

$$\max_{\{(q_n,p_n)\}_{n\in[N]}} \sum_{n=1}^{N} f_n p_n \tag{DP}$$

subject to $U_n(q_n,p_n) \geq U_n(q_{n'},p_{n'})$ for all $n \geq n'$ in $[N]$ and $U_1(q_1,p_1) \geq 0$. Furthermore, the two problems have the same set of solutions.

DCP does not imply that the optimal allocations are type-increasing. However, it still ensures monotonicity of the less convex good (good 1) as well as the price. A straightforward application of Matthews and Moore (1987) yields the following.

Proposition 1 *Under relative convexity, DCP holds. Then, an optimal mechanism* $M = \{(q_n,p_n)\}_{n\in[N]}$ *satisfies the following:*

(i) All the local DIC constraints are binding, i.e. $U_{n+1}(q_{n+1},p_{n+1}) = U_{n+1}(q_n,p_n)$ $\forall n = 1,2,\ldots,N-1$.

(ii) The IR constraint for the lowest type is binding, i.e. $U_1(q_1,p_1) = 0$.

(iii) q_n^1 *and* p_n *are non-decreasing* $\forall n \in [N]$.

(iv) (OP) and (DP) share the same set of solutions.

Proof. See Appendix A.2 and Online Appendix OA.1.1. ■

These properties also hold under SCP. They carry over to our DCP setting because of monotone valuations. The difference lies in *non-local* DIC. In a single-crossing environment, there cannot be a binding non-local incentive constraint unless bunching occurs, but this is no longer true under double-crossing. We shall see how this leads to non-monotone allocations, separation, and stochastic bundles.

[8] When $q > \tilde{q}$ ($q \geq \tilde{q}$ but $q \neq \tilde{q}$), $U_n(q,p) - U_n(\tilde{q},\tilde{p}) = v_n \cdot (q - \tilde{q}) + (p - \tilde{p})$ is strictly increasing in n.

3 Extending the Myersonian Approach

We reformulate (DP) as a virtual surplus maximization problem *à la* Myerson (1981), capturing binding non-local DIC constraints. For good i and type n, define the individual profit as $\Pi_n^i := v_n^i \cdot (1 - F_{n-1})$, and refer to $m^i := \arg\max_{n \in [N]} \Pi_n^i$ as the "monopoly type".[9] Then, the (probability-weighted) virtual valuation can be expressed as the change in profit when the price decreases from v_{n+1}^i to v_n^i, i.e.

$$V_n^i := \Pi_n^i - \Pi_{n+1}^i = f_n v_n^i - (1 - F_n)(v_{n+1}^i - v_n^i).$$

Let $V_N^i := f_N v_N^i$. Imposing the binding local DIC and IR constraints in Proposition 1(i) and (ii), the seller's expected revenue is given by

$$\sum_{n=1}^{N} f_n p_n = \sum_{n=1}^{N} (V_n^1 q_n^1 + V_n^2 q_n^2).^{10} \tag{3}$$

Non-local DIC as majorization constraint In addition to the monotonicity of q_n^1 and p_n (Proposition 1(iii)), monotone MRS above n^* (Lemma 1) and the requirements of DIC further imply the following allocation structure for types above n^*.

Lemma 2 *Let $M = \{(q_n, p_n)\}_{n \in [N]}$ be a solution to (DP). For all $n \geq n^*$, (i) either $q_n^1 = 1$ or $q_n^2 = 0$, and (ii) both q_n^1 and q_n^2 are non-decreasing.*

Proof. See Appendix A.3. ∎

To economize on information rents, the seller first exhausts the provision of less convex good 1 and then adds more convex good 2. Since the relative attractiveness of good 1 falls, it is less costly to raise q^1 than q^2 to prevent downward deviations between types above n^*, while keeping their payoffs unchanged. A one-dimensional structure is maintained in the optimized quantity schedule, and given type-increasing

[9] Since the monopoly price is equal to some v_n^i, it is without loss to define Π_n^i only over $n \in [N]$.

[10] By binding local DIC, $p_{n+1} - p_n = \sum_{i=1}^2 v_{n+1}^i (q_{n+1}^i - q_n^i)$, and since $U_1(q_1, p_1) = 0$, $p_1 = \sum_{i=1}^2 v_1^i q_1^i$. Hence, by using summation by parts,

$$\sum_{n=1}^{N} f_n p_n = p_1 + \sum_{n=1}^{N-1} (1 - F_n)(p_{n+1} - p_n) = \sum_{i=1}^{2} \left[v_1^i q_1^i + \sum_{n=1}^{N-1} (1 - F_n)\, v_{n+1}^i (q_{n+1}^i - q_n^i) \right]$$
$$= \sum_{i=1}^{2} \left[\sum_{n=1}^{N-1} \left(f_n v_n^i - (1 - F_n)(v_{n+1}^i - v_n^i) \right) q_n^i + f_N v_N^i q_N^i \right] = \sum_{i=1}^{2} \sum_{n=1}^{N} V_n^i q_n^i.$$

prices, the quantities of both goods are non-decreasing in this region.[11] SCP holds if we restrict attention to $\{(q_n, p_n)\}_{n \geq n^*}$.

The quantity of good 2 may decrease below n^*, where MRS moves in the opposite direction. But just as above n^*, SCP holds below n^* due to the monotonicity of q_n^1 and p_n, with good 2's relative attractiveness falling. This implies the essentiality of local DIC also within this region. Thus, the exact nature of the non-monotonicity depends on non-local DIC *across* the critical type n^*. These constraints can be represented by comparing the information rent required to deter a direct, *non-local* downward deviation between a pair of such types with the sum of rents implied by the binding *local* DIC constraints for all adjacent type pairs lying in between.

For any pair of types n, ℓ such that $n > n^* > \ell$, we have

$$U_n(q_n, p_n) \geq U_n(q_\ell, p_\ell) \iff \sum_{j=\ell}^{n-1} \sum_{i=1}^{2} \Delta v_j^i q_j^i \geq \sum_{i=1}^{2} (v_n^i - v_\ell^i) q_\ell^i,$$

where $\Delta v_j^i := v_{j+1}^i - v_j^i$ for all $j < N$.[12] This is equivalent to, for all $n > n^*$,

$$R(n) := \sum_{j=n^*}^{n-1} \sum_{i=1}^{2} \Delta v_j^i \, q_j^i \geq L(n; \{q_\ell\}_{\ell < n^*}) := \max_{0 \leq \ell < n^*} L^{(\ell)}(n; \{q_\ell\}_{\ell < n^*}), \tag{4}$$

where

$$L^{(\ell)}(n; \{q_\ell\}_{\ell < n^*}) := \sum_{i=1}^{2} (v_n^i - v_\ell^i) q_\ell^i - \sum_{j=\ell}^{n^*-1} \sum_{i=1}^{2} \Delta v_j^i q_j^i.$$

The left-hand side, $R(n)$, represents the *cumulative rents under binding local DIC from n^* to n*. The right-hand side, $L(n; \{q_\ell\}_{\ell < n^*})$, is the maximum of $L^{(\ell)}(n; \{q_\ell\}_{\ell < n^*})$ over $\ell < n^*$, where each $L^{(\ell)}(n; \{q_\ell\}_{\ell < n^*})$ is the difference between (i) the information rent given to type n to deter mimicking of type ℓ and (ii) the cumulative rents under binding local DIC from ℓ to n^*, both computed under the same schedule $\{q_\ell\}_{\ell < n^*}$.

We can interpret (4) together with Lemma 2 as a (weak) *majorization constraint*, following Kleiner, Moldovanu, and Strack (2021). For a pair of non-decreasing functions $g, g' \in L^1$, we say that g is weakly majorized by g', or $g \preceq g'$, if $\int_{\underline{x}}^{x} g(s)ds \geq \int_{\underline{x}}^{x} g'(s)ds$ for all $x \in X := [\underline{x}, \overline{x}]$.[13] Our discrete analogue replaces the integral $\int_{\underline{x}}^{x}$

[11] This is also the key intuition for Kwak (2025), who derives the "prefix" structure and type-increasing allocations under monotone MRS in the same model but with arbitrarily many goods.

[12] By successively applying the binding local DIC constraints, we obtain $U_n(q_n, p_n) - U_\ell(q_\ell, p_\ell) = \sum_{j=\ell}^{n-1} \sum_{i=1}^{2} (v_{j+1}^i - v_j^i) q_j^i$, while $U_n(q_\ell, p_\ell) - U_\ell(q_\ell, p_\ell) = \sum_{i=1}^{2} (v_n^i - v_\ell^i) q_\ell^i$.

[13] In Kleiner et al. (2021), g is weakly majorized by g' if $\int_x^{\overline{x}} g(s)ds \leq \int_x^{\overline{x}} g'(s)ds$ for all $x \in [\underline{x}, \overline{x}]$.

with the sum $\sum_{n=n^*}^{n-1}$ under an appropriate change of measure. Moreover, instead of real-valued non-decreasing functions, it orders functions whose two-dimensional values are non-decreasing along the one-dimensional line from $(\cdot, 0)$ to $(1, \cdot)$.

To see the connection, let $\Delta R(n) \coloneqq R(n+1) - R(n)$ with $R(n^*) = 0$, and given Lemma 2(i), define the right derivative

$$R_v(n) : \{n^*, \dots, N-1\} \to \{q \in [0,1]^2 \mid q^1 = 1 \text{ or } q^2 = 0\}$$

such that $\Delta R(n) = R_v(n) \cdot \Delta v_n$, where $\Delta v_n \coloneqq (\Delta v_n^1, \Delta v_n^2)$.[14] Clearly, $R_v(n)$ is uniquely determined by q_n, which is component-wise non-decreasing by Lemma 2(ii). It is then natural to say that $R(n)$ is *convex* along the valuation path $\{v_n\}_{n=n^*}^N$. We can similarly define $L_v(n;\cdot)$ with $L(n^*;\cdot) = 0$ and show that by relative convexity, it is non-decreasing in n (Lemma 6 in Appendix A.7). Thus, given Lemma 2, (4) is equivalent to requiring $R_v(n)(= q_n)$ to be (weakly) majorized by $L_v(n;\cdot)$, or $R_v(n) \preceq_v L_v(n;\cdot)$.

Virtual valutaion per rent Due to non-local DIC, the central object of our analysis is not the virtual valuation V_n^i *per se* but the ratio $\frac{V_n^i}{\Delta v_n^i}$, reflecting the coefficients on q_n^i in the original objective function (3) and non-local DIC constraints (4).

For good i and type n, define *virtual valuation per (information) rent* as

$$W_n^i \coloneqq \frac{V_n^i}{\Delta v_n^i} = f_n \frac{v_n^i}{\Delta v_n^i} - (1 - F_n).$$

This represents the slope of the profit function Π^i with respect to valuation, indicating how to efficiently provide information rent to higher types with type n's allocation under binding local DIC constraints. Let B denote an interval of types, or a "bunch", and define the corresponding (rent-weighted) *average* virtual valuation per rent as

$$W_B^i \coloneqq \frac{\sum_{n \in B} W_n^i \, \Delta v_n^i}{\sum_{n \in B} \Delta v_n^i}.$$

Given the non-monotonicity of MRS (Lemma 1), increasing the quantity of more

[14] Specifically,

$$R_v(n) \coloneqq \begin{cases} \left(\dfrac{\Delta R(n)}{\Delta v_n^1}, 0\right) & \text{if } \Delta R(n) \le \Delta v_n^1; \\ \left(1, \dfrac{\Delta R(n) - \Delta v_n^1}{\Delta v_n^2}\right) & \text{if } \Delta R(n) > \Delta v_n^1. \end{cases}$$

convex good 2 is more (less) desirable than increasing the quantity of less convex good 1 below (above) n^* in terms of economizing on information rents under binding local DIC. The next lemma directly follows from a reformulation of the original definition of W_B^i.[15] It is consistent with Lemma 2 and illustrated by Figure 2 for $B = \{n\}$.

Lemma 3 *For any $B \subseteq [N]$, $W_B^1 < W_B^2$ if $\max B < n^*$ and $W_B^1 > W_B^2$ if $\min B \geq n^*$.*

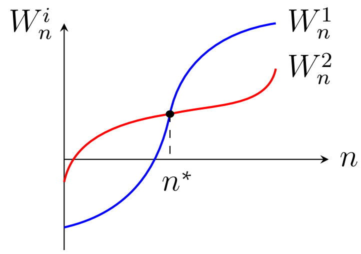

Figure 2: Virtual valuations per rent

Now, consider

$$\max_{\{q_n\}_{n \in [N]}} \sum_{n=1}^{N} \left(W_n^1 \Delta v_n^1 q_n^1 + W_n^2 \Delta v_n^2 q_n^2 \right) \tag{VP}$$

subject to

- q_n^1 is non-decreasing for $n < n^*$;
- p_n, determined by $\{q_n\}$ under binding local DIC constraints and $U_1(q_1, p_1) = 0$, is non-decreasing for $n < n^*$;[16]
- (Majorization constraint) For $n \geq n^*$, $R_v(n)(= q_n) \preceq_v L_v(n; \{q_\ell\}_{\ell < n^*})$ (i.e. either $q_n^1 = 1$ or $q_n^2 = 0$, (q_n^1, q_n^2) is non-decreasing, and $R(n) \geq L(n; \{q_\ell\}_{\ell < n^*})$).

Note that if $n^* \in \{1, N\}$, the non-local DIC constraints, $R(n) \geq L(n; \{q_\ell\}_{\ell < n^*})$ for $n > n^* > \ell$, are redundant and the majorization constraint reduces to the conditions in Lemma 2. Also, as for (OP) and (DP), the existence of a solution to (VP) is

[15] Note that

$$W_B^i = \frac{\Pi_{\min B}^i - \Pi_{\max B+1}^i}{v_{\max B+1}^i - v_{\min B}^i} = (F_{\max B} - F_{\min B-1}) \cdot \frac{v_{\max B+1}^i}{v_{\max B+1}^i - v_{\min B}^i} - (1 - F_{\max B}).$$

[16] For $n < n^*$, if $U_n(q_n, p_n) = U_n(q_{n-1}, p_{n-1})$ and $q_n^1 \geq q_{n-1}^1$, then $p_n \geq p_{n-1}$ if and only if either $q_n^2 \geq q_{n-1}^2$ or $\frac{v_n^2}{v_n^1} \leq \frac{q_n^1 - q_{n-1}^1}{q_{n-1}^2 - q_n^2}$.

guaranteed by a compactness argument, given the compact allocation set. We refer to the maximand of (VP) as the *total* virtual surplus and to its component $W_n^1 \Delta v_n^1 q_n^1 + W_n^2 \Delta v_n^2 q_n^2$ as the virtual surplus (for type n).

Theorem 1 *Under relative convexity,* (OP)*,* (DP)*, and* (VP) *share the same set of solutions.*

Proof. See Appendix A.4. ∎

The relaxation from (DP) preserves the set of optimal mechanisms because the constraints in (VP) continue to imply DIC. For types above n^*, every DIC constraint is captured by the majorization constraint, which ensures both non-decreasing quantities among these types and non-local DIC between them and types below n^*. SCP also holds among types below n^*. Although the monotonicity of q_n^1 alone does not imply SCP, it holds when combined with monotone prices, due to decreasing MRS.

Note that in (VP), we impose monotonicity only for $n < n^*$, despite the solution ultimately maintaining the monotonicity of q_n^1 and p_n up to n^*, as implied by the theorem and Proposition 1(iii).[17] The separation of constraints is by design. It ensures that the allocations above n^* are pinned down entirely by the majorization constraint for a given quantity schedule (strictly) below n^*. This, in turn, enables us to identify distinct patterns of optimal allocations on either side of n^*.

4 Properties of the Optimal Mechanism

Before solving (VP), we establish several implications of non-monotone MRS and non-local DIC constraints. These properties reveal central features of the underlying incentive structure and help us characterize optimal allocations below the critical type n^*. Throughout this section, fix an optimal mechanism, $M = \{(q_n, p_n)\}_{n \in [N]}$. Since local DIC constraints bind and $U_1(q_1, p_1) = 0$, $\{p_n\}$ are uniquely determined by $\{q_n\}$.

4.1 Non-monotone Allocations

Due to non-monotone MRS, non-monotonicity may arise in the quantity of good 2 for types below n^*. To locate this phenomenon, the next lemma characterizes when the seller prefers to prioritize the more convex good.

[17] In fact, this extended monotonicity holds for every feasible quantity schedule in (VP) under the majorization constraint, thereby ensuring SCP on both sides of n^*, as well as at n^*.

Lemma 4 *For $n < n^*$, either $q_n^1 = 0$ or $q_n^2 = 1$ if $U_h(q_h, p_h) > U_h(q_n, p_n)$ for all $h > n^*$ such that $\frac{v_h^2}{v_h^1} > \frac{v_n^2}{v_n^1}$.*

Proof. See Appendix A.5. ■

This observation contrasts with Lemma 2, which says that the seller prioritizes less convex good 1 above n^*. For a type below n^*, if we exclude those above n^* with higher MRS, the arguments are largely analogous, except that on this side of n^*, it is good 2 that has falling MRS and must be prioritized to maximize profit.

Unlike above n^*, the one-dimensional structure below n^* is conditional and does not guarantee a fully type-increasing quantity schedule. However, if the quantity schedule is strictly increasing at some type, it must be (weakly) increasing for all lower types. Therefore, there can be at most one interval of types immediately below n^*, where q^1 and q^2 move in opposite directions.

Proposition 2 *Let $n_0 \leq n^*$ be the highest type below n^* such that $q_{n_0-1} < q_{n_0}$,[18] and define $D := \{n_0, n_0{+}1, \ldots, n^*\}$. Then, q_n^2 is non-decreasing in $n \leq n_0$, and $\{(q_n, p_n)\}_{n \in D}$ can be represented as a finite menu $M_D := \{(q_{n_k}, p_{n_k})\}_{k=0,\ldots,K}$ for some $K \geq 0$, where $n_0 < n_1 < \cdots < n_K \leq n^* < n_{K+1}$, such that:*

- *$(q_n, p_n) = (q_{n_k}, p_{n_k})$ for all $k = 0, \ldots, K$ and all $n = n_k, n_k + 1, \ldots, n_{k+1} - 1$;*
- *$q^1_{n_{k+1}} - q^1_{n_k} > 0$ and $q^2_{n_{k+1}} - q^2_{n_k} < 0$ for all $k = 0, \ldots, K-1$ and $q_{n_{K+1}} > q_{n_{K+1}-1}$.*[19]

Proof. See Appendix A.6. ■

Following this result, we henceforth refer to D as the "decreasing interval" and K as the number of "steps" in this interval such that $K+1$ is the number of distinct bundles, or bunches, associated with D. The corresponding menu is M_D. We use $k = 0, \ldots, K$ to index the cutoff type located at the beginning of each bunch, consisting of $\{n_k, \ldots, n_{k+1} - 1\}$. When $K > 0$, n_1 is the lowest type who receives strictly less of good 2, and strictly more of good 1, than the type immediately below. Below n_1, both q_n^1 and q_n^2 are non-decreasing. Hence, if $K > 1$, q_{n_k} assigns a lottery over both goods for $k = 1, \ldots, K-1$. There may be bunching between n_K and some types above n^* (up to n_{K+1}). Figure 3 illustrates the optimal schedule for q_n^2 when $K = 2$.

The optimal mechanism may still be fully monotone. If so, selling each good at its respective monopoly price maximizes profit.

[18] To ensure n_0 well-defined, we set $n_0 = 1$ if there is no $n_0 > 1$ such that $q_{n_0-1} \leq q_{n_0}$ and $q_{n_0-1} \neq q_{n_0}$.
[19] To ensure n_{K+1} well-defined, we set $n_{K+1} = N+1$ if $q_n = q_N = (1,1)$ $\forall n \geq n_K$ (so that $K = 0$).

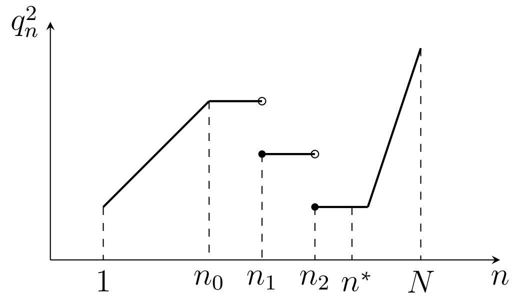

Figure 3: Decreasing interval for the more convex good

Proposition 3 *If $K = 0$, "separate monopoly pricing (SMP)," i.e. for $i \in \{1, 2\}$, $q_n^i = 1$ if $n \geq m^i$ and $q_n^i = 0$ otherwise, is also optimal. If m^i is unique for $i \in \{1, 2\}$, the optimal mechanism is SMP.*

To see this, suppose that $K = 0$. This happens when $n^* \in \{1, N\}$ and MRS is monotone (Lemma 2), but is also possible more generally. Then, given Proposition 1(i)-(ii), a solution to (DP) can be obtained simply by solving $\max_{\{q_n\}} \sum_{n=1}^N V_n^1 q_n^1 + V_n^2 q_n^2$ subject to q_n^i being non-decreasing for $i \in \{1, 2\}$. By the additivity of valuations, this is in turn equivalent to solving $\max_{\{q_n^i\}} \sum_{n=1}^N V_n^i q_n^i$ for each i, and by Myerson (1981), SMP is optimal and uniquely so when each monopoly type m^i is unique.[20] The monopoly price is $v_{m^i}^i$ for each i, with the grand bundle $(1, 1)$ priced at $v_{m^1}^1 + v_{m^2}^2$.

By Lemma 2, we then obtain a case in which SMP is suboptimal, and hence, the optimal mechanism features a decreasing interval.

Corollary 1 *If $m^1 > n^* > m^2$, $K > 0$.*

This can hold when the slope of MRS is sufficiently steeper below than above n^*. For example, given any $\{v_n^1\}_{n \in [N]}$ and f such that $m^1 > n^*$ and $\frac{F_{n^*-1}}{1-F_{n^*-1}} \geq \frac{\Pi_{m^1}^1 - \Pi_{n^*}^1}{\Pi_{n^*}^1}$, $m^2 = 1 < n^*$ if $v_n^2 = v_{n^*}^2$ for $n \leq n^*$ and MRS is constant for $n > n^*$.

4.2 Binding Non-local DIC Constraints

We are interested in cases with $K > 0$, where some non-local DIC constraints are essentially binding. By Theorem 1, these constraints must involve pairs of types on either side of n^*, the type with minimum MRS. Moreover, by SCP, the paired types

[20] This observation, that type-increasing quantities imply the optimality of separate sales, extends to any number of goods under monotone valuations (Kwak, 2025).

below n^* must belong to the decreasing interval D. Relative convexity and DCP further generate a well-behaved structure for how such pairs are *matched.*

Proposition 4 *Suppose that $K > 0$. Fix any $k \in \{1, \ldots, K\}$, and let $\hat{n}_k > n^*$ denote the highest type such that the non-local DIC constraint against n_k binds. Then, the following hold:*

(i) $\hat{n}_k$ is well-defined, non-increasing in k, and satisfies $\hat{n}_K = n_{K+1}$.

(ii) $\frac{v^2_{n_k}}{v^1_{n_k}} < \frac{v^2_{\hat{n}_k}}{v^1_{\hat{n}_k}}$.

(iii) There is no binding non-local DIC constraint between a pair of types $n \in \{n_k, \ldots, \hat{n}_k - 1\}$ and $\ell \notin \{n_k, \ldots, \hat{n}_k - 1\}$ such that $n < n^* < \ell$ or $n > n^* > \ell$, except when $\ell = \hat{n}_k$.

(iv) If the non-local DIC constraint binds between $n > n^*$ and n_{k-1}, then $q_n > q_{n-1}$.

Proof. See Appendix A.7. ■

Part (i) means that for every $k \in \{1, \ldots, K\}$, there exists a binding non-local DIC constraint between n_k and a type above n^*. The highest such type is denoted by $\hat{n}_k$. Due to DCP, $\hat{n}_k$ must be non-increasing, and since q^1_n and q^2_n are both non-decreasing for $n > n^*$, $\hat{n}_K = n_{K+1}$. Also, since $q^1_{n_k} \neq 0$ and $q^2_{n_k} \neq 1$ given that $q^1_{n_0} < q^1_{n_k}$ and $q^2_{n_0} > q^2_{n_k}$, Lemma 4 implies that MRS is lower at n_k than at $\hat{n}_k$, as in part (ii). Part (iii) says that if $n_k \in D$ is matched to $\hat{n}_k > n^*$ via binding non-local DIC, then there cannot be another type located between n_k and $\hat{n}_k$, who can be matched to a type outside this interval. Part (iv) implies that if n_{k-1} ($\neq n_K$) is matched to some $n > n^*$, then the matched type n must itself lie at the beginning of a bunch.

Properties (i) and (iii) hold for all $k \leq K$, and also, for $k = 0$ if $\hat{n}_0$ is well-defined. Thus, we obtain an *assortative* matching structure for binding non-local DIC between types in D and those above n^*, as illustrated by Figure 4 for $K = 2$.

For every $k \in \{0, \ldots, K\}$, the bunch $\{n_k, \ldots, n_{k+1} - 1\} \subseteq D$ (with the bundle (q_{n_k}, p_{n_k})) is matched to the interval $\{\hat{n}_{k+1}, \hat{n}_{k+1} + 1, \ldots, \hat{n}_k\}$ above n^*.[21] The non-local DIC constraint binds between $\hat{n}_k$ and each type in the bunch. For a type in the interior of the latter interval (depicted by the empty dot), non-local DIC may bind against the matched bunch but never a type outside it. Thus, under DCP, there exists a nested sequence of type intervals surrounding the threshold type n^*, namely

[21] For k and $k+1$, the two corresponding intervals above n^* are disjoint except at the endpoints.

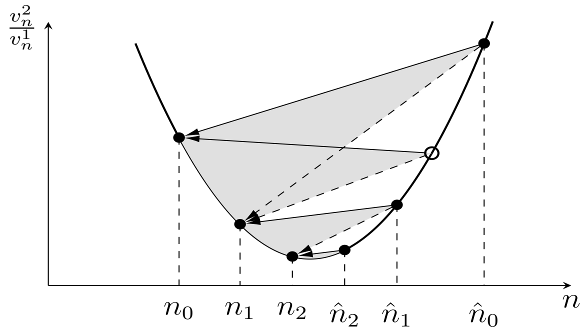

Figure 4: Structure of binding non-local DIC

$\{n_k, \ldots, \hat{n}_k\}_{k\in\{0,\ldots,K\}}$, within each of which non-local DIC is exclusively relevant; moreover, it suffices to consider non-local deviations only across the matched segments, $\{n_k, \ldots, n_{k+1} - 1\}$ and $\{\hat{n}_{k+1}, \hat{n}_{k+1} + 1, \ldots, \hat{n}_k\}$. Given this structure, even a very fine separation schedule above n^* can be optimally supported by a minimal number of bunches in D, as will be shown in the next section.

It is also immediate from Proposition 4(ii) that the decrease in q_n^2 cannot be too large relative to the increase in q_n^1, as this would stop the price from strictly increasing.

Corollary 2 *For all $n < N$, $p_n = p_{n+1}$ if and only if $q_n = q_{n+1}$.*

To see this, for each $k \in \{1, \ldots, K\}$, the binding DIC constraints between $\hat{n}_k$ and n_k and between $n_k - 1$ and n_k imply that

$$\frac{q^1_{n_k} - q^1_{n_k-1}}{q^2_{n_k-1} - q^2_{n_k}} \geq \frac{v^2_{\hat{n}_k} - v^2_{n_k}}{v^1_{\hat{n}_k} - v^1_{n_k}} > \frac{v^2_{n_k}}{v^1_{n_k}}$$

under part (iv), which in turn implies that $p_{n_k-1} < p_{n_k}$. Within D, where q_n^2 is decreasing, the price increases strictly after bunching, just as it does outside D, where both quantities are non-decreasing.

5 Low-Type Bunching and High-Type Stochastic Separation

We now develop ironing techniques to address (VP). Combined with the results of the previous section, these techniques deliver a specific form of the optimal mechanism. We first establish the general existence of such a mechanism, and then, present a regular case in which every optimal mechanism exhibits this form.

5.1 Ironing Preliminaries

The reduced problem (VP) is constructed to separate the monotonicity constraints on the quantity schedule across the critical type n^*. Also, non-local DIC constraints are imposed only between types above and below n^*. Therefore, we can isolate the derivation of the optimal schedule for $n \geq n^*$, while for $n < n^*$, the possibility of non-monotonicity must be addressed. Our method involves three distinct procedures:

- For $n \geq n^*$, we apply ironing under the majorization constraint $R_v(n)(= q_n) \preceq_v L_v(n, \cdot)$, which incorporates type-increasing quantities along the one-dimensional line from $(\cdot, 0)$ to $(1, \cdot)$, with a kink at (1,0).
- For types below the decreasing interval D, standard Myersonian ironing applies under type-increasing quantities.
- For D, we apply ironing under the majorization constraint and different monotonicity constraints on each side of n^* (i.e. q_n^1 is non-decreasing for all n, but q_n^2 may be strictly decreasing below n^*).

To this end, define a *truncated* monopoly type for good i as a type that maximizes the corresponding profit function Π^i over a subset of $[N]$. In particular, we want to define the truncated monopoly types above n^* and below D. Let $n_1^* := \min\{n_1, n^*\}$, where n_1 corresponds to the cutoff type with $k = 1$ in Proposition 2. Note that $n_1 \leq n^*$ when $K > 0$ and $n_1 > n^*$ when $K = 0$. Then, for each $i = 1, 2$, define $\underline{m}_1^i := \arg\max_{n \leq n_1^*} \Pi_n^i$ and $\overline{m}^i := \arg\max_{n \geq n^*} \Pi_n^i$. If the maximizer is not unique, choose the highest such type. When $n_1^* = n^*$, we simply write $\underline{m}_1^i = \underline{m}^i$. The following relationship is immediate from monotone MRS on either side of n^* (or Lemma 3).

Lemma 5 *For any n_1, $\underline{m}_1^1 \geq \underline{m}_1^2$ and $\overline{m}^1 \leq \overline{m}^2$.*

The majorization constraint sets a lower bound on the cumulative rents under binding local DIC from n^* up to each $n \geq n^*$. This constraint implies that when the efficiency of rent provision (measured by virtual valuation per rent, W_n^i) fails to increase with type, total virtual surplus can be increased by bunching types and front-loading rent, as in the standard Myersonian ironing procedure.

To ensure the monotonicity of the virtual valuation per rent above n^*, each profit function Π^i is truncated to the domain $\{n \geq n^*\}$ and then ironed with respect to the valuation axis v_n^i. This yields the smallest concave function lying above the truncated profit function, or its *concave closure*, which will be denoted by $\overline{\Pi}^{i*}$. A type $n \geq n^*$ is called a "concavity type" for good i if $\overline{\Pi}^{i*}_n = \Pi^i_n$. The set of such types, with cardinality $S_i + 1$ for some $S_i \geq 0$, is denoted by N^{i*} with ordering $n^* = N^{i*}_0 < N^{i*}_1 < \cdots < N^{i*}_{S_i} = N$. Then, the s-th (minimal) "ironing interval" for good i is defined by

$$I^{i*}_s := \{N^{i*}_{s-1}, N^{i*}_{s-1} + 1, \ldots, N^{i*}_s - 1\}.$$

The dashed vertical lines in Figure 5 illustrate ironing intervals when $S_i = 4$.

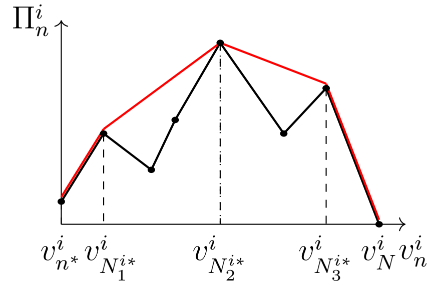

Figure 5: Profit function, concave closure, and ironing intervals

For each ironing interval I^{i*}_s, define the ironed virtual valuation per rent as the (rent-weighted) interval average. By construction, this equals the slope of the concave closure $\overline{\Pi}^{i*}_n$ with respect to v^i_n, i.e. for all $n \in I^{i*}_s$,

$$\overline{W}^{i*}_s := W^i_{I^{i*}_s} = \frac{\overline{\Pi}^{i*}_{n+1} - \overline{\Pi}^{i*}_n}{\Delta v^i_n},$$

which is non-decreasing in $s = 1, \ldots, S_i$.

5.2 Main Result

We focus on the optimal mechanism in which the decreasing interval below n^* features *maximal* bunching. Recall from Proposition 2 that K refers to the number of steps in this interval, which begins at type n_0, and n_1 marks the beginning of the first downward step such that $q^2_{n_1} - q^2_{n_1 - 1} < 0$ (when $K > 0$).

Definition 3 (LBHSS) *A mechanism $\{(q_n, p_n)\}_{n \in [N]}$ exhibits "Low-type Bunching and High-type Stochastic Separation (LBHSS)" if the following hold (with $\{p_n\}$ determined by $\{q_n\}$, the binding local DIC constraints, and $U_1(q_1, p_1) = 0$):*

(a) For $n < n^*$:

(i) $K \le 2$ with either $q^1_{n_0} = 0$ or $q^2_{n_0} = 1$ when $K = 1$, and both when $K = 2$.

(ii) For $i \in \{1,2\}$, $q^i_n = 0$ if $n < \underline{m}^i_1$ and $q^i_n = q^i_{n_0} \in [0,1]$ if $\underline{m}^i_1 \le n < n_1$.

(b) For $n \ge n^*$:

(i) q^1_n and q^2_n are non-decreasing, and either $q^1_n = 1$ or $q^2_n = 0$.

(ii) For $i \in \{1,2\}$, q^i_n is constant over every ironing interval I^{i*}_s, and $q^i_n = 1$ for all $n \ge \overline{m}^i$; moreover, if $q^i_n = q^i_{n'} \in (0,1)$ for some $n \in I^{i*}_s$ and $n' \in I^{i*}_{s'}$ with $q_n \ne q_{n^*-1}$, then $\overline{W}^{i*}_s = \overline{W}^{i*}_{s'}$.

Definition 3(a) describes *Low-type Bunching* (LB). Part (i) says that there are at most two steps in D, i.e. $K \le 2$, with q_{n_0} lying on the edge $(0,\cdot)$ or $(\cdot,1)$. Moreover, if $K = 2$, the bunch containing n_0 must be assigned the deterministic bundle $(0,1)$, highlighting the particularity of this case. Part (ii) means that the allocations below n_1 correspond to truncated separate monopoly pricing, possibly allowing for randomization over a single good. For each good i, types below $\underline{m}^i_1$ are excluded, and types between $\underline{m}^i_1$ and n_1 are assigned i with identical probability.

Definition 3(b) describes *High-type Stochastic Separation* (HSS). Part (i) restates Lemma 2 and implies that if one good is allocated stochastically, the other must be fixed at 0 or 1. If $q^1_n \in [0,1)$, then $q^2_n = 0$; if $q^2_n \in (0,1]$, then $q^1_n = 1$. The first part of (ii) requires the quantity of good i to be constant over each ironing interval I^{i*}_s, with *no distortion at the top* (i.e. $q^i_n = 1$ above $\overline{m}^i$). The second part is a partial converse that implies stochastic separation above n^*. If a stochastic quantity is assigned uniformly across distinct ironing intervals, which means bunching by part (i), the intervals must share the same ironed virtual valuation per rent (unless the bunch begins below n^*).[22]

Figure 6 plots an example of LBHSS for $K = 1$. As type rises and the bundle becomes more expensive, the less convex good increases monotonically, while the more convex good first increases, then decreases, and rises again. Very low types are excluded; very high types receive the grand bundle.

LBHSS permits randomization of only one good, except for the bunch $\{n_1, \ldots, n_2 - 1\} \subseteq int(D)$ when $K = 2$. Note also that the latter "only if" part of Definition 3(b)(ii) does not apply to deterministic bundles. Thus, LBHSS includes any deterministic

[22] If $q_n = q_{n^*-1}$ for some $n \ge n^*$ and feasible $\{q_n\}_{n\in[N]}$ in (VP), either $q^1_{n^*-1} = 1$ or $q^2_{n^*-1} = 0$. Moreover, non-local DIC between $n^* + 1$ and $n^* - 1$ (or $n^* = N$) requires $q_{n^*} \ge q_{n^*-1}$, implying $q_n \ge \cdots \ge q_{n^*} \ge q_{n^*-1}$. Thus, $q_n = q_{n^*-1}$ means that a bunch includes all types from $n^* - 1$ up to n, i.e. $n_K \le n^* - 1 < n < n_{K+1}$.

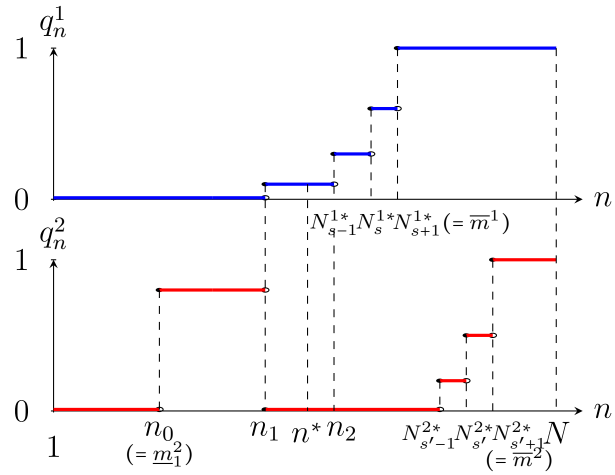

Figure 6: LBHSS

bundling that satisfies the other conditions of Definition 3. For example, by Lemma 5, SMP satisfies LBHSS with $K = 0$ unless $m^1 > n^* > m^2$. Another deterministic LBHSS with $K = 1$ is the following "mixed bundling": $q_n = (0,0)$ for $n < \underline{m}_1^1$, $(0,1)$ for $\underline{m}_1^1 \leq n < n_1$, $(1,0)$ for $n_1 \leq n < \overline{m}^2$, and $(1,1)$ for $n \geq \overline{m}^2$, where $n_1 \notin \{\underline{m}_1^1, \overline{m}^2\}$.

We now present our main result.

Theorem 2 *Under relative convexity, there exists an optimal mechanism that exhibits LBHSS. Moreover, every optimal mechanism satisfies HSS (Definition 3(b)).*

Proof. See Appendix A.8. ■

There may be other optimal mechanisms with $K > 2$. In such a case, redundant separation occurs below n^*, which we rule out. However, above n^*, stochastic separation is always an optimality feature. If adjacent ironing intervals with distinct ironed virtual valuations per rent are bunched and given a stochastic bundle, they can be profitably separated along the edge $(1,\cdot)$ or $(0,\cdot)$ where the bundle lies.

HSS is obtained by maximizing total virtual surplus over the *restricted* schedule $\{q_n\}_{n\geq n^*}$ under the majorization constraint (Section A.8.1):

$$\max_{\{q_n\}_{n\geq n^*}} \sum_{n=n^*}^{N} \left(W_n^1 \Delta v_n^1 q_n^1 + W_n^2 \Delta v_n^2 q_n^2 \right) \tag{VP*}$$

subject to $R_v(n)(= q_n) \preceq_v L_v(n; \{q_\ell\}_{\ell<n^*})$. By the construction of (VP) and Theorem 1, an optimal mechanism must be consistent with a solution to this problem, defined

by its restricted quantity schedule for $n < n^*$. More generally, we solve (VP*) for any $\{q_\ell\}_{\ell<n^*}$ feasible in (VP) (i.e. q^1_ℓ and p_ℓ are non-decreasing under binding local DIC).

This extends Kleiner, Moldovanu, and Strack (2021), who show that in the single-good problem, Myersonian ironing can be applied under a majorization constraint. Types must be bunched within each minimal ironing interval, and the resulting monotonicity of the ironed virtual valuation, combined with the majorization constraint, implies separation across maximal ironing intervals below the monopoly type, where the ironed virtual valuation remains constant.[23] If we apply a change of measure from 1 to Δv^i_n, and replace the virtual valuation V^i_n with the virtual valuation per rent W^i_n, the same result corresponds to Definition 3(b)(ii) for each good i, with the other quantity fixed at $q^2_n = 0$ or $q^1_n = 1$.[24]

However, our problem is two-dimensional, even though types and allocations are both embedded along a one-dimensional line. Depending on whether $q^2_n = 0$ or $q^1_n = 1$, the virtual surplus per unit of rent along the edge is given by either W^1_n or W^2_n for type n. This means that the virtual surplus function may have a kink, where the bundle $(1,0)$ is assigned. We show that using Lemma 3, ironing can be performed across the bunch at the kink to deliver type-increasing virtual surplus per rent and optimal separation across ironing intervals.

Two further procedures are applied to demonstrate LB. Consider first the interval $\{1,\dots,n_0,\dots,n^*_1-1\}$ (Section A.8.2), where $n^*_1 := \min\{n_1, n^*\}$ is the type n below n^* who first receives strictly less of good 2 than q^2_{n-1} (if such a type exists) or n^*. Since both q^1_n and q^2_n are non-decreasing, the majorization constraint is redundant for this region, and it suffices to solve $\max_{\{q^i_n\}_{n<n^*_1}} \sum_{n=1}^{n^*_1-1} V^i_n q^i_n$ under type-increasing $q^i_n \le q^i_{n_0}$ for $i \in \{1,2\}$. By standard Myersonian ironing, SMP is optimal, except that each q^i_n offered above the truncated monopoly type $\underline{m}^i_1$ is not necessarily 1, but must equal $q^i_{n_0} \in [0,1]$. This ties the region with the rest of $[N]$. The pair $(q^1_{n_0}, q^2_{n_0})$ is derived as part of the allocations for D, where non-local DIC matters. By Lemma 5, $\underline{m}^1_1 \ge \underline{m}^2_1$, and hence, $n_0 = \underline{m}^1_1$ if $q^1_{n_0} > 0$ and $n_0 = \underline{m}^2_1$ otherwise, provided that $q_{n_0} \ne (0,0)$.

To apply ironing for the decreasing interval D (Section A.8.3), we must recognize

[23] Kleiner et al. (2021) also show that the quantity assigned to each ironing interval is pinned down such that either the majorization constraint binds or it equals an extreme value. In Online Appendix OA.2.1, we derive a solution to (VP*) with a similar property.

[24] Specifically, $R_v(n)$ and $L_v(n; \{q_\ell\}_{\ell<n^*})$ are in a majorization relation along the valuation axis v_n, rather than the type axis n. Given the monotonicity of valuations with a one-dimensional allocation structure, the axis can be converted separately on either side of the kink at $(1,0)$.

that for each $k \leq K$, q_{n_k} can move along any two-dimensional vector. This contrasts with the other regions of types, where the direction of allocation shift is fixed along a one-dimensional line. Moreover, due to the majorization constraint, the quantities assigned to types below and above n^* must move *jointly*. The key challenge in implementing an ironing procedure for D is finding a directional vector that adjusts the entire quantity schedule while preserving all the binding non-local DIC constraints.

The nested structure of binding non-local DIC under DCP (Proposition 4) enables us to construct such a directional vector within the feasibility set. Recall that there is no binding non-local DIC between a type inside the interval $\{n_k, \ldots, \hat{n}_k - 1\}$ and a type outside it, except for $\hat{n}_k$. This means that allocations within the interval can move freely without violating any non-local DIC against types outside, provided that the binding non-local DIC constraint between n_k and $\hat{n}_k$ is preserved. From (4), we can see that by shifting $\{q_n\}_{n \in \{n_k, \ldots, \hat{n}_k - 1\}}$ uniformly in the positive or negative direction of

$$\Delta_k^* := \left(-\frac{1}{v_{\hat{n}_k}^1 - v_{n_k}^1}, \frac{1}{v_{\hat{n}_k}^2 - v_{n_k}^2} \right),$$

it is possible to maintain the binding non-local DIC constraint between any pair of types in the interval $\{n_k, \ldots, \hat{n}_k - 1\} \cup \{\hat{n}_k\}$. The shift for $n \geq n^*$ can be adjusted to occur along $(1, \cdot)$ or $(\cdot, 0)$, provided that the change in rent is kept at $\Delta v_n \cdot \Delta_k^*$.

When there is bunching at the kink $(1, 0)$, the directional vector must be chosen more restrictively so that the bunch remains fixed at $(1, 0)$. Nevertheless, such a vector always exists within the feasibility set when $K > 2$ (or more generally, whenever LB fails). Depending on whether the marginal virtual surplus with respect to this vector is positive or negative, a move along the vector compresses the allocations in D towards the edges $(0, \cdot)$ and $(\cdot, 1)$, or the edges $(1, \cdot)$ and $(\cdot, 0)$, until there is additional bunching or a newly binding non-local DIC constraint. In other words, starting from a solution with the coarsest bunch structure and largest number of binding non-local DIC constraints, which must exist in our finite setting, this procedure reduces the number of steps K maximally. The extreme case of LB is $K = 2$, with $q_{n_0} = (0, 1)$.

5.3 A Regular Case

Under a regular type distribution, we can strengthen our LBHSS results. To rule out redundant multiplicity, let us also assume that the truncated monopoly types $\overline{m}^i$ and $\underline{m}^i$ are uniquely defined for $i = 1, 2$.

Definition 4 (Concave pasting at the MRS minimum) *Type distribution f is regular if for every $n, n' \in [N]$ such that $n < n^* < n'$,*

$$\frac{\Pi^1_{n^*} - \Pi^1_n}{v^1_{n^*} - v^1_n} \geq \frac{\Pi^2_{n'} - \Pi^2_{n^*}}{v^2_{n'} - v^2_{n^*}}.$$

This condition requires *local* concavity of profit functions. It says that if Π^1 and Π^2 are pasted together at the MRS-minimizing type n^*, where their virtual valuations per rent intersect (Lemma 3), the resulting function must be concave at n^*.[25] Figure 7 illustrates the concave pasting (dashed curve). The single-crossing of W^1_n and W^2_n implies that the two profit functions are tangent at n^* in the continuous-type limit. Under monotone MRS (i.e. when $n^* \in \{1, N\}$), Definition 4 is vacuously satisfied; under non-monotone MRS, it holds, for example, if f is regular for good 1 in the standard sense (i.e. Π^1_n is concave) and the slope of MRS above n^* is not too steep.[26]

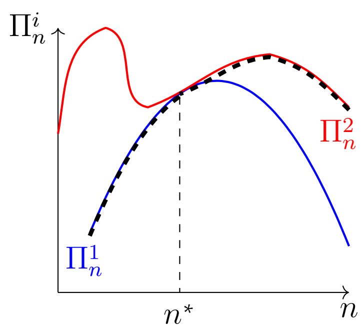

Figure 7: Concave pasting

Theorem 3 *Suppose that f is regular. Then, under relative convexity, every optimal mechanism exhibits LBHSS with $K \leq 1$. Moreover, $n_1 = n^*$ if $K = 1$, and either $q^1_{n_0} = 0$ or $q^2_{n_0} = 1$ if $n_0 < n^*$.*[27]

Proof. See Online Appendix OA.1.2. ■

To understand how regularity leads to a sharper characterization of LBHSS, note first that it bounds the average virtual valuations per rent of the two goods around

[25]Consider Π^1 for $n \leq n^*$ and Π^2 for $n \geq n^*$, defined over $\{v^1_n\}_{n \leq n^*}$ and $\{v^2_n\}_{n \geq n^*}$. The functions can be joined by leveling and rescaling such that $\Pi^1_{n^*} = \Pi^2_{n^*}$ and $V^i_n = W^i_n$ for all n and $i \in \{1, 2\}$.

[26]If f is regular for good 2, which is more convex, Definition 4 always holds.

[27]For $K = 1$, $n_0 < n^*$ trivially. If $K = 0$ and $n_0 < n^*$, by LBHSS, it follows that $q_{n_0} = (0, 0)$ or $(1, 1)$, and thus, $q_n = (0, 0)$ $\forall n \leq n^*$ or $q_n = (1, 1)$ $\forall n \geq n^*$.

n^*. Specifically, for a pair of types n, n' such that $n < n^* < n'$, we have $W^1_{\underline{B}} \leq W^2_{\overline{B}}$, where $\underline{B} \coloneqq \{n, \ldots, n^* - 1\}$ and $\overline{B} \coloneqq \{n^*, \ldots, n' - 1\}$. By Lemma 3, this implies

$$W^1_{\underline{B}} = \min_{i \in \{1,2\}} W^i_{\underline{B}} \leq \min_{i \in \{1,2\}} W^i_{\overline{B}} = W^2_{\overline{B}}. \tag{5}$$

Then, there cannot be a bunch that contains n^* in its *interior* (i.e. $n_K = n^*$). Increasing q^1 for the bunch just below n^* is less costly to the seller than increasing q^2 for the bunch just above n^* in terms of the provision of cumulative rents under binding local DIC. By (5), this is true even when the shift below n^* also involves decreasing q^2, while the shift above n^* increases either q^2 or q^1 along $(1, \cdot)$ or $(\cdot, 0)$.[28] Thus, the seller can strictly improve total virtual surplus by separating the types around n^*. This argument further eliminates the possibility of $K = 2$. If non-monotonicity arises in the optimal allocations, the decrease in q^2 takes place exactly at n^*.

In the regular case, monopoly types cannot be ordered such that $m^1 < n^* < m^2$, since then Definition 4 would fail for $(n, n') = (m^1, m^2)$. Therefore, by Corollary 1, Theorem 3 immediately implies the following conditions that rule out separate sales.

Corollary 3 *Under regularity, $K = 1$ unless $m^1, m^2 \leq n^*$ or $m^1, m^2 \geq n^*$.*

In Online Appendix OA.2, we extend Theorem 3 to fully derive the optimal mechanism. This shows that with regularity, the condition in Corollary 1 (i.e. $m^1 > n^* > m^2$) further implies the generic suboptimality of deterministic bundling.

6 Discussion

6.1 Interpretations

Our results are driven by the relative convexity assumption (Definition 1). In Remark 1, we offered an example, where consuming one of the goods (good 1) generates disutility that is convexly increasing in type. One way to interpret this preference structure is opportunity cost of time. If n corresponds to income, low- and high-type consumers may value time more highly than intermediate types.

Consider a firm selling internet and TV packages. The marginal cost of providing any internet speed or number of TV programs is constant. While internet can be

[28] Without the regularity condition, we cannot rule out the possibility of more efficient rent provision from decreasing the quantities above n^*, combined with a coordinated increase in q^1 and decrease in q^2 below n^*.

consumed at any moment, watching a TV program requires time commitment. TV is then the less convex good, and according to our results, it will be offered monotonically with higher-priced bundles containing a greater number of programs, while the internet speed varies non-monotonically, as depicted in Figure 6.

We indeed observe real-world menus that include low-priced internet-only bundles alongside high-priced internet-and-TV bundles, where the internet speed is lower in some of the internet-and-TV bundles compared to the internet-only bundles. Another application is credit cards offering multiple reward schemes. Relative convexity may emerge, for example, between travel discounts and simple cash-back, due to the opportunity cost of time from travel.

6.2 More than Two Goods

A natural next step is to consider many goods. We conjecture that LBHSS offers a collection of properties that can be generalized by the following arguments. Consider $i = 1, 2, 3$. Note first that while relative convexity can be defined for more than two goods (such that v_n^i is more convexly increasing than v_n^j whenever $i > j$), it alone does not imply DCP. To ensure that relative convexity implies DCP, for any pair of bundles, one must contain (weakly) larger quantities of the higher-indexed goods than the other. This holds trivially with two goods but fails with three (or more).

To guarantee DCP, an additional convexity-based preference ordering is needed. Suppose that for each $n \in [N]$, there exists $r_n = (r_n^1, r_n^2, r_n^3) \in \mathbb{R}_+^3$ such that

$$r_n^1 v_n^1 + r_n^3 v_n^3 \le r_n^2 v_n^2,$$

while for all $n' > n$,

$$r_n^1 v_{n'}^1 + r_n^3 v_{n'}^3 > r_n^2 v_{n'}^2.$$

This is a relative convexity property over components rather than types. Kwak (2024) shows that under such a condition, a solution to (DP) only uses allocations with "interval structure", in which for each type n, there exist component indices $i' < i''$ such that $q_n^i = 1$ for all $i \in \{i'+1, \ldots, i''-1\}$ and $q_n^i = 0$ for $i < i'$ or $i > i''$.

Under relative convexity over both types and components, DCP holds among interval structures, and hence, (OP) and (DP) would admit the same solutions. With DCP, the type-increasing property of q_n^1 (Proposition 1(iii)) extends to monotonicity of interval allocations under a lexicographic order that prioritizes lower-indexed goods.

Also, we can extend Lemma 2 by considering each pair of goods. For $i, i' \in \{1, 2, 3\}$ such that $i < i'$, let $n^*_{ii'}$ denote the type that minimizes MRS between the pair.[29] Then, for $n \geq n^*_{ii'}$, we must have $q^i_n = 1$ or $q^{i'}_n = 0$. Using these arguments, an extension of (VP) and Theorem 1 should be possible.

One desirable feature of interval structures is that except for the two boundary components, all other quantities are pinned down deterministically, equalling 0 or 1. We conjecture that despite the presence of multiple MRS minima, the properties in Section 4 carry over; moreover, the decreasing intervals for different pairs of goods do not overlap, and the quantities assigned to types between two such successive decreasing intervals lie on a one-dimensional line, similarly to Lemma 4.

The optimal mechanism would then display a tree structure of LBHSS. Starting from the highest MRS-minimum type, an LBHSS forms locally around it; then, another LBHSS forms around the next highest MRS-minimum, nesting the previously identified LBHSS, and so on. While the exact characterization will be complex, this structure further suggests how DCP can potentially explain a wide range of bundling practices, even with a single-dimensional type space.

7 Conclusion

This paper extends the canonical Myersonian approach to the monopoly screening problem from a single-good to a two-good setting. While the type space remains one-dimensional, consumer preferences are additive and satisfy a double-crossing property. The assumptions of single-dimensionality and additivity crystallize a fundamental force behind bundling. The relative convexity property that generates double-crossing preferences has a clear economic interpretation and helps explain a variety of selling practices, including stochastic bundling. Our methods follow a well-trodden path, uncover a detailed incentive structure, and offer scope for further extension.

A Proofs

A.1 Proof of Lemma 1

Since n^* is assumed to be the unique type with the minimum MRS, we have $\frac{v^2_{n^*}}{v^1_{n^*}} < \frac{v^2_{n^*-1}}{v^1_{n^*-1}}$ and $\frac{v^2_{n^*}}{v^1_{n^*}} < \frac{v^2_{n^*+1}}{v^1_{n^*+1}}$, which implies $\frac{v^2_{n^*}-v^2_{n^*-1}}{v^1_{n^*}-v^1_{n^*-1}} < \frac{v^2_{n^*}}{v^1_{n^*}} < \frac{v^2_{n^*+1}-v^2_{n^*}}{v^1_{n^*+1}-v^1_{n^*}}$. Under relative

[29] Under relative convexity over components, $n^*_{ii'}$ is non-increasing in both i and i'.

convexity, this further implies that for every $n < n^* \leq n'$,

$$\frac{v^2_{n+1} - v^2_n}{v^1_{n+1} - v^1_n} < \cdots < \frac{v^2_{n^*} - v^2_{n^*-1}}{v^1_{n^*} - v^1_{n^*-1}} < \frac{v^2_{n^*}}{v^1_{n^*}} < \frac{v^2_{n^*+1} - v^2_{n^*}}{v^1_{n^*+1} - v^1_{n^*}} < \cdots < \frac{v^2_{n'+1} - v^2_{n'}}{v^1_{n'+1} - v^1_{n'}}.$$

It follows that

$$\begin{aligned} \frac{v^2_{n+1} - v^2_n}{v^1_{n+1} - v^1_n} &< \frac{v^2_{n^*} - \sum_{\ell=n}^{n^*-1}(v^2_{\ell+1} - v^2_\ell)}{v^1_{n^*} - \sum_{\ell=n}^{n^*-1}(v^1_{\ell+1} - v^1_\ell)} = \frac{v^2_n}{v^1_n}; \\ \frac{v^2_{n'+1} - v^2_{n'}}{v^1_{n'+1} - v^1_{n'}} &> \frac{v^2_{n^*} + \sum_{\ell=n^*}^{n'-1}(v^2_{\ell+1} - v^2_\ell)}{v^1_{n^*} + \sum_{\ell=n^*}^{n'-1}(v^1_{\ell+1} - v^1_\ell)} = \frac{v^2_{n'}}{v^1_{n'}}. \end{aligned}$$

Therefore, $\frac{v^2_n}{v^1_n} > \frac{v^2_{n+1}}{v^1_{n+1}}$ and $\frac{v^2_{n'}}{v^1_{n'}} < \frac{v^2_{n'+1}}{v^1_{n'+1}}$ for every $n < n^* \leq n'$.

A.2 Proof of Proposition 1

Here, we show that relative convexity implies DCP. Online Appendix OA.1.1 provides the remainder of the proof, which adapts Matthews and Moore (1987). Note that $U_n(q,p) - U_n(\tilde{q},\tilde{p}) = U_n(q-\tilde{q}, p-\tilde{p})$. By (1) in Definition 2, we have

$$\begin{aligned} & U_{n'}(q-\tilde{q},p-\tilde{p}),\ U_{n''}(q-\tilde{q},p-\tilde{p}) \leq 0 \leq U_{n^0}(q-\tilde{q},p-\tilde{p}) \\ \Rightarrow\ & U_{n''}(q-\tilde{q},p-\tilde{p}) - U_{n^0}(q-\tilde{q},p-\tilde{p}) \leq 0 \leq U_{n^0}(q-\tilde{q},p-\tilde{p}) - U_{n'}(q-\tilde{q},p-\tilde{p}). \end{aligned} \tag{6}$$

Then, by the form of $U_n(q,p) := v_n \cdot q - p$, where $v_n = (v^1_n, v^2_n)$, (6) is equivalent to

$$(v_{n''} - v_{n^0}) \cdot (q - \tilde{q}) \leq 0 \leq (v_{n^0} - v_{n'}) \cdot (q - \tilde{q}). \tag{7}$$

Let us show that this implies $q^1 > \tilde{q}^1$ and $q^2 < \tilde{q}^2$ as in Definition 2.

Suppose not. So, suppose first that $q \leq \tilde{q}$ or $\tilde{q} \leq q$. But then, since $q \neq \tilde{q}$ and $v_{n''} - v_{n^0}, v_{n^0} - v_{n'} > 0$, we have a contradiction against (7). Next, suppose that $q^1 < \tilde{q}^1$ and $q^2 > \tilde{q}^2$. But then, by relative convexity (Definition 1),

$$\begin{aligned} 0 \leq \frac{v_{n^0} - v_{n'}}{v^1_{n^0} - v^1_{n'}} \cdot (q - \tilde{q}) = \frac{v^2_{n^0} - v^2_{n'}}{v^1_{n^0} - v^1_{n'}}(q^2 - \tilde{q}^2) + (q^1 - \tilde{q}^1) &< \frac{v^2_{n''} - v^2_{n^0}}{v^1_{n''} - v^1_{n^0}}(q^2 - \tilde{q}^2) + (q^1 - \tilde{q}^1) \\ &= \frac{v_{n''} - v_{n^0}}{v^1_{n''} - v^1_{n^0}} \cdot (q - \tilde{q}), \end{aligned}$$

which contradicts (7) again.

To show that relative convexity implies (2) in Definition 2, note that by relative convexity, for every $n > n''$, $U_n(q - \tilde{q}, p - \tilde{p}) < U_{n''}(q - \tilde{q}, p - \tilde{p})$, or

$$\begin{aligned}\frac{1}{v_n^1 - v_{n''}^1}(U_n(q - \tilde{q}, p - \tilde{p}) - U_{n''}(q - \tilde{q}, p - \tilde{p})) &= \frac{v_n - v_{n''}}{v_n^1 - v_{n''}^1} \cdot (q - \tilde{q}) \\ &= (q^1 - \tilde{q}^1) - \frac{v_n^2 - v_{n''}^2}{v_n^1 - v_{n''}^1}(\tilde{q}^2 - q^2) < (q^1 - \tilde{q}^1) - \frac{v_{n''}^2 - v_{n^0}^2}{v_{n''}^1 - v_{n^0}^1}(\tilde{q}^2 - q^2) \\ &= \frac{1}{v_{n''}^1 - v_{n^0}^1}(U_{n''}(q - \tilde{q}, p - \tilde{p}) - U_{n^0}(q - \tilde{q}, p - \tilde{p})) \leq 0.\end{aligned}$$

Since $U_{n''}(q - \tilde{q}, p - \tilde{p}) \leq 0$, this means that $U_n(q - \tilde{q}, p - \tilde{p}) < 0$, yielding the first part of (2). The second part is obtained by similar arguments.

A.3 Proof of Lemma 2

Part (i) Suppose not. So, for some $n \geq n^*$, $q_n^1 < 1$ and $q_n^2 > 0$. Then, replace $q_n := (q_n^1, q_n^2)$ with $\tilde{q}_n := (q_n^1 + \frac{\epsilon}{v_n^1}, q_n^2 - \frac{\epsilon}{v_n^2})$ for small $\epsilon > 0$. Keep p_n unchanged. This does not alter type n's utility, but each higher type's DIC constraint against n becomes slack since $\frac{v_h^2}{v_h^1}$ is strictly increasing for $h \geq n^*$ by Lemma 1, i.e. for $n' > n \geq n^*$,

$$U_{n'}(q_{n'}, p_{n'}) - U_{n'}(\tilde{q}_n, p_n) \geq U_{n'}(q_n, p_n) - U_{n'}(\tilde{q}_n, p_n) = \epsilon\left(\frac{v_{n'}^2}{v_n^2} - \frac{v_{n'}^1}{v_n^1}\right) > 0.$$

Now, raise q_n^2 by a sufficiently small $\epsilon' > 0$ and p_n by $\epsilon' v_n^2$, so that type n's utility remains equal to $U_n(\tilde{q}_n, p_n)$. This strictly increases revenue without violating any DIC or IR constraint, contradicting that M is a solution to (DP).

Part (ii) By part (i), for every $n' > n \geq n^*$, it must be that either $q_{n'} \geq q_n$ or $q_{n'} \leq q_n$, where $\geq$ denotes component-wise order. To ensure DIC under type-increasing prices (Proposition 1(iii)), it must be that $q_{n'} \geq q_n$.

A.4 Proof of Theorem 1

Let $\{q_n\}_{n \in [N]}$ solve (VP), with $\{p_n\}_{n \in [N]}$ determined by binding local DIC and $U_1(q_1, p_1) = 0$. (VP) imposes monotonicity of q_n^1 and p_n for $n \leq n^* - 1$. We first show that the monotonicity extends to n^*, and then, show the solution's feasibility in (DP).

Step 1: $q_{n^*-1}^1 \leq q_{n^*}^1$ and $p_{n^*-1} \leq p_{n^*}$.

Proof. This is trivially satisfied if $q_{n^*-1} \leq q_{n^*}$. Also, we only need to consider the case $n^* < N$, since when $n^* = N$, it is straightforward to verify that $q_{n^*} = (1, 1)$, as

$V_N^i = v_N^i f_N > 0$ and q_N is unconstrained in (VP).

The majorization constraint rules out the possibility that $q_{n^*-1} > q_{n^*}$ ($q_{n^*-1} \geq q_{n^*}$ but $q_{n^*-1} \neq q_{n^*}$). If this were the case, SCP holds, so the binding (local) DIC constraint between n^* and n^*-1 would imply that all types above n^* strictly prefer (q_{n^*-1}, p_{n^*-1}) over (q_{n^*}, p_{n^*}). This contradicts the binding DIC constraint between n^* and n^*+1.

Then, consider two cases: (a) $q^1_{n^*-1} > q^1_{n^*}$ and $q^2_{n^*-1} < q^2_{n^*}$; (b) $q^1_{n^*-1} < q^1_{n^*}$ and $q^2_{n^*-1} > q^2_{n^*}$. The first case cannot arise. By Lemma 2(i), which is embedded in the majorization constraint, $q^1_{n^*-1} > q^1_{n^*}$ would imply that $q^2_{n^*} = 0$, contradicting $q^2_{n^*-1} < q^2_{n^*}$.

In case (b), we need to show that $p^1_{n^*-1} \leq p^1_{n^*}$. Since type n^*+1 weakly prefers (q_{n^*}, p_{n^*}) over (q_{n^*-1}, p_{n^*-1}), while type n^* is indifferent, it follows that

$$\frac{q^1_{n^*} - q^1_{n^*-1}}{q^2_{n^*-1} - q^2_{n^*}} \geq \frac{v^2_{n^*+1} - v^2_{n^*}}{v^1_{n^*+1} - v^1_{n^*}}.$$

Then, since MRS is strictly increasing above n^*, we have

$$\frac{q^1_{n^*} - q^1_{n^*-1}}{q^2_{n^*-1} - q^2_{n^*}} > \frac{v^2_{n^*}}{v^1_{n^*}},$$

which implies that $p_{n^*} - p_{n^*-1} = v^1_{n^*}(q^1_{n^*} - q^1_{n^*-1}) - v^2_{n^*}(q^2_{n^*-1} - q^2_{n^*}) > 0$. □

Step 2: $\{(q_n, p_n)\}_{n\in[N]}$ satisfies all the constraints in (DP).

Proof. Since we assume that all the local DIC constraints, the IR constraint for the lowest type, and the non-local DIC constraints between types on opposite sides of n^* are satisfied, it suffices to verify that non-local DIC holds among types on the same side of n^*, i.e. $\{1, \ldots, n^*\}$ and $\{n^*, \ldots, N\}$.

The case of $n \geq n^*$ follows directly from Lemma 2(ii), which implies that SCP holds for $\{(q_n, p_n)\}_{n\geq n^*}$. For $n \leq n^*$, only q^1_n and p_n are monotone (Step 1), but SCP also holds. To see this, for any $n \leq n^*$ and any pair (q, p) and $(\tilde{q}, \tilde{p})$ with $(q^1, p) \leq (\tilde{q}^1, \tilde{p})$,

$$\begin{aligned} U_n(\tilde{q}, \tilde{p}) - U_n(q, p) &= v^1_n(\tilde{q}^1 - q^1) + v^2_n(\tilde{q}^2 - q^2) - (\tilde{p} - p) \\ &= v^2_n\left[\frac{v^1_n}{v^2_n}(\tilde{q}^1 - q^1) + (\tilde{q}^2 - q^2) - \frac{1}{v^2_n}(\tilde{p} - p)\right]. \end{aligned}$$

Since $(\tilde{q}^1 - q^1, \tilde{p} - p) \geq 0$ and both $\frac{v^1_n}{v^2_n}$ and $-\frac{1}{v^2_n}$ are (strictly) increasing for $n \leq n^*$, once $U_n(\tilde{q}, \tilde{p}) \geq (>) U_n(q, p)$, $U_{n'}(\tilde{q}, \tilde{p}) \geq (>) U_{n'}(q, p)$ for every $n' \geq n$. □

A.5 Proof of Lemma 4

Suppose not. So, for some $n < n^*$, $U_h(q_h, p_h) > U_h(q_n, p_n)$ for all $h > n^*$ s.t. $\frac{v_h^2}{v_h^1} > \frac{v_n^2}{v_n^1}$, but $q_n^1 > 0$ and $q_n^2 < 1$. Then, change q_n to $\tilde{q}_n := \left(q_n^1 - \frac{\epsilon}{v_n^1}, q_n^2 + \frac{\epsilon}{v_n^2}\right)$ for small $\epsilon > 0$. Since type n's utility is unchanged, all DIC constraints for n, and all IR constraints, hold.

Consider $h > n$ and the DIC constraint against n. If $\frac{v_h^2}{v_h^1} > \frac{v_n^2}{v_n^1}$, $h > n^*$, so the originally non-binding constraint still holds for ϵ close to 0; if $\frac{v_h^2}{v_h^1} \leq \frac{v_n^2}{v_n^1}$, it holds since

$$U_h(q_n, p_n) - U_h(\tilde{q}_n, p_n) = \epsilon\left(\frac{v_h^1}{v_n^1} - \frac{v_h^2}{v_n^2}\right) \geq 0.$$

The inequality is strict for $h = n+1$ because MRS is strictly decreasing below n^*. Thus, local DIC between n and $n+1$ is non-binding. Since the modified mechanism leaves total revenue unchanged, it remains optimal, which contradicts Proposition 1(i).

A.6 Proof of Proposition 2

It suffices to prove the following claim. Then, set $n_0 := d$, and let $k = 1, \ldots, K$ index $n \in \{n_0 + 1, \ldots, n^*\}$ with $q_n^2 < q_{n-1}^2$. Define n_{K+1} as the lowest $n > n^*$ with $q_n \neq q_{n^*}$, which implies $q_n > q_{n^*}$ by Lemma 2 if such n exists; otherwise, set $n_{K+1} = N + 1$.

Claim: There exists $d \leq n^*$ with $q_{d-1} < q_d$ ($q_{d-1} \leq q_d$ and $q_{d-1} \neq q_d$) such that:

(a) q_n^2 is non-decreasing for $n \leq d$;

(b) q_n^2 is non-increasing for $n \in \{d, d+1, \ldots, n^*\}$ s.t.

$$q_n^2 > q_{n+1}^2 \implies q_n^1 < q_{n+1}^1; \quad q_n^2 = q_{n+1}^2 \implies q_n^1 = q_{n+1}^1.$$

Proof. Define d as the largest $n \leq n^*$ s.t. $q_{n-1} < q_n$. Since this is assumed to hold at $n = 1$, such d always exists.

(a) By the definition of d, SCP holds such that every $n > d$ strictly prefers (q_d, p_d) over (q_{d-1}, p_{d-1}). Hence, Lemma 4 implies that $q_{d-1}^1 = 0$ or $q_{d-1}^2 = 1$. If $q_{d-1}^1 = 0$, then $q_n^1 = 0$ for all $n \leq d-1$ since q_n^1 is non-decreasing. Since p_n is non-decreasing, it follows that q_n^2 is also non-decreasing for all $n \leq d - 1$.

If $q_{d-1}^2 = 1$, let $n' < d - 1$ be the highest type s.t. $q_{n'}^2 < 1$. Then, for all $n = n' + 1, \ldots, d - 1$, $q_n^2 = 1$, so $q_{n'} < q_{n'+1}$ due to non-decreasing q_n^1 for all $n \in [N]$. Then, repeat the above argument to $n' + 1$. It follows that either $q_{n'}^1 = 0$ or $q_{n'}^2 = 1$, but the latter contradicts the definition of n'. Hence, $q_{n'}^1 = 0$, and $q_n^1 = 0$ for all $n \leq n'$. Then, non-decreasing p_n implies that q_n^2 is non-decreasing for $n \le n'$, completing the proof.

(b) By the definition of d, q_n^2 is non-increasing for $d \le n \le n^*$, and the second part follows immediately. For the first part, note that if $q_n^2 > q_{n+1}^2$ and $q_n^1 \ge q_{n+1}^1$, then $p_n > p_{n+1}$ by binding local DIC. This contradicts that p_n is non-decreasing. □

A.7 Proof of Proposition 4

Part (i) We proceed as follows. Note first that $n_{K+1} \le N$ for $K > 0$ (footnote 19).

Step 1: For $k \in \{1, \ldots, K\}$, there exists $n > n^*$ s.t. $U_n(q_n, p_n) = U_n(q_{n_k}, p_{n_k})$, and hence, $\hat{n}_k$ is well-defined and $\hat{n}_K = n_{K+1}$.

Proof. For $k \notin \{0, K\}$, $q_{n_k}^1, q_{n_k}^2 \in (0,1)$ by Proposition 2. By Lemma 4, this implies that the non-local DIC constraint between n_k and some type strictly above n^* is binding. For $k = K$, by the definition of n_K and $n_{K+1}(> n^*)$, DIC between n_K and n_{K+1} is binding. Moreover, since $q_{n_K} = q_{n^*} = q_{n_{K+1}-1} < q_{n_{K+1}}$, SCP holds, so binding local DIC between n_{K+1} and $n_{K+1} - 1$ imply that all types above n_{K+1} strictly prefer $(q_{n_{K+1}}, p_{n_{K+1}})$ over (q_{n_K}, p_{n_K}). Thus, $\hat{n}_K = n_{K+1}$. □

Step 2: We have:

(a) For $n_k < n < \hat{n}_k$, $U_n(q_{n_{k'}}, p_{n_{k'}}) < U_n(q_{n_k}, p_{n_k})$ $\forall k' < k$;

(b) For $n > \hat{n}_k$, $U_n(q_{n_{k'}}, p_{n_{k'}}) < U_n(q_{n_k}, p_{n_k})$ $\forall k' > k$.

Proof. By Theorem 1 (or Proposition 1(iv)), the optimal mechanism satisfies both upward and downward IC constraints between types $n_{k'}$ and n_k. Also, by definition, non-local DIC between $\hat{n}_k$ and n_k binds. Thus,

$$U_{n_{k'}}(q_{n_{k'}}, p_{n_{k'}}) \ge U_{n_{k'}}(q_{n_k}, p_{n_k});$$
$$U_{n_k}(q_{n_{k'}}, p_{n_{k'}}) \le U_{n_k}(q_{n_k}, p_{n_k});$$
$$U_{\hat{n}_k}(q_{n_{k'}}, p_{n_{k'}}) \le U_{\hat{n}_k}(q_{n_k}, p_{n_k}).$$

For $k' < k$, since $q_{n_{k'}}^1 < q_{n_k}^1$ and $q_{n_{k'}}^2 > q_{n_k}^2$, DCP implies that all types between n_k and $\hat{n}_k$ strictly prefer the allocation of type n_k to that of type $n_{k'}$, proving the first part. Similarly, for $k' > k$, since $q_{n_{k'}}^1 > q_{n_k}^1$ and $q_{n_{k'}}^2 < q_{n_k}^2$, DCP implies that all types above $\hat{n}_k$ strictly prefer the allocation of type n_k to that of type $n_{k'}$. □

By Step 2, $\{\hat{n}_k\}$ is non-increasing in k. Given Step 1, this establishes part (i).

Part (ii) This follows from Lemma 4, since $q_{n_k}^1 \neq 0$ and $q_{n_k}^2 \neq 1$ for $k \in \{1, \ldots, K\}$, given that $q_{n_0}^1 < q_{n_k}^1$ and $q_{n_0}^2 > q_{n_k}^2$.

Part (iii) By Step 2(b) and the definition of $\hat{n}_k$, the non-local DIC constraint between any $\ell > \hat{n}_k$ and $n \in \{n_k, \dots, n^* - 1\}$ cannot bind. Similarly, Step 2(a) rules out binding DIC between $\ell < n_k$ and $n \in \{n^* + 1, \dots, \hat{n}_k - 1\}$.

Part (iv) Suppose not. By part (iii), there cannot be a binding non-local DIC constraint between n_{k-1} and n s.t. $n_K \le n^* < n < \hat{n}_K = n_{K+1}$. Hence, there exists a binding non-local DIC constraint between n_{k-1} and $n > n_{K+1}$ s.t. $q_n = q_{n-1}$. Since $q_n = q_{n-1} \neq q_{n^*-1}$ (footnote 22), the following lemma leads to a contradiction.

Lemma 6 *Under relative convexity, for any $\{q_\ell\}_{\ell<n^*}$ feasible in (VP) (i.e. non-decreasing q^1_ℓ and p_ℓ under binding local DIC), $L(n; \{q_\ell\}_{\ell<n^*})$ is convex along the path $\{v_n\}_{n=n^*}^N$. Moreover, the convexity is strict at some $n' \in \{n^* + 1, \dots, N - 1\}$, unless:*

(a) either $q^1_{n^*-1} = 1$ or $q^2_{n^*-1} = 0$, and

(b) $L(n'; \{q_\ell\}_{\ell<n^*}) = L^{(\ell)}(n'; \{q_\ell\}_{\ell<n^*})$ only for ℓ s.t. $q_\ell = q_{n^*-1}$.[30]

Therefore, for any $\{q_n\}_{n\in[N]}$ feasible in (VP) s.t. $q_N = (1, 1)$, there cannot be a binding non-local DIC constraint between any $\ell < n^*$ and any $n > n^*$ s.t. $q_n = q_{n-1} \neq q_{n^*-1}$.

Proof. Note that the definition $L(n; \{q_\ell\}_{\ell<n^*}) := \max_{0\le\ell<n^*} L^{(\ell)}(n; \{q_\ell\}_{\ell<n^*})$ for $n > n^*$ can be extended to $n = n^*$ if $\{q_\ell\}_{\ell<n^*}$ is feasible in (VP). That is,

$$\max_{0\le\ell<n^*} L^{(\ell)}(n^*; \{q_\ell\}_{\ell<n^*}) = 0 = L(n^*; \{q_\ell\}_{\ell<n^*}),$$

which holds since SCP below n^* (Step 2 of Section A.4) implies that $L^{(\ell)}(n^*; \{q_\ell\}_{\ell<n^*}) = U_{n^*}(q_\ell, p_\ell) - U_{n^*}(q_{n^*-1}, p_{n^*-1}) \le 0$, with equality for $\ell = n^* - 1$.

For the first part, since the maximum of convex functions is convex, it suffices to show that each $L^{(\ell)}(n; \{q_\ell\}_{\ell<n^*})$ is convex along the valuation path $\{v_n\}_{n=n^*}^N$. This follows since its right derivative

$$L^{(\ell)}_v(n; \{q_\ell\}_{\ell<n^*}) := \begin{cases} \left(q^1_\ell + q^2_\ell \dfrac{\Delta v^2_n}{\Delta v^1_n}, 0\right) & \text{if } \dfrac{\Delta v^2_n}{\Delta v^1_n} \le \dfrac{1 - q^1_\ell}{q^2_\ell}; \\ \left(1, q^2_\ell - \dfrac{(1 - q^1_\ell)\Delta v^1_n}{\Delta v^2_n}\right) & \text{otherwise,} \end{cases} \tag{8}$$

is non-decreasing in $n \ge n^*$ by relative convexity.

[30] For any $\{q_\ell\}_{\ell<n^*}$ feasible in (VP), if $q_\ell = q_{\ell'}$ for some $\ell, \ell' < n^*$, then $(q_\ell, p_\ell) = (q_{\ell'}, p_{\ell'})$.

To ensure that $L_v(n; \{q_\ell\}_{\ell<n^*})$ is well-defined on $\{q \in [0,1]^2 \mid q^1 = 1 \text{ or } q^2 = 0\}$, we show that $(0,0) \le L_v(n; \{q_\ell\}_{\ell<n^*}) \le (1,1)$. This follows from

$$(0,0) \le L_v^{(\ell)}(n; \{q_\ell\}_{\ell<n^*}) \le (1,1) \quad \text{for all } \ell < n^*, \tag{9}$$

by (8). If ℓ and ℓ' satisfy $L^{(\ell)}(n; \{q_\ell\}_{\ell<n^*}) = L(n; \{q_\ell\}_{\ell<n^*})$ and $L^{(\ell')}(n+1; \{q_\ell\}_{\ell<n^*}) = L(n+1; \{q_\ell\}_{\ell<n^*})$, then $L_v^{(\ell)}(n; \{q_\ell\}_{\ell<n^*}) \le L_v(n; \{q_\ell\}_{\ell<n^*}) \le L_v^{(\ell')}(n; \{q_\ell\}_{\ell<n^*})$.

For the second part on strict convexity, note first that if either $q_\ell^1 = 1$ or $q_\ell^2 = 0$ for some $\ell < n^* - 1$, then $q_{\ell+1} \ge q_\ell$. If $q_\ell^1 = 1$, feasibility in (VP) implies $q_{\ell+1}^1 = q_\ell^1 = 1$, so for $p_{\ell+1} \ge p_\ell$, $q_{\ell+1}^2 \ge q_\ell^2$. If $q_\ell^2 = 0$, then $q_{\ell+1}^1 \ge q_\ell^1$, so again $q_{\ell+1} \ge q_\ell$. Therefore, unless $q_{\ell+1} = q_\ell$, it must be that for all $n' > n^*$,

$$L^{(\ell+1)}(n'; \{q_\ell\}_{\ell<n^*}) - L^{(\ell)}(n'; \{q_\ell\}_{\ell<n^*}) = (q_{\ell+1} - q_\ell) \cdot (v_{n'} - v_{\ell+1}) > 0.$$

Thus, if for some $\ell < n^* - 1$ s.t. $q_\ell^1 = 1$ or $q_\ell^2 = 0$, $L(n'; \{q_\ell\}_{\ell<n^*}) = L^{(\ell)}(n'; \{q_\ell\}_{\ell<n^*})$ for some $n' > n^*$, then necessarily $q_\ell = q_{n^*-1}$.

Consequently, if either condition (a) or (b) in the lemma fails, there exists some $\ell < n^*$ s.t. $q_\ell^1 \ne 1$, $q_\ell^2 \ne 0$, and $L(n'; \{q_\ell\}_{\ell<n^*}) = L^{(\ell)}(n'; \{q_\ell\}_{\ell<n^*})$. Then, by (8),

$$L_v^{(\ell)}(n'; \{q_\ell\}_{\ell<n^*}) > L_v^{(\ell)}(n'-1; \{q_\ell\}_{\ell<n^*}),$$

and since $L(n'; \cdot)$ is defined as the maximum of $L^{(\ell)}(n'; \cdot)$ over $\ell < n^*$,

$$L_v(n'; \{q_\ell\}_{\ell<n^*}) \ge L_v^{(\ell)}(n'; \{q_\ell\}_{\ell<n^*}); \; L_v^{(\ell)}(n'-1; \{q_\ell\}_{\ell<n^*}) \ge L_v(n'-1; \{q_\ell\}_{\ell<n^*}).$$

This implies

$$L_v(n'; \{q_\ell\}_{\ell<n^*}) > L_v(n'-1; \{q_\ell\}_{\ell<n^*}),$$

establishing strict convexity at n'.

Lastly, if $q_n = q_{n-1} \ne q_{n^*-1}$ for some $N > n > n^*$, $L(\tilde{n}; \cdot)$ is strictly convex at $\tilde{n} = n$. Even when (a) holds, the non-local DIC constraint between $n^* + 1$ and $n^* - 1$ implies $q_n \ge q_{n^*} \ge q_{n^*-1}$, and hence, $q_n = q_{n-1} > q_{n^*-1}$. By strict SCP, DIC between $n-1$ and $n^* - 1$ further implies that DIC between n and $n^* - 1$ cannot be binding, so (b) fails.

Now, consider $B = \{\underline{n}, \dots, \overline{n}\}$ containing n s.t. $N > \overline{n} \ge n > \underline{n}$.[31] Since the

[31] If $\overline{n} = N$, then $q_n = q_{n-1} = q_N = (1,1) > q_{n^*-1}$, so $(1,1) > q_\ell$ for any $\ell < n^*$. Then, again by strict SCP, the non-local DIC constraint between any such ℓ and n cannot be binding.

allocation is constant over the bunch B, $R(\tilde{n})$ must be affine for $\tilde{n} \in B \cup \{\overline{n}+1\}$. Over this interval, the majorization constraint requires the affine function $R(\tilde{n})$ to lie above the convex function $L(\tilde{n};\cdot)$. If the non-local DIC constraint for n against some $\ell < n^*$ is binding, the two functions must intersect at $\tilde{n} = n$: $R(n) = L(n;\{q_\ell\}_{\ell<n^*}) = L^{(\ell)}(n;\{q_\ell\}_{\ell<n^*})$. However, since $L(\tilde{n};\cdot)$ is strictly convex at $\tilde{n} = n$, this would imply that $R(\tilde{n}) < L(\tilde{n};\cdot)$ at either $\tilde{n} = n-1$ or $\tilde{n} = n+1$, a contradiction. ∎

A.8 Proof of Theorem 2

Throughout this proof, we restrict attention to mechanisms with binding local DIC constraints and binding IR constraint for the lowest type (Proposition 1(i)-(ii)). Under these binding constraints, $\{p_n\}$ are determined by $\{q_n\}$ (even when $\{(q_n,p_n)\}_{n\in[N]}$ is not feasible in (VP)), so let $M = \{q_n\}_{n\in[N]}$ with slight abuse of notation. Given M, define $\mathcal{B}(M)$ as the set of its bunches, i.e. the partition of $[N]$ into intervals on which q_n is constant and takes different values across successive intervals. Also, let $B_1(M)$ be the union of the bunches in $\mathcal{B}(M)$ assigned the bundle $(1,0)$, i.e. $q_n = (1,0)$ if and only if $n \in B_1(M)$.

A.8.1 Stochastic separation above n^* (Definition 3(b))

Part (i) of Definition 3(b) follows directly from the constraint. For part (ii), if $M = \{q_n\}_{n\in[N]}$ is a solution to (VP), $\{q_n\}_{n\geq n^*}$ must be a solution to (VP*) given $\{q_n\}_{n<n^*}$. More generally, we show that for any $\{q_n\}_{n<n^*}$ feasible in (VP), a solution $\{q_n\}_{n\geq n^*}$ to (VP*) satisfies all the conditions in the claim. Let $\overline{\mathcal{B}}(M)$ be the set of bunches for the restricted schedule $\{q_n\}_{n\geq n^*}$ of $M = \{q_n\}_{n\in[N]}$. Note that $q_N = (1,1)$.

"If" part Suppose not. So, there exists an ironing interval for good 1, $I_s^{1*} := \{N_{s-1}^{1*}, N_{s-1}^{1*}+1, \ldots, N_s^{1*}-1\}$ s.t. $q^1_{N^{1*}_{s-1}} < q^1_{N^{1*}_s-1}$. In other words, the (minimal) ironing interval I_s^{1*} contains distinct values of q^1. Note that $q^1_{N^{1*}_{s-1}} < 1$ and $q^2_{N^{1*}_{s-1}} = 0$. A similar argument applies to any I_s^{2*} containing distinct values of q^2. More specifically:

- $\exists$ distinct $B, B' \in \overline{\mathcal{B}}(M)$ with $N_{s-1}^{1*} \in B$ and $N_s^{1*}-1 \in B'$, and $q^1_{N^{1*}_{s-1}} < q^1_{N^{1*}_s-1}$.

Consider $B \cap I_s^{1*}$ and $B' \cap I_s^{1*}$. By the definition of I_s^{1*}, and by Lemma 3, we have

$$W^1_{B\cap I_s^{1*}} > W^1_{I_s^{1*}} > \max_{j\in\{1,2\}} W^j_{B'\cap I_s^{1*}}. \tag{10}$$

The last term accounts for the possibility that the allocations to $B \cap I_s^{1*}$ and $B' \cap I_s^{1*}$ lie on different edges $(\cdot,0)$ and $(1,\cdot)$, respectively.

But then, consider increasing the quantity of good 1 assigned to $B \cap I_s^{1*}$ by a sufficiently small $\epsilon > 0$, while decreasing the quantity of one of the goods given to $B' \cap I_s^{1*}$ along either $(\cdot, 0)$ or $(1, \cdot)$ by $\epsilon' > 0$ s.t. $R(N_s^{1*}) - R(N_{s-1}^{1*})$ remains unaffected. The resulting allocations from front-loading rent raise $R(n)$ point-wise for all $n > n^*$, and thus, maintain the majorization constraint in (VP*). Moreover, by (10), this adjustment strictly increases total virtual surplus, contradicting that $\{q_n\}$ is optimal.

Moreover, the average virtual valuation per rent over $B \cap \{\overline{m}^1, \cdots, N\}$ for $B \in \overline{\mathcal{B}}(M)$ with $\overline{m}^1 \in B$ is strictly positive for good 1, since we define $\overline{m}^1$ as the highest of the truncated monopoly types. Hence, if $q^1_{\overline{m}^1} < 1$ (so $q^2_{\overline{m}^1} = 0$), increasing good 1's quantity for $B \cap \{\overline{m}^1, \cdots, N\}$ strictly increases total virtual surplus without violating the majorization constraint. Since $\overline{m}^2 \geq \overline{m}^1$, the same reasoning applies to good 2.

"Only if" part Suppose not. So, suppose that there exists $B = \bigcup_{s=\underline{s}}^{\overline{s}} I_s^{1^*} \in \overline{\mathcal{B}}(M)$ with $\underline{s} < \overline{s}$ such that the allocation is $(q^1, 0) \neq q_{n^*-1}$ for some $q^1 \in (0,1)$ and $\overline{W}_{\underline{s}}^{1*} < \overline{W}_{\overline{s}}^{1*}$. In other words, multiple adjacent ironing intervals with different average virtual valuations per rent are bunched. A similar argument applies if the bunch is given $(1, q^2) \neq q_{n^*-1}$ for some $q^2 \in (0,1)$.

Note first that for $n \in B$ s.t. $n > \min B$, the non-local DIC constraint against any $\ell < n^*$ is slack by Lemma 6. Then, consider reducing each $q_n^1 (= q^1)$ for $n \in I_{\underline{s}}^{1*}$ by small $\epsilon > 0$ s.t. non-local DIC between any $n \in B \backslash \min B$ and $\ell < n^*$ remains slack, while raising each $q_{n'}^1 (= q^1)$ for $n' \in I_{\overline{s}}^{1*}$ uniformly by $\epsilon' > 0$ s.t. $R(\max B + 1) - R(\min B)$ remains unchanged. This rent back-loading preserves non-local DIC between any $n > \min B$ and $\ell < n^*$, as well as that between any $n \leq \min B$ and $\ell < n^*$. Hence, the new allocations are feasible in (VP*). However, since $\overline{W}_{\underline{s}}^{1*} < \overline{W}_{\overline{s}}^{1*}$, this reallocation strictly increases total virtual surplus, contradicting that $\{q_n\}$ is optimal.

A.8.2 Separate monopoly pricing below D (Definition 3(a), part (ii))

Recall that $n_1^* := \min\{n_1, n^*\}$.

Lemma 7 *Suppose that $M = \{q_n\}_{n \in [N]}$ is a solution to (VP). Then, $\tilde{M} = \{\tilde{q}_n\}_{n \in [N]}$ s.t. $\tilde{q}_n = q_n$ for all $n \geq n_1^*$ is also a solution if for each $i \in \{1, 2\}$,*

$$\tilde{q}_n^i = \begin{cases} 0 & \text{if } n < \underline{m}_1^i; \\ q_{n_0}^i \in [0,1] & \text{if } \underline{m}_1^i \leq n < n_1^*. \end{cases} \tag{11}$$

Moreover, if $\arg\max_{n \le n_1^*} \Pi_n^i = \{\underline{m}_1^i\}$ *for each* $i \in \{1,2\}$*, then* M *and* $\tilde{M}$ *coincide.*

Proof. Since q_ℓ is non-decreasing for $\ell < n_1^*$, SCP holds, and all higher types prefer the allocation of $n_1^* - 1$ to that of ℓ. Hence, as long as q_ℓ is non-decreasing for $\ell < n_1^*$, the non-local DIC constraints in (4) hold against any $\ell < n_1^* - 1$ whenever they hold against $\ell = n_1^* - 1$.

Moreover, $q_{n_1^*-1}$ can be reduced to a level below q_{n_0} without violating any non-local DIC constraint against $\ell < n^*$ s.t. $\ell \ge n_1^* - 1$. For $\ell \in \{n_1^*, \dots, n^* - 1\}$, the reduction does not affect (4). For $\ell = n_1^* - 1$, if $K = 0$ ($n_1^* = n^*$), this trivially holds by SCP with $q_n \ge q_{n_1^*} = q_{n_0}$ $\forall n \ge n_1^*$. For $K > 0$ ($n_1^* = n_1$), $q_\ell = q_{n_0}$. So, when q_ℓ is decreased by $(\Delta^1, \Delta^2) \ne (0,0)$ s.t. $\Delta^1, \Delta^2 \ge 0$, the function $R(n)$ remains unchanged, while $L^{(\ell)}(n;\cdot)$ decreases by $\sum_{i=1}^2 \Delta^i \big(v_n^i - v_{\ell+1}^i\big) > 0$, ensuring that (4) continues to hold.

Consequently, any $\tilde{M} = \{\tilde{q}_n\}_{n \in [N]}$ s.t. $\tilde{q}_n = q_n$ for all $n \ge n_1^*$ is feasible in (VP) whenever, for each $i \in \{1,2\}$, the sequence q_n^i is non-decreasing and satisfies $q_n^i \le q_{n_0}^i$ for all $n < n_1^*$. Since M satisfies these conditions, $\tilde{M}$ is also a solution if it maximizes total virtual surplus for types below n_1^*:

$$\max_{\{q_n\}_{n<n_1^*}} \sum_{n=1}^{n_1^*-1} \left(W_n^1 \Delta v_n^1 q_n^1 + W_n^2 \Delta v_n^2 q_n^2\right) = \max_{\{q_n\}_{n<n_1^*}} \sum_{n=1}^{n_1^*-1} \left(V_n^1 q_n^1 + V_n^2 q_n^2\right)$$

subject to q_n^1 and q_n^2 being non-decreasing and, for all $i \in \{1,2\}$ and $n < n_1^*$, $q_n^i \le q_{n_0}^i$.

This problem separates into two standard single-good problems (Myerson, 1981), with maximum quantities scaled from 1 to $q_{n_0}^i$. Hence, the mechanism satisfying (11) is optimal. Moreover, it is uniquely optimal if $\arg\max_{n \le n_1^*} \Pi_n^i = \{\underline{m}_1^i\}$ for $i \in \{1,2\}$, in which case $\{q_n\}_{n<n_1^*}$ of M must satisfy (11). ■

We now focus on a solution to (VP) satisfying (11). This restiction does not increase the *smallest* K since $\{\tilde{q}_n^i\}_{n<n_1^*}$ is non-decreasing and bounded above by $q_{n_0}^i$. For $K > 0$, by Lemma 5 with (11), the cutoff type n_0 is a truncated monopoly type:

$$n_0 = \begin{cases} \underline{m}_1^1 & \text{if } q_{n_0}^1 > 0; \\ \underline{m}_1^2 & \text{if } q_{n_0}^1 = 0 \text{ and } q_{n_0}^2 > 0. \end{cases} \tag{12}$$

It then follows that $q_n^1 = 0$ for all $n < n_0$.

A.8.3 Maximal bunching in D (Definition 3(a), part (i))

Outline We show that $K \le 2$ and build on the arguments to establish the properties of q_{n_0}. The arguments proceed by way of contradiction. We first define a particular class of solutions that we call "maximal bunching DIC mechanism", which must exist in our finite environment. We then show that if $K > 2$ for such a solution, it contradicts either optimality or the maximality property. The key step in deriving the contradiction is defining a directional vector in $\mathbb{R}^{2N}$ such that shifting an optimal mechanism along the vector preserves the binding non-local DIC constraints.

Maximal bunching DIC mechanism For a pair of solutions to (VP), M and M', we say that $M' \succeq_B M$ if (i) for each $B \in \mathcal{B}(M)$, there exists $B' \in \mathcal{B}(M')$ such that $B \subseteq B'$, i.e. the set of bunches is coarser under M' than under M and (ii) $B_1(M) \subseteq B_1(M')$, i.e. the bunch with the bundle $(1,0)$ is (weakly) larger under M'. For any solution M, $B_1(M)$ is a single bunch, i.e. $B_1(M) \in \mathcal{B}(M)$,[32] and it must be located above n_K by Proposition 2 with $K > 0$. The strict order $M' \succ_B M$ holds if $M' \succeq_B M$ and $(\mathcal{B}(M'), B_1(M')) \neq (\mathcal{B}(M), B_1(M))$.

Define $\mathcal{C}(M) \subseteq \{(\ell, n) \mid 1 \le \ell < n^* < n \le N\}$ as the set of type pairs across n^* such that the non-local DIC constraint binds between them. We write $M' \succeq_C M$ if $\mathcal{C}(M') \supseteq \mathcal{C}(M)$, i.e. M' has more binding non-local DIC constraints than M. The strict order $\succ_C$ is defined by proper superset inclusion.

We write $M' \succeq M$ if *both* $M' \succeq_B M$ and $M' \succeq_C M$ hold; we write $M' \succ M$ if $M' \succeq M$ and at least one of the two relations is strict.

Definition 5 *Among solutions to* (VP) *that satisfy* (11) *with the smallest* K, $M = \{q_n\}_{n \in [N]}$ *is a "maximal bunching DIC mechanism (MBM)" if there exists no other such solution* $M' = \{q'_n\}_{n \in [N]}$ *with* $M' \succ M$.

This refinement incorporates the optimal properties below D (Section A.8.2). Also, a solution satisfying MBM always exists, since $\succeq_B$ and $\succeq_C$ are induced by set-inclusion orders on the finite sets of partitions of $[N]$ and $\{(\ell, n) \mid 1 \le \ell < n^* < n \le N\}$.

DIC-preserving directional vectors Fix a solution $M = \{q_n\}_{n \in [N]}$ that satisfies (11). Let $B_1 = B_1(M) \in \mathcal{B}(M)$ for notational simplicity. Recall from Proposition 4

[32] For a solution M to (VP), each $B \in \mathcal{B}(M)$ satisfies $q_n = q_{n'}$ for all $n, n' \in B$ and $q_n \neq q_{n'}$ whenever $n \in B$ and $n' \notin B$. The latter requires that distant bunches are assigned different quantities, which holds due to type-increasing q_n^1 and p_n by Proposition 1(iii) and Theorem 1. Thus, $B_1(M) \in \mathcal{B}(M)$.

that for $k = 1, \dots, K$, $\hat{n}_k$ is the highest type above n^* s.t. the non-local DIC constraint binds against $n_k \in D$, which must be the lowest type in the bunch containing it:

$$q_{\hat{n}_k - 1} < q_{\hat{n}_k} \leq (1,1). \tag{13}$$

If there is a binding non-local DIC constraint against n_0, then $\hat{n}_0$ is well-defined. Given M, define the following vector.

Definition 6 *Given $\Delta := (\Delta^1, \Delta^2) \in \mathbb{R}^2$ and $0 < k < K$, the "DIC-preserving directional vector", denoted by $D^k(\Delta) := (D_1^k(\Delta), D_2^k(\Delta), \dots, D_N^k(\Delta)) \in \mathbb{R}^{2N}$, is defined as*

$$D_n^k(\Delta) = \begin{cases} (0,0) & \forall n \text{ s.t. } m < n_k \text{ or } n \geq \hat{n}_k; \\ \Delta & \forall n \text{ s.t. } n_k \leq n < n_K; \\ L_v(B; \Delta) & \forall n \text{ s.t. } n_K \leq n < \hat{n}_k \text{ and } n \in B \in \mathcal{B}(M), \end{cases}$$

where

$$L_v(B; \Delta) := \begin{cases} \left(\dfrac{\Delta v_B^1 \Delta^1 + \Delta v_B^2 \Delta^2}{\Delta v_B^1}, 0 \right) & \text{if } q_n < (1,0); \\ \left(0, \dfrac{\Delta v_B^1 \Delta^1 + \Delta v_B^2 \Delta^2}{\Delta v_B^2} \right) & \text{if } q_n \geq (1,0), \end{cases}$$ [33]

and $\Delta v_B^i := v_{\max B + 1}^i - v_{\min B}^i$ for $i = 1, 2$. For $k = 0$, if $\hat{n}_0$ is well-defined, $D^k(\Delta)$ is defined analogously, except that $D_n^k(\Delta) = (0, \Delta^2)$ for $n \in \{\underline{m}_1^2, \dots, n_0 - 1\}$.[34]

By construction, perturbing M along $D^k(\Delta)$ preserves the bunch structure $\mathcal{B}(M)$, including $\{n_K, \dots, n_{K+1} - 1\}$, as well as the the extreme form of allocations required by Lemma 2 (i.e. $q_n^1 = 1$ or $q_n^2 = 0$ for $n \geq n^*$). Furthermore:

Lemma 8 *Fix any $\Delta \in \mathbb{R}^2$ and $k \in \{1, \dots, K-1\}$. Consider $\{\tilde{q}_n\}_{n \in [N]} := \{q_n\}_{n \in [N]} \pm \epsilon D^k(\Delta)$ for $\epsilon > 0$. Then, for any $\ell, n \in [N]$ s.t. $n_k \leq \ell < n^* < n \leq \hat{n}_k$, if the non-local*

[33] For Δ with sufficiently small norm $\|\Delta\|$ and $B = \{n\}$ with $q_n < (\geq)(1,0)$, $L_v(B;\Delta) = L_v^{(\ell)}\big(n; \{q_\ell + \Delta\}_{\ell < n^*}\big) - L_v^{(\ell)}\big(n; \{q_\ell\}_{\ell < n^*}\big)$ if $\frac{\Delta v_n^2}{\Delta v_n^1} < (>) \frac{1 - q_\ell^1}{q_\ell^2}$ by (8).

[34] By (12), $\{\underline{m}_1^2, \dots, n_0 - 1\}$ is a nonempty set in $\mathcal{B}(M)$ if $\underline{m}_1^2 < \underline{m}_1^1$ and $q_{n_0}^1 > 0$.

DIC constraint binds under $\{q_n\}$, it binds under $\{\tilde{q}_n\}$, i.e.

$$\sum_{j=\ell}^{n-1}\sum_{i=1}^{2}\Delta v_j^i \tilde{q}_j^i \;=\; \sum_{i=1}^{2}(v_n^i - v_\ell^i)\tilde{q}_\ell^i. \tag{14}$$

If $\hat{n}_0$ is well-defined, this also holds for $k = 0$, and hence, if the non-local DIC constraint binds between any ℓ, n s.t. $\ell < n^* < n$, it remains binding under $\{\tilde{q}_n\}$.

Proof. Given Proposition 4(iii)-(iv), consider a binding non-local DIC constraint between ℓ and n such that $n_k \le \ell < n_K < n_{K+1} \le n \le \hat{n}_k$ and $n = \max B + 1$ for some $B \in \mathcal{B}(M)$. For the right-hand side of (14), we have

$$\sum_{i=1}^{2}(v_n^i - v_\ell^i)\tilde{q}_\ell^i - \sum_{i=1}^{2}(v_n^i - v_\ell^i)q_\ell^i = \Delta \cdot (v_n^1 - v_\ell^1, v_n^2 - v_\ell^2) = \Delta \cdot \sum_{j=\ell}^{n-1}\Delta v_j. \tag{15}$$

For the left-hand side, note that for any $B \in \mathcal{B}(M)$ s.t. $\min B \ge n_K$, the change in cumulative rents under binding local DIC is equal to

$$\Delta v_B^1 \Delta^1 + \Delta v_B^2 \Delta^2 = \Delta \cdot \sum_{n \in B}\Delta v_n.$$

Summing this over all such B with $\max B < n$, and adding the changes in cumulative rents under binding local DIC from ℓ to n_K, yields the last term in (15).

This also holds for $k = 0$ if $\hat{n}_0$ is well-defined, since Proposition 4(i) and (iii) extend to all $k \in \{0, \dots, K\}$ by DCP; that is, every binding non-local DIC constraint between types ℓ and n with $n_0 \le \ell < n^* < n \le \hat{n}_0$ remains binding. Moreover, by Proposition 4(iii), together with $q_n < q_{n_0}$ for $n < n_0$ and the resulting strict SCP between q_n and q_{n_0}, no other non-local DIC constraint in (4) is binding. Hence, this property extends to all pairs ℓ, n s.t. $\ell < n^* < n$. ■

Given the definition of $L_v(B;\Delta)$, it is possible that $q_n^2 + \epsilon D_n^k(\Delta)$ or $q_n^2 - \epsilon D_n^k(\Delta)$ is always negative for any sufficiently small $\epsilon > 0$ if $n \in B_1$. This violates feasibility since we cannot have a negative allocation. Later, we augment $D^k(\Delta)$ to preserve B_1. For $k = 1, \dots, K-1$, to preserve all the binding non-local DIC constraints, let (as above)

$$\Delta_k^* := \left(-\frac{1}{v_{\hat{n}_k}^1 - v_{n_k}^1},\ \frac{1}{v_{\hat{n}_k}^2 - v_{n_k}^2}\right). \tag{16}$$

Also, let $\Delta R_k := \Delta v_{B_1}^1 \Delta_k^{1*} + \Delta v_{B_1}^2 \Delta_k^{2*}$. If $\Delta R_k = 0$, $D_n^k(\Delta_k^*) = 0$ for all $n \in B_1$.

Lemma 9 *Fix any $k = 1, \dots, K-1$, and consider $\{\tilde{q}_n\} := \{q_n\} \pm \epsilon D^k(\Delta_k^*)$ for $\epsilon > 0$. For any $\ell, n \in [N]$ s.t. $\ell < n^* < n$, if the non-local DIC constraint binds under $\{q_n\}$, it binds under $\{\tilde{q}_n\}$. Also, for sufficiently small ϵ, $\{\tilde{q}_n\}$ is feasible in (VP) if $B_1 = \varnothing$, $\min B_1 \geq \hat{n}_k$, or $\Delta R_k = 0$ so that B_1 is preserved under the perturbation.*

Proof. For the first part, by Proposition 4(iii) and Lemma 8, it suffices to derive (14) for any $\ell < n_k$ and $n \geq \hat{n}_k$. To see this, note first that Δ_k^* is defined as a shift in n_k's allocation that keeps $\hat{n}_k$'s payoff from deviating to n_k's allocation unchanged:

$$\Delta_k^* \cdot \left(v_{\hat{n}_k}^1 - v_{n_k}^1,\, v_{\hat{n}_k}^2 - v_{n_k}^2\right) = 0.$$

By Lemma 8, shifting $\{q_n\}$ along $D^k(\Delta_k^*)$ also leaves the cumulative rents from n_k to $\hat{n}_k$ unchanged. Then, (14) holds since $D_\ell^k(\Delta_k^*) = 0 = D_n^k(\Delta_k^*)$ for $\ell < n_k$ and $n \geq \hat{n}_k$.

For feasibility, it suffices to verify the monotonicity constraints on the quantity schedule, provided that the bunch B_1 at $(1,0)$ is unaffected. By (13), for sufficiently small ϵ, $\pm\epsilon D^k(\Delta_k^*)$ shifts every bunch above n_K and below $\hat{n}_k$ along the edge $q_n^1 = 1$ or $q_n^2 = 0$, while maintaining type-increasing quantities. Moreover, by Corollary 2, and since $q_{n_k}^1 < q_{n_{k+1}}^1$ $\forall k \neq K$, $\{p_n\}$ and $\{q_n^1\}$ for $n < n^*$ remain monotone. ■

We are now ready to proceed to the proof of the claim.

Proof of $K \leq 2$ Suppose not. So, $K > 2$ for every solution to (VP). Among the solutions, let $M = \{q_n\}_{n \in [N]}$ be an MBM (Definition 5). We proceed as follows.

Step 1: M lies in the interior of a line segment, $\mathcal{L}$, in the feasibility set of (VP).

Proof. Given M, define

$$D_0^* = \begin{cases} D^{K-1}(\Delta_{K-1}^*) & \text{if } B_1 = \varnothing,\ \min B_1 \geq \hat{n}_{K-1},\ \text{or } \Delta R_{K-1} = 0; \\ D^{K-2}(\Delta_{K-2}^*) & \text{if } B_1 \neq \varnothing,\ \min B_1 < \hat{n}_{K-1}, \Delta R_{K-1} \neq 0,\ \text{and } \Delta R_{K-2} = 0; \\ D^{K-2,K-1} & \text{otherwise,} \end{cases}$$

where

$$D^{K-2,K-1} := \frac{D^{K-2}(\Delta_{K-2}^*)}{\Delta R_{K-2}} - \frac{D^{K-1}(\Delta_{K-1}^*)}{\Delta R_{K-1}}.$$

$D^{K-2,K-1}$ is a linear combination of $D^{K-1}(\Delta_{K-1}^*)$ and $D^{K-2}(\Delta_{K-2}^*)$, and it is well-defined since it applies when $\Delta R_{K-1} \neq 0$ and $\Delta R_{K-2} \neq 0$. Moreover, it preserves B_1. Hence, Lemma 9 applies to D_0^*, which is summarized below.

Lemma 10 *Suppose that $K > 2$, and consider $\{\tilde{q}_n\} := \{q_n\} \pm \epsilon D_0^*$ for $\epsilon > 0$. We have:*

- *For any $\ell, n \in [N]$ s.t. $\ell < n^* < n$, if the non-local DIC constraint binds under $\{q_n\}$, it binds under $\{\tilde{q}_n\}$.*
- *$\mathcal{B}(\{\tilde{q}_n\})$ is coarser than $\mathcal{B}(\{q_n\})$ and $B_1(\{\tilde{q}_n\}) \supseteq B_1(\{q_n\})$.*
- *$\tilde{q}_n^2 = 1$ or $\tilde{q}_n^1 = 0$ for $n \geq n_K$.*

Therefore, for sufficiently small $\epsilon > 0$, $\{\tilde{q}_n\}$ is feasible in (VP).

Note that the perturbation is allowed in both directions of the line segment. □

The next two steps establish a contradiction against M being an MBM.

Step 2: If $M' \in \mathcal{L}$, then M' solves (VP).

Proof. Let $V = (V_1^1, V_1^2, \ldots, V_N^1, V_N^2)$ be the $2N$-dimensional vector where V_n^i is type n's virtual valuation of good i. Then, the difference in total virtual surplus between M and $M' = M \pm \epsilon D_0^* \in \mathcal{L}$ is $\pm\epsilon D_0^* \cdot V$. If $D_0^* \cdot V \neq 0$, it would contradict the optimality of M. Hence, total virtual surplus must be constant on $\mathcal{L}$. □

Under M, there must be non-binding feasibility constraints by Step 1. They are either non-local DIC or monotonicity constraints (with $q_n^i \in [0,1]\ \forall i, n$).[35] As we shift M along $\mathcal{L}$, by Lemma 10 and the compactness of the feasibility set, at least one must eventually bind while preserving all previously binding constraints. Let $\overline{\mathcal{L}}$ and $\underline{\mathcal{L}}$ be the two endpoints of $\mathcal{L}$, where this happens for the first time in each direction.

Step 3: If $M' \in \mathcal{L}$, $M' \geq M$; if $M' \in \{\underline{\mathcal{L}}, \overline{\mathcal{L}}\}$, $M' \succ M$ under (11) (Definition 5).

Proof. By Lemma 10, and by the definition of $\mathcal{L}$ with Step 2, $M' \geq M$ if $M' \in \mathcal{L}$. Suppose that $M' \in \{\underline{\mathcal{L}}, \overline{\mathcal{L}}\}$.

Step 3(a): If the constraint that becomes binding under M' is a monotonicity constraint, then it is for some $n \geq n_K$. Thus, M' satisfies (11) with the same K.

Suppose not. By the definition of D_0^* with $K > 2$, the constraint is then between n_k and $n_k - 1$ for some $k \in \{1, \ldots, K\}$. Under M, we have $q_{n_{k-1}}^1 < q_{n_k}^1$, $q_{n_{k-1}}^2 > q_{n_k}^2$, and $p_{n_{k-1}} < p_{n_k}$ for $k \in \{1, \ldots, K\}$. Thus, in this case, the binding constraint under M' takes the form $q_{n_{k-1}}^{1'} = q_{n_k}^{1'}$, $q_{n_{k-1}}^{2'} = q_{n_k}^{2'}$, or $p'_{n_{k-1}} = p'_{n_k}$ for some $k \in \{1, \ldots, K\}$.

If $q_{n_{k-1}}^{1'} = q_{n_k}^{1'}$ or $p'_{n_{k-1}} = p'_{n_k}$, then by feasibility or by optimality (Corollary 2), it must also be that $q_{n_{k-1}}^{2'} = q_{n_k}^{2'}$. If $q_{n_{k-1}}^{2'} = q_{n_k}^{2'}$, then either K decreases by one (when

[35] The monotonicity constraints require that $0 \leq q_{n_0}^1 < q_{n_1}^1 < \cdots < q_{n_K}^1$, $1 \geq q_{n_0}^2 > q_{n_1}^2 > \cdots > q_{n_K}^2$, and $p_{n_0} < p_{n_1} < \cdots < p_{n_K}$, while for all $n \geq n_K$, the pair (q_n^1, q_n^2) is non-decreasing along $(\cdot, 0)$ and $(1, \cdot)$.

$q^{1'}_{n_{k-1}} = q^{1'}_{n_k}$ or $k = 1$), contradicting K being the smallest; otherwise, we obtain a contradiction to Proposition 2, since then there are two separated decreasing intervals (when $k \neq K$) or a decreasing interval not immediately below n^* (when $k = K$).

Step 3(b): $M' \succ_B M$ or $M' \succ_C M$.

By Step 3(a), it is either a non-local DIC constraint or a monotonicity constraint above n_K that may become binding under M'. Also, both may occur simultaneously. If the constraints involve a non-local DIC constraint, then $M' \succ_C M$, and hence, $M' \succ M$. Consider a monotonicity constraint for some $n \geq n_K$.

For $n \in B \in \mathcal{B}(M)$ s.t. $n \geq n_K$ and $q_n \neq (1,1)$ (by (13)), we have:

- if $n_K \leq n < \min B_1$, then $0 < q^1_n < q^1_{\max B+1} < 1$ with $q^2_n = 0$;
- if $n > \max B_1$, then $0 < q^2_n < q^2_{\max B+1}$ with $q^1_n = 1$.

Thus, if one of the strict inequalities becomes an equality under M', it means that there is either additional bunching or a larger bunch with the bundle $(1,0)$. This implies that $M' \succ_B M$. When $q^1_{\max B} = 0$, we have $q_{\max B} = 0$, and hence, another monotonicity constraint must bind for n below n_K, contradicting Step 3(a). □

Proof of either $q^1_{n_0} = 0$ or $q^2_{n_0} = 1$ when $K = 1$ or $K = 2$ Suppose not, so no such MBM exists. Fix an MBM, $M = \{q_n\}_{n \in [N]}$, such that $K = 1$ or $K = 2$, and $q^1_{n_0} \neq 0$ and $q^2_{n_0} \neq 1$. Let us proceed as follows.

Step 1: $\hat{n}_0$ is well-defined; hence, $D^0(\Delta)$ is well-defined for any $\Delta \in \mathbb{R}^2$.

Proof. Suppose not. So, there is no binding non-local DIC constraint between n_0 and any type above n^*. Then, by Lemma 4, $q^1_{n_0} = 0$ or $q^2_{n_0} = 1$, a contradiction. □

Step 2: M lies in the interior of a line segment of the feasibility set of (VP).

Proof. Consider the following directional vector

$$D^*_1 = \begin{cases} D^0(\Delta^*_0) & \text{if } B_1 = \varnothing \text{ or } \min B_1 \geq \hat{n}_0; \\ D^0(\Delta_{B_1}) & \text{otherwise,} \end{cases} \tag{17}$$

where $\Delta_{B_1} = (\Delta^1_{B_1}, \Delta^2_{B_1}) := \left(-\frac{1}{\Delta v^1_{B_1}}, \frac{1}{\Delta v^2_{B_1}}\right)$ so that D^*_1 always preserves B_1 by Definition 6. Moreover, Lemma 8 applies to D^*_1.

Since $0 < q^1_{n_0} < q^1_{n_1}$ and $1 > q^2_{n_0} > q^2_{n_1}$, the perturbed schedule $\{\tilde{q}_n\} = \{q_n\} \pm \epsilon D^*_1$ satisfies the feasibility constraints above n_0 for sufficiently small $\epsilon > 0$, as in Lemma 9. Since D^*_1 also moves q_{n_0}, we must also check feasibility below n_0. Note that $q^1_{n_0} > 0$, while $q^1_n = 0$ for $n < n_0$ by (11) and (12). Thus, the inequalities $q^1_{n_0} > q^1_{n_0-1}$ and $p_{n_0} > p_{n_0-1}$ persist under small perturbations along D^*_1. $\square$

By the same reasoning as in Step 2 of the proof of $K \leq 2$, the total virtual surplus must be constant on the line segment. To show that MBM is violated at each endpoint, say, M', note that unlike D^*_0, D^*_1 also shifts q_{n_0}. But, since it preserves (11) and K, and since $q^1_{n_0} \neq 0$ and $q^2_{n_0} \neq 1$ for any MBM, monotonicity constraints below n_0 cannot become binding under M'. Thus, Step 3 of the proof of $K \leq 2$ still holds.

Proof of both $q^1_{n_0} = 0$ and $q^2_{n_0} = 1$ when $K = 2$ Suppose not, so no such MBM exists. By the above arguments, there exists an MBM, $M = \{q_n\}_{n\in[N]}$, such that $K = 2$, and $q_{n_0} = (0, a)$ or $q_{n_0} = (b, 1)$ for some $0 < a, b < 1$ (since $q_{n_0} \notin \{(0, 0), (1, 1)\}$).

Step 1: $\hat{n}_0$ is well-defined; hence, $D^0(\Delta)$ is well-defined for any $\Delta \in \mathbb{R}^2$.

Proof. Suppose not. So, there is no binding non-local DIC constraint between n_0 and any type above n^*. Consider the case $q_{n_0} = (0, a)$ for some $0 < a < 1$. The other case can be addressed via a symmetric argument.

In this case, $\underline{m}^2_1 < n_1$, i.e. $\Pi^2_{\underline{m}^2_1} > \Pi^2_{n_1}$. Otherwise, $q^2_{n_0} = 0$ by Definition 3(a)(ii), contradicting $K > 0$. Now, consider the following perturbation: raise q^2_n from $q^2_{n_0}$ to $q^2_{n_0} + \epsilon$ for all $n \in \{\underline{m}^2_1(= n_0), \ldots, n_1 - 1\}$ (by (12)). Since every non-local DIC constraint in (4) is slack against $\ell < n_1$, it continues to hold, while the one against $\ell \geq n_1$ is unaffected. Moreover, by Corollary 2, type-increasing p_n for $n < n^*$ is ensured, while type-increasing q^1_n holds trivially. Thus, the perturbation preserves feasibility while strictly increasing total virtual surplus, contradicting the optimality of M. $\square$

Step 2: M lies in the interior of a line segment of the feasibility set of (VP).

Proof. For each $i = 1, 2$, define

$$D^*_2(e_i) = \begin{cases} D^1(\Delta^*_1) & \text{if } B_1 = \varnothing \text{ or } \min B_1 \geq \hat{n}_1 \text{ or } \Delta R_1 = 0; \\ D^{0,1}_i & \text{otherwise,} \end{cases}$$

where

$$D^{0,1}_i := \frac{D^0(e_i)}{\Delta v^i_{B_1}} - \frac{D^1(\Delta^*_1)}{\Delta R_1}.$$

$D^*_2(e_i)$ preserves B_1 by Definition 6. Moreover, since $D^{0,1}_i$ is a linear combination of $D^0(e_i)$ and $D^1(\Delta^*_1)$, Lemma 8 and the first part of Lemma 9 apply to $D^*_2(e_i)$.

Once good i corresponding to $D^*_2(e_i)$ is selected so that $q^i_{n_0} \in (0, 1)$, by the same reasoning as in Lemma 9, the perturbed schedule $\{\tilde{q}_n\} = \{q_n\} \pm \epsilon D^*_2(e_i)$ satisfies the feasibility conditions above n_0 for sufficiently small $\epsilon > 0$. Below n_0, since $q_n^i = 0$ for all $n < n_0$ by (11) and (12), the stochastic component $q_{n_0}^i \in (0,1)$ can be slightly adjusted while preserving type-increasing quantities and price. □

We then invoke the preceding arguments to obtain a contradiction against MBM.

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