Optimal Auction Mechanisms in the Presence of Regret

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Abstract

Bidders often regret the outcome of an auction. The winner might regret leaving money on the table, and the losers might regret not bidding more to win at a favorable price. Most theoretical literature on auctions assumes that a bidder's preferences depend only on her expected profit. In contrast, I incorporate a penalty for regret into bidder utility functions and study the effects on equilibrium bidding and seller revenue. I prove that second price auctions are always optimal, and I classify the full family of optimal mechanisms. I then characterize equilibrium bidding formulae for first price and soft floor auctions, and provide sufficient conditions for soft floor auctions to revenue-dominate first price and second price auctions. In addition, I determine the optimal reserve prices for first price and soft floor auctions in the case of two bidders with uniform distributions.

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1. Introduction

Bidders often regret the outcome of an auction, either because they failed to win or because they overpaid. In hindsight, a winning bidder wishes she could change her bid to a hair above the next highest bid to leave essentially no money on the table. Similarly, a bidder regrets losing if she could have bid higher and won at a price beneath her value for the object. Traditional analysis of auctions assumes that each bidder's utility depends only on her profit, but expected profit alone fails to predict bidding behavior in many auctions. Engelbrecht-Wiggans (1989) introduces winner's and loser's regret into bidder utility functions and analyzes their effects on equilibrium bidding strategies in first price auctions. In contrast, I consider a broader notion of regret and study its effect on general auction mechanisms in which the highest bidder wins. I then characterize the manner in which bidders change their bidding behavior, analyze the resulting effects on seller revenue, and identify the class of revenue-maximizing mechanisms.

When bidders anticipate experiencing regret upon seeing the results of the auction, they may change their bidding strategy. This affects the seller's expected revenue and incentivizes her to choose rules for the auction that exploit the change in bidding. A bidder who weighs regret heavily may opt not to participate in the auction. To encourage bidder participation, the seller wants to choose a mechanism that increases a bidder's expected utility as her value for the object increases. Theorem 2 proves that among such mechanisms, the seller's expected revenue depends only on how heavily bidders weigh regret, the lowest type v^* for which bidders choose to participate, and the expected maximal profit of a bidder of type v^*. Theorem 2 thus allows me to classify the full range of optimal mechanisms. Proposition 3 proves that a second price auction with a well-chosen reserve price is always optimal. I then partition regret into winner's and loser's regret and determine their effects on the equilibrium bidding strategies of first price, second price, and soft floor auctions. Proposition 9 proves that when bidders only weigh loser's regret, a soft floor auction outperforms both first and second price auctions.

To isolate the effects of regret on equilibrium bidding strategies, I choose a setting similar to that of the Revenue Equivalence Theorem. The Revenue Equivalence Theorem states that for independent private value auctions, Dutch, English, first price, and second price auctions produce the same expected revenue facing risk-neutral and profit-maximizing bidders. In a symmetric Nash Equilibrium, revenue equivalence is guaranteed if (i) the highest bidder wins and (ii) a bidder who minimally values the item has 0 expected profit. I maintain these assumptions with two modifications. First, I assume that bidders maximize expected utility, where utility is profit less regret. Second, I require a sealed-bid auction that publishes all bids after the fact. A sealed-bid auction ensures that the seller knows all bids after the auction and therefore that regret is well-defined; publishing bids enables bidders to anticipate regret. This simple framework makes it easy to compare the revenue generated by different auction mechanisms.

Twenty years after Vickrey (1961) proves the Revenue Equivalence Theorem, Myerson (1981) generalizes the proof to fully classify optimal mechanisms for the seller, resolving a significant question in mechanism design. Empirical work, however, shows that in first price auctions, bidders tend to overbid compared to rational expectations (see e.g. Cox, Roberson, et al. 1982 or the overview in Kagel and Roth 1995). Possible explanations include risk aversion (Cox, Smith, et al. 1992), spite bidding (Morgan et al. 2003), and joy of winning (Goeree et al. 2002). The most common explanation is risk aversion, but examination of third-price auctions in Kagel and Levin (1993) contradicts the expected findings if risk aversion were the primary culprit. Filiz-Ozbay and Ozbay (2007) instead proposes anticipated regret. If people aim not only to gain profit, but also to avoid regret, their utility functions change, perturbing the Revenue Equivalence Theorem.

Regret has a rich history in auction literature. Savage (1951) pioneers the mini-max regret approach in the context of statistical decision theory. Loomes and Sugden (1982) and Bell (1982) generalize Savage's ideas to form modern regret theory, defining regret as the difference between what is and what could have been. This stems from the observation that in games with incomplete information, the optimal strategy ex-ante may not be the best strategy ex-post (see Harsanyi 1967). Engelbrecht-Wiggans (1989) introduces regret in the auction setting, studying a sealed-bid first price auction. Following his example, I assume that regret is linear and additively separable.[1] Regret in auction literature has appeared only sparingly beyond first price auctions, perhaps because it is not well-defined for many common mechanisms (such as Dutch auctions), and there is no regret in the truth-telling Nash Equilibrium of second price auctions and English auctions (see Filiz-Ozbay and Ozbay 2007).[2] Since the truth-telling Nash Equilibrium of a second price auction has no regret, if the seller implements a second price auction, she can do no worse than if bidders only maximize profit. Using the conceptual framework of Riley and Samuelson (1981), I prove that even if the seller knows the utility form of bidders, she cannot earn more revenue; a second price auction is therefore optimal. I then classify the family of optimal auctions.

Following the work of Engelbrecht-Wiggans (1989) and later Filiz-Ozbay and Ozbay (2007), I then partition regret into two categories—winner's regret and loser's regret. They have the

[1]A plausible belief given the additive utility in Bell (1982).

[2]Winner's regret is not well-defined in Dutch auctions because the winner never learns whether she would have won had she waited longer. Nonetheless, in expectation, Dutch auctions maintain strategic equivalence to first price auctions (Filiz-Ozbay and Ozbay 2007).

same definition as classical regret conditional on winning or losing the auction, respectively. Engelbrecht-Wiggans (1989) shows that in a first price auction, loser's regret increases bids, whereas winner's regret shaves bids. This is intuitive: if a bidder fears losing, she will bid more aggressively at the cost of some potential profit, whereas if she worries about overpaying, she will bid more conservatively and win less often. Engelbrecht-Wiggans (1989) demonstrates that if risk-neutral bidders weigh both types of regret equally, and if the seller announces the winning bid, then the equilibrium bidding strategy doesn't depend on how heavily bidders weigh regret. However, the theory breaks down in empirical experiments (see e.g. Katuščák et al. 2015). Filiz-Ozbay and Ozbay (2007) explains the phenomenon of overbidding in a first price auction by arguing that in a one-shot auction, bidders anticipate loser's regret but do not anticipate winner's regret, and thus tend to overbid. The empirical experiments therefore suggest that the theory overstates the emphasis on winner's regret.

My paper builds upon the findings of Engelbrecht-Wiggans (1989) by explicitly calculating the optimal bidding behaviour in first price auctions. Let α be the weight on winner's regret and β be the weight on loser's regret. Proposition 5 proves that (i) if $\alpha > \beta$, bidders shave bids and a second price auction outperforms a first price auction; (ii) if $\beta > \alpha$, bidders increase bids and a first price auction outperforms a second price auction; and (iii) if $\alpha = \beta$, the optimal mechanism produces the same revenue as if bidders did not consider regret. This confirms the comparative statics result in Engelbrecht-Wiggans (1989) and allows for the direct computation of seller revenue for given distributions.

I then analyze the effects of regret on a soft floor auction and characterize the equilibrium bidding strategy. Zeithammer (2019) proves that a soft floor auction with symmetric bidders preserves revenue equivalence. The seller announces a standard hard floor r and soft floor s; if the largest bid exceeds s, the winner pays the second highest bid as in a second price auction (she pays s if the second highest bid is beneath s). If the highest bid is beneath s, the winner pays her bid as in a first price auction. While bidders react to the implementation of a soft floor, monotonic symmetric bidding strategies still characterize the resulting equilibrium. Thus soft floor auctions do not change a bidder's chance of winning relative to the second price auction with the same hard floor. Because the highest bidder still wins and because bidders who minimally value the item still have zero expected utility, the Revenue Equivalence Theorem applies.[3] Zeithammer aims to extract bidder surplus, so to perturb revenue equivalence, he introduces high-value bidders that occasionally join the auction. In contrast, when I implement Zeithammer's soft floor, I maintain symmetric bidders and perturb revenue equivalence by

[3]He also shows that even an unannounced soft floor level maintains the Revenue Equivalence Theorem.

introducing winner's and loser's regret. By taking $\beta > 0$ and $\alpha = 0$, thereby eliminating winner's regret, I prove that the soft floor auction generates strictly more revenue than a first price auction which generates strictly more revenue than a second price auction.

2. Auctions with Regret

In this section I describe the model and I define classical regret.

2.1. **Model.** Consider an independent private value setting with n risk-neutral bidders indexed $i \in \{1, 2, \cdots, n\}$. The seller has one unit of an indivisible good she values at v_0, normalized to $v_0 = 0$. Each bidder i has type v_i pulled from distribution $F(\cdot)$ with full support on the interval $[\underline{v}, \bar{v}]$. I assume $F(\cdot)$ is non-decreasing and differentiable on its support and has density function $f(\cdot)$. Bidders bid above the reserve price announced by the seller, the highest bid wins, and ties are broken by randomly assigning the object to one of the highest bidders. At the end of the auction, all bids are disclosed. The utility of a bidder is the weighted average of profit less regret.

I assume the existence of a common equilibrium bidding strategy $b(\cdot)$ with bid $b_i = b(v_i)$ strictly increasing in v_i. Given that all others bid in the range of $b(\cdot)$, bidder i optimally bids in that range as well. Thus $b_i = b(x)$, where bidder i chooses x, and if this is a Nash Equilibrium, $b(x) = b(v_i)$. When all others bid truthfully, bidder i has transfer

$$\bar{t}(b(x), b(v_{-i})),$$

where $b(v_{-i})$ denotes the vector of the $n-1$ bids $b(v_j)$ for $j \neq i$. To extrapolate from bids, define

$$t(x, v_{-i}) = \bar{t}(b(x), b(v_{-i})).$$

A negative transfer is a payment and a positive transfer is a subsidy. Denote the expected transfer $T(x)$ by

$$T(x) = \mathop{\mathbb{E}}_{v_{-i}}[t(x, v_{-i})].$$

Throughout this paper, when a lower case letter represents a function, its uppercase counterpart represents its expected value (over a specified set of variables). Now, the profit of bidder i is

$$\pi(v_i, x, v_{-i}) = \begin{cases} v_i + t(x, v_{-i}) & \text{if i wins} \\ t(x, v_{-i}) & \text{if i loses.} \end{cases}$$

For $j \neq i$, the probability $v_j < x$ is $F(x)$. Since bidder values are independent, bidder i wins with probability $Q(x) = F(x)^{n-1}$. She has expected profit

$$\Pi(v_i, x) = \mathop{\mathbb{E}}_{v_{-i}}[\pi(v_i, x, v_{-i})] = v_i Q(x) + T(x).$$

The maximum profit of a bidder is the profit they could have made choosing x optimally under complete information. Denote the expected value of the maximal profit by[1]

$$\Pi^*(v_i) = \mathop{\mathbb{E}}_{v_{-i}}[\sup_x \pi(v_i, x, v_{-i})].$$

I use the supremum because $\pi(\cdot)$ is often non-continuous and need not be monotone, and there's no guarantee that the maximum value is obtained. Since this is a point-wise maximization, $\Pi^*(\cdot)$ cannot generally be expressed in terms of $Q(\cdot)$ and $T(\cdot)$.

2.2. **Classical Regret.** I follow the classical definition of regret laid out in Bell (1982) and Loomes and Sugden (1982). In the context of auctions, regret is the difference between profit under complete information and incomplete information. Thus expected regret is

$$R(v_i, x) = \Pi^*(v_i) - \Pi(v_i, x).$$

Let $\mu \in [0, 1]$ be the weight bidders place on regret, so $(1 - \mu)$ is the weight bidders place on profit. Then bidder i has expected utility

$$\begin{aligned} U(v_i, x) &= (1-\mu)\Pi(v_i, x) - \mu R(v_i, x) \\ U(v_i, x) &= \Pi(v_i, x) - \mu \Pi^*(v_i). \end{aligned} \tag{1}$$

3. Optimal Mechanisms with Regret

This section derives the expected revenue of a mechanism in general terms for auctions with regret. I then classify the family of optimal mechanisms. Finally, I partition regret into winner's and loser's regret and derive a bidder's expected transfer.

[1] I skip writing $\pi^*(v_i, v_{-i})$ explicitly because if there's a tie, optimal profit can only be described in expectation. However, a basic continuity argument proves it's sufficient to write

$$\pi^*(v_i, v_{-i}) = \sup_x \pi(v_1, x, v_{-i}) = \sup_x \begin{cases} v_i + t(x, v_{-i}) & \text{if } x > v_j \text{ for all } j \neq i \\ t(x, v_{-i}) & \text{if } x < v_j \text{ for some } j \neq i. \end{cases}$$

Let y be the maximal of the other $n-1$ bids. Unless there is a discontinuity at $x = y$ and the mechanism rewards bidders for tying, choosing $x = y$ is not optimal because either winning or losing with bid $y + \epsilon$ or $y - \epsilon$ is weakly better. Such a mechanism would be extremely contrived, and since the set of ties has measure 0, this detail is unimportant in expectation.

3.1. **Optimal Mechanisms with Classical Regret.** Individual rationality requires that a type v bidder participates in the auction if and only if she has non-negative expected utility. Depending on the auction mechanism and the value of μ, equation (1) will be positive for only some $v \in [\underline{v}, \bar{v}]$, and possibly for no v at all. Consider the case $\mu = 1$, so that bidders only minimize regret. Then

$$U(v_i, x) = -R(v_i, x) = \Pi(v_i, x) - \Pi^*(v_i) \leq 0.$$

The inequality is strict except in the special case in which bidders never experience regret. Since regret is the difference between expected maximal profit with complete and incomplete information, bidders never experience regret if and only if the Nash Equilibrium of the mechanism is ex-post incentive compatible. Second price auctions are famously ex-post incentive compatible, and the second price auction with Myerson reserve is the unique revenue-maximizing, ex-post incentive compatible mechanism (Myerson 1981). Since $v_0 = 0$, the optimal reserve is $r = 0$. In less extreme cases for which $\mu \in [0, 1)$, a type v bidder participates if and only if

$$U(v_i, x) = \Pi(v_i, x) - \mu\Pi^*(v_i) \geq 0.$$

To derive the seller's expected revenue, I need some $v^* \in [\underline{v}, \bar{v}]$ such that $U(v^*, v^*) = 0$ and $\sup_x U(v_i, x)$ is increasing in v_i. That guarantees that all bidders of type $v \in [v^*, \bar{v}]$ participate in the auction, and will eventually allow us to calculate expected revenue. Most common mechanisms satisfy this constraint, such as first and second price auctions. One sufficient condition is to assume that losing bidders make no profit and the transfer function is continuous conditional on winning the auction. These constraints greatly reduce the scope of possible mechanisms and are most likely much stronger than what's necessary. Future work can relax these conditions; I conjecture that the mechanism can be generalized to allow for small positive transfers to losing bidders so long as the winning bidder makes strictly greater profit.

Proposition 1 (Utility Regularity)**.**
Assume that (i) losing bidders make no profit and (ii) the transfer function is continuous conditional on winning the auction. Then there exists $\gamma \in (0, 1]$ such that for all $\mu \in [0, \gamma]$, the supremum of the utility function $\sup_x U(v_i, x)$ is increasing in v_i.

Proof. Let $v^m = \max_{j \neq i} v_j$ and let $\mathbb{1}_{x > v^m}$ be the indicator function conditional on x being the largest reported type, so that if bidder i reports type x, she wins the auction. By equation (1),

$$\begin{aligned}\sup_x \Big[U(v_i, x)\Big] &= \sup_x \Big[\Pi(v_i, x) - \mu \Pi^*(v_i)\Big] \\ &= \sup_x \Big[\mathop{\mathbb{E}}_{v_{-i}}[\pi(v_i, x, v_{-i})]\Big] - \mu \mathop{\mathbb{E}}_{v_{-i}}\Big[\sup_x [\pi(v_i, x, v_{-i})]\Big] \\ &= \sup_x \Big[\mathop{\mathbb{E}}_{v_{-i}}[v_i \mathbb{1}_{x > v^m} + t(x, v_{-i})]\Big] - \mu \mathop{\mathbb{E}}_{v_{-i}}\Big[\sup_x [v_i \mathbb{1}_{x > v^m} + t(x, v_{-i})]\Big]\end{aligned}$$

I fix x in an ϵ-neighborhood of v_i and look to apply the Envelope Theorem. I aim to prove that locally, the following expression increases in v_i.

$$\mathop{\mathbb{E}}_{v_{-i}}[v_i \mathbb{1}_{x > v^m} + t(x, v_{-i})] - \mu \mathop{\mathbb{E}}_{v_{-i}}\Big[\sup_x [v_i \mathbb{1}_{x > v^m} + t(x, v_{-i})]\Big]. \tag{2}$$

Taking the derivative of expression (2) with respect to v_i gives

$$\begin{aligned}&\mathop{\mathbb{E}}_{v_{-i}}[\mathbb{1}_{x > v^m}] - \mu \frac{\partial}{\partial v_i} \mathop{\mathbb{E}}_{v_{-i}}\Big[\sup_x [v_i \mathbb{1}_{x > v^m} + t(x, v_{-i})]\Big] = \\ &\mathop{\mathbb{E}}_{v_{-i}}[\mathbb{1}_{x > v^m}] - \mu \mathop{\mathbb{E}}_{v_{-i}}\Big[\frac{\partial}{\partial v_i} \sup_x [v_i \mathbb{1}_{x > v^m} + t(x, v_{-i})]\Big]\end{aligned} \tag{3}$$

Taking the expected value is equivalent to integrating over the relevant bounds. The Leibniz Integration Rule states that to switch the order of derivation and integration, I need regularity conditions on $\sup_x [v_i \mathbb{1}_{x > v^m} + t(x, v_{-i})]$. By assumption (i), if

$$\sup_x [v_i \mathbb{1}_{x > v^m} + t(x, v_{-i})] > 0,$$

then $x > v^m$. Further, conditioning bidder i winning the auction, $v_i \mathbb{1}_{x > v^m} = v_i$ is constant and by assumption (ii), $t(x, v_{-i})$ is continuous. It follows that $\sup_x [v_i \mathbb{1}_{x > v^m} > t(x, v_{-i})]$ is zero when bidder i loses and continuous when bidder i wins. I may therefore switch the order of derivation and integration. Next I apply the Envelope Theorem to hold x fixed. Define x^* such that

$$x^*(v_{-i}) = \arg\max_x (v_i \mathbb{1}_{x > v^m} + t(x, v_{-i})).$$

Then expression (3) gives

$$\begin{aligned}&\mathop{\mathbb{E}}_{v_{-i}}[\mathbb{1}_{x > v^m}] - \mu \mathop{\mathbb{E}}_{v_{-i}}\Big[\frac{\partial}{\partial v_i} [v_i \mathbb{1}_{x^* > v^m} + t(x, v_{-i})]\Big] = \\ &\mathop{\mathbb{E}}_{v_{-i}}[\mathbb{1}_{x > v^m}] - \mu \mathop{\mathbb{E}}_{v_{-i}}[\mathbb{1}_{x^* > v^m}].\end{aligned} \tag{4}$$

The Envelope Theorem applies because conditions (i) and (ii) provide continuity conditional on winning the auction. The first term of expression (4) is the probability that bidder i gets the good with incomplete information, whereas the second term is the probability i chooses to get the good with complete information. Finally, we need expression (4) to be non-negative,

$$\mathop{\mathbb{E}}_{v_{-i}}[\mathbb{1}_{x>v^m}] \geq \mu \mathop{\mathbb{E}}_{v_{-i}}[\mathbb{1}_{x^*>v^m}]. \tag{5}$$

The mechanism the auctioneer employs determines the sign of equation (5). If $\mu = 0$, equation (5) holds trivially and if $\mu = 1$, it holds only in the case of a second price auction. It follows that there exists some $\gamma \in [0, 1]$ such that for all $\mu \in [0, \gamma]$, equation (5) is non-negative. ■

For the rest of the paper, assume (i) that the mechanism is sufficiently well behaved and (ii) that μ is sufficiently small to satisfy Proposition 1. This leads to Theorem 2.

Theorem 2 (Revenue Equivalence with Regret)**.**
Consider a sealed-bid auction in which the highest bidder wins and the mechanism satisfies Proposition 1. Say n risk-neutral bidders maximize a convex combination of expected profit minus expected regret. Each bidder i has type v_i drawn independently from $F(\cdot)$ with full support $[\underline{v}, \bar{v}]$, $\underline{v} \neq \bar{v}$. Assume $F(\cdot)$ is non-decreasing and differentiable on its support with strictly positive density. Let v^* be the lowest value for which a bidder gets 0 expected utility participating in the auction. Then the seller's revenue in a symmetric Nash Equilibrium is determined only by the weight on regret μ, v^*, and the expected maximal profit of a bidder of type v^*.

Proof. In a Nash Equilibrium, bidder i must optimally select $x = v_i$. To satisfy first order conditions,

$$\frac{\partial U(v_i, x)}{\partial x} = v_i Q'(x) + T'(x) = 0 \text{ when } x = v_i.$$

This means

$$-T'(v_i) = v_i Q'(v_i). \tag{6}$$

By definition, a type v^* bidder has zero expected utility.

$$\begin{aligned} U(v^*, v^*) = 0 &= v^* Q(v^*) + T(v^*) - \mu \Pi^*(v^*) \\ -T(v^*) &= v^* Q(v^*) - \mu \Pi^*(v^*) \qquad (7) \end{aligned}$$

Integrate equation (6) from v^* to the valuation of bidder i and integrate by parts.

$$\begin{aligned} \int_{v^*}^{v_i} -T'(v)dv &= \int_{v^*}^{v_i} vQ'(v)dv \\ -T(v_i) + T(v^*) &= v_i Q(v_i) - v^* Q(v^*) - \int_{v^*}^{v_i} Q(v)dv. \end{aligned}$$

Plugging in equation (7), the expected transfer simplifies to

$$-T(v_i) = v_i Q(v_i) - \int_{v^*}^{v_i} Q(v)dv - \mu\Pi^*(v^*). \tag{8}$$

Equation (8) holds for any $v_i \geq v^*$. To find the expected revenue earned from bidder i, integrate equation (8) from v^* to $\bar{v}$.

$$\begin{aligned}\mathbb{E}[Rev_i] &= \int_{v^*}^{\bar{v}} -T(v_i)dF(v_i)\\ &= \int_{v^*}^{\bar{v}} \left(v_i Q(v_i) - \int_{v^*}^{v_i} Q(v)dv\right) dF(v_i) - \int_{v^*}^{\bar{v}} \mu\Pi^*(v^*)dF(v_i).\end{aligned}$$

Integrating by parts and simplifying,

$$\mathbb{E}[Rev_i] = \int_{v^*}^{\bar{v}} Q(v)\left(v - \frac{1-F(v)}{f(v)}\right) dF(v) - \mu(1-F(v^*))\Pi^*(v^*). \tag{9}$$

Since all bidders are symmetric, the seller has total expected revenue

$$\mathbb{E}[Rev] = n\int_{v^*}^{\bar{v}} Q(v)\left(v - \frac{1-F(v)}{f(v)}\right) f(v)dv - n\mu(1-F(v^*))\Pi^*(v^*). \tag{10}$$

The second term is Equation (10) depends only on μ, v^*, and $\Pi^*(v^*)$, as claimed. ■

As discussed, there is no regret in a second price auction. Further, when bidders maximize profit, a second price auction is optimal for the seller. Thus in this model, the seller can guarantee herself at least as much revenue as in the standard, profit-maximizing model. Applying Theorem 2 proves she cannot do better. To see this, first define the virtual valuation function

$$J(v) = v - \frac{1-F(v)}{f(v)}. \tag{11}$$

Assume $J(v)$ is non-decreasing in v.

Proposition 3 (Second Price Auction is Optimal)**.**
Let $z \in [\underline{v}, \bar{v}]$ be the smallest value such that $J(z) \geq 0$. Then a second price auction with reserve price z is an optimal mechanism.

Proof. By Theorem 2, the seller receives expected revenue

$$\mathbb{E}[Rev] = n\int_{v^*}^{\bar{v}} Q(v)\left(v - \frac{1-F(v)}{f(v)}\right) f(v)dv - n\mu(1-F(v^*))\Pi^*(v^*).$$

The first term is the expected revenue from a symmetric auction in which bidders only maximize profit. The second term is

$$-n\mu(1-F(v^*))\Pi^*(v^*) \leq 0. \tag{12}$$

The seller does best if equation (12) is zero *and* the integral is maximized. In a second price auction there is no regret, $v^* = \underline{v}$, and $\Pi^*(\underline{v}) = 0$. The seller therefore maximizes

$$n \int_{v^*}^{\bar{v}} Q(v) J(v) f(v) dv.$$

The integrand is positive for $v \geq z$, so choosing a reserve price of z maximizes revenue. ■

A second price auction is not necessarily the unique optimal mechanism. An optimal mechanism must be revenue-maximizing in the standard model and must satisfy $\Pi(v^*) = 0$. By equation (7), this requires

$$-T(v^*) = v^* Q(v^*).$$

One sufficient criterion is

$$0 = -T(v^*) = Q(v^*),$$

meaning that if bidder i cannot win, she has an expected payment of zero. In a standard first price auction with no entry fee, $v^* = \underline{v}$ and bidder i wins with probability zero. Hence $\Pi^*(v^*) = \Pi(\underline{v}) = 0$. So long as $J(\underline{v}) \geq 0$, this auction is optimal as well. If $z > \underline{v}$, then the seller must impose an appropriately chosen reserve price. In a perhaps unsurprising result, when bidders experience regret, first price auctions with no entry fee outperform those with an entry fee.

3.2. **Winner's and Loser's Regret.** I partition regret into winner's regret and loser's regret. The definition of each aligns with the classical definition of regret conditional on winning and losing the auction, respectively. As in Proposition 1, let v^m denote the maximal element of v_{-i},

$$v^m = \max_{j \neq i} v_j.$$

Since the highest bidder always win, bidder i has expected winner's regret

$$\begin{aligned} R^W(v_i, x) &= \mathop{\mathbb{E}}_{v_{-i}} [\Pi^*(v_i | x > v^m) - \Pi(v_i, x | v_i > v^m)] \\ &= \Pi^*(v_i | x > v^m) - T(x | x > v^m) - v_i Q(x). \end{aligned}$$

She has expected loser's regret

$$\begin{aligned} R^L(v_i, x) &= \mathop{\mathbb{E}}_{v_{-i}} [\Pi^*(v_i | x < v^m) - \Pi(v_i, x | v_i < v^m)] \\ &= \Pi^*(v_i | x < v^m) - T(x | x < v^m). \end{aligned}$$

For $\alpha, \beta \in [0,1)$ such that $(1-\alpha-\beta) \in [0,1]$, define the expected utility of bidder i as

$$\begin{aligned} U(v_i, x) &= (1-\alpha-\beta)\Pi(v_i, x) - \alpha R^W(v_i, x) - \beta R^L(v_i, x) \\ &= (1-\alpha-\beta)(v_i Q(x) + T(x)) - \alpha \Pi^*(v_i | x > v^m) + \alpha T(x | x > v^m) + \alpha v_i Q(x) \\ &\qquad - \beta \Pi^*(v_i | x < v^m) + \beta T(x | x < v^m). \end{aligned}$$

If $\alpha = 1$, bidders only minimize winner's regret and thus select $x = \underline{v}$ so that they never overpay, but also never win. If $\beta = 1$, bidders only minimize loser's regret and choose $x = \bar{v}$ so that they never lose. Using the fact $T(x) = T(x|x < v^m) + T(x|x > v^m)$ and $\Pi^*(v_i) = \Pi^*(v_i|x < v^m) + \Pi^*(v_i|x > v^m)$, I rewrite all relevant terms conditional on losing the auction.

$$\begin{aligned} U(v_i, x) = (1-\beta)v_i Q(x) + (1-\beta)T(x) + (\beta-\alpha)T(x|x < v^m) \\ - \beta \Pi^*(v_i) + (\beta - \alpha)\Pi^*(v_i | x < v^m). \end{aligned}$$

In a Nash Equilibrium, bidder i optimally selects $x = v_i$, and the following first order condition holds.

$$\begin{aligned} \frac{\partial U(v_i, x)}{\partial x} &= 0 = (1-\beta)v_i Q'(x) + (1-\beta)T'(x) \\ &+ (\beta-\alpha)\frac{\partial T(x|x < v^m)}{\partial x} + (\beta-\alpha)\frac{\partial \Pi^*(v_i|x < v^m)}{\partial x} \text{ when } x = v_i. \end{aligned}$$

Although $\Pi^*(v_i)$ is a function of v_i, conditioning on $x < v^m$ changes the bounds of integration of the expected value since $v_j < x$ for all $j \neq i$. That means $\Pi^*(v_i|x < v^m)$ is also a function of x. Thus

$$-T'(v_i) = v_i Q'(v_i) + \frac{\beta-\alpha}{1-\beta}\left(\frac{\partial T(x|x < v^m)}{\partial x}\right)\bigg|_{x=v_i} + \frac{\beta-\alpha}{1-\beta}\left(\frac{\partial \Pi^*(v_i|x < v^m)}{\partial x}\right)\bigg|_{x=v_i}. \tag{13}$$

Assume that the mechanism and levels of regret α, β are such that lowest biddable type $v^* \in [\underline{v}, \bar{v}]$, as in Proposition 1, and that all types $v \geq v^*$ have non-negative expected utility when bidders bid optimally. Then equation (13) holds true for $v_i \geq v^*$. Further,

$$\begin{aligned} 0 = U(v^*, v^*) = (1-\beta)v^* Q(v^*) + (1-\beta)T(v^*) + (\beta-\alpha)T(v^*|v^* < v^m) - \beta\Pi^*(v^*) \\ + (\beta-\alpha)\Pi^*(v^*|v^* < v^m). \end{aligned}$$

Rearranging terms and dividing by $1-\beta$,

$$-T(v^*) = v^* Q(v^*) + \frac{\beta-\alpha}{1-\beta}T(v^*|v^* < v^m) + \frac{\beta-\alpha}{1-\beta}\Pi^*(v^*|v^* < v^m) - \frac{\beta}{1-\beta}\Pi^*(v^*). \tag{14}$$

Integrate (13) from v^* to v_i (replacing the dummy variable with v) to obtain

$$\int_{v^*}^{v_i} -T'(v)dv = \int_{v^*}^{v_i} \left[vQ'(v) + \frac{\beta-\alpha}{1-\beta}\left(\frac{\partial T(x|x<v^m)}{\partial x}\right)\bigg|_{x=v} + \frac{\beta-\alpha}{1-\beta}\left(\frac{\partial \Pi^*(v|x<v^m)}{\partial x}\right)\bigg|_{x=v}\right] dv$$

$$-T(v_i) = -T(v^*) + v_iQ(v_i) - v^*Q(v^*) - \int_{v^*}^{v_i} Q(v)dv + \frac{\beta-\alpha}{1-\beta}\int_{v^*}^{v_i}\left(\frac{\partial T(x|x<v^m)}{\partial x}\right)\bigg|_{x=v} dv + \frac{\beta-\alpha}{1-\beta}\int_{v^*}^{v_i}\left(\frac{\partial \Pi^*(v|x<v^m)}{\partial x}\right)\bigg|_{x=v} dv.$$

Plugging in (14) and simplifying yields

(15)

$$-T(v_i) = v_iQ(v_i) - \int_{v^*}^{v_i} Q(v)dv + \frac{\beta-\alpha}{1-\beta}T(v^*|v^*<v^m) + \frac{\beta-\alpha}{1-\beta}\Pi^*(v^*|v^*<v^m) - \frac{\beta}{1-\beta}\Pi^*(v^*) + \frac{\beta-\alpha}{1-\beta}\int_{v^*}^{v_i}\left(\frac{\partial T(x|x<v^m)}{\partial x}\right)\bigg|_{x=v} dv + \frac{\beta-\alpha}{1-\beta}\int_{v^*}^{v_i}\left(\frac{\partial \Pi^*(v|x<v^m)}{\partial x}\right)\bigg|_{x=v} dv$$

The second line of equation (15) is complicated. The bounds of integration for the expectation of the transfer conditional on losing also involve x. The same is true for the expectation of the maximal profit conditional on losing. It vanishes when $\alpha = \beta$, which is why it does not appear in the classical formulation of regret. Evaluating the derivative and then integrating the result yields a large and difficult to interpret expression. I begin by rewriting

$$\frac{\beta-\alpha}{1-\beta}\int_{v^*}^{v_i}\left(\frac{\partial T(x|x<v^m)}{\partial x}\right)\bigg|_{x=v} dv. \tag{16}$$

For ease of notation and without loss of generality, assume $v_i = v_1$. First evaluate the integrand by dividing $T(x|x<v^m)$ into $n-1$ (symmetric) pieces, one each for $v^m = v_2, \cdots, v_n$.

$$\begin{aligned} T(x|x<v^m) &= \int_x^{\bar{v}}\left[\int_{\underline{v}}^{v_2}\dots\int_{\underline{v}}^{v_2} t(x,v_{-1})dF(v_n)\cdots dF(v_3)\right]dF(v_2) \\ &+ \int_x^{\bar{v}}\left[\int_{\underline{v}}^{v_3}\dots\int_{\underline{v}}^{v_3} t(x,v_{-1})dF(v_n)\cdots dF(v_2)\right]dF(v_3) \\ &\vdots \\ &+ \int_x^{\bar{v}}\left[\int_{\underline{v}}^{v_n}\dots\int_{\underline{v}}^{v_n} t(x,v_{-1})dF(v_{n-1})\cdots dF(v_2)\right]dF(v_n) \\ &= (n-1)\int_x^{\bar{v}}\left[\int_{\underline{v}}^{v_2}\dots\int_{\underline{v}}^{v_2} t(x,v_{-1})dF(v_n)\cdots dF(v_3)\right]dF(v_2). \end{aligned}$$

Thus

$$\frac{\partial T(x|x<v^m)}{\partial x} = (n-1)\int_x^{\bar{v}}\left[\int_{\underline{v}}^{v_2}\cdots\int_{\underline{v}}^{v_2}\frac{\partial t(x,v_{-1})}{\partial x}dF(v_n)\cdots dF(v_3)\right]dF(v_2)$$
$$- f(x)\int_{\underline{v}}^{x}\cdots\int_{\underline{v}}^{x} t(x,x,v_{-1,-2})dF(v_n)\cdots dF(v_3).$$

Therefore

$$\left(\frac{\partial T(x|x<v^m)}{\partial x}\right)\bigg|_{x=v_1} = (n-1)\int_{v_1}^{\bar{v}}\left[\int_{\underline{v}}^{v_2}\cdots\int_{\underline{v}}^{v_2}\frac{\partial t(v_1,v_{-1})}{\partial v_1}dF(v_n)\cdots dF(v_3)\right]dF(v_2)$$
$$-(n-1)f(v_1)\int_{\underline{v}}^{v_1}\cdots\int_{\underline{v}}^{v_1} t(v_1,v_1,v_{-1,-2})dF(v_n)\cdots dF(v_3).$$

I use $v_{-1,-2}$ to denote the vector of values excluding both the first and second values. Replace v_1 with the dummy variable v and integrate from v^* to v_1.

$$\int_{v^*}^{v_1}\left(\frac{\partial T(x|x<v^m)}{\partial x}\right)\bigg|_{x=v}dv = (n-1)\int_{v^*}^{v_1}\int_{v}^{\bar{v}}\left[\int_{\underline{v}}^{v_2}\cdots\int_{\underline{v}}^{v_2}\frac{\partial t(v,v_{-1})}{\partial v}dF(v_n)\cdots dF(v_3)\right]dF(v_2)dv$$
$$-(n-1)\int_{v^*}^{v_1}\int_{\underline{v}}^{v}\cdots\int_{\underline{v}}^{v} t(v,v,v_{-1,-2})dF(v_n)\cdots dF(v_3)dF(v).$$

Although complicated, this form looks promising. The first term has an integration by v on the outside and a function with a derivative with respect to v on the inside. However, changing the bounds of integration breaks the integral into 2^{n-2} terms (with some symmetry). Creative techniques may resolve this problem, but one potential way forward is to write

$$\begin{aligned}T(x|x<v^m) &= T(x) - T(x|x>v^m)\\ &= T(x) - \int_{\underline{v}}^{x}\cdots\int_{\underline{v}}^{x} t(x,v_{-i})dF(v_n)\cdots dF(v_2),\end{aligned}$$

taking the derivative with respect to x gives.

$$\frac{\partial T(x|x>v^m)}{\partial x} = (n-1)\int_{\underline{v}}^{x}\cdots\int_{\underline{v}}^{x}\left[f(x)t(x,x,v_{-1,-2}) + \int_{\underline{v}}^{x}\frac{\partial t(x,v_{-1})}{\partial x}dF(v_2)\right]dF(v_n)\cdots dF(v_3).$$

However, this expression appears equally difficult to handle and I see no easy way to approach it. Further, a similar analysis on

$$\frac{\beta-\alpha}{1-\beta}\int_{v^*}^{v_i}\left(\frac{\partial \Pi^*(v|x<v^m)}{\partial x}\right)\bigg|_{x=v}dv \tag{17}$$

gives equally complicated results. Although equation (15) is complicated, Section 5 shows that a first price auction gives a manageable expression, whereas a general soft floor auction gives an intractable integral. This suggests that I need strong assumptions about $T(\cdot)$ and $\Pi^*(\cdot)$ to simplify equation (15). If the auction mechanism gives zero expected transfer conditional upon losing the auction, $T(x|x < v^m) = 0$ and expression (16) vanishes. No similar restriction works to eliminate expression (17). In the case $\alpha = \beta$, both expression (16) and expression (17) vanish, and equation (15) simplifies to

$$-T(v_i) = v_i Q(v_i) - \int_{v^*}^{v_i} Q(v)dv - \frac{\beta}{1-\beta}\Pi^*(v^*). \tag{18}$$

This makes sense, since $\alpha = \beta$ corresponds to the case where bidders face general regret. Indeed, equations (18) and (8) are conceptually equivalent; they only differ by the relative weight on $\Pi^*(v^*)$. By initially separating winner's and loser's regret, I place relatively more weight on regret, since profit is also discounted by both α and β. For equations (18) and (8) to be identical, I need

$$\mu = \frac{\beta}{1-\beta}$$
$$\beta = \frac{\mu}{1+\mu}.$$

Since $\mu \in [0,1]$, this requires $\alpha = \beta \in [0, 1/2]$, as expected.

4. Notions of Regret

Regret theory is a popular alternative to the expected utility theory of von Neumann and Morgenstern developed in 1947 (von Neumann and Morgenstern 2007).[4] Regret theory first appears in the economic literature in the simultaneous works of Bell (1982), Loomes and Sugden (1982), and Fishburn (1982), although it appears as early as Savage (1951) in the context of statistical decision making. When agents make decisions based on incomplete information, they base their decision in expectation of what they think will happen. After the fact, when all information is revealed, an agent evaluates the results of her choices not just based on the actualized outcome, but also based on the optimal outcome that could have transpired had she acted differently. The comparison between what happened and what could have happened causes regret and reduces the satisfaction she feels with the results of her decision. Agents might anticipate experiencing regret and choose their subsequent actions to minimize the amount of regret they experience. Psychologists have investigated the role of regret in decision making and have strong empirical evidence that suggests anticipated regret affects how people make strategic choices (see e.g. Connolly and Zeelenberg 2002 or Connolly and Butler 2006).

Regret theory models choice under uncertainty when an agent's utility function depends negatively on the best possible outcome. There are many ways to incorporate regret into decision

[4] I thank Diecidue and Somasundaram (2017) for their thorough overview of the history of regret theory, from which I borrow.

making. Savage (1951) invents the mini-max approach in which agents choose their actions to minimize the maximum amount of regret they could experience. The mini-max approach also appears frequently in the economic literature in a wide range of applications. For example, Linhart and Radner (1989) employs the mini-max approach to provide a range of efficient outcomes in bilateral trade negotiations, Bergemann and Schlag (2008) uses it to derive pricing policies with minimal information about the buyer's distributions, and Bergemann and Schlag (2011) uses it to determine a robust monopoly pricing scheme. In a statistical context, Eldar, Ben-Tal, et al. (2004) and Eldar and Merhav (2004) bound mean-squared error with mini-max regret techniques. The implementation of the mini-max regret function violates the Independence of Irrelevant Alternatives that Arrow proposes in social choice theory. To incorporate regret into decision theory, Milnor (1954) creates a rigorous axiomization of regret, with a more recent adaptation by Stoye (2011).

In classic utility theory, an agent has utility function u and prospective choices a and b with outcomes $\{a_1, \cdots, a_n\}$ and $\{b_1, \cdots, b_n\}$. Then a is preferred to b if and only if

$$a \succeq b \iff \sum_{i=1}^{n} p_i(s_i)(u(a_i) - u(b_i)) \geq 0, \tag{19}$$

where $p_i(s_i)$ is an agent's subjective probability that state s_i occurs. This can be rewritten as

$$a \succeq b \iff \sum_{i=1}^{n} p_i(s_i)u(a_i) \geq \sum_{i=1}^{n} p_i(s_i)u(b_i), \tag{20}$$

which allows for convenient comparison across states. Bell (1982), Loomes and Sugden (1982), and Fishburn (1982) simultaneously generalize modern regret theory as an alternative to expected utility theory. Bell (1982) and Loomes and Sugden (1982) introduce a regret function Q that modifies equation (19) such that a is preferred to b if and only if

$$a \succeq b \iff \sum_{i=1}^{n} p_i(s_i)Q(u(a_i) - u(b_i)) \geq 0. \tag{21}$$

The introduction of Q is particularly restrictive because equation (21) cannot be written in a form analogous to equation (20). Changing Q describes the various possible attitudes toward risk an agent might have, and a linear Q is equivalent to the expected utility theory of von Neumann and Morgenstern. When Q is convex, equation (21) is similar to (but still distinct from) risk aversion, and it accounts for some empirical violations of expected utility theory. Fishburn (1982) takes a different approach of skew-symmetric bi-linear preferences, but his work leads to a similar result. A decade later, Fishburn (1989) and Sugden (1993) propose preference foundations for regret theory that do not separate the utility and regret functions. Although more general, comparing utilities in such a broad setting is often intractable and even basic techniques such as comparative statics can be demanding. As a result, while the formalized axioms of Fishburn (1989) and Sugden (1993) provide a rigorous framework, the theory is restrictive enough to prevent wide-spread implementation. It is often desirable, and even necessary, to make regret additively separable from utility and to have the foundations for continuous regret functions, as in Diecidue and Somasundaram (2017).

Regret aversion appears in many contexts in economics, such as regret-matching in correlated equilibria (Hart and Mas-Colell 2000), investments and currency hedging (Michenaud and Solnik 2008), and most recently, climate change (DeCanio et al. 2022). Whenever agents face a random state of the world, regret can model their behavior under uncertainty. Nekipelov et al. (2015) uses an algorithmic approach to minimizing regret as a way to study the interaction between agents who have not reached a Nash Equilibrium. When bidders employ no-regret learning, advertisers can infer the values of players based on their bids and use those inferences to improve the pricing scheme in sponsored search ad auctions. Auctions generally provide a convenient framework for studying regret because agents make strategic decisions from a controlled set of choices. The random state of the world reduces to the values bidders have for the object, and each agent's choice set reduces to her bidding behavior.

Engelbrecht-Wiggans (1989) first introduces regret in auctions, in which he defines regret as the difference between profit under complete and incomplete information. To maintain simplicity and allow for comparative statics, he assumes that bidders have objective functions that are linear combinations of profit and regret and studies how bidders change their behavior in first price auctions. Further, he divides regret into two categories conditional on whether the bidder wins or loses the auction. The winner of the auction regrets overbidding and therefore leaving money on the table, and a loser regrets not winning so long as the winning bid does not exceed her value for the object. Engelbrecht-Wiggans (1989) assumes that bidders estimate the value of the object based on their private information, but also allows for their estimation to change based on the private information of the other bidders. He shows that, in this broad framework, if bidders weigh both types of regret equally, their equilibrium bidding strategy is independent of regret. He further shows that if winner's regret is weighed more heavily, then the seller earns less expected revenue; conversely, if loser's regret is weighed more heavily, the seller increases expected revenue. These results require that the seller publishes the winning price. Otherwise, the winning price might exceed a loser's value, in which case she would not experience regret, and she thereby anticipates less regret when bidding. Since the seller determines the mechanism of the auction, it is in her favor to announce the winning price.

Filiz-Ozbay and Ozbay (2007) builds on the results of Engelbrecht-Wiggans, maintaining the assumption that regret is additively separable from profit, but differing by assuming regret is not necessarily linear. They reach the same qualitative comparative statics conclusion that winner's regret induces more conservative bidding and loser's regret induces aggressive bidding. They then translate the theory to a laboratory setting in an attempt to explain the empirical phenomenon of overbidding in first price auctions. They use a one-shot auction to avoid the possibility that bidders accumulate regret through repeated experiments that bias the direction of their bids. They conclude that bidders anticipate loser's regret but do not anticipate winner's regret, which causes overbidding. Kagel and Levin (2016) questions the statistical significance of their findings and Neugebauer and Selten (2006) fails to replicate their results. That said, studies from Ockenfels and Selten (2005) and Cason and Friedman (1997) suggest that people regret losing an opportunity to profit more than they regret overpaying, supporting the conclusion in Filiz-Ozbay and Ozbay (2007).

5. Examples

This section provides two examples—the standard first price auction, and the more nuanced soft floor auction. I maintain the assumptions of section 3.2: there are n symmetric risk-neutral bidders in a sealed-bid independent private value auction maximizing profit minus regret. For convenience, I normalize the support of $F(\cdot)$ to the unit interval $[0, 1]$. At the end of the auction, the seller discloses the winning bid and the second highest bid. I derive monotonic bidding functions for first price and soft floor auctions and use them to calculate the seller's expected revenue. In a soft floor auction, winner's regret introduces an intractable integral, so I consider the case in which bidders only experience loser's regret. So long as the weight on loser's regret $\beta > 0$, a soft floor auction revenue-dominates a first price auction which revenue-dominates a second price auction.

5.1. **First Price Auction.** I conjecture the existence of a symmetric bidding strategy $b(\cdot)$ that is monotonically increasing and continuously differentiable. As before, let $Q(v) = F(v)^{n-1}$ denote the distribution of the maximal value of the $n-1$ other bidders, and let $Q'(v) = q(v)$. Bidder i has value v_i and bids $b_i = b(v_i)$. She has expected profit

$$\Pi(v_i, b_i) = (v_i - b_i)Q(v_i).$$

She has expected winner's regret

$$R^W(v_i, b_i) = \int_0^{v_i} (b_i - b_j)dQ(v_j),$$

and expected loser's regret

$$R^L(v_i, b_i) = \int_{b_i}^{v_i} (v_i - b_j) dQ(v_j).$$

As in section 3.2, let $\alpha, \beta \in [0,1]$ be the weights on winner's and loser's regret such that $(1-\alpha-\beta) \in [0,1]$. Define the expected utility of bidder i as

$$\begin{aligned} U(v_i, b_i) &= (1-\alpha-\beta)\Pi(v_i, b_i) - \alpha R^W(v_i, b_i) - \beta R^L(v_i, b_i) \\ &= (1-\alpha-\beta)(v_i - b_i)Q(v_i) - \alpha \int_0^{v_i} (b_i - b_j) dQ(v_j) - \beta \int_{b_i}^{v_i} (v_i - b_j) dQ(v_j). \end{aligned}$$

Bidder i chooses b_i to maximize expected utility. Since the bidding function is monotonically increasing and continuously differentiable, it has a well-defined inverse $\phi(b_i) = v_i$. Plug it in to find first order conditions.

$$\max_{b_i} (1-\alpha-\beta)(v_i - b_i)Q(\phi(b_i)) - \alpha \int_0^{v_i} (b_i - b_j) dQ(\phi(b_j)) - \beta \int_{b_i}^{v_i} (v_i - b_j) dQ(\phi(b_j))$$

$$0 = -(1-\beta)Q(\phi(b_i)) + (1-\alpha-\beta)(v_i - b_i)q(\phi(b_i))\phi'(b_i) + \beta(v_i - b_i)q(\phi(b_i))\phi'(b_i)$$

$$\phi'(b_i) = \frac{(1-\beta)Q(\phi(b_i))}{(1-\alpha)(v_i - b_i)q(\phi(b_i))}.$$

Since $v_i = \phi\left(b_i\left(v_i\right)\right)$ and $\phi'\left(b_i\left(v_i\right)\right) = 1/b_i'\left(v_i\right)$,

$$\frac{db}{dv} = \frac{1}{\phi'(b)} = \frac{1-\alpha}{1-\beta}\left(v-b\right)\frac{q(v)}{Q(v)}. \tag{22}$$

I drop the subscript i to emphasize that the bidding formula is symmetric. Solving equation (22) leads to the following bidding function.

Proposition 4 (First Price Auction Bidding Function)**.**

The optimal bidding function for a type v bidder in a first price auction with hard reserve r is

$$b(v) = v - \int_r^v \left(\frac{F(z)}{F(v)}\right)^{(n-1)\frac{1-\alpha}{1-\beta}} dz. \tag{23}$$

Proof. Equation (22) is a linear first order ordinary differential equation. For simplification, let $b_i = y$ and $v_i = x$. Further, define

$$P(x) = \frac{(1-\alpha)q(x)}{(1-\beta)Q(x)},\ R(x) = x\frac{(1-\alpha)q(x)}{(1-\beta)Q(x)},\ y = uv \text{ and } y' = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Then by equation (22),

$$R(x) = y' + P(x)y$$

(24) $$R(x) = u\frac{dv}{dx} + v\left(\frac{du}{dx} + P(x)u\right).$$

Since $y = uv$, equation (24) has a degree of freedom. Set $\frac{du}{dx} + P(x)u = 0$ to obtain

$$0 = \frac{du}{dx} + P(x)u$$

$$\frac{du}{u} = -P(x)dx$$

$$\ln(u) = -\int P(x)dx$$

(25) $$u = e^{-\int P(x)dx}.$$

Plug equation (25) into equation (24).

$$R(x) = \frac{dv}{dx} \times e^{-\int P(x)dx}$$

$$dv = \frac{R(x)}{e^{-\int P(x)dx}}dx$$

(26) $$v = \int \left(\frac{R(x)}{e^{-\int P(x)dx}}\right) dx.$$

Substituting equations (25) and (26) for u and v gives

(27) $$y = uv = e^{-\int P(x)dx} \int \left(\frac{R(x)}{e^{-\int P(x)dx}}\right) dx.$$

Plug $P(x)$ and $R(x)$ into equation (27) to get

$$y = e^{-\int P(x)dx} \times \int \left(\frac{R(x)}{e^{-\int P(x)dx}}\right) dx$$

(28) $$y = e^{-\int \frac{(1-\alpha)q(x)}{(1-\beta)Q(x)}dx} \times \int \left(\frac{\frac{(1-\alpha)xq(x)}{(1-\beta)Q(x)}}{e^{-\int \frac{(1-\alpha)q(x)}{(1-\beta)Q(x)}dx}}\right) dx.$$

Observe that

$$-\int \frac{(1-\alpha)q(x)}{(1-\beta)Q(x)}dx = -\frac{(1-\alpha)\log(Q(x))}{(1-\beta)} + c = \log\left(Q(x)^{-\frac{1-\alpha}{1-\beta}}\right) + c.$$

Thus equation (28) becomes

$$
\begin{aligned}
y &= e^{\log\left(Q(x)^{-\frac{1-\alpha}{1-\beta}}\right)+c} \times \int \left(\frac{\frac{(1-\alpha)xq(x)}{(1-\beta)Q(x)}}{e^{\log\left(Q(x)^{-\frac{1-\alpha}{1-\beta}}\right)+c}} \right) dx \\
&= e^c \times Q(x)^{-\frac{1-\alpha}{1-\beta}} \times \int \left(\frac{\frac{(1-\alpha)xq(x)}{(1-\beta)Q(x)}}{e^c \times Q(x)^{-\frac{1-\alpha}{1-\beta}}} \right) dx \\
&= Q(x)^{-\frac{1-\alpha}{1-\beta}} \times \int \left(\frac{(1-\alpha)xq(x)}{(1-\beta)Q(x)^{1-\frac{1-\alpha}{1-\beta}}} \right) dx.
\end{aligned}
$$

Integrate by parts to obtain

$$
\begin{aligned}
y &= Q(x)^{-\frac{1-\alpha}{1-\beta}} \times \int \left(\frac{(1-\alpha)xq(x)}{(1-\beta)Q(x)^{1-\frac{1-\alpha}{1-\beta}}} \right) dx \\
&= Q(x)^{-\frac{1-\alpha}{1-\beta}} \left(xQ(x)^{\frac{1-\alpha}{1-\beta}} - \int Q(x)^{\frac{1-\alpha}{1-\beta}} dx \right) \\
&= x - Q(x)^{-\frac{1-\alpha}{1-\beta}} \times \int Q(x)^{\frac{1-\alpha}{1-\beta}} dx.
\end{aligned}
$$

Finally let $y = b$ and $x = v$, and replace the integral variable with a dummy variable z integrated from our lower bound r to the value v to obtain equation (23) in Proposition 4. ■

When $\alpha = \beta$, equation (23) recovers the standard bidding formula for first price auctions and confirms that the effects of winner's and loser's regret offset. Even if $\alpha = \beta = 1/2$ and bidders ignore profit and only aim to minimize expected regret, they still bid as if maximizing expected profit. To find the seller's expected revenue in first and second price auctions, I follow Matthews (1995) and define the density functions of the largest and second largest values. By independence, the largest value v_1 has distribution

$$F_1(x) = Pr[v_1 \leq x] = F(x)^n,$$

and density function

$$f_1(x) = nF(x)^{n-1}f(x). \tag{29}$$

To derive the distribution of the second largest value v_2, break it into two mutually exclusive cases. The probability that all values are less than x is $Pr[v_2 \leq x] = F(x)^n$, and the probability that exactly one value exceeds x is $(1 - F(x))F(x)^{n-1}$. There are n choices for which value exceeds x. Together,

$$F_2(x) = Pr[v_2 \leq x] = F(x)^n + nF(x)^{n-1}(1 - F(x)).$$

This yields density function

$$f_2(x) = n(n-1)F(x)^{n-2}(1-F(x))f(x). \tag{30}$$

It is now easy to compute the revenues generated from first and second price auctions. The revenue generated from a first price auction with reserve price r is given by

$$\int_r^1 b(v)f_1(v)dv = \int_r^1 \left(v - \int_r^v \left(\frac{Q(z)}{Q(v)}\right)^{\frac{1-\alpha}{1-\beta}} dz\right) nF(v)^{n-1}f(v)dv. \tag{31}$$

The revenue of a second price auction with reserve r is the probability only one person participates and thus pays r plus the probability at least two people participate and the winner pays the second highest value.

$$n(1-F(r))F(r)^{n-1}r + \int_r^1 vf_2(v)dv. \tag{32}$$

Proposition 5 (First Price vs. Second Price Auction)**.**
When bidders face regret, a first price auction revenue-dominates a second price auction if $\beta > \alpha$, and is revenue-dominated by a second price auction if $\alpha > \beta$.

Proof of Proposition 5. A second price auction has the same expected revenue with or without regret. By the Revenue Equivalence Theorem, without regret, a first price auction and second price auction generate the same expected revenue. Hence I compare the expected revenue of a first price auction with regret to that of a first price auction without regret. The expected revenue of a first price auction without regret is given by

$$\int_r^1 \left(v - \int_r^v \left(\frac{Q(z)}{Q(v)}\right) dz\right) nF(v)^{n-1}f(v)dv. \tag{33}$$

Subtracting equation (33) from equation (31) gives

$$\int_r^1 \left(\int_r^v \left(\frac{Q(z)}{Q(v)}\right) - \left(\frac{Q(z)}{Q(v)}\right)^{\frac{1-\alpha}{1-\beta}} dz\right) nF(v)^{n-1}f(v)dv$$

Since $Q(\cdot)$ is non-decreasing, $Q(z) \le Q(v)$. If $\beta > \alpha$, then $1-\alpha > 1-\beta$ and

$$\frac{Q(z)}{Q(v)} > \left(\frac{Q(z)}{Q(v)}\right)^{\frac{1-\alpha}{1-\beta}},$$

so the integrand is positive. Conversely, $\beta < \alpha$ means $1-\alpha < 1-\beta$ and

$$\frac{Q(z)}{Q(v)} < \left(\frac{Q(z)}{Q(v)}\right)^{\frac{1-\alpha}{1-\beta}},$$

so the integrand is negative. ■

Conceptually the proof is easy, and Engelbrecht-Wiggans (1989) and Filiz-Ozbay and Ozbay (2007) both provide non-explicit versions. Loser's regret induces aggressive bidding whereas winner's regret causes conservative bidding. Since these effects exactly offset when $\alpha = \beta$, if one outweighs the other, profit either rises or falls. But I appear to provide the first explicit calculation to prove it, which has the added practical benefit that it is then easy to compare revenues for given α and β.

5.2. **Soft Floor Auction.** The seller implements a soft floor auction with hard reserve r and soft floor $s \geq r$. Bidders submit sealed bids exceeding r or decline to participate. If the winning bid exceeds s, the winner pays the maximum of the second highest bid and s, as in a second price auction. Otherwise, the winner pays her bid. I take $\alpha = 0$ so that bidders only weigh profit and loser's regret. I make this simplifying assumption because winner's regret introduces an intractable integral into the seller's revenue function. It is the same term that prevented further progress on the transfer function in equation (15). In a soft floor auction, the expected profit conditional on losing has a convenient closed form, but the expected profit conditional on winning involves integrating the bidding function.

When $\alpha = 0$, the soft floor auction also has the following interpretation as a non-sealed-bid auction. The auction begins as an English auction with opening bid s. If no bidder chooses to participate, the auction switches to a Dutch auction with opening bid s. In a Dutch auction, loser's regret is well-defined since the winning bid is known. Winner's regret, however, is not well-defined because the winner does not know if she would have won had she waited longer. Filiz-Ozbay and Ozbay (2007) argues that bidders anticipate loser's regret but do not anticipate winner's regret, and therefore tend to bid more aggressively. This effect would only be magnified if the winner never knew the second highest bid; in fact, Engelbrecht-Wiggans (1989) demonstrates the diminished effect of winner's regret when the losing bid is not announced. This provides a theoretical framework for the elimination of winner's regret.

Let v be the maximal value of the $n-1$ participants other than bidder i, so that v has distribution $Q(\cdot)$. Assume there exists a a symmetric bidding strategy $b(\cdot)$ that is monotonically increasing and continuously differentiable. A symmetric Nash Equilibrium implies that the type v bidder also has the highest bid; call it b. Let the second highest bid of the $n-1$ other bidders be b'. Say bidder i has type v_i and bids $b_i = b(v_i)$. Considering the possibility of winning in both the first and second price auctions, bidder i earns profit

$$\pi_i\left(v_i, b_i, v, b\right) = \begin{cases} v_i - v & \text{if} \quad s < b = v < b_i = v_i \\ v_i - s & \text{if} \quad b < s < b_i = v_i \\ v_i - b_i & \text{if} \quad b < b_i < s \\ 0 & \text{else.} \end{cases}$$

She has loser's regret

$$r_i^L\left(v_i, b_i, v, b\right) = \begin{cases} v_i - b & \text{if} \quad b_i < b < v_i \\ v_i - s & \text{if} \quad b', b_i < s < v_i, s < b = v \\ 0 & \text{else.} \end{cases}$$

The second condition for loser's regret says that the winning bidder is alone in the second price auction and therefore pays the soft floor s, but that the value of bidder i exceeds s. In the symmetric equilibrium, there is a threshold value $w > s$ that determines if a bidder participates in the first or second price auction. If $v_i > w$, she enters the second price auction, and if $v_i < w$, she waits for the first price auction. If $v_i = w$, she is indifferent between the two auctions. Let $\beta \in [0, 1]$ be the weight on profit and $(1 - \beta)$ be the weight on loser's regret.[5] Then by equation (23), the equilibrium bidding strategy in a first price auction with hard reserve r is given by

$$B(v) = v - \int_r^v \left(\frac{F(z)}{F(v)}\right)^{\frac{n-1}{\beta}} dz. \tag{34}$$

This leads to the following proposition.

Conjecture 6 (Soft Floor Auction Bidding Function)**.**
Let $w < B(1)$. Then the soft floor auction has a unique symmetric monotone pure strategy equilibrium given by

$$b(v) = \begin{cases} B(v) & \textit{if } v \le \phi(w) \\ v & \textit{if } v > \phi(w), \end{cases} \tag{35}$$

where $B(v)$ corresponds to equation (34) and $\phi(\cdot)$ is its inverse. If $w \geq B(1)$, the soft floor auction reverts to a first price auction and $b(v) = B(v)$.

I assume Conjecture 6 holds. The chosen soft floor s and the exogenous level of regret determine the threshold w. When bidders have no regret, $b(w) = s$, but introducing regret causes w to rise. To determine the exact relationship between s and w, consider when bidder i

[5]Notice that β now weighs profit, not regret. When only loser's regret is present, this makes the comparative statics on β more intuitive.

has value $v_i = w$, and is therefore indifferent between participating in the first or second price auction. By equating her expected utility in each portion of the soft floor auction, I derive a relationship between s and w.

Proposition 7 (Formula for s)**.**
The relationship between s and w is given by

$$s(w) = w - \frac{\beta(w - b(w))F(w)}{\beta F(w) + (1-\beta)(n-1)(1-F(w))}. \tag{36}$$

Further, $w \geq s$, with equality if and only if $w = s = r$ or $\beta = 0$.

Proof of Proposition 7. Let bidder i have type $v_i = w$. Participating in the first price auction yields expected profit

$$(w - b_i)F(w)^{n-1}.$$

Bidder i has three potential sources of loser's regret when she participates in the first price auction.

(1) If $v_j \leq w$ for all $j \neq i$, bidder i experiences regret if she loses but the winning bid is beneath her value. Since $v_i = w$, any winning bid exceeds her value, and this term vanishes.
(2) If $v_j \geq w, v_k < w$ for all $k \neq j$, by default j wins the second price auction at price s, and bidder i has regret of $w - s$.
(3) At least two other bidders participate in the second price auction. But then they both have values of at least w, so the winning price is at least w. Hence this term also vanishes.

Thus bidder i only experiences loser's regret if exactly one bidder has a value greater than w and the other $n-2$ people have values less than w. There are $n-1$ choices for who has the value greater than w, so the probability this occurs is

$$(n-1)(1-F(w))F(w)^{n-2}.$$

Expected loser's regret is

$$(n-1)(1-F(w))F(w)^{n-2}(w-s).$$

The expected utility bidder i receives from participating in the first price auction is

$$U(w, b(w)) = \beta(w - b(w)F(w)^{n-1} - (1-\beta)(n-1)(1-F(w))F(w)^{n-2}(w-s). \tag{37}$$

If bidder i instead participates in the second price auction, she faces two possible scenarios.

(1) Bidder i is alone in the second price auction. This occurs with probability $Q(w) = F(w)^{n-1}$ and her profit is $w - s$.

(2) Bidder i is not alone in the second price auction. Then the winning bid is no smaller than w, so bidder i cannot profit.

There is no regret in a second price auction, so her expected utility is just her expected profit.

$$U(w, b(w)) = \beta F(w)^{n-1}(w - s). \tag{38}$$

Since a type w bidder is indifferent between participating in the first or second price auction, set equation (37) equal to equation (38) to obtain

$$\beta F(w)^{n-1}(w - s) = \beta(w - b(w))F(w)^{n-1} - (1 - \beta)(n - 1)(1 - F(w))F(w)^{n-2}(w - s).$$

Isolating $s(w)$ gives

$$s(w) = w - \frac{\beta(w - b(w))F(w)}{\beta F(w) + (1 - \beta)(n - 1)(1 - F(w))}.$$

To prove $w \geq s$, rearrange terms to get

$$w - s = \frac{\beta(w - b(w))F(w)}{\beta F(w) + (1 - \beta)(n - 1)(1 - F(w))}.$$

Thus $w = s$ if $\beta = 0$. For $\beta \neq 0$,

$$w = s \iff \frac{\beta(w - b(w))F(w)}{\beta F(w) + (1 - \beta)(n - 1)(1 - F(w))} = 0.$$

This requires $F(w) = 0$ or $w = b(w)$. In the former case, $s = w = 0 \leq r$ and in the latter, by equation (23), $w = r$ so that $b(r) = r$. Since $s \geq r$, $s = r$. ■

The structure of Proposition 7 is counter-intuitive in that s is exogenous and w is endogenous, but it is in general impossible to solve for w, and isolating $s(w)$ allows for the direct comparison of revenues between auctions. Evaluating equation (36) when $w = 1$ gives

$$s(1) = b(1). \tag{39}$$

To calculate the expected revenue in a soft floor auction, consider three cases.

(1) $v_j \leq w$ for all j. Bidders participate in the first price auction and the item goes to the highest bidder.

$$\int_r^w b(v)f_1(v)dv = \int_r^w b(v)nF(v)^{n-1}f(v)dv.$$

(2) Exactly one bidder j has a value higher than w, so j wins at price s. There are n choices for who wins, the probability the winner has value greater than w is $1-F(w)$, and the probability all other bidders have value less than w is $F(w)^{n-1}$. Expected revenue is

$$n(1-F(w))F(w)^{n-1}s(w).$$

(3) At least two bidders have values greater than w, and the winner pays the second highest bid.

$$\int_w^1 vf_2(v)dv = \int_w^1 vn(n-1)F(v)^{n-2}(1-F(v))f(v)dv.$$

Combining these three possible scenarios gives expected revenue as a function of w.

$$\begin{aligned} R(w) = \int_r^w b(v)nF(v)^{n-1}f(v)dv + n(1-F(w))F(w)^{n-1}s(w) \\ + \int_w^1 vn(n-1)F(v)^{n-2}(1-F(v))f(v)dv. \end{aligned} \tag{40}$$

I use equation (40) to prove that a soft floor auction is revenue-equivalent to a first and second price auction when bidders only maximize profit.

Proposition 8 (Soft Floor Auction No Regret)**.**
Assume that $\beta = 1$ and thus bidders only maximize profit. Then a soft floor auction with hard reserve r and soft floor $s \geq r$ is revenue equivalent to a first price auction with hard reserve r.

Zeithammer (2019) gives a conceptual proof that appeals to the Revenue Equivalence Theorem, whereas I provide an explicit calculation.

Proof of Proposition 8. Equation (33) gives the revenue of a first price auction without regret. Set $\beta = 1$ and subtract equation (33) from equation (40) to obtain

$$\begin{aligned} &\int_r^1 b(v)f_1(v)dv - \left(\int_r^w b(v)f_1(v)dv + n(1-F(w))F(w)^{n-1}s(w) + \int_w^1 vf_2(v)dv\right) = \\ &\int_w^1 b(v)f_1(v)dv - \left(n(1-F(w))F(w)^{n-1}s(w) + \int_w^1 vf_2(v)dv\right) \end{aligned} \tag{41}$$

By equation (36), since $\beta = 1$, $s = b(w)$. The term in parentheses in expression (41) simplifies to

$$n(1-F(w))F(w)^{n-1}b(w) + \int_w^1 vf_2(v)dv. \tag{42}$$

If a first price auction has hard reserve w, then $b(w) = w$. By equation (32), if $b(w) = w$, then equation (42) is precisely the revenue of a second price auction with hard reserve w. Similarly,

$$\int_w^1 b(v)f_1(v)dv$$

is the revenue of a first price auction with hard reserve w. By the Revenue Equivalence Theorem, they are equal. Therefore expression (41) equals zero, proving that a soft floor auction is revenue equivalent to a first price auction. ∎

Equation (40) also allows for the direct comparison between the revenues of a soft floor auction and first and second price auctions in the presence of regret.

Proposition 9 (Soft Floor Auction Beats First and Second Price Auctions)**.** *The expected revenue generated by a soft floor auction dominates that of a first price auction and that of a second price auction.*

Proof of Proposition 9. Proposition 5 proves that a first price auction dominates a second price auction. It is therefore sufficient to show that a soft floor auction dominates a first price auction. Use equation (40) to take the derivative of $R(w)$ with respect to w and evaluate it at $w = 1$.

$$\begin{aligned}
\frac{\partial R}{\partial w} &= b(w)nF(w)^{n-1}f(w) + s'(w)n(1-F(w))F(w)^{n-1} \\
&\quad + ns(w)F(w)^{n-1}f(w)\,(n-1-nF(w)) - wn(n-1)F(w)^{n-2}(1-F(w))f(w) \\
\left.\frac{\partial R}{\partial w}\right|_{w=1} &= nb(1)f(1) + ns(1)f(1)(-1) \\
&= 0.
\end{aligned}$$

The penultimate line uses equation (39) to substitute $s(1) = b(1)$. Hence $w = 1$ is a critical point. To verify it is a minimum, take the derivative again and evaluate at $w = 1$.

$$\left.\frac{\partial^2 R}{\partial w^2}\right|_{w=1} = \frac{nf(1)}{\beta} \times \left(-\beta b'(1) + 2(n-1)(1-b(1))f(1) - \beta(n-1)(1-b(1))f(1)\right).$$

Use equation (22) to obtain an expression for $b'(1)$.

$$\begin{aligned}
b'(v) &= \frac{v-b}{\beta} \times \frac{f(v)}{F(v)}\,(n-1) \\
b'(1) &= \frac{1-b(1)}{\beta} f(1)\,(n-1).
\end{aligned}$$

Hence

$$\begin{aligned}
\left.\frac{\partial^2 R}{\partial w^2}\right|_{w=1} &= \frac{nf(1)^2}{\beta}\left(-\beta\left(\frac{1-b(1)}{\beta}\right)(n-1) + 2(n-1)(1-b(1)) - \beta(n-1)(1-b(1))\right) \\
&= \frac{n(n-1)(1-\beta)(1-b(1))f(1)^2}{\beta} > 0.
\end{aligned}$$

Since $w = 1$ is a minimum, a slightly smaller w increases revenue. Since $w = 1$ is the revenue with only the first price auction, a soft floor auction dominates a first price auction. ∎

I now demonstrate that the optimal soft floor auction not only generates more revenue than first and second price auctions, but also is more efficient.

Proposition 10.

For $\beta \in (0,1)$, a soft floor auction outperforms a first price and second price auctions with optimal reserve price with respect to revenues and efficiency.

Proof. Proposition 9 proves that introducing a soft floor increases revenue. To establish efficiency, it suffices to show that the optimal hard reserve r in a soft floor auction is lower than the optimal reserve in both first and second price auctions. Indeed, the optimal reserve price in a first price auction increases with the support of the values. Conversely, if the range over which bidders enter the first price auction decreases, then so does the reserve price. ■

Proposition 10 says nothing about the optimal choice of hard reserve r. Figure 1 illustrates Proposition 10 in the case of two bidders with uniform distributions. Even in this simple case, calculating the optimal hard and soft reserves is difficult.

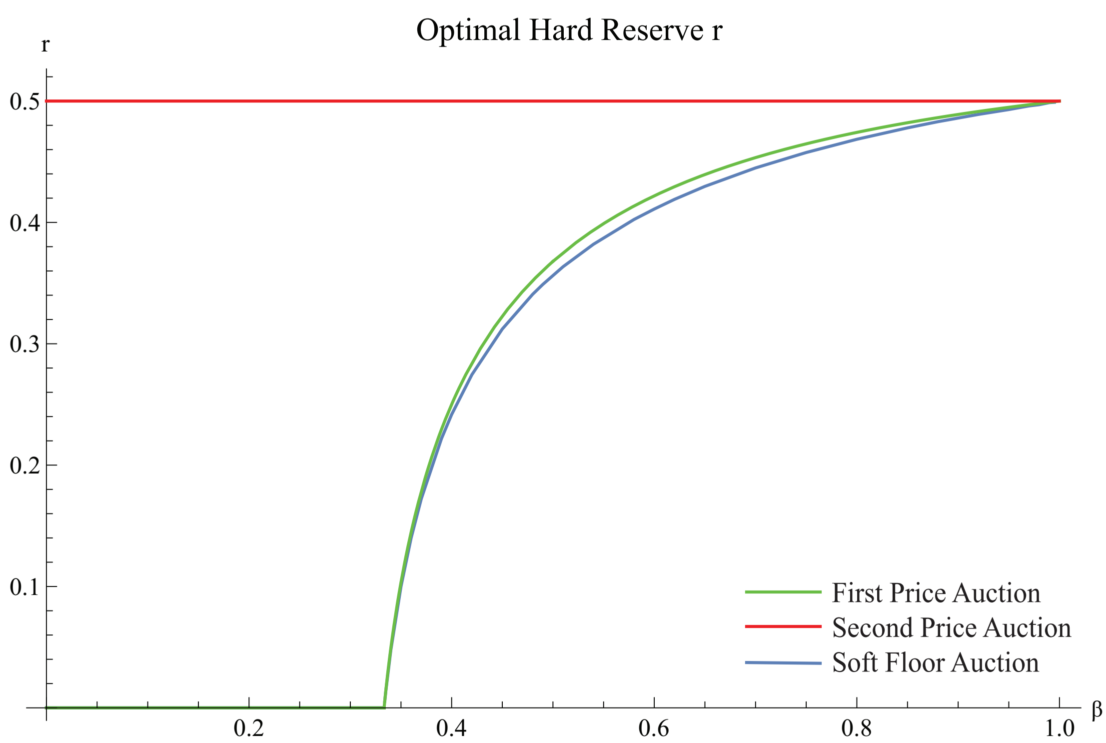

FIGURE 1. Optimal r for Two Bidders, Uniform Distribution

6. Two Bidders with Uniform Distribution

I consider the case of two bidders with values drawn from the uniform distribution. Since β is determined exogenously, the seller chooses the reserve price(s) as a function of β. In the first price auction, I solve explicitly for the optimal hard reserve $r^*(\beta)$ and show that it strictly increases in β for $\beta \in (1/3, 1]$. Then I determine the corresponding maximal revenue. In the soft floor auction, the optimal hard and soft reserves are interdependent. I solve for $r^*(\beta)$ in terms of w, and then I solve several implicit functions to graph the optimal reserves $r^*(\beta)$ and $s^*(\beta)$ and the corresponding threshold w^* and maximal expected revenue.

6.1. **First Price Auction.** Plugging $F(v) = v$ and $n = 2$ into equation (23) gives the bidding function in a first price auction with hard reserve r.

$$b(v) = \frac{v}{1+\beta} + r\left(\frac{r}{v}\right)^{1/\beta}\left(\frac{\beta}{1+\beta}\right). \tag{43}$$

By equation (29), $f_1(v) = 2v$. Plug equation (43) into equation (40) to find the seller's expected revenue.

$$\begin{aligned} RF(r) &= \int_r^1 b(v) f_1(v) dv \\ &= \int_r^1 \left(\frac{v}{1+\beta} + r\left(\frac{r}{v}\right)^{1/\beta}\frac{\beta}{1+\beta}\right) 2v dv \end{aligned}$$

$$RF(r) = \frac{2}{3(1+\beta)} + \frac{2r^{\frac{\beta+1}{\beta}}\beta^2}{(1+\beta)(2\beta-1)} + \frac{2r^3(1-3\beta)}{3(2\beta-1)}. \tag{44}$$

For given β, the seller picks r to maximize equation (44).

Proposition 11 (Optimal r in a First Price Auction)**.**
The optimal hard reserve $r^*(\beta)$ is given by

$$r^*(\beta) = \begin{cases} 0 & \text{if } \beta \in [0, 1/3] \\ r(\beta) = \left(3 - \frac{1}{\beta}\right)^{\frac{\beta}{1-2\beta}} & \text{if } \beta \in (1/3, 1]. \end{cases} \tag{45}$$

Further, $r(\beta)$ is strictly increasing for $\beta \in (1/3, 1]$.

Proof. The first term of equation (44) is independent of r. For $\beta \in [0, 1/3]$, the second and third terms of equation (44) are weakly negative and their sum is strictly negative.

$$\frac{2r^{\frac{\beta+1}{\beta}}\beta^2}{(1+\beta)(2\beta-1)} + \frac{2r^3(1-3\beta)}{3(2\beta-1)} = -\frac{2r^{\frac{\beta+1}{\beta}}\beta^2}{(1+\beta)(1-2\beta)} - \frac{2r^3(1-3\beta)}{3(1-2\beta)} < 0.$$

Setting $r = 0$ is thus maximal and gives total expected revenue

$$\frac{2}{3(1+\beta)}. \tag{46}$$

Now let $\beta \in (1/3, 1]$. Taking the derivative of equation (44) with respect to r gives

$$\frac{\partial RF}{\partial r} = \frac{2\left(\beta r^{1/\beta} + (1-3\beta)r^2\right)}{2\beta - 1} = 0. \tag{47}$$

Equation (47) has two solutions,

$$r \in \left\{0, \left(\frac{\beta}{3\beta - 1}\right)^{\frac{\beta}{2\beta-1}}\right\}.$$

If $r = 0$, the revenue is given by equation (46). Rewrite the non-trivial solution as

$$r(\beta) = \left(3 - \frac{1}{\beta}\right)^{\frac{\beta}{1-2\beta}}. \tag{48}$$

For $\beta \in (1/3, 1]$, equation (48) strictly increases (see Figure 2). Plug equation (48) into equation (44) to obtain the expected revenue.

$$RF(r(\beta)) = \frac{2}{3(\beta+1)} + \frac{2\left(\frac{3\beta^2\left(\left(\left(\frac{\beta}{3\beta-1}\right)^{\frac{1}{2\beta-1}}\right) - \left(\frac{\beta}{3\beta-1}\right)^{\frac{2\beta}{2\beta-1}}\right)\left(\frac{\beta}{3\beta-1}\right)^{\frac{\beta}{2\beta-1}}}{2\beta-1} - \left(\frac{\beta}{3\beta-1}\right)^{\frac{3\beta}{2\beta-1}}\right)}{3(\beta+1)}$$

$$RF(r(\beta)) = \frac{2}{3(\beta+1)} + \frac{2\beta\left(\frac{\beta}{3\beta-1}\right)^{\frac{\beta+1}{2\beta-1}}}{3(\beta+1)}. \tag{49}$$

The second term in equation (49) is strictly positive for $\beta > 1/3$, and the first term is the revenue when $r = 0$. Hence equation (48) gives the optimal r for $\beta \in (1/3, 1]$.[6] ■

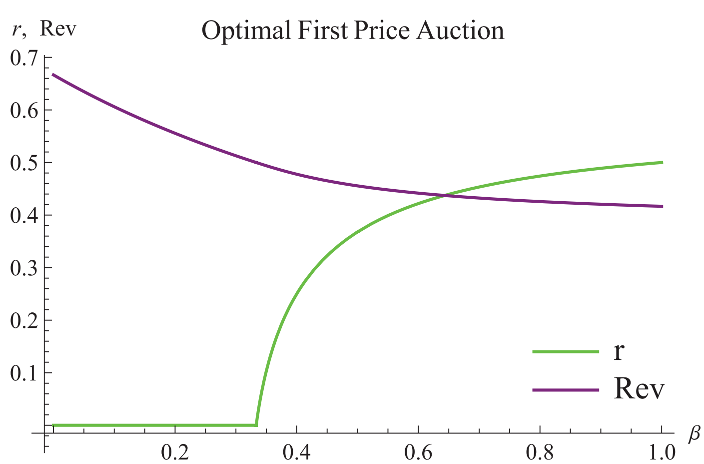

Figure 2. First Price Auction for Two Bidders, Uniform Distribution

[6] Second order conditions are difficult to prove explicitly but are easily verified graphically.

Proposition 12 (First Price Auction vs. Second Price Auction)**.**
For any $\beta \in [0,1)$, the maximal revenue of a first price auction is strictly greater than the maximal revenue of a second price auction.

Proof of Proposition 12. By equation (30), $f_2(v) = 2(v - v^2)$. The revenue of a second price auction with reserve r is independent of β and given by

$$\int_r^1 v f_2(v) dv = \frac{1}{3}(4r^2 + r + 1)(1 - r),$$

which achieves its maximal value at $r = 1/2$ and gives a revenue $5/12$. In the first price auction, the revenue function is strictly decreasing in β and is exactly equal to $5/12$ when $\beta = 1$. ■

Proposition 12 also follows from Proposition 5.

6.2. **Soft Floor Auction.** In a soft floor auction, the seller optimizes r and s simultaneously. However, the revenue also depends on w which, in turn, depends on s. By equation (40), the revenue as a function of r, s, and w is given by

$$R(r,s,w) = 2sw(1-w) + \frac{2w^3 - 3w^2 + 1}{3} + \frac{2\left((w^3 - r^3) + \frac{3\beta^2 r\left(r^2 - \left(\frac{r}{w}\right)^{\frac{1}{\beta}} w^2\right)}{1-2\beta}\right)}{3(1+\beta)} \tag{50}$$

By equation (36),

$$s(w) = \frac{w\left(1 - w + w\beta - \beta^2 + r\left(\frac{r}{w}\right)^{\frac{1}{\beta}} \beta^2 + w\beta^2\right)}{(1+\beta)(1 - w - \beta + 2w\beta)}.$$

Eliminating s from equation (50) gives

$$\begin{aligned} R(r,s(w),w) = {} & \frac{w\left(1 - w + w\beta - \beta^2 + r\left(\frac{r}{w}\right)^{\frac{1}{\beta}} \beta^2 + w\beta^2\right)}{(1+\beta)(1 - w - \beta + 2w\beta)}(2w - 2w^2) \\ & + \frac{2w^3 - 3w^2 + 1}{3} + \frac{2\left((w^3 - r^3) + \frac{3\beta^2 r\left(r^2 - \left(\frac{r}{w}\right)^{\frac{1}{\beta}} w^2\right)}{1-2\beta}\right)}{3(1+\beta)}. \end{aligned} \tag{51}$$

First order conditions on r give the optimal hard reserve in terms of w.

Proposition 13 (Soft Floor Auction r^*)**.**
In a soft floor auction with two bidders with uniform values, the optimal hard reserve is given by

$$r^* = \begin{cases} 0 & \text{if } \beta \in [0, 1/3] \\ w\left(\frac{\beta^2}{(3\beta-1)(-\beta+2\beta w - w + 1)}\right)^{\frac{\beta}{2\beta-1}} & \text{if } \beta \in (1/3, 1]. \end{cases}$$

The proof is in the appendix and solves for r^* in the slightly more general case of n bidders with values drawn from the uniform distribution. As anticipated, the optimal hard reserve depends on the threshold value w, which depends on the chosen soft floor s. The seller's choice of s uniquely determines w, and each w corresponds to a unique s. It is therefore sufficient for the seller to choose w, solve the first order conditions of equation (51) for r and w, and then use equation (36) to determine the corresponding s. Unfortunately, the first order conditions of equation (51) for w prove complicated, but I use Mathematica to construct Figure 3, which plots the optimal r, s, and w, and the corresponding maximal expected revenue, as a function of β.

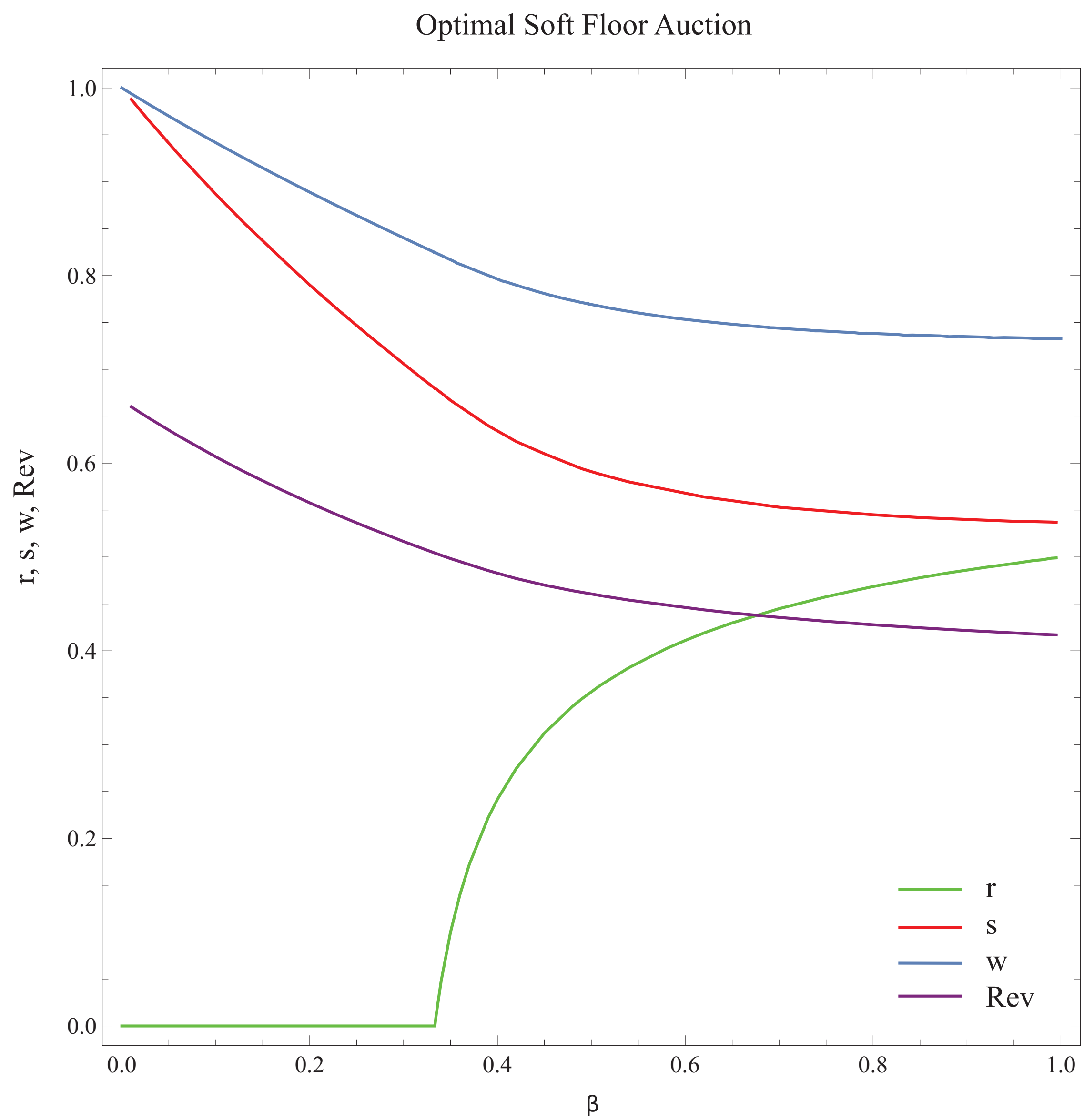

FIGURE 3. Soft Floor Auction for Two Bidders with Uniform Distribution

Another interesting extension considers a soft floor auction with no hard reserve $r = 0$. For $\beta \in [0, 1/3]$, the optimal s, w and expected revenue are unchanged since $r = 0$. For $\beta \in (1/3, 1]$, set $r = 0$ and take first order conditions of equation (50) to get

$$w^* = \frac{2\beta^2 + \sqrt{5\beta^2 - 4\beta^4} + \beta - 2}{2\left(2\beta^2 + \beta - 1\right)}.$$

Substituting into equation (53) gives

$$s^* = \frac{-4\beta^6 - 2\beta^5 + 10\beta^4 - 6\beta^2 - 2\sqrt{5\beta^2 - 4\beta^4}\beta + \sqrt{5\beta^2 - 4\beta^4} + \left(2\sqrt{5\beta^2 - 4\beta^4} + 2\right)\beta^3 + \beta}{\left(2\beta^2 + \beta - 1\right)^2\left(\sqrt{5\beta^2 - 4\beta^4} + \beta\right)}.$$

The optimal revenue in a soft floor auction with no hard reserve is given by substituting $r = 0$, s^*, and w^* into equation (50). The result yields Figure 4.

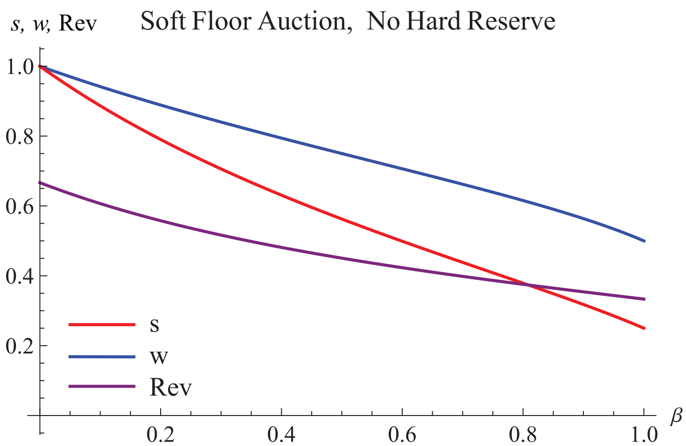

FIGURE 4. Optimal s, w, and rev for Two Bidders, Uniform Distribution

Combining the revenues from the optimal first price auction, soft floor auction with no hard reserve, and soft floor auction allows for the comparison of revenues across auctions.

The revenues of the soft floor auction with and without a hard reserve exactly coincide for $\beta \in [0, 1/3]$ because the optimal hard reserve is $r = 0$ in that range. For $\beta > 1/3$, the performance of the soft floor auction with no hard reserve sharply decreases. By contrast, the first price auction always generates strictly lower revenue than the soft floor auction, confirming Proposition 9, but by relatively small margins. The introduction of a soft floor does indeed increase expected revenue, but only for β near 1/2 is the effect apparent.

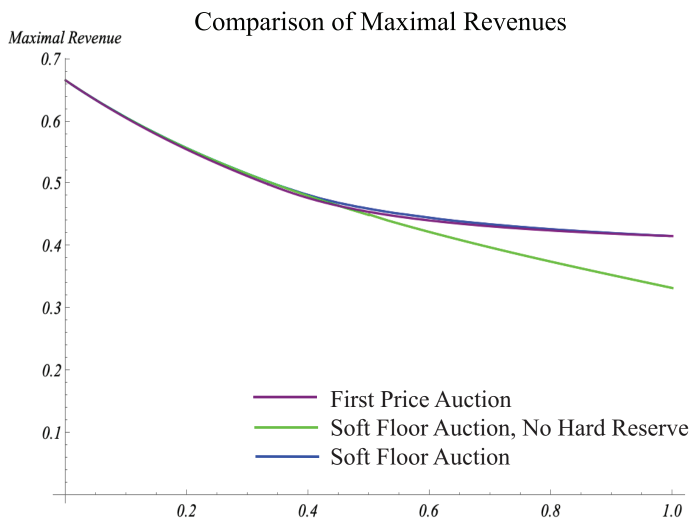

FIGURE 5. Comparison of Revenues for Two Bidders with Uniform Distribution

7. Discussion and Future Work

In this section, I provide commentary on my work, discuss the difficulties I encountered, and consider potential ways to address the problems and generalize the results. I then highlight interesting extensions and areas for future research.

7.1. **Discussion.** There are many ways to generalize and improve upon this project. The most obvious extension is to derive an expression for expected revenue when bidders experience both winner's regret and loser's regret, and then to classify the family of optimal mechanisms. This requires either sufficient restrictions on equation (15) to simplify the calculations for expected revenue or finding another approach to avoid the explicit calculation. Section 5 shows that even in the relatively simple case of a soft floor auction, including winner's regret introduces an intractable integral into the revenue expression that makes it difficult to even compare the expected revenue with that of a first or second price auction. The difficulties in both cases stem from the same cause: $\Pi^*(v_i)$ is not generally a well-behaved function. Unfortunately, $\Pi^*(v_i)$ is the key feature of regret.

The erratic nature of $\Pi^*(v_i)$ complicates the regularity conditions of the utility function. The individual rationality constraint in Proposition 1 most likely requires substantially weaker assumptions than those provided in the paper. Milgrom and Segal (2002) provides sufficient regularity conditions for application of the Envelope Theorem, and various continuity constraints allow for switching the order of derivation and integration, per the Leibniz Integration Rule. A point-wise maximized function such as $\Pi^*(v_i)$ does not generally satisfy these conditions, but restricting to a specific class of well-behaved mechanisms does. This is evidenced by the fact that both first price auctions and soft floor auctions are easy to work with when $\alpha = \beta$.

Classical regret in general mechanisms enables an auction to have some interesting but not-necessarily-desirable features. For example, consider an auction in which the seller encourages higher bids by providing each losing bidder a rebate proportional to the size of her bid. A losing bidder might then experience loser's regret not because she didn't win the auction, but because she would have preferred to lose with a higher bid and therefore obtain a larger rebate. Similarly, if that rebate is large enough, and if a winning bidder only made a small profit, she might have preferred to lose to instead capture the larger rebate. These paradoxical features are integral to deriving a general result, but often cause the utility function to violate the conditions of Proposition 1. One way to address the problem is to restrict the mechanism so that losing bidders make no profit, and the winning bidder pays no more than her bid. This, along with monotonicity conditions that assume the utility of a bidder increases with her type, satisfy Proposition 1, but greatly reduce the scope of implementable mechanisms. I suspect that the mechanism can be generalized to allow for small positive transfers to losing bidders so long as the winning bidder makes strictly greater profit.

It is also worth studying how the model changes as the number of bidders increases. The density of valuations increases and the expected gap between any two adjacent bidders shrinks. That means winning bidders tend to overbid less, and losing bidders could have won at a favorable price less frequently. Together, this decreases the maximal profit any bidder could have earned under complete information and thereby decreases expected regret. For example, in a first price auction, equation (23) shows that as $n \to \infty$, bidders bid their valuation for any $\alpha < 1$. For classical regret, equation (10) shows that expected revenue is decreased by

$$n\mu(1 - F(v^*))\Pi^*(v^*).$$

Expected revenue therefore depends on $O(n\Pi^*(v^*))$. If

$$n \to \infty \implies n\Pi^*(v^*) \to 0,$$

then the seller's revenue converges to that of an auction without regret. If instead $n\Pi^*(v^*)$ blows up, regret hurts the seller.

An important extension is to identify the necessary and sufficient conditions to ensure a mechanism is incentive compatible when bidders experience regret. In the appendix, Proposition 14 provides one attempt at a proof, but the restrictions on $\Pi^*(\cdot)$ prove too strong to be useful. It's possible that classifying incentive compatible mechanisms is unfeasible in general due to the complications that $\Pi^*(\cdot)$ introduce, or else that an entirely novel approach is needed.

7.2. **Future Work.** There are many areas for future research that generalize the results on classical regret. I only consider the case of symmetric bidders because asymmetric equilibria are difficult to classify, even in the case of two bidders (see e.g. Lizzeri and Persico 2000). Nonetheless, there are rich dynamics to explore with two bidders. The natural starting point is consider two bidders whose distributions have the same support. If the distribution of bidder 1 stochastically dominates that of bidder 2, then I conjecture bidder 1 will shave her bids to avoid overpaying when she wins. However, if she shaves her bids too much and loses, the magnitude of her regret will be larger. On the other hand, bidder 2 will experience less expected regret: when she wins, it will likely be by a small margin, and when she loses, the price will be more likely to exceed her value. The overall effect on her bid is unclear, but a comparative statics approach could prove fruitful. In the symmetric case, Theorem 2 proves that sellers earn strictly less revenue when bidders face regret, but in the asymmetric case, it's possible the seller can choose a mechanism that increases her expected profit. More generally, when one bidder does not stochastically dominate the other, the dynamics are even less clear, and regret could increase or decrease the seller's expected revenue.

It would also be interesting to relax the requirement that the highest bidder always wins the auction. While this guarantees ex-post efficiency in the symmetric case, Myerson (1981) famously proves that the optimal mechanism in standard auctions need not be ex-post efficient. For auctions with regret, a variation of a random allocation rule might prove optimal for the seller. A mechanism that randomly allocates the object makes it less clear how bidders evaluate regret since regret depends on the outcome of the auction. A bidder might bid poorly and get lucky, or bid well and get unlucky. To account for this, bidders might optimally implement a mixed strategy rather than a pure strategy, as sometimes occurs in regret minimization (see e.g. Hart and Mas-Colell (2000)).

8. Appendix

The appendix contains several auxiliary results and proofs.

Proof of Proposition 13. I solve the slightly more general problem of implicitly solving for the optimal r in terms of w and β for the uniform distribution $F(v) = v$ with n bidders. By equation (23), the bidding function is.

$$b(v) = v - \int_r^v \left(\frac{z^{\frac{n-1}{\beta}}}{v^{\frac{n-1}{\beta}}} \right) dz = v - \frac{\beta \left(v - r^{\frac{\beta+n-1}{\beta}} v^{\frac{1-n}{\beta}} \right)}{\beta + n - 1} = \frac{\left(v(n-1) - \beta r^{\frac{\beta+n-1}{\beta}} v^{\frac{1-n}{\beta}} \right)}{\beta + n - 1}. \tag{52}$$

Equation (36) gives the relationship between s and w.

$$s(w) = \frac{w \left(\beta - \beta n - \frac{\beta^2 \left(w - r^{\frac{\beta+n-1}{\beta}} w^{\frac{1-n}{\beta}} \right)}{\beta+n-1} + \beta n w - n w + n + w - 1 \right)}{\beta + (\beta-1)n(w-1) + w - 1}$$

$$s(w) = w - \frac{w \left(\frac{\beta^2 \left(w - r^{\frac{\beta+n-1}{\beta}} w^{\frac{1-n}{\beta}} \right)}{\beta+n-1} \right)}{(n-1)(1-\beta)(1-w) + \beta w} \tag{53}$$

Plug equations (52) and (53) into equation (40) to obtain the revenue as a function of r and w.

$$R(r,w) = \frac{n w \left(\frac{\beta^2 r^{\frac{\beta+n-1}{\beta}} w^{\frac{(\beta-1)(n-1)}{\beta}}}{(\beta-1)n+1} + \frac{(n-1)w^n}{n+1} \right)}{\beta + n - 1} + \frac{(n-1)\left((n(w-1)-1)w^n + 1 \right)}{n+1} - \frac{n(\beta + (\beta-1)n + 1) r^{n+1}}{(n+1)((\beta-1)n+1)} + n(1-w)w^n - \frac{n(1-w)w^n \left(\frac{\beta^2 \left(w - r^{\frac{\beta+n-1}{\beta}} w^{\frac{1-n}{\beta}} \right)}{\beta+n-1} \right)}{(n-1)(1-\beta)(1-w) + \beta w}$$

Although complicated, the first order conditions on r give a manageable expression.

$$\frac{\partial R}{\partial r} = \frac{n \left(\frac{\beta^2 r^{\frac{n-1}{\beta}} w^{\frac{(\beta-1)n+1}{\beta}}}{\beta + (\beta-1)n(w-1) + w - 1} - (\beta + (\beta-1)n + 1) r^n \right)}{(\beta-1)n+1} = 0$$

$$r^{\frac{n\beta - n + 1}{\beta}} = w^{\frac{n\beta - n + 1}{\beta}} \frac{\beta^2}{(1 + \beta + n\beta - n)((n-1)(1-\beta)(1-w) + w\beta}$$

$$r^* = w \left(\frac{\beta^2}{(1 + \beta + n\beta - n)((n-1)(1-\beta)(1-w) + w\beta)} \right)^{\frac{\beta}{1 + n\beta - n}}.$$

When $n = 2$,

$$r^* = w\left(\frac{\beta^2}{(3\beta-1)((1-\beta)(1-w)+\beta w)}\right)^{\frac{\beta}{2\beta-1}}.$$

Hence if $\beta \in [0, 1/3]$, $3\beta - 1 < 0$ and optimally $r = 0$. If $\beta \in (1/3, 1]$, r^* is as above. ■

This is my attempt to derive necessary and sufficient conditions for IC. Unfortunately, it doesn't appear fruitful, as it simply restates the classic result. Further, it requires that $\Pi^*(\cdot)$ is differentiable, which is almost always too strong of a condition to be helpful.

Proposition 14 (Incentive Compatibility)**.**
Assume $\Pi^*(\cdot)$ is differentiable. Then our social choice function is Bayesian incentive compatible if and only if

(i) $Q(\cdot)$ is non-decreasing

(ii)
$$U(v_i) = \int_{\underline{v}}^{v_i} Q(v)dv + U(\underline{v}) - \mu\Pi^*(v_i) + \mu\Pi^*(\underline{v}).$$

Recall that $U(v_i) = v_iQ(v_i) + T(v_i) - \mu\Pi^*(v_i)$.

Proof. (i) *Necessity.* Assume our social choice function is incentive compatible. Let $v_i > x$. Then

$$\begin{aligned} U(v_i) &\geq v_iQ(x) + T(x) - \mu\Pi^*(v_i) \\ &= U(x) + (v_i - x)Q(x) - \mu(\Pi^*(v_i) - \Pi^*(x)) \\ Q(x) &\leq \mu\left(\frac{\Pi^*(v_i) - \Pi^*(x)}{v_i - x}\right) + \frac{U(v_i) - U(x)}{v_i - x}. \end{aligned}$$

Similarly,

$$\begin{aligned} U(x) &\geq xQ(v_i) + T(v_i) - \mu\Pi^*(x) \\ &= U(v_i) + (x - v_i)Q(v_i) - \mu(\Pi^*(x) - \Pi^*(v_i)) \\ Q(v_i) &\geq \mu\left(\frac{\Pi^*(v_i) - \Pi^*(x)}{v_i - x}\right) + \frac{U(v_i) - U(x)}{v_i - x}. \end{aligned}$$

Hence we have

$$Q(v_i) \geq \mu\left(\frac{\Pi^*(v_i) - \Pi^*(x)}{v_i - x}\right) + \frac{U(v_i) - U(x)}{v_i - x} \geq Q(x).$$

This proves that $Q(\cdot)$ is non-decreasing. Let $h(v_i) = \mu\Pi^*(v_i) + U(v_i)$. Then taking the limit as $x \to v_i$,

$$\begin{aligned} Q(v_i) &= h'(v_i) \\ &= \mu\Pi^{*'}(v_i) + U'(v_i). \end{aligned}$$

Integrating,

$$\begin{aligned} \int_{\underline{v}}^{v_i} Q(v)dv &= \int_{\underline{v}}^{v_i} h'(v)dv \\ &= U(v_i) - U(\underline{v}) + \mu\Pi^*(v_i) - \mu\Pi^*(\underline{v}) \\ U(v_i) &= \int_{\underline{v}}^{v_i} Q(v)dv + U(\underline{v}) - \mu\Pi^*(v_i) + \mu\Pi^*(\underline{v}). \end{aligned}$$

(ii) *Sufficiency.* Say $x > v_i$. Then

$$\begin{aligned} U(x) - U(v_i) &= \mu(\Pi^*(v_i) - \Pi^*(x)) + \int_{v_i}^{x} Q(v)dv \\ &\geq \mu(\Pi^*(v_i) - \Pi^*(x)) + (x - v_i)Q(v_i) \\ U(x) &\geq \mu(\Pi^*(v_i) - \Pi^*(x)) + v_iQ(v_i) + T(v_i) - \mu\Pi^*(v_i) + (x - v_i)Q(v_i) \\ U(x) &\geq xQ(v_i) + T(v_i) - \mu\Pi^*(x). \end{aligned}$$

Similarly,

$$\begin{aligned} U(v_i) - U(x) &= \mu(\Pi^*(x) - \Pi^*(v_i)) + \int_{x}^{v_i} Q(v)dv \\ &\geq \mu(\Pi^*(x) - \Pi^*(v_i)) + (v_i - x)Q(x) \\ U(v_i) &\geq \mu(\Pi^*(x) - \Pi^*(v_i)) + xQ(x) + T(x) - \mu\Pi^*(x) + (v_i - x)Q(x) \\ U(v_i) &\geq v_iQ(x) + T(x) - \mu\Pi^*(v_i). \end{aligned}$$

This proves our claim.

■

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