April 1, 2015 Econometrics Seminar – Emmanuel Guerre

Abstract:

"Parametric and nonparametric quantile regression methods for first-price auction: A signal approach"
(joint with Nathalie Gimenes, University of Sao Paulo)

This paper considers a quantile signal framework for first-price auction. Under the independent private value paradigm, a key stability property is that a linear specification for the private value conditional quantile function generates a linear specification for the bids one, from which it can be easily identified. This applies in particular for standard quantile regression models but also to more flexible additive sieve specification which are not affected by the curse of dimensionality. A combination of local polynomial and sieve methods allows to estimate the private value quantile function with a fast optimal rate and for all quantile levels in [0,1] without boundary effects. This allows to estimate the optimal bidding strategy and all bidder's private values near the boundaries with a fast rate. The choice of the smoothing parameters is also discussed. Extension to the case where only the winning bid is observed is straightforward. The linear quantile specification also allows for unobserved heterogeneity. An extension to a new interdependent value additive specification including bidder specific variables is also considered.

Parametric and Nonparametric Quantile Regression Methods for First-Price Auctions: A Signal Approach

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Work in Progress

Plan of the talk

Notations

Quantile regression and additive/interactive quantile models

Quantile and auction

Identification of linear (sieve) quantile specification

Augmented (Sieve) Quantile Regression: dimension reduction and boundary free estimation

A small simulation experiment

Extension to interdependent values

Sealed bids first-price auction

- Auctioned good, with characteristics known to the bidders and econometrician
- Bidder forms a bid which is not observed by his opponents
- Bids are sealed and collected
- Bids are opened
- Winner = largest bid
- Paid price = bid of the winner = largest bid

Notations

ℓ = auction, $\ell = 1, \ldots, L$
x_ℓ = auction good covariate
I_ℓ = number of bidders
i = bidders, $i = 1, \ldots, I_\ell$

Notations (cont'd): private value case

Private value $V_{i\ell}$: iid given (x_ℓ, I_ℓ)

- Common knowledge cdf $F(v|x_\ell, I_\ell)$, continuous pdf $f(v|x_\ell, I_\ell) > 0$ over its compact support
- Conditional quantile $V(\alpha|x_\ell, I_\ell)$, quantile level $\alpha \in [0, 1]$
- Private value rank $A_{i\ell} = F(V_{i\ell}|x_\ell, I_\ell)$: prob. that an opponent has a pv smaller than $V_{i\ell}$

Important property of $A_{i\ell}$: $[0, 1]$-uniform and independent of (x_ℓ, I_ℓ)

Notations (cont'd): observed bids

Observed bids $B_{i\ell}$: iid bids given (x_ℓ, I_ℓ)

- Cdf $G(b|x_\ell, I_\ell)$, pdf $g(b|x_\ell, I_\ell)$
- Conditional quantile $B(\alpha|x_\ell, I_\ell)$, $\alpha \in [0, 1]$
- Bid rank $U_{i\ell} = G(B_{i\ell}|x_\ell, I_\ell)$

Why quantiles?

Fondamental Simulation Theorem: *The private value rank $A_{i\ell}$ is independent of (x_ℓ, I_ℓ) with a $[0,1]$-uniform distribution, and satisfies,*

$$V_{i\ell} = V\left(A_{i\ell}|x_\ell, I_\ell\right)$$

- Allow to simulate $V_{i\ell}$ in full generality

- Since, in most case,

$$\left(V_{i\ell}, B_{i\ell}\right) = \left(V\left(A_{i\ell}|x_\ell, I_\ell\right), B\left(A_{i\ell}|x_\ell, I_\ell\right)\right)$$

  counterfactuals as the expected revenue $\mathbb{E}\left[V_{i\ell} - B_{i\ell}|x_\ell\right]$ for a given mechanism

- Since $V\left(\alpha|x_\ell, I_\ell\right) = F^{-1}\left(\alpha|x_\ell, I_\ell\right)$, quantile can be estimated nonparametrically as fast as a c.d.f. and with faster rates than a p.d.f

Private value rank and Milgrom-Weber model

- In Milgrom & Weber (1982),

$$W_{i\ell} = W_i\left(\widetilde{A}_{0\ell}, \widetilde{A}_{1\ell}, \ldots, \widetilde{A}_{I_\ell \ell}, x_\ell\right), \quad \widetilde{A}_{1\ell}, \ldots, \widetilde{A}_{I\ell} \text{ bidders signals}$$

  where the ith bidder knows $\widetilde{A}_{i\ell}$ but not $\widetilde{A}_{j\ell}$, and $\left(\widetilde{A}_{1\ell}, \ldots, \widetilde{A}_{I\ell}\right)$ independent of x_ℓ

- In the private value case

$$V_{i\ell} = V_i\left(\widetilde{A}_{i\ell}, x_\ell\right), \quad \widetilde{A}_{i\ell} \text{ independent r.v}$$

$\Rightarrow$ The private value rank $A_{i\ell}$ can be viewed as a standardized signal

- Issue: the general model is not identified (Laffont & Vuong, 1996)
  $\Rightarrow$ Extension of the paper: a new additive specification for the general model

An additive specification

$$
\begin{aligned}
W_{i\ell} &= W_i\left(A_{1\ell}, \ldots, A_{I_\ell \ell}; x_\ell\right) \\
&= \sum_{j=0}^{I_\ell} \pi_{ij} V_j\left(A_{j\ell}; x_\ell\right), \quad \pi_{ii} = 1
\end{aligned}
$$

- $V_i\left(A_{i\ell}, x_\ell\right) = V_i\left(A_{i\ell}, x_\ell, z_{i\ell}\right)$, $z_{i\ell}$ individual characteristic ("capacity")
- $V_i\left(A_{i\ell}, x_\ell\right)$: intrinsic private value of the good for the ith bidder
- Interactions $\Rightarrow$ the final value must aggregate the intrinsic private values ("prestige", trading after auction, etc...).

Differs from Somaini (2014), $W_i = W_i\left(A_{1\ell}, \ldots, A_{I_\ell \ell}; x_\ell, z_{i\ell}\right)$

Dimension reduction issues

Many covariate available in auction datasets:

- Athey, Levin & Seira (2011) or Li & Zheng (2009, 2012): 5 to 15 covariates for 1,000 observations
- Haile & Tamer (2003), Aradillas-López, Ghandi & Quint (2013): 5-6 covariates for few thousands observations

Not many dimension reduction methods for first-price auction

- Paarsch & Hong (2006): implement p.d.f. estimation as in G., Perrigne & Vuong (2000) using an index assuming $V_{i\ell} = g\left(x_\ell'\beta\right) + \varepsilon_{i\ell}$. A quantile approach as in Chaudhuri, Doksum & Samarov (1997) would be less restrictive
- Haile, Hong & Shum (2003), Rezende (2008): $V_{i\ell} = x_\ell'\beta + v_{i\ell}$ implies $B_{i\ell} = x_\ell'\beta + b_{i\ell}$ where the p.d.f of $v_{i\ell}$ can be estimated from the ones of $b_{i\ell}$ as in G., Perrigne & Vuong (2000)

Additive quantile specification

Various lower dimensional models have been proposed to restrict the general quantile specification

$$V_{i\ell} = V\left(A_{i\ell} | x_\ell, I_\ell\right)$$

- Quantile regression (Koenker & Bassett, 1978);

$$V_{i\ell} = x_{i\ell}' \beta_1 \left(A_{i\ell} | I_\ell\right) + \beta_0 \left(A_{i\ell} | I_\ell\right) = X_{i\ell}' \beta \left(A_{i\ell} | I_\ell\right)$$

  Nests Haile, Hong & Shum (2003), Rezende (2008) ($\beta_1 \left(A_{i\ell} | I_\ell\right) = \beta_1$) and allows for interactions between signal and covariates

- Additive specification (Horowitz & Lee, 2005): for $x_\ell = \left[x_{1\ell}, \ldots, x_{d\ell}\right]$,

$$V_{i\ell} = V_1 \left(A_{i\ell} | x_{1\ell}, I_\ell\right) + \cdots + V_d \left(A_{i\ell} | x_{1d}, I_\ell\right)$$

- Additive interactive specification (Andrews & Whang (1990) for regression)

$$V_{i\ell} = \sum_{k=1}^{D} \sum_{j_1 < \cdots < j_k} V_{j_1,\dots,j_D}\left(A_{i\ell} | x_{j_1\ell}, \dots, x_{j_k\ell}, I_\ell\right)$$

$\Rightarrow$ A wide class of models ranging from parametric to nonparametric

The general linear quantile specification of the paper

All previous specifications can be nested in the linear sieve specification with D interactions ($0 \leq D \leq \dim x$)

$$V_{i\ell} = \sum_{k=0}^{\infty} P_k(x_\ell)\,\gamma_k(A_{i\ell}|I_\ell), \quad P_k(x_\ell) = P_k\left(x_{j_1(k)\ell}, \ldots, x_{j_D(k)\ell}\right),$$

and where $\gamma_k(\alpha|I) = \langle V(\alpha|x, I), P_k(x)\rangle_x$ for orthonormal sieve

$\Rightarrow$ An infinite dimensional version of Koenker & Bassett (1978) quantile regression

Other econometric issues

Econometric issues with G., Perrigne & Vuong (2000) two step kernel density estimation method

- Boundary bias for the upper and lower tails distribution (Hickman & Hubbard, 2014)

- Lack of clearcut bandwidth choice (Henderson, List, Millmet, Parmeter & Price, 2012)

The proposed new quantile methodology is helpful regarding these issues

Quantile and auction in the econometric literature

- Haile, Hong & Shum (2003): Quantile, dimension reduction using a regression model. See also Rezende (2008)
- Marmer & Shneyerov (2012): avoids estimation of private values
- G. & Sabbah (2012), Fan, Li & Pesendorfer (2013,WP): LP quantile estimation
- Menzel & Morganti (2013): order statistic (sample quantile) approach for second-price auction
- Gimenes (2013, WP): QR for ascending auction

Rest of the talk

- Quantile identification
- A key property: Stability of linear quantile specification
- Augmented (Sieve) Quantile regression
- Interdependent value extension

Quantile identification: a preliminary lemma

Lemma *Suppose that the values W_i are such,*

$$W_i = W_i\left(A_0, A_1, \ldots, A_I, x, I\right), \; i = 1, \ldots, I,$$

where $(A_0, A_1, \ldots, A_I)$ is independent of (x, I), each A_i are $[0,1]$ uniform, and that each bidder plays a strictly increasing strategy,

$$B_i = s_i\left(A_i | x, I\right), \; s_i\left(\cdot | x, I\right) \; \uparrow \; \textit{for all} \; \left(x, I\right).$$

- $\Rightarrow$ No equilibrium assumption. Increasing strategy assumption strong enough to identify A_i and $s_i\left(\cdot|\cdot,\cdot\right)$ in a constructive way
- Bayesian Nash Equilibrium generates increasing strategies (Reny & Zhamir, 2004)

Lemma cont'd: Signal identification

(i) The signal A_i, $i \geq 1$, *can be recovered from the observed bids with,*

$$A_i = G_i\left(B_i | x, I\right),$$

where $G_i\left(\cdot | x, I\right)$ *is the conditional c.d.f of* B_i;

- $\Rightarrow$ the joint distribution of $\left(A_1, \ldots, A_I\right)$ is identified
- The signal A_i can be estimated (known identity or $G_i\left(B_i | x, I\right) = G\left(B_i | x, I\right)$)

Lemma cont'd: strategy identification

$$A_i = G_i\left(B_i|x, I\right) \Rightarrow B_i = B_i\left(A_i|x, I\right)$$

(ii) the strategy $s_i\left(\cdot|x, I\right)$ *is identified by the conditional bid quantile function,*

$$s_i\left(A|x, I\right) = B_i\left(A|x, I\right), \text{ for any } A \in [0, 1];$$

- Contrasts with strategies depending upon the private value for symmetric IPV.

Lemma cont'd: Probability of Winning

(iii) under symmetric IPV, that is if A_i *independent,* $V_i = V(A_i, x, I)$ *and* $B_i = B(A_i|x, I)$ *for all* $i = 2, \ldots, I$*, the probability that a bid* $B(A|x, I)$ *wins is* A^{I-1}.

- Under asymmetry or interdependent value, the probability that a bid $B_1(A|z, I)$ is also identified since it is

$$\mathbb{P}\left(B_1(A|x, I) > \max_{i=2,\ldots,I} B_i(A_i|x, I) \,|A_1, x, I\right)$$

  which depends upon the identified $B_i(\cdot|x, I)$ and the identified joint distribution of $(A_1, \ldots, A_I)'$. But no close form expression in general

Quantile under symmetric IPV and Bayesian Nash Equilibrium

$\Rightarrow$ Under symmetric IPV and BNE, $B(\cdot|x, I)$ is the optimal strategy

This identifies $V(\alpha|x, I)$ in a simple linear way under risk neutrality

The risk neutral expected utily of a bid $B(a|x, I)$ given first bidder signal $A_1 = A$ is

$$(V_1 - B(a|x, I))\, a^{I-1} = (V(A|x, I) - B(a|x, I))\, a^{I-1}$$

Since the optimal bid is $B\left(A|x,I\right)$,

$$\left(V\left(A|x,I\right)-B\left(a|x,I\right)\right)a^{I-1}\leq\left(V\left(A|x,I\right)-B\left(A|x,I\right)\right)A^{I-1}$$

for all $a\in\left[0,1\right]$.

Hence, for all $A\in\left(0,1\right)$,

$$\frac{\partial}{\partial a}\left[\left(V\left(A|x,I\right)-B\left(a|x,I\right)\right)a^{I-1}\right]\Big|_{a=A}=0$$
$$\Leftrightarrow -B^{(1)}\left(A|x,I\right)A^{I-1}+\left(I-1\right)\left(V\left(A|x,I\right)-B\left(A|x,I\right)\right)A^{I-2}=0$$

Rearranging gives the differential equation,

$$V(A|x,I) = B(A|x,I) + \frac{A \times B^{(1)}(A|x,I)}{I-1}, \quad B(0|x,I) = V(0|x,I)$$

which is the quantile version of the identification method in G., Perrigne & Vuong (2000)

$$V_{i\ell} = B_{i\ell} + \frac{1}{I_\ell - 1} \frac{G(B_{i\ell}|x_\ell, I_\ell)}{g(B_{i\ell}|x_\ell, I_\ell)}$$

Suggests to estimate $V(\alpha|x, I)$ using

$$\widehat{V}(\alpha|x, I) = \widehat{B}(\alpha|x, I) + \frac{\alpha \widehat{B}^{(1)}(\alpha|x, I)}{I - 1}$$

as in G. & Sabbah (2012) or Fan et al. (2013).

However,

- Not so many good estimators of $B^{(1)}(\alpha|x, I)$ in the literature
- It may be fruitful to solve the linear differential equation before estimating

A key lemma: (i) stability of linear specification

(i) The conditional quantile function of optimal bids is given by the linear operator,

$$B(\alpha|x,I) = \frac{I-1}{\alpha^{I-1}} \int_0^{\alpha} t^{I-2} V(t|x,I)\, dt.$$

linear specification for $V(\cdot|x,I)$

$\Rightarrow$

linear specification for $B(\cdot|x,I)$

as noted in Haile et al. (2003) and Rezende (2008) for the particular case of regression.

Example: if, for some $\gamma_k(\alpha|I) = \langle V(\alpha|\cdot, I), P_k(\cdot)\rangle$

$$V(\alpha|x, I) = \sum_{k=0}^{\infty} P_k(x)\,\gamma_k(\alpha|I),$$

then

$$B(\alpha|x, I) = \sum_{k=0}^{\infty} P_k(x)\,\beta_k(\alpha|I)$$

with

$$\beta_k(\alpha|I) = \frac{I-1}{\alpha^{I-1}} \int_0^{\alpha} t^{I-2}\gamma_k(t|I)\,dt.$$

A key lemma (ii): identification

(ii) The conditional private values quantile function can be recovered from the bid one,

$$V(\alpha|x, I) = B(\alpha|x, I) + \frac{\alpha}{I-1} B^{(1)}(\alpha|x, I).$$

Example (Cont'd): since

$$B(\alpha|x,I) = \sum_{k=0}^{\infty} P_k(x)\,\beta_k(\alpha|I)$$

$$\text{with } \beta_k(\alpha|I) = \frac{I-1}{\alpha^{I-1}} \int_0^{\alpha} t^{I-2}\gamma_k(t|I)\,dt,$$

then

$$V(\alpha|x,I) = \sum_{k=0}^{\infty} P_k(x)\,\gamma_k(\alpha|I)$$

$$\text{with } \gamma_k(\alpha|I) = \beta_k(\alpha|I) + \frac{\alpha}{I-1}\beta_k^{(1)}(\alpha|I).$$

Estimation methology

1. Postulate a quantile regression specification for the private values or set $X = (P_1(x), \ldots, P_{k_L}(x))$,

$$\Rightarrow V(\alpha|x, I) = X'\gamma(\alpha|I) + bias_V \text{ (no bias for QR)}$$

2. By the stability property

$$B(\alpha|x, I) = X'\beta(\alpha|I) + bias_B \text{ (no bias for QR)}$$

3. Given an estimation of $\widehat{\beta}(\alpha|I)$ and $\widehat{\beta}^{(1)}(\alpha|I)$, set

$$\widehat{\gamma}(\alpha|I) = \widehat{\beta}(\alpha|I) + \frac{\alpha\widehat{\beta}^{(1)}(\alpha|I)}{I-1}, \quad \widehat{V}(\alpha|x, I) = X'\widehat{\gamma}(\alpha|I)$$

$\Rightarrow$ Needs new techniques to find good estimation of $\beta^{(1)}(\alpha|I)$, an issue mostly ignored in the literature.

Standard quantile regression

Check function $\rho_\alpha(t) = t(\alpha - \mathbb{I}(t \leq 0))$

$$\beta(\alpha|I) = \arg\min_\beta \mathbb{E}\left[\mathbb{I}(I_\ell = I)\,\rho_\alpha\left(B_{i\ell} - X_\ell'\beta\right)\right]$$

$$\Rightarrow \widehat{\beta}(\alpha|I) = \arg\min_\beta \frac{1}{L}\sum_{\ell=1}^{L} \mathbb{I}(I_\ell = I)\,\rho_\alpha\left(B_{i\ell} - X_\ell'\beta\right)$$

- Does not give an estimator of $\beta^{(1)}(\alpha|I)$
- Difficult to define for $\alpha = 0$ or $\alpha = 1$

Augmented quantile regression

- Allow small variation of α in the check function $\rho_\alpha(t)$
- Expand $\beta(\alpha + ht)$ to estimate $\beta^{(1)}(\alpha|I)$ by local polynomial smoothing

For $a = \alpha + ht$, $h > 0$ bandwidth, and $\beta(\cdot|I)$ $s+2$ differentiable,

$$\begin{aligned} X'\beta(a|I) \\ &= X'\left\{\beta(\alpha|I) + (a-\alpha)\beta^{(1)}(\alpha|I) + \cdots + \frac{(a-\alpha)^{s+1}}{(s+1)!}\beta^{(s+1)}(\alpha|I)\right\} \\ &\quad + O\left((a-\alpha)^{s+2}\right) \\ &= X(a-\alpha)'\boldsymbol{\beta}(\alpha|I) + O\left((a-\alpha)^{s+2}\right),\ X(t) = \left[\begin{array}{c} 1 \\ \vdots \\ \frac{t^{s+1}}{(s+1)!} \end{array}\right] \otimes X \end{aligned}$$

where $\boldsymbol{\beta}(\alpha|I) = \left[\beta(\alpha|I)', \beta^{(1)}(\alpha|I)', \ldots, \beta^{(s+1)}(\alpha|I)'\right]'$.

Objective function of the augmented quantile regression

$K(\cdot)$ kernel, h bandwidth $\Rightarrow$ objective function $\widehat{\mathcal{R}}(\boldsymbol{\beta};\alpha,I)$ is

$$\frac{1}{LIh}\sum_{\ell=1}^{L}\mathbb{I}(I_\ell = I)\sum_{i=1}^{I_\ell}\int_0^1 \rho_a\left(B_{i\ell} - X_\ell (a-\alpha)'\boldsymbol{\beta}\right)K\left(\frac{a-\alpha}{h}\right)da$$

$$= \frac{1}{LI}\sum_{\ell=1}^{L}\mathbb{I}(I_\ell = I)\sum_{i=1}^{I_\ell}\int_{-\frac{\alpha}{h}}^{\frac{1-\alpha}{h}} \rho_{\alpha+ht}\left(B_{i\ell} - X_\ell (ht)'\boldsymbol{\beta}\right)K(t)\,dt.$$

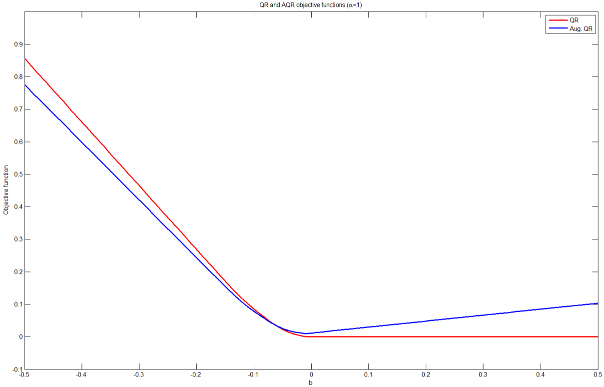
The augmented quantile regression estimator is

$$\widehat{\boldsymbol{\beta}}\left(\alpha|I\right) = \arg\min_{\boldsymbol{\beta}} \widehat{\mathcal{R}}\left(\boldsymbol{\beta}; \alpha, I\right), \quad \widehat{\boldsymbol{\beta}}\left(\alpha|I\right) = \begin{bmatrix} \widehat{\beta}^{(0)}\left(\alpha|I\right) \\ \widehat{\beta}^{(1)}\left(\alpha|I\right) \\ \vdots \\ \widehat{\beta}^{(s+1)}\left(\alpha|I\right) \end{bmatrix}$$

$$\widehat{V}\left(\alpha|x, I\right) = X' \left[\widehat{\beta}^{(0)}\left(\alpha|I\right) + \frac{\alpha}{I-1}\widehat{\beta}^{(1)}\left(\alpha|I\right)\right]$$

Boundary behavior

Smoothing gives a convex AQR function for $\alpha = 0, 1 \Rightarrow \widehat{\boldsymbol{\beta}}(0|I)$ and $\widehat{\boldsymbol{\beta}}(1|I)$ are well defined



Assumptions (QR case)

1. X in a compact set, $-\infty < X'\gamma\left(0|I\right) < X'\gamma\left(1|I\right) < \infty$, $\sup_{\alpha} X'\gamma^{(1)}\left(\alpha|I\right) < \infty$, $\inf_{\alpha} X'\gamma^{(1)}\left(\alpha|I\right) > 0$

   $\Rightarrow$ boundary bias for kernel estimation

2. $\alpha \in [0,1] \mapsto \gamma\left(\alpha|I\right)$ $(s+1)$th continuously differentiable $\Rightarrow$ $\beta\left(\alpha|I\right)$ $(s+2)$ cont. diff. except at $\alpha = 0$.

Theoretical results for quantile regression models

Theorem *Suppose the private value quantile regression specification is correct. Then if* $h \to 0$ *with* $\log^3 L / \left(Lh^2\right) = O(1)$

$$\sup_{(\alpha,x)\in[0,1]\times\mathcal{X}} \left| \widehat{V}(\alpha|x,I) - V(\alpha|x,I) \right| = O_{\mathbb{P}}\left(\left(\frac{\log L}{LIh} \right)^{1/2} + h^{s+1} \right).$$

It also holds that

$$\sup_{(\alpha,x)\in[0,1]\times\mathcal{X}} \left| \widehat{B}(\alpha|x,I) - B(\alpha|x,I) \right| = O_{\mathbb{P}}\left(\left(\frac{1}{LI} \right)^{1/2} + h^{s+2} \right).$$

Uniform consistency rate for private values

$$\sup_{(\alpha,x)\in[0,1]\times\mathcal{X}} \left|\widehat{V}(\alpha|x,I) - V(\alpha|x,I)\right| = O_{\mathbb{P}}\left(\left(\frac{\log L}{LIh}\right)^{1/2} + h^{s+1}\right)$$

- Rate given by $\widehat{\beta}^{(1)}(\alpha|I)$. No boundary bias at $\alpha = 0$ or 1.
- Optimal rate $= \left(\frac{\log L}{LI}\right)^{\frac{s+1}{2(s+1)+1}}$ = minimax optimal rate of G., Perrigne & Vuong (2000) with no covariate and for all $s > 0$. Achieved when
$$h \asymp \left(\frac{\log L}{LI}\right)^{\frac{1}{2(s+1)+1}}.$$
- CLT + MSE decomposition allowing for plug in bandwidth choice

Private value estimation

$$\widehat{A}_{i\ell} = \arg\min_{\alpha \in [0,1]} \left| B_{i\ell} - \widehat{B}\left(\alpha | x_\ell, I_\ell\right) \right|,$$
$$\widehat{V}_{i\ell} = \widehat{V}\left(\widehat{A}_{i\ell} | x_\ell, I_\ell\right).$$

Lemma *It holds that,*

$$\max_{\ell=1,\dots,L} \max_{i=1,\dots,I_\ell} \left| \widehat{V}_{i\ell} - V_{i\ell} \right| = O_{\mathbb{P}}\left(\left(\frac{\log L}{LIh} \right)^{1/2} + h^{s+1} \right).$$

$$\Rightarrow O_{\mathbb{P}}\left(\frac{\log L}{L} \right)^{\frac{s+1}{2(s+1)+1}} \text{ for optimal bandwidth choice}$$

Holds for all private values due to the absence of boundary bias

Sieve interactive specification

With D interactions and localized sieve as wavelets of cardinal B splines and under suitable bandwdith $\left(K = h^{-D}\right)$ and smoothness assumptions,

$$\sup_{(\alpha,x)\in[0,1]\times\mathcal{X}} \left|\widehat{V}\left(\alpha|x,I\right) - V\left(\alpha|x,I\right)\right| = O_{\mathbb{P}}\left(\left(\frac{\log L}{LIh^{D+1}}\right)^{1/2} + h^{s+1}\right)$$

under conditions which imposes $s > \frac{3}{2}\left(D-1\right)$ for the optimal $h = \left(L/\ln L\right)^{-1/(2s+D+3)}$

IMSE, MSE expansions and CLT

Simulation example

$L = 50$ and $I = 2$

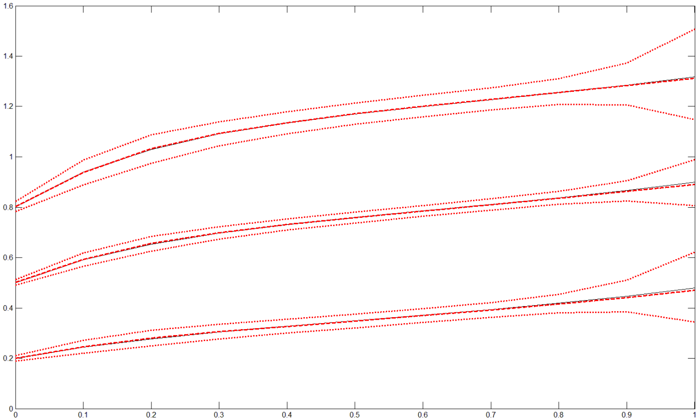
Second-order LP ($s + 1 = 2$), Epachnikov kernel, data-driven $\widehat{h}$ computed from a regression model with truncated exponential error

10,000 replications

$$V(\alpha|x) = \gamma_0(\alpha) + x_1 + \gamma_2(\alpha)\, x_2,$$
$$\gamma_0(\alpha) = -0.1 \times \log\left(1 - \frac{\alpha}{e}\right),$$
$$\gamma_2(\alpha) = 1 - \exp(-\alpha).$$

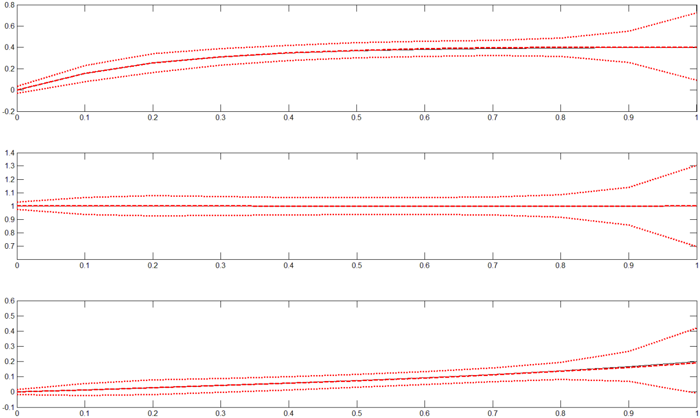
The covariates x_1 and x_2 are two independent uniform variables. x_2 inactive for small α

Quantiles



$x_1 = x_2 \in \{0.2, 0.5.0.8\}$

Slope coefficients



$$\gamma_0(\alpha) = -0.1 \times \log\left(1 - \frac{\alpha}{e}\right),\ \gamma_1(\alpha) = 1 \text{ and}$$
$$\gamma_2(\alpha) = 1 - \exp(-\alpha)$$

Extension to additive interdependent value

- I bidders with known identity from now on
- z_i: characteristic of ith bidder observed by all ("capacity" variable as distance to the project, labor force, cash flow,...) $z = (1, z_1, \ldots, z_I)'$ full-rank

The general additive specification

$$W_i\left(A;x,z\right)=W_0\left(x,z\right)+V_0\left(A_0;x\right)+\sum_{j=1}^{I}\pi_{ij}V_j\left(A_j;x,z_j\right),$$

$\pi_{ii}=1,\pi_{ij}\geq 0$

$V_j\left(\cdot;x,z_j\right)\uparrow,\ V_j\left(0;x,z_j\right)=0$, and

$$V_j\left(A_j;x,z_j\right)=v_j\left(A_j;z_j\right)+\int_0^{z_j}\frac{\partial V_j\left(A_j;x,t\right)}{\partial z_j}dt$$

$\Rightarrow V_j\left(A_j;x,z_j\right)\neq V_{1j}\left(A_j;x\right)+V_{2j}\left(A_j;x,z_j\right)$

$\Rightarrow$ force an interaction between A_j and z_j

A simple interdependent value specification

$$W_i = \gamma_0 (A_0) + \sum_{j=1}^{I} \pi_{ij} z_j \gamma_j (A_j)$$

- A_j: jth bidder private signal with a $U_{[0,1]}$ distribution
- $z_j \gamma_j (A_j)$: jth bidder "private" component of the ith bidder value W_i, $i = 1, \ldots, I$

  Weighted by π_{ij} in W_i
- $\gamma_0 (A_0)$: common component of the values W_i, $i = 1, \ldots, I$
  A_0: $U_{[0,1]}$ distribution
  Not identified without a completness assumption

Parameter of interest: slope coefficients $\gamma_1 (\cdot), \ldots, \gamma_I (\cdot)$

Assumption

1. *The signals $A_0, A_1, \ldots, A_I$ are affiliated with a conditional c.d.f which is bounded away from 0 over $[0,1]^{I+1}$. The signals are independent of z*
2. *The slope cofficients $\gamma_j(\cdot)$ are strictly increasing with $\gamma_j(0) = 0$ and*
$$\pi_{ii} = 1,\ \pi_{ij} \geq 0$$
3. *Each bidder plays a best-response strictly increasing and differentiable strategy $s_i(A_i; z)$ (Reny and Zamir, 2004)*

$$\Rightarrow s_i(A_i; z) = B_i(A_i; z)$$

Expected profit and best response condition

- Expected profit of a bid $B_i(a|z)$ given $A_i = \alpha$

$$\mathbb{E}\left[\left(W_i - B_i(a|z)\right)\mathbb{I}\left\{B_i(a|z) \geq \max_{j \neq i} B_j\right\} |A_i = \alpha, z\right]$$
$$= \overline{W}_i(a|\alpha, z) - B_i(a|z)\,\omega_i(a|\alpha, z)$$

where

$$\omega_i(a|\alpha, z) = \mathbb{E}\left[\mathbb{I}\left\{B_i(a|z) \geq \max_{j \neq i} B_j\right\} |A_i = \alpha, z\right]$$
$$= \mathbb{P}\left(B_i(a|z) \text{ wins} | A_i = \alpha, z\right)$$
$$\overline{W}_i(a|\alpha, z) = \mathbb{E}\left[W_i \mathbb{I}\left\{B_i(a|z) \geq \max_{j \neq i} B_j\right\} |A_i = \alpha, z\right]$$

- Identification issue: $\overline{W}_i(a|\alpha, z) \neq W_i$

$$\overline{W}_i\left(a|\alpha,z\right) = \mathbb{E}\left[\gamma_0\left(A_0\right)\mathbb{I}\left\{B_i\left(a|z\right) \geq \max_{j\neq i} B_j\right\}|A_i = \alpha, z\right] + \sum_{j=1}^{I}\pi_{ij}z_j\mathbb{E}\left[\gamma_j\left(A_j\right)\mathbb{I}\left\{B_i\left(a|z\right) \geq \max_{j\neq i} B_j\right\}|A_i = \alpha, z\right]$$

$$\Rightarrow \overline{W}_i\left(a|\alpha,z\right) = \overline{\gamma}_{i0}\left(a|\alpha,z\right) + \sum_{j=1}^{I}\pi_{ij}z_j\overline{\gamma}_{ij}\left(a|\alpha,z\right)$$

with

$$\overline{\gamma}_{ij}\left(a|\alpha,z\right) = \mathbb{E}\left[\gamma_j\left(A_j\right)\mathbb{I}\left\{B_i\left(a|z\right) \geq \max_{j\neq i} B_j\right\}|A_i = \alpha, z\right]$$

$$\overline{\gamma}_{ii}\left(a|\alpha,z\right) = \gamma_i\left(\alpha\right)\mathbb{P}\left(B_i\left(a|z\right) \geq \max_{j\neq i} B_j|A_i = \alpha, z\right) = \gamma_i\left(\alpha\right)\omega_i\left(a|\alpha,z\right)$$

Best response condition

$$\alpha = \arg\max_{\alpha} \left\{ \overline{W}_i \left(a|\alpha, z\right) - B_i \left(a|z\right) \omega_i \left(a|\alpha, z\right) \right\}$$

$$\begin{aligned} \text{FOC} \Rightarrow \frac{\partial \overline{W}_i}{\partial a} \left(\alpha|\alpha, z\right) \\ = B_i \left(\alpha|z\right) \frac{\partial \omega_i}{\partial a} \left(\alpha|\alpha, z\right) + B_i^{(1)} \left(\alpha|z\right) \omega_i \left(\alpha|\alpha, z\right) \end{aligned}$$

$\Leftrightarrow$

$$\mathbf{W}_i \left(\alpha; z\right) = B_i \left(\alpha|z\right) + \Omega_i \left(\alpha; z\right) B_i^{(1)} \left(\alpha|z\right) \text{ with I.C. } B_i \left(0|z\right) = 0$$

where

$$\Omega_i \left(\alpha; z\right) = \frac{1}{\frac{\partial \omega_i}{\partial a} \left(\alpha|\alpha, z\right)} \omega_i \left(\alpha|\alpha, z\right)$$

$$\mathbf{W}_i \left(\alpha; z\right) = \frac{1}{\frac{\partial \omega_i}{\partial a} \left(\alpha|\alpha, z\right)} \frac{\partial \overline{W}_i}{\partial a} \left(\alpha|\alpha, z\right)$$

Comparison with IPV

- **Symmetric IPV case**:

$$\text{PV Quantile } (\alpha|I) = B(\alpha) + \frac{\alpha}{I-1} B^{(1)}(\alpha)$$

- **Interdependent value case:**

$$\mathbf{W}_i(\alpha; z) = B_i(\alpha|z) + \Omega_i(\alpha; z) B_i^{(1)}(\alpha|z)$$

$$\text{where } \Omega_i(\alpha; z) = \frac{1}{\frac{\partial \omega_i}{\partial a}(\alpha|\alpha, z)} \omega_i(\alpha|\alpha, z)$$

is identified as

$$\omega_i(a|\alpha, z) = \mathbb{P}\left(B_i(a|z) \geq \max_{j \neq i} B_j | A_i = \alpha, z \right).$$

$$\Rightarrow \mathbf{W}_i(\alpha; z) = \frac{1}{\frac{\partial \omega_i}{\partial a}(\alpha|\alpha, z)} \frac{\partial \overline{W}_i}{\partial a}(\alpha|\alpha, z) \text{ is identified}$$

but again $\mathbf{W}_i(\alpha; z) \neq W_i$

Stability property for additive interdependent values (1)

$$\overline{W}_i\left(a|\alpha,z\right)=\overline{\gamma}_{i0}\left(a|\alpha,z\right)+\sum_{j=1}^{I}\pi_{ij}z_j\overline{\gamma}_{ij}\left(a|\alpha,z\right)$$

$$\text{with } \overline{\gamma}_{ii}\left(a|\alpha,z\right)=\gamma_i\left(\alpha\right)\omega_i\left(a|\alpha,z\right),$$

$$\mathbf{W}_i\left(\alpha;z\right)=\frac{1}{\frac{\partial\omega_i}{\partial a}\left(\alpha|\alpha,z\right)}\frac{\partial\overline{W}_i}{\partial a}\left(\alpha|\alpha,z\right)$$

$$\Rightarrow\mathbf{W}_i\left(\alpha;z\right)=\gamma_{i0}\left(\alpha;z\right)+\sum_{j=1}^{I}\pi_{ij}z_j\gamma_{ij}\left(\alpha;z\right)\text{ with}$$

$$\gamma_{ii}\left(\alpha;z\right)=\gamma_i\left(\alpha\right)\ \textbf{(invariance of }\gamma_i\left(\alpha\right)\textbf{)}$$

$$\gamma_{ij}\left(\alpha;z\right)=\frac{1}{\frac{\partial\omega_i}{\partial a}\left(\alpha|\alpha,z\right)}\frac{\partial\overline{\gamma}_{ij}}{\partial a}\left(\alpha|\alpha,z\right)$$

Stability property (2) and identification

$$\mathbf{W}_i(\alpha; z) = \gamma_{i0}(\alpha; z) + \sum_{j=1}^{I} \pi_{ij} z_j \gamma_{ij}(\alpha; z) \text{ with } \gamma_{ii}(\alpha; z) = \gamma_i(\alpha)$$

and solving the differential equation in $B_i(\alpha|xz)$

$$\mathbf{W}_i(\alpha; z) = B_i(\alpha|z) + \Omega_i(\alpha; z) B_i^{(1)}(\alpha|z), \quad B_i(0|z) = 0$$

gives the "random coefficient" quantile regression model

$$B_i(\alpha|z) = \beta_{i0}(\alpha; z) + \sum_{j=1}^{I} z_j \beta_{ij}(\alpha; z) \text{ with}$$

$$\pi_{ij}\gamma_{ij}(\alpha; z) = \beta_{ij}(\alpha; z) + \Omega_i(\alpha; z) \beta_{ij}^{(1)}(\alpha; z) \text{ (with } \pi_{i0} = 1)$$

$$\gamma_i(\alpha) = \beta_{ii}(\alpha; z) + \Omega_i(\alpha; z) \beta_{ii}^{(1)}(\alpha; z)$$

- $\Rightarrow \gamma_i(\alpha)$ is identified for all $i \geq 1$

- $\Rightarrow \pi_{ij}\gamma_{ij}(\alpha; z)$ is identified for all $j \geq 0$

But $\gamma_{ij}(\alpha; z)$ is also identified for all $j \geq 1$, since

$$\gamma_{ij}(\alpha; z) = \frac{1}{\frac{\partial \omega_i}{\partial a}(\alpha|\alpha, z)} \frac{\partial \overline{\gamma}_{ij}}{\partial a}(\alpha|\alpha, z) \text{ where}$$

$$\overline{\gamma}_{ij}(a|\alpha, z) = \mathbb{E}\left[\gamma_j(A_j)\,\mathbb{I}\left\{B_i(a|z) \geq \max_{k \neq i} B_k\right\}|A_i = \alpha, z\right]$$

$$\omega_i(\alpha|\alpha, z) = \mathbb{P}\left(B_i(a|z) \geq \max_{k \neq i} B_k|A_i = \alpha, z\right)$$

- $\Rightarrow \pi_{ij}$ is identified

Estimation method

- **Localised Augmented Quantile**: For each i, estimate $\beta_{ij}(\alpha; z)$ and $\beta_{ij}^{(1)}(\alpha; z)$ from the "random coefficient" quantile regressions

$$B_i(\alpha|z) = \beta_{i0}(\alpha; z) + \sum_{j=1}^{I} z_j \beta_{ij}(\alpha; z)$$

using a kernel weighted AQR with weights $K\left(\frac{z_\ell - z}{h}\right)$

- **Estimate**

$$\Omega_i\left(\alpha; z\right) = \frac{1}{\frac{\partial \omega_i}{\partial a}\left(\alpha | \alpha, z\right)} \omega_i\left(\alpha | \alpha, z\right) \text{ where}$$

$$\omega_i\left(\alpha | \alpha, z\right) = \mathbb{P}\left(B_i\left(a | z\right) \geq \max_{j \neq i} B_j | A_i = \alpha, z\right)$$

from $\widehat{B}_i\left(\alpha | z\right)$ and $\widehat{A}_{i\ell}$

- **Structural parameters:** compute

$$\widehat{\gamma}_{ii}\left(\alpha; z\right) = \widehat{\beta}_{ii}\left(\alpha; z\right) + \widehat{\Omega}_i\left(\alpha; z\right)\widehat{\beta}_{ii}^{(1)}\left(\alpha; z\right)$$

and average to improve convergence rate

$$\widehat{\gamma}_i\left(\alpha\right) = \frac{1}{L}\sum_{\ell=1}^{L}\widehat{\gamma}_{ii}\left(\alpha; z_\ell\right)$$

$\widehat{\pi}_{ij}$ from $\widehat{\gamma}_i\left(\alpha\right)$ and $\widehat{\gamma}_{ij}\left(\alpha; z_\ell\right)$

Final remarks

- Flexible quantile regression specifications including nonparametric components which can be estimated with fast rate
- A class of additive interactive specification ranging from quantile regression to the general nonparametric quantile model. Can be tested from the data
- Address the curse of dimensionality
- No boundary bias. Allows estimation of all private values
- Bandwidth choice
- Additive interdependent value specification
- Statistical extension: dimension reduction for conditional p.d.f