SEQUENTIAL AUCTIONS WITH AMBIGUITY: DECLINING PRICES, IDENTIFICATION AND REVENUE RANKINGS

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ABSTRACT. We study sealed-bid sequential first and second price auctions in which multiple units of a good are sold to bidders who have single-unit demand, independent private values and are ambiguity averse. We prove the existence of, and characterize, the unique symmetric equilibrium that satisfies sequential rationality appropriately defined in our context with ambiguity. We discuss the equilibrium implications with dynamic consistency as well as implications without it. Dynamic inconsistency can lead to history dependence which is absent in independent private-value sequential auctions without ambiguity. We provide sufficient conditions on the primitives that ensure dynamic consistency and show that under dynamic consistency prices are a supermartingale for both auction formats, thus providing an explanation for the 'declining price anomaly' observed in many sequential auctions. We also provide an identification result and an empirical strategy to estimate variables of interest, in the presence of ambiguity, from bidding data of sequential auctions. Finally, we provide a revenue ranking between various auction formats and show that sequential auctions dominate their static counterparts.

Keywords: sequential auctions, declining prices, ambiguity, dynamic consistency, identification
JEL classification: C57, C72, C73, D44, D81

1. INTRODUCTION

Payoff uncertainty exists in most economic environments where decisions are made by an agent or groups of agents. In dynamic strategic situations this uncertainty is more prominent as agents have to make inferences about the state of the world based on past observations of actions of others. Hence, how uncertainty is modeled can have profound influence on the outcomes. In this paper, we study sequential auctions where uncertainty is modeled as ambiguity and the bidders are ambiguity averse.

1.1. **Motivation.** The aim of this paper is fourfold. First, we define an appropriate equilibrium concept that captures sequential rationality and generalizes the monotone weak Perfect Bayesian Equilibrium (wPBE) in auctions without ambiguity. Belief updating under ambiguity, in general, is difficult to deal with: based on past observations it is not obvious how an ambiguous averse agent

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would form beliefs to find her optimal strategy. But in our setting, we show that it is particularly tractable. In addition, to the best of our knowledge, ours is the first attempt to bring ambiguity aversion into a sequential auction setting. Existing works on ambiguity aversion in auctions focus on the static setting.[1]

Second, we prove the existence of an equilibrium that has these features and check for uniqueness. Since belief updating can lead to may possibilities, the equilibrium, in theory, could be quite complex to characterize. Furthermore, issues related to existence of monotone equilibrium in sequential auction models with risk-aversion are well documented.[2] Despite these issues, our framework provides a relatively 'clean' characterization. An interesting feature of the equilibrium is that it can involve history dependence even in an IPV setting. This history dependence comes from belief updating and sequential rationality, and it is different from the history dependence in an interdependent and affiliated values setting. In the standard IPV setting, as studied in Milgrom and Weber (2000) (MW henceforth), bids in a round do not affect bids in the next.

Third, from a practical standpoint, we also wish to understand the implications of ambiguity and aversion to it on price sequences and revenues. Sequential auctions are often used for selling identical or similar goods. Auctions of cattle, condominiums, fish, flowers, fur, lobsters, mussels, paintings by the same artist, rare stamps, used cars, and wines are some examples. In addition, online auctions for identical goods that have different closing times can also be viewed as examples of sequential auctions. It is often the case that these sequential auctions occur and are concluded within a matter of hours. Given the nature of the environment, casual intuition suggests that prices in these auctions should remain, on average, constant due to a simple no-arbitrage condition. If prices are expected to be lower (higher) in later auctions then demand should shift to (away from) those rounds till prices are equalized in expectation. Indeed, as shown by Weber (1983) and MW, in a private values model of sequential auctions, prices are a martingale. Therefore, in the absence of frictions, the law of one price holds.

However, as first noted by Ashenfelter (1989), prices in sequential auctions for identical items (rare wines) on average seem to show a declining trend. Since then this phenomenon was substantiated by other authors for a wide variety of goods and auction mechanisms.[3] Due to a lack of a satisfactory explanation for such price trends it has been dubbed the 'declining price anomaly' or the 'afternoon effect' as prices seem to be lower later in the day. We show that our model with ambiguity averse bidders can generate prices that are a supermartingale for both sequential Second Price Auctions (sSPAs) and sequential First Price Auctions (sFPAs). For the former in general and

[1] See Lo (1998) and Levin and Ozdenoren (2004).

[2] See McAfee and Vincent (1993).

[3] See Ashenfelter and Genovese (1992) (condominiums) McAfee and Vincent (1993) (wines), Beggs and Graddy (1997) (art), Van den Berg et al. (2001) (roses), Lambson and Thurston (2006) (fur) and Snir (2006) (computers) among others.

for the latter under some standard conditions. The reason our model can generate declining prices is quite intuitive. In the presence of ambiguity, bidders who are averse to it tend to over-estimate competition, both in the current round and in the future. Over-estimating competition in the current round means bidding more aggressively than they would in the absence of ambiguity in the current round of a sequential auction. Furthermore, over-estimating competition in the future lowers the *option-value*, from the point of view of a bidder, of moving to the next round conditional on losing, an effect that also makes bidders bid more aggressively in the current round thereby causing prices to be higher in the current round than the next.

Finally, our analysis of sequential auctions is motivated by empirical estimations in the presence of ambiguity. Starting from Paarsch (1992) and following the seminal contribution of Guerre et al. (2000) (GPV henceforth) there is a rich literature on estimation of variables of interest from auction data.[4] The clever non-parametric estimation strategy developed in the latter rests, like much of the auction theory literature, on the assumption of a common-prior. Without this assumption, or more precisely, in the presence of ambiguity, the econometrician may not be able correctly estimate the variables of interest, such as bidder valuations, from the bidding data of single-unit auctions. In fact, based on bidding data of single-unit auctions the econometrician may not even be able to *identify* whether indeed there is ambiguity present from the point of view of the bidder.

Our results show that considering data from dynamic auctions may provide identification techniques to ascertain the presence of ambiguity and correctly estimate the variables of interest in case ambiguity is present. These techniques, due to their dynamic nature, are not available if we consider data from single-unit auctions. Furthermore, our methodology also provides a tool with which one can test our theory using auction data.

1.2. **Model and Results.** We study a model where K units of a good are sold to $N \geq K+1$ ambiguity averse bidders who have single unit demand and private values. We study two forms of ambiguity: ambiguity regarding the valuation distribution and ambiguity regarding the number of bidders. In the main body of the paper we study the former where the true distribution of values is not known, but bidders are aware of the set of possible distributions. Bidders are *maxmin* expected utility maximizers à la Gilboa et al. (1989). We prove the existence and uniqueness of a symmetric equilibrium that satisfies sequential rationality suitably adapted to our setting. In equilibrium, in each round, bidders use the strongest conditional distribution from the set of all conditional distributions they can possibly compete against to calculate their payoffs. Crucially, the strongest conditional distribution can depend on previous round prices. We show that for sFPAs this means that bids can be history-dependent unlike the case without ambiguity. We also provide sufficient

[4]See Athey and Haile (2007) and Hickman et al. (2012) for surveys.

conditions under which bids in sFPAs are history-independent. For sSPAs, bids are always history-independent. This novel divergence between sFPAs and sSPAs comes from the fact the final round bids in sSPAs are always history-independent, and this feeds back into previous rounds, whereas this is not always the case for sFPAs.

For sFPAs we show that for any set of distributions, if prices are high enough in a round, then they will be expected to decline in the next. Thus, prices are not a martingale in the presence of ambiguity aversion. Furthermore, if bidder behavior is *Dynamically Consistent* (which we discuss below), which it will be if the set of priors is *Rectangular*, prices are shown to be a supermartingale. For sSPAs prices are always a supermartingale. Our findings indicate that the martingale property of prices in the classical setting is not robust in the presence of multiple priors and ambiguity aversion. Furthermore, prices are more likely to decline, which is an implication of the uncertainty premium from over-estimation of competition: bidders are more pessimistic than the actual distribution of values with ambiguity aversion; that is, even if the actual price path declines overtime, bidders do not want to take the "risk of waiting" because they evaluate their chances of winning from a distribution that gives them the lowest probability of winning.

With ambiguity it is no surprise that the revenue equivalence theorem no longer holds. We show that the revenue from sFPAs dominates the revenue from sSPA under the same conditions that guarantee DC in sFPAs. In addition, we show that sequential auctions in which prices are a supermartingale revenue dominate a commonly used static multi-unit auction: the uniform price auction. Also, virtually all of our results remain the same when we consider ambiguity regarding number of bidders.

Finally, using our model and results we suggest an empirical strategy and methodology to identify and estimate variables of interest and test model predictions. The identification strategy essentially exploits price differences, or more precisely, bidder specific bid differences between consecutive rounds of a sequential auctions to identify (a) presence of ambiguity, (b) the distribution of variables of interest such as valuations and (c) the distribution bidders use to evaluate their payoffs. By recovering the 'true' distribution *and* the distribution bidder's use, one can crudely measure the 'amount' of ambiguity present in the environment. Our identification strategy is an extension of the seminal methodology of GPV to a sequential auction setting with ambiguity. The crucial aspect of our procedure is the use of dynamic bidding data, in particular, declining prices, to recover more than one 'primitive' distributions.

1.3. **Related Literature.** Our paper is related to four strands of literature. The first is auctions with ambiguity. Most of the papers in this literature study the effects of ambiguity in a single-unit auction.[5] In particular, many papers analyze single-unit sealed-bid auctions, thus dynamic

[5] Salo and Weber (1995) show that over-bidding in single-unit auction experiments may be a consequence of ambiguity aversion. Lo (1998) studies single unit FPA and SPA under ambiguity about valuation distributions and MMEU.

considerations were not part of their analysis.[6] To the best of our knowledge the current paper is the first to study sequential auctions where multiple units are sold in an environment with ambiguity. Some other papers propose optimal mechanisms for selling a single good to ambiguity averse bidders. Bose et al. (2006) show that full insurance auctions can increase seller revenue and that first and second price may not be optimal. While we do not undertake a design approach and we have multiple units, we do show that sFPAs and sSPAs may perform better than a standard static multi-unit auction. Bodoh-Creed (2012) proves a payoff equivalence result for mechanisms with a single-unit and ambiguity averse bidders and shows that revenue ranking between FPA and SPA may be sensitive to the kind of ambiguity. In the current paper, again, in a multi-unit and dynamic setting, we provide sufficient conditions under which sFPA revenue dominates sSPA.

Second, our paper relates to the literature on dynamic games with ambiguity. In a dynamic environment issues surrounding how agents update and *Dynamic Consistency* (DC) are now familiar.[7] Our contribution to this literature is as follows. Our approach shows how a practical application of ambiguity can be carried out by appropriately defining an equilibrium concept that generalizes sequential rationality to a multiple-prior setting. In particular, we focus on equilibria that are monotone in valuations, which allows us to restrict attention to a simple belief updating rule. This in turn leads to some practical implications of DC or lack thereof in a dynamic auction setting.

Loosely, DC requires a decision maker prefer an act a to b using updated preferences of a later period if and only if she prefers a to b evaluated using preferences of an earlier period. While some papers choose an axiomatic approach to guarantee dynamic consistency, some others argue that dynamic consistency might be too strong of a requirement.[8] To wit, we first prove the existence and uniqueness of an equilibrium under general conditions that may involve dynamic inconsistency. Then, we provide sufficient conditions, similar to those made in previous studies, to guarantee dynamic consistency. An implication of dynamic consistency is that prices are a supermartingale in sFPAs, whereas when there is dynamic inconsistency, prices may increase. In addition, we also show that dynamic consistency in a sequential auction model with ambiguity can be guaranteed by semi-lattice structure (under an appropriate partial order) of the set of priors which to the best of our knowledge is not a known result.

Levin and Ozdenoren (2004) study FPA and SPA when there is ambiguity about the number of bidders. Chen et al. (2007) study FPA and SPA with IPV and ambiguity experimentally.

[6] Bose and Daripa (2009), in a design setting, show that a dutch auction can perform better than static auctions when a single unit is sold.

[7] See Gilboa and Schmeidler (1993) for some updating rules. In recent paper Gul and Pesendorfer (2017) introduce a new updating rule. See Al-Najjar and Weinstein (2009) for a critical assessment of the ambiguity literature.

[8] Paper such as Epstein and Schneider (2003), Wang (2003) and Hayashi (2005) provide sufficient conditions for DC. Ozdenoren and Peck (2008) argue that DC might be too strong of a requirement and behavior that may seem like dynamic inconsistency can be thought of as consistency in a sub-game perfect sense.

Third, our result on prices provides an explanation for the declining price anomaly. There have been a few other explanations for this empirical observation. One set of explanations suggest that specifics of an auction mechanism can account for the anomaly.[9] Another suggests that good specific features can account for the price trend.[10] While interesting and important in furthering our understanding of certain auction settings, the above explanations are specific to those settings. Declining prices are quite general and has been observed across a wide variety of auctions and goods.

Another set of explanations, which is more general in some sense than the ones mentioned above, relies on alternate preference structure. McAfee and Vincent (1993) show that a model with risk-averse bidders can generate declining prices.[11] However, the model relies on non-decreasing absolute risk aversion in wealth, a non-standard assumption, to guarantee existence of a monotone equilibrium in pure strategies. This also suggests that a more conventional form of risk aversion, such as DARA, would lead to inefficient allocations. More recently, Mezzetti (2011) shows a different kind of risk aversion, called "aversion to price risk", which is tantamount to an additively separable utility function with a convex cost function, can generate declining prices in sequential auctions. Hu and Zou (2015) further generalize the result in Mezzetti (2011) by considering a more general utility function. Finally, Rosato (2017) shows that expectations-based reference-dependent preferences and loss aversion can also account for such price trends in a two round auction. Our approach sits within this set and complements the above studies by considering a preference structure involving ambiguity which has not been explored in a sequential auction setting.[12]

Finally, due to our identification result and empirical strategy, our paper is also related to the empirical auction literature. However, since we do not undertake an actual empirical exercise we do not claim contributions to this field beyond the strategy we propose. Within the empirical auctions

[9] Ashenfelter (1989) suggested that the 'buyer's option' wherein the winner of the first auction has the possibility of buying the remaining units at the same price she won the first unit may cause bidders to bid more aggressively initially. Ginsburgh (1998) and Menezes and Monteiro (1997) suggest that absentee bidding and participation fees, respectively, can account for the anomaly. Lebrun (2017) studies sequential second price auctions and shows that some asymmetric equilibria can lead to declining prices. Jeitschko (1999) considers supply side uncertainty where number of units for sale is unknown to bidders. Using a design approach Branco (1996) shows that an optimal mechanism can be implemented using a sequential auction format where reserve prices depend on the round number. The author shows that the expected price decreases over time.

[10] For example some studies have found that some degree of heterogeneity between the objects for sale can lead to declining prices. See Bernhardt and Scoones (1994), Engelbrecht-Wiggans (1994), Gale and Hausch (1994) and Kittsteiner et al. (2004).

[11] The intuition is that prices in later rounds are more variable than the current round and hence bidders require a risk premium for later rounds which manifests as lower bids and hence declining prices.

[12] We depart from these studies with respect to identification and revenue results. Furthermore, Athey and Haile (2007) discuss the difficulties in identification with risk aversion. Our approach suggests that these issues can be overcome by considering a different kind of uncertainty.

literature our paper is related to a small set of papers that study dynamic auctions. Most empirical work on auctions considers single-unit auctions or static multi-unit auctions.[13] Exceptions are Jofre-Bonet and Pesendorfer (2003), Donald et al. (2006) and Donna and Espín-Sánchez (2018) who study dynamic auctions using models with capacity constraints, multi-unit demand and complementarities respectively. They develop techniques, similar to GPV and Athey and Haile (2002), to recover valuation distributions. All papers assume common-priors which is different from our setting.

In addition our paper offers an estimation strategy to recover variables of interest from auction data in the presence of ambiguity. Most papers that have a similar goal do so in an experimental setting since recovering preference parameters (from data) in the presence of ambiguity in static settings is not feasible.[14] A notable exception is Aryal et al. (2018) who identify and estimate valuation distribution in static FPAs in the presence of ambiguity using variation in the number of bidders. To the best of our knowledge this is the only other paper to offer an identification strategy under ambiguity. Our identification approach in contrast is to use dynamic bidding data from sequential auctions to recover unobservables in the presence of ambiguity.

1.4. **Roadmap.** The rest of the paper is organized as follows. In the next section we begin with an example that explores the major themes of the paper using a parametrized distribution space. Section 3 lays out the benchmark model with ambiguity averse bidders. In Section 4 we study sFPAs, where we define an equilibrium concept for our setting, prove the existence and uniqueness of a symmetric equilibrium, show that prices are not a martingale and then provide sufficient conditions under which prices are a supermartingale. In Section 5 we provide an identification result and empirical strategy in the presence of ambiguity to recover the distributions of valuations. In Section 6 we analyze sSPAs. In Section 7 we compares sFPAs and sSPAs in terms of revenue and also compare their revenue to a static multi-unit auction. Section 8 provides some concluding remarks. While some shorter proofs of results are presented in the main body of the paper, all other proofs can be found in Appendix A. Finally, in Appendix C we use an alternate formulation of ambiguity, in terms of number of bidders, and show that results regarding prices and revenue remain the same as the benchmark model.

2. An Illustrative Example

To set ideas, we introduce the following example where some major themes of the paper, specifically, price-paths, revenue rankings and identification are explored.[15] Suppose three bidders are

[13]See Hortaçsu and McAdams (2018) for a survey on multi-unit auctions.

[14]See Chen et al. (2007) for an example of testing for ambiguity in laboratory setting using auctions.

[15]In this example we are glossing over some important aspects of the analysis. Namely, an appropriate definition of an equilibrium in this game and proving its existence, both of which require careful and subtle analysis. Interested

competing for two units of a good that are sold in two sequential auctions. Each bidder has single unit demand characterized by a valuation $v \in [0,1]$. Bidders draw that values independently according to a bid distribution $F(v)$ that is equal to v^s where $s \in [\underline{s}, 1]$ with $0 < \underline{s} < 1$.[16] While bidders know that the distribution function is v^s, they do not know s.[17] Bidders are ambiguity averse and are modeled as MMEU maximizers.

Sequential FPAs: Suppose the units are sold in sequential FPAs. Further, suppose an equilibrium exists that is monotone in valuations. That is bidders use strictly increasing bidding functions $\beta_1, \beta_2 : [0,1] \to \mathbb{R}_+$. Then, after observing the first round winning bid bidders will know the valuation of the winner of the first round, say y_1, and hence the upper bound of the valuation of the remaining bidders. Suppose a bidder bids $b \leq \beta_2(y_1)$ in the second auction. The payoff from such a bid in the second round to an ambiguity averse bidder is given by

$$\Pi_2(v,b,y_1) = \min_{s \in [\underline{s},1]} (v-b) \left(\frac{\beta_2^{-1}(b)}{y_1} \right)^s = (v-b) \left(\frac{\beta_2^{-1}(b)}{y_1} \right)$$

Maximizing with respect to b, we get the familiar optimal response of $\beta_2(v) = v/2$. As is the case in the standard expected utility environment, round one prices do not affect bidding behavior in the second round.[18] Given bidding in the second round, a bidder's payoff in the first round as a function of a bid $b \geq \beta_1(v)$ is

$$\begin{aligned}\Pi_1(v,b) &= \min_{s \in [\underline{s},1]} (v-b) \left(\beta_1^{-1}(b)^s\right)^2 + 2(1-\beta_1^{-1}(b)^s)\beta_1^{-1}(b)^s \left(\frac{v}{\beta_1^{-1}(b)} \right)^s (v - \beta_2(v)) \\ &= \min_{s \in [\underline{s},1]} (v-b)\beta_1^{-1}(b)^{2s} + (1-\beta_1^{-1}(b)^s)v^{s+1}.\end{aligned}$$

A bidder's problem is to maximize the above payoff with respect to b. Suppose the above is minimized at some s. Then optimizing the above and using the fact that in equilibrium $b = \beta_1(v)$, the equilibrium bidding function is

$$\beta_1(v) = \frac{s}{2s+1} v.$$

Substituting back, the equilibrium payoff is

$$\Pi_1^* = \left(1 - \frac{s}{2s+1} v\right) v^{s+1}.$$

readers should turn to the model section and beyond for the analysis. In this example the equilibrium definition and existence are relatively straightforward and we instead discuss the practical aspects of our results.

[16] The upper limit can be replaced with an arbitrary $\overline{s} < \infty$.

[17] Technically, the set of priors in this example is not convex. However, we can take the convex hull of the set of priors and the equilibrium will remain the same. We discuss this point in detail in Appendix A.3.

[18] Later on we show that bids may be history dependent. We also give sufficient conditions under which bids *are* history independent.

Since the term inside the parentheses is decreasing in s and v^{s+1} is also decreasing in s since $v < 1$, the payoff is minimized at $s = 1$. Thus the equilibrium bidding functions are given by

$$\beta_1(v) = \frac{v}{3}; \;\; \beta_2(v) = \frac{v}{2}.$$

Now, suppose the true state of the world is given by $s_0 < 1$. Given price p_1 in the first auction the expected price in the second auction is

$$\mathbb{E}\left[P_2 \middle| p_1\right] = \int_0^{3p_1} \Bigg(\Bigg(\frac{x}{2} \frac{dx^{2s_0}}{(3p_1)^{2s_0}} = \frac{3s_0}{2s_0+1} p_1 < p_1.$$

for all $s_0 \in [\underline{s}, 1)$. Thus the equilibrium generates a downward price trend. Furthermore, this also implies that $\mathbb{E}[P_1] > \mathbb{E}[P_2]$.

Note that the equilibrium generates a downward price trend irrespective of $\underline{s}$ as long as s_0 is bounded away from 1. Thus declining prices do not depend on the ‘level’ of ambiguity. The level of ambiguity can affect the degree to which prices may decline.

$$p_1 - \mathbb{E}\left[P_2 \middle| p_1\right] = p_1 \left(\frac{1 - s_0}{2s_0 + 1} \right) \Bigg($$

Thus higher the true state of the world s_0, lower will be the price decline. For example, for a 5% price decline $s_0 \approx 0.87$ and for a 1% price decline $s_0 \approx 0.97$.

Sequential SPAs: Suppose the units are sold in sSPAs. In this case bidders have a weakly dominant strategy in the second round which is to bid their valuation, much like the static auction case. Given this we can follow an identical procedure as the sFPAs to derive the optimal strategy in the first round. The equilibrium bidding functions in sSPAs are given by

$$\beta_1^{II}(v) = \frac{v}{2}; \;\; \beta_2^{II}(v) = v.$$

Given the bidding functions and given a price p_1 in the first round of the auction, the expected price in the second round is given by

$$\mathbb{E}\left[P_2 \middle| p_1\right] = \int_0^{2p_1} \frac{x}{2} \frac{dx^{s_0}}{(2p_1)^{s_0}} = \frac{2s_0}{s_0+1} p_1 < p_1.$$

for all $s_0 \in [\underline{s}, 1)$. In deriving the expected price in the second round we explicitly use the fact that price will be given by the second highest bid in the final round. Note that the final inequality implies that the equilibrium generates a downward price trend. And as before, $\mathbb{E}[P_1] > \mathbb{E}[P_2]$.

Revenue Ranking: Let β_k^I represent the bidding functions in sFPAs. First, note from the closed form of the bidding functions that $\beta_1^{II}(v) = \beta_2^I(v)$. Now, let $V_l^{(m)}$ be the random variable denoting the l-th highest draw out of m draws from the distribution v^{s_0}. Then,

$$\mathbb{E}\left[P_1^I\right] > \mathbb{E}\left[P_2^I\right] = \mathbb{E}\left[\beta_2^I\left(V_2^{(3)}\right)\right] = \mathbb{E}\left[\beta_1^{II}\left(V_2^{(3)}\right)\right] = \mathbb{E}\left[P_1^{II}\right] > \mathbb{E}\left[P_2^{II}\right] \tag{1}$$

where the first and last inequality follows from the fact that prices are a supermartingale in both auction formats. The final inequality follows from the fact the price in SPA is determined by the bid of the bidder with the second highest valuation in a monotone equilibrium. The above equation implies that the revenue in sFPAs is greater than sSPAs as long as $s_0 < 1$.

Finally, we compare the revenue from sequential auctions with a uniform price auction (UPA) where the selling price is given by the highest losing bid. In this auction all bidders simultaneously submit bids and the two highest bidders win the two units and pay the bid of the third bidder. In this format, much like a single-unit SPA or the final round of sSPAs, it is weakly dominant to bid one's valuation. Thus the revenue generated in this formant is two times the expected valuation of the lowest valued bidder, $\mathbb{E}[V_3^{(3)}]$. Note that $\mathbb{E}[V_3^{(3)}] = E[P_2^{II}]$. And given the fact that sSPAs generate prices that are a supermartingale, sSPAs generate more revenue than UPA. Therefore, using equation (1) we get that the revenue from sFPAs> sSPAs>UPA.

Identification and Estimation: Consider sFPAs again. Let the true state of the world be given by s_0. Suppose $s_0 \in [\underline{s}, \bar{s}]$. That is the econometrician is not aware of the the exact set of priors. Carrying out an identical exercise as above we get that the equilibrium bidding functions are given by

$$\beta_1(v) = \frac{2\bar{s}^2 v}{(2\bar{s}+1)(\bar{s}+1)} = \frac{v}{\eta(\bar{s})};\ \ \beta_2(v) = \frac{\bar{s}v}{\bar{s}+1}.$$

Again let the true state of the world be given by $s_0 < \bar{s}$. Given price p_1 in the first auction the expected price in the second auction is

$$\mathbb{E}\left[P_2 | p_1\right] = \int_0^{\eta(\bar{s})p_1} \frac{\bar{s}x}{\bar{s}+1} \frac{dx^{2s_0}}{(\eta(\bar{s})p_1)^{2s_0}} = \frac{s_0}{2s_0+1} \frac{2\bar{s}+1}{\bar{s}} p_1 < p_1.$$

Note that from the observable winning prices in both rounds, we have

$$\mathbb{E}[P_1] = \frac{1}{\eta(\bar{s})} \int_0^1 v dv^{3s_0} = \frac{3s_0}{(3s_0+1)\eta(\bar{s})}$$

and

$$\mathbb{E}[P_1 - P_2] = \left(1 - \frac{s_0}{2s_0+1} \frac{2\bar{s}+1}{\bar{s}}\right) \mathbb{E}[P_1].$$

This system has a unique solution: $(s_0, \bar{s})$. To see this, define

$$A(x) = \frac{3x}{3x+1},\ B(y) = \frac{2y^2}{(2y+1)(y+1)},\ C(x) = \frac{x}{2x+1},\ D(y) = \frac{2y+1}{y}.$$

Then the system can be rewritten as

$$A(s_0)B(\bar{s}) = c_1,$$
$$C(s_0)D(\bar{s}) = c_2,$$

where $c_1 = \mathbb{E}[P_1] > 0$ is the expected winning price for the first auction and $c_2 = 1 - \mathbb{E}[P_1 - P_2]/\mathbb{E}[P_1] > 0$ is the percentage of price drop from the first to the second auction.

Note that $A' > 0$, $B' > 0$, $C' > 0$ and $D' < 0$. Therefore, the first equation gives a one-to-one mapping from s_0 to $\bar{s}$ with a negative slope and the second equation is a one-to-one mapping from s_0 to $\bar{s}$ with a positive slope. Moreover, if $s_0 \simeq 0$, the solution $\bar{s}$ to the first equation is very large, whereas the solution $\bar{s}$ to the second equation is close to zero. Therefore, there is a unique solution $(s_0, \bar{s})$ to the system; furthermore both s_0 and $\bar{s}$ are positive.[19]

3. Model

$K \geq 2$ units of a good are sold one at a time, sequentially, in sealed bid auctions with a reserve price of zero.[20] We study both sequential first-price (sFPA) and sequential second-price auctions (sSPA). There are $N \geq K+1$ bidders competing for the units. A bidder i values each unit at $v_i \in [\underline{v}, \overline{v}]$ and has single-unit demand. A bidder draws her valuation from a common, atom-less and differentiable distribution F with density f. The distribution F belongs to a compact and convex set of distributions Δ where $\Delta \subset \{G \in C_1 | G(\underline{v}) = 0, G(\overline{v}) = 1, G' > 0\}$.[21] Bidders do not know the distribution the valuations are drawn from, however, they are aware of Δ.

To model bidders sensitivity to uncertainty we will follow the multiple-priors approach of Gilboa et al. (1989). Under this framework bidders' aversion to ambiguity manifests as a *maxmin expected utility* (MMEU) that determines their preferences. Thus bidders are assumed to maximize the minimum utility they expect to obtain. In this framework, a bidder's payoff as a function her bid in any around is the minimum payoff she expects to get where the minimum is taken over Δ after conditioning on all available information. Conditioning takes the form of a prior-by-prior Bayesian updating. We provide more details in the next section.

The timing is as follows. In each auction bidders submit sealed bids. The bidder who submits the highest bid wins the auction and pays her bid in the case of FPA and the second highest bid in the case of SPA. The winning bidder leaves the auction and the winning bid is announced. The remaining bidders compete in the next auction using the same procedure. Let p_t be winning bid in

[19] Since we know the parametric form of the distribution here, the estimation technique presented in this example is a little different from the general technique we describe in section 5. The point here is to show how the dynamic nature of the auction can be used to identify parameters of interest.

[20] This assumption is not essential and only made for simplicity.

[21] Restriction to strictly positive densities is to ensure the existence of a strictly monotone equilibrium which makes the analysis tractable.

round $t < k$. A history in round k is made up of a sequence of winning bids $\tilde{p}_{k-1} = (p_1, \dots, p_{k-1})$. Ties are broken via a fair coin-flip.

Let $b_{i,k}$ be bidder i's bid in auction k. If bidder i wins auction k then $b_{i,k+l} = 0, l \in \{1, 2 \dots, K - k\}$. A strategy for a player i given by $\beta_i = \{\beta_{i,1}, \dots, \beta_{i,K}\}$ is a sequence of bid functions where $\beta_{i,k}(v_i, \tilde{p}_{k-1})$ is the bid bidder i places in auction k given the history of winning bids. Let $\boldsymbol{\beta}_k = \{\beta_{i,k}\}_{i=1}^N$,$\boldsymbol{\beta} = \{\beta_{i,1}, \beta_{i,2}, \dots, \beta_{i,K}\}_{i=1}^N$ and $\boldsymbol{\beta}_k = \{\beta_{i,k}\}_{i=1}^N$.

We impose the following regularity condition on the set of priors. Let Δ_y be the set of conditional distributions derived from the set of priors Δ. That is

$$\Delta_y = \left\{ F_y(\cdot) = \frac{F(\cdot)}{F(y)} : [\underline{v}, y] \to [0,1] \middle| F \in \Delta \right\} \Bigg(\tag{2}$$

In addition we need the following definitions which are slight variations of the definitions in Topkis (2011).[22] Let Δ be partially ordered (p.o.) by $\geq_{FOSD}$. That is $F_1 \geq_{FOSD} F_2$ if and only if $F_1(x) \leq F_2(x)$ for all $x \in [\underline{v}, \bar{v}]$. For any $F_1, F_2 \in \Delta$ let $F_1 \vee F_2$ represent the *join* of F_1 and F_2.[23] That is $F_1 \vee F_2 \geq_{FOSD} F_1, F_2$. If every pair of distributions in Δ has a join in Δ then X is a *semi-lattice*. A semi-lattice Δ is *complete* if every nonempty subset S of Δ has a join $\vee S$ in X.

Assumption 3.1. For each $y \in [\underline{v}, \bar{v}], \Delta_y$ is a compact and complete semi-lattice.[24]

Assumption 3.1 implies that the lower envelope of Δ_y is always contained in Δ_y. The latter condition will allow us to obtain an equilibrium in closed form.[25] We conjecture that one can prove the existence of an equilibrium under weaker conditions, however, one may not be able to get a closed form of the equilibrium as we do under this assumption.

3.1. **Dynamic Consistency.** Before stating and discussing the equilibrium we discuss the issue of dynamic consistency. This follows our discussion on this topic in the introduction. In dynamic decision problems and games DC is an important property since it allows the use of tools such backward induction. While it can be relatively easily established in EU models, DC is not so easy to come by in dynamic environments with ambiguity.[26] To wit, Epstein and Schneider (2003) show that dynamic consistency can be insured by a sufficient condition on the set of priors.

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[22]For applications see Milgrom and Shannon (1994) and Reny (2011).

[23]One can find the join here by finding the point-wise lower envelope of F_1 and F_2.

[24]Since the distribution function is over (a subset of) $\mathbb{R}_+$, FOSD can lead to a lattice structure. See Müller and Scarsini (2006).

[25]This assumption holds trivially in the case without ambiguity, i.e., Δ is a singleton set. Note that the set Δ in Example 2 satisfies this assumption. Furthermore, since Δ_y is a space of c.d.fs the lower envelope is also a c.d.f.

[26]In dynamic models where decision makes are ambiguity averse, failures of dynamic consistency are not too difficult to find. As shown by Ghirardato (2002) and Siniscalchi (2011) dynamic consistency, in many settings, is equivalent to Savage's *sure-thing principle*, violations of which are abundant in theoretical as well as experimental literature. See Loomes and Sugden (1986), Ozdenoren and Peck (2008) and Esponda and Vespa (2017).

However there are at least two reasons we do not assume DC from the outset. First, one of the goals of the current paper is to prove the existence of an equilibrium in the most general setting possible for MMEU bidders with private values. Thus, our approach is to solve for an equilibrium with a general set of priors and a solution concept that generalizes sequential rationality under ambiguity. Second, another goal of the paper is to understand the practical, and empirical, importance of DC. Thus, later on when we do impose the condition used by Epstein and Schneider (2003) to ensure DC, we will be able to illustrate exactly where DC bites.

4. First Price Auctions

4.1. Equilibrium. Our equilibrium concept is essentially a generalization of sequential rationality to a sequential auction environment with ambiguity. To see this, let $\Pi_K(v,b,\boldsymbol{\beta}_{-i},\tilde{p}_{K-1},F(\cdot|\tilde{p}_{K-1}))$ be the expected payoff to bidder i with valuation v_i from bidding b in round K given she has not won a unit yet and the bidder thinks the valuation of other (remaining) bidders is governed by some conditional distribution $F(\cdot|\tilde{p}_{K-1})$.[27] Let $\Delta_{\tilde{p}_{K-1}}$ be the set of all conditional distributions of valuations, found using some update rule, given the price sequence $\tilde{p}_{K-1}$. Given this payoff function, we can find the optimal response of bidder i to $\beta_{-i,K}$ as a function of her valuation and the price sequence that maximizes the minimum payoff she recieves where the minimum is taken over all distributions in $\Delta_{\tilde{p}_{K-1}}$. A strategy profile $\boldsymbol{\beta}_K$ represents an 'equilibrium' in the final round of the auction if and only if for all i,

$$\beta_{i,K} \in \arg\max_{b} \min_{F(\cdot|\tilde{p}_{K-1})\in\Delta_{\tilde{p}_{K-1}}} \Pi_K(v,b,\boldsymbol{\beta}_{-i},\tilde{p}_{K-1},F(\cdot|\tilde{p}_{K-1})$$

Suppose an equilibrium exists in the final round of the auction. Then, we define a payoff function of round $K, \Pi^*_K(v,\beta_{i,K},\boldsymbol{\beta}_{-i},\tilde{p}_{K-1})$, that maps from valuations and price sequences to reals given the strategy profile $\boldsymbol{\beta}_{-i}$ and the bidder's optimal bid (function) in the final round β_K. Fixing this function, we can move to round $K-1$ and define a *continuation* payoff function given by $\Pi_{K-1}(v,b,\beta_{i,K},\boldsymbol{\beta}_{-i},\tilde{p}_{K-2},F(\cdot|\tilde{p}_{K-2}))$ which is essentially a sum of the round $K-1$ payoff for bidder i conditional on winning with a bid b and the payoff from the final round conditional on losing round $K-1$ with bid b. That is

$$\begin{aligned}\Pi_{K-1}(v,b,\beta_{i,K},\boldsymbol{\beta}_{-i},F(\cdot|\tilde{p}_{K-2})) = &\left(\text{ Pr of winning with } b|F(\cdot|\tilde{p}_{K-2})\right)(v-b)\\ &+\mathbb{E}_{F(\cdot|\tilde{p}_{K-2})}\left[\Pi^*_K\left(v,\beta_{i,K},\boldsymbol{\beta}_{-i},\tilde{p}_{K-2},p_{K-1}\right)\middle| p_{K-1}>b\right],\end{aligned}$$

where the expectation in the second term is taken over prices in round $K-1$ higher than b using the distribution $F(\cdot|\tilde{p}_{K-2})$. Again, we can define payoff function that calculates the minimum

[27]The the third argument in the above payoff function is the *entire* strategy profile of other players since what a player learns from the price sequences about other players valuations depends on the strategies followed by other players in previous rounds.

payoff for a given bid and one that calculates the maximum minimum payoff. Then using the same method we can move to the previous round and so on. With this idea in place we can define an equilibrium in our environment as follows.

Definition 4.1. A collection of strategies $\boldsymbol{\beta}$ constitutes an *sequentially optimal equilibrium* of a sFPA game with ambiguity if for each i and k

$$\beta_{i,k}(\cdot,\tilde{p}_{k-1}) \in \arg\max_{b} \min_{F(\cdot|\tilde{p}_{k-1})\in\Delta_{\tilde{p}_{k-1}}} \Pi_k(\cdot,b,\beta_{i,k+2},\ldots,\beta_{i,K},\boldsymbol{\beta}_{-i},F(\cdot|\tilde{p}_{k-1}))).$$ [28]

Our definition of an equilibrium is essentially a generalization of wPBE to a model with ambiguity and prior-by-prior updating. Note that if Δ is a singleton then the equilibrium is equivalent to a wPBE. From now on we refer to such an equilibrium simply as an equilibrium. Bidders update their beliefs prior-by-prior and then define their payoffs as the minimum they can achieve.[29]

We focus on equilibrium in symmetric strategies. Thus from now on we drop the subscript with respect to the players. Suppose there exists an equilibrium characterized by $\beta = \{\beta_1,\ldots,\beta_K\}$ where β_k strictly increasing in v for any k. This will be proved to be true in equilibrium. Since the bid functions are increasing, a bidder with the k-th highest valuation will win the k-th round. Therefore, in equilibrium $p_1,\ldots,p_{k-1}$, i.e. the winning bids, can be mapped back to the realized values, $y_1 \geq \ldots \geq y_{k-1}$, of the winners. Let $\tilde{y}_k = (y_k,\ldots,y_1)$ be the vector of past winners' valuations. Then the bidding function can be re-written as $\beta_k^I(v,\tilde{y}_{k-1})$. The monotonic nature of the equilibrium also means that after the conclusion of a round bidders can calculate the set of conditional distribution of valuations of the remaining bidders by applying Bayes rule prior-by-prior.

Let $\Pi_k(v,z,\tilde{y}_{k-1})$ be the payoff of a bidder with value v who bids $\beta_k(z,\tilde{y}_{k-1})$ in round k and follows the equilibrium strategy from $k+1$ onwards. If she loses then she will know the valuation of the winner, x, due to monotonicity o the bidding functions. Suppose all other bidders follow the strategy $\boldsymbol{\beta}$. Then a bidder's payoff in round k is

(3)

$$\Pi_k(v,z,\tilde{y}_{k-1}) = \min_{F_{y_{k-1}}\in\Delta_{y_{k-1}}} F_{y_{k-1}}(z)^{N-k}(v-\beta_k(z,\tilde{y}_{k-1})) + \int_z^{y_{k-1}} \Pi_{k+1}(v,v,x,\tilde{y}_{k-1})dF_{y_{k-1}}(x)^{N-k}.$$

A bidder's problem is choose a bid, or equivalently, z in each round to maximize the above payoff function, given the history of prices and equilibrium bidding next round onwards. Our first result is the existence and uniqueness of a symmetric equilibrium where bidders do precisely this.

[28] Hanany et al. (2018) also define an equilibrium concept with the same name but in an environment with smooth ambiguity aversion.

[29] In Appendix B we show that the equilibrium we find using our equilibrium concept is equivalent to an equilibrium in which bidders are assumed to follow *consistent planning* as defined by Strotz (1955).

Proposition 4.2. *There exists a unique symmetric equilibrium where bidders follow the strategy*

$$\beta_k(v,\tilde{y}_{k-1}) = \frac{1}{\bar{F}(v|\,y_{k-1})^{N-k}} \int\limits_{\underline{v}}^{v} \beta_{k+1}(x,x,\tilde{y}_{k-1})d\bar{F}(x|\,y_{k-1})^{N-k}, \text{ for } k \leq K-1, \text{ and}$$

$$\beta_K(v,\tilde{y}_{K-1}) = \frac{1}{\bar{F}(v|\,y_{K-1})^{N-K}} \int\limits_{\underline{v}}^{v} x d\bar{F}(x|\,y_{K-1})^{N-K},$$

where

$$\bar{F}(v|\,y_{k-1}) = \min_{F\in\Delta} \frac{F(v)}{F(y_{k-1})}.$$

The equilibrium bid in the final round for a bidder with valuation v is the expected value of the highest order statistic out of $N-K$ draws conditional on it being less than v, where the expectation is taken using the lower envelope of the set of conditional distributions, $\Delta_{y_{K-1}}$. Thus the equilibrium bid function is precisely the bid function in a single unit auction with ambiguity averse bidders where $\bar{v} = y_{K-1}$ and $\Delta = \Delta_{y_{K-1}}$.[30] The closed form of the equilibrium bid function in a single-unit or the final round of a sequential auction is exactly the bid function of a single unit auction without ambiguity but where the distribution of valuations is given by the lower envelope of $\Delta_{y_{K-1}}, \bar{F}(v|y_{K-1})$. Intuitively, in the final round, as in a single-unit auction, MMEU maximizing bidders evaluate their payoffs using the 'strongest' distribution they could be competing against. This is the lower envelope of the set of distributions since it first order stochastically dominates all other distributions. With this in place the rest of derivation is straightforward.

The equilibrium bidding function in each round has a similar feature. However, from equation (3) it is not obvious how the above logic extends to each round of the auction. In equation (3), for a given b a distribution function $F_{y_{k-1}}$ that minimizes the first term essentially maximizes the second. However, in the proof of the proposition we show that $\Pi_{k+1}(v,x) \leq v - \beta_k(v,\tilde{y}_{k-1})$ and is decreasing in x where x is the current round winner's valuation. Therefore, the lower envelope of the set of conditional distributions in each round is the minimizer in the bidder's problem.

In the absence of ambiguity and with EU maximizing bidders, a model that is equivalent to ours if Δ is a singleton, the symmetric equilibrium bidding strategy, as derived by MW is

$$\beta_k^{MW}(v) = \frac{1}{F(v)^{N-k}} \int\limits_{\underline{v}}^{v} \beta_{k+1}(x)dF(x)^{N-k}, \text{ for } k \leq K-1,$$

$$\beta_K^{MW} = E\left[V_1^{(N-K)} \middle| V_1^{(N-K)} \leq v\right] \tag{4}$$ [31]

30 See Lo (1998).

Clearly, the bidding function in our model and the MW model bear similarities as well as differences. The closed form of the equilibrium bidding function look remarkably similar. However, a crucial difference is in terms of history-dependence. In the MW the equilibrium is always history independent. However in our framework, bids may or may not depend on previous round prices. History dependence depends on whether the lower envelope of the set of conditional distributions is history independent or dependent. Consider the following examples.

Example 4.3. Suppose $[\underline{v},\overline{v}] = [0,1]$ and Δ be the set of distributions F_ε where

$$F_\varepsilon(v) = \begin{cases} (1-\varepsilon)\,v, \text{ for } v \in \left[0,\frac{1}{2}\right) \\ (1+\varepsilon)\left(v-\frac{1}{2}\right)+\frac{1-\varepsilon}{2}, \text{ for } v \in \left[\frac{1}{2},1\right], \end{cases}$$

where $\varepsilon \in [0,1/2]$. Note that $\bar{F} = F_{1/2}$ and F_0 is a uniform distribution over $[0,1]$.[32] For $y \leq 1/2$ the the set of conditional distributions Δ_y collapses to a single distribution: $F_0(x)/F_0(y) = F_\varepsilon(x)/F_\varepsilon(y) = x/y$. For $y > 1/2$ the lower envelope of Δ_y is given by $F_{1/2}(\cdot)/F_{1/2}(y)$.[33]

Example 4.4. Suppose $[\underline{v},\overline{v}] = [0,1]$. Let F_0 be the uniform distribution over $[0,1]$ and F_1 be

$$F_1(v) = \begin{cases} \frac{v}{2}, \text{ for } v \in [0,\frac{1}{3}), \\ \frac{1}{6}, \text{ for } v \in [\frac{1}{3},\frac{2}{3}), \\ \frac{5v}{2}-\frac{3}{2}, \text{ for } v \in [\frac{2}{3},1]. \end{cases}$$

Moreover, for each $\kappa \in [2/3,3/4]$, let F_κ be

$$F_\kappa(v) = \begin{cases} v, \text{ for } v \in [0,\frac{\kappa}{3(5\kappa-3)}), \\ \frac{\kappa}{3(5\kappa-3)}, \text{ for } v \in [\frac{\kappa}{3(5\kappa-3)},\frac{2}{3}), \\ \frac{(\kappa-\frac{\kappa}{3(5\kappa-3)})(v-\frac{2}{3})}{\kappa-\frac{2}{3}}+\frac{\kappa}{3(5\kappa-3)}, \text{ for } v \in [\frac{2}{3},\kappa), \\ v, \text{ for } v \in [\kappa,1]. \end{cases}$$

Finally, let Δ to be the convex hull of the set $\{F_\kappa : \kappa \in [2/3,3/4] \cup \{0,1\}\}$. Note that for any $y \in [3/4,1]$, the lower envelope of Δ_y is $F_1(\cdot)/F_1(y)$. However, for any $y \in [1/3,2/3]$, the lower envelope of Δ_y is given by $F_0(\cdot)/F_0(y)$.

[32] For all $\varepsilon > 0$ these distributions have a point of non-differentiability. However, we can approximate these functions with a polynomial with arbitrary accuracy.

[33] For $y \geq 1/2$ note that

$$\frac{x}{y} \geq \frac{(1-\varepsilon)x}{(1+\varepsilon)\left(y-\frac{1}{2}\right)+\frac{1-\varepsilon}{2}} \text{ for } x \leq \frac{1}{2} \text{ and } \frac{x}{y} \geq \frac{(1+\varepsilon)\left(x-\frac{1}{2}\right)+\frac{1-\varepsilon}{2}}{(1+\varepsilon)\left(y-\frac{1}{2}\right)+\frac{1-\varepsilon}{2}} \text{ for } \frac{1}{2} \leq x \leq y$$

and the right hand side in the above inequalities is minimized at $\varepsilon = 1/2$.

4.2. **Non-Martingale Price Path.** Using the bid functions β^{MW} we can see that the prices in a model without ambiguity are a martingale. However, as we discussed in introduction, in reality prices seem to show a declining trend more often than they would if prices on average remained constant. In this section we show that a model with ambiguity can generate prices that show a declining trend. Let p_k be the price in round k. In a monotone equilibrium the bidder with the k-th highest valuation, y_k say, wins the k-th round. Thus,

$$p_k(y_k,\tilde{y}_{k-1}) = \beta_k(y_k,\tilde{y}_{k-1}) = \int_{\underline{v}}^{y_k} \beta_{k+1}(x,x,\tilde{y}_{k-1})d\frac{\bar{F}(x|y_{k-1})^{N-k}}{\bar{F}(y_k|y_{k-1})^{N-k}}.$$

In order to compute the conditional price in the next round one requires knowledge of the distribution of valuations. However in our framework the bidders and perhaps even the econometrician is unaware of the true distribution of values. Thus, the expectation of price in the next round conditional on the price history will be a range rather than a number. That is, we can calculate the conditional expectation of prices in the next round conditional on some distribution $F \in \Delta$, which is given by

$$\mathbb{E}_F\left[P_{k+1}|\, p_k,\tilde{p}_{k-1}\right] = \int_{\underline{v}}^{y_k} \beta_{k+1}(x,y_k,\tilde{y}_{k-1})d\frac{F(x)^{N-k}}{F(y_k)^{N-k}}.$$

Here again we use the fact that the history of prices, $\tilde{p}_{k-1}$ map to a history of valuations $\tilde{y}_{k-1}$.

Proposition 4.5. *Prices are not a martingale. For any k and any $F \neq \bar{F}$ there exists p_k such that for all $p'_k > p_k$*

$$p'_k > \mathbb{E}_F\left[P_{k+1}|\, p'_k,\tilde{p}_{k-1}\right].$$

Proof. Fix a k and F. To prove this result we need to show that for $y_k = y_{k-1}$ the price in round k is higher than the conditional expectation of price in the next round. Now, we have

$$p_k(y_{k-1},\tilde{y}_{k-1}) = \int_{\underline{v}}^{y_{k-1}} \beta_{k+1}(x,x,\tilde{y}_{k-1})d\frac{\bar{F}(x|y_{k-1})^{N-k}}{\bar{F}(y_{k-1}|y_{k-1})^{N-k}} = \int_{\underline{v}}^{y_{k-1}} \beta_{k+1}(x,x,\tilde{y}_{k-1})d\bar{F}(x|y_{k-1})^{N-k},$$

and

$$\mathbb{E}_F\left[P_{k+1}|\, p_k,\tilde{p}_{k-1}\right] = \int_{\underline{v}}^{y_{k-1}} \beta_{k+1}(x,y_{k-1},\tilde{y}_{k-1})d\frac{F(x)^{N-k}}{F(y_{k-1})^{N-k}}.$$

By definition we know that $\bar{F}(\cdot|y_{k-1})$ FOSD $\frac{F(\cdot)}{F(y_{k-1})}$ for any $F \in \Delta$ with a strict inequality if $\frac{F(\cdot)}{F(y_{k-1})} \neq \bar{F}(\cdot|y_{k-1})$. Furthermore we know that $\beta_{k+1}(x,y,\tilde{y}_{k-1})$ is increasing in x for any k. Finally, note that

$$\beta_{k+1}(x,x,\tilde{y}_{k-1}) = \int_{\underline{v}}^{x} \beta_{k+2}(z,z,\tilde{y}_{k-1})d\bar{F}(z|x)^{N-k-1}; \beta_{k+1}(x,y,\tilde{y}_{k-1}) = \int_{\underline{v}}^{x} \beta_{k+2}(z,z,\tilde{y}_{k-1})d\frac{\bar{F}(z|y)^{N-k-1}}{\bar{F}(x|y)^{N-k-1}}.$$

Again, since $\bar{F}(\cdot|x)$ FOSD $\frac{\bar{F}(\cdot|y)}{\bar{F}(x|y)}$ in the space of distributions Δ_x, we get that $\beta_{k+1}(x,x,\tilde{y}_{k-1}) \geq \beta_{k+1}(x,y,\tilde{y}_{k-1})$. Therefore, $p_k(y_{k-1},\tilde{y}_{k-1}) \geq \mathbb{E}_F\left[P_{k+1}\,|\,\tilde{p}_k\right]$ with strict inequality if $\frac{F(\cdot)}{F(y_{k-1})} \neq \bar{F}(\cdot|y_{k-1})$. □

A bidder's bid in the current round depends on several factors. Two important factors are the probability of winning the current round and the option value of competing in the next round. In our framework a bidder always use the lower envelope of the set of conditional distributions to calculate their payoff as a function of their current round bid. This implies that bidders overestimate competition in the current round. Over-estimation of competition in the current round makes a bidder bid more aggressively in the current round. Also, if prices are high enough then bidders will also overestimate competition in the next round as well which lowers the option value of competing in the next round which again pushes up bids, and hence prices, in the current round. Thus, if prices are high enough in the current round they are expected to fall in the next round.[34]

The above result shows that irrespective of the set of priors, when prices are high enough in one auction they will be expected to decline in the next round. This result can be extended to a general result on declining prices under some conditions. However, before we do so, we present in example that will show that prices can rise from one auction to the next. The construction of the example will also pave the way for the conditions under which we can provide a general result on declining prices.

Example 4.6. Consider a two round sequential FPA where three bidders are competing for the goods. The bidders valuation for the good come from a support $[0,1]$. Then,

$$p_1(y_1) = \int_0^{y_1} \beta_2(x,x)d\frac{\bar{F}(x)^2}{\bar{F}(y_1)^2} = \int_0^{y_1}\int_0^{x} zd\bar{F}(z|x)d\frac{\bar{F}(x)^2}{\bar{F}(y_1)^2},$$

[34] This logic does not apply for all prices for all sets of priors. The reason being if conditional distributions in the next round depend on current round prices, then a bidder may not underestimate option value in the next round for low enough prices. We explain this further in example 4.6.

$$\mathbb{E}_F\left[P_2|\,p_1\right]=\int_0^{y_1}\beta_2(x,y_1)d\frac{F(x)^2}{F(y_1)^2}=\int_0^{y_1}\int_0^{x} zd\frac{\bar{F}(z|y_1)}{\bar{F}(x|y_1)}d\frac{F(x)^2}{F(y_1)^2}.$$

Now let $F_{y_1}^{min}\in\Delta$ be such that $\bar{F}(z|y_1)=\frac{F_{y_1}^{min}(z)}{F_{y_1}^{min}(y_1)}$ which we know exists due to Assumption 3.1. Suppose $F=F_{y_1}^{min}$. Then,

$$\mathbb{E}_{F_{y_1}^{min}}\left[P_2|\,p_1\right]=\int_{\underline{v}}^{y_1}\int_{\underline{v}}^{x} zd\frac{F_{y_1}^{min}(z)}{F_{y_1}^{min}(x)}d\frac{F_{y_1}^{min}(x)^2}{F_{y_1}^{min}(y_1)^2}.$$

We will show that for a set of distributions the expected price in second round calculated using some distributions in the set is higher than the price in the first round. Consider a set of distributions from example 4.4. The lower envelope of the original set of distributions Δ was given by F_1. That is $\bar{F}=F_1$. Now, consider the case $y_1\leq\frac{1}{3}$. In this case $F_{y_1}^{min}=F_0=F_\kappa=F_1$. Substituting in the equation for prices, it is straightforward to see that

$$p_1(y_1)=\mathbb{E}_{F_0}\left[P_2|\,p_1(y_1)\right],\text{ for all } y_1<\frac{1}{3}.$$

Now consider $y_1\in\left[\frac{1}{3},\frac{2}{3}\right]$. From example 4.4 we know that $F_{y_1}^{min}=F_0$. Thus $F_x^{min}=F_0$ for all $x\in[0,2/3]$. Substituting in the equation for prices, we have

$$p_1(y_1)=\int_{\underline{v}}^{y_1}\int_{\underline{v}}^{x} zd\frac{F_0(z)}{F_0(x)}d\frac{F_1(x)^2}{F_1(y_1)^2},$$

$$\mathbb{E}_{F_0}\left[P_2|\,p_1\right]=\int_{\underline{v}}^{y_1}\int_{\underline{v}}^{x} zd\frac{F_0(z)}{F_0(x)}d\frac{F_0(x)^2}{F_0(y_1)^2}.$$

Since $\frac{F_0(\cdot)}{F_0(y_1)}$ FOSD $\frac{F_1(\cdot)}{F_1(y_1)}$, we get that $p_1<\mathbb{E}_{F_0}\left[P_2|\,p_1\right]$. Hence, price may increase for some distributions.[35]

An important aspect of the above example is that the bid function in the second round is history-dependent. In contrast, bids in a model in which the set of priors is given by the distributions in example 4.3 would be history-independent. This provides us with some clues as to under what conditions might we expect prices to be a supermartingale. We explore this theme in the next section.

[35] Due to the strict inequality, we can find distributions close to F_0 that will also satisfy the inequality.

4.3. **Dynamic Consistency and Supermartingale Prices.** An important aspect of example 4.6 is that conditional on certain prices bidders use different distributions to evaluate their payoffs in the second round versus the first. This is indicative of dynamic inconsistency: a bidder's optimal bid in the second round when calculated using the lower envelope of Δ in round one is different from her optimal bid in round two conditional on certain prices in the first round. That is optimal bids depend on previous round prices.

In this section we explore the case where bidder's behavior is dynamically consistent and provide sufficient conditions for DC. First, DC in an IPV sequential auctions context implies history-independence of bids. To see this recall DC requires a decision maker not to change her preference ranking over two acts between two rounds. That is if she prefers a second round bid b to b' before the start of round one, then after the conclusion of round one this ranking should not change.[36]

Furthermore, the bidding function depends on previous rounds through the distribution $\bar{F}(v|y)$ as can be see from the closed form expression of the equilibrium. Thus, history-independence of bids is equivalent to history-independence of the lower envelope of the conditional distributions in each round. Thus, DC will imply the invariance of the lower envelope of the conditional distributions in each round. That is the case where the lower envelope of Δ_y is always given by the lower envelope of the parent family $\bar{F}$. Hence we have the following result that follows from proposition 4.2.

Proposition 4.7. *In the unique dynamically consistent symmetric equilibrium the bidding functions are given by*

$$\beta_k(v) = \frac{1}{\bar{F}(v)^{N-k}} \int_{\underline{v}}^{v} \beta_{k+1}(x) d\bar{F}(x)^{N-k}, \text{ for } k \leq K-1, \text{ and}$$

$$\beta_K(v) = \frac{1}{\bar{F}(v)^{N-K}} \int_{\underline{v}}^{v} x d\bar{F}(x)^{N-K}.$$

Whether the above equilibrium exists will depend on whether the lower envelopes of conditional distribution sets are history-independent. We will elaborate more on this later. An implication of the above equilibrium bidding functions is that prices are a supermartingale.

Proposition 4.8. *In the unique dynamically consistent equilibrium of sFPAs prices are a supermartingale. That is $p_k \geq E_F[P_{k+1}|p_k]$ with strict inequality for $F \neq \bar{F}$.*

Proof. The proof follows immediately from definitions. Note that

$$p_k = \frac{1}{\bar{F}(v)^{N-k}} \int_{\underline{v}}^{y_k} \beta_{k+1}(x) d\bar{F}(x)^{N-k} \geq \frac{1}{F(v)^{N-k}} \int_{\underline{v}}^{y_k} \beta_{k+1}(x) dF(x)^{N-k} = E_F[P_{k+1}|p_k],$$

[36]Where b and b' are evaluated using the lower envelope of Δ in round one.

where the inequality will be strict for $F \neq \bar{F}$ since $\frac{\bar{F}(\cdot)}{\bar{F}(y_k)}$ FOSD $\frac{F(\cdot)}{F(y_k)}$. □

DC implies that bidders use the same distribution, $\bar{F}$, to evaluate their payoff function in each round. If they do so then bidders over-estimate competition in each round and also know that they will use the same distribution to evaluate competition in the next round. That is, they will also over-estimate competition in the next round and thus under-estimate the option value of competing in the next round. Both over-estimation of competition in the current round and under-estimation of option value in the next round pushes prices up in the current round compared to the next.[37]

Under what conditions is bidder behavior dynamically consistent? Or in other words, under what conditions does the equilibrium stated in proposition 4.7 exist? From Epstein and Schneider (2003) we know that a condition for DC is 'rectangularity of priors'. In what follows we confirm this finding in our context and show that rectangularity of priors implies an 'invariance to conditioning' condition in our set up. That is, rectangularity implies the lower envelope of the conditional set of priors is always given by the conditional lower envelope of the parent set of priors.

4.3.1. *Rectangularity of Priors.* Let $\Omega = [\underline{v}, \bar{v}]^N$ be the space of states of nature. Let $\mathscr{P}$ be the set of all priors (joint pdfs) over Ω. That is

$$\mathscr{P} = \left\{ \prod_{i=1}^{N} f(x_i) \,\middle|\, x \in \Omega \text{ and } f = F' \,\middle|\, F \in \Delta \right\}.^{38}$$

The information bidder i receives at the beginning of round k is modeled as a filtration $\mathscr{F}_k^i$ where $k \in \{0, 1, \ldots, K\}$. Here $\mathscr{F}_0^i = \Omega$ and $\mathscr{F}_1^i = [\underline{v}, \bar{v}]^{N-1} \times v_i$, that is at the start of the sFPAs a bidders knows her own valuation. As the auction pogresses bidders observe previous round prices. Thus, depending on bidder's beliefs about strategies employed by others, filtrations can take many forms. However, if we restrict attention to strategies that are monotone in valuations, then the filtration can only take one form: each price maps back to a unique valuation. Thus, after relabeling the final $k-1$ bidders as previous winners, the filtration in round k is given by

$$\mathscr{F}_k^i(\omega) = [\underline{v}, y_{k-1}]^{N-k} \times v_i \times y_1 \ldots \times y_{k-1},$$

where ω is the true state of the world and $y_1 \geq y_2 \ldots \geq y_{k-1}$. Restricting attention to monotone strategies, we only consider such filtrations from now on.

Let $\mathscr{F}_k$ represent the filtration at the beginning of round k. The only difference between between $\mathscr{F}_k$ and $\mathscr{F}_k^i$ is that the observer does not know v_i and only knows that $v_i \in [\underline{v}, y_{k-1}]$ where i is a bidder who has not won yet.

[37]This logic is confirmed when we consider sSPA. For sSPA prices are show to always be declining.

[38]Our approach to define a measure using density functions is done only for ease as it makes the proofs cleaner. Furthermore, for any Borel measurable set the conditional measure of the set is equal to the integral over the conditional density. Thus there is no loss of generality by using this definition.

Let $m \in \mathscr{P}$ represent a candidate measure. Let $m_k(\omega) = m(\cdot | \mathscr{F}_k(\omega))$ be the $\mathscr{F}_k$-conditional of m found by Bayesian updating. Let m_k^{+1}, called the 'one-step-ahead' measure, be the restriction of m_k to $\mathscr{F}_{k+1}$. That is m_k^{+1} is the marginal of m_k in round $k+1$. Now, we know that for any $m \in \mathscr{P}$

$$m_k(\omega) = \int_{\Omega} m_{k+1} dm_k^{+1}(\omega).$$

That is a measure can always be decomposed into its conditional and marginal. Rectangularity requires that this property applies to any combination of marginals and conditionals. Precisely, we have the following definition from Epstein and Schneider (2003).

Definition 4.9. $\mathscr{P}$ is $\mathscr{F}_k$-rectangular if for all k and ω

$$\mathscr{P}_k(\omega) = \int_{\Omega} \mathscr{P}_{k+1} d\mathscr{P}_k^{+1}(\omega), \tag{5}$$

where $\mathscr{P}_k$ and $\mathscr{P}_k^{+1}$ are collections of all Bayesian updated measure and conditional one-step-ahead measures.

The set equality in the above definition implies that any combination of conditionals and marginals, the right hand side, produces a measure that is contained in $\mathscr{P}_k$. That is, rectangularity essentially implies that the set of priors is 'large enough'. It imposes no restrictions on the set of *one-step-ahead* marginals, $\mathscr{P}_k^{+1}$, which can be an arbitrary set of measures.

Next, we show that an implication of rectangularity is that the lower envelope of any set of conditional distributions Δ_y will always be given by the lower envelope of the set of unconditional distributions Δ. Based on our earlier discussion, this would intern imply that bidding behavior is dynamically consistent. [39]

Lemma 4.10. *If $\mathscr{P}$ is $\mathscr{F}_k$-rectangular then for any $v \leq y, \bar{F}(v|y) = \frac{\bar{F}(v)}{\bar{F}(y)}$.*

In Appendix A.3 we also show that instead of using rectangularity to guarantee DC, we can also use a stronger condition on the set of priors that is based on the Likelihood Ratio partial order. The latter condition might be easier to check in applications. From lemma 4.10 we get the following corollary.

Corollary 4.11. *Bidding in sFPAs is dynamically consistent, i.e. equilibrium stated in proposition 4.7 exists if $\mathscr{P}$ is rectangular .*

[39] From Theorem 3.1 in Epstein and Schneider (2003) we should expect this, however we prove it here for completeness and for pointing the exact effect of DC in our model

Due to proposition 4.8 the above corollary provides a sufficient condition on the primitives, that is the set of priors, that lead to declining prices in sFPAs.

An interesting aspect of the above analysis is that with DC, bidders act as if they are EU maximizing agents who use the lower envelope of Δ as their 'belief' about distribution of valuations. This observation is in line with the findings of Ellis (2018) who shows that if players are dynamically consistent and respect consequentialism then players act as if they have expected utility for uncertainty over types. However, our analysis shows that despite this, ambiguity still has considerable 'bite' as it drives a well observed empirical phenomenon of declining prices in sequential auctions. Furthermore, since we prove the existence and uniqueness of an equilibrium in the general setting we can see the practical impact of dynamic consistency beyond the theoretical understanding.

Remark 4.12. While the above conditions are sufficient for prices to be a supermartingale, the unconditional prices may show a declining trend even if the above sufficient conditions are not satisfied. For example in a 2-unit 3-bidder example the unconditional prices in the first and second round, where the expectations are calculate using some $F \in \Delta$, are

$$\mathbb{E}_F[P_1] = \int_{\underline{v}}^{\bar{v}} \int_{\underline{v}}^{y} \beta_2(x,x) d\frac{\bar{F}(x)^2}{\bar{F}(y)^2} dF(y)^3; \quad \mathbb{E}_F[P_2] = \int_{\underline{v}}^{\bar{v}} \int_{\underline{v}}^{y} \beta_2(x,y) d\frac{F(x)^2}{F(y)^2} dF(y)^3.$$

Now, if the bids are history independent then $\bar{F}(x)/\bar{F}(y)$ FOSD $F(x)/F(y)$ for any $F \neq \bar{F}$ and $F \in \Delta$. Furthermore, with history independence, $\beta_2(x,x) = \beta_2(x,y) = \beta_2(x)$. Thus, $\mathbb{E}_F[P_1] > \mathbb{E}_F[P_2]$. This result immediately follows from the fact that with history independence, prices are a supermartingale.

Now, suppose prices are not a supermartingale. In this case we know $\beta_2(x,x) \geq \beta_2(x,y)$ for all $x \leq y$ with strict inequality for some y. Thus, we get

$$\mathbb{E}_{\bar{F}}[P_1] > \mathbb{E}_{\bar{F}}[P_2].$$

Note that the above inequality would still hold for some F 'close enough' to $\bar{F}$. Thus, for certain sets of Δ, where the distributions are not too distinct from $\bar{F}$ the unconditional prices will show a declining trend without any assumptions.

5. Identification in the Presence of Ambiguity

An important and practical distinction between our framework and results, and those from static single-unit auctions is with regards to identification in the presence of ambiguity. In single-unit auction models, such as those studied in Lo (1998), ambiguity averse bidders over-bid in the presence of ambiguity since they use the strongest distribution to measure competition, similar to our results. Thus bids are affected by the valuation distribution bidders *use* to calculate their payoffs.[40] However, the distribution of observable bids is also affected by the *true* distribution of valuations since that governs the actual realization of bidder types. Thus, the data-generating process in a single-unit auction with ambiguity is dependent on *two* distributions: the true distribution of valuations and the distribution of valuations bidders use to calculate their optimal bids. In the absence of ambiguity, the two are the same and one can identify the distribution of valuations from the distribution of bids as was done in the seminal work of Guerre et al. (2000) (GPV hereafter). However, in the presence of ambiguity, and aversion to it, the two distributions may not be the same and can not be identified from a single distribution of bids. What this means is that from the empirical data of single-unit auctions the econometrician (1) can not identify whether there is indeed ambiguity present from the point of view of the bidders, and, (2) may incorrectly estimate the distribution of valuations if she assumes a model with a common-prior when indeed ambiguity is present.

To formally state the above concerns, consider the same framework as in our benchmark model introduced in Section 3, but with $K = 1$. Then our model is equivalent to the model in Lo (1998). Following the analysis there, a MMEU bidder's payoff from bidding b is given by $\min_{F\in\Delta}(v - b)F(\beta^{-1}(b))^{N-1}$, where β is the strictly monotone bidding function followed by other bidders. Thus, the differential equation describing a bidder's optimal bid b is given by

$$(v-b)\frac{d\bar{F}(\beta^{-1}(b))^{N-1}}{db} = \bar{F}(\beta^{-1}(b))^{N-1} \implies v = b + \frac{\bar{F}(\beta^{-1}(b))}{(N-1)\bar{f}(\beta^{-1}(b))}\beta'(\beta^{-1}(b)).$$

Let $\bar{G}(b)$ represent the lowest probability that a bidder's bid will be lower than b. Note that is precisely $\bar{F}(\beta^{-1}(b))$. Let $\bar{g}$ be the density function of $\bar{G}$. Then the above equation can be re-written as

$$v = b + \frac{\bar{G}(b)}{(N-1)\bar{g}(b)}. \tag{6}$$

As is usually the case in auctions, bids are observable and valuations are not. Thus, in order to infer v from b using the above equation, one needs to know the ratio $\bar{G}/\bar{g}$. If Δ is a singleton, or, $\bar{F} = F$ then the above equation becomes identical to equation (3) in GPV on page 529. In this case, $G(b)$ is precisely the distribution of bids, which can be empirically estimated.

However, with ambiguity, and in the generic case where $\bar{F} \neq F$, the distribution $\bar{G}$ is not the same as the distribution of observable bids. In this case there are two important concerns. One, the econometrician may not be able to ascertain if they are using the incorrect model, as the GPV method will produce a distribution of valuations that rationalizes the observable bids in an EU setting.[41] And two, if the econometrician uses the GPV method to estimate the distribution of

[40] Abusing nomenclature, one can also call these the bidder's *beliefs* about the distribution of valuations.

[41] See Theorem 1 in Guerre et al. (2000).

valuations, they might end up with a distribution that does not correspond to the true distribution of valuations.[42] Aryal et al. (2018) offer a solution to the above concern. Using variation in the number of bidders, the authors show that the econometrician can recover both F and $\bar{F}$ under an assumption called the *exclusion restriction*. This assumption states that set of priors does not vary with N.

Experiments offer another way to circumvent the above noted issues. In a laboratory setting an econometrician would know the true distribution of valuations and thus be able to 'identify' the distribution of valuations the bidders use to calculate their optimal bids as was done in Chen et al. (2007). If the two are not the same, one can identify the presence of ambiguity and the level of it (by measuring the amount of overbidding or underbidding). However this does not alleviate the issue of non-identification of ambiguity from real-world data in single-unit auctions. [43]

In contrast, our framework suggests that the econometrician can identify the presence of ambiguity and the level of it from the price sequence and vector bidder specific bids. Our results show that the price sequence, which is easily observable from data, can help the econometrician identify whether there is ambiguity or not: a declining price sequence is an indication of ambiguity and aversion to it.[44] Thus, the dynamic auction framework, and data from it, provides us with additional tools to identify important factors that can affect bidding, compared to static auctions outside of an experimental set up.[45] In what follows we develop a strategy for identifying the true distribution of valuations (F in our model) from the data of sequential auctions in the presence of ambiguity. We also suggest a method for estimating the 'amount' of ambiguity by backing out the distribution of valuations bidder's use in the optimization problem ($\bar{F}$) and then testing the validity of our model.

Besides being a test for our model, the identification strategy we suggest below has practical applications as well. For example, we suggest that if some identical, or almost, identical units of a good are sold sequentially, it could be beneficial to consider the data from the individual auctions

[42]See Proposition 1 in Aryal et al. (2018) for a formal result of non-identification.

[43]Even outside of an auction setting, most papers that carry out estimation exercises in the presence of ambiguity do so in an experimental (laboratory or field) setting. See Cabantous (2007), Abdellaoui et al. (2011) and Ahn et al. (2014) among others. Please note due to space considerations we are not citing any papers from the extensive literature on qualitatively testing for attitudes towards ambiguity.

[44]While we have not proved the following formally, intuition developed in our work suggests that 'ambiguity loving' preferences should lead to an increasing price sequence since bidders under-estimate competition. It is interesting that an increasing price sequence can also be caused by dynamic inconsistency in our framework. Thus, increasing price sequences, which are much rarer than decreasing price sequences, require further investigation and may be a fruitful area of future research.

[45]While in the paper we assume that Δ does not depend on N this is not a crucial assumption. As long as bidders know N, we can allow Δ to vary with N. Thus our identification does not require the exclusion restriction in contrast to Aryal et al. (2018). However, our approach does require the auction data to be dynamic.

as being connected in a dynamic sense.[46] That is, to consider the history of bids as data points. Then, using our suggested strategy, the econometrician can ascertain the presence of ambiguity.

Another important practical application is with respect to revenue. In the presence of ambiguity the revenue equivalence theorem no longer holds.[47] Thus, switching auction mechanism can change the expected revenue. In order to do any counter-factual exercises to gauge the impact of changing an auction format, a designer needs to have the correct valuation distribution. Our methodology suggests a way to get the correct distribution in the presence of ambiguity.

5.1. **An Empirical Strategy.** For simplicity let us consider the case of N bidders competing for two units in two sFPAs. All assumptions and notations from the benchmark model are adopted here. We know that if Assumption 3.1 is satisfied there exists a unique symmetric equilibrium in which bidders use monotone strategies. In addition, let us assume that the equilibrium is dynamically consistent. While DC may not be guaranteed in reality, we think this is reasonable first step in providing some results on identification of ambiguity. With DC and a single parameter, i.e. v, governing a bidder's preferences, the equilibrium bid functions in round $k = 1,2$ can simply be written as $\beta_k : [\underline{v}, \bar{v}] \to \mathbb{R}_+$. Let $\phi_k = \beta_k^{-1}$ be the inverse bidding strategy.

Our model suggests that a simple (first) test for the presence of ambiguity is the slope of the average price-path. If prices, on average, show a declining trend then it is likely that the standard IPV model, as in MW (IPV and a common-prior), is inadequate to explain the data. Furthermore, our results (Proposition 4.8) suggest it would be prudent to formulate the bidders' problem as one with ambiguity which nests the standard model. In fact our identification strategy will allow the researcher to carry out tests to check which model can explain the data better. Of course, our first simple test would suggest whether a model with ambiguity (Δ has more than element) is the closer to the real environment or not. Next, we develop the identification strategy.

First let us consider the second round of the auction. Given the monotonicity of β_1 the valuation of the winner of the first round, y_1 would be revealed by the winning bid. Then, the remaining bidders will calculate their payoff in the final round using a distribution that minimizes their payoff for any bid. As we showed in the paper, with DC this means that they will use the lower envelope of the parent set of distributions, $\bar{F}$, appropriately conditioned to calculate their payoff. That is, the payoff to a bidder as a function of their bid b_2 in second round of the auction given their valuation and y_1 is given by

$$\Pi_2(v, b_2, y_1) = (v - b_2)\frac{\bar{F}(\phi(b_2))^{N-2}}{\bar{F}(y_1)^{N-2}}.$$

[46]Most auctions for livestock, flowers and other animal products take place sequentially.

[47]See Lo (1998) and Bodoh-Creed (2012).

First order conditions imply that optimal b_2 must solve

$$v = b_2 + \frac{\bar{F}(\phi_2(b_2))}{(N-2)\bar{f}(\phi_2(b_2))}\frac{1}{\phi_2'(b_2)} \implies v = b_2 + \frac{\bar{F}(v)}{(N-2)\bar{f}(v)}\frac{1}{\phi_2'(b_2)}, \tag{7}$$

where the implication follows from the equilibrium condition $\phi_2(b_2) = v$.

Now, consider the first round of the auction. We assume that the bidders will play the equilibrium strategy in the next round. Let $\Pi_2(v,x)$ be a bidder's payoff in the next round given that valuation of the winner of the first round x. Then her payoff in the first round as a function of her bid b_1 is given by

$$\Pi_1(v,b_1) = \bar{F}(\phi_1(b_1))^{N-1}(v-b_1) + \int_{\phi_1(b_1)}^{\bar{v}} \Pi_2(v,x)d\bar{F}(x)^{N-1}.$$

Taking first order conditions, the optimal b_1 must solve

$$\begin{aligned}&(N-1)\bar{f}(\phi_1(b_1))\bar{F}(\phi_1(b_1))^{N-2}\phi_1'(b_1)(v-b_1)\\ &\quad = \bar{F}(\phi_1(b_1))^{N-1} + \Pi_2(v,\phi_1(b_1))(N-1)\bar{F}(\phi_1(b_1))^{N-2}\bar{f}(\phi_1(b_1))\phi_1'(b_1).\end{aligned}$$

Substituting the equilibrium condition $\phi_1(b_1) = v$ in the above equation we get

$$(v - b_1 - \Pi_2(v,v)) = \frac{\bar{F}(v)}{(N-1)\bar{f}(v)}\frac{1}{\phi_1'(b_1)} \implies b_2 - b_1 = \frac{\bar{F}(v)}{(N-1)\bar{f}(v)}\frac{1}{\phi_1'(b_1)}. \tag{8}$$

Now, substituting equation (8) in (7) we get an equation that identifies a bidder's valuation from her two bids in the presence of ambiguity.

$$v = b_2 + \left(\frac{N-1}{N-2}\right)\left(b_2 - b_1\right)\frac{\phi_1'(b_1)}{\phi_2'(b_2)}. \tag{9}$$

In the above equation in order to get v we need the slopes of the inverse bid functions. These are not directly observable from data. However, can get at them indirectly by using the bid distributions from the two auctions. Let $G_k(\cdot)$ be the distribution of bids in round k of the auction and let $g_k(\cdot)$ be the probability density. Using bidding data we can compute G_k using the same methods as GPV. Now, note that

$$G_1(b_1) = F(\phi_1(b_1));\;\; G_2(b_2) = F(\phi_2(b_2)).$$

This is the same transformation as in GPV. Rearranging the above equations and taking derivatives we find

$$g_1(b_1) = f(\phi_1(b_1))\phi_1'(b_1);\;\; g_2(b_2) = f(\phi_2(b_2))\phi_2'(b_2).$$

As was the case in equation (9), let b_1 and b_2 be the bids of a bidder in round one and two respectively. Then $\phi_1(b_1) = \phi_2(b_2) = v$. Substituting in the above equation and then substituting in equation (9) we get

$$v = b_2 + \left(\frac{N-1}{N-2}\right)\left(b_2 - b_1\right)\frac{g_1(b_1)}{g_2(b_2)} = \zeta(b_1,b_2,G_1,G_2). \tag{10}$$

Thus, given any bidder's bids in both the rounds and the number of bidders, we can calculate their valuation. Furthermore, the above procedure, much like the identification methodology of GPV, does not assume any parametric form of the distribution function or the bid function. Thus, our identification strategy can be considered non-parametric. With ample data the above equation can be used to recover a distribution of 'pseudo valuations' in the same way as GPV. That is we can recover the 'true' distribution of valuations, F.

We are also interested in estimating the distribution bidders use to calculate their payoffs: $\bar{F}$. To calculate this distribution, we rewrite equation (8) to get

$$\frac{\bar{f}(v)}{\bar{F}(v)} = \frac{d\log\bar{F}(v)}{dv} = \frac{1}{(N-1)(b_2-b_1)\phi_1'(b_1)} = \frac{f(v)}{(N-1)(b_2-b_1)g_1(b_1)}. \tag{11}$$

Using numerical integration we can calculate $\bar{F}$ from the above equation. Thus under our assumptions and using our methodology we can recover both the true distribution of values as well as the distribution that bidders' use to calculate payoffs.

Once we recover $\bar{F}$ we can check whether it is the same as F or not.[48] Specifically, if the data is generated from the standard model with a common prior, then our identification procedure will produce $F=\bar{F}$.[49] On the other hand, if $\bar{F}\neq F$, and more importantly $\bar{F}\leq_{FOSD} F$ then a model with ambiguity explains the data better. Thus our identification strategy provides a second test (along with the simple test based on average price-path) to check for ambiguity.

We collect the arguments above in the following result, the proof of which can be found in the Appendix. Let $[\underline{b}_1,\overline{b}_1],[\underline{b}_2,\overline{b}_2]\subset\mathbb{R}_+$. Let $G_k(\cdot)$ be a distribution over $[\underline{b}_k,\overline{b}_k]$ such that $G_k(\underline{b}_k)=0$ and $G_k(\overline{b}_k)=0$ for $k=1,2$. Finally, let $\boldsymbol{G}_k:[\underline{b}_k,\overline{b}_k]^{N-k+1}\to[0,1]$ be joint distribution of bids in round k of an auction.

Proposition 5.1. *For any distributions $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$, there exist distributions $F,\bar{F}:[\underline{v},\overline{v}]\to[0,1]$ such that $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ are the equilibrium joint distribution of bids in rounds 1 and 2 of a sequence of FPAs with ambiguity if and only if*

C1 : The joint distribution of bids in any round is given by

$$\boldsymbol{G}_k(b_1,\ldots,b_{N-k+1}) = \prod_{i=1}^{N-k+1} G_k(b_i).$$

C2 : $\underline{b}_1=\underline{b}_2$ and $\overline{b}_1<\overline{b}_2$. Furthermore, there exists an increasing function $\psi(\cdot)$ such that $\psi(b_{i,2})=b_{i,1}\leq b_{i,2}$ for $i\in\{1,\ldots,N\}$, and $b_{i,k}$ is bidder i's bid in auction round k. Finally, $G_2(b)\leq G_1(b)$ for any b.

[48] Using a two-sample Kolmogorov-Smirnov test.

[49] In fact, if $\bar{F}=F$ equation (11) collapses to FOC from round one in equation (8).

C3 : The function $\zeta(\psi(b_2), b_2, G_1, G_2)$ *is strictly increasing on* $[\underline{b}_2, \overline{b}_2]$.

Moreover when $F, \bar{F}$ *exist they are unique with support* $[\underline{v}, \overline{v}]$ *and satisfy the equations* $F(v) = G_2(\zeta^{-1}(v, G_1, G_2))$ *and (11).*

Conditions C1 and C3 are similar to conditions C1 and C2 in Theorem 1 of GPV. Condition C2 is an extra condition we have due to the dynamic nature of the auction. The function ψ is equal to $\beta_1 \circ \beta_2^{-1}$ the monotonicity of which follows from the monotonicity of the equilibrium.[50] Furthermore, with DC we know that $\beta_2(v) \geq \beta_1(v)$[51] thus $\psi(b) \leq b$.

Similar to Theorem 1 in GPV, the above result gives us additional tools to check the validity of the model. For example, an implication of the model is that both ψ and ζ are increasing. These conditions can be checked using auction data in which bid vectors of a bidder can be identified.

Remark 5.2. We claim that the above methodology can be extended to K rounds. In our model of ambiguity there are always two distributions of interest: F and $\bar{F}$. Thus, as long as we have rich enough data (highest losing bid in round $k-1$ and winning bid in round k) for any two sequential rounds, we can use our procedure to estimate the objects of interest.

6. Second Price Auctions

In this section we study sequential second-price auctions. The environment is the same as before except now the winners pay the second highest bid in each round. This means that the winning bid and price in each round are, potentially, different. Therefore, bidding functions could be different based on whether the auctioneer announces the winning bid or the price from each round. If only the winning bid is announced then all active bidders in the next round have the same information. If on the other hand prices are announced, then the bidder who had the second highest bid has potentially valuable information. If bidders follow monotone strategies, then she knows she now has the highest valuation. Other active bidders are not privy to this, making the game asymmetric. However, we show that price announcements do not matter in equilibrium as the equilibrium for sSPA with ambiguity is *always* history-independent, unlike sFPAs as we showed previously. For now, we assume that only the winning bids are announced. Suppose, β_k^{II} be the bidding function in round k. The next result establishes the existence and uniqueness of a symmetric equilibrium for sSPAs.

Proposition 6.1. *In the unique symmetric equilibrium bidders follow the strategy*

$$\beta_k^{II}(v) = \int_{\underline{v}}^{v} \beta_{k+1}^{II}(x) d\bar{F}(x|v)^{N-k-1}, \text{ for } k \leq K-1, \text{ and } \beta_K^{II}(v) = v,$$

[50] Such a function exists with or without ambiguity.

[51] From proposition 4.7 this is straightforward.

where

$$\bar{F}(x|v) = \min_{F \in \Delta} \frac{F(x)}{F(v)}.$$

The equilibrium bidding functions in sSPA are always history-independent. This is in contrast to the sFPA where bids can depend on previous round prices. Why this discrepancy? In the final round, it is weakly dominant for bidders to bid their valuations in sSPA. So while they payoffs in the final round may depend on previous round prices, their strategy is independent of the previous round prices. This is not always the case for sFPA as we showed previously.

A bidder's optimal bid in round $K-1$ must be such that the marginal gain in $K-1$ due to bidding a little more equals the marginal loss from not competing in round K. The latter, written as $\Pi_K(v,v)$ in the proof, is the payoff the bidder would get in the final round if the winner in the current round had a valuation of v (in the limit). Now, since bids in the final round are independent of previous round prices, $\Pi_K(v,v)$ is independent of previous round prices. Therefore, the marginal gain from bidding a little more in round $K-1$ must also be independent of previous round prices. This makes the bid in round $K-1$ independent of previous prices. This logic extends to any round k if bids in round $k+1$ are independent of prices.

With regard to prices the next result shows that prices are always declining in a sSPAs. This result is stronger than the result we obtained for sFPAs and is due to the fact that bids in sSPA are history-independent. Recall that we found that is bids were history-independent in sFPAs then prices were a supermartingale.

Proposition 6.2. *Prices are a supermartingale in sequential SPAs. That is $p_k \geq E_F\left[P_{k+1}|p_k\right]$ with strict inequality for $F \neq \bar{F}$.*

Proof. In second price auctions the price in a round is the second highest bid. Thus, in equilibrium the price in round k will be the bid placed by the bidder with the second highest valuation in that round. That is,

$$p_k = \beta_k^{II}(y_{k-1}) = \int_{\underline{v}}^{y_{k-1}} \beta_{k+1}^{II}(x) d\bar{F}(x|y_{k-1})^{N-k-1}.$$

Bidders in the next round will observe the winning bid and hence the valuation of the winner, y_k. However, recall that in equilibrium the previous round prices do not affect the bidding functions in sSPAs. They only scale the payoffs. Thus, the expected price in the next round calculated at some $F \in \Delta$ is given by expected bid of the second highest valued bidder out of $N-k$ bidders conditional on the value being less than y_{k-1}. This is equivalent to the expected bid of the highest valued bidder out of $N-k-1$ bidders.

$$\mathbb{E}_F\left[P_{k+1}|\, p_k\right] = \int_{\underline{v}}^{y_{k-1}} \beta^{II}_{k+1}(x)d\frac{F(x)^{N-k-1}}{F(y_{k-1})^{N-k-1}} \leq p_k,$$

where the inequality follows since $\bar{F}(\cdot|y_{k-1})$ FOSD $\frac{F(\cdot)}{F(y_{k-1})}$. □

The bidding function in sSPAs are independent of the history. This is in contrast to sFPAs which can be history-dependent. Despite this difference there are traits that the bidding functions share.

Corollary 6.3. *Suppose β^I_k is the equilibrium bidding function in round k of sFPAs. Then for $k<K$*

$$\beta^{II}_k(v) = \beta^I_{k+1}(v,v) \geq \beta^I_{k+1}(v,y_{k-1}),$$

where the weak inequality is an equality if $\mathscr{P}$ is rectangular or Δ is a semi-lattice under $\geq_{lr}$.

Without ambiguity there is no history-dependence and the equilibrium bid in round k of sSPA equals the bid in round $k+1$ of sFPAs. This equivalence in-fact is also implied by the revenue equivalence theorem in a model without ambiguity. Thus, we are also interested in the effects of ambiguity on revenue ranking between various auction formats.

7. Revenue Comparisons

In this section we compare the revenue across the two sequential auction formats we have studied as well as a commonly used static multi-unit auction: the uniform price auction. We begin by comparing the sequential auction formats. Let R^{format}_F be the expected revenue calculated using $F\in\Delta$ and $format \in \{I, II, up\}$ where up stands for uniform price.

We know that $R^I = R^{II}$ if Δ is a singleton. In this case there is no ambiguity and the revenue equivalence theorem holds. Under ambiguity, from corollary 6.3 we know that the bids in round k of sSPA are weakly higher than the bids in round $k+1$ of sFPA. However the revenue in sSPA is the sum of second highest bids and there is no direct ranking between the second highest bids in sSPA and the highest bids in sFPA in the general case. However, if the set of priors satisfy one of the regularity conditions (rectangularity or semi-lattice) we know that the weak inequality in the statement of corollary 6.3 is an equality and we can then compare revenues more easily. Thus we have the following result.

Proposition 7.1. *If $\mathscr{P}$ is rectangular then $R^I_F \geq R^{II}_F$ with strict inequality for $F\neq\bar{F}$.*

Proof. Under rectangularity we know that $\bar{F}(\cdot|x) = \frac{\bar{F}(\cdot)}{\bar{F}(x)}$. Therefore, for $k < K$,

$$\beta_k^{II}(v) = \int\limits_{\underline{v}}^{v} \beta_{k+1}^{II}(x) d\frac{\bar{F}(x)^{N-k-1}}{\bar{F}(v)^{N-k-1}} = \int\limits_{\underline{v}}^{v} \beta_{k+2}^{I}(x) d\frac{\bar{F}(x)^{N-k-1}}{\bar{F}(v)^{N-k-1}} = \beta_{k+1}^{I}(v),$$

where the first equality follows from substituting $\bar{F}(\cdot|x) = \frac{\bar{F}(\cdot)}{\bar{F}(x)}$ in the bidding function of sSPAs. The second from corollary 6.3 and proposition 4.7 and the third again from proposition 4.7.

Now, let $V_l^{(m)}$ be the random variable denoting the l-th highest draw out of m draws from some distribution F. Let P_k^I and P_k^{II} be the price in round k in sFPAs and sSPAs respectively. Then, since the equilibrium is monotone, for $k < K$ we get,

$$\mathbb{E}_F\left[P_{k+1}^I\right] = \mathbb{E}_F\left[\beta_{k+1}^I\left(V_{k+1}^{(N)}\right)\right] = \mathbb{E}_F\left[\beta_k^{II}\left(V_{k+1}^{(N)}\right)\right] = \mathbb{E}_F\left[P_k^{II}\right].$$

Now, since prices are a supermartingale in sFPAs, as was shown in proposition 4.8, we know that for $k < K$

$$\mathbb{E}_F\left[P_k^I\right] \geq \mathbb{E}_F\left[P_{k+1}^I\right] = \mathbb{E}_F\left[P_k^{II}\right].$$

and for $k = K$, since prices are a supermartingale in sSPAs, as was shown in proposition 6.2 we know that

$$\mathbb{E}_F\left[P_K^{II}\right] \leq \mathbb{E}_F\left[P_{K-1}^{II}\right] = \mathbb{E}_F\left[P_K^I\right].$$

Thus the revenue from sFPAs is higher than sSPAs. □

Is the 'declining price anomaly' bad for auctioneers? Despite frequently observing price declines, why would auctioneers choose to sell their goods using a sequential format?[52] In the context of affiliated values the linkage principle would suggest that a sequential auction should generate higher revenue than static multi-unit auctions. However, as shown in MW, affiliated values lead to increasing not decreasing price sequences. Thus, the linkage principle and affiliated values do not seem to provide us with a satisfactory answer to the above questions.[53]

Here we provide a partial answer to these questions by comparing the revenue between a static multi-unit auction and sequential auctions. For this purpose we use a commonly used static multi-unit auction format, the Uniform Price Auction (UPA) with the highest losing bid as the selling price. This auction format is the multi-unit analogue of a single-unit SPA. This auction also has a symmetric equilibrium in weakly dominant strategies. Bidders bid truthfully and bid their own valuations since their bid only affects their probability of winning and not the price they pay. Thus the price in equilibrium will be the $K+1$-th highest valuation out of N valuations. Then it is

[52] For a discussion see Ashenfelter and Graddy (2006) and Ginsburgh and van Ours (2007).

[53] Furthermore, as Hausch (1986) and Mezzetti (2011) show the revenue dominance of sequential auctions is not obvious and fails under many instances.

straightforward to show that the expected revenue calculated using some $F \in \Delta$ in this auction format is

$$R^{up} = K\mathbb{E}_F\left[V_{K+1}^{(N)}\right] = K\mathbb{E}_F\left[V_1^{(N-K)}\right]$$

The expected revenue in final round of sequential SPA is the second highest valuation of the remaining $N-K+1$ bidders. This is equal to $\mathbb{E}_F\left[V_1^{(N-K)}\right]$. Thus the final round of sequential SPA generates $\frac{R^{up}}{K}$ expected revenue. Furthermore, due to proposition 6.2 we know that the expected revenue (in a round) progressively declines in sSPA. Thus sSPAs generate more revenue than UPA. Furthermore, if the set of priors satisfies rectangularity or is a semi-lattice under $\geq_{lr}$ then due to 7.1, we get a revenue ranking between all three formats.

Proposition 7.2. *Sequential SPA generate higher revenue compared to UPA. Furthermore, if $\mathscr{P}$ is rectangular or Δ is a a semi-lattice under $\geq_{lr}$ then the revenue ranking between the three auction formats is $R^I > R^{II} > R^{up}$.*

The above proposition shows that in many cases the sequential auctions generate higher revenue than UPA. This is particularly apparent when prices decline. This suggests that declining prices lead to higher revenues for the auctioneer.

8. Conclusion

Since Ellsberg (1961), understanding the causes and effects of ambiguity has attracted a lot of work in economics. In this paper we introduce ambiguity into a sequential auction environment and theoretically studied the behavior of bidders who are MMEU maximizers instead of the standard assumption of EU maximizers. Our main results are the following: (1) characterization of the unique symmetric equilibrium that satisfies consistent planning, (2) prices are a supermartingale for sSPAs in general, and for sFPAs if bidder behavior was dynamically consistent, (3) providing sufficient conditions for DC in sequential auctions, (4) nonparametric identification in the presence of ambiguity and an empirical strategy to estimate variables of interest using dynamic auction data, and (5) providing a revenue ranking between sSPAs and sFPAs as well as showing that sequential auctions with declining prices can generate higher revenue than a static multi-unit auction. Points (1) and (2) illustrate the importance of modeling ambiguity in this environment as it provides an explanation of an old puzzle in game-theory. Points (2) and (3) show some practical implications of DC. Point (4) highlights the importance of considering data that is dynamic in nature in order to carry out estimation in environments that generalize the standard EU framework. Finally, point (5) provides new results on revenue comparisons across different auction formats.

Studying ambiguity in dynamic environments, while extremely crucial, is not quite straightforward due to the reasons we discussed in the introduction as well as throughout the paper. Indeed existence of an equilibrium is not straightforward to establish in these environments. Nevertheless, we take a standard model of utility (MMEU) along with a simple and widely adopted updating rule (prior-by-prior) and obtain an intuitive monotone equilibrium in close form. Monotonicity of the bidding strategies implies that bidders' updated beliefs are always in the form of a 'right curtailment' of the valuation set. With this insight in place, we conjecture that while other models of ambiguity (alternate utilities or update rules) may lead to different form of an equilibrium (if they exist) the flavor will remain the same. Finally, we hope that our approach can be further generalized to study ambiguity in dynamic problems with monotone decision rules.

Appendix A. Proofs

A.1. **Proof of proposition 4.2.** Let us assume bidders are following some strictly increasing bidding strategy β so that prices can be mapped back to valuations of winners. Let Δ_y be the set of conditional distributions derived from the primitive set Δ as stated in equation (2).

A.1.1. *Round K.* : Consider the final round. Consider a bidder with valuation v who has not won a unit yet and was following the equilibrium strategy till round $K-1$. That is $v \leq y_{K-1}$. This bidder's payoff from bidding $\beta_K(z, \tilde{y}_{K-1})$ where $z \leq y_{K-1}$ is

$$\begin{aligned}
\Pi_K(v, z, \tilde{y}_{K-1}) &= \min_{F_{y_{K-1}} \in \Delta_{y_{K-1}}} F_{y_{K-1}}(z)^{N-K}(v - \beta_K(z, \tilde{y}_{K-1})) \\
&= \bar{F}(z|\, y_{K-1})^{N-K}(v - \beta_K(z, \tilde{y}_{K-1})).
\end{aligned} \tag{12}$$

Note that $\bar{F}(\cdot|y_{K-1})$ is a strictly increasing and continuous distribution function. Moreover, in some sense, it represents the remaining bidders' 'belief' in the final round about the valuation of other bidders. Therefore, for any y_{K-1}, there is a unique symmetric equilibrium strategy in the K-th auction, $\beta_K(\cdot, \tilde{y}_{K-1}) : [\underline{v}, y_{K-1}] \to \mathbb{R}$, where

$$\beta_K(v, \tilde{y}_{K-1}) = \frac{1}{\bar{F}(v|y_{K-1})^{N-K}} \int_{\underline{v}}^{v} x d\bar{F}(x|y_{K-1})^{N-K} = v - \frac{\int_{\underline{v}}^{v} \bar{F}(x|y_{K-1})^{N-K} dx}{\bar{F}(v|y_{K-1})^{N-K}},$$

which can found by optimizing the payoff function in round K with respect to z and then substituting $z = v$ in the resulting differential equation. In the equation for the bidding function, note that in equilibrium, bids in round K only depend on previous round prices through the conditional distribution $\bar{F}(\cdot|y_{K-1})$. For notational parsimony, from now on let us refer to bidding functions as functions of valuations and the previous round winner's valuation. For notational parsimony, from now on let us refer to bidding functions as functions of valuations and the previous round winner's valuation.

Let $\Pi_K(v, y_{K-1})$ be a bidder's expected payoff from participating in the K-th round of the auction given y_{K-1}, in equilibrium. Then, given the equilibrium strategy,

$$\Pi_K(v, y_{K-1}) = \bar{F}(v|\, y_{K-1})^{N-K}(v - \beta_K(v, y_{K-1}))\,.$$

A.1.2. *Monotonicity of* $\beta_K(\cdot,\cdot)$ *and* $\Pi_K(v,\cdot)$. Consider the function $\beta_K(v,v)$. Note that

$$\beta_K(v,v) = \frac{1}{\bar{F}(v|v)^{N-K}} \int\limits_{\underline{v}}^{v} x d\bar{F}(x|v)^{N-K} = \int\limits_{\underline{v}}^{v} x d\bar{F}(x|v)^{N-K}.$$

The above integral can also be re-written as

$$\beta_K(v,v) = \max_{F\in\Delta} \int\limits_{\underline{v}}^{v} x \frac{dF(x)^{N-K}}{F(v)^{N-K}} = \int\limits_{\underline{v}}^{v} x \frac{dF_v(x)^{N-K}}{F_v(v)^{N-K}},$$

where

$$F_{v,K} = \arg\max_{F\in\Delta} \int\limits_{\underline{v}}^{v} x \frac{dF(x)^{N-K}}{F(v)^{N-K}}.$$

Next consider $v' > v$. We have,

$$\beta_K(v',v') = \max_{F\in\Delta} \int\limits_{\underline{v}}^{v'} x \frac{dF(x)^{N-K}}{F(v')^{N-K}} \geq \int\limits_{\underline{v}}^{v'} x \frac{dF_{v,K}(x)^{N-K}}{F_{v,K}(v')^{N-K}} \geq \int\limits_{\underline{v}}^{v} x \frac{dF_{v,K}(x)^{N-K}}{F_{v,K}(v)^{N-K}} = \beta_K(v,v).$$

Hence, $\beta_K(v,v)$ is non decreasing in v.

Now, consider the payoff in the final round, $\Pi_K(v,x)$. Note that for any $v \leq x$, we have

$$\begin{aligned}
\Pi_K(v,x) &= \max_{z} \min_{F_x\in\Delta_x} F_x(v)^{N-K}(v - \beta_K(z,x)) \\
&= \min_{F\in\Delta} \left(\frac{F(v)}{F(x)}\right)^{N-K} \frac{\int_{\underline{v}}^{v} \bar{F}(w|x)^{N-K} dw}{\bar{F}(v|x)^{N-K}} \\
&= \frac{1}{\bar{F}(x|x)^{N-K}} \int\limits_{\underline{v}}^{v} \bar{F}(w|x)^{N-K} dw \\
&= \min_{F\in\Delta} \int\limits_{\underline{v}}^{v} \left(\frac{F(w)}{F(x)}\right)^{N-K} dw.
\end{aligned}$$

Applying the envelope theorem, we conclude that $\Pi_K(v,x)$ is non-increasing in x.

A.1.3. *Round $K-1$.* Now consider round $K-1$. Suppose in this round bidders follow a strategy given by $\beta_{K-1}(v,\tilde{y}_{K-2})$ which is strictly increasing in v. Suppose a bidder with valuation v bids $\beta_{K-1}(z,\tilde{y}_K-1)$ where $z \geq v$ and $z \leq y_{K-2}$. Assuming she bids according to the equilibrium strategy β_K in the next round, her payoff is

(13)

$$\Pi_{K-1}(v,z,y_{K-2}) = \min_{F_{y_{K-2}} \in \Delta_{y_{K-2}}} F_{y_{K-2}}(z)^{N-K+1}\left(v - \beta_{K-1}(z,y_{K-2})\right) + \int_{z}^{y_{K-2}} \Pi_K(v,x)dF_{y_{K-2}}(x)^{N-K+1}.$$

Let $\hat{F}_{y_{K-2}}$ be the minimizer of the above payoff function. Given the payoff function, it is not clear that there is (1) a unique $\hat{F}_{y_{K-2}}$ that minimizes the above payoff function and (2) the minimizer is independent of z and v. We will prove that both of these conditions are indeed true. We will prove (2) for a general round k to avoid repetition. We will show (1) for all rounds. Thus, for now, let us assume that $\hat{F}_{y_{K-2}}$ does not depend on z or v. Then

$$\Pi_{K-1}(v,z,y_{K-2}) = \hat{F}_{y_{K-2}}(z)^{N-K+1}\left(v - \beta_{K-1}(z,y_{K-2})\right) + \int_{z}^{y_{K-2}} \Pi_K(v,x)d\hat{F}_{y_{K-2}}(x)^{N-K+1} \tag{14}$$

$$= \hat{F}_{y_{K-2}}(z)^{N-K+1}\left(v - \beta_{K-1}(z,y_{K-2})\right) + \int_{z}^{y_{K-2}} \bar{F}(v|x)^{N-K}(v - \beta_K(v,x))d\hat{F}_{y_{K-2}}(x)^{N-K+1}.$$

Taking derivative of the payoff function with respect to z, we get

$$\begin{aligned}\frac{\partial \Pi_{K-1}(v,z,y_{K-2})}{\partial z} = \frac{d\hat{F}_{y_{K-2}}(z)^{N-K+1}}{dz}\left(v - \beta_{K-1}(z,y_{K-2})\right) - \hat{F}_{y_{K-2}}(z)^{N-K+1}\frac{\partial \beta_{K-1}(z,y_{K-2})}{\partial z} \\ - \bar{F}(v|z)^{N-K}(v - \beta_K(v,z))\frac{d\hat{F}_{y_{K-2}}(z)^{N-K+1}}{dz}.\end{aligned} \tag{15}$$

In equilibrium, the bidder would choose $z = v$. Thus, the first order condition for the bidder is

$$\frac{d\hat{F}_{y_{K-2}}(v)^{N-K+1}}{dv}\left(\beta_K(v,v) - \beta_{K-1}(v,y_{K-2})\right) - \hat{F}_{y_{K-2}}(v)^{N-K+1}\frac{\partial \beta_{K-1}(v,y_{K-2})}{\partial v} = 0, \tag{16}$$

since $\bar{F}(v|v) = 1$. From the above differential equation, we have

$$\beta_{K-1}(v,y_{K-2}) = \frac{1}{\hat{F}_{y_{K-2}}(v)^{N-K+1}} \int_{\underline{v}}^{v} \beta_K(x,x)d\hat{F}_{y_{K-2}}(x)^{N-K+1}.$$

Note that

$$\frac{\partial \beta_{K-1}(v,y_{K-2})}{\partial v} = \left(\beta_K(v,v) - \int_{\underline{v}}^{v} \beta_K(x,x) d\frac{\hat{F}_{y_{K-2}}(x)^{N-K+1}}{\hat{F}_{y_{K-2}}(v)^{N-K+1}}\right) \left(\frac{N-K+1}{\hat{F}_{y_{K-2}}(v)}\right) \frac{d\hat{F}_{y_{K-2}}(v)}{dv}$$

$$= \left(\int_{\underline{v}}^{v} \frac{\hat{F}_{y_{K-2}}(x)^{N-K+1}}{\hat{F}_{y_{K-2}}(v)^{N-K+1}} \frac{d\beta_K(x,x)}{dx}\right) \left(\frac{N-K+1}{\hat{F}_{y_{K-2}}(v)}\right) \left(\frac{d\hat{F}_{y_{K-2}}(v)}{dv} > 0,\right.$$

where the second equality follows from integration by parts and the final inequality follows from the fact that $\beta_K(x,x)$ is non-decreasing in x as proved previously. Thus the bid function in round $K-1$ is increasing in valuation. Along with the first order condition in equation (26) this implies that $\beta_{K-1}(v,y_{K-2}) \leq \beta_K(v,v)$.

Thus, what remains to be show for this round is that $\hat{F}_{y_{K-2}}(v) = \bar{F}(v|y_{K-2})$. Recall that in equilibrium a bidder's payoff in this round is

$$\Pi_{K-1}(v) = \hat{F}_{y_{K-2}}(v)^{N-K+1}(v - \beta_{K-1}(v,y_{K-2})) + \int_{v}^{y_{K-2}} \Pi_K(v,x) d\hat{F}_{y_{K-2}}(x)^{N-K+1}. \tag{17}$$

Now, note that $v - \beta_{K-1}(v,y_{K-2}) \geq v - \beta_K(v,v) = \Pi_K(v,v)$ since $\beta_{K-1}(v,y_{K-2}) \leq \beta_K(v,v)$ as the former is the expectation of $\beta_K(x,x)$ conditional on $x \leq v$ and $\beta_K(x,x)$ is increasing in x. Let

$$r(x) = \begin{cases} v - \beta_{K-1}(v,y_{K-2}), \text{ for } x \in [\underline{v}, v), \\ \Pi_K(v,x), \text{ for } x \in [v, \bar{v}]. \end{cases}$$

Note that $r(x)$ is weakly decreasing in x. In equation (17) we are calculating the expected value of $r(x)$ using the distribution $\hat{F}_{y_{K-2}}(\cdot)^{N-K+1}$. Since $r(x)$ is weakly decreasing the payoff function is minimized by using the strongest distribution FOSD sense. That is

$$\hat{F}_{y_{K-2}}(v) = \bar{F}(v|y_{K-2}),$$

which, by Assumption 3.1, is in the feasible set of distributions.

A.1.4. *Monotonicity of $\beta_{K-1}(v,v)$ and $\Pi_{K-1}(v,x)$ in v and x respectively.* Since

$$\beta_{K-1}(v,v) = \int_{\underline{v}}^{v} \beta_K(x,x) d\bar{F}(x|v)^{N-K+1} = \max_{F \in \Delta} \int_{\underline{v}}^{v} \beta_K(x,x) d\frac{F(x)^{N-K+1}}{F(v)^{N-K+1}},$$

using the same argument we used in showing $\beta_K(v,v)$ is increasing in v,

$$\beta_{K-1}(v',v') = \max_{F\in\Delta} \int_{\underline{v}}^{v'} \beta_K(x,x) d\frac{F(x)^{N-K+1}}{F(v')^{N-K+1}} \geq \int_{\underline{v}}^{v'} \beta_K(x,x) d\frac{F_{v,K-1}(x)^{N-K+1}}{F_{v,K-1}(v')^{N-K+1}}$$
$$\geq \int_{\underline{v}}^{v} \beta_K(x,x) d\frac{F_{v,K-1}(x)^{N-K+1}}{F_{v,K-1}(v)^{N-K+1}} = \beta_{K-1}(v,v),$$

where the second inequality follows since $\beta_K(x,x)$ is increasing in x. Hence $\beta_{K-1}(v,v)$ is non decreasing in v. Now, consider $\Pi_{K-1}(v,x)$.

$$\begin{aligned}
\Pi_{K-1}(v,x) &= \min_{F\in\Delta} \left(\frac{F(v)}{F(x)}\right)^{N-K+1} (v - \beta_{K-1}(v,x)) + \int_{v}^{x} \Pi_K(v,z) d\left(\frac{F(z)}{F(x)}\right)^{N-K+1} \\
&= \min_{F\in\Delta} \left(\frac{F(v)}{F(x)}\right)^{N-K+1} \left(v - \beta_K(v,v) + \int_{\underline{v}}^{v} \frac{\bar{F}(z|x)^{N-K+1}}{\bar{F}(v|x)^{N-K+1}} d\beta_K(z,z)\right) \Bigg(\\
&\quad + \int_{v}^{x} \Pi_K(v,z) d\left(\frac{F(z)}{F(x)}\right)^{N-K+1} \\
&= \min_{F\in\Delta} \left(\frac{F(v)}{F(x)}\right)^{N-K+1} \Pi_K(v,v) + \int_{v}^{x} \Pi_K(v,z) d\left(\frac{F(z)}{F(x)}\right)^{N-K+1} \\
&\quad + \left(\frac{F(v)}{F(x)}\right)^{N-K+1} \int_{\underline{v}}^{v} \frac{\bar{F}(z|x)^{N-K+1}}{\bar{F}(v|x)^{N-K+1}} d\beta_K(z,z)
\end{aligned}$$

Now, let

$$r(z) = \begin{cases} \Pi_K(v,v), \text{ for } z \in [\underline{v}, v), \\ \Pi_K(v,z), \text{ for } z \in [v,x]. \end{cases}$$

Note that $r(z)$ is non-increasing. Then,

$$\Pi_{K-1}(v,x) = \min_{F\in\Delta} \int_{\underline{v}}^{x} r(z) d\left(\frac{F(z)}{F(x)}\right)^{N-K+1} + \left(\frac{F(v)}{F(x)}\right)^{N-K+1} \int_{\underline{v}}^{v} \frac{\bar{F}(z|x)^{N-K+1}}{\bar{F}(v|x)^{N-K+1}} d\beta_K(z,z) \tag{18}$$

Since $\beta_K(z,z)$ is increasing in z, as we have shown previously, by the Envelope Theorem the second term in the above equation is non-increasing in x. What needs to be shown is that the first term is also non-increasing in x. To see this, let $F_x \in \Delta$ be such that $\bar{F}(\cdot|x) = F_x(\cdot)/F_x(x)$. Fixing

F_x, note that

$$\frac{d}{dx}\int_{\underline{v}}^{x} r(z)d\frac{F_x(z)^{N-K+1}}{F_x(x)^{N-K+1}} = \frac{f_x(x)}{F_x(x)}\left(r(x)-\int_{\underline{v}}^{x} r(z)d\frac{F_x(z)^{N-K+1}}{F_x(x)^{N-K+1}}\right)\left(\leq 0\right.$$

where the inequality follows from the fact that $r(z)$ is non-increasing. Thus, again by the Envelope Theorem the first term in equation (18) is also non-increasing in x.

A.1.5. *Induction.* Next, we proceed by induction to find the equilibrium strategy in the k-th auction. By induction, suppose $\beta_{k+1}(v,v)$ is non-decreasing in v and $\Pi_{k+1}(v,x)$ is non-increasing in x. Suppose bidders follow the strategy $\beta_k(v,y_{k-1})$ that is increasing in v. Consider a bidder with value v who bids $\beta_k(z,y_{k-1})$ where $z\leq y_{k-1}$. This bidder's expected payoff is

$$\Pi_k(v,z,y_{k-1}) = \min_{F_{y_{k-1}}\in\Delta_{y_{k-1}}} F_{y_{k-1}}(z)^{N-k}(v-\beta_k(z,y_{k-1})) + \int_{z}^{y_{k-1}} \Pi_{k+1}(v,x)dF_{y_{k-1}}(x)^{N-k},$$

where $\Pi_{k+1}(v,x)$ is the bidder's continuation payoff if she loses in the k-th auction and the value of the winner for this auction is $x(\geq z)$. Let $\hat{F}_{y_{k-1}}(\cdot,z)$ denote a minimizer of the above problem. In deriving the equillibrium in this round we will also show that the aforementioned condition (2) is also true in equilibrium. Consider the payoff difference between $\Pi_k(v,v,y_{k-1})$ and $\Pi_k(v,v+\varepsilon,y_{k-1})$:

$$\begin{aligned}(19)\quad &\hat{F}_{y_{k-1}}(v,v)^{N-k}(v-\beta_k(v,y_{k-1})) + \int_{v}^{y_{k-1}} \Pi_{k+1}(v,x)d\hat{F}_{y_{k-1}}(x,v)^{N-k}\\ &-\hat{F}_{y_{k-1}}(v+\varepsilon,v+\varepsilon)^{N-k}(v-\beta_k(v+\varepsilon,y_{k-1})) - \int_{v+\varepsilon}^{y_{k-1}} \Pi_{k+1}(v,x)d\hat{F}_{y_{k-1}}(x,v+\varepsilon)^{N-k}.\end{aligned}$$

Dividing the above expression by ε and letting $\varepsilon\to 0$, the limit must be zero if β_k is an equilibrium. Therefore, we have the first order condition

(20)

$$\begin{aligned}\hat{F}_{y_{k-1}}(v,v)^{N-k}\beta_k'(v,y_{k-1}) = &\frac{\partial}{\partial_1 v}\left(\hat{F}_{y_{k-1}}(v,v)^{N-k}\right)\left(\beta_{k+1}(v,v)-\beta_k(v,y_{k-1})\right)\\ &+\frac{\partial\hat{F}_{y_{k-1}}(v,v)^{N-k}}{\partial_2 v}(v-\beta_k(v,y_{k-1})) + \int_{v}^{y_{k-1}} \Pi_{k+1}(v,x)d\frac{\partial\hat{F}_{y_{k-1}}(x,v)^{N-k}}{\partial_2 v}.\end{aligned}$$

We first establish that the sum of the final two terms in the above equation is weakly negative. To see this, note that for any $\varepsilon > 0$, we have

$$A = \hat{F}_{y_{k-1}}(v+\varepsilon, v+\varepsilon)^{N-k}(v - \beta_k(v+\varepsilon, y_{k-1})) - \hat{F}_{y_{k-1}}(v+\varepsilon, v)^{N-k}(v - \beta_k(v+\varepsilon, y_{k-1})) \\ + \int_{v+\varepsilon}^{y_{k-1}} \Pi_{k+1}(v,x) d\hat{F}_{y_{k-1}}(x, v+\varepsilon)^{N-k} - \int_{v+\varepsilon}^{y_{k-1}} \Pi_{k+1}(v,x) d\hat{F}_{y_{k-1}}(x,v)^{N-k}$$

as an approximation of the final two terms of (29). Since $\hat{F}_{y_{k-1}}(v+\varepsilon, v+\varepsilon)$ is a minimizer of the payoff function in round k, we have

$$\hat{F}_{y_{k-1}}(v+\varepsilon, v+\varepsilon)^{N-k}(v+\varepsilon - \beta_k(v+\varepsilon, y_{k-1})) + \int_{v+\varepsilon}^{y_{k-1}} \Pi_{k+1}(v+\varepsilon, x) d\hat{F}_{y_{k-1}}(x, v+\varepsilon)^{N-k} \\ \leq \hat{F}_{y_{k-1}}(v+\varepsilon, v)^{N-k}(v+\varepsilon - \beta_k(v+\varepsilon, y_{k-1})) + \int_{v+\varepsilon}^{y_{k-1}} \Pi_{k+1}(v+\varepsilon, x) d\hat{F}_{y_{k-1}}(x,v)^{N-k}.$$

Define the difference between the two sides as B.

$$B = \hat{F}_{y_{k-1}}(v+\varepsilon, v+\varepsilon)^{N-k}(v+\varepsilon - \beta_k(v+\varepsilon, y_{k-1})) - \hat{F}_{y_{k-1}}(v+\varepsilon, v)^{N-k}(v+\varepsilon - \beta_k(v+\varepsilon, y_{k-1})) \\ + \int_{v+\varepsilon}^{y_{k-1}} \Pi_{k+1}(v+\varepsilon, x) d\hat{F}_{y_{k-1}}(x, v+\varepsilon)^{N-k} - \int_{v+\varepsilon}^{y_{k-1}} \Pi_{k+1}(v+\varepsilon, x) d\hat{F}_{y_{k-1}}(x,v)^{N-k} \leq 0.$$

The difference between the equations is

$$B - A = \varepsilon \left(\hat{F}_{y_{k-1}}(v+\varepsilon, v+\varepsilon)^{N-k} - \hat{F}_{y_{k-1}}(v+\varepsilon, v)^{N-k}\right) \\ + \int_{v+\varepsilon}^{y_{k-1}} (\Pi_{k+1}(v+\varepsilon, x) - \Pi_{k+1}(v,x))\, d\left(\hat{F}_{y_{k-1}}(x, v+\varepsilon)^{N-k} - \hat{F}_{y_{k-1}}(x,v)^{N-k}\right) \tag{21}$$

Now, since

$$\lim_{\varepsilon \to 0} \frac{B-A}{\varepsilon} = 0 \text{ and } \lim_{\varepsilon \to 0} \frac{B}{\varepsilon} \leq 0,$$

we have $\lim_{\varepsilon \to 0} \frac{A}{\varepsilon} \leq 0$. That is the sum of the last two terms in equation (29) is weakly negative. Now, since β_k is increasing in v the left hand side of equation (29) is positive. Therefore, the right hand side must be positive which along with A being weakly negative implies $\beta_{k+1}(v,v) \geq \beta_k(v, y_{k-1})$. This implies $v - \beta_k(v, y_{k-1}) \geq v - \beta_{k+1}(v,v) = \Pi_{k+1}(v,v)$. By induction, $\Pi_{k+1}(v,x)$ is weakly decreasing in x. Then, using the same argument we used in round $K-1$ the distribution that minimizes the payoff in round $k, \hat{F}_{y_{k-1}}$ must be the lower envelope $\bar{F}(\cdot|y_{k-1})$ which is independent of v. This implies that the final two terms in equation (29) must be zero. Thus, the differential equation implies

$$\beta_k(v, y_{k-1}) = \frac{1}{\bar{F}(v|y_{k-1})^{N-k}} \int_{\underline{v}}^{v} \beta_{k+1}(x,x) d\bar{F}(x|y_{k-1})^{N-k}.$$

Note that $\beta_k(v, y_{k-1})$ is increasing in v if $\beta_{k+1}(x,x)$ is increasing in x which is true by induction.[54]

Finally, we verify that the above bidding functions constitute an equilibrium. In the last auction, for a bidder with value $v \le y_{K-1}$ and any z, her payoff difference from bidding $\beta_K(v, y_{K-1})$ and $\beta_K(z, y_{K-1})$ is

$$\begin{aligned}\Pi_K(v,v,y_{K-1}) - \Pi_K(v,z,y_{K-1}) &= \int_{\underline{v}}^{v} (v-x) d\bar{F}(x|y_{K-1})^{N-K} - \int_{\underline{v}}^{z} (v-x) d\bar{F}(x|y_{K-1})^{N-K} \\ &= \int_{z}^{v} (v-x) d\bar{F}(x|y_{K-1})^{N-K} \ge 0,\end{aligned}$$

where we used the fact that the distribution $\bar{F}(\cdot|y_{K-1})$ is independent of z. Consider the k-th auction. For a bidder with value $v \le y_{k-1}$ and any z, assuming that she follows the equilibrium bidding functions from the $(k+1)$-th auction onward, her payoff difference from bidding $\beta_k(v, y_{k-1})$ and $\beta_k(z, y_{k-1})$ is

$$\Pi_k(v,v,y_{k-1}) - \Pi_k(v,z,y_{k-1}) = \int_{z}^{v} \left(v - \beta_{k+1}(x,x) - \Pi_{k+1}(v,v,x)\right) d\bar{F}(x|y_{k-1})^{N-k}.$$

First, let us consider the case, $z \ge v$. Then,

$$\Pi_k(v,v,y_{k-1}) - \Pi_k(v,z,y_{k-1}) = \int_{v}^{z} \left(\Pi_{k+1}(v,v,x) - v + \beta_{k+1}(x,x)\right) d\bar{F}(x|y_{k-1})^{N-k} \ge 0.$$

where the inequality follows from the fact that $\Pi_{k+1}(v,v,x)$ is the maximum continuation payoff that this bidder can obtain when $x \ge v$. Next, for $z \le v$

$$\Pi_k(v,v,y_{k-1}) - \Pi_k(v,z,y_{k-1}) = \int_{z}^{v} \left(v - \beta_{k+1}(x,x) - \Pi_{k+1}(v,v,x)\right) d\bar{F}(x|y_{k-1})^{N-k} \ge 0.$$

where the inequality follows from the fact that the bidder can guarantee a win by bidding $\beta_{k+1}(x,x) \le \beta_{k+1}(v,x)$. $\square$

A.2. **Proof of lemma 4.10.** Let ω, a vector of valuations, be the true state of the world. After the conclusion of round k the bidders know the valuations of the winners of the first $k-1$ rounds, since we restrict attention to monotone strategies. Without loss of generality, suppose bidders 1 to $k-1$ are the winners of the first $k-1$ rounds. Then,

$$m_k(\omega)(x) = m\left(x|\mathscr{F}_k(\omega)\right) = \begin{cases} \prod_{j=k}^{N} \frac{f(x_j)}{F(y_{k-1})}, & \text{if } (x_1,\dots,x_N) \in (y_1, y_2, \dots, y_{k-1}) \times [\underline{v}, y_{k-1}]^{N-k+1}, \\ 0, & \text{otherwise.} \end{cases}$$

[54]We proved that $\beta_K(v,v)$ is increasing in v which in tern implied that $\beta_{K-1}(v,v)$ is increasing in v.

For any $m \in \mathscr{P}$ the above constitutes a measure over remaining bidder types in round k. Let $\mathscr{P}_k$ be the set of all measures m_k. Without loss of generality let $x_k = y_k$. That is x_k is the highest valuation in round k of the auction. At the conclusion of round k the filtration $\mathscr{F}_{k+1}$ will inform us of y_k. Let the the one-step-ahead marginal of m_k be given by $m(\mathscr{F}_{k+1}|\mathscr{F}_k)$. Given some measure $n \in \mathscr{P}$ the conditional measure in round $k+1$ is given by $n(x|\mathscr{F}_{k+1})$. Precisely,

$$m(\mathscr{F}_{k+1}|\mathscr{F}_k) = \frac{f(y_k)}{F(y_{k-1})}\frac{F(y_k)^{N-k}}{F(y_{k-1})^{N-k}}; \quad n(x|\mathscr{F}_{k+1}) = \prod_{j=k+1}^{N}\frac{g(x_j)}{G(y_k)},$$

where $G \in \Delta$ and $g = G'$. Rectangularity, as defined in equation (5), requires that any combination of a one-step ahead marginal and a, possibly, distinct conditional produces a measure that is contained within $\mathscr{P}_k$. That is

$$m(\mathscr{F}_{k+1}|\mathscr{F}_k)\, n(x|\mathscr{F}_{k+1}) = r(x|\mathscr{F}_k) \in \mathscr{P}_k.$$

Clearly, if $m = n$ then $r = m$ and the above equation is satisfied. Expanding the above,

$$\frac{f(y_k)}{F(y_{k-1})}\frac{F(y_k)^{N-k}}{F(y_{k-1})^{N-k}}\prod_{j=k+1}^{N}\frac{g(x_j)}{G(y_k)} = \prod_{j=k}^{N}\frac{h(x_j)}{H(y_{k-1})},$$

for some $H \in \Delta$. Integrating both sides with respect to $x_j, j \neq t$ over $[\underline{v}, x]$ where $x \le y_k$.

$$\int_{\underline{v}}^{x}\cdots\int_{\underline{v}}^{x}\frac{f(y_k)}{F(y_{k-1})}\frac{F(y_k)^{N-k}}{F(y_{k-1})^{N-k}}\prod_{j=k+1}^{N}\frac{g(x_j)}{G(y_k)}dx_1\ldots dx_N = \int_{\underline{v}}^{x}\cdots\int_{\underline{v}}^{x}\prod_{j=k}^{N}\frac{h(x_j)}{H(y_{k-1})}dx_1\ldots dx_N$$

$$\Longrightarrow \frac{f(y_k)}{F(y_{k-1})}\frac{F(y_k)^{N-k}}{F(y_{k-1})^{N-k}}\frac{G(x)^{N-k}}{G(y_k)^{N-k}} = \frac{H(x)^{N-k}}{H(y_{k-1})^{N-k}}\frac{h(y_k)}{H(y_{k-1})}. \tag{22}$$

In the above let $x = y_k$, then we get

$$\left(\frac{1}{N-k+1}\right)\left(\frac{d}{dy_k}\frac{F(y_k)^{N-k+1}}{F(y_{k-1})^{N-k+1}} = \left(\frac{1}{N-k+1}\right)\left(\frac{d}{dy_k}\frac{H(y_k)^{N-k+1}}{H(y_{k-1})^{N-k+1}},\right.\right.$$

for all $y_k \le y_{k-1}$. This implies,

$$\frac{f(y_k)}{F(y_{k-1})} = \frac{h(y_k)}{H(y_{k-1})},$$

where the implication follows since the first equality is true for all $y_k \le y_{k-1}$. Substituting in equation (22) implies

$$\frac{F(y_k)^{N-k}}{F(y_{k-1})^{N-k}}\frac{G(x)^{N-k}}{G(y_k)^{N-k}} = \frac{H(x)^{N-k}}{H(y_{k-1})^{N-k}}.$$

Now consider $k = 1$. Then $y_{k-1} = \bar{v}$. The right hand side of the above equation is a distribution in Δ. Now fixing x let us take the minimum on both sides. Thus,

$$\min_{F,G\in\Delta} F(y_1)^{N-1}\frac{G(x)^{N-1}}{G(y_1)^{N-1}} = \bar{F}(y_1)^{N-1}\bar{G}(x|y_1)^{N-1} = \min_{H\in\Delta} H(x)^{N-1} = \bar{F}(x)^{N-1}$$

$$\implies \bar{G}(x|y_1)^{N-1} = \frac{\bar{F}(x)^{N-1}}{\bar{F}(y_1)^{N-1}},$$

for any $y_1 \geq x$. Thus the lower envelope of the set of conditional distributions Δ_{y_1}, $\bar{G}(x|y_1)$ in the above, is given by the lower envelope of the unconditional distributions.

□

A.3. **Likelihood Ratio Partial Order.** While rectangularity is an important theoretical construct, it is not the easiest condition to check in applications. Thus, we provide another, more familiar condition that guarantees DC in our model. We establish that for a set of priors that is a semi-lattice under a stronger partial order, the Likelihood Ratio p.o. ($\geq_{lr}$), the lower envelope of conditional distributions is given by the lower envelope of the parent Δ.

Definition A.1. Let $F_1, F_2 \in \Delta$ be two differentiable distribution functions with density functions f_1, f_2 respectively. Then $F_1 \geq_{lr} F_2$ if and only if

$$f_1(x)f_2(y) \geq f_1(y)f_2(x);\text{ for all } x \geq y.$$ [55]

From Müller and Scarsini (2006) we know that $\geq_{lr}$ creates a lattice in the space of distributions that have strictly positive Lebesgue densities whose logs are of bounded variation (Theorem 3.13). This assumption is satisfied in our set-up since all $F \in \Delta$ are differentiable and have strictly positive densities. Furthermore, from Whitt (1980) we know that on totally ordered spaces, like $\mathbb{R}$ and it's subsets, $\geq_{lr}$ is equivalent to Uniform Conditional Stochastic Order (UCSO). We define UCSO in our set-up.

Definition A.2. Let $A = [a, b] \subseteq [\underline{v}, \bar{v}]$ and $F(A) = F(b) - F(a)$. Then $F_1 \geq_{UCSO} F_2$ if

$$\frac{F_1(\cdot) - F_1(a)}{F_1(A)} \geq_{FOSD} \frac{F_2(\cdot) - F_2(a)}{F_2(A)} \text{ over } A \text{ for all } A.$$ [56]

Based on the above definitions we have the following result.

Lemma A.3. *Suppose Δ is a complete semi-lattice under the partial order $\geq_{lr}$. Then, for any $v \leq y, \bar{F}(v|y) = \frac{\bar{F}(v)}{\bar{F}(y)}$.*

[55] The general definition of $\geq_{lr}$ requires $f_1(x \wedge y)f_2(x \vee y) \geq f_1(x)f_2(y)$ for all x, y. When $x, y \in \mathbb{R}$ the definitions are equivalent.

[56] In the original definition in Whitt (1980) A is required to be a sub-lattice of the domain space.

Proof. Since Δ is a complete semi-lattice under $\geq_{lr}$ there exists $\bar{F} \in \Delta$ such that $\bar{F} \geq_{lr} F$ for all $F \in \Delta$. Furthermore, we know that if $\bar{F} \geq_{lr} F$ then $\bar{F} \geq_{FOSD} F$.[57] Thus, $\bar{F}$ is the lower envelope of Δ. Furthermore, since $\geq_{lr} \iff \geq_{UCSO}$ when the domain is in $\mathbb{R}$, by setting $A = [\underline{v}, y]$ we get that $\bar{F}(\cdot)/\bar{F}(y)$ is the lower envelope of Δ_y and hence the desired result. $\square$

An example of a set of priors that is a semi-lattice under $\geq_{lr}$ is the set in example 2. The set in that example belongs to a family of cdfs $\{F(\cdot;\theta)\}_{\theta\in\Theta} = \Delta$, indexed by a parameter θ that takes values in an ordered set Θ such that for any $\theta' > \theta''$

$$\frac{f(v,\theta')}{F(v,\theta')} \geq \frac{f(v,\theta'')}{F(v,\theta'')}.$$

Now, technically, the set of priors in example 2 is not rectangular. However, this is so because Δ in that example is not convex. If we take the convex hull of the set of priors in 2, the equilibrium and hence the analysis remains the same, and rectangularity is restored. We discuss this point next.

A.3.1. *Failure of Rectangularity and Nonconvexity of the Set of Distributions.* Let $S = [0,1]$ Let $P \subset \Delta(S)$ be a subset of distributions with generic elements p.

Rectangularity (to be slightly more precise, rectangularity w.r.t. a filtration) requires that for any p and q, we have

$$q(A|B)p(B) = r(A \cap B),$$

for some $r \in P$.

Suppose the set P of distributions is $\{F(v,\theta) = v^{\theta},\ \theta \in [1,2]\}$ Note that P is not a convex set even though θ is from a convex set. Consider $A = \{v \leq 1/4\}$ and $B = \{v \leq 1/2\}$. For every θ, let p^{θ} be the corresponding probability measure of F^{θ}. Note that we have

$$p^1(A|B)p^2(B) = \frac{1}{2} \cdot \left(\frac{1}{2}\right)^2 = \frac{1}{8}.$$

However, there does not exist a $\theta \in [1,2]$ such that

$$\left(\frac{1}{4}\right)^{\theta} = \frac{1}{8}.$$

Therefore, the set P is not rectangular, even though its lower envelope, which is v^2, is invariant to conditioning.

This issue arises because P is not convex. Suppose we instead consider the convex hull of P, co$P$. The lower envelope of co$P$, which is v^2, is again invariant to conditioning. And for the above

[57] See Shaked and Shanthikumar (2007) Theorem 6.E.8.

events, we can pick $\lambda = 1/3$ so that

$$\lambda\left(\frac{1}{4}\right)^1 + (1-\lambda)\left(\frac{1}{4}\right)^2 = \frac{1}{8}.$$

A.4. **Proof of Proposition 5.1.** *Necessity.* Suppose given F and $\bar{F}$, the distributions $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ are the equilibrium bid distributions. Condition C1 must hold since the valuations are drawn independently. Thus, the bids in each round of the auction must also be independent.

Consider the first statement in C2. Given DC, we can use proposition 4.7 to see that $\beta_1(\underline{v}) = \underline{b}_1 = \underline{b}_2 = \beta_2(\underline{v}) = \underline{v}$. Using proposition 4.7 again, note that $\beta_2(v) > \beta_1(v)$ for $v \in (\underline{v}, \overline{v}]$. Thus, $\beta_1(\overline{v}) = \overline{b}_1 < \overline{b}_2 = \beta_2(\overline{v})$. Next, if $b_{i,1}$ and $b_{i,2}$ are equilibrium bids of bidder i then there must exist v_i such that $b_{i,1} = \beta_1(v_i)$ and $b_{i,2} = \beta_2(v_i)$. Thus, $b_{i,1} = \beta_1(\phi_2(b_{i,2}))$. Note that $\psi(\cdot) = \beta_1(\phi_2(\cdot))$ is strictly increasing since β_1, ϕ_2 are strictly increasing. And $\psi(b_{i,2}) = b_{i,1} \leq b_{i,2}$ follows from $\beta_2 \geq \beta_1$. Lastly, the final part of C2 follows from $G_2(b) = F(\phi_2(b)) \leq F(\phi_1(b)) = G_1(\phi_1(b)$ since $\phi_2(b) \leq \phi_1(b)$. The final inequality follows from $\beta_2(v) > \beta_1(v)$.

Finally, we need to show that $\zeta(\psi(b_2), b_2, G_1, G_2)$ is strictly increasing in b_2. First note that in equilibrium $v_i = \phi_2(b_{i,2})$. Thus, rewriting equation (10) we get

$$\phi_2(b_{i,2}) = b_{i,2} + \left(\frac{N-1}{N-2}\right)\left(b_{i,2} - \psi(b_{i,2})\right)\frac{g_1(\psi(b_{i,2}))}{g_2(b_{i,2})} = \zeta(\psi(b_{i,2}), b_{i,2}, G_1, G_2)$$

for all $b_{i,2} \in [\underline{b}_2, \overline{b}_2]$. Now, we know that $\phi_2(\cdot) = \beta_2^{-1}(\cdot)$ is strictly increasing. Thus, the right hand side of the above equation must also be strictly increasing in $b_{i,2}$.

Sufficiency. To prove sufficiency we use similar arguments as in the proof of Theorem 1 in GPV. Suppose $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$ are such that conditions C1,C2 and C3 are satisfied. First, due to condition C1 each $b_{i,k}$ for a fixed k is independent and distributed according to some G_k. Next, note that due to condition C2 and the functional form of ζ, $\lim_{b_2 \to \underline{b}_2} \zeta(\psi(b_2), b_2, G_1, G_2) = \underline{b}_2$. Next, define $F(\cdot) = F(\zeta^{-1}(\cdot, G_1, G_2))$ over $[\underline{v}, \overline{v}]$ where $\underline{v} = \underline{b}_1 = \underline{b}_2$ and $\overline{v} = \zeta(\overline{b}_1, \overline{b}_2, G_1, G_2)$. Since $\zeta(\psi(b_2), b_2, G_1, G_2)$ is strictly increasing in b_2 by condition C3, F is increasing and thus a distribution over $[\underline{v}, \overline{v}]$. Once we derive F we can use equation (11) to compute $\bar{F}$.

What remains to be shown is that F and $\bar{F}$ can rationalize G_1 and G_2. First, note that given F and $\bar{F}$ the distribution G_1 must satisfy equation (11). Now, let $\xi_2(b_2, G_1, G_2) = \zeta(\psi(b_2), b_2, G_1, G_2)$. By construction $G_2(\cdot) = F(\xi_2(\cdot, G_1, G_2))$. Thus it suffices to show that $\xi_2^{-1}(\cdot, G_1, G_2)$ gives the equilibrium bidding function in round 2 of the auction. Or $\xi_2(\cdot, G_1, G_2) = \phi_2(\cdot)$. From equation (10) we know that this is true. Finally, $G_1(\cdot) = F(\xi_2(\psi^{-1}(\cdot), G_1, G_2)) = F(\xi_1(\cdot, G_1, G_2))$. Thus the proof is complete if $\xi_1^{-1}(\cdot, G_1, G_2)$ gives the equilibrium bidding function in round 1 of the auction. From equation (10) and the definition of ψ we know that

$$\xi_1(b_1, G_1, G_2) = v = \phi_2(b_2) = \phi_2(\psi^{-1}(b_1)) = \phi_2(\phi_2^{-1}(\beta_1^{-1}(b_1))) = \phi_1(b_1)$$

where (b_1, b_2) is a bidder specific bid-pair. Thus, ξ_1^{-1} is an optimal bidding function in round 1 of the auction since ϕ_1 is the inverse of the equilibrium bid function.

Finally, the monotonicity of ξ, the monotonicity of ψ and equation (11) imply the uniqueness of F and $\bar{F}$.

$\square$

A.5. **Proof of Proposition 6.1.** Bidders have a weakly dominant strategy in the last round which is to bid their valuation. Thus their payoff from participating in the final round conditional on not winning any previous rounds is given by

$$\Pi_K(v, y_{K-1}) = \min_{F_{y_{K-1}} \in \Delta_{y_{K-1}}} F_{y_{K-1}}(v)^{N-K} v - \int_{\underline{v}}^{v} x dF_{y_{K-1}}(x)^{N-K}$$

$$= \min_{F_{y_{K-1}} \in \Delta_{y_{K-1}}} \int_{\underline{v}}^{v} (v - x) dF_{y_{K-1}}(x)^{N-K} + (1 - F_{y_{K-1}}(v)^{N-K}).$$

Since $v - x$ is a decreasing function, the above payoff function is minimized by the lower envelope of the family of distributions $\Delta_{y_{K-1}}$. Thus, in equilibrium,

$$\Pi_K(v, y_{K-1}) = \bar{F}(v|y_{K-1})^{N-K} v - \int_{\underline{v}}^{v} x d\bar{F}(x|y_{K-1})^{N-K}.$$

Now consider the payoff in round $K-1$. Suppose bidders are following some strategy β^{II}_{K-1} in this round that possibly depends on previous round prices. Then for $z \geq v$,

$$\Pi_{K-1}(v, z, y_{K-2}) = \int_{\underline{v}}^{z} \left(v - \beta^{II}_{K-1}(x, y_{K-2})\right) d\hat{F}_{y_{K-2}}(x)^{N-K+1} + \int_{z}^{y_{K-2}} \Pi_K(v, x) d\hat{F}_{y_{K-2}}(x)^{N-K+1}.$$

First order conditions and $z = v$ in equilibrium imply

$$v - \beta^{II}_{K-1}(v, y_{K-2}) = \Pi_K(v, v) = v - \int_{\underline{v}}^{v} x d\bar{F}(x|v)^{N-K} \implies$$

$$\beta^{II}_{K-1}(v, y_{K-2}) = \int_{\underline{v}}^{v} x d\bar{F}(x|v)^{N-K} = \int_{\underline{v}}^{v} \beta^{II}_{K}(x) d\bar{F}(x|v)^{N-K}.$$

From the above we can see that the bid in round $K-1$ is independent of the price in round $K-2$. Also, the bid is increasing in v. Now consider the equilibrium payoff function in round $K-1$.

$$\Pi_{K-1}(v,y_{K-2})$$
$$=\min_{F_{y_{K-2}}\in\Delta_{y_{K-2}}}\int_{\underline{v}}^{v}\left(v-\beta^{II}_{K-1}(x,y_{K-2})\right)dF_{y_{K-2}}(x)^{N-K+1}+\int_{v}^{y_{K-2}}\Pi_K(v,x)dF_{y_{K-2}}(x)^{N-K+1}.$$

The term $v-\beta^{II}_{K-1}(x,y_{K-2})$ is decreasing in x. Consider the payoff function in the final round $\Pi_K(v,x)$. Let

$$F_x=\arg\min_{F\in\Delta}\frac{F(v)^{N-K}}{F(x)^{N-K}}v-\int_{\underline{v}}^{v}zd\frac{F(z)^{N-K}}{F(x)^{N-K}}.$$

Now consider $x'>x$. We have

$$\Pi_K(v,x')=\frac{F_{x'}(v)^{N-K}}{F_{x'}(x')^{N-K}}v-\int_{\underline{v}}^{v}zd\frac{F_{x'}(z)^{N-K}}{F_{x'}(x')^{N-K}}\leq\frac{F_x(v)^{N-K}}{F_x(x')^{N-K}}v-\int_{\underline{v}}^{v}zd\frac{F_x(z)^{N-K}}{F_x(x')^{N-K}}$$
$$\leq\frac{F_x(v)^{N-K}}{F_x(x)^{N-K}}v-\int_{\underline{v}}^{v}zd\frac{F_x(z)^{N-K}}{F_x(x)^{N-K}}.=\Pi_K(v,x).$$

Thus the payoff function is decreasing in x and hence the distribution that minimizes the payoff function in round $K-1$ is the lower envelope of $\Delta_{y_{K-2}}$. Therefore, in equilibrium.

$$\Pi_{K-1}(v,y_{K-2})=\int_{\underline{v}}^{v}\left(v-\beta^{II}_{K-1}(x,y_{K-2})\right)d\bar{F}(x|y_{K-2})^{N-K+1}+\int_{v}^{y_{K-2}}\Pi_K(v,x)d\bar{F}(x|y_{K-2})^{N-K+1}.$$

Next consider round k. Suppose the bid function in $k+1$ is given by the candidate bidding function. Note that

$$\Pi_{k+1}(v,y_k)=\int_{\underline{v}}^{v}\left(v-\beta^{II}_{k+1}(x)\right)d\bar{F}(x|y_k)^{N-k-1}+\int_{v}^{y_k}\Pi_{k+2}(v,x)d\bar{F}(x|y_k)^{N-k-1}.$$

Then for $z\geq v$,

$$\Pi_k(v,z,y_{k-1})=\int_{\underline{v}}^{z}\left(v-\beta^{II}_k(x,y_{k-1})\right)d\hat{F}_{y_{k-1}}(x)^{N-k}+\int_{z}^{y_{k-1}}\Pi_{k+1}(v,x)d\hat{F}_{y_{k-1}}(x)^{N-k}.$$

First order conditions and $z=v$ in equilibrium imply

$$v-\beta^{II}_k(v,y_{k-1})=\Pi_{k+1}(v,v)=v-\int_{\underline{v}}^{v}\beta^{II}_{k+1}(x,y_k)d\bar{F}(x|v)^{N-k-1}\implies$$

$$\beta_k^{II}(v) = \int_{\underline{v}}^{v} \beta_{k+1}^{II}(x) d\bar{F}(x|v)^{N-k-1}.$$

□

APPENDIX B. CONSISTENT PLANNING

In defining the equilibrium concept in the benchmark model we made the following conceptual assumption about how bidders view the future rounds. A bidder in round k views her payoff from round $k+1$ as a function of her valuation and price sequence. Embedded in this function are beliefs (lower envelope of conditional distributions) that she will use in the next round.

Clearly, it is possible to model bidder's view of future rounds under different assumptions. One such alternative is to model bidders as following *consistent planning* (CP). This is a variant of backward induction (in certain cases exactly that) that pins down what future selves of a decision maker would do conditional on current information and then keep this as fixed when making decisions now. The second approach was introduced by Strotz (1955). Siniscalchi (2011) provided behavioral foundations for CP which were essentially a level of sophistication on the part of the player.

A bidder in our framework is a consistent planner if she (i) holds correct beliefs about her own future choices, that is she knows her own future strategies and (ii) evaluates her stream of payoffs using her current beliefs about types. A crucial difference between our approach in the main body of the paper and CP is point (ii) above. Precisely, while (i) allows the bidder to work backwards to find her optimal strategy in each round, (ii) implies that the payoff does not have a recursive structure under CP as it did using our assumption about how bidders view the future rounds. Despite this seemingly crucial difference we will show that both the methods produce the same outcome.

Let $\Pi_k^{cp}(v, b, \beta_{i,k+1}, \ldots, \beta_{i,K}, \boldsymbol{\beta}_{-i}, F(\cdot|\tilde{p}_{k-1}))$ be a bidder's continuation payoff (conditional on not winning a unit yet) from bidding b in round k when others are following $\boldsymbol{\beta}_{-i}$, she will be using the equilibrium strategy next round onwards and the bidder thinks the valuation of other (remaining) bidders is governed by some conditional distribution $F(\cdot|\tilde{p}_{k-1})$. A CP bidder will evaluate the entire sequence of payoffs using this distribution function. This means that a CP bidder will think about all eventualities using the distribution $F(\cdot|\tilde{p}_{k-1})$: the chances of winning the current round, next round and so on. This is where our approach differed. In our approach a bidder in round k evaluates her chances of winning the current round using $F(\cdot|\tilde{p}_{k-1})$. The probability of winning future rounds may depend on the outcome of the current round and is encoded in the payoff function of the next round. This will become clearer when we write down the payoff function explicitly.

Definition B.1. A collection of strategies $\boldsymbol{\beta}^{cp}$ constitutes a *consistently planned equilibrium* of a sFPA game with ambiguity if for each i and k

$$\beta^{cp}_{i,k}(\cdot,\tilde{p}_{k-1}) \in \arg\max_b \min_{F(\cdot|\tilde{p}_{k-1})\in\Delta_{\tilde{p}_{k-1}}} \Pi^{cp} i_k(\cdot,b,\beta_{i,k+2},\ldots,\beta_{i,K},\boldsymbol{\beta}_{-i},F(\cdot|\tilde{p}_{k-1}))).$$

B.1. **Equivalence of sequentially optimal equilibrium and CP equilibrium.** The definitions of a sequentially optimal (SO) equilibrium and CP equilibrium are essentially identical. This is because both are found using backward induction. However, the crucial difference is with regards how bidders evaluate future payoffs. In this section we explicitly define the CP equilibrium in our auction environment and show that it leads to the same closed form as the SO equilibrium. To do this we go through the proof of Proposition 4.2 and show that CP leads to same equilibrium.

B.1.1. *Round K*. In the final round both equilibrium concepts will give the same equilibrium bidding function, as long as the equilibrium is monotone. That is in a CP equilibrium, the bidding function in the final round is also

$$\beta^{cp}_K(v,y_{K-1}) = \frac{1}{\bar{F}(v|y_{K-1})^{N-K}} \int_{\underline{v}}^{v} x d\bar{F}(x|y_{K-1})^{N-K}.$$

This is so because in the final round the payoff function in a CP equilibrium coincides with the payoff function of our candidate SO equilibrium. Define the following 'payoff' function for round K.

$$\Gamma_K(v,x,F_{y_{K-2}}) = \frac{F_{y_{K-2}}(v)^{N-K}}{F_{y_{K-2}}(x)^{N-K}}(v-\beta^{cp}_K(v,x))$$

This is a CP bidder's evaluation of her round K payoff in round $K-1$ given a winning bidder's valuation equals x and her 'belief' about the distribution of values in round K, $F_{y_{K-2}} \in \Delta_{y_{K-2}}$.

B.1.2. *Monotonicity of $\beta^{cp}_K(v,v)$ and $\Gamma_K(v,x,F_{y_{K-2}})$ in v and x respectively.* Since β^{cp}_K is the same as β_K the monotonicity of $\beta^{cp}_K(v,v)$ follows since we proved the monotonicity of $\beta_K(v,v)$. Let

$$\xi(x) = \frac{F_{y_{K-2}}(v)^{N-K}}{F_{y_{K-2}}(x)^{N-K}},$$

$$\eta(x) = \frac{F_x(x)^{N-K}}{F_x(v)^{N-K}} = \max_{F\in\Delta} \frac{F(x)^{N-K}}{F(v)^{N-K}},$$

$$\zeta(x) = \frac{\int_{\underline{v}}^{v} F_x(w)^{N-K} dw}{F_x(x)^{N-K}} = \min_{F\in\Delta} \frac{\int_{\underline{v}}^{v} F(w)^{N-K} dw}{F(x)^{N-K}}.$$

Now, consider $\Gamma_K(v,x,F_{y_{K-2}})$. Note that for any $v \leq x$, we have

$$\begin{aligned}
\Gamma_K(v,x,F_{y_{K-2}}) =& \frac{F_{y_{K-2}}(v)^{N-K}}{F_{y_{K-2}}(x)^{N-K}} \frac{\int_{\underline{v}}^{v} \bar{F}(w|x)^{N-K} dw}{\bar{F}(v|x)^{N-K}} \\
=& \frac{F_{y_{K-2}}(v)^{N-K}}{F_{y_{K-2}}(x)^{N-K}} \frac{\int_{\underline{v}}^{v} F_x(w)^{N-K} dw}{F_x(v)^{N-K}}; \text{ where } F_x(\cdot) \in \arg\min_{F\in\Delta} \frac{F(\cdot)}{F(x)} \\
=& \frac{F_{y_{K-2}}(v)^{N-K}}{F_{y_{K-2}}(x)^{N-K}} \frac{F_x(x)^{N-K}}{F_x(v)^{N-K}} \frac{\int_{\underline{v}}^{v} F_x(w)^{N-K} dw}{F_x(x)^{N-K}} \\
=& \xi(x)\eta(x)\zeta(x),
\end{aligned}$$

Note that ξ, η and ζ are all positive and $\xi'(x) < 0$. Applying the Envelope Theorem to $\eta(x)$ and $\zeta(x)$, we have

$$\eta'(x) = \frac{(N-K)F_x(x)^{N-K-1} f_x(x)}{F_x(v)^{N-K}}$$

and

$$\zeta'(x) = -\frac{(N-K)F_x(x)^{N-K-1} f_x(x) \int_{\underline{v}}^{v} F(w)^{N-K} dw}{F_x(x)^{2(N-K)}},$$

where f_x is the density of F_x. Therefore, applying the chain rule gives

$$\frac{\partial \Gamma_K(v,x,F_{y_{K-2}})}{\partial x} = \xi'(x)\eta(x)\zeta(x) + \xi(x)\eta'(x)\zeta(x) + \xi(x)\eta(x)\zeta'(x).$$

Since $\xi'(x)\eta(x)\zeta(x) < 0$ and

$$\begin{aligned}
\xi(x)\eta'(x)\zeta(x) + \xi(x)\eta(x)\zeta'(x) = \xi(x) \Bigg[& \frac{(N-K)F_x(x)^{N-K-1} f_x(x)}{F_x(v)^{N-K}} \cdot \frac{\int_{\underline{v}}^{v} F_x(w)^{N-K} dw}{F_x(x)^{N-K}} \\
& - \frac{F_x(x)^{N-K}}{F_x(v)^{N-K}} \cdot \frac{(N-K)F_x(x)^{N-K-1} f_x(x) \int_{\underline{v}}^{v} F(w)^{N-K} dw}{F_x(x)^{2(N-K)}} \Bigg]
\end{aligned}$$

$$= \xi(x)\left[\frac{(N-K)f_x(x)\int_{\underline{v}}^{v}F_x(w)^{N-K}dw}{F_x(v)^{N-K}F_x(x)} - \frac{(N-K)f_x(x)\int_{\underline{v}}^{v}F_x(w)^{N-K}dw}{F_x(v)^{N-K}F_x(x)}\right] = 0,$$

it follows that

$$\frac{\partial \Gamma_K(v,x,F_{y_{K-2}})}{\partial x} < 0.$$

B.1.3. *Round* $K-1$. : Now consider round $K-1$. Suppose in this round bidders follow a strategy given by $\beta^{cp}_{K-1}(v,y_{K-2})$ which is strictly increasing in v. Suppose a bidder with valuation v bids $\beta^{cp}_{K-1}(z,y_{K-2})$ where $z \geq v$ and $z \leq y_{K-2}$. Assuming she bids according to the equilibrium strategy β^{cp}_{K} in the next round, her payoff is

$$\begin{aligned}\Pi_{K-1}(v,z,y_{K-2}) = \min_{F_{y_{K-2}}\in\Delta_{y_{K-2}}} & F_{y_{K-2}}(z)^{N-K+1}\left(v-\beta^{cp}_{K-1}(z,y_{K-2})\right) \\ & + \int_{z}^{y_{K-2}} \frac{F_{y_{K-2}}(v)^{N-K}}{F_{y_{K-2}}(x)^{N-K}}\left(v-\beta^{cp}_{K}(v,x)\right)dF_{y_{K-2}}(x)^{N-K+1}.\end{aligned} \tag{23}$$

Let $\hat{F}_{y_{K-2}}$ be the minimizer of the above payoff function. As before, let us assume that $\hat{F}_{y_{K-2}}$ does not depend on z or v. Then

$$\begin{aligned}\Pi_{K-1}(v,z,y_{K-2}) = & \hat{F}_{y_{K-2}}(z)^{N-K+1}\left(v-\beta^{cp}_{K-1}(z,y_{K-2})\right) \\ & + \int_{z}^{y_{K-2}} \frac{\hat{F}_{y_{K-2}}(v)^{N-K}}{\hat{F}_{y_{K-2}}(x)^{N-K}}(v-\beta^{cp}_{K}(v,x))d\hat{F}_{y_{K-2}}(x)^{N-K+1}.\end{aligned} \tag{24}$$

Taking derivative of the payoff function with respect to z, we get

$$\begin{aligned}\frac{\partial \Pi_{K-1}(v,z,y_{K-2})}{\partial z} = & \frac{d\hat{F}_{y_{K-2}}(z)^{N-K+1}}{dz}\left(v-\beta^{cp}_{K-1}(z,y_{K-2})\right) - \hat{F}_{y_{K-2}}(z)^{N-K+1}\frac{\partial \beta^{cp}_{K-1}(z,y_{K-2})}{\partial z} \\ & - \frac{\hat{F}_{y_{K-2}}(v)^{N-K}}{\hat{F}_{y_{K-2}}(z)^{N-K}}(v-\beta^{cp}_{K}(v,z))\frac{d\hat{F}_{y_{K-2}}(z)^{N-K+1}}{dz}.\end{aligned} \tag{25}$$

In equilibrium, the bidder would choose $z = v$. Thus, the first order condition for the bidder is

$$\frac{d\hat{F}_{y_{K-2}}(v)^{N-K+1}}{dv}\left(\beta^{cp}_{K}(v,v)-\beta^{cp}_{K-1}(v,y_{K-2})\right) - \hat{F}_{y_{K-2}}(v)^{N-K+1}\frac{\partial \beta^{cp}_{K-1}(v,y_{K-2})}{\partial v} = 0, \tag{26}$$

Note the FOC for a CP equilibrium is exactly the same as the FOC for the SO equilibrium. Thus, for the CP equilibrium as well, the bid function in round $K-1$ is given by

$$\beta^{cp}_{K-1}(v,y_{K-2}) = \frac{1}{\hat{F}_{y_{K-2}}(v)^{N-K+1}} \int_{\underline{v}}^{v} \beta^{cp}_{K}(x,x)d\hat{F}_{y_{K-2}}(x)^{N-K+1}.$$

The monotonicity of the bid functions follows in the same way as for the SO equilibrium. Thus, what remains to be show for this round is that $\hat{F}_{y_{K-2}}(v) = \bar{F}(v|y_{K-2})$. Note the the maximum payoff function in round $K-1$ can be written as

$$\Pi^*_{K-1}(v,y_{K-2}) = \hat{F}_{y_{K-2}}(v)^{N-K+1}\left(v - \beta^{cp}_{K-1}(v,y_{K-2})\right) + \int_{v}^{y_{K-2}} \Gamma_K(v,x,\hat{F}_{y_{K-2}})d\hat{F}_{y_{K-2}}(x)^{N-K+1}. \tag{27}$$

Note that $v-\beta^{cp}_{K-1}(v,y_{K-2}) \geq v-\beta_K(v,v) = \Gamma_K(v,v,y_{K-2})$ using the same argument as the SO equilibrium. Furthermore, since $\Gamma_K(v,x,\hat{F}_{y_{K-2}})$ is non-increasing in x we can conclude that $\hat{F}_{y_{K-2}}(v) = \bar{F}(v|y_{K-2})$ since the latter minimizes the above function. This also shows that $\beta^{cp}_{K-1} = \beta_{K-1}$.

B.1.4. *Monotonicity of $\beta^{cp}_{K-1}(v,v)$ and $\Gamma_{K-1}(v,x,F_{y_{K-3}})$ in v and x respectively.* As was the case for K, before since $\beta^{cp}_{K-1} = \beta_{K-1}$, the monotonicity of $\beta^{cp}_{K-1}(v,v)$ follows from the monotonicity of the latter. Consider $\Gamma_{K-1}(v,x,F_{y_{K-3}})$. The notation follows the familiar pattern.

$$\begin{aligned}
\Gamma_{K-1}(v,x,F_{y_{K-3}}) &= \left(\frac{F_{y_{K-3}}(v)}{F_{y_{K-3}}(x)}\right)^{N-K+1}\left(v - \beta^{cp}_{K-1}(v,x)\right) \\
&\quad + \int_{v}^{x} \frac{F_{y_{K-3}}(v)^{N-K}}{F_{y_{K-3}}(z)^{N-K}}(v - \beta^{cp}_{K}(v,z))d\frac{F_{y_{K-3}}(z)^{N-K+1}}{F_{y_{K-3}}(x)^{N-K+1}} \\
&= \left(\frac{F_{y_{K-3}}(v)}{F_{y_{K-3}}(x)}\right)^{N-K+1}\left(v - \beta^{cp}_{K}(v,v) + \int_{\underline{v}}^{v} \frac{\bar{F}(z|x)^{N-K+1}}{\bar{F}(v|x)^{N-K+1}}d\beta_K(z,z)\right) \\
&\quad + \int_{v}^{x} \Gamma_K(v,z,F_{y_{K-3}})d\frac{F_{y_{K-3}}(z)^{N-K+1}}{F_{y_{K-3}}(x)^{N-K+1}}
\end{aligned}$$

Now, let

$$r(z) = \begin{cases} \Gamma_K(v,v,F_{y_{K-3}}), & \text{for } z \in [\underline{v},v), \\ \Gamma_K(v,z,F_{y_{K-3}}), & \text{for } z \in [v,x]. \end{cases}$$

Then,

$$\Gamma_{K-1}(v,x,F_{y_{K-3}}) = \int_{\underline{v}}^{x} r(z)d\frac{F_{y_{K-3}}(z)^{N-K+1}}{F_{y_{K-3}}(x)^{N-K+1}} + \left(\frac{F_{y_{K-3}}(v)}{F_{y_{K-3}}(x)}\right)^{N-K+1}\int_{\underline{v}}^{v} \frac{\bar{F}(z|x)^{N-K+1}}{\bar{F}(v|x)^{N-K+1}}d\beta_K(z,z)$$

Using the same argument as in the SO equilibrium, we can show that the derivative of the first term in the above equation is negative. Thus, we need to show that the second term is also deceasing in x. Since $\beta_K(z,z)$ is increasing in z we can use the exact same method we used in proving the monotonicity of Γ_K.

B.1.5. *Induction.* So far we have shown that the CP equilibrium is equivalent to SO equilibrium in the final two rounds of the auction. Now we complete the proof of equivalence using induction. Suppose $\beta_{k+1}^{cp}(v,v)$ is non-decreasing in v and $\Gamma_{k+1}(v,x,y_{k-1})$ is non-increasing in x. Suppose bidders follow the strategy $\beta_k^{cp}(v,y_{k-1})$ that is increasing in v. Consider a bidder with value v who bids $\beta_k^{cp}(z,y_{k-1})$ where $z \leq y_{k-1}$. This bidder's expected payoff is

$$\Pi_k(v,z,y_{k-1}) = \min_{F_{y_{k-1}} \in \Delta_{y_{k-1}}} F_{y_{k-1}}(z)^{N-k}(v - \beta_k^{cp}(z,y_{k-1})) + \int_z^{y_{k-1}} \Gamma_{k+1}(v,x,F_{y_{k-1}})\, dF_{y_{k-1}}(x)^{N-k},$$

Till now we have assumed that $\hat{F}$ is independent of v. As we did in the SO equilibrium, we now establish that this is the case in the CP equilibrium as well. Let $\hat{F}_{y_{k-1}}(\cdot,z)$ denote a minimizer of the above problem. Consider the payoff difference between $\Pi_k(v,v,y_{k-1})$ and $\Pi_k(v,v+\varepsilon,y_{k-1})$:

$$\begin{aligned}
&\hat{F}_{y_{k-1}}(v,v)^{N-k}(v - \beta_k^{cp}(v,y_{k-1})) + \int_v^{y_{k-1}} \Gamma_{k+1}(v,x,\hat{F}_{y_{k-1}}(\cdot,v))\, d\hat{F}_{y_{k-1}}(x,v)^{N-k} \\
&- \hat{F}_{y_{k-1}}(v+\varepsilon,v+\varepsilon)^{N-k}(v - \beta_k^{cp}(v+\varepsilon,y_{k-1})) \\
&- \int_{v+\varepsilon}^{y_{k-1}} \Gamma_{k+1}(v,x,\hat{F}_{y_{k-1}}(\cdot,v+\varepsilon))\, d\hat{F}_{y_{k-1}}(x,v+\varepsilon)^{N-k}.
\end{aligned} \tag{28}$$

Dividing the above expression by ε and letting $\varepsilon \to 0$, the limit must be zero if β_k^{cp} is an equilibrium. Therefore, we have the first order condition

$$\begin{aligned}
&\hat{F}_{y_{k-1}}(v,v)^{N-k} \frac{\partial \beta_k^{cp}(v,y_{k-1})}{\partial v} \\
&= \frac{\partial}{\partial_1 v}\left(\hat{F}_{y_{k-1}}(v,v)^{N-k}\right)(\beta_{k+1}^{cp}(v,v) - \beta_k^{cp}(v,y_{k-1})) \\
&+ \frac{\partial \hat{F}_{y_{k-1}}(v,v)^{N-k}}{\partial_2 v}(v - \beta_k^{cp}(v,y_{k-1})) + \int_v^{y_{k-1}} \Gamma_{k+1}(v,x,\hat{F}_{y_{k-1}}(\cdot,v))\, d\frac{\partial \hat{F}_{y_{k-1}}(x,v)^{N-k}}{\partial_2 v} \\
&+ \int_v^{y_{k-1}} \lim_{\varepsilon \to 0} \frac{\Gamma_{k+1}(v,x,\hat{F}_{y_{k-1}}(\cdot,v)) - \Gamma_{k+1}(v,x,\hat{F}_{y_{k-1}}(\cdot,v+\varepsilon))}{\varepsilon}\, d\hat{F}_{y_{k-1}}(x,v)^{N-k}
\end{aligned} \tag{29}$$

Just as in SO case, the final three term of the equation can be shown to be weakly negative. This leads us to the same conclusion as in SO equilibrium: the distribution that minimizes the payoff in round $k, \hat{F}_{y_{k-1}}$ must be the lower envelope $\bar{F}(\cdot|y_{k-1})$ which is independent of v due the semi-lattice structure of $\Delta_{y_{k-1}}$. This implies that the final three terms in the above equation must be zero. Thus, the differential equation implies

$$\beta_k^{cp}(v, y_{k-1}) = \frac{1}{\bar{F}(v|y_{k-1})^{N-k}} \int_{\underline{v}}^{v} \beta_{k+1}^{cp}(x,x) d\bar{F}(x|y_{k-1})^{N-k}.$$

Finally we verify that the bid functions constitute an equilibrium. Using the same argument as SO we can establish that bidding $\beta_K^{cp}(v, y_{K-1})$ in the final round is optimal if others are doing so. Let us consider the k-th round. The payoff difference from bidding $\beta_k^{cp}(v, y_{k-1})$ and $\beta_k^{cp}(z, y_{k-1})$ where $z \geq v$ is

$$\Pi_k(v,v,y_{k-1}) - \Pi_k(v,z,y_{k-1}) = \int_v^z \left(\left(\Gamma_{k+1}(v,x,\bar{F}(\cdot|y_{k-1}) - v + \beta_{k+1}^{cp}(x,x) \right) \right) d\bar{F}(x|y_{k-1})^{N-k}.$$

Now, since $\frac{\bar{F}(\cdot|x)}{\bar{F}(x|x)}$ FOSD $\frac{\bar{F}(\cdot|y_{k-1})}{\bar{F}(x|y_{k-1})}$ it is the case that $\Gamma_{k+1}(v,x,\bar{F}(\cdot|y_{k-1}) \geq \Pi_{k+1}^{so}(v,v,x)$ where Π^{so} is the continuation payoff in the SO equilibrium. It is straightforward to see this for $k+1 = K-1$. For any other case the inequality follows inductively. The intuition is that that Γ evaluates a bidder's continuation payoff using the current round's beliefs (lower envelope). Whereas Π^{so} evaluates the payoffs using the beliefs of future rounds which will necessarily be (weakly) below the current beliefs. Then,

$$\Pi_k(v,v,y_{k-1}) - \Pi_k(v,z,y_{k-1}) \geq \int_v^z \left(\left(\Pi_{k+1}^{so}(v,v,x) - v + \beta_{k+1}^{cp}(x,x) \right) \right) d\bar{F}(x|y_{k-1})^{N-k} \geq 0$$

where the inequality follows from the proof of sufficiency of the SO equilibrium. The case of $z \leq v$ follows similarly.

Appendix C. Extension: Uncertain Number of Bidders

In the model we have studied till now ambiguity takes the form of an uncertain distribution of values. Another way to model ambiguity is to assume that the number of bidders is not known as was the case in the single-unit environment of Levin and Ozdenoren (2004). In this section we study a variation of the original model where the number of bidders are uncertain and bidders are ambiguity averse.

The model is the same as before with few modification. Now there is a common prior (distribution of valuations) F with continuous density f. That is Δ is a singleton. In addition, there is uncertainty about the the number of bidders actively participating in the auction. To model this, suppose there N *potential* bidders in the auction. Before the first round of the auction, a potential bidder is *active* with probability $\rho \in (\underline{\rho}, \overline{\rho})$ where $\underline{\rho} > 0$ and $\overline{\rho} \leq 1$. Active bidders bid seriously, where as *inactive* bidders do not. A bid is considered serious if it is greater than the reserve price $w < \underline{v}$. For simplicity, we assume that inactive bidders bid the reserve price.[58] Active and inactive bidders remain so through out the auction. Suppose bidders only know that $\rho \in (\underline{\rho}, \overline{\rho})$ however they do not know its true value. As in the benchmark model, bidders are MMEU maximizers.

C.1. **First Price Auctions.** To begin our analysis, we first consider sFPAs. As before, we will look for symmetric strategies that are monotone in valuations. An equilibrium is defined in definition B.1. All notations from the benchmark model are adopted here. The following proposition states the equilibrium in this environment. The proof can be found in appendix D.

Proposition C.1. *There exists a unique symmetric equilibrium of sFPAs, β^I where,*

$$\beta_k^I(v) = \alpha_{0,k}(v)w + \sum_{l=1}^{N-k} \alpha_{l,k}(v)\mathbb{E}\left[\beta_{k+1}^I\left(V_1^{(l)}\right) \middle| V_1^{(l)} < v\right] \text{ for } k \leq K-1 \text{ and}$$

$$\beta_K^I(v) = \alpha_{0,K}(v)w + \sum_{l=1}^{N-K} \alpha_{l,K}(v)\mathbb{E}\left[V_1^{(l)} \middle| V_1^{(l)} < v\right]$$

where

$$\alpha_{l,k}(v) = \frac{\binom{N-k}{l}(1-\overline{\rho})^{N-k-l}\overline{\rho}^l F(v)^l}{\sum_{s=0}^{N-k}\binom{N-k}{s}(1-\overline{\rho})^{N-k-s}\overline{\rho}^s F(v)^s}.$$

Note that the above equilibrium is history-independent, much like the equilibrium in MW. In MW a bidder's bid in round k is equal to her bid in round $k+1$ conditional on *just* losing round k. This intuition essentially suggests that a bidder equates her marginal gain and loss from bidding a little more in round k. The intuition behind the equilibrium in the current model is very similar. Again, a bidder's bid in round k is equal to her bid in round $k+1$ conditional on just losing round k. However, a bidder can lose in multiple ways depending on the number of active bidders. Therefore, the bid is a weighted average. The probability weight $\alpha_{l,k}$ is the *highest* probability of facing l bidders all of whom have a lower valuation than the bidder's valuation of v. The bid function has these weights since a bidder is conditioning on just losing the current round. The bid in the last round for a bidder with valuation v is the expected valuation of the highest valued bidder who will lose to this bidder.

Finally, these probability weights are calculated using the $\overline{\rho}$. This is an effect of the bidders being MMEU maximizers. Recall, in the benchmark model bidders use the lower envelope of the set of distributions in each round to evaluate their payoffs. This was because the lower envelope, or the strongest possible competition, minimizes their payoffs. Similarly, with ambiguity about

[58] We can let $w = 0$ and it does not affect the results.

the number of competitors, bidders evaluate their payoffs using $\overline{\rho}$ as their payoffs are minimized at this value of the parameter ρ. From the closed form of the equilibrium we can also make the following observation.

Remark C.2. Bidders become more aggressively as the auction proceeds. That is $\beta_k^I(v) < \beta_{k+1}^I(v)$.

The functional form of the equilibrium strategies and the argument above also suggest similarities between the bidding strategies in the standard IPV environment of MW and our set up. To wit, the following corollary confirms that our model is in fact a generalization of the MW model of sequential auctions.

Corollary C.3. *As $\underline{\rho} \to \overline{\rho} \to 1$ the bidding strategy $\beta^I \to \beta^{MW}$ given in equation (4).*

When $\rho = 1$, and bidders know that there are N bidders in the auction. Thus it is no surprise that the equilibrium converges to the MW equilibrium with a known number of bidders.

C.1.1. *Price-path.* As in the bench mark model, the true value of the parameter ρ is not known. However, we can calculate expectation of future prices given current prices for some ρ. In a monotone equilibrium, the price in round k will be given by the bid of the highest valued bidder in round k. That is, using previous notation, the bid of the bidder with value y_k is

$$p_k = \beta_k^I(y_k) = \alpha_{0,k}(y_k)w + \sum_{l=1}^{N-k} \alpha_{l,k}(y_k)\mathbb{E}\left[\beta_{k+1}\left(V_1^{(l)}\right)\middle| V_1^{(l)} < y_k\right]$$

Let $\mathbb{E}_\rho\left[P_{k+1}|p_k\right]$ be the expectation of price in round $k+1$ given previous prices. Let $\alpha_{\rho,l,k}$ be the probability there are l active bidders in round $k+1$ given ρ and y_k. Then,

$$\alpha_{\rho,l,k} = \frac{\binom{N-k}{l}(1-\rho)^{N-k-l}\rho^l F(y_k)^l}{\sum_{s=0}^{N-k}\binom{N-k}{s}(1-\rho)^{N-k-s}\rho^s F(y_k)^s},$$

$$\mathbb{E}_\rho\left[P_{k+1}|p_k\right] = \alpha_{\rho,0,k}w + \sum_{l=1}^{N-k} \alpha_{\rho,l,k}\mathbb{E}\left[\beta_{k+1}^I\left(V_1^{(l)}\right)\middle| V_1^{(l)} < y_k\right]$$

Now, the probability mass function $\{\alpha_{l,k}(y_k)\}$ first order stochastically dominates $\{\alpha_{\rho,l,k}\}$ since $\rho \le \overline{\rho}$. Thus, since the bid functions are monotonically increasing, we have the following result on the price path.

Proposition C.4. *With uncertain number of bidders, prices are a supermartingale in sFPAs. That is $p_k \ge \mathbb{E}_\rho\left[P_{k+1}|p_k\right]$ with strict inequality for $\rho < \overline{\rho}$.*

The intuition behind the result again rests on the over-estimation of competition by bidders as we discussed in the benchmark model. Thus, this over-estimation of competition in the presence of ambiguity and aversion to it in the part of bidders can account for the declining price anomaly.

C.2. **Second Price Auctions.** In this section we turn to second-price auctions (SPA). The framework and model remains the same as the previous case, but now the winners in each round pay the second highest bid. As before, we assume that the winning bids are announced. However, again, prices will not affect future round bids.

Let $\beta_k^{II}(v, \tilde{p}_{k-1})$ represent their equilibrium strategy in auction k as a function of their valuations and past winning bids. For now we assume that $\beta_k^{II}(v, \tilde{p}_{k-1}) > w$. As before, assume that $\beta_k^{II}(v, \tilde{p}_{k-1})$ is strictly increasing in v for any k. Since the bid functions are increasing, $p_1, \ldots, p_{k-1}$ correspond to the realized values, $y_1, \ldots, y_{k-1}$ of the winners, with $y_1 \geq \ldots \geq y_{k-1}$.Therefore the bidding function can be re-written as $\beta_k^{II}(v, \tilde{y}_{k-1})$. The equilibrium can be derived in a similar fashion to the derivation in the first price auction.

Proposition C.5. *There exists a unique symmetric equilibrium of sSPAs, β^{II} where,*

$$\beta_k^{II}(v) = \delta_{0,k}(v)w + \sum_{l=1}^{N-k-1} \delta_{l,k}(v)\mathbb{E}\left[\beta_{k+1}^{II}\left(V_1^{(l)}\right)\middle| V_1^{(l)} < v\right] \text{ for } k \leq K-1 \text{ and}$$

$$\beta_K^{II}(v) = v,$$

where

$$\delta_{l,k}(v) = \frac{\binom{N-k-1}{l}\left(1-\overline{\rho}\right)^{N-k-1-l}\overline{\rho}^l F(v)^l}{\sum_{s=0}^{N-k-1}\binom{N-k-1}{s}\left(1-\overline{\rho}\right)^{N-k-1-s}\overline{\rho}^s F(v)^s}.$$

The derivation of the equilibrium and the proof of existence and monotonicity is similar to the sFPAs derivation and hence omitted.

C.2.1. *Price Path.* Price in any round of a sSPA is the bid placed by the bidder with the second highest valuation. That is, price in round k is

$$p_k = \beta_k^{II}(y_{k-1}) = \delta_{0,k}(y_{k-1})w + \sum_{l=1}^{N-k-1} \delta_{l,k}(y_{k-1})\mathbb{E}\left[\beta_{k+1}^{II}\left(V_1^{(l)}\right)\middle| V_1^{(l)} < y_{k-1}\right]$$

for $k \leq K-1$. If $p_k > w$ then we know that there exist at least two active bidders in round k one of whom would still be present in the next round. In this case, fixing ρ, let $\delta_{\rho,l,k}$ be the probability that there are l additional active bidders competing for the unit in round $k+1$ of auction, other the price setting active bidder. Then we have

$$\delta_{\rho,l,k} = \frac{\binom{N-k-1}{l}\left(1-\rho\right)^{N-k-1-l}\rho^l F(y_{k-1})^l}{\sum_{s=0}^{N-k-1}\binom{N-k-1}{s}\left(1-\rho\right)^{N-k-1-s}\rho^s F(y_{k-1})^s}.$$

Given ρ, the expected price in round $k+1$ is

$$\mathbb{E}_\rho\left[P_{k+1}|p_k\right] = \delta_{\rho,0,k}w + \sum_{l=1}^{N-k-1} \delta_{\rho,l,k}\mathbb{E}\left[\beta_{k+1}^{II}\left(V_1^{(l)}\right)\middle| V_1^{(l)} < y_{k-1}\right].$$

Since $\{\delta_{\rho,l,k}\}$ FOSD $\delta_{l,k}(y_{k-1})$, we get the following result.

Proposition C.6. *With uncertain number of bidders, prices are a supermartingale in sSPAs. That is $p_k \geq \mathbb{E}_\rho\left[P_{k+1}|p_k\right]$ with strict inequality for $\rho < \overline{\rho}$.*

C.3. **Revenue Comparisons.** Having computed the equilibrium strategies and shown that prices decline in our model, we now compare revenues across auction formats. To do so, let us first compare the bid functions. In the standard model studied in MW, the authors found an equivalence between the bid functions between the two auction formats which can be stated as $\beta^I_{k+1}(v) = \beta^{II}_k(v)$. Bidders in the standard auction can thought of as ambiguity neutral. It turns out that in our model with ambiguity averse bidders, the equilibrium bidding functions in sFPA and sSPA show the same relationship.

Corollary C.7. *The equilibrium bidding functions in sSPA and sFPA are such that $\beta^{II}_k(v) = \beta^I_{k+1}(v)$*

Without ambiguity the above condition implied a revenue equivalence between the two auction formats. However, just as was the case in the benchmark model, with ambiguity aversion the two auction formats are not revenue equivalent. Note that

$$\mathbb{E}_\rho\left[P^I_{k+1}\right] = \mathbb{E}_\rho\left[\beta^I_{k+1}\left(V^{(N)}_{k+1}\right)\right] = \mathbb{E}_\rho\left[\beta^{II}_k\left(V^{(N)}_{k+1}\right)\right] = \mathbb{E}_\rho\left[P^{II}_k\right] \geq \mathbb{E}_\rho\left[P^{II}_{k+1}\right],$$

where the first equality follows from definition, the second from corollary C.7, the third from definition and the final inequality from the fact that the prices are supermartingale in sSPAs. The inequality is strict if $\rho < \overline{\rho}$. Thus we have that sFPA generate higher revenue than sSPAs.

Finally, comparing the revenue in the sequential auction formats with the uniform price auction we again find that the sFPAs and sSPAs generate more revenue that the uniform price auction. The proof and intuition for the result is exactly the same as before. In uniform price auction bidders bid their valuations thus, the market clear price is the $k+1$-st highest valuation which is precisely the price in the final sound of a sSPA. And with declining prices we get that the revenue from sSPA is higher than the uniform price auction. Thus we have the following result. Let R^{format}_ρ be the expected revenue in an auction format calculate using a fixed ρ.

Proposition C.8. *With uncertain number of bidders the revenue from ranking between the sequential auction formats and uniform price auction is, $R^I_\rho \geq R^{II}_\rho \geq R^{up}_\rho$, where the inequality is strict for $\rho < \overline{\rho}$.*

Appendix D. Proof of proposition C.1

Let ρ_k be the probability that a given potential bidder who did not win in any of the previous rounds is active in the round k. Since bidders are following a monotone strategy, this probability is given by

$$\rho_k = \frac{\rho_{k-1}\frac{F(y_{k-1})}{F(y_{k-2})}}{1-\rho_{k-1}+\rho_{k-1}\frac{F(y_{k-1})}{F(y_{k-2})}}.$$

Bidders, however, do not know the value of ρ_k. They only know that $\rho_k \in [\underline{\rho}_k, \overline{\rho}_k]$. Now, let us derive the equilibrium backwards. Consider round K and payoff of a bidder with a value $v < y_{K-1}$, who bids $\beta_K^I(z, y_{K-1})$ with $z \leq y_{K-1}$:

$$\begin{aligned}\Pi_K(v,z,y_{K-1}) = \min_{\rho_K \in [\underline{\rho}_K, \overline{\rho}_K]} \Bigg(& (1-\rho_K)^{N-K}\left(v-\beta_K^I(z,y_{K-1})\right) \\ & + \sum_{l=1}^{N-K}\binom{N-K}{l}(1-\rho_K)^{N-K-l}\rho_K^l\left(\frac{F(z)^l}{F(y_{K-1})^l}\left(v-\beta_K^I(z,y_{K-1})\right)\right)\Bigg).\end{aligned}$$

where the first term is the event the bidder is only facing inactive bidders. In this case, a bidder believes that she will defeat all of them. The remaining terms correspond to her facing l active bidders. In these cases she wins conditional on z being the highest of l valuations, conditional on all being smaller than y_{K-1}. Note that the payoff function can be rewritten as

$$\Pi_K(v,z,y_{K-1}) = \min_{\rho_K \in [\underline{\rho}_K, \overline{\rho}_K]} \left(\left(1-\rho_K+\rho_K\frac{F(z)}{F(y_{K-1})}\right)^{N-K}\left(v-\beta_K^I(z,y_{K-1})\right)\right).$$

From the above, $\overline{\rho}_K$ minimizes a bidder's expected payoff since $z \leq y_{K-1}$. Substituting, and taking derivatives, the corresponding first order condition is

$$\begin{aligned}&-(1-\overline{\rho}_K)^{N-K}\frac{\partial\beta_K^I(v,y_{K-1})}{\partial v} \\ &+\sum_{l=1}^{N-K}(1-\overline{\rho}_K)^{N-K-l}\overline{\rho}_K^l\binom{N-K}{l}\left[\left(-\frac{F(v)^l}{F(y_{K-1})^l}\frac{\partial\beta_K^I(v,y_{K-1})}{\partial v}+\frac{lF(v)^{l-1}f(v)}{F(y_{K-1})^l}\left(v-\beta_K^I(v,y_{K-1})\right)\right)\right]=0,\end{aligned}$$

where we use the fact that in equilibrium z must equal v for β_K^I to be an equilibrium. The boundary condition of the differential equation is $\beta_K^I(\underline{v}, y_{K-1}) = w$. Therefore, we have

$$\beta_K^I(v,y_{K-1}) = \frac{(1-\overline{\rho}_K)^{N-K}w+\sum_{l=1}^{N-K}\binom{N-K}{l}\left(1-\overline{\rho}_K\right)^{N-K-l}\overline{\rho}_K^l\frac{F(v)^l}{F(y_{K-1})^l}\mathbb{E}\left[V_1^{(l)}|V_1^{(l)}<v\right]}{(1-\overline{\rho}_K)^{N-K}+\sum_{l=1}^{N-K}\binom{N-K}{l}(1-\overline{\rho}_K)^{N-K-l}\overline{\rho}_K^l\frac{F(v)^l}{F(y_{K-1})^l}}.$$

Now, substituting the value for $\overline{\rho}_K$ we get

$$\beta_K^I(v,y_{K-1}) = \frac{(1-\overline{\rho}_{K-1})^{N-K}w+\sum_{l=1}^{N-K}\binom{N-K}{l}\left(1-\overline{\rho}_{K-1}\right)^{N-K-l}\overline{\rho}_K^l\frac{F(v)^l}{F(y_{K-2})^l}\mathbb{E}\left[V_1^{(l)}|V_1^{(l)}<v\right]}{(1-\overline{\rho}_{K-1})^{N-K}+\sum_{l=1}^{N-K}\binom{N-K}{l}(1-\overline{\rho}_K)^{N-K-l}\overline{\rho}_{K-1}^l\frac{F(v)^l}{F(y_{K-2})^l}}.$$

We again substitute the value of ρ_{K-1}. Continuing this process, we get

$$\beta_K^I(v) = \frac{(1-\overline{\rho})^{N-K}w + \sum_{l=1}^{N-K}\binom{N-K}{l}(1-\overline{\rho})^{N-K-l}\overline{\rho}^l F(v)^l \mathbb{E}\left[V_1^{(l)} | V_1^{(l)} < v\right]}{(1-\overline{\rho})^{N-K} + \sum_{l=1}^{N-K}\binom{N-K}{l}(1-\overline{\rho})^{N-K-l}\overline{\rho}^l F(v)^l}.$$

Let, $\Pi_K(v, y_{K-1})$ be the equilibrium payoff in round K to a bidder with valuation v when the price in the previous round revealed the winner's valuation to be y_{K-1}. Then, clearly, $v - \beta_K(v) = \Pi_K(v, v) \geq \Pi_K(v, y_{K-1})$ with strict inequality if $v < y_{K-1}$. Now, consider a bidder's payoff in round $K-1$ if she pretends to be bidder with value $z \leq v$

$$\begin{aligned}\Pi_{K-1}(v,z,y_{K-2}) = \min_{\rho_{K-1}\in\left[\underline{\rho}_{K-1},\overline{\rho}_{K-1}\right]} & \left(\left(1-\rho_{K-1}+\rho_{K-1}\frac{F(z)}{F(y_{K-2})}\right)^{N-K+2}\left(v-\beta_{K-1}^I(z,y_{K-2})\right)\right. \\ & \left. + \quad 1-\left(1-\rho_{K-1}+\rho_{K-1}\frac{F(z)}{F(y_{K-2})}\right)^{N-K+2}\right)\int_z^{y_{K-2}}\Pi_K(v,x)d\frac{F(x)^{N-K+1}}{F(y_{K-2})^{N-K+1}}.\end{aligned}$$

Let $\hat{\rho}_{K-1}$ minimize the bidder's payoff over the feasible set of ρ_{K-1}. Using similar arguments as in the benchmark model we can show that $\hat{\rho}_{K-1}$ is independent of z. Substituting, the corresponding first order condition can be solve to give us the bidding function in round $K-1$ as

$$\beta_{K-1}^I(v, y_{K-2})$$
$$= \frac{(1-\hat{\rho}_{K-1})^{N-K+1}w + \sum_{l=1}^{N-K+1}\binom{N-K+1}{l}(1-\hat{\rho}_{K-1})^{N-K+1-l}\hat{\rho}_{K-1}^l\frac{F(v)^l}{F(y_{K-1})^l}\mathbb{E}\left[\beta_K\left(V_1^{(l)}\right)\middle| V_1^{(l)} < v\right]}{(1-\hat{\rho}_{K-1})^{N-K} + \sum_{l=1}^{N-K+1}\binom{N-K+1}{l}(1-\hat{\rho}_{K-1})^{N-K+1-l}\hat{\rho}_{K-1}^l\frac{F(v)^l}{F(y_{K-1})^l}}.$$

This implies that $\beta_{K-1}(v, y_{K-2}) < \beta_K(v)$. Therefore, $v - \beta_{K-1}(v, Y_{K-2}) > v - \beta_K(v) = \Pi_K(v, v)$. The implication of this inequality is that the in the payoff function in round $K-1$ the value of ρ_{K-1} that minimizes a bidders payoff is the one that minimizes the probability winning round $K-1$. That is, $\hat{\rho}_{K-1} = \overline{\rho}_{K-1}$. Substituting in the bidding function we get

$$\beta_{K-1}^I(v) = \frac{(1-\overline{\rho})^{N-K+1}w + \sum_{l=1}^{N-K+1}\binom{N-K+1}{l}(1-\overline{\rho})^{N-K+1-l}\overline{\rho}^l F(v)^l\mathbb{E}\left[\beta_K\left(V_1^{(l)}\right)\middle| V_1^{(l)} < v\right]}{\sum_{l=0}^{N-K+1}\binom{N-K+1}{l}(1-\overline{\rho})^{N-K+1-l}\overline{\rho}^l F(v)^l}.$$

Now consider the k-th auction, where $k = 1, \ldots, K-1$. By induction, let the bidding function in round $k+1$ be given by $\beta_{k+1}^I(\cdot)$. Given the realized prices $p_1, \ldots, p_{k-1}$ and the corresponding realized values $y_1, \ldots, y_{k-1}$, $\beta_k^I(\cdot, y_{k-1})$ can be solved heuristically as follows. Let $\hat{\rho}_k$ be the ρ_k that provides the lower bound for a bidder's payoff in round k. For a bidder who has value v and is active, she bids $\beta_k^I(v, y_{k-1})$ in equilibrium. Consider her payoff change if she bids $\beta_k^I(v+\varepsilon, y_{k-1})$.

Note that she faces $N-k$ opponents in the k-th auction. If all her opponents are inactive or have valuations that are lower than v, then she pays more without affecting the winning probability (which is one). On the other hand, if the highest valuation of her opponents is between v and $v+\varepsilon$ and the second highest valuation of her opponents is below v, then she wins the k-th auction rather than the $(k+1)$-th auction. Hence the payoff difference between bidding $\beta_k^I(v,y_{k-1})$ and $\beta_k^I(v+\varepsilon,y_{k-1})$ is given by

$$\begin{aligned}
&-(1-\hat{\rho}_k)^{N-k}\left(\beta_k^I(v+\varepsilon,y_{k-1})-\beta_k^I(v,y_{k-1})\right)\\
&-\sum_{l=1}^{N-k}(1-\hat{\rho}_k)^{N-k-l}\hat{\rho}_k^l\binom{N-k}{l}\left(\frac{F(v)^l}{F(y_{k-1})^l}\left(\beta_k^I(v+\varepsilon,y_{k-1})-\beta_k^I(v,y_{k-1})\right)\right.\\
&\qquad\qquad\left.+\int_v^{v+\varepsilon}\left(\beta_k^I(v+\varepsilon,y_{k-1})-\beta_{k+1}^I(v,x)\right)d\left(\frac{F(x)^l}{F(y_{k-1})^l}\right)\right)
\end{aligned}$$

Dividing the above expression by ε, and letting $\varepsilon\to 0$, the above expression must converge to zero if β_k^I is an equilibrium. The resulting unique solution is

$$\beta_k^I(v,y_{k-1})=\frac{(1-\hat{\rho}_k)^{N-k}w+\sum_{l=1}^{N-k}\binom{N-K}{l}(1-\hat{\rho}_k)^{N-k-l}\hat{\rho}_k^l\frac{F(v)^l}{F(y_{k-1})^l}\mathbb{E}\left[\beta_{k+1}^I\left(V_1^{(l)}\right)\middle|V_1^{(l)}<v\right]}{(1-\hat{\rho}_k)^{N-k}+\sum_{l=1}^{N-k}\binom{N-K}{l}(1-\hat{\rho}_k)^{N-k-l}\hat{\rho}_k^l\frac{F(v)^l}{F(y_{k-1})^l}}.$$

Again, $\beta_k^I(v,y_{k-1})<\beta_{k+1}(v)$ and therefore using the same argument as round $K-1$ it must be that $\hat{\rho}_k=\overline{\rho}_k$. Therefore,

$$\beta_k^I(v)=\frac{(1-\overline{\rho})^{N-k}w+\sum_{l=1}^{N-k}\binom{N-k}{l}(1-\overline{\rho})^{N-k-l}\overline{\rho}^lF(v)^l\mathbb{E}\left[\beta_{k+1}^I\left(V_1^{(l)}\right)\middle|V_1^{(l)}<v\right]}{\sum_{l=0}^{N-k}\binom{N-k}{l}(1-\overline{\rho})^{N-k-l}\overline{\rho}^lF(v)^l}.$$

Now, we verify that the above bidding functions are (1) monotone in v and (2) constitute an equilibrium. First let us show (1). Note that for any k the probability mass function $\{\alpha_{l,k}(x)\}$ first order stochastically dominates $\{\alpha_{l,k}(x')\}$ where $x>x'$. To see this, taking the derivative of $\alpha_{l,k}(x)$ with respect to x, we have

$$\frac{d\alpha_{l,k}(x)}{dlx}=\alpha_{l,k}(x)\frac{f(x)}{F(x)}\left(l-\sum_{s=0}^{N-k}s\alpha_{s,k}(x)\right).$$

This derivative must be positive for some $l'>0$. Let $l'=\inf\left\{l\,|\,l-\sum_{s=0}^{N-k}s\alpha_{s,k}(x)\geq 0\right\}$. Then for all $l<l'$, $\frac{d\alpha_{l,k}(x)}{dx}<0$ and for all $l>l'$, $\frac{d\alpha_{l,k}(x)}{dx}>0$. Note $0<l'<N-k$. Therefore, for all $x'<x$, $\{\alpha_{l,k}(x)\}$ FOSD $\alpha_{l,k}(x')$. Now, consider β_K. The expectation term in the closed form of the bidding function is increasing in l. Thus, $\beta_K(v)$ must be increasing in v. Using induction and the result on stochastic dominance of $\{\alpha_{l,k}(x)\}$ we get that the bid function in any round is increasing in v.

Now, we prove (2). We proceed by induction. First consider the last auction. It is straightforward to establish that this bidding function is indeed an equilibrium. Suppose, all bidders follow this strategy in round K. Then a bidder's payoff as a function of her valuation is

$$\begin{aligned}
\Pi_K(v,v,y_{K-1}) &= \sum_{l=0}^{N-K}\binom{N-K}{l}(1-\overline{\rho}_K)^{N-K-l}\overline{\rho}_K^l\frac{F(v)^l}{F(y_{K-1})^l}\left(v-\beta_K^I(v)\right)\\
&= \left(\frac{1-\overline{\rho}+\overline{\rho}F(v)}{1-\overline{\rho}+\overline{\rho}F(y_{K-1})}\right)^{N-K}\quad \alpha_{0,K}(v)(v-w)+\sum_{l=1}^{N-K}\alpha_{s,K}(v)\left(v-\mathbb{E}\left[V_1^{(l)}\middle| V_1^{(l)}<v\right]\right)\\
&= \frac{(1-\overline{\rho})^{N-K}}{(1-\overline{\rho}+\overline{\rho}F(y_{K-1}))^{N-K}}(v-w)+\sum_{l=1}^{N-K}\frac{\binom{N-k}{l}(1-\overline{\rho})^{N-K-l}\overline{\rho}^lF(v)^l}{(1-\overline{\rho}+\overline{\rho}F(y_{K-1}))^{N-K}}\int_{\underline{v}}^{v}\frac{F(x)^l}{F(v)^l}dx\\
&= (1-\overline{\rho}_K)^{N-K}(v-w)+\sum_{l=1}^{N-K}\binom{N-K}{l}(1-\overline{\rho}_K)^{N-K-l}\overline{\rho}_K^l\frac{F(v)^l}{F(y_{K-1})^l}\int_{\underline{v}}^{v}\frac{F(x)^l}{F(v)^l}dx.
\end{aligned}$$

Now, suppose the bidder deviates from the equilibrium and bids $\beta_K^I(z)$. Then her payoff will be

$$\begin{aligned}
\Pi_K(z,v,y_{K-1}) = (1-\overline{\rho}_K)^{N-K}(v-w) &+ \sum_{l=1}^{N-K}\binom{N-K}{l}(1-\overline{\rho}_K)^{N-K-l}\overline{\rho}_K^l\frac{F(z)^l}{F(y_{K-1})^l}\int_{\underline{v}}^{z}\frac{F(x)^l}{F(z)^l}dx\\
&+ \sum_{l=1}^{N-K}\binom{N-K}{l}(1-\overline{\rho}_K)^{N-K-l}\overline{\rho}_K^l\frac{F(z)^l}{F(y_{K-1})^l}(v-z).
\end{aligned}$$

Note that for any $z \neq v$, we have

$$\begin{aligned}
&\Pi_K(v,v,y_{K-1})-\Pi_K(z,v,y_{K-1})\\
&\quad= \sum_{l=1}^{N-K}\binom{N-K}{l}(1-\overline{\rho}_K)^{N-K-l}\overline{\rho}_K^l\frac{1}{F(y_{K-1})^l}\left(\int_z^v F(x)^l dx-F(z)^l(v-z)\right)>0.
\end{aligned}$$

Next consider the k-th auction. The payoff of a bidder with value v who bids $\beta_k^I(v)$ is

$$\begin{aligned}
\Pi_k(v,v,y_{k-1}) = &\sum_{l=0}^{N-k}\binom{N-k}{l}(1-\overline{\rho}_k)^{N-k-l}\overline{\rho}_k^l\frac{F(v)^l}{F(y_{k-1})^l}\left(v-\beta_k^I(v)\right)\\
&+\sum_{l=1}^{N-k}\binom{N-k}{l}(1-\overline{\rho}_k)^{N-k-l}\overline{\rho}_k^l\int_v^{y_{k-1}}\Pi_{k+1}(v,v,x)d\frac{F(x)^l}{F(y_{k-1})^l},
\end{aligned}$$

and whereas her payoff from bidding $\beta_k^I(z)$ for some $z \in (v, y_{k-1})$ is

$$\begin{aligned}
\Pi_k(z,v,y_{k-1}) = &\sum_{l=0}^{N-k}\binom{N-k}{l}(1-\overline{\rho}_k)^{N-k-l}\overline{\rho}_k^l\frac{F(z)^l}{F(y_{k-1})^l}\left(v-\beta_k^I(z)\right)\\
&+\sum_{l=1}^{N-k}\binom{N-k}{l}(1-\overline{\rho}_k)^{N-k-l}\overline{\rho}_k^l\int_z^{y_{k-1}}\Pi_{k+1}(v,v,x)d\frac{F(x)^l}{F(y_{k-1})^l}.
\end{aligned}$$

Therefore, we have

$$
\begin{aligned}
&\Pi_k(v,v,y_{k-1}) - \Pi_k(z,v,y_{k-1}) \\
&= \sum_{l=1}^{N-k} \binom{N-k}{l} (1-\overline{\rho}_k)^{N-k-l} \overline{\rho}_k^l \frac{F(v)^l}{F(y_{k-1})^l} \left(v - \mathbb{E}\left[\beta_{k+1}^I \left(V_1^{(l)} \right) \middle| V_1^{(l)} < v \right] \right) \\
&\quad - \sum_{l=1}^{N-k} \binom{N-k}{l} (1-\overline{\rho}_k)^{N-k-l} \overline{\rho}_k^l \frac{F(z)^l}{F(y_{k-1})^l} \left(v - \mathbb{E}\left[\beta_{k+1}^I \left(V_1^{(l)} \right) \middle| V_1^{(l)} < z \right] \right) \\
&\quad + \sum_{l=1}^{N-k} \binom{N-k}{l} (1-\overline{\rho}_k)^{N-k-l} \overline{\rho}_k^l \int_v^z \Pi_{k+1}(v,v,x) d\frac{F(x)^l}{F(y_{k-1})^l} \\
&= \sum_{l=1}^{N-k} \binom{N-k}{l} (1-\overline{\rho}_k)^{N-k-l} \overline{\rho}_k^l \frac{1}{F(y_{k-1})^l} \left(\int_v^z \Pi_{k+1}(v,v,x) dF(x)^l - F(z)^l v \right. \\
&\qquad\qquad \left. + \int_{\underline{v}}^z \beta_{k+1}^I(x) dF(x)^l + F(v)^l v - \int_{\underline{v}}^v \beta_{k+1}^I(x) dF(x)^l \right) \\
&= \sum_{l=1}^{N-k} \binom{N-k}{l} (1-\overline{\rho}_k)^{N-k-l} \overline{\rho}_k^l \frac{1}{F(y_{k-1})^l} \int_v^z \left(\Pi_{k+1}(v,v,x) - (v - \beta_{k+1}^I(x)) \right) dF(x)^l .
\end{aligned}
$$

Note that if the highest possible value in the $(k+1)$-th auction is $x \in (z,v)$ then the bidder wins this auction with probability one if she bids $\beta_k^I(x)$. Since by induction $\Pi_{k+1}(v,v,x)$ is the maximum continuation payoff that this bidder can obtain, it follows that $\Pi_{k+1}(v,v,x) \geq v - \beta_{k+1}^I(x)$. Hence, $\Pi_k(v,v,y_{k-1}) - \Pi_k(z,v,y_{k-1}) \geq 0$. The proof for the case of bidding $\beta_k^I(z,y_{k-1})$ where $z < v$ is similar and thus omitted. □

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