Efficient Multi-unit Auctions for Normal Goods

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Abstract

I study efficient multi-unit auction design when bidders have private values, multi-unit demands, and non-quasilinear preferences. Without quasilinearity, the Vickrey auction loses its desired incentive and efficiency properties. Instead of assuming that bidders have quasilinear preferences, I assume that bidders have positive wealth effects. This nests cases where bidders are risk averse, face financial constraints, or have budgets.

With two bidders, I show that there is a mechanism that retains the desirable properties of the Vickrey auction if and only if bidders have single dimensional types. If bidders have multi-dimensional types, there is no mechanism that satisfies (1) individual rationality, (2) dominant strategy incentive compatibility, (3) ex-post Pareto efficiency, and (4) weak budget balance. When there are more than two bidders, I show that there is no mechanism with desirable incentive and efficiency properties, even if bidders have single dimensional types.

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1 Introduction

1.1 Motivation

Understanding how to design auctions with desirable incentive and efficiency properties is a central question in mechanism design. The Vickrey-Clarke-Groves (hereafter, VCG) mechanism is celebrated as a major achievement in the field because it performs well in both respects - agents have a dominant strategy to truthfully report their private information and the mechanism implements a Pareto efficient allocation of resources. However, the VCG mechanism loses its desired incentive and efficiency properties without the quasilinearity restriction. Moreover, there are many well-studied cases where the quasilinearity restriction is violated: bidders may be risk averse, have wealth effects, face financing constraints or be budget constrained. Indeed, observed violations of quasilinearity are frequently cited as reasons for why we do not see multi-unit Vickrey auctions used in practice.[1]

In this paper, I study multi-unit auctions for indivisible homogenous goods when bidders have private values, multi-unit demands, and non-quasilinear preferences. I provide conditions under which we can construct an auction that retains the desired incentive and efficiency properties of the Vickrey auction. Instead of quasilinearity, I assume only that bidders have positive wealth effects; i.e. the goods being auctioned are normal goods. My environment nests well-studied cases where bidders are risk averse, have budgets, or face financing constraints. I show that there exists a mechanism that retains the desirable properties of the Vickrey auction if there are two bidders and bidders have single dimensional types. However, I construct two impossibility theorems showing there is no mechanism that retains the desired incentive and efficiency properties of the Vickrey auction when bidders have multi-dimensional types or when there are three or more bidders.

I begin by studying a setting with two bidders who have single dimensional types. I use the taxation principle (see Rochet (1985)) to construct a mechanism that is ex-post Pareto efficient and dominant strategy incentive compatible. The taxation principle simplifies the problem of finding an efficient mechanism to a problem of finding an efficient pricing rule that ensures an ex-post Pareto efficient allocation of resources. In my efficient mechanism, each bidder faces a pricing rule that states the price of acquiring additional units. The price a bidder pays for additional units is determined by her rival's reported type. I find an efficient price vector by defining a transformation that maps an arbitrary allocation rule to a more efficient allocation rule. I use Schauder's fixed point theorem to show that this transformation

[1] For example, Ausubel and Milgrom (2006), Rothkopf (2007), and Nisan et al. (2009) all cite budgets and financing constraints as salient features of real-world auction settings that inhibits the use of the Vickrey auctions. Che and Gale (1998) note that bidders often face increasing marginal costs of expenditures when they have access to imperfect financial markets.

has a fixed point that defines an efficient mechanism (Theorem 1).

I then construct two impossibility theorems showing that the positive result obtained in Theorem 1 does not generalize beyond the two bidder single dimensional types case. My first impossibility theorem (Theorem 2) is for a setting where bidders have multi-dimensional types. I show there is no mechanism that satisfies: (1) individual rationality, (2) weak budget balance, (3) dominant strategy incentive compatibility, and (4) ex-post Pareto efficiency. To prove this Theorem, it suffices to consider a setting where two bidders compete for two units and bidders have two-dimensional types. The first dimension of a bidder's type describes her willingness to pay for her first unit - I call this her intercept. The second dimension of a bidder's type is binary - it is either flat or steep. All else equal, a bidder with a flat demand type has relatively greater demand for her second unit than a bidder with a steep demand type.

I prove my first impossibility theorem by contradiction. If a mechanism satisfies the four aforementioned properties, then I show that the price a bidder pays to win one unit must vary with both dimensions of her rival's reported type. However, wealth effects imply that a bidder's demand for her second unit depends on how much she pays to win her first unit. Thus, the solution to the efficient design problem endogenously causes a bidder's demand for her second unit to vary with her rival's reported type, even when bidders have private values. The endogenous interdependence of bidder demands in the design problem without quasilinearity yields an impossibility theorem that is similar to the well-known impossibility theorems in interdependent value settings, when bidders have quasilinear preferences and multi-dimensional types (see Dasgupta and Maskin (2000) and Jehiel and Moldovanu (2001)).

Because my proof shows that it is impossible to construct a mechanism that satisfies the four properties with two-dimensional types (where the second dimension is binary), it follows that there is no mechanism that satisfies the four properties on richer type spaces. The increase in dimensionality only increases the number of incentive constraints that an efficient mechanism must satisfy.

I then return to the single dimensional types case and I consider a setting with three or more bidders. To this end, my second impossibility theorem shows that there is no mechanism that satisfies (a) individual rationality, (b) no subsides for bidders, (c) dominant strategy incentive compatibility, (d) ex-post Pareto efficient, and (e) a monotonicity constraint on the allocation rule. The monotonicity constraint states that a bidder wins a weakly greater number of units if her demand increases while her rivals' reported demands decreases. While the impossibility theorem holds in a general setting, it provides insights on the particular case of bidders with private values and public budgets.[2] The theorem shows that the clinching auc-

[2]This setting is studied by Laffont and Robert (1996) and Maskin (2000) with a single good, and is one

tion studied by Dobzinski, Lavi, and Nisan (2012) loses its desirable incentive and efficiency properties when assume bidders have strictly decreasing marginal values instead of constant marginal values.

The rest of the paper proceeds as follows. The remainder of the introduction discusses how my work relates to the existing literature in mechanism design. Section 2 describes my model and mechanisms in the single dimensional types setting. Section 3 constructs an efficient mechanism for a setting with two bidders and two units. Section 4 considers the efficient auction design problem in a general setting with 2 bidders and k units. Section 5 then studies the multi-dimensional type setting, and section 6 studies the single dimensional type setting with more than 2 bidders.

1.2 Related Literature

This paper studies a central question of mechanism design: how to design an auction that allocates many homogenous goods to many different buyers? With quasilinearity, Holmström (1979) shows that the Vickrey auction is the unique dominant strategy mechanism that implements a Pareto efficient allocation of goods. However, the Vickrey auction is not dominant strategy implementable or efficient without quasilinearity (see Ausubel and Milgrom (2006)).

There is a small literature that studies efficient auction design without quasilinearity. Most of this literature studies cases where bidders have hard budget constraints. In the single unit context, Maskin (2000) and Pai and Vohra (2014) study the design of constrained efficient auctions that are Bayesian implementable. However, this paper studies efficient mechanisms that are dominant strategy implementable. Dobzinski, Lavi, and Nisan (2012) is more similar to this paper. They study dominant strategy implementable auctions in a setting where bidders have private values and multi-unit demands. They study the case where bidders have hard budgets.

This paper is distinct from Dobzinski, Lavi, and Nisan in a few key respects. First, I give necessary and sufficient conditions for efficient auction design without making functional form restrictions on bidder preferences. I only assume bidder preferences are such that goods are normal, and hard budgets are nested as a special case. Second, my results show that the privacy of bidder budgets do not necessarily inhibit efficient auction design.[3] In fact, Theorem 1 shows that if bidders' private budget are perfectly correlated with their private values - and hence bidder private information can be summarized by a single-dimensional type - then

of the cases studied by Dobzinski, Lavi, and Nisan (2012) a multi-unit setting.

[3]Dobzinski, Lavi, and Nisan (2012) show efficient auction design is possible if and only if bidders have private budgets.

we can construct an efficient auction for two bidders.[4] In addition, Theorem 2 shows that it is impossible to construct an efficient auction if bidder types are multi-dimensional, even if bidder budgets are public information.[5]

While most literature on auctions without quasilinearity focuses on the case of hard budgets, there is a small literature that studies auction design without strong function form restrictions on bidder preferences. The most notable paper in this literature is Maskin and Riley (1984). Maskin and Riley study the design of revenue maximizing auctions in a general setting where bidders are risk averse and have single dimensional types. More recently, Baisa (2016a) studies the design of revenue maximizing auctions when bidders have positive wealth effects.

On efficiency, Saitoh and Serizawa (2008) and Morimoto and Serizawa (2015) study the auction design with non-quasilinear bidders, but in a single-unit demand settings. However, this paper focuses on a multi-unit demand setting. Both of my impossibility theorems show that the presence of multi-unit demands inhibits efficient auction design. Hence, we get results that are distinct from the positive results seen in the two aforementioned papers. Most recently, Serizawa and Kazumura (2016) study the problem of selling heterogenous goods to buyers with non-quasilinear preferences, but allow for multi-unit demands. They show that there is no mechanism that is dominant strategy incentive compatible and Pareto efficient, even if only one bidder has multi-unit demands. In contrasts, this paper studies the sale of homogenous goods. This is distinction is important for my results because an impossibility result in a heterogenous good setting does not imply that efficient design is impossible with homogenous goods. In fact, Theorem 1 illustrates cases where efficient design is possible with multi-unit demand. In addition, I focus on how the dimensionality of bidder types changes the auction design problem, while the above Serizawa and Kazumura do not consider dimensionality restrictions on bidder types.

Outside of the auctions literature, Garratt and Pycia (2014) investigate a problem similar to the one studied here. Garratt and Pycia study how positive wealth effects influence the possibility of efficient bilateral trade in a Myerson and Satterthwaite (1983) setting. In contrast to this paper, Garratt and Pycia show that the presence of wealth effects and can help to induce efficient trade when there is two-sided private information.

[4]Bidders have perfectly correlated values and budgets in auctions where bidders endogenously determine their budgets after observing their value (see, Baisa and Rabinovich (2016)).

[5]An example of a setting with public budgets and multi-dimensional types is setting where bidders demands two units of the good, and the first dimension of a bidder's type is her value for her first unit, and the second dimension of a bidder's type is her value for her second unit.

2 Model

2.1 Bidder Preferences - The Single Dimensional Types Case

A seller has k units of an indivisible homogenous good. There are N bidders, who are vNM expected utility maximizers. Bidders have private values and multi-unit demands. Bidder i's preferences are described by her type $\theta_i \in [0, \overline{\theta}] := \Theta \subset \mathbb{R}_+$. If bidder i wins $q \in \{0, 1, \ldots, k\}$ units and receives $m \in \mathbb{R}$ in monetary transfers, her utility is

$$u(q, m, \theta_i) \in \mathbb{R}.$$

A bidder's utility is continuous in her type θ_i and continuous and strictly increasing in monetary transfers m.

If $\theta_i = 0$, then bidder i has no demand for units,

$$u(q, m, 0) = u(q', m, 0) \;\forall q, q' \in \{0, 1, \ldots, k\},\; m \in \mathbb{R}.$$

If $\theta_i > 0$, then bidder i has positive demand for units,

$$q' > q \iff u(q', m, \theta_i) > u(q, m, \theta_i),\; \forall \theta_i \in \Theta,\; m \in \mathbb{R}.$$

Without loss of generality, I assume that $u(0, 0, \theta_i) = 0 \;\forall \theta_i$. Bidders have bounded demand for units of the good. Thus, there exists a $p > 0$ such that

$$0 > u(q, -p, \theta_i) \;\forall q \in \{0, 1, \ldots, k\},\; \theta_i \in \Theta.$$

I make three additional assumptions on bidder preferences. First, I assume that bidder preferences satisfy the law of demand. That is, if a bidder is unwilling to pay p for her q^{th} unit, then she is unwilling to pay p for her $(q+1)^{st}$ unit. This implies that bidders have downward sloping demand curves and generalizes the declining marginal values assumption imposed in the benchmark quasilinear setting.

Assumption 1. *(Law of Demand)*

$$u(q-1, m, \theta_i) \geq u(q, m-p, \theta_i) \implies u(q, m, \theta_i) > u(q+1, m-p, \theta_i),\; \forall q \in \{1, \ldots, k-1\},\; \theta_i > 0.$$

Second, I assume that bidders have positive wealth effects. This means a bidder's demand does not decrease as her wealth increases. To be more concrete, suppose that bidder i was faced with the choice between two bundles of goods. The first bundle has x goods and costs p_x total, and the second bundle has y goods and costs p_y total, where we assume $x > y$. If bidder i prefers the first bundle with more goods, then positive wealth effects state that she also prefers the first bundle with more goods if we increased her wealth prior to her purchasing decision. This is a multi-unit generalization of Cook and Graham's (1977) definition of an indivisible, normal good. I define two versions of positive wealth effects, weak and strict. When constructing an efficient mechanism, I assume the weak version, which nests quasilinearity. When presenting my impossibility theorems, I assume the strict version that rules out the quasilinear setting where the benchmark Vickrey auction is already known to solve the efficient auction design problem.

Assumption 2. *(Positive wealth effects)*
Suppose $x > y$ where $x, y \in \{0, 1, \dots k\}$. Bidders have weakly positive wealth effects if

$$u(x, -p_x, \theta_i) \geq u(y, -p_y, \theta_i) \implies u(x, m - p_x, \theta_i) \geq u(y, m - p_y, \theta_i) \ \forall m > 0,\ \theta_i > 0,$$

and strictly positive wealth effects if

$$u(x, -p_x, \theta_i) \geq u(y, -p_y, \theta_i) \implies u(x, m - p_x, \theta_i) > u(y, m - p_y, \theta_i) \ \forall m > 0,\ \theta_i > 0.$$

Finally, I assume that bidders with higher types have greater demands.

Assumption 3. *(Single-crossing)*
Suppose $x > y$ where $x, y \in \{0, 1, \dots k\}$. Then,

$$u(x, -p_x, \theta) \geq u(y, -p_y, \theta) \implies u(x, -p_x, \theta') > u(y, m - p_y, \theta') \ \forall \theta' > \theta.$$

I let $b_1(\theta_i)$ be the amount that bidder i is willing to pay for her first unit of the good. Thus, $b_1(\theta_i)$ implicitly solves

$$u(1, -b_1(\theta_i), \theta_i) = 0.$$

It is without loss of generality to assume types are such that $b_1(\theta) = \theta \ \forall \theta \in \Theta$. Thus, θ_i parametrizes the intercept of bidder i's demand curve.

I similarly define $b_j(\theta_i, x)$ as bidder i's willingness to pay for her j^{th} unit, conditional on winning her first $j-1$ units for a cost of $x \in \mathbb{R}$. Thus, $b_j(\theta_i, x)$ is implicitly defined as solving,

$$u(j, -x - b_j(\theta_i, x), \theta_i) = u(j-1, -x, \theta_i).$$

I analogously define $s_j(\theta_i, x)$ as bidder i's willingness to sell her j^{th} unit, conditional on having paid x in total. Thus, a bidder's willingness to sell her j^{th} unit s_j is implicitly defined as solving

$$u(j, -x, \theta_i) = u(j-1, -x + s_j, \theta_i).$$

Note that by construction,

$$s_j(\theta_i, x) = b_j(\theta_i, x + s_j(\theta_{i,} x)).$$

Assumptions 1, 2, and 3 imply:

1. $b_j(\theta, x) > b_{j+1}(\theta, x)$ and $s_j(\theta, x) > s_{j+1}(\theta, x)$ for all $\theta \in \Theta,\ x \in \mathbb{R},\ j \in \{1, \ldots, k\}$.

2. b_j and s_j are continuous and decreasing in the second argument.[6]

3. b_j and s_j are continuous and strictly increasing in the first argument.

The first point is implied by the law of demand. The second point is implied by positive wealth effects. The final point is implied by single crossing.

2.2 Mechanisms

By the revelation principle, it is without loss of generality to consider direct revelation mechanisms. A direct revelation mechanism maps the profile of reported types to an outcome. An outcome specifies a feasible allocation of goods and payments. An allocation of goods $y \in \{0, 1, \ldots, k\}^N$ is feasible if $\sum_{i=1}^N y_i \leq k$. I let Y be the set of all feasible allocations. A (deterministic) allocation rule q maps the profile of reported types to a feasible allocation $q : \Theta^N \to Y$. The payment rule maps the profile of reported types to payments $x : \Theta^N \to \mathbb{R}^N$. A direct revelation mechanism Γ consists of an allocation rule and a payment rule. For simplicity, I let $\Gamma_i(\theta_i, \theta_{-i})$ represent that allocation and payment for bidder i when she reports type θ_i and her rivals report types θ_{-i}.

I study direct revelation mechanisms that satisfy the following properties.

Definition 1. (Ex-post individual rationality)
A mechanism Γ is ex-post individually rational if

$$u(\Gamma_i(\theta_i, \theta_{-i}), \theta_i) \geq 0 \ \forall (\theta_i, \theta_{-i}) \in \Theta^N.$$

Thus, a mechanism is ex-post individually rational (hereafter, individually rational) if a bidder's utility never decreases from participating in the mechanism.

[6] b_j and s_j are weakly decreasing under weakly positive wealth effects and strictly decreasing under strictly positive wealth effects.

Definition 2. (Weak budget balance)
A mechanism Γ satisfies weak budget balance if

$$\sum_{i=1}^{N} x_i(\theta_i, \theta_{-i}) \geq 0 \ \forall(\theta_i, \theta_{-i}) \in \Theta^N.$$

The weak budget balance condition is an individual rationality constraint on the auctioneer. A mechanism that satisfies weak budget balance always yields weakly positive revenue.

When I study the single dimensional types setting with $N \geq 3$ bidders, I impose a stronger but related requirement - no subsidies. A mechanism provides no subsidies if it never pays a bidder a positive amount to participate. Morimoto and Serizawa (2015) impose the same condition when studying efficient auctions in a setting where bidders have unit demand.

Definition 3. (No subsidies)
A mechanism Γ gives no subsidies if $x_i(\theta_i, \theta_{-i}) \geq 0 \ \forall(\theta_i, \theta_{-i}) \in \Theta^N$.

We study mechanisms that implement efficient allocation of goods. With quasilinearity, efficiency implies that goods are assigned to bidders with the highest values. Yet, without quasilinearity, there is no clear analog for a bidder's value. In this paper, I look at mechanisms that satisfy a weak notion of efficiency, ex-post Pareto efficient. Our notion of ex-post Pareto efficiency is the same notion used by Dobzinski, Lavi, and Nisan (2012) and Morimoto and Serizawa (2015).

Definition 4. (ex-post Pareto efficient)
An outcome $(y, x) \in Y \times \mathbb{R}^N$ is ex-post Pareto efficient if $\forall(\tilde{y}, \tilde{x}) \in Y \times \mathbb{R}^N$ such that

$$u(\tilde{y}_i, -\tilde{x}_i, \theta_i) > u(y_i, -x_i, \theta_i),$$

for some $i \in \{1, \ldots, N\}$, then either

$$\sum_{i=1}^{N} x_i > \sum_{i=1}^{N} \tilde{x}_i,$$

or

$$u(y_j, -x_j, \theta_j) > u(\tilde{y}_j, -\tilde{x}_j, \theta_j),$$

for some $j \in \{1, \ldots, N\}$.

Thus, an outcome is ex-post Pareto efficient, if any reallocation of resources that makes bidder i strictly better off necessarily makes her rival strictly worse off, or strictly decreases revenue. We say that the mechanism Γ is an ex-post Pareto efficient mechanism (hereafter, Pareto efficient) if $\Gamma(\theta_1, \ldots, \theta_N) \in Y \times \mathbb{R}^N$ is an ex-post Pareto efficient allocation $\forall (\theta_1, \ldots, \theta_N) \in \Theta^N$.

In addition, we study mechanisms that can be implemented in dominant strategies.

Definition 5. (Dominant strategy incentive compatibility)
A mechanism Γ is dominant strategy incentive compatible if

$$u(\Gamma_i(\theta_i, \theta_{-i}), \theta_i) \geq u(\Gamma_i(\theta_i', \theta_{-i}), \theta_i) \ \forall \theta_i, \theta_i' \in \Theta, \ \theta_{-i} \in \Theta^{N-1}.$$

Since we study mechanisms that are dominant strategy implementable (hereafter, incentive compatible), there is no need to model bidder beliefs.

When I study the single dimensional types setting with $N \geq 3$ bidders, I also consider mechanisms that satisfy a monotonicity condition on the allocation rule.

Definition 6. (Monotone allocation rule)
A mechanism Γ has a monotone allocation rule if $\theta_i^h > \theta_i^\ell$ and $\theta_{-i}^h \geq \theta_{-i}^\ell$ in the coordinate wise sense implies that

$$q_i(\theta_i^h, \theta_{-i}^\ell) \geq q_i(\theta_i^\ell, \theta_{-i}^h).$$

Thus, a mechanism has a monotone allocation rule if bidder i always wins a weakly greater number of units when she increases her reported demand and her rivals decrease their reported demands. Note that with two bidders, any incentive compatible mechanism has a monotone allocation rule.

It will also be useful to note the taxation principle of Rochet (1985). The taxation principle shows that if Γ is incentive compatible, then there exists a corresponding set of pricing rules $\{p_1, \ldots, p_N\}$ where $p_i : \Theta^{N-1} \to \mathbb{R}^{k+1}$ that implements mechanism Γ. The corresponding pricing rule that implements mechanism Γ is such that

$$q_i(\theta_i, \theta_{-i}) = m \implies x_i(\theta_i, \theta_{-i}) = \sum_{j=0}^{m} p_{i,j}(\theta_{-i}),$$

and

$$q_i(\theta_i, \theta_{-i}) = m \implies m \in \arg \max_{m \in \{0,1,\ldots,k\}} u(m, -\sum_{j=0}^{m} p_{i,j}(\theta_{-i}), \theta_i).$$

3 Single Dimensional Types - the 2x2 case

In order to motivate the construction of an efficient mechanism in a general two bidder, single dimensional type environment, I first construct a mechanism that retains the desired incentive and efficiency properties of the Vickrey auction in the two bidder and two unit case. The mechanism is symmetric and gives no subsidies.

The mechanism's allocation function can be described (almost everywhere) by a cut-off rule d, where $d : \Theta \to \Theta$ is such that

$$\theta_i > d(\theta_j) \implies q_i(\theta_i, \theta_j) \geq 1 \textit{ and } d(\theta_j) > \theta_i \implies q_i(\theta_i, \theta_j) = 0.$$

The cut-off rule states the minimal type that bidder i must report to win at least one unit, given her rival's type. The rule divides the two-dimensional space of bidder reports into three regions: a region where bidder 1 wins both units, a region where each bidder wins one unit, and a region where bidder 2 wins both units. I construct a cut-off rule that characterizes a mechanism that is incentive compatible and efficient. I assume that a bidder's cut-off is strictly increasing and continuous in her rival's type.[7]

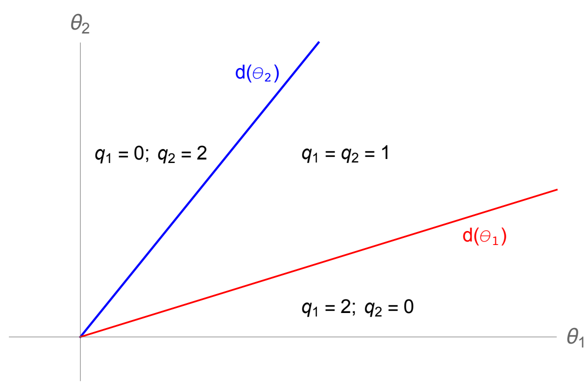

Figure 1: Allocations implied by an arbitrary symmetric cut-off rule d.

The corresponding (symmetric) pricing rule is such that $p_1(\theta_j) = d(\theta_j)$, because bidder i wins at least one unit ($\theta_i \geq d(\theta_j)$) when her willingness to pay for her first unit θ_i exceeds the amount she must pay for the first unit $p_1(\theta_j)$.[8]

My goal is to construct a cut-off rule that implements a Pareto efficient outcome. The mechanism assigns units to bidders based upon a bidder's willingness to pay for an additional unit. More precisely, bidder i wins both units if and only if her willingness to pay for her last unit, conditional on paying $p_1(\theta_j)$ for her first unit, exceeds her rival's willingness to pay for

[7]Proposition 3 shows that a continuous and strictly increasing first unit cut-off is a necessary condition for any mechanism that satisfies individual rationality, budget balance, incentive compatibility, and Pareto efficiency.

[8]There is no bidder specific subscript in the description of the pricing rule and the cut-off rule because we construct a symmetric mechanism.

her first unit. Thus, I construct a mechanism where the price a bidder pays for her second unit equals her rival's willingness to pay for her first unit,

$$p_2(\theta_j) = \theta_j.$$

Therefore, if bidder i's conditional willingness to pay for her second unit $b_2(\theta_i, p_1(\theta_j))$ exceeds bidder j's willingness to pay for her first unit θ_j, the cut-off rule is such that bidder i wins both units $d(\theta_i) \geq \theta_j$. Thus, the cut-off rule d is such that

$$0 \geq \theta_j - b_2(\theta_i, p_1(\theta_j)) = \theta_j - b_2(\theta_i, d(\theta_j)) \implies d(\theta_i) \geq \theta_j,$$

and

$$\theta_j - b_2(\theta_i, p_1(\theta_j)) = \theta_j - b_2(\theta_i, d(\theta_j)) \geq 0 \implies \theta_j \geq d(\theta_i).$$

If the cut-off rule d is continuous and increasing then $\theta_j - b_2(\theta_i, d(\theta_j))$ is continuous and strictly increasing in θ_j.[9] Then,

$$p_1(\theta_i) = d(\theta_i) = \theta_j \iff \theta_j - b_2(\theta_i, p_1(\theta_j)) = \theta_j - b_2(\theta_i, d(\theta_j)) = 0.$$

In other words, if bidder j is indifferent between winning zero and one unit, then bidder j's willingness to pay for her first unit equals bidder i's conditional willingness to pay for her second unit. When we substitute $d(\theta_i)$ for θ_j, the right hand side of the above expression simplifies to,

$$d(\theta_i) = b_2(\theta_i, d(d(\theta_i))). \tag{1}$$

Lemma 1 shows that there exists a cut-off d that satisfies the above equation for all $\theta_i \in [0, \overline{\theta}]$.

Lemma 1. *There exists a continuous and strictly increasing function $d : [0, \overline{\theta}] \to [0, \overline{\theta}]$ such that $\theta > d(\theta)$ $\forall \theta > 0$, and*

$$d(\theta) = b_2(\theta, d(d(\theta))).$$

Proposition 1 confirms that a cut-off rule that satisfies Equation (1) characterizes a mechanism which retains the desirable properties of the Vickrey auction. In particular, the cut-off rule d defines a feasible mechanism that satisfies (1) individual rationality, (2) no subsidy, (3) incentive compatibility, (3) no subsidies, and (4) Pareto efficiency. The proof follows from the construction of the cut-off rule d. The associated pricing rule is such that bidder i demands both units if and only if she was willingly to pay more for her final unit than her rival was willing to pay for her first unit. Or in other words, the pricing rule associated with d is such that the two highest conditional willingness to pays are winning bids.

[9]This is because b_2 is decreasing in the second argument and $d(\theta_j)$ is increasing in θ_j.

Proposition 1. *There exists a mechanism that satisfies (1) individual rationality, (2) no subsidies, (3) incentive compatibility, and (4) Pareto efficiency. The mechanism has cut-off rule d that solves Equation (1) and pricing rule p, where*

$$p_1(\theta_j) = d(\theta_j),\ p_2(\theta_j) = \theta_j.$$

The proof of Proposition 1 is a Corollary to Proposition 2 in the k unit setting. Hence, it is omitted.

4 The two bidder k unit case

In this section we show that there exists a mechanism that retains the desirable incentive and efficiency properties of the Vickrey auction when two bidders compete for k units. Just as in the last section, we characterize the mechanism by constructing a symmetric cut-off rule $d : \Theta \to \Theta^k$. The m^{th} dimension of the cut-off rule $d_m(\theta_j)$ gives the lowest type that bidder i must report to win at least m units, given that her rival reports type θ_j.[10] Therefore, a mechanism Γ has cut-off rule d if

$$\theta_i > d_m(\theta_j) \implies q_i(\theta_i, \theta_j) \geq m,$$

and

$$d_m(\theta_j) > \theta_i \implies m > q_i(\theta_i, \theta_j).$$

Incentive compatibility implies $q_i(\theta_i, \theta_j)$ is weakly increasing in θ_i and weakly decreasing in θ_j.[11] Thus, the cut-off rule $d_m(\theta)$ is weakly increasing in θ and weakly increasing in m. We let $\mathcal{D} \subset \{d | d : \Theta \to \Theta^k\}$ be the set of all cut-off rules that are weakly increasing in θ and m. Note that a cut-off rule $d \in \mathcal{D}$ does not necessarily correspond to a feasible mechanism.

I prove that there exists an efficient mechanism by constructing a transformation T that maps an arbitrary cut-off rule d to a more efficient cut-off rule $T(d)$. I argue that the fixed point of this transformation defines a feasible mechanism that satisfies (1) individual rationality, (2) no subsidy, (3) incentive compatibility, and (4) Pareto efficiency.

In order to define the transformation, I first define a pricing rule that corresponds to a cut-off rule d. The corresponding pricing rule ensures that bidder i demands at least m units if and only if her type θ_i exceeds $d_m(\theta_j)$. Therefore, the price of bidder i's first unit is

[10] Note that if a direct revelation mechanism is such that $q_i(\theta_i, \theta_j) \geq m$, then dominant strategy incentive compatibility implies that $q_i(\theta_i', \theta_j) \geq m\ \forall \theta_i' \geq \theta_i$.

[11] $q_i(\theta_i, \theta_j)$ is weakly decreasing in θ_j because Pareto efficiency implies $q_j(\theta_i, \theta_j) = k - q_i(\theta_i, \theta_j)$ and $q_j(\theta_i, \theta_j)$ is weakly increasing in θ_j.

$p_1(\theta_j, d) = d_1(\theta_j)$. This is because bidder i demands at least one unit if and only if her type θ_i exceeds the cut-off for her first unit $d_1(\theta_j)$.[12]

I determine the price of bidder i's m^{th} unit inductively. If bidder i pays $\sum_{n=1}^{m-1} p_n(\theta_j, d)$ to win $m-1$ units, then she is willing to pay $b_m(\theta_i, \sum_{n=1}^{m-1} p_n(\theta_j, d))$ to win her m^{th} unit. In the pricing rule that corresponds to cut-off rule d, bidder i demands her m^{th} unit if and only if her type is above the cut-off for the m^{th} unit. Or equivalently,

$$b_m(\theta_i, \sum_{n=1}^{m-1} p_n(\theta_j, d)) > p_m(\theta_j, d) \iff \theta_i > d_m(\theta_j).$$

Thus, the price of the m^{th} unit is,

$$p_m(\theta_j, d) = b_m(d_m(\theta_j), \sum_{n=1}^{m-1} p_n(\theta_j, d)).$$

Lemma 2 shows bidder i pays a higher total price for m units when bidder j has a higher type.

Lemma 2. *$\sum_{n=1}^{m} p_n(\theta_j, d)$ is weakly increasing in θ_j for all $\theta_j \in \Theta$, $m \in \{1, \dots, k\}$, $d \in \mathcal{D}$.*

The transformed cut-off rule is such that bidder i wins her m^{th} unit if and only if her type θ_i is such that her willingness to pay for her m^{th} unit exceeds her rival's willingness to pay for her $k-m+1^{st}$ unit. In other words, the transformed cut-off rule is such that bidder i wins at least m units if and only if her willingness to pay for her m^{th} unit ranks among the top k willingness to pays of the two bidders.

We calculate a bidder's willingness to pay for her m^{th} unit under the untransformed pricing rule that corresponds to cut-off rule d. Thus, bidder i's willingness to pay for her m^{th} unit is

$$b_m(\theta_i, \sum_{n=1}^{m-1} p_n(\theta_j, d)).$$

And bidder j's willingness to pay for her $k-m+1^{st}$ unit is

$$b_{k-m+1}(\theta_j, \sum_{n=1}^{k-m} p_n(\theta_i, d)).$$

Thus, the transformed cut-off rule is such that

$$\theta_i > T(d_m)(\theta_j) \implies b_m(\theta_i, \sum_{n=1}^{m-1} p_n(\theta_j, d)) > b_{k-m+1}(\theta_j, \sum_{n=1}^{k-m} p_n(\theta_i, d)).$$

[12] I show that there is a mechanism that satisfies the no subsidy condition. Hence, we assume $p_0(\theta_j) = 0$.

For ease of notation, we let

$$f(\theta_i, \theta_j, m, d) := b_m(\theta_i, \sum_{n=1}^{m-1} p_n(\theta_j, d)) - b_{k-m+1}(\theta_j, \sum_{n=1}^{k-m} p_n(\theta_i, d)).$$

Thus, $f(\theta_i, \theta_j, m, d)$ represents the amount that bidder i's willingness to pay for her m^{th} unit exceeds her rival's willingness to pay for her $k-m+1^{st}$ unit, when we evaluate willingness to pays under the pricing rule induced by cut-off rule d.

Lemma 3. *f is strictly increasing in the first argument, and strictly decreasing in the second and third arguments.*

Lemma 3 implies that if $f(m, \overline{\theta}, \theta_j, d) \leq 0$, then bidder i's willingness to pay for her m^{th} unit is always less than her rival's willingness to pay for her $k-m+1^{st}$ unit. Therefore, the transformed cut-off is such that bidder i wins less than m units for any reported type. Hence, let $T(d_m)(\theta_j) = \overline{\theta}$.

If $f(m, \overline{\theta}, \theta_j, d) > 0$, then bidder i's willingness to pay for her m^{th} unit exceeds her rival's willingness to pay for her $k-m+1^{st}$ unit when θ_i is sufficiently large. Then, let $T(d_m)(\theta_j)$ be the lowest type where bidder i's willingness to pay for her m^{th} unit exceeds her rival's willingness to pay for her $k-m+1^{st}$ unit,

$$T(d_m)(\theta_j) := \inf\{\theta \in [0, \overline{\theta}] | f(m, \theta, \theta_j, d) \geq 0\}.$$

Thus, the transformed cut-off rule assigns bidder i at least m units if and only if her willingness to pay for her m^{th} unit exceeds her rival's willingness to pay for her $k-m+1^{st}$ unit. We calculate a bidder's willingness to pay for her m^{th} unit by assuming that the price she paid for her first $m-1$ units was determined by the pricing rule corresponding to the (untransformed) cut-off rule d. This is stated in the remark below.

Remark 1. If $d \in \mathcal{D}$, then

$$b_m(\theta_i, \sum_{n=1}^{m-1} p_n(\theta_j, d)) \geq b_{k-m+1}(\theta_j, \sum_{n=1}^{k-m} p_n(\theta_i, d)) \implies \theta_j \geq T(d_m)(\theta_j),$$

$$b_{k-m+1}(\theta_j, \sum_{n=1}^{k-m} p_n(\theta_i, d)) \geq b_m(\theta_i, \sum_{n=1}^{m-1} p_n(\theta_j, d)) \implies T(d_m)(\theta_j) \geq \theta_i.$$

A consequence of Remark 1 is that if T has a fixed point $d^* \in \mathcal{D}$, then d^* defines a feasible mechanism that satisfies individual rationality, no subsidy, incentive compatibility, and Pareto efficiency. This is because the pricing rule corresponding to d^* is such that (1) bidder i demands m units if and only if her rival demands $k-m$ units, and (2) bidder i wins her m^{th} unit if and only if that her willingness to pay for her m^{th} unit exceeds her rival's willingness to pay for her $k-m+1^{st}$ unit. Thus, there are no Pareto improving trades where bidder i sells one of her units to bidder j.

Proposition 2. *If $d^* \in \mathcal{D}$ is a fixed point of the mapping T, then d^* corresponds to a feasible mechanism that satisfies (1) individual rationality, (2) no subsidy, (3) incentive compatibility, and (4) Pareto efficiency.*

I use Schauder's fixed point theorem to show that the mapping T has a fixed point $d^* \in \mathcal{D}$. In particular, I argue that (1) if $d \in \mathcal{D}$, then $T(d) \in \mathcal{D}$; (2) T is a continuous mapping; and (3) $\mathcal{D}$ is compact. These three conditions guarantee the existence of a fixed point according to Schauder's fixed point theorem (see Aliprantis and Border (2006, pg.. 583)).

Theorem 1. *There exists a $d^* \in \mathcal{D}$ such that $T(d^*) = d^*$.*

Thus, Theorem 1 shows that in the $2 \times k$ setting, there is a mechanism that retains the desirable properties of the Vickrey auction. The mechanism satisfies (1) individually rationality (2) no subsidy, (3) incentive compatibility, and (4) Pareto efficiency.

5 Multi-dimensional Types and An Impossibility Theorem

While we obtain a positive result with two bidders who have single dimensional types, in this section I show that the positive result does not carry over to the multi-dimensional type case. I study a setting where bidders have two-dimensional types, and the second dimension of a bidder's type is binary. I show that there is no mechanism that satisfies (1) individual rationality (2) weak budget balance, (3) incentive compatibility, and (4) Pareto efficiency (for the remainder of this section, Properties (1)-(4)). Since there is no mechanism that satisfies Properties (1)-(4) in the two-dimensional types, it follows that there is no mechanism that satisfies Properties (1)-(4) when we have a richer type space - the increase in dimensionality only adds to the number of incentive constraints that our mechanism must satisfy.

5.1 Setting with Multi-dimensional Types

We study a setting where there are two bidders who compete for two homogenous goods. A bidder's type is described by a two dimensional variable $\gamma_i = (\theta_i, t_i) \in [0, \overline{\theta}] \times \{f, s\}$. If bidder i has type γ_i, wins $q \in \{0, 1, 2\}$ units, and receives transfer $m \in \mathbb{R}$, then her utility is

$u(q, m, \gamma_i) \in \mathbb{R}$, where u is continuous and strictly increasing in m. Again, we assume that bidder i has no demand for units if the first dimension of her type $\theta_i = 0$,

$$u(q, -x, (0, t_i)) = u(q', -x, (0, t_i)) \; \forall q, q' \in \{0, 1, 2\}, \; x \in \mathbb{R}, \; t_i \in \{s, f\},$$

and a bidder has positive demand if $\theta_i > 0$,

$$u(q, -x, (\theta_i, t_i)) > u(q', -x, (\theta_i, t_i)) \; \forall q > q' \in \{0, 1, 2\}, \; x \in \mathbb{R}, \; t_i \in \{s, f\}.$$

The second dimension of bidder i's type $t_i \in \{f, s\}$ represents the steepness of her demand curve - it can either be flat (f) or steep (s). Bidders with steeper demand curves have relatively lower demand for their second unit. Thus, if $\theta_i > 0$,

$$u(2, -x-y, (\theta_i, s)) \geq u(1, -x, (\theta_i, s)) \implies u(2, -x-y, (\theta_i, f)) > u(1, -x, (\theta_i, f)).$$

Hence, if $b_2(\gamma_i, x)$ is bidder i's willingness to pay for her second unit when she has type γ_i and paid x for her first unit, then b_2 is such that

$$b_2((\theta_i, f), x) > b_2((\theta_i, s), x) > 0 \; \forall \theta_i \in (0, \overline{\theta}].$$

Again, we assume u is continuous in θ_i, and it is without loss of generality to assume that θ_i represents bidder i's willingness to pay for her first unit of the good. I refer to θ_i as bidder i's intercept. We assume bidder preferences satisfy (1) the law of demand, (2) strictly positive wealth effects, and (3) single-crossing in θ (Assumptions 1-3, as defined in Section 2). Thus,

1. $\theta_i > b_2(\gamma_i, x) > 0$, and $s_1(\gamma_i, x) \geq s_2(\gamma_i, x), \; \forall x \geq 0, \; \gamma_i = (\theta_i, t_i) \in (0, \overline{\theta}] \times \{f, s\}$.

2. $b_n(\gamma_i, x)$ and $s_n(\gamma_i, x)$ are continuous and strictly decreasing in x, $\forall x \in \mathbb{R}, \; \gamma_i \in (0, \overline{\theta}] \times \{f, s\}$.

3. $b_n((\theta_i', t_i), x) > b_n((\theta_i, t_i), x)$ and $s_n((\theta_i', t_i), x) > s_n((\theta_i, t_i), x) \; \forall \theta_i' > \theta_i, \; x \in \mathbb{R}, \; t_i \in \{f, s\}$.

Points 1, 2, and 3 above are direct implications of Assumptions 1, 2, and 3, respectively.

5.2 An impossibility theorem for the multi-dimensional type case

I prove the impossibility theorem in the multi-dimensional type setting by contradiction. The proof proceeds roughly as follows. If a mechanism satisfies Properties (1)-(4), then the price a bidder pays to win her first unit depends on her rival's demand for a second unit. Thus, all else equal, bidder i faces a relatively higher price for her first unit when bidder j reports a flatter demand curve.[13]Consequently, positive wealth effects imply that bidder i has lower demand for a second unit (conditional on buying her first unit) when her rival reports a flatter demand curve (bidder i pays more for a first unit when bidder j reports a flatter demand curve). Yet, if bidder i has lower demand for a second unit, then bidder j faces a relatively lower price to win her first unit. This is because bidder i's demand for a second unit determines the price bidder j pays for a first unit. Thus, bidder j is able to pay a lower price for her first unit by overstating her demand for her second unit. When she overstates demand for her second unit, she is able to lower her rival's demand for her second unit, and thus lower the price she pays for her first unit. This violates incentive compatibility and hence, we obtain a contradiction.

In order to formalize this line of argument, it is useful to invoke the taxation principle, which states that changes in bidder i's reported type change her payment only if they change her allocation.

Remark 2. (Taxation principle) If Γ satisfies Properties (1)-(4), then there exists pricing rules $p_{i,0}, p_{i,1}$, and $p_{i,2}$ such that

$$x_i(\gamma_i, \gamma_j) = p_{i,0}(\gamma_j) \iff q_i(\gamma_i, \gamma_j) = 0,$$

$$x_i(\gamma_i, \gamma_j) = p_{i,0}(\gamma_j) + p_{i,1}(\gamma_j) \iff q_i(\gamma_i, \gamma_j) = 1,$$

and

$$x_i(\gamma_i, \gamma_j) = p_{i,0}(\gamma_j) + p_{i,1}(\gamma_j) + p_{i,2}(\gamma_j) \iff q_i(\gamma_i, \gamma_j) = 2.$$

Lemma 4 simplifies the proof further. It shows that any mechanism that satisfies Properties (1)-(4) also satisfies the no subsidy condition. Individual rationality ensures that $p_{i,0}(\gamma_j) \leq 0$, because a bidder never regrets participating in the mechanism, even if she wins no units. In addition, I show that we violate weak budget balance if $p_{i,0}(\gamma_j) < 0$. This is because efficiency and incentive compatibility imply that bidder j wins both units for a negligible payment when bidder i reports a sufficiently low intercept θ_i. Thus, bidder i must be paid at most an arbitrarily small amount, or else we would violated weak budget balance.

Lemma 4. *If Γ satisfies Properties (1)-(4), then*

$$q_i(\gamma_i, \gamma_j) = 0 \implies x_i(\gamma_i, \gamma_j) = 0.$$

[13]When bidder j reports flat demand, she is indicating that she has greater demand for her second unit of the good relative to the case where she has steep demand.

I form the proof by contradiction by placing necessary conditions on a mechanism's allocation rule. It is useful to describe a mechanism's allocation rule by a cut-off rules. The first cut-off rule specifies the minimal intercept a bidder must report to win at least one unit, for a given reported steepness and her rival's type,

$$d_{i,1}^{t_i}(\gamma_j) := \begin{cases} \inf\{\theta \in \Theta | q_i((\theta, t_i), \gamma_j) \geq 1\} & if\ \exists \theta \in \Theta\ s.t.\ q_i((\theta, t_i), \gamma_j) \geq 1 \\ \overline{\theta} & else. \end{cases}.$$

Therefore, given bidder j's reported type γ_j, bidder i wins at least one unit if she reports type $\gamma_i = (\theta_i, t_i)$, where $\theta_i > d_{i,1}^{t_i}(\gamma_j)$. Similarly, the second cut-off function that states the minimal intercept a bidder must report in order to win both units,

$$d_{i,2}^{t_i}(\gamma_j) = \begin{cases} \inf\{\theta \in \Theta | q_i((\theta, t_i), \gamma_j) = 2\} & if\ \exists \theta \in \Theta\ s.t.\ q_i((\theta, t_i), \gamma_j) \geq 2 \\ \overline{\theta} & else. \end{cases}.$$

Remark 3 gives necessary conditions on the cut-off rules of mechanism that satisfy Properties (1)-(4).

Remark 3. If Γ satisfies Properties (1)-(4), then

(1) $q_1(\gamma_1, \gamma_2) + q_2(\gamma_1, \gamma_2) = 2\ if\ \theta_i > 0\ for\ some\ i = 1, 2.$

(2) $d_{i,1}^{t_i}(0, t_j) = d_{i,2}^{t_i}(0, t_j) = 0.$

(3) $d_{i,2}^{t_i}(\theta_j, t_j)$, and $d_{i,1}^{t_i}(\theta_j, t_j)$ are weakly increasing in θ_j.

(4) $d_{i,2}^{t_i}(\gamma_j) \geq d_{i,1}^{t_i}(\gamma_j)$.

The first point states that both units are sold if at least one bidder has positive demand. The second point states that a bidder wins both units if she reports positive demand and her rival reports no demand. The third point states that a bidder faces a greater intercept cut-off when her rival reports greater demand. The final point states the cut-off intercept for winning both units is weakly greater than the cut-off intercept for winning a single unit. The first two points directly follow from Pareto efficiency, and the latter two points follow from incentive compatibility.

Proposition 3 places additional restrictions on the cut-off rules. The Proposition shows that a bidder's first cut-off is continuous and strictly increasing in her rival's intercept. Thus, it becomes continuously more difficult for bidder i to win a single unit as her rival increases her reported intercept. In addition, bidder i has a strictly greater cut-off for her second unit than she does for her first unit. Or in other words, given bidder j's reported type γ_j, there is always an interval of reported intercepts for which bidder i wins exactly one unit.

Proposition 3. *Fix $t_i, t_j \in \{s, f\}$. If Γ satisfies Properties (1)-(4), then $d_{i,1}^{t_i}(\theta_j, t_j)$ is continuous and strictly increasing in θ_j, for all $\theta_j \in \Theta$, and*

$$d_{i,2}^{t_i}(\theta_j, t_j) > d_{i,1}^{t_i}(\theta_j, t_j) \; \forall \theta_j \in (0, \overline{\theta}].$$

While the formal proof of Proposition 3 is long, the intuition behind the proof is straightforward. I first show that $d_{i,2}^{t_i}(\theta_j, t_j) > d_{i,1}^{t_i}(\theta_j, t_j)$. The proof is by contradiction. If the two cutoffs are equal at a point, then there is a point where bidder i can increase her reported intercept by an arbitrarily small amount and change her allocation from winning zero units to winning both units. Yet, if it is the case that there is a Pareto efficient outcome where bidder i wins no units (and bidder j wins both units) when she reports the lower intercept, then it must be the case that bidder i's demand for her first unit is sufficiently low relative to bidder j's demand for her second unit. Declining demand then implies that her demand for her second unit is non-negligibly lower than bidder j's demand for her first unit. Thus, if we increase bidder i's intercept by an arbitrarily small amount, her demand for her second unit remains lower than her rival's demand for her first unit. Hence, it is Pareto inefficient to assign bidder i both units.

I then use a similar style of proof to show that a small increase in bidder j's reported intercept must lead to a small increase in bidder i's first cut-off. Hence, the first cut-off is continuous and strictly increasing.

The taxation principle and Proposition 3 imply that a bidder's first cut-off is independent of her steepness. This is intuitive, because a bidder demands at least one unit if and only if her willingness to pay for her first unit θ_i exceeds the price of the first unit $p_{i,1}(\gamma_j)$. Bidder i's demand for her first unit is independent of her steepness, thus her first intercept cut-off is independent of her first unit.

Corollary 1. *If Γ satisfies Properties (1)-(4)*

$$p_{i,1}(\gamma_j) = d_{i,1}^{f}(\gamma_j) = d_{i,1}^{s}(\gamma_j) \; \forall \gamma_j \in \{s, f\} \times \Theta.$$

Given the Corollary 1, I drop the superscript on a bidder's first cut-off $d_{i,1}^{t_i}$.

We also use Proposition 3 to get more precise information on the pricing rule. Corollary 2 shows that the price a bidder pays for her first unit equals her rival's willingness to pay for her second unit, conditional on having already won her first unit. In addition, the price a bidder pays for her second unit equals her rival's willingness to pay for her first unit. Thus, the mechanism assigns one bidder both units if and only if her willingness to pay for her second unit, conditional on winning her first unit, exceeds her rival's willingness to pay for her first unit. To prove the Corollary, consider a case where bidder i intercept equals her first cut-off $\theta_i = d_{i,1}(\gamma_j) = p_{i,1}(\gamma_j)$. In this case, both bidders' intercept reports are locally pivotal. If bidder i increases her reported intercept by a negligible amount, then she wins one unit. Similarly if bidder j increases her reported intercept by a negligible amount, she wins both units. Incentive compatibility then implies that the pricing rule is such that bidder i is indifferent between winning zero and one units and bidder j is indifferent between winning one and two units. In addition, efficiency implies bidder i's willingness to pay for her first unit equals bidder j's willingness to pay for her second unit. If the two quantities were unequal, then we could find a Pareto improving reallocation where the bidder with the lower willingness to pay sells a unit to the bidder with the higher willingness to pay.

Corollary 2. *If* Γ *satisfies Properties (1)-(4), then* $\theta_i = d_{i,1}(\gamma_j)$*, implies that*

$$\theta_i = p_{i,1}(\gamma_j) = p_{j,2}(\gamma_i) = b_2(\gamma_j, p_{j,1}(\gamma_i)). \tag{2}$$

A consequence of Corollary 2 is that bidder i pays more to win her first unit when her rival has a flat demand curve. This is because her rival has a relatively higher willingness to pay for her second unit when her demand curve is flat, and the price bidder i pays for her first unit is her rival's (conditional) willingness to pay for her second unit.

Corollary 3. *If* Γ *satisfies Properties (1)-(4), then*

$$p_{i,1}(\theta_j, f) = d_{i,1}(\theta_j, f) > d_{i,1}(\theta_j, s) = p_{i,1}(\theta_j, s)\ \forall \theta_j \in (0, \overline{\theta}].$$

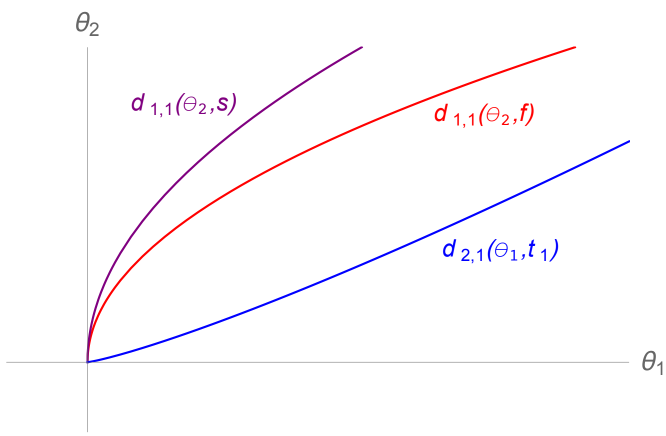

Figure 2: First Unit Cut-off rules for a fixed t_1.

Thus, if bidder j reports a flat demand curve (as opposed to a steep demand curve), then bidder i pays more to win her first unit. If bidder i pays more to win her first unit, then she has lower demand for her second unit. If bidder i has lower demand for her second unit of the good when bidder j reports a flat demand curve, then Corollary 3 implies bidder j pays a lower price to win her first unit of the good. Thus, bidder j is able to lower the price that she pays for her first unit by reporting a flat demand curve instead of a steep demand curve. This violates incentive compatibility and yields a contradiction.

More formally, Corollary 2 states that if $\theta_i = d_{i,1}(\gamma_j)$, then Equation 2 implies that

$$\theta_i = b_2(\gamma_j, p_{j,1}(\theta_i, s)) = b_2(\gamma_j, p_{j,1}(\theta_i, f)).$$

Yet Corollary 3 shows that

$$p_{j,1}(\theta_i, f) = d_{j,1}(\theta_i, f) > d_{j,1}(\theta_i, s) = p_{j,1}(\theta_i, s) \implies b_2(\gamma_j, p_{j,1}(\theta_i, s)) > b_2(\gamma_j, p_{j,1}(\theta_i, f)).$$

Hence, the final inequality above contradicts the implication of Equation 2. Thus, there is no mechanism that satisfies Properties (1)-(4).

Theorem 2. *There is no mechanism that satisfies Properties (1)-(4).*

The proof of Theorem 2 illustrates how the presence of wealth effects impedes efficient auction design. In the quasilinear setting, there are no wealth effects and the Vickrey auction satisfies Properties (1)-(4). In a 2×2 setting, the Vickrey auction is such that the price bidder i pays for her first unit equals her rival's willingness to pay for her second unit. Corollary 2 shows that this is a necessary condition in the non-quasilinear setting as well. Yet, in the non-quasilinear setting, the presence of wealth effects implies that the price a bidder pays for her first unit affects her demand for her second unit. This inhibits efficient auction design because a bidder can lower the price she pays for her first unit by misreporting her demand for her second unit. By stating a high demand for her second unit, a bidder forces her rival to pay more for her first unit. This benefits the bidder in a non-quasilinear setting because when the bidder's rival pays more for her first unit, her rival has lower demand for her second unit. Moreover, a bidder pays less to win her first unit when her rival has lower demand for her second unit. Thus, no mechanism can simultaneous satisfy (1)-(4) when we introduce wealth effects and multi-dimensional heterogeneity. A similar incentive to overreport demand for later units is present when bidders with positive wealth effects bid in the standard Vickrey auction (see Baisa (2016b)).

The proofs of Theorems 1 and 2 also illustrate the connection between efficient multi-unit auction design problem with private values and non-quasilinear preferences, and efficient single-unit auction design problem with interdependent values and quasilinear preferences. In the latter setting, Dasgupta and Maskin (2000) and Jehiel and Moldovanu (2001) show that efficient design is possible if and only if bidders have single dimensional types. Theorems 1 and 2 combine to give similar result in my setting, because the presence of wealth effects endogenously causes interdependence of bidders' demands. In the efficient auction design problem, wealth effects imply that a bidder's demand for her second unit is a function of the price she pays for her first unit. In addition, the price of a bidder's first unit depends on her rival's type. Thus, a bidder's willingness to pay for her second unit depends on her rival's type. Thus, when types are multi-dimensional, a bidder's demand for her second unit depends on both dimensions of her rival's type, and thus we can obtain an impossibility result. We do not see similar impossibility results in single unit demand settings, because the interdependence is only in the demand of later units.[14]

While Theorem 2 is obtained in a 2 bidders and 2 objects setting, the proof generalizes to an n bidder k object case. To see this, consider a proof similar to the one presented above. Suppose that there is a mechanism that satisfies Properties (1)-(4) in the $n \times k$ setting. Furthermore, suppose that bidder 2 has a relatively high intercept and steepness $t_2 \in \{s, f\}$, and bidders $3, \ldots, n$ all have arbitrarily small intercepts. Then, the mechanism assigns no units to bidder j where $j \geq 3$. If a bidder j won a positive number of units, there is a Pareto improving trade where bidder j sells a unit to bidder 2.

I look at the decision problem of bidder 1. Incentive compatibility implies that the price bidder 1 pays for her first unit $p_{1,1}(\gamma_{-1})$ is independent of her type. In addition, $p_{1,1}(\gamma_{-1})$ is determined by bidder 2's willingness to pay for her last unit. This is proved using same argument given in Corollary 2. Bidder j's report does not change the price bidder 1 pays for a unit because bidder j's demand for units is arbitrarily small.

Yet, then bidder 1 can lower bidder 2's willingness to pay for her last unit, and hence the price of her first unit, by reporting a flat demand versus a steep demand. When bidder 1 reports a flat demand and has greater demand for her later units, then bidder 2 pays more to win her first $k-1$ units. Positive wealth effects then imply that bidder 2 has lower demand for her final unit when she pays more for her first $k-1$ units. Thus, bidder 1 pays less for her first unit when she reports a flat demand. Yet this contradicts with incentive compatibility that requires that the price bidder 1 pays for her first unit is independent of her type.

[14]Morimoto and Serizawa (2015) show there is a mechanism that satisfies Properties (1)-(4) when bidders have single unit demands.

6 An impossibility theorem for the single dimensional type case with $N \geq 3$ bidders.

In this section, we return to the single-dimensional type case. We show that there is no mechanism that retains the desirable properties of the Vickrey auction, even in the single dimensional type setting, when there are at least three bidders. In particular, we show that there is no mechanism that satisfies (a) individual rationality, (b) no subsidies, (c) incentive compatibility, (d) Pareto efficiency, and (e) monotonicity (for the remainder of this section, Properties (a)-(e)).

We consider a setting where there are three bidders who compete for two goods and $N \geq 3$ bidders. The proof is by contradiction. We begin by assuming that there is a mechanism Γ that satisfies Properties (a)-(e). Let $q_i(\theta_i, \theta_{-i})$ be the number of units won by bidder i in mechanism Γ. Incentive compatibility implies that $q_i(\theta_i, \theta_{-i})$ is weakly increasing in bidder i's type θ_i. In addition, Pareto efficiency implies that

$$\sum_{i=1}^{N} q_i(\theta_i, \theta_{-i}) = 2 \; if \; \theta_i > 0 \; for \; some \; i \in \{1, \ldots, N\}.$$

Again, it is useful to define a cut-off rule that corresponds to mechanism Γ. The cut-off rule for bidder i's first unit $d_{i,1}$ is

$$d_{i,1}(\theta_{-i}) = \begin{cases} \inf\{\theta \in \Theta | q_i(\theta, \theta_{-i}) \geq 1\} & if \; q_i(\overline{\theta}, \theta_{-i}) \geq 1 \\ \overline{\theta} & if \; q_i(\overline{\theta}, \theta_{-i}) = 0 \end{cases}.$$

Similarly, the cut-off rule for bidder i's second unit $d_{i,2}$ is

$$d_{i,2}(\theta_{-i}) = \begin{cases} \inf\{\theta \in \Theta | q_i(\theta, \theta_{-i}) = 2\} & if \; q_i(\overline{\theta}, \theta_{-i}) = 2 \\ \overline{\theta} & if \; q_i(\overline{\theta}, \theta_{-i}) \leq 1 \end{cases}.$$

Incentive compatibility implies that $d_{i,2}(\theta_{-i}) \geq d_{i,1}(\theta_{-i}) \; \forall \theta_{-i} \in \Theta^2$.

Again, Pareto efficiency implies that if bidder i's rivals report zero demand, then bidder i wins both units if she reports positive demand.

Remark 4. $d_{i,1}(\theta_{-i}) = 0 \iff \theta_j = 0 \; \forall j \neq i$.

To condense notation, for the remainder of this section, I study the decision problem from the perspective of bidder 1, and I assume that $\theta_2 \geq \theta_3 \geq \theta_j \; \forall j \neq 1, 2, 3$.

Lemma 5 uses the implication of Pareto efficiency to show that a bidder wins a positive number of units only if her reported demand is among the two highest reports. If this property did not hold true, then there would be a Pareto improving trade where a bidder with a lower type sells a unit to a bidder with a higher type who did not win any units.

Lemma 5. *For any mechanism that satisfies, then*

$$d_{1,1}(\theta_{-1}) \geq \theta_3.$$

The next Lemma shows that if a bidder's two highest rivals have types that are sufficiently close together, then bidder i wins at least one unit if and only if her report type is among the two highest reported types. This follow from the law of demand.

To see the intuition, suppose $\theta_2 > \theta_3$. If bidders 2 and 3 have types that are sufficiently close together, then declining demand implies that $\theta_3 \geq b_2(\theta_2, 0)$. In other words, bidder 3 is willing to pay more for her first unit than bidder 2 is willing to pay for her second unit. Thus if bidder 1 reports a type that is in the interval (θ_3, θ_2), Lemma 5 implies that bidder 3 wins zero units, because her type is not among the two highest demands. Then, if bidder 1 wins no units, it means that bidder 2 must win both units. Yet, if bidder 2 wins both units, there is a Pareto improving trade where bidder 1 buys one of bidder 2's units. The trade is Pareto improving because the law of demand implies that bidder 2's demand for her second unit is low relative to bidder 1's demand for her first unit.

Lemma 6. *For any mechanism that satisfies (1)-(5), if $b_2(\theta_2, 0) < \theta_3 \leq \theta_2$, then*

$$d_{1,1}(\theta_{-1}) = \theta_3.$$

Thus, when bidders 2 and 3 have types that are sufficiently close together, bidder i wins at least one unit if and only if her type ranks among the two highest types. This means that if θ_2 and θ_3 are sufficiently close together, bidder 1 must pay θ_3 to win one unit of the good.

Furthermore, if we continue to assume bidder 1's two highest rivals report demands that are sufficiently close together, then Pareto efficiency implies that bidder 1 wins both units of the good if and only if her willingness to pay for her second unit (conditional on having paid θ_3 for her first unit) exceeds her highest rival's willingness to pay for her first unit of the good θ_2. Or in other words, bidder 1 wins both units only if $b_2(\theta_1, \theta_3)$ is greater than θ_2. This is summarized in Lemma 7 below.

Lemma 7. *For any mechanism that satisfies (1)-(5), if $b_2(\theta_2, 0) < \theta_3 < \theta_2$, then*

$$d_{1,2}(\theta_{-1}) = \theta_1^*$$

where θ_1^* is defined as solving

$$b_2(\theta_1^*, \theta_3) = \theta_2.$$

Lemmas 6 and 7 combine to yield a violation of the monotonicity constraint on the allocation rule. Suppose that $\theta_1 > \theta_2 > \theta_3$, and that θ_2 is sufficiently close to θ_3. Then if θ_3 increases slightly, Lemma 6 implies that it becomes more difficult for bidder 1 to win a second unit of the good, as the price she pays for her first unit increases. Thus, it becomes relatively easier for bidder 2 to win her first unit of the good when bidder 3 increases her reported type slightly.[15] In fact, bidder 2 faces a lower first unit cut-off when her rival bidder 3 stated a higher demand. This contradicts with the monotonicity constraint on the allocation rule because monotonicity implies that a bidder faces a higher cut-off when her rivals report greater demands.

Theorem 3. *There is no mechanism that satisfies properties (1)-(5).*

Thus, to recap the argument, we see that if $\theta_1 > \theta_2 > \theta_3 \geq \theta_j \ \forall j \neq 1, 2, 3$, then Lemma 5 shows us that bidder 3 never wins any units. Yet, if bidder 3 reports a demand that is sufficiently close to bidder 2, then bidder 3 can still change the allocation. This is because the price bidder 1 pays to win her first unit equals bidder 3's willingness to pay for her first unit when θ_3 is sufficiently close to θ_2. Thus, by increasing her report, bidder 3 increases the price that bidder 1 pays for her first unit, and lowers bidder 1's demand for her second unit. When bidder 1 has lower demand for her second unit, bidder 2 then faces a lower cut-off for her first unit. However, we have a violation of monotonicity if bidder 2's first cut-off is lower when her rivals increase their demands.

[15]Indeed, bidder 2 faces a lower cut-off for her first unit of the good, even if bidder 1 reports a slightly higher type.

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7 Proofs

Proof of Lemma 1.

Proof. I show that there exists a solution d to Equation 1 by showing that for any $n \in \mathbb{N}$, there exists a unique function $d^n : [0, \overline{\theta}] \to [0, \overline{\theta}]$ such that

$$d^n(\theta) = \max\{\frac{1}{n}, b_2(\theta, d^n(d^n(\theta))\}. \tag{3}$$

Then I show that d^n converges uniformly to a function d that solves Equation 1 .

Equation 3 uniquely defines a function over the interval $[0, \frac{1}{n}]$ because

$$d^n(\theta) = \frac{1}{n} = \max\{\frac{1}{n}, b_2(\theta, d^n(d^n(\theta))\} > b_2(\theta, 0) \geq b_2(\theta, d^n(d^n(\theta))).$$

Thus, there is an interval of the form $[0, x]$, where $x > 0$, over which there exists unique function d^n that satisfies Equation 3. Let $\overline{x}$ be the supremum x such that there is a function that is uniquely defined by Equation 3 over the interval $[0, x]$. We know from the above construction that $\overline{x} \geq \frac{1}{n}$.

I show d^n is weakly increasing over $(0, \overline{x})$ by contradiction. If d^n was strictly decreasing, then $\exists \theta^* \in (0, \overline{x})$ such that

$$\theta^* = \inf\{\theta | \exists \theta' > \theta \ s.t. \ d^n(\theta') < d^n(\theta)\}.$$

Thus, for any $\epsilon > 0$ there exists a θ_ℓ, θ_h such that $\theta_\ell \leq \theta^* \leq \theta_h$, $\theta_h, \theta_\ell \in (\theta^* - \epsilon, \theta^* + \epsilon)$, and $d^n(\theta_\ell) > d^n(\theta_h) \geq \frac{1}{n}$. Thus,

$$d^n(\theta_\ell) > \frac{1}{n} \implies \theta_\ell > d^n(\theta_\ell) = b_2(\theta_\ell, d^n(d^n(\theta_\ell))) > \frac{1}{n}.$$

In addition, $d^n(\theta_\ell) > d^n(\theta_h)$ implies that

$$b_2(\theta_h, d^n(d^n(\theta_h)) < b_2(\theta_\ell, d^n(d^n(\theta_\ell))).$$

Since b_2 is increasing in the first argument and $\theta_h > \theta_\ell$, then we must have that

$$d^n(d^n(\theta_h)) > d^n(d^n(\theta_\ell)).$$

However, the above inequality can not hold because

$$d^n(\theta_h) < d^n(\theta_\ell) < \theta_\ell \leq \theta^* \implies d^n(d^n(\theta_h)) \leq d^n(d^n(\theta_\ell)),$$

where the final inequality holds because d^n is weakly increasing when $\theta < \theta^*$. Thus, we have a contradiction that shows d^n is weakly increasing.

A similar proof shows that d^n is continuous over $(0, \overline{x})$. Let θ^* be the point of the first discontinuity. By construction d^n is continuous when θ is such that $b_2(\theta, \frac{1}{n}) < \frac{1}{n}$. Thus, $\lim_{\theta \to^+ \theta^*} d(\theta) > \frac{1}{n}$. Yet, when ϵ is small, $d^n(\theta^* - \epsilon) \approx d^n(\theta^* + \epsilon)$ because $d^n(d^n(\theta^* - \epsilon)) \approx d^n(d^n(\theta^* + \epsilon)) \leq d^n(\theta^* - \epsilon)$. Since b_2 is continuous in both arguments, this implies that $d^n(\theta^* + \epsilon) \approx d^n(\theta^* - \epsilon)$, which contradicts our assumption that d^n is discontinuous at θ^*.

Next, I show that $\overline{x} = \overline{\theta}$ by contradiction. Suppose that $x < \overline{\theta}$. Then, for any $\epsilon > 0$, there exists a $\tilde{\theta} \in [\overline{x}, \overline{x} + \epsilon)$ such that $d^n(\tilde{\theta})$ is not uniquely defined by Equation 3. Note that $b_2(\tilde{\theta}, \frac{1}{n}) > \frac{1}{n}$. If not, then $d^n(\tilde{\theta})$ would be uniquely defined by Equation 3.

If $d^n(\tilde{\theta} - \epsilon) = \frac{1}{n}$, then

$$\frac{1}{n} = d^n(\tilde{\theta} - \epsilon) \geq d^n(b_2(\tilde{\theta}, \frac{1}{n})) \geq \frac{1}{n} \implies d^n(b_2(\tilde{\theta}, \frac{1}{n})) = \frac{1}{n},$$

where the first inequality holds because $\tilde{\theta} - \epsilon > b_2(\tilde{\theta}, \frac{1}{n})$ when ϵ is sufficiently small by declining demand, and the second inequality holds because d^n is bounded below by $\frac{1}{n}$. Thus,

$$b_2(\tilde{\theta}, \frac{1}{n}) = b_2(\tilde{\theta}, d^n\left(b_2(\tilde{\theta}, \frac{1}{n})\right)).$$

Thus, if we let $d^n(\tilde{\theta}) = b_2(\tilde{\theta}, \frac{1}{n})$, then

$$d^n(\tilde{\theta}) = b_2(\tilde{\theta}, d^n(d^n(\tilde{\theta})).$$

This is the unique solution to Equation 3 because $x - b_2(\tilde{\theta}, d^n(x))$ is weakly increasing in x as b_2 is decreasing in the second argument and d^n is increasing in x. Thus, it can not be the case that $d^n(\tilde{\theta} - \epsilon) = \frac{1}{n}$.

Next suppose that $d^n(\tilde{\theta} - \epsilon) > \frac{1}{n}$. Recall that $x - b_2(\tilde{\theta}, d^n(x))$ is strictly increasing in x. If $x = \frac{1}{n}$, then

$$\frac{1}{n} - b_2(\tilde{\theta}, d^n(\frac{1}{n})) < \tilde{\theta} - \epsilon - b_2(\tilde{\theta} - \epsilon, d^n(d^n(\tilde{\theta} - \epsilon))) = 0,$$

where the inequality follows because (1) $\tilde{\theta} - \epsilon > \frac{1}{n}$ because $d^n(\tilde{\theta} - \epsilon) > \frac{1}{n} \implies \tilde{\theta} - \epsilon > b_2(\tilde{\theta} - \epsilon, \frac{1}{n}) > \frac{1}{n}$; and (2) b_2 is decreasing in the second argument, while d^n is increasing and $d^n(\tilde{\theta} - \epsilon) > \frac{1}{n}$.

In addition, if $x = \tilde{\theta} - \epsilon$, then

$$0 < \tilde{\theta} - \epsilon - b_2(\tilde{\theta}, d^n(\tilde{\theta} - \epsilon)).$$

The above inequality holds because $d^n(\tilde{\theta} - \epsilon) > \frac{1}{n}$ implies that

$$0 = \tilde{\theta} - \epsilon - b_2(\tilde{\theta} - \epsilon, d^n(d^n(\tilde{\theta} - \epsilon))) < \tilde{\theta} - \epsilon - b_2(\tilde{\theta}, d^n(\tilde{\theta} - \epsilon)),$$

where the final inequality holds because (1) $d^n(\tilde{\theta} - \epsilon) \geq d^n(d^n(\tilde{\theta} - \epsilon))$ by construction and (2) b_2 is increasing in the first argument and decreasing in the second market. Thus,

$$x - b_2(\tilde{\theta}, d^n(x))$$

is strictly increasing, less than zero when $x = \frac{1}{n}$ and greater than zero when $x = \tilde{\theta} - \epsilon$. Since d^n is continuous, then the above expression is continuous and there is a unique x that solves

$$x - b_2(\tilde{\theta}, d^n(x)) = 0.$$

Thus, if $x = d^n(\tilde{\theta})$, then there is a unique solution defined by Equation 3. This contradicts our assumption that d^n is not uniquely defined by Equation 3 at $\tilde{\theta}$, and we have shown that $\overline{x} = \overline{\theta}$.

The last step of the proof shows $\{d^n\}_{n=1}^{\infty}$ is uniformly Cauchy. I show that

$$m > n > N^* \implies \max_{\theta \in [0,\overline{\theta}]} |d^n(\theta) - d^m(\theta)| \leq \frac{2}{N}.$$

First, I show that the above property holds if θ is such that $b_2(\theta, \frac{1}{N}) \leq \frac{1}{N}$.

$$b_2(\theta, \frac{1}{N}) \leq \frac{1}{N} \implies d^n(\theta) = \max\{b_2(\theta, d^n(d^n(\theta))), \frac{1}{n}\} \leq \max\{b_2(\theta, 0), \frac{1}{N}\},$$

because $\frac{1}{n} > \frac{1}{N}$ and $d^n(d^n(\theta)) > 0$ and b_2 is weakly decreasing in the second argument. In addition the proof of Lemma 2 shows that

$$b_2(\theta, 0) < b_2(\theta, \frac{1}{N}) + \frac{1}{N}.$$

Thus, $b_2(\theta, \frac{1}{N}) \leq \frac{1}{N} \implies b_2(\theta, 0) < \frac{2}{N}$, and

$$d^n(\theta) \leq \max\{b_2(\theta, 0), \frac{1}{N}\} \leq \max\{\frac{2}{N}, \frac{1}{N}\} = \frac{2}{N}.$$

The same argument shows $d^m(\theta) \leq \frac{2}{N}$ when $b_2(\theta, \frac{1}{N}) \leq \frac{1}{N}$. Since $d^m(\theta), d^n(\theta) \in [0, \frac{2}{N}]$, then

$$b_2(\theta, \frac{1}{N}) \leq \frac{1}{N} \implies |d^n(\theta) - d^m(\theta)| < \frac{2}{N}.$$

Next, suppose that θ is such that $b_2(\theta, \frac{1}{N}) > \frac{1}{N}$. I first show $d^n(\theta) > \frac{1}{n}$, by contradiction. If $d^n(\theta) = \frac{1}{n}$, then by the construction of d^n, then $b_2(\theta, d^n(\frac{1}{n})) \leq \frac{1}{n}$. But $d^n(\frac{1}{n}) = \frac{1}{n}$, because

$$d^n(\theta) \geq \frac{1}{n} \ \forall\theta \implies d^n(\frac{1}{n}) = \max\{\frac{1}{n}, b_2(\frac{1}{n}, \frac{1}{n})\} = \frac{1}{n},$$

since $b_2(\frac{1}{n}, \frac{1}{n}) \leq b_2(\frac{1}{n}, 0) < \frac{1}{n}$ by declining demand. Therefore, $b_2(\theta, d^n(\frac{1}{n})) = b_2(\theta, \frac{1}{n}) \leq \frac{1}{n}$. Yet,

$$n > N \implies b_2(\theta, \frac{1}{n}) \geq b_2(\theta, \frac{1}{N}) > \frac{1}{N} > \frac{1}{n},$$

which contradicts, $b_2(\theta, \frac{1}{n}) \leq \frac{1}{n}$. Thus, $d^n(\theta) > \frac{1}{n}$. I use this to show

$$b_2(\theta, \frac{1}{N}) > \frac{1}{N} \implies |d^n(\theta) - d^m(\theta)| \leq \frac{2}{N}.$$

The proof is by contradiction. If there is a θ such that

$$|d^n(\theta) - d^m(\theta)| > \frac{2}{N},$$

then since both d^n and d^m are continuous in θ, there exists a θ^*,

$$\theta^* = \min\{\theta^* | d^n(\theta^*) - d^m(\theta^*)| = \frac{2}{N}\}.$$

We know that $\theta^* > b_2(\theta^*, \frac{1}{N})$ because $|d^n(\theta) - d^m(\theta)| < \frac{1}{N} \ \forall\theta \ s.t. \ b_2(\theta, \frac{1}{N}) \leq \frac{1}{N}$. Without loss of generality, assume $d^n(\theta^*) > d^m(\theta^*)$. Since

$$\theta^* > b_2(\theta^*, \frac{1}{N}) \implies d^n(\theta^*) > \frac{1}{n}, \ d^m(\theta^*) > \frac{1}{m},$$

then,

$$d^n(\theta^*) = b_2(\theta^*, d^n(d^n(\theta^*))) = b_2(\theta^*, d^n(d^m(\theta)^* + \frac{2}{N})) = b_2(\theta^*, d^m(d^m(\theta^*))) + \frac{2}{N} = d^m(\theta^*) + \frac{2}{N}.$$

In addition, the proof of Lemma 2 implies

$$b_2(\theta^*, d^n(d^m(\theta)^* + \frac{2}{N})) = b_2(\theta^*, d^m(d^m(\theta^*))) + \frac{2}{N} \implies d^m(d^m(\theta^*)) > d^n(d^m(\theta^*) + \frac{2}{N}) + \frac{2}{N}.$$

Since d^n is weakly increasing this implies

$$d^m(d^m(\theta^*)) > d^n(d^m(\theta^*)) + \frac{2}{N}.$$

Yet, $\tilde{\theta} := d^m(\theta^*) = b_2(\theta^*, d^m(d^m(\theta^*))) < \theta^*$. Thus,

$$d^m(\tilde{\theta}) - d^n(\tilde{\theta}) > \frac{2}{N}.$$

Since d^m and d^n are continuous and $d^m(0) - d^m(0) < \frac{2}{N}$, then there exists a $\theta' < \tilde{\theta}$ such that

$$d^m(\theta') - d^n(\theta') = \frac{2}{N}.$$

Since $\theta' < \tilde{\theta} < \theta^*$, the above inequality contradicts the definition of θ^*. Thus, we have shown that $\{d^n\}$ is uniformly Cauchy because

$$\max_{\theta \in [0,\bar{\theta}]} |d^n(\theta) - d^m(\theta)| \leq \frac{2}{N}.$$

Thus, there exists a function d such that $d^n \to d$ uniformly. Since d^n is continuous and increasing for all $n \in \mathbb{N}$, uniform convergence implies that d is continuous, increasing and

$$d(\theta) = b_2(\theta, d(d(\theta))).$$

In addition, d is strictly increasing because b_2 is strictly increasing in the first argument.

□

Proof of Lemma 2.

Proof. The proof is by induction. When $m = 1$, $p_1(\theta_j, d)$ is weakly increasing in θ_j because $p_1(\theta_j, d) = d_1(\theta_j)$ and $d_1(\theta_j)$ is weakly increasing in θ_j for all $d \in \mathcal{D}$.

Before showing the inductive step, it is useful to note that

$$z \geq y \geq 0 \implies b_m(\theta, z) + z \geq b_m(\theta, y)) + y \;\forall m \in \{1, \dots, k\}.$$

This is because

$$z \geq y \geq 0 \implies u(m-1, -y-b_m(\theta, y), \theta) = u(m, -y, \theta) \geq u(m, -z, \theta) = u(m-1, -z-b_m(\theta, z), \theta).$$

Because u is increasing in the second argument, the final inequality shows

$$z \geq y \geq 0 \implies z + b_m(\theta, z) \geq y + b_m(\theta, y) \tag{4}$$

Returning to the proof, suppose that $\sum_{n=1}^{m-1} p_n(\theta_j, d)$ is weakly increasing in θ_j. Let

$\theta_j^h > \theta_j^\ell$. Then,

$$\sum_{n=1}^{m} p_n(\theta_j^h, d) = \sum_{n=1}^{m-1} p_n(\theta_j^h, d)+b_m(d_m(\theta_j^h), \sum_{n=1}^{m-1} p_n(\theta_j^h, d)) \geq \sum_{n=1}^{m-1} p_n(\theta_j^h, d)+b_m(d_m(\theta_j^\ell), \sum_{n=1}^{m-1} p_n(\theta_j^h, d)),$$

where the equality follows from the definition of p_m, and the inequality follows because b_m is increasing in the first argument and $d_m(\theta_j^h) \geq d_m(\theta_j^\ell)$. Then,

$$\sum_{n=1}^{m-1} p_n(\theta_j^h, d)+b_m(d_m(\theta_j^\ell), \sum_{n=1}^{m-1} p_n(\theta_j^h, d)) \geq \sum_{n=1}^{m-1} p_n(\theta_j^\ell, d)+b_m(d_m(\theta_j^\ell), \sum_{n=1}^{m-1} p_n(\theta_j^\ell, d)) = \sum_{n=1}^{m} p_n(\theta_j^\ell, d),$$

where the inequality holds by Equation 4 where we let $z = \sum_{n=1}^{m-1} p_n(\theta_j^h, d) \geq y = \sum_{n=1}^{m-1} p_n(\theta_j^\ell, d) \geq 0$. The final equality holds from the construction of p_m. □

Proof of Lemma 3

Proof. f is strictly increasing in θ_i because $b_m(\theta_i, \sum_{n=1}^{m-1} p_n(\theta_j, d))$ is strictly increasing in θ_i. In addition, $b_{k-m+1}(\theta_j, \sum_{n=1}^{k-m} p_n(\theta_i, d))$ is decreasing in the second argument and Lemma 2 shows $\sum_{n=1}^{k-m} p_n(\theta_i, d)$ is increasing in θ_i. The same argument shows that f is strictly decreasing in θ_j. The law of demand and positive wealth effects imply that f is strictly decreasing in m. □

Proof of Proposition 2.

Proof. I construct a mechanism that follows from the symmetric cut-off rule d_m^*. I assume ties (in terms of willingness to pays for additional units) are broken in favor of bidder 1. Thus,

$$q_1(\theta_1, \theta_2) = \max\{m \in \{0, 1, \ldots, k\} | b_m(\theta_1, \sum_{n=1}^{m-1} p_n(\theta_2, d^*)) \geq b_{k-m+1}(\theta_2, \sum_{n=1}^{k-m} p_n(\theta_1, d^*))\},$$

and $q_2(\theta_1, \theta_2) = k - q_1(\theta_1, \theta_2)$. We let $x_i(\theta_1, \theta_2) = \sum_{n=1}^{q_i(\theta_i,\theta_j)} p_n(\theta_j, d^*)$. By construction, the mechanism satisfies weak budget balance and individual rationality.

I show that mechanism is incentive compatible. If $q_i(\theta_i, \theta_j) \geq m$ then the construction of the mechanism implies that

$$b_m(\theta_i, \sum_{n=1}^{m-1} p_n(\theta_j, d^*)) \geq b_{k-m+1}(\theta_j, \sum_{n=1}^{k-m} p_n(\theta_i, d^*) \implies \theta_i \geq d_m^*(\theta_j),$$

where the implication follows from Remark 1. And since b_m is increasing in the first argument, $\theta_i \geq d_m^*(\theta_j)$ implies that

$$b_m(\theta_i, \sum_{n=1}^{m-1} p_n(\theta_j, d^*)) \geq b_m(d_m^*(\theta_j), \sum_{n=1}^{m-1} p_n(\theta_j, d^*)) = p_m(\theta_j, d^*).$$

In other words, the price of bidder i's m^{th} unit is below her willingness to pay for her m^{th} unit. Thus, bidder i has no incentive to deviate by reporting a lower type and winning fewer units.

Similarly, $m > q_i(\theta_i, \theta_j)$, then the construction of of mechanism implies that

$$b_{k-m+1}(\theta_j, \sum_{n=1}^{k-m} p_n(\theta_i, d^*) \geq b_m(\theta_i, \sum_{n=1}^{m-1} p_n(\theta_j, d^*)) \implies d_m^*(\theta_j) \geq \theta_i.$$

Thus, $d_m^*(\theta_j) \geq \theta_i$ implies that

$$p_m(\theta_j, d^*) = b_m(d_m^*(\theta_j), \sum_{n=1}^{m-1} p_n(\theta_j, d^*)) \geq b_m(\theta_i, \sum_{n=1}^{m-1} p_n(\theta_j, d^*)).$$

Thus, the price of winning an m^{th} unit where $m > q_i(\theta_i, \theta_j)$ exceeds bidder i's willingness to pay for her m^{th} unit, conditional on having won $m-1$ units under pricing rule $p(\theta_j, d^*)$. Therefore, bidder i does not increase her utility by reporting a type θ_i' that allows her to win more units. Thus, the mechanism is incentive compatible.

Last, I show that the mechanism is Pareto efficient. For ease of notation, let $q_i = q_i(\theta_1, \theta_2)$, $x_i = x_i(\theta_1, \theta_2)$, and ρ be

$$\rho := \max\{b_{q_1+1}(\theta_1, x_1), b_{q_2+1}(\theta_2, x_2)\}.$$

I show that the mechanism is Pareto efficient by first showing that

$$u(\tilde{q}_i, -\tilde{x}_i, \theta_i) \geq u(q_i, -x_i, \theta_i) \implies x_i + \rho(q_i - \tilde{q}_i) \geq \tilde{x}_i.$$

If $q_i = \tilde{q}_i$, the above inequality holds trivially. If $\tilde{q}_i < q_i$, and

$$u(\tilde{q}_i, -\tilde{x}_i, \theta_i) \geq u(q_i, -x_i, \theta_i).$$

Let y be such that,

$$u(\tilde{q}_i, -x_i + y, \theta_i) = u(q_i, -x_i, \theta_i).$$

Thus, y is bidder i's willingness to sell $q_i - \tilde{q}_i$ units when her initial allocation is having q_j units and paying x_j. I show that $y \geq \rho(q_j - \tilde{q}_j)$. That is, bidder i must be compensated by at least ρ per unit to ensure that she is made no worse off. To show this, it is useful to note that

$$b_n(\theta, x) = s_n(\theta, x + b_n(\theta, x)) \leq s_n(\theta, x + c) \; \forall c < b_n(\theta, x), \tag{5}$$

where the first equality follows directly from the definitions of willingness to pay/sell, and the final inequality follows from positive wealth effects.

Next, note that bidder i's willingness to pay for each of her first q_i units when facing pricing rule $p(\theta_j, d^*)$ exceeds ρ, because the willingness to pays associated with any winning bids exceeds ρ. Thus, bidder i pays an amount below her willingness to pay for each of her first q_j units when the pricing rule is $p(\theta_j, d^*)$. Thus, Equation 5 implies that her willingness to sell each unit exceeds ρ, and

$$u(\tilde{q}_i, -\tilde{x}_i, \theta_i) \geq u(q_i, -x_i, \theta_i) = u(\tilde{q}_i, -x_i + y, \theta_i) \implies x_i - y \geq \tilde{x}_i \implies x_i - \tilde{x}_i \geq y \geq \rho(q_i - \tilde{q}_i).$$

Or equivalently

$$u(\tilde{q}_i, -\tilde{x}_i, \theta_i) \geq u(q_i, -x_i, \theta_i) \implies x_i + (\tilde{q}_i - q_i)\rho \geq \tilde{x}_i.$$

If $\tilde{q}_i > q_i$, we show that

$$u(\tilde{q}_i, -\tilde{x}_i, \theta_i) \geq u(q_i, -x_i, \theta_i) \implies x_i + \rho(\tilde{q}_i - q_i) \geq \tilde{x}_i.$$

This follows because declining demand and positive wealth effects imply that $b_m(\theta_i, y) \leq b_{q_i+1}(\theta_i, x_i) \leq \rho$ for all $m \geq q_i + 1, \; y \geq x_i$. Thus, if a bidder wins more than q_i units, and her utility increases, the marginal price of additional units must be less than ρ,

$$u(\tilde{q}_i, -\tilde{x}_i, \theta_i) \geq u(q_i, -x_i, \theta_i) \implies x_i + (\tilde{q}_i - q_i)\rho \geq \tilde{x}_i.$$

Thus, any reallocation that makes bidder i strictly better and bidder j no worse gives strictly lower revenue. This is because

$$u(\tilde{q}_i, -\tilde{x}_i, \theta_i) > u(q_i, -x_i, \theta_i) \implies x_i + (\tilde{q}_i - q_i)\rho > \tilde{x}_i,$$

and

$$u(\tilde{q}_j, -\tilde{x}_j, \theta_j) \geq u(q_j, -x_j, \theta_j) \implies x_j + (\tilde{q}_j - q_j)\rho \geq \tilde{x}_i.$$

Feasibility implies that $q_1 + q_2 \geq \tilde{q}_1 + \tilde{q}_2$. Thus,

$$x_i + x_j \geq x_i + (\tilde{q}_i - q_i)\rho + x_j + (\tilde{q}_j - q_j)\rho > x_1 + x_2.$$

Similarly, any reallocation that makes both bidders no worse off, generates weakly lower revenue. Thus, $\{q_i, x_i\}_{i=1}^2$ is a Pareto efficient allocation. $\square$

Proof of Theorem 1.

Proof. First, we show $d \in \mathcal{D} \implies T(d) \in \mathcal{D}$. To do this, I show that $T(d_m)(\theta)$ is weakly increasing. If $T(d_m)(\theta_j^h) = \overline{\theta}$, then $T(d_m)(\theta_j^\ell) \leq \overline{\theta}$ because $T(d_m)(\theta) \in [0, \overline{\theta}]\ \forall \theta \in \Theta$. If $T(d_m)(\theta_j^h) < \overline{\theta}$, then

$$\inf\{\theta | f(\theta, \theta_j^h, m, d) > 0\} \geq \inf\{\theta | f(\theta, \theta_j^\ell, m, d) > 0\},$$

because Lemma 3 shows f is strictly increasing in the first argument and $f(\theta_i, \theta_j^\ell, m, d) > f(\theta_i, \theta_j^h, m, d)\ \forall \theta_i \in \Theta$. Thus, $T(d_m)(\theta_j^h) \geq T(d_m)(\theta_j^\ell)\ \forall \theta_j^h > \theta_j^\ell$.

Next, we show that

$$T(d_{m+1})(\theta_j) \geq T(d_m)(\theta_j)\ \forall m \in \{1, \ldots, k-1\}.$$

If $T(d_{m+1})(\theta_j) = \overline{\theta}$, the condition holds because $T(d_m)(\theta) \in [0, \overline{\theta}]\ \forall \theta \in \Theta$. If $T(d_{m+1})(\theta_j) < \overline{\theta}$. Note that Lemma 3 shows that

$$f(\theta_i, \theta_j, m, d) > f(\theta_i, \theta_j, m+1, d)\ \forall \theta_i, \theta_j \in \Theta,\ d \in \mathcal{D}.$$

In addition f is strictly increasing in the first argument. Therefore,

$$\inf\{\theta | f(\theta, \theta_j, m+1, d) > 0\} \geq \inf\{\theta | f(\theta, \theta_j, m, d) > 0\} \implies T(d_{m+1})(\theta_j) \geq T(d_m)(\theta_j).$$

Thus, $T(d) \in \mathcal{D}\ \forall d \in \mathcal{D}$, because $T(d_m)(\theta)$ is weakly increasing in θ and $T(d_{m+1})(\theta) \geq T(d_m)(\theta)$.

Next, we show that T is a continuous mapping. Since $\mathcal{D}$ is a metric space (under the uniform norm), it suffices to show that if $\{d^k\}_{k=1}^\infty$ is such that $d^k \in \mathcal{D}\ \forall k \in \mathbb{N}$ and $\lim d^k = d$, then $\lim T(d^k) = T(d)$ (see Aliprantis and Border (2006), pg. 36).

Fix θ_j. By assumption $d^k(\theta_j) \to d(\theta_j)$. First, I show that $\sum_{n=1}^m p_n(\theta_j, d^k) \to \sum_{n=1}^m p_n(\theta_j, d)\ \forall m \in \{1, \ldots, k\}$. The proof is by induction. It holds when $m = 1$, because

$$p_1(\theta_j, d^k) = d_1^k(\theta_j),$$

thus, $\lim p_1(\theta_j, d^k) = \lim d_1^k(\theta_j) = d_1(\theta_j) = p_1(\theta_j, d)$. Next, suppose that $\lim \sum_{n=1}^{m-1} p_n(\theta_j, d^k) = \sum_{n=1}^{m-1} p_n(\theta_j, d)$. Then,

$$\sum_{n=1}^{m} p_n(\theta_j, d^k) = b_m(d_m^k(\theta_j), \sum_{n=1}^{m-1} p_n(\theta_j, d^k)) + \sum_{n=1}^{m-1} p_n(\theta_j, d^k).$$

Since b_m is continuous in both arguments, $d_m^k(\theta_j) \to d_m(\theta_j)$, and $\sum_{n=1}^{m-1} p_n(\theta_j, d^k) \to \sum_{n=1}^{m-1} p_n(\theta_j, d)$, then

$$b_m(d_m^k(\theta_j), \sum_{n=1}^{m-1} p_n(\theta_j, d^k)) + \sum_{n=1}^{m-1} p_n(\theta_j, d^k) \to b_m(d_m(\theta_j), \sum_{n=1}^{m-1} p_n(\theta_j, d)) + \sum_{n=1}^{m-1} p_n(\theta_j, d) = \sum_{n=1}^{m} p_n(\theta_j, d).$$

Recall that

$$f(\theta_i, \theta_j, m, d) = b_m(\theta_i, \sum_{n=1}^{m-1} p_n(\theta_j, d)) - b_{k-m+1}(\theta_j, \sum_{n=1}^{k-m} p_n(\theta_i, d)).$$

Since $\sum_{n=1}^{m} p_n(\theta_j, d^k) \to \sum_{n=1}^{m} p_n(\theta_j, d)$ and b_n is continuous in the second argument, it follows that

$$f(\theta_i, \theta_j, m, d^k) \to f(\theta_i, \theta_j, m, d).$$

Thus, if $\lim f(\overline{\theta}, \theta_j, m, d^k) = f(\overline{\theta}, \theta_j, m, d) \leq 0$, then $T(d_m)(\theta_j) = \overline{\theta}$. In addition, for any $\epsilon > 0$, there exists a k^* such that for all $k > k^*$,

$$f(\overline{\theta} - \epsilon, \theta_j, m, d^k) < 0 \implies T(d_m^k)(\theta_j) \geq \overline{\theta} - \epsilon.$$

Since $\epsilon > 0$ is arbitrary, this implies that $T(d_m^k)(\theta_j) \to \overline{\theta}$.

If $f(\overline{\theta}, \theta_j, m, d) > 0$, then $T(d_m)(\theta_j) < \overline{\theta}$, and

$$\lim T(d_m^k)(\theta_j) = \lim \inf\{\theta | f(\theta, \theta_j, m, d^k) > 0\} = \inf\{\theta | f(\theta, \theta_j, m, d) > 0\} = T(d_m)(\theta_j)$$

because $f(\theta, \theta_j^h, m, d^k) \to f(\theta, \theta_j^h, m, d)$ and f is strictly increasing in the second argument. Since m and θ_j are arbitrary, it follows that T is a continuous mapping because

$$d^k \to d \implies T(d^k) \to T(d).$$

In order to show that $\mathcal{D}$ is compact, I show that it is complete and totally bounded. The set $\mathcal{D}$ is complete because every Cauchy sequence $\{d^n\}_{n=1}^{\infty}$ converges to an element $d \in \mathcal{D}$ when we use the L^1 norm as our metric.

In addition, $\mathcal{D}$ is totally bounded, as $d \in \mathcal{D} \subset \{d | d : [0, \overline{\theta}] \to [0, \overline{\theta}]^k\}$. More formally, $\mathcal{D}$ is totally bounded under the L^1 norm because any weakly increasing and bounded function can be approximated by a series of simple functions. Thus, for any $\epsilon > 0$, we can construct a finite set of simple functions $\{d_1, \dots, d_n\}$, where $d_i \in \mathcal{D}$ and for any $d \in \mathcal{D}$, there is an i such that $|d - d_i| < \epsilon$ according to the L^1 norm. Thus, $\mathcal{D}$ is compact (see Theorem 3.28 in Aliprantis and Border (2006)).

Thus, we have shown that $T : \mathcal{D} \to \mathcal{D}$ is a continuous mapping from a compact space $\mathcal{D}$ into itself. Schauder's fixed point theorem then states that the mapping T has a fixed point $d^* \in \mathcal{D}$. □

Proof of Lemma 4.

Proof. Individual rationality implies that if $q_i(\gamma_i, \gamma_j) = 0$, then $x_i(\gamma_i, \gamma_j) = p_{i,0}(\gamma_j) \leq 0$. When $q_j(\gamma_i, \gamma_j) = 2$, individual rationality implies that

$$u(2, -x_j(\gamma_i, \gamma_j), \gamma_j) \geq u(0, 0, \gamma_j).$$

Thus, $x_j(\gamma_i, \gamma_j) \leq \theta_j + b_2(\gamma_j, \theta_j)$ because

$$u(0, 0, \gamma_j) = u(1, -\theta_j, \gamma_j) = u(2, -\theta_j - b_2(\gamma_j, \theta_j), \gamma_j) \leq u(2, -x_j(\gamma_i, \gamma_j), \gamma_j).$$

If $\gamma_i = (0, t_i)$ and $\gamma_j = (\theta_j, t_j)$ where $\theta_j > 0$ then $q_j(\gamma_i, \gamma_j) = 2$. Thus, the above inequality implies

$$x_j(\gamma_i, \gamma_j) = p_{j,0}(\gamma_i) + p_{j,1}(\gamma_i) + p_{j,2}(\gamma_i) \leq \theta_j + b_2(\gamma_j, \theta_j).$$

The above inequality holds for all $\theta_j > 0$. Taking the limit of this expression as $\theta_j \to^+ 0$, gives that

$$p_{j,0}(\gamma_i) + p_{j,1}(\gamma_i) + p_{j,2}(\gamma_i) \leq \lim_{\theta_j \to^+ 0} \theta_j + b_2(\gamma_j, \theta_j) = 0.$$

Thus, if $\gamma_i = (0, t_i)$ and $\gamma_j = (\theta_j, t_j)$ where $\theta_j > 0$, then weak budget balance implies

$$x_i(\gamma_i, \gamma_j) + x_j(\gamma_i, \gamma_j) = p_{i,0}(\gamma_j) + (p_{j,0}(\gamma_i) + p_{j,1}(\gamma_i) + p_{j,2}(\gamma_i)) \geq 0.$$

However we have already shown that $(p_{j,0}(\gamma_i) + p_{j,1}(\gamma_i) + p_{j,2}(\gamma_i)) \leq 0$ and $p_{i,0}(\gamma_j) \leq 0$. Thus,

$$p_{i,0}(\gamma_j) + (p_{j,0}(\gamma_i) + p_{j,1}(\gamma_i) + p_{j,2}(\gamma_i)) \geq 0 \implies p_{i,0}(\gamma_j) = 0 \ if \ \theta_j > 0.$$

This yields our result,

$$q_i(\gamma_i, \gamma_j) = 0 \implies x_i(\gamma_i, \gamma_j) = p_{i,0}(\gamma_j) = 0.$$

$\square$

Proof of Proposition 3.

I prove Proposition 3 by establishing Remarks 5-6 and proving Lemmas 8-9. In the proof I take t_i and t_j as fixed, and thus I will proceed with the an abuse of notation by dropping t_i and t_j from the description of bidder types. Thus, the functions $d_{i,1}^{t_i}$ and $d_{i,2}^{t_i}$ are written as $d_{i,1}$ and $d_{i,2}$ to condense notation. Similarly, I will refer to bidder $i's$ type as θ_i assuming that her steepness is fixed and is $t_i \in \{s, f\}$.

Remark 5. Pareto efficiency implies that

$$q_i(\theta_i, \theta_j) = 2 \iff s_2(\theta_i, x_i(\theta_i, \theta_j)) \geq \theta_j,$$

and

$$q_i(\theta_i, \theta_j) = 1 \iff b_2(\theta_i, x_i(\theta_i, \theta_j)) \leq s_1(\theta_j, x_j(\theta_i, \theta_j)).$$

If $d_{i,2}(\theta_j) > d_{i,1}(\theta_j)$, then incentive compatibility implies,

$$p_{i,1}(\theta_j) = d_{i,1}(\theta_j)$$

and

$$d_{i,1}(\theta_j) = \lim_{\theta_i \to^- d_{i,1}(\theta_j)} s_1(\theta_i, p_{i,1}(\theta_j)).$$

Lemma 8. *If* $\theta_j > 0$*, then*

$$\lim_{\theta_i \to^+ d_{i,2}(\theta_j)} s_2(\theta_i, x_i(\theta_i, \theta_j)) = \lim_{\theta_i \to^- d_{i,2}(\theta_j)} b_2(\theta_i, d_{i,1}(\theta_j)).$$

Proof. Fix θ_j. Let $\theta_1^* := d_{i,1}(\theta_j)$, $\theta_2^* := d_{i,2}(\theta_j)$, and $x_2^* := x_i(\theta_i, \theta_j)$ if $q_i(\theta_i, \theta_j) = 2$, and $x_1^* := x_i(\theta_i, \theta_j)$ if $q_i(\theta_i, \theta_j) = 1$.

If $\theta_2^* = \theta_1^*$, then bidder i has utility $u(2, -x_2^*, \theta_i)$ if $\theta_i > \theta_2^*$. If $\theta_2^* > \theta_i$, bidder i has utility $u(0, 0, \theta_i)$. Incentive compatibility implies that bidder i utility is continuous and increasing in her type θ_i. Thus,

$$u(2, -x_2^*, \theta_2^*) = \lim_{\theta_i \to^+ \theta_2^*} u(2, -x_2^*, \theta_i) = \lim_{\theta_i \to^- \theta_2^*} u(0, 0, \theta_i) = u(0, 0, \theta_2^*).$$

And, $u(2, -x_2^*, \theta_2^*) = u(0, 0, \theta_2^*)$ implies that $x_2^* = \theta_2^* + b_2(\theta_2^*, \theta_2^*)$ because

$$u(0,0,\theta_2^*) = u(1, -\theta_1^*, \theta_2^*) = u(2, -\left(\theta_1^* + b_2(\theta_2^*, \theta_1^*)\right), \theta_2^*).$$

Thus,

$$\lim_{\theta_i \to^+ d_{i,2}(\theta_j)} s_2(\theta_i, -x_i(\theta_i, \theta_j)) = s_2(\theta_i^*, -x_i^*) = b_2(\theta_2^*, \theta_1^*) = \lim_{\theta_i \to^- d_{i,2}(\theta_j)} b_2(\theta_i, d_{i,1}(\theta_j)).$$

Next, suppose that $\theta_2^* > \theta_1^*$. Recall, incentive compatibility implies that a bidder's utility is continuous in her type,

$$u(2, -x_2^*, \theta_2^*) = \lim_{\theta_i \to^+ d_{i,2}(\theta_j)} u(2, -x_i(\theta_i, \theta_j), \theta_i) = \lim_{\theta_i \to^- d_{i,2}(\theta_j)} u(1, -x_i(\theta_i, \theta_j), \theta_i) = u(1, -x_1^*, \theta_2^*).$$

Thus,

$$u(2, -x_2^*, \theta_2^*) = u(1, -x_1^*, \theta_2^*) \implies s_2(\theta_2^*, x_2^*) = x_2^* - x_1^* = b_2(\theta_2^*, x_1^*).$$

□

Remark 6. Remark 5 and Lemma 8 show

$$b_2(d_{i,2}(\theta_j), d_{i,1}(\theta_j)) = \lim_{\theta_i \to^- d_{i,2}(\theta_j)} b_2(\theta_i, d_{i,1}(\theta_j)) = \lim_{\theta_i \to^+ d_{i,2}(\theta_j)} s_2(\theta_i, x_i(\theta_i, \theta_j)) \geq \theta_j.$$

Lemma 9. *If $\theta_j > 0$, then*

$$d_{i,2}(\theta_j) > d_{i,1}(\theta_j).$$

Proof. My proof is by contradiction. Suppose that $d_{i,2}(\theta_j') = d_{i,1}(\theta_j')$ for some $\theta_j' > 0$. Let $\tilde{\theta}_i := d_{i,2}(\theta_j') = d_{i,1}(\theta_j')$. Then,

$$q_i(\theta_i, \theta_j') = \begin{cases} 2 & if\ \theta_i > \tilde{\theta}_i \\ 0 & if\ \theta_i < \tilde{\theta}_i \end{cases}.$$

Thus, Remark 6 implies

$$b_2(\tilde{\theta}_i, \tilde{\theta}_i) \geq \theta_j' \ \forall \theta_j' \ s.t.\ d_{i,1}(\theta_j') = d_{i,2}(\theta_j') = \tilde{\theta}_i. \tag{6}$$

Let $\theta_j^* := \inf\{\theta_j : d_{i,1}(\theta_j) = d_{i,2}(\theta_j) = \tilde{\theta}_i\}$. Then, $d_{i,1}(\theta_j) < \tilde{\theta}_i$ for all $\theta_j < \theta_j^*$, because $d_{i,1}$

and $d_{i,2}$ are weakly increasing. For a given $\epsilon > 0$, then $\theta_i \in (d_{i,1}(\theta_j^* - \epsilon), \tilde{\theta}_i)$ implies

$$q_i(\theta_i, \theta_j) = \begin{cases} \geq 1 & if\ \theta_j < \theta_j^* - \epsilon \\ 0 & if\ \theta_j > \theta_j^* \end{cases}.$$

Or equivalently if $\theta_i \in (d_{i,1}(\theta_j^* - \epsilon), \tilde{\theta}_i)$,

$$q_j(\theta_i, \theta_j) = \begin{cases} \leq 1 & if\ \theta_j < \theta_j^* - \epsilon \\ 2 & if\ \theta_j > \theta_j^* \end{cases}.$$

Thus, $d_{j,2}(\theta_i) \in [\theta_j^* - \epsilon, \theta_j^*]$ if $\theta_i \in (d_{i,1}(\theta_j^* - \epsilon), \tilde{\theta}_i)$, and Remark 6 implies

$$b_2(d_{j,2}(\theta_i), d_{j,1}(\theta_i)) \geq \theta_i.$$

Recall that Equation 6 implies that

$$b_2(\tilde{\theta}_i, \tilde{\theta}_i) \geq \theta_j' \geq \theta_j^*\ \forall \theta_j'\ s.t.\ d_{i,1}(\theta_j') = d_{i,2}(\theta_j'),$$

where the final inequality follows from the definition of θ_j^*. Combining the above two expressions gives

$$b_2(\tilde{\theta}_i, \tilde{\theta}_i) \geq \theta_j^* \geq d_{j,2}(\theta_i) \geq b_2(d_{j,2}(\theta_i), d_{j,1}(\theta_i)) \geq \theta_i\ \forall \theta_i \in (d_{i,1}(\theta_j^* - \epsilon), \tilde{\theta}_i),$$

where the second inequality holds because $d_{j,2}(\theta_i) \in [\theta_j^* - \epsilon, \theta_j^*]$ if $\theta_i \in (d_{i,1}(\theta_j^* - \epsilon), \tilde{\theta}_i)$, and the third inequality holds because of declining demand. Thus,

$$b_2(\tilde{\theta}_i, \tilde{\theta}_i) \geq \theta_i\ \forall \theta_i \in (d_{i,1}(\theta_j^* - \epsilon), \tilde{\theta}_i) \implies b_2(\tilde{\theta}_i, \tilde{\theta}_i) \geq \tilde{\theta}_i.$$

Yet $b_2(\tilde{\theta}_i, \tilde{\theta}_i) \geq \tilde{\theta}_i$, contradicting the law of demand assumption. Hence, $d_{i,2}(\theta_j) > d_{i,1}(\theta_j)$. □

Now that I have established Lemmas 8-9 and Remarks 5-6, I complete the proof of Proposition 3.

Proof. First, I show that $d_{i,1}$ is continuous in θ_j. The proof is by contradiction. Incentive compatibility implies that $d_{i,1}(\theta_j)$ is weakly increasing. Thus, if $d_{i,1}(\theta_j)$ is discontinuous, then there exists a $\theta_j^* > 0$ such that

$$\lim_{\theta_j \to^- \theta_j^*} d_{i,1}(\theta_j) < \lim_{\theta_j \to^+ \theta_j^*} d_{i,1}(\theta_j).$$

Let $\theta_i^\ell := \lim_{\theta_j \to^- \theta_j^*} d_{i,1}(\theta_j)$ and $\theta_i^h := \lim_{\theta_j \to^+ \theta_j^*} d_{i,1}(\theta_j)$. Thus, $\theta_i \in (\theta_i^\ell, \theta_i^h)$ implies that

$$q_j(\theta_i, \theta_j) = 2 - q_i(\theta_i, \theta_j) = \begin{cases} \leq 1 & if\ \theta_j < \theta_j^* \\ 2 & if\ \theta_j > \theta_j^* \end{cases}.$$

Therefore, $d_{j,2}(\theta_i) = \theta_j^*\ \forall \theta_i \in (\theta_i^\ell, \theta_i^h)$, and Remark 6 shows

$$b_2(\theta_j^*, d_{j,1}(\theta_i)) \geq \theta_i\ \forall \theta_i \in (\theta_i^\ell, \theta_i^h) \implies \lim_{\theta_i \to^- \theta_i^h} b_2(\theta_j^*, d_{j,1}(\theta_i)) \geq \theta_i^h. \tag{7}$$

Similarly, Lemmas 8 and Remark 5 show that,

$$\lim_{\theta_i \to^+ d_{i,1}(\theta_j)} b_2(\theta_j, d_{j,1}(\theta_i)) \leq \lim_{\theta_i \to^+ d_{i,1}(\theta_j)} s_1(\theta_i, d_{i,1}(\theta_j)) = d_{i,1}(\theta_j) \leq \lim_{\theta_j \to^- \theta_j^*} d_{i,1}(\theta_j) = \theta_i^\ell\ \forall \theta_j < \theta_j^*, \tag{8}$$

where the final inequality follows because $d_{i,1}(\theta_j)$ is weakly increasing. Thus, positive wealth effects imply that

$$b_2(\theta_j, d_{j,1}(\theta_i^\ell)) \leq \lim_{\theta_i \to^+ d_{i,1}(\theta_j)} b_2(\theta_j, d_{j,1}(\theta_i)) \leq \theta_i^\ell\ \forall \theta_j < \theta_j^*, \tag{9}$$

where the final inequality from Equation 8. Combining Equations 8 and 7 gives,

$$\lim_{\theta_i \to^- \theta_i^h} b_2(\theta_j^*, d_{j,1}(\theta_i)) \geq \theta_i^h > \theta_i^\ell \geq b_2(\theta_j^*, d_{j,1}(\theta_i^\ell)).$$

This yields a contradiction as $\lim_{\theta_i \to^- \theta_i^h} d_{j,1}(\theta_i) \geq d_{j,1}(\theta_i^\ell)$ and positive wealth effects imply

$$b_2(\theta_j^*, d_{j,1}(\theta_i^\ell)) \geq \lim_{\theta_i \to^- \theta_i^h} b_2(\theta_j^*, d_{j,1}(\theta_i))$$

Thus, $d_{i,1}$ is continuous.

Next, I prove that $d_{i,1}(\theta_j)$ is strictly increasing in θ_j by contradiction. Incentive compatibility requires that $d_{i,1}(\theta_j)$ is weakly increasing. If $d_{i,1}(\theta_j)$ is not strictly increasing, there exists an interval $(\theta_j^\ell, \theta_j^h)$ such that $d_{i,1}(\theta_j') = d_{i,1}(\theta_j'')\ \forall \theta_j', \theta_j'' \in (\theta_j^\ell, \theta_j^h)$. Let $\tilde{\theta}_i := d_{i,1}(\theta_j)\ \forall \theta_j \in (\theta_j^\ell, \theta_j^h)$, $\theta_j^\ell := \inf\{\theta_j : \tilde{\theta}_i = d_{i,1}(\theta_j)\}$, and $\theta_j^h := \sup\{\theta_j : \tilde{\theta}_i = d_{i,1}(\theta_j)\}$. If $\theta_j \in (\theta_j^\ell, \theta_j^h)$

$$\tilde{\theta}_i = d_{i,1}(\theta_j) = \lim_{\theta_i \to^+ d_{i,1}(\theta_j)} s_1(\theta_i, d_{i,1}(\theta_j)) \geq \lim_{\theta_i \to^+ d_{i,1}(\theta_j)} b_2(\theta_j, d_{j,1}(\theta_i)) = b_2(\theta_j, d_{j,1}(\tilde{\theta}_i)),$$

where the second equality and the inequality holds from Remark 5, and final equality holds because we showed that $d_{j,1}$ is continuous. Using the above expression we see that

$$\tilde{\theta}_i \geq b_2(\theta_j, d_{j,1}(\tilde{\theta}_i)) \; \forall \theta_j \in (\theta_j^\ell, \theta_j^h) \implies \tilde{\theta}_i \geq b_2(\theta_j^h, d_{j,1}(\tilde{\theta}_i)). \tag{10}$$

In addition, if $\theta_j > \theta_j^\ell$, then $d_{i,1}(\theta_j) \geq \tilde{\theta}$. Thus, if $\theta_j > \theta_j^\ell$ and $\theta_i < \tilde{\theta}_i$, then $q_i(\theta_i, \theta_j) = 0 \implies q_j(\theta_i, \theta_j) = 2$. Thus, if $\theta_i < \tilde{\theta}_i$, then $d_{j,2}(\theta_i) \leq \theta_j^\ell$ and Remark 6 implies

$$b_2(d_{j,2}(\theta_i), d_{j,1}(\theta_i)) \geq \theta_i \; \forall \theta_i < \tilde{\theta}_i \implies \lim_{\theta_i \to^+ \tilde{\theta}_i} b_2(d_{j,2}(\theta_i), d_{j,1}(\theta_i)) \geq \lim_{\theta_i \to^+ \tilde{\theta}_i} \theta_i.$$

Recall that $d_{j,1}(\theta_i)$ is continuous and $d_{j,2}(\theta_i) \leq \theta_j^\ell \; \forall \theta_i < \tilde{\theta}_i$. As such,

$$\lim_{\theta_i \to^+ \tilde{\theta}_i} b_2(d_{j,2}(\theta_i), d_{j,1}(\theta_i)) \geq \lim_{\theta_i \to^+ \tilde{\theta}_i} \theta_i \implies b_2(\theta_j^\ell, d_{j,1}(\tilde{\theta}_i)) \geq \tilde{\theta}_i.$$

I combine this with Equation 10 to show that

$$b_2(\theta_j^\ell, d_{j,1}(\tilde{\theta}_i)) \geq \tilde{\theta}_i \geq b_2(\theta_j^h, d_{j,1}(\tilde{\theta}_i)) \implies \theta_j^\ell \geq \theta_j^h.$$

However, this contradicts the fact that $\theta_j^h > \theta_j^\ell$. Hence, $d_{i,1}(\theta_j)$ is strictly increasing. □

Proof of Corollary 2.

Proof. Let $\gamma_i^* := (\theta_i^*, t_i^*)$ and $\gamma_j^* := (\theta_j^*, t_j^*)$. Suppose that $\theta_i^* = d_{i,1}(\gamma_j^*)$. Then, $d_{i,1}(\theta_j, t_j^*) > \theta_i^* \iff \theta_j > \theta_j^*$ and if $\gamma_j = (\theta_j, t_j^*)$,

$$q_j(\gamma_i^*, \gamma_j) = 2 - q_i(\gamma_i^*, \gamma_j) = \begin{cases} 2 & if \; \theta_j > \theta_j^* \\ \leq 1 & if \; \theta_j < \theta_j^* \end{cases}.$$

The above expression implies that $d_{j,2}^{t_j^*}(\gamma_i^*) = \theta_j^*$ when $\theta_i^* = d_{i,1}(\gamma_j^*)$. Thus, $\gamma_i^* := (\theta_i^*, t_i^*)$ and $\gamma_j^* := (\theta_j^*, t_j^*)$ implies both bidders' intercept reports are locally pivotal. Incentive compatibility then implies

$$\theta_j^* = d_{j,2}^{t_j^*}(\gamma_i^*) \implies p_{j,2}(\gamma_i^*) = b_2(\gamma_j^*, p_{j,,1}(\gamma_i^*)) \geq \theta_i^* = d_{i,1}(\gamma_j^*) = p_{i,1}(\gamma_j^*) \tag{11}$$

where the final inequality follows from Remark 6. In addition,

$$\theta_i^* = \lim_{\theta_i \to^+ \theta_i^*} s_1((\theta_i, t_i^*), p_{i,1}(\gamma_j^*)) \geq \lim_{\theta_i \to^+ \theta_i^*} b_2(\gamma_j^*, p_{j,1}(\theta_i, t_i^*)) = b_2(\gamma_j^*, p_{j,1}(\gamma_i^*)), \tag{12}$$

where the first inequality holds because $q_i(\gamma_i, \gamma_j^*) = 1$ if $\gamma_i = (\theta_i, t_i^*)$ where $\theta_i > \theta_i^*$. The final equality holds because $p_{j,1}$ is continuous in θ_i. Combining the above two 11 and 12 gives

$$b_2(\gamma_j^*, p_{j,1}(\gamma_i^*)) \geq \theta_i^* \geq b_2(\gamma_j^*, p_{j,1}(\gamma_i^*)) \implies \theta_i^* = b_2(\gamma_j^*, p_{j,1}(\gamma_i^*)).$$

This is our desired result because $p_{i,1}(\gamma_j^*) = \theta_i^*$ and $b_2(\gamma_j^*, p_{j,1}(\gamma_i^*)) = p_{j,2}(\gamma_i^*)$. □

For the remaining proofs recall that we assume $\theta_2 \geq \theta_3 \geq \theta_j \; \forall j \neq 1, 2, 3$.

Proof of Lemma 5.

Proof. I show that $d_{1,1}(\theta_{-1}) \geq \theta_3$. The proof is by contradiction. Suppose that there exists θ_2, θ_3 such that $\theta_3 > d_{1,1}(\theta_{-1})$. This implies that if $\theta_1 = d_{1,1}(\theta_{-1}) + \epsilon$, then $q_1(\theta_1, \theta_{-1}) \geq 1$ and if $\theta_1 = d_{1,1}(\theta_{-1}) - \epsilon$, then $q_1(\theta_1, \theta_{-1}) = 0$. Thus, as θ_1 approaches $d_{1,1}(\theta_{-1})$ from above, bidder 1 is willing to sell one of her units for at most θ_1 (if bidder 1 wins 2 units when $\theta_1 > d_{1,1}(\theta_{-1})$, then her willingness to sell an additional unit is lower). Thus, there is a Pareto improving trade where bidder 1 sells one unit to bidder 3 for a price in the interval (θ_1, θ_3). □

Proof of Lemma 6.

Proof. Lemma 5 shows that $d_{1,1}(\theta_{-1}) \geq \theta_3$. I show this holds with equality when $\theta_3 \geq b_2(\theta_2, 0)$. The proof is by contradiction. Suppose $d_{1,1}(\theta_{-1}) > \theta_3$. Let θ_1 be such that $\theta_1 \in (\theta_3, d_{1,1}(\theta_{-1}))$. Then $q_2(\theta_1, \theta_2, \theta_3, \theta_{-1,2,3}) = 2$. This holds because (1) $q_1(\theta_1, \theta_{-1}) = 0$ because $d_{1,1}(\theta_{-1}) > \theta_1$ and (2) $q_i(\theta_i, \theta_{-i}) = 0 \forall i \neq 1, 2$ because Lemma 5 shows $d_{i,1}(\theta_{-i}) \geq \min\{\theta_1, \theta_2\}$ and $\min\{\theta_1, \theta_2\} > \theta_i$. Thus,

$$\theta_2 \geq d_{2,2}(\theta_{-2}) \geq d_{2,1}(\theta_{-2}) \geq \theta_3.$$

Let $\tilde{\theta}_2 = d_{2,2}(\theta_{-2}) + \epsilon$, and $\hat{\theta}_2 = d_{2,2}(\theta_{-2}) - \epsilon$. Incentive compatibility and continuity of bidder 2's preferences imply that when $\epsilon > 0$ is sufficiently small,

$$s_2(\tilde{\theta}_2, x_2(\theta_1, \theta_3)) \approx b_2(\hat{\theta}_2, d_{2,1}(\theta_1, \theta_3)) \leq b_2(\theta_2, 0) < \theta_1.$$

Thus, $s_2(\tilde{\theta}_2, x_2(\theta_1, \theta_3) < \theta_1$. This implies that there is a Pareto improving trade when bidder 2 is type $\tilde{\theta}_2$. Bidder 2 sells one unit to bidder 1 for a price in the interval $(s_2\left(\tilde{\theta}_2, x_2(\theta_2, \theta_{-2})\right), \theta_1)$. □

Proof of Lemma 7.

Proof. The proof is by contradiction. First suppose $d_{1,2}(\theta_{-1}) > \theta_1^*$. Then

$$q_1(\theta_1, \theta_{-1}) = 1 \; if \; \theta_1 \in (\theta_1^*, d_{1,2}(\theta_{-1})),$$

because $\theta_1 > \theta_1^* > \theta_2 > \theta_3 = d_{1,1}(\theta_{-1})$ where the final equality holds because Lemma 6 shows that $\theta_3 = d_{1,1}(\theta_{-1})$ if $\theta_3 \in (b_2(\theta_2, 0), \theta_2)$. In addition, $q_2(\theta_1, \theta_2, \theta_3, \theta_{-1,2,3}) = 1$ because both units are sold and bidder $i \neq 1, 2$ wins zero units when her type is not among the two highest types reported. Thus, $\theta_2 \geq d_{2,1}(\theta_{-2})$.

Let $\tilde{\theta}_2 = \min\{\theta_2, d_{2,1}(\theta_{-2}) + \epsilon\}$ where $\epsilon > 0$ is small. Note that $\tilde{\theta}_2 > \theta_3$ because $d_{2,1}(\theta_{-1}) \geq \theta_3$ and $\theta_2 > \theta_3$. Thus, $\theta_2 \geq \tilde{\theta}_2 > \theta_3 \implies \theta_3 \in (b_2(\tilde{\theta}_2, 0), \tilde{\theta}_2)$, which follows because we assume $\theta_3 \in (b_2(\theta_2, 0), \theta_2)$. Thus, Lemma 6 shows $d_{1,1}(\tilde{\theta}_2, \theta_{-1,2}) = \theta_3$, and bidder 1 is willing to pay $b_2(\theta_1, \theta_3)$ for an additional unit. Note that

$$b_2(\theta_1, \theta_3) > b_2(\theta_1^*, \theta_3) = \theta_2 \geq \tilde{\theta}_2.$$

Where the first inequality holds because $\theta_1 > \theta_1^*$ and the equality holds from the definition of θ_1^*.

In addition, since $\tilde{\theta}_2 - 2\epsilon < d_{2,1}(\theta_{-2}) \leq \tilde{\theta}_2$, where $\epsilon > 0$ is arbitrarily small, incentive compatibility implies that bidder $s_1(\tilde{\theta}_2, d_{2,1}(\theta_{-2})) \approx d_{2,1}(\theta_{-2})$ because $\tilde{\theta}_2 \approx d_{2,1}(\theta_{-2})$. Yet $q_2(\tilde{\theta}_2, \theta_{-2}) = 1$ by construction. Thus, there is a Pareto improving trade where bidder 1 buys the unit from bidder 2 for a price in the interval $(\tilde{\theta}_2, b_2(\theta_1, \theta_3))$. Thus, if $d_{1,2}(\theta_{-1}) > \theta_1^*$, there exists a Pareto improving trade and the mechanism does not satisfy Properties (a)-(e).

Next, suppose that $\theta_1^* > d_{1,2}(\theta_{-1})$. Then, $d_{1,2}(\theta_{-1}) \geq d_{1,1}(\theta_{-1}) = \theta_3$, where the final inequality holds by Lemma 6. Let $\tilde{\theta}_1 = d_{1,2}(\theta_{-1}) + \epsilon$, where $\epsilon > 0$ is sufficiently small. Thus, $q_1(\tilde{\theta}_1, \theta_{-1}) = 2$. Incentive compatibility implies that bidder 1 is approximately indifferent between buying her second unit when her type is near $d_{1,2}(\theta_{-1})$. Thus, bidder 1 is willing to sell her second unit for approximately $b_2(\tilde{\theta}_1, \theta_3)$ (this follows from Remark 6). In addition, bidder 2 is willing to pay θ_2 for her first unit and $\theta_2 > b_2(\tilde{\theta}_1, \theta_3)$ because I assumed that $\theta_1^* > \tilde{\theta}_1 \approx d_{1,2}(\theta_{-1})$. Thus, there is a Pareto improving trade where bidder 1 sells her second unit to bidder 2 for a price in the interval $\Big(b_2(\tilde{\theta}_1, \theta_3), \theta_2\Big)$. $\square$

Proof of Theorem 3.

Proof. Let θ_2^h and θ_2^ℓ be such that $\theta_2^h = \theta_2^\ell + \epsilon$ where $\epsilon > 0$ is sufficiently small. Then,

$$b_2(\theta_2^h, 0) < \theta_2^\ell < \theta_2^h.$$

Let $\theta_3^h, \theta_3^\ell$ be such that $\theta_3^\ell < \theta_3^h$ where

$$\theta_3^\ell = b_2(\theta_2^h, 0) + \epsilon$$

and

$$\theta_3^h = \theta_2^\ell - \epsilon,$$

where $\epsilon > 0$ is sufficiently small.

Finally, let θ_1^ℓ, θ_1^h be such that $\theta_1^\ell < \theta_1^h$ and

$$b_2(\theta_1^h, \theta_3^h) < \theta_2^\ell < \theta_2^\ell + \epsilon = \theta_2^h < b_2(\theta_1^\ell, \theta_3^\ell).$$

We know such $\theta_1^h, \theta_1^\ell$ exist when $\epsilon > 0$ is sufficiently small, because $\theta_3^h > \theta_3^\ell$ implies that $b_2(\theta_1, \theta_3^\ell) > b_2(\theta_1, \theta_3^h)\ \forall \theta_1 > 0$, and b_2 is continuous in the first argument. Thus $\theta_1^h > \theta_1^\ell > \theta_2^h > \theta_2^\ell > \theta_3^h > \theta_3^\ell$. In addition suppose $\theta_i < \theta_3^\ell \forall i \neq 1, 2, 3$.

Then,

$$q_2(\theta_1^h, \theta_2^\ell, \theta_3^h, \theta_{-1,2,3}) \geq 1.$$

This is because $q_1(\theta_1^h, \theta_2^\ell, \theta_3^h, \theta_{-1,2,3}) \leq 1$, as $\theta_1^h < \theta_1^* = d_{1,2}(\theta_{-1})$ where

$$b_2(\theta_1^*, \theta_3^h) = \theta_2^\ell > b_2(\theta_1^h, \theta_3^h).$$

In addition $q_i(\theta_1^h, \theta_2^\ell, \theta_3^h, \theta_{-1,2,3}) = 0\ \forall i \neq 1, 2$ by Lemma 5 because $\theta_1^h, \theta_2^\ell > \theta_i$

Yet, our assumptions also imply $q_1(\theta_1^\ell, \theta_2^h, \theta_3^\ell, \theta_{-1,2,3}) = 2$, because $\theta_1^\ell > \theta^*$ where

$$b_2(\theta_1^*, \theta_3^\ell) = \theta_2^h < b_2(\theta_1^\ell, \theta_3^\ell).$$

Thus, Lemma 7 implies $q_1(\theta_1^\ell, \theta_2^h, \theta_3^\ell, \theta_{-1,2,3}) = 2 \implies q_2(\theta_1^\ell, \theta_2^h, \theta_3^\ell, \theta_{-1,2,3}) = 0$. Yet this violates the monotonicity, because monotonicity implies

$$q_2(\theta_1^\ell, \theta_2^h, \theta_3^\ell, \theta_{-1,2,3}) \geq q_2(\theta_1^h, \theta_2^\ell, \theta_3^h, \theta_{-1,2,3}) \geq 1.$$

□