

Survival and long-run dynamics with heterogeneous beliefs under recursive preferences*

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Abstract

I study the long-run behavior of an economy with two types of agents who differ in their beliefs and are endowed with homothetic recursive preferences of the Duffie–Epstein–Zin type. Contrary to models with separable preferences in which the wealth of agents with incorrect beliefs vanishes in the long run, recursive preference specifications lead to long-run outcomes where both agents survive, or more incorrect agents dominate. In this respect, the market selection hypothesis is not robust to deviations from separability. I derive analytical conditions for the existence of nondegenerate long-run equilibria in which agents with differently accurate beliefs coexist in the long run, and show that these equilibria exist for broad ranges of plausible parameterizations when risk aversion is larger than the inverse of the intertemporal elasticity of substitution. The results highlight a crucial interaction between risk sharing, speculative behavior and the consumption-saving choice of agents with heterogeneous beliefs.

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1 Introduction

The market selection hypothesis first articulated by [Alchian \(1950\)](#) and [Friedman \(1953\)](#) is one of the supporting arguments for the plausibility of the rational expectations theory. The hypothesis states that agents who systematically evaluate the distributions of future quantities incorrectly (and are therefore called ‘irrational’) lose wealth on average, and will ultimately be driven out of the market. Thus, in a long-run equilibrium, the dynamics of the economy are only determined by the behavior of the rational agents whose beliefs about the future are in line with the true probability distributions.

Survival of agents with incorrect beliefs has been studied extensively in complete market models populated by agents endowed with separable preferences, and the literature has indeed found widespread support for the market selection hypothesis. If agents have identical preferences and relative risk aversion is bounded, then only agents with the most accurate beliefs survive in the long run in the sense that their wealth share does not converge to zero, regardless of the specification of the aggregate endowment process.

While these results look appealing, separable preferences are inconsistent with many features of the asset pricing data, a fact reflected in different asset pricing ‘puzzles’. Since the market selection hypothesis focuses on wealth dynamics of agents who trade in financial markets, the goal of this paper is to study the survival problem for preferences which provide a better account of the patterns observed in asset return data. In fact, the analysis in this paper is interesting not only from the perspective of long-run outcomes but also because it sheds light on the market interaction between agents with heterogeneous beliefs and the resulting equilibrium asset price dynamics.

I find that the above survival results are overturned when the assumption of separability of preferences is relaxed. I analyze a class of homothetic recursive preferences axiomatized by [Kreps and Porteus \(1978\)](#), and developed by [Epstein and Zin \(1989\)](#) and [Weil \(1990\)](#) in discrete time, and by [Duffie and Epstein \(1992b\)](#) in continuous time. These preferences allow one to disentangle the risk aversion with respect to intratemporal gambles from the intertemporal elasticity of substitution (IES), and include the separable, constant relative risk aversion (CRRA) utility as a special case. Thanks to the additional degree of flexibility, this class of preferences became the workhorse model used in the asset pricing literature.

In particular, I study an endowment economy populated by two classes of competitive agents (called, for simplicity, two agents) who differ in their beliefs about the growth rate of the stochastic aggregate endowment that follows a geometric Brownian motion. Agents are endowed with identical recursive preferences and trade in complete markets which, in the Brownian information setup, corresponds to dynamic trading in an infinitesimal risk-free bond and a risky claim to the aggregate endowment.

The decoupling of risk aversion and IES effectively separates essential forces that shape long-run wealth dynamics. Agents in an economy can accumulate wealth, and thus avoid extinction, by holding portfolios with high expected *logarithmic* returns and by choosing a high saving rate. The portfolio mechanism, which is driven by the interaction of risk aversion and belief distortions,

is in line with the well-established literature on the growth-optimal portfolio, initiated by Kelly (1956) and Breiman (1960, 1961). The saving mechanism, featured in Blume and Easley (1992), is generated through the interaction between IES and perceived expected returns on the agent’s portfolio. While the portfolio and saving mechanisms have been known in the literature, I establish how they emerge from equilibrium price dynamics and determine novel long-run outcomes in the recursive preference setup.

I show that in the class of recursive preferences, there exist broad ranges of empirically plausible values for preference parameters under which agents with less accurate beliefs survive or even dominate the economy. Perhaps most interestingly, agents with arbitrarily large belief distortions can coexist with rational agents in the long-run equilibrium under preference parameterizations typically estimated in asset pricing models, with risk aversion sufficiently higher than the inverse of IES; see, e.g., the long-run risk literature initiated by Bansal and Yaron (2004). Belief heterogeneity should thus be viewed as a natural long-run outcome.

From the perspective of individual decisions, this paper uncovers a crucial interaction between the use of risky assets for risk sharing and saving on the one hand and, on the other hand, as a speculative tool to trade on belief differences. Specifically, I identify four channels through which individual choices vis-à-vis equilibrium prices influence long-run wealth accumulation:

1. *Speculative bias channel*: Agents base their risk-return tradeoff on the *perceived* (subjective) expected returns. This biases their portfolio choice away from the optimal objective risk-return tradeoff toward assets that pay off in states which are less likely to materialize.
2. *Speculative volatility channel*: Speculative behavior arising from differences in beliefs makes agents choose portfolios with more volatile returns which lowers expected logarithmic returns due to Jensen’s inequality.
3. *Risk premium channel*: Optimistic agents hold larger positions in risky assets and thus benefit from high risk premia.
4. *Saving channel*: Agents with a high perceived expected return on their portfolio choose a high (low) saving rate when IES is high (low).

Figure 1 provides an illustration of the results derived in the paper. Consider an economy where agent 2 has correct beliefs while agent 1 is optimistic (left panel) or pessimistic (right panel). Each panel shows the long-run survival outcome in the economy as the preference parameters (which are identical for both agents) vary. Risk aversion is displayed on the horizontal axis, while the inverse of IES on the vertical axis. The dotted upward sloping line represents parameter combinations that correspond to CRRA preferences.

For preference parameters in the neighborhood of CRRA preferences, the *speculative bias channel* dominates. In line with the findings of an extensive literature that studied the case of separable preferences, the rational agent 2 always dominates the economy in the long run. Under separable preferences, agents with incorrect beliefs buy assets that provide high payoffs in states which they

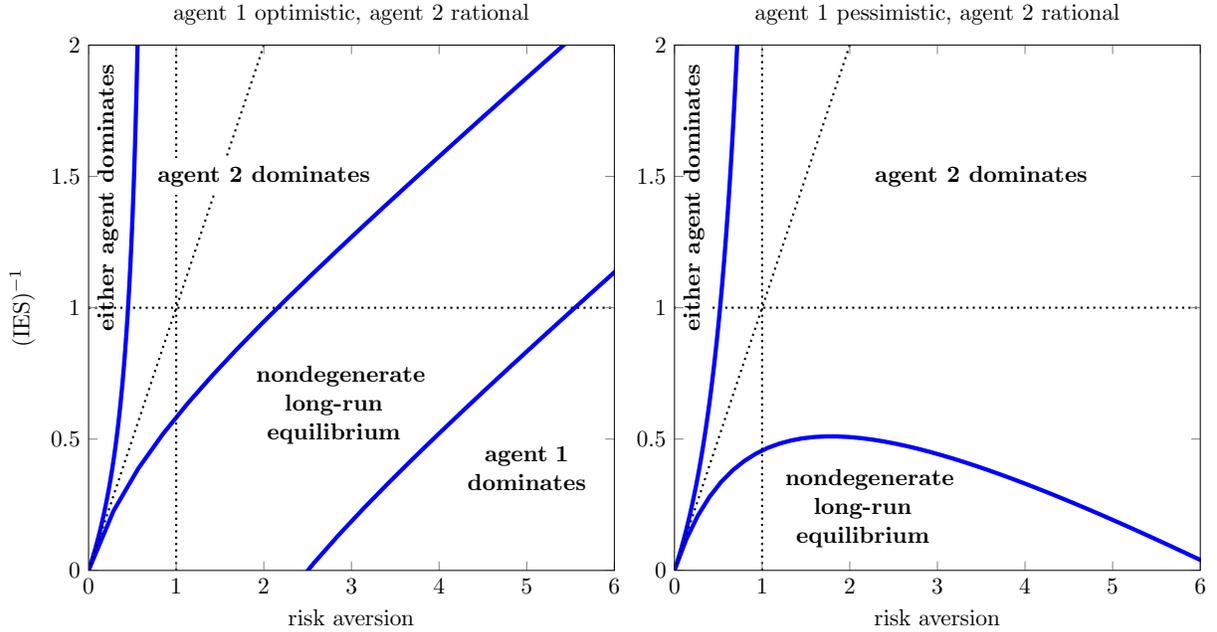


Figure 1: Survival regions for two pairs of agents' beliefs.

believe are likely. This implies that over time, as belief distortions accumulate, these agents preserve their wealth in states which have a vanishing probability under the true probability measure. While they subjectively believe they will accumulate wealth in equilibrium, the states that are actually realized lead to their extinction. With nonseparable preferences, this simple logic ceases to hold because agents' valuation of future states also depends on the realized path of consumption choices between today and the state in the distant future.

An increase in risk aversion (holding IES fixed, corresponding to a move to the right in the figure) discourages speculative behavior, as agents dislike volatile returns. At the same time, equilibrium risk premia in the economy increase. This increases relatively more the return on the portfolio of the more optimistic agent. When risk aversion is sufficiently high, the *risk premium channel* dominates. More precisely, holding other parameters fixed, there is always a level of risk aversion above which the more optimistic agent dominates. In the left panel of Figure 1 this is the optimistic agent 1, while in the right panel the relatively more optimistic agent is the rational agent 2.

As the wealth share of the optimistic agent in the economy increases, the price of the risky asset increases and the risk premium declines. The general equilibrium price dynamics thus acts as a balancing force, slowing down the rate of wealth accumulation of the optimistic agent. For a nontrivial set of moderately high values of the risk aversion parameter, this mechanism preserves a nondegenerate wealth distribution in the long run.

On the other hand, when risk aversion decreases (a move to the left in the figure), the *speculative volatility channel* determines the long-run outcomes. Low risk aversion incentivizes risk taking, and agents choose 'speculative' portfolios with volatile returns that reflect the differences in their assessment of probabilities of future states. While the optimal [Markowitz \(1952\)](#)–[Merton \(1971\)](#)

portfolio choice is determined by the tradeoff between the expected level return and the underlying volatility, survival chances depend on the expected *logarithmic* growth rate of wealth, and thus on the expected *logarithmic* return on the agent’s portfolio. Due to Jensen’s inequality, volatile portfolios are detrimental to survival.¹

Equilibrating prices again play a crucial role in this mechanism. Consider the situation when the wealth share of one of the agents becomes negligible. Equilibrium prices have to adjust in a way that the large agent holds the market portfolio, which prevents her from taking risky asset positions with volatile returns. The negligible agent then chooses an investment portfolio that overweighs positions in assets that are, according to her own beliefs, cheap and earn high expected *level* returns relative to their risk. When risk aversion is low (along the left edge of each of the panels in Figure 1), this ‘speculative’ position in the negligible agent’s portfolio will be large, the portfolio return very volatile, and the expected *logarithmic* return on such a portfolio very low. Contrary to the risk premium channel, the interaction of equilibrium price dynamics with the speculative volatility channel thus acts as a diverging force — it tends to decrease the wealth growth rate of agents whose wealth share is already small, driving them to extinction, regardless of their belief distortion. More precisely, holding other parameters fixed, there is always a level of risk aversion sufficiently low such that, depending on the sequence of shock realizations, one of the agents will vanish in the long run but it can be either of the two agents with a strictly positive probability.

While the risk aversion parameter determines the portfolio choice decision, the IES parameter is crucial for the consumption-saving decision and impacts the *saving channel* of wealth accumulation. When IES is higher than one, then the saving rate is an increasing function of the subjective expected level return on the agent’s portfolio.

Consider again the situation when one agent has a negligible wealth share. Equilibrium asset prices have to be such that the large agent consumes the aggregate endowment. As long as the negligible agent chooses a portfolio with a higher *subjective* expected level return than her large counterpart, a high IES is conducive to her survival. She will choose a high saving rate, which can compensate for the potentially inferior choice of her portfolio, and in this way ‘outsave’ her extinction. Since this consumption-saving mechanism under a high IES operates for the negligible agent, regardless of her identity, it acts as a converging force that preserves a nondegenerate distribution of wealth, with both agents surviving in the long run.

Figure 1 captures the survival outcomes in the plane of risk aversion / IES parameters for a particular choice of the magnitude of the belief distortions and aggregate uncertainty in the economy. I provide a complete analytical characterization of long-run outcomes for the whole parameter space and isolate the four channels described above. Several conclusions stand out.

First, survival of agents with distorted beliefs is a generic outcome. Agents with incorrect beliefs can survive or dominate in an economy populated by rational agents for a wide range of preference

¹To illuminate the difference between level and logarithmic returns, consider a simple case of a risk-neutral agent who starts with a given wealth of k dollars and engages in a sequence of \$1 coin flip bets that win with probability 0.5. While the net *level* return is zero and the agent views these bets as fair, she ultimately ends up with zero wealth with probability one. The expected *logarithmic* return on a sequence of these bets is negative and converges to minus infinity.

parameters. Moreover, these results do not hinge on belief distortions being small; in fact, as we will see, they hold for agents with arbitrarily large belief distortions.

Second, equilibria in which agents with heterogeneous beliefs coexist in the long run occur for parameter combinations that are empirically relevant. In particular, risk aversion has to be sufficiently high to prevent the speculative volatility channel to dominate, and at the same time IES has to be sufficiently high to incentivize agents with a small wealth share to choose a high saving rate vis-à-vis the high subjective expected return on her portfolio, thus out-saving her extinction.

Finally, the channels for the survival mechanism highlight the critical role of the endogenously determined equilibrium price dynamics. In order for the two agents to coexist, equilibrium prices always have to be conducive to the survival of the negligible agent, and thus have to adjust when the roles of the two agents switch.

Relative to the existing literature, I analyze in more detail the role of the portfolio choice and consumption-saving decision mechanism for survival. The separable utility case that the existing literature primarily focused on can be solved by computing optimal allocations using a planner's problem without the need for a decentralization and computation of equilibrium prices. The interaction of equilibrium prices with individual decisions provides additional insights to the mechanism that determines long-run allocations.

The paper also contributes to the literature along the methodological and technical dimension. First, I provide a rigorous proof of the existence and properties of the continuous-time optimal allocation problem with heterogeneous agents endowed with recursive preferences, formulated as a dynamic problem with stochastic Pareto weights. Second, I prove that this class of problems can be studied by focusing on the boundary behavior of the economy when one of the agents becomes negligible, which is often significantly simpler — in my case, I obtain analytical answers for the survival problem despite the fact that the economy does not have an analytical solution. Finally, I explicitly link the optimal allocations to decentralized portfolios and equilibrium price dynamics. Since the methodological approach used in this paper differs significantly from the existing approaches in much of the survival literature, I provide a more detailed discussion and comparison to the literature later in the paper, in Section 6.

The rest of the paper is organized as follows. Section 2 outlines the economic environment and derives the planner's problem that is central to the analysis. The proof of the existence and differentiability of the solution is deferred to Appendix A. Sections 3 and 4 present the survival results. I provide in analytical form tight sufficient conditions for survival and extinction and discuss the economic interpretation of the results. This analytical part is followed by numerical analysis of consumption and price dynamics for economies with nondegenerate long-run equilibria in Section 5. Section 6 revisits the methodological contribution and compares the approach in this paper to the existing literature. Section 7 summarizes the findings and outlines extensions of the developed framework. Appendix B contains further proofs omitted from the main text. Additional material that provides more details and extends the analysis is available in the online appendix.²

² https://files.nyu.edu/jb4457/public/files/research/survival_heterogeneous_beliefs_online_appendix.pdf

2 Optimal allocations under heterogeneous beliefs

I analyze the dynamics of equilibrium allocations in a continuous-time endowment economy populated by two types of infinitely-lived agents endowed with identical recursive preferences. I call an economy where both agents have strictly positive wealth shares a heterogeneous economy. A homogeneous economy is populated by a single agent only. The term ‘agent’ refers to an infinitesimal competitive representative of the particular type.

Agents differ in their subjective beliefs about the distribution of future quantities but are firm believers in their probability models and ‘agree to disagree’ about their beliefs as in [Morris \(1995\)](#). Since they do not interpret their belief differences as a result of information asymmetries, there is no strategic trading behavior.

Without introducing any specific market structure, I assume that markets are dynamically complete in the sense of [Harrison and Kreps \(1979\)](#). This allows me to sidestep the problem of directly calculating the equilibrium by considering a planner’s problem. The discussion of market survival then amounts to the analysis of the dynamics of Pareto weights associated with this planner’s problem. Optimal allocations and continuation values generate a valid stochastic discount factor and a replicating trading strategy for the decentralized equilibrium.

In this section, I specify agents’ preferences and belief distortions, and lay out the planner’s problem. I utilize the framework introduced by [Dumas, Uppal, and Wang \(2000\)](#), and exploit the observation that belief heterogeneity can be analyzed in their framework without increasing the degree of complexity of the problem. The method then leads to a Hamilton–Jacobi–Bellman equation for the planner’s value function.

2.1 Information structure and beliefs

The stochastic structure of the economy is given by a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ with an augmented filtration defined by a family of σ -algebras $\{\mathcal{F}_t\}$, $t \geq 0$ generated by a univariate Brownian motion W . Given the continuous-time nature of the problem, equalities are meant in the appropriate almost-sure sense. I also assume that all processes, in particular belief distortions and permissible trading strategies, satisfy regularity conditions like square integrability over finite horizons, so that stochastic integrals are well defined and pathological cases are avoided (see, e.g., [Huang and Pagès \(1992\)](#)). Under the parameter restrictions below, constructed equilibria satisfy these assumptions.

The scalar aggregate endowment process Y follows a geometric Brownian motion

$$\frac{dY_t}{Y_t} = \mu_y dt + \sigma_y dW_t, \quad Y_0 > 0 \tag{1}$$

with given parameters μ_y and σ_y .

Agents of type $n \in \{1, 2\}$ are endowed with identical preferences but differ in their subjective probability measures that they use to assign probabilities to future events. I model the belief distortion of agent n using an adapted process u^n such that the process

$$M_t^n \doteq \left(\frac{dQ^n}{dP} \right)_t = \exp \left(-\frac{1}{2} \int_0^t |u_s^n|^2 ds + \int_0^t u_s^n dW_s \right), \quad (2)$$

is a martingale under P . The martingale M^n is called the Radon–Nikodým derivative or the belief ratio and defines the subjective probability measure Q^n that characterizes the beliefs of agent n . The Radon–Nikodým derivative measures the disparity between the subjective and true probability measures.

In order for the belief heterogeneity not to vanish in the long run, the measures P and Q^n cannot be mutually absolutely continuous.³ However, given the construction of M^n , the restrictions of the measures P and Q^n , $n \in \{1, 2\}$ to \mathcal{F}_t for every $t \geq 0$ are equivalent.⁴ In other words, the agents agree with the data generating measure on zero-probability finite-horizon events. While a likelihood evaluation of past observed data reveals that the view of an agent with distorted beliefs becomes less and less likely to be correct as time passes, absolute continuity of the measure Q^n with respect to P over finite horizons implies that she cannot refute her view of the world as impossible in finite time. From now on, I assume that both agents have constant belief distortions u^n , a frequently considered case in the survival literature. Possible extensions are discussed in concluding remarks.

The belief distortion process u^n has a clear economic interpretation. The Girsanov theorem implies that agent n , whose deviation from rational beliefs is described by u^n , views the evolution of the Brownian motion W as distorted by a drift component u^n , i.e., $dW_t = u^n dt + dW_t^n$, where W^n is a Brownian motion under Q^n . Consequently, the aggregate endowment is perceived to contain an additional drift component $u^n \sigma_y$, and u^n can be interpreted as a degree of optimism or pessimism about the growth rate of Y . When $\sigma_y = 0$, the distinction between optimism and pessimism loses its meaning but the survival problem is still nondegenerate, as long as the agents can contract upon the realizations of the process W .

2.2 Recursive utility

Agents endowed with separable preferences reduce intertemporal compound lotteries (different payoff streams allocated over time) to atemporal simple lotteries that resolve uncertainty at a single point in time. In the Arrow–Debreu world with separable preferences, once trading of state-contingent securities for all future periods is completed at time 0, uncertainty about the realized path of the economy can be resolved immediately without any consequences for the ex-ante preference ranking of the outcomes by the agents.

Kreps and Porteus (1978) relaxed the separability assumption by axiomatizing discrete-time preferences where temporal resolution of uncertainty matters and preferences are not separable. While intratemporal lotteries in the Kreps–Porteus axiomatization still satisfy the von Neumann–

³Sandroni (2000) and Blume and Easley (2006) link absolute continuity of the subjective probability measures to merging of agents' beliefs.

⁴See, for example, Revuz and Yor (1999), Section VIII for details. The construction prevents arbitrage opportunities in finite-horizon strategies. The martingale representation theorem (e.g., Øksendal (2007), Theorem 4.3.4) implies that modeling belief distortions under Brownian information structures using martingales of the form (2) is essentially without loss of generality.

Morgenstern expected utility axioms, intertemporal lotteries cannot in general be reduced to atemporal ones. The work by [Epstein and Zin \(1989, 1991\)](#) extended the results of [Kreps and Porteus \(1978\)](#), and initiated the widespread use of recursive preferences in the asset pricing literature. [Duffie and Epstein \(1992a,b\)](#) formulated the continuous-time counterpart of the recursion.⁵

I utilize a characterization based on the more general variational utility approach studied by [Geoffard \(1996\)](#) in the deterministic case and [El Karoui, Peng, and Quenez \(1997\)](#) in a stochastic environment.⁶ They show that recursive preferences can be represented as a solution to the maximization problem

$$\lambda_t^n V_t^n(C^n) = \sup_{\nu^n} E_t^{Q^n} \left[\int_t^\infty \lambda_s^n F(C_s^n, \nu_s^n) ds \right] \quad (3)$$

subject to

$$\frac{d\lambda_t^n}{\lambda_t^n} = -\nu_t^n dt, \quad \lambda_0^n > 0, \quad t \geq 0. \quad (4)$$

where ν^n is called the discount rate process, and λ^n the discount factor process. The felicity function $F(C, \nu)$ encodes the contribution of the consumption stream C to present utility. This representation closely links recursive preferences to the literature on endogenous discounting, initiated by [Koopmans \(1960\)](#) and [Uzawa \(1968\)](#).

For the case of the Duffie–Epstein–Zin preferences, the felicity function is given by

$$F(C, \nu) = \beta \frac{C^\gamma}{\gamma} \left(\frac{\gamma - \rho \frac{\nu}{\beta}}{\gamma - \rho} \right)^{1 - \frac{\gamma}{\rho}}, \quad (5)$$

with parameters satisfying $\gamma, \rho < 1$, and $\beta > 0$. Preferences specified by this felicity function⁷ are homothetic and exhibit a constant relative risk aversion with respect to intratemporal wealth gambles $\alpha = 1 - \gamma$ and (under intratemporal certainty) a constant intertemporal elasticity of substitution $\eta = \frac{1}{1-\rho}$. Parameter β is the time preference coefficient. Assumption [A.1](#) below restricts parameters to assure sufficient discounting for the continuation values to be finite in both homogeneous and heterogeneous economies. In the case when $\gamma = \rho$, the utility reduces to the separable CRRA utility with the coefficient of relative risk aversion α .

Formula (3), together with an application of the Girsanov theorem, suggests that it is advantageous to combine the contribution of the discount factor process λ^n and the martingale M^n that

⁵[Duffie and Epstein \(1992b\)](#) provide sufficient conditions for the existence of the recursive utility process for the infinite-horizon case but these are too strict for the preference specification utilized in this paper. Similarly, the results from [Duffie and Lions \(1992\)](#) for Markov environments do not apply for all cases considered here. [Schroder and Skiadas \(1999\)](#) establish conditions under which the continuation value is concave, and provide further technical details. [Skiadas \(1997\)](#) shows a representation theorem for the discrete time version of recursive preferences with subjective beliefs.

⁶[Hansen \(2004\)](#) offers a tractable summary of the link between the recursive and variational utility. Interested readers may refer to the online appendix for a more detailed discussion.

⁷The cases of $\rho \rightarrow 0$ and $\gamma \rightarrow 0$ can be obtained as appropriate limits. The maximization problem (3) assumes that the felicity function is concave in its second argument. When it is convex, the formulation becomes a minimization problem.

specifies the belief distortion in (2):

Definition 2.1 *A modified discount factor process $\bar{\lambda}^n$ is a discount factor process that incorporates the martingale M^n arising from the belief distortion, $\bar{\lambda}^n \doteq \lambda^n M^n$.*

Applying Itô's lemma to $\bar{\lambda}^n$ leads to a maximization problem under the true probability measure

$$\bar{\lambda}_t^n V_t^n(C^n) = \sup_{\nu^n} E_t \left[\int_t^\infty \bar{\lambda}_s^n F(C_s^n, \nu_s^n) ds \right] \quad (6)$$

subject to

$$\frac{d\bar{\lambda}_t^n}{\bar{\lambda}_t^n} = -\nu_t^n dt + u^n dW_t, \quad \bar{\lambda}_0^n > 0, \quad t \geq 0. \quad (7)$$

The problem (6)–(7) indicates that $F(C, \nu)$ can be viewed as a generalization of the period utility function with a potentially stochastic rate of time preference ν that depends on the properties of the consumption process and thus arises endogenously in a market equilibrium. Moreover, belief distortions are now fully incorporated in the framework of [Dumas, Uppal, and Wang \(2000\)](#) — the only difference is that the modified discount factor process is not locally predictable.

The diffusion term $u^n dW_s$ has an intuitive interpretation. Consider an optimistic agent with $u^n > 0$. This agent's beliefs are distorted in that the mass of the distribution of dW_s is shifted to the right — the agent effectively overweighs good realizations of dW_s . Formula (7) indicates that under the true probability measure, positive realizations of dW_s increase the term $d\bar{\lambda}_s^n/\bar{\lambda}_s^n$, which implies that the optimistic agent discounts positive realizations of dW_s less than negative ones.

From the perspective of the utility-maximizing agent, assigning a higher probability to an event and a lower discounting of the utility contribution of this event have the same effect. In fact, equation (3) suggests that we can understand the belief distortion as a preference shock and view $\bar{\lambda}^n F(C^n, \nu^n)$ as a state-dependent felicity function. However, interpreting the martingale M^n as a belief distortion is more appealing since it bears a clearer economic meaning, separating the structure of beliefs and preferences.

2.3 Planner's problem and optimal allocations

The problem of an individual agent (3)–(4) is homogeneous degree one in the modified discount factors and homogeneous degree γ in consumption. In the homogeneous economy, there exists a closed-form solution for the continuation value $V_t^n(Y) = Y_t^\gamma \bar{V}^n$, with \bar{V}^n and the associated constant discount rate $\bar{\nu}^n$ given in [Appendix A](#) in equations (20) and (21), respectively.

In the heterogeneous economy, I follow [Dumas, Uppal, and Wang \(2000\)](#) and introduce a fictitious planner who maximizes a weighted average of the continuation values of the two agents.⁸ Given that the initial conditions on the discount factor processes $\bar{\lambda}^n$ are free, we can choose $\bar{\lambda}_0 \doteq (\bar{\lambda}_0^1, \bar{\lambda}_0^2)$ to be a pair of strictly positive initial Pareto weights.

⁸The validity of this approach for a finite-horizon economy is discussed in [Dumas, Uppal, and Wang \(2000\)](#) and [Schroder and Skiadas \(1999\)](#). The infinite-horizon problem in (8) is a straightforward extension when individual continuation values are well-defined.

Definition 2.2 *The planner’s value function is the solution to the problem*

$$J(\bar{\lambda}_0, Y_0) \doteq \sup_{(C^1, C^2)} \sum_{n=1}^n \bar{\lambda}_0^n V_0^n(C^n) = \sup_{(C^1, C^2, \nu^1, \nu^2)} \sum_{n=1}^2 E_0 \left(\int_0^\infty \bar{\lambda}_t^n F(C_t^n, \nu_t^n) dt \right) \quad (8)$$

subject to the law of motion for the modified discount factors (7) with initial conditions $(\bar{\lambda}_0^1, \bar{\lambda}_0^2)$, and the feasibility constraint $C^1 + C^2 \leq Y$.

The planner’s problem is well-defined under a simple restriction on the parameters of the economy, imposed in Assumption A.1 in Appendix A. The restriction effectively states that the agents have to be sufficiently impatient (β is sufficiently high). Since the survival results do not depend on β , Assumption A.1 does not introduce substantial restrictions for the analysis of the problem.

The planner’s problem (8) suggests that we can interpret the modified discount factor processes $\bar{\lambda}^n$ as stochastic Pareto weights. Indeed, if $\bar{\lambda}_0^n$ are the initial weights, then $\bar{\lambda}_t^n$ are the consistent state-dependent weights for the continuation problem of the planner at time t .^{9,10}

The evolution of the weights involves the drift component ν^n and thus can only be determined in equilibrium unless agent’s preferences are separable, in which case $\nu^n = \beta$. The variation in Pareto weights arises from the interaction of two components in the model — the nonseparable preference structure and the belief distortion that drives the diffusion component in (7). Belief heterogeneity introduces an additional risk component $u^n dW_t$ arising through the stochastic reweighing of wealth shares which will have a direct impact on local risk prices.

Observe that the introduction of belief heterogeneity kept the structure of the problem unchanged. For instance, Dumas, Uppal, and Wang (2000) show that in a Markov environment, the discount factor processes λ^n serve as new state variables that allow a recursive formulation of the problem using the Hamilton-Jacobi-Bellman (HJB) equation. The same conclusion is true for the modified discount factor processes $\bar{\lambda}^n$, once belief heterogeneity is incorporated. Belief distortions thus do not introduce any additional state variables into the problem, as long as the distorting processes u^n are functions of the existing state variables.

2.4 Hamilton–Jacobi–Bellman equation

The planner’s problem has an appealing Markov structure. Lemma A.3 in Appendix A shows that the value function (8) for the planner’s problem at time t is homogeneous degree 1 in $(\bar{\lambda}^1, \bar{\lambda})$,

⁹Similar techniques, which extend the formulation of the representative agent provided by Negishi (1960) to representations with nonconstant Pareto weights, can be used to study models with incomplete markets where changes in the Pareto weights reflect the tightness of the binding constraints. See Cuoco and He (2001) for a general approach in discrete time and Basak and Cuoco (1998) for a model with restricted stock market participation in continuous time.

¹⁰Jouini and Napp (2007) approach the problem from a different angle to show that a planner’s problem formulation with constant Pareto weights is in general not feasible under heterogeneous beliefs. Given an equilibrium with heterogeneous beliefs, they define a hypothetical representative agent with a utility function constructed as a weighted average of individual utility functions, with weights given by the inverses of marginal utilities of wealth. The implied consensus belief of the representative agent that would replicate the equilibrium allocation is not a proper belief but can be decomposed into the product of a proper belief and a discount factor. This discount factor would mimic the dynamics of the Pareto shares in problem (8).

homogeneous degree γ in Y and can be written as

$$J(\bar{\lambda}_t, Y_t) = (\bar{\lambda}_t^1 + \bar{\lambda}_t^2) Y_t^\gamma \tilde{J}(\theta_t)$$

where $\theta = \bar{\lambda}^1 / (\bar{\lambda}^1 + \bar{\lambda}^2)$ represents the Pareto share of agent 1 and acts as the only relevant state variable in the problem.

As we will see in the next section, the dynamics of θ are central to the study of survival in this paper. Obviously, θ is bounded between zero and one. It will become clear that for strictly positive initial weights, the boundaries are unattainable, so that θ evolves on the open interval $(0, 1)$. Moreover, the planner's problem can be characterized as a solution to the Hamilton–Jacobi–Bellman equation for $\tilde{J}(\theta)$. The proof of the following proposition together with further technical discussion is in Appendix A.

Proposition 2.3 *The Hamilton–Jacobi–Bellman equation*

$$\begin{aligned} 0 = & \sup_{(\zeta, \nu^1, \nu^2)} \theta F(\zeta, \nu^1) + (1 - \theta) F(1 - \zeta, \nu^2) + \\ & + \left[-\theta \nu^1 - (1 - \theta) \nu^2 + (\theta u^1 + (1 - \theta) u^2) \gamma \sigma_y + \gamma \mu_y + \frac{1}{2} \gamma (\gamma - 1) \sigma_y^2 \right] \tilde{J}(\theta) \\ & + \theta (1 - \theta) [\nu^2 - \nu^1 + (u^1 - u^2) \gamma \sigma_y] \tilde{J}_\theta(\theta) + \frac{1}{2} \theta^2 (1 - \theta)^2 (u^1 - u^2)^2 \tilde{J}_{\theta\theta}(\theta) \end{aligned} \quad (9)$$

with boundary conditions $\tilde{J}(0) = \bar{V}^2$ and $\tilde{J}(1) = \bar{V}^1$ has a unique bounded twice continuously differentiable solution such that $J(\bar{\lambda}_t, Y_t) = (\bar{\lambda}_t^1 + \bar{\lambda}_t^2) Y_t^\gamma \tilde{J}(\theta_t)$ is the planner's value function.

Unfortunately, equation (9) does not in general have a closed-form solution. However, the Pareto share θ of agent 1 remains the only state variable, so that the survival problem can be stated in terms of the boundary behavior of a scalar Itô process. Indeed, the crucial part of the solution is the law of motion for the state variable θ that dictates how the planner adjusts the weights of the two agents, and thus their current consumption and wealth, over time. In this respect, the only relevant force for survival is the willingness of the planner to increase the Pareto weight of the agent that becomes negligible and faces the risk of becoming extinct, and thus only the boundary behavior of $\tilde{J}(\theta)$ matters. Despite the nonexistence of a closed-form solution for $\tilde{J}(\theta)$, this boundary behavior can be characterized analytically by studying the limiting behavior of the objective function.¹¹

¹¹Equation (9) is not specific to the planner's problem (8). For instance, [Gârleanu and Panageas \(2010\)](#) use the martingale approach to directly analyze the equilibrium in an economy with agents endowed with heterogeneous recursive preferences, and show that they can derive their asset pricing formulas in closed form up to the solution of a nonlinear ODE that has the same structure as (9), which they have to solve for numerically. The analytical characterization of the boundary behavior of the ODE derived in this paper is thus applicable to a wider class of recursive utility models, and can aid numerical calculations which are often unstable in the neighborhood of the boundaries in this type of problems.

3 Survival

I show in this section that in order to evaluate the survival chances of individual agents, a complete solution for the consumption allocation, continuation values, and the implied discount rate processes is not necessary. In fact, it is sufficient to characterize the wealth dynamics in the limiting cases when the wealth share of one of the agents becomes negligible, and this limiting behavior can be solved for in closed form. The central result is a combination of Proposition 3.2, which determines survival conditions in terms of the endogenous dynamics of the Pareto share θ , and Proposition 3.6, which derives analytical formulas for these dynamics at the boundaries.

This characterization of survival requires taking an approach that is different from the majority of the literature, which typically analyzes the global properties of relative entropy as a measure of disparity between subjective beliefs and the true probability distribution, and its convergence as $t \nearrow \infty$. I return to a more detailed comparison with this literature in Section 4.3.

Instead, I derive the local dynamics of the Pareto share θ and rely on its ergodic properties, which allow me to investigate the existence of a unique stationary distribution for θ that is closely related to survival. The derived sufficient conditions are tightly linked to the behavior of the difference of endogenous discount rates of the two agents. In a decentralized economy, these *relative patience* conditions can be reinterpreted in terms of the difference in expected logarithmic growth rates of individual wealth.

Since the analyzed model includes growing and decaying economies, I am interested in a measure of relative survival. The following definition distinguishes between survival along individual paths and almost-sure survival.

Definition 3.1 *Agent 1 becomes extinct along the path $\omega \in \Omega$ if $\lim_{t \rightarrow \infty} \theta_t(\omega) = 0$. Otherwise, agent 1 survives along the path ω . Agent 1 dominates in the long run along the path ω if $\lim_{t \rightarrow \infty} \theta_t(\omega) = 1$.*

Agent 1 becomes extinct (under measure P) if $\lim_{t \rightarrow \infty} \theta_t = 0$, P -a.s. Agent 1 survives if $\limsup_{t \rightarrow \infty} \theta_t > 0$, P -a.s. Agent 1 dominates in the long run if $\lim_{t \rightarrow \infty} \theta_t = 1$, P -a.s.

Kogan, Ross, Wang, and Westerfield (2011) or Yan (2008) use the consumption share $\zeta \doteq C^1/Y$ as a measure of survival. Since the consumption share is continuous and strictly increasing in θ and the limits are $\lim_{\theta \searrow 0} \zeta(\theta) = 0$ and $\lim_{\theta \nearrow 1} \zeta(\theta) = 1$ (see equation (33)) the two measures are equivalent in this setting.

3.1 Dynamics of the Pareto share and long-run distributions

Recall the dynamics of the modified discount factor processes $\bar{\lambda}^n$ in (7). An application of Itô's lemma to $\theta = \bar{\lambda}^1 / (\bar{\lambda}^1 + \bar{\lambda}^2)$ yields

$$\begin{aligned} \frac{d\theta_t}{\theta_t} &= (1 - \theta_t) [\nu_t^2 - \nu_t^1 + (\theta_t u^1 + (1 - \theta_t) u^2) (u^2 - u^1)] dt + \\ &\quad + (1 - \theta_t) (u^1 - u^2) dW_t. \end{aligned} \tag{10}$$

Both heterogeneous beliefs and heterogeneous recursive preferences lead to nonconstant dynamics of the Pareto share, although with different implications. Under nonseparability, preference heterogeneity induces a smooth evolution of the Pareto weights, while belief heterogeneity leads to dynamics with a nonzero volatility term. Identical belief distortions ($u^1 = u^2$) under separable preferences with identical time preference coefficients or under identical recursive preferences imply a constant Pareto share $\theta_t \equiv \bar{\lambda}_0^1 / (\bar{\lambda}_0^1 + \bar{\lambda}_0^2)$, $\forall t \geq 0$.

Under nonseparable preferences, the discount rates $\nu_t^n = \nu^n(\theta_t)$ are determined endogenously in the model as part of the solution to problem (8) and are given in equations (41)–(42). Intuitively, one would expect a stationary distribution for θ to exist if the process exhibits sufficient pull toward the center of the interval when close to the boundaries. This is formalized in the following Proposition:

Proposition 3.2 *Define the following ‘repelling’ conditions (i) and (ii), and their ‘attracting’ counterparts (i’) and (ii’).*

$$\begin{aligned} \text{(i)} \quad \lim_{\theta \searrow 0} [\nu^2(\theta) - \nu^1(\theta)] &> \frac{1}{2} [(u^1)^2 - (u^2)^2] & \text{(i')} &< \\ \text{(ii)} \quad \lim_{\theta \nearrow 1} [\nu^2(\theta) - \nu^1(\theta)] &< \frac{1}{2} [(u^1)^2 - (u^2)^2] & \text{(ii')} &> \end{aligned}$$

Then the following statements are true:

- (a) *If conditions (i) and (ii) hold, then both agents survive under P .*
- (b) *If conditions (i) and (ii’) hold, then agent 1 dominates in the long run under P*
- (c) *If conditions (i’) and (ii) hold, then agent 2 dominates in the long run under P .*
- (d) *If conditions (i’) and (ii’) hold, then there exist sets $S^1, S^2 \subset \Omega$ which satisfy*

$$S^1 \cap S^2 = \emptyset, \quad P(S^1) \neq 0 \neq P(S^2), \quad \text{and} \quad P(S^1 \cup S^2) = 1$$

such that agent 1 dominates in the long run along each path $\omega \in S^1$ and agent 2 dominates in the long run along each path $\omega \in S^2$.

The conditions are also the least tight bounds of this type.

Given the dynamics of the Pareto share (10), conditions (i) and (ii) are jointly sufficient for the existence of a unique stationary density $q(\theta)$. The proof of Proposition 3.2 is based on the classification of boundary behavior of diffusion processes, discussed in Karlin and Taylor (1981). The four ‘attracting’ and ‘repelling’ conditions are only sufficient and their combinations stated in Proposition 3.2 are not exhaustive. However, the only unresolved cases are knife-edge cases involving equalities in the conditions of the Proposition, which are only of limited importance in the analysis below.

I call the difference in the discount rates $\nu^2(\theta) - \nu^1(\theta)$ *relative patience* because it captures the difference in discounting of future felicity in the variational utility specification (3) between the two agents. Conditions in Proposition 3.2 have an intuitive interpretation. Survival condition (i) states that agent 1 survives under the true probability measure even in cases when her beliefs are more distorted, $|u^1| > |u^2|$, as long as her relative patience becomes sufficiently high to overcome the distortion when her Pareto share vanishes.

Lucas and Stokey (1984) impose a similar condition called *increasing marginal impatience* that is sufficient to guarantee the existence of a nondegenerate steady state as an exogenous restriction on the preference specification. This condition requires the preferences in their framework to be nonhomothetic, and rich agents must discount future more than poor ones. In this model, preferences are homothetic, and variation in relative patience arises purely as a response to the market interaction of the two agents endowed with heterogeneous beliefs. The discount rate ν^n encodes not only a pure time preference but also the interaction of current discounting with the dynamics of the continuation values that reflects the behavior of the equilibrium consumption streams.

3.2 Decentralization

Proposition 3.2 states the survival conditions in terms of the endogenous discount rates ν^n . In this section, I derive closed-form formulas for the boundary behavior of ν^n , and evaluate analytically the region in the parameter space in which these conditions hold.

The proof strategy in this section relies on a decentralization argument and utilizes the asymptotic properties of the differential equation (9) for the planner's continuation value. The economy is driven by a single Brownian shock, and two suitably chosen assets that can be continuously traded are therefore sufficient to complete the markets in the sense of Harrison and Kreps (1979). Let the two traded assets be an infinitesimal risk-free bond in zero net supply that yields a risk-free rate $r_t = r(\theta_t)$ and a claim on the aggregate endowment with price $A_t = Y_t \xi(\theta_t)$, where $\xi(\theta)$ is the aggregate wealth-consumption ratio. Individual wealth levels are denoted $A_t^n = Y_t \zeta^n(\theta_t) \xi^n(\theta_t)$, where $\xi^n(\theta)$ are the individual wealth-consumption ratios.

The results reveal that as the Pareto share of one of the agents converges to zero, the infinitesimal returns associated with the two assets converge to those which prevail in a homogeneous economy populated by the agent with the large Pareto share. This implies that an agent that becomes extinct in the long run also has no long-run *price impact* on the two assets that are traded, see also Kogan, Ross, Wang, and Westerfield (2011).

These results are, however, even stronger because they also state that when the wealth of an agent becomes negligible, she has no impact on the current prices of the two assets even if she survives in the long run and her wealth recovers in the future. The ability to pin down asset returns when the wealth of one agent is negligible even though she may survive in the long run plays a crucial role in the analysis because it allows me to determine the wealth dynamics of the two agents in the proximity of the boundary by solving two straightforward portfolio choice problems.

The solutions then yield in closed form the required limiting behavior of the discount rates ν^n from Proposition 3.2, and thus determine the survival outcomes. Further, the link between the decentralized solution and the planner's problem conditions from Proposition 3.2 reveals that the survival conditions can be directly restated as conditions on the limiting expected growth rates of the logarithm of individual wealth levels in a decentralized economy.

3.2.1 Equilibrium prices

The following Proposition summarizes the limiting pricing implications as the wealth share of one of the agents becomes arbitrarily small. Without loss of generality, it is sufficient to focus on the case $\theta \searrow 0$.

Proposition 3.3 *As $\theta \searrow 0$, the infinitesimal risk-free rate $r(\theta)$, the aggregate wealth-consumption ratio $\xi(\theta)$, and the drift and volatility coefficients of the aggregate wealth process $dA_t/A_t = \mu_A(\theta_t) dt + \sigma_A(\theta_t) dt$ converge to their homogeneous economy counterparts:*

$$\begin{aligned} \lim_{\theta \searrow 0} r(\theta) &= r(0) = \beta + (1 - \rho)(\mu_y + u^2 \sigma_y) - \frac{1}{2}(2 - \rho)(1 - \gamma)\sigma_y^2, \\ \lim_{\theta \searrow 0} A(\theta) &= A(0) = \left[\beta - \rho \left(\mu_y + u^2 \sigma_y - \frac{1}{2}(1 - \gamma)\sigma_y^2 \right) \right]^{-1}, \\ \lim_{\theta \searrow 0} \mu_A(\theta) &= \mu_y, \quad \text{and} \quad \lim_{\theta \searrow 0} \sigma_A(\theta) = \sigma_y. \end{aligned}$$

Further, the infinitesimal return on the claim on aggregate wealth,

$$\frac{dR_t}{R_t} \doteq \left[[\xi(\theta_t)]^{-1} + \mu_A(\theta_t) \right] dt + \sigma_A(\theta_t) dW_t, \quad (11)$$

has coefficients that converge as well.

The proof is provided in Appendix B and is based on the characterization of the dynamics of the equilibrium stochastic discount factor as $\theta \searrow 0$. Notice that the convergence of the coefficients $\mu_A(\theta)$ and $\sigma_A(\theta)$ of the wealth process is not an immediate consequence of the convergence of the aggregate wealth-consumption ratio. It may be that the wealth-consumption ratio $\xi(\theta)$ converges as $\theta \searrow 0$, yet the price dynamics are such that $\mu_A(\theta)$ and $\sigma_A(\theta)$ do not converge to μ_y and σ_y , respectively. The fact that this does not happen is closely linked to the dynamics of $\log \theta$ which has bounded drift and volatility coefficients. This ensures that the local variation in $\xi(\theta)$ becomes irrelevant as $\log \theta \searrow -\infty$.

The results in Proposition 3.3 are sufficient to proceed with the construction of the main result. As a side note, prices of finite-horizon risk-free claims and individual cash flows from the aggregate endowment converge as well:

Corollary 3.4 *For every fixed maturity t , the prices of a zero-coupon bond and a claim to a payout from the aggregate endowment stream (a consumption strip) converge to their homogeneous economy counterparts as $\theta \searrow 0$.*

The agent with negligible wealth therefore has no price impact not only on the two assets that dynamically complete the market but also on every finite-maturity bond and consumption strip. Recall that the results from Proposition 3.3 and Corollary 3.4 do not assume that the agent with negligible wealth vanishes in the long run. The reason is that even if the agent survives, the logarithmic growth rates of her wealth are always bounded in this economy. This implies that the distribution of her wealth at a given future date t can be driven arbitrarily close to zero by driving to zero her current wealth level. Of course, ultimately, the surviving agent will recover from an arbitrarily small wealth level, so that the convergence in Corollary 3.4 is not uniform in the maturity horizon $t \in (0, \infty)$.

3.2.2 Decision problem of an agent with negligible wealth

Proposition 3.3 establishes that the actual *general equilibrium* price dynamics in the proximity of the boundary are locally the same as those in an economy populated only by agent 2. To conclude the argument, we need to infer the wealth dynamics for agent 1 that has negligible wealth. The marginal utility under recursive preferences is forward-looking and depends on agent's continuation value (see the stochastic discount factor specification (50)). If agent 1 ultimately survives, then she will always have a nontrivial price impact *in the future*, even if her current Pareto share is negligible. The forward looking nature of the optimization problem then implies that she should take this price impact into account when making her current portfolio and consumption-saving decisions. However, the result below shows that this impact of future price dynamics becomes immaterial as $\theta \searrow 0$.

Proposition 3.5 *The consumption-wealth ratio of agent 1 converges to*

$$\lim_{\theta \searrow 0} [\xi^1(\theta)]^{-1} = [\xi(0)]^{-1} - \frac{\rho}{1-\rho} \left[(u^1 - u^2) \sigma_y + \frac{1}{2} \frac{(u^1 - u^2)^2}{1-\gamma} \right] \quad (12)$$

and the wealth share invested into the claim on aggregate consumption to

$$\lim_{\theta \searrow 0} \pi^1(\theta) = 1 + \frac{u^1 - u^2}{(1-\gamma) \sigma_y}. \quad (13)$$

It follows that the asymptotic coefficients for the evolution of agent's 1 wealth, $dA_t^1/A_t^1 = \mu_{A^1}(\theta_t) dt + \sigma_{A^1}(\theta_t) dt$, are

$$\begin{aligned} \lim_{\theta \searrow 0} \mu_{A^1}(\theta) &= \mu_y + \frac{1}{1-\rho} (u^1 - u^2) \sigma_y + \frac{1}{2} \frac{2-\rho}{1-\rho} \frac{(u^1 - u^2)^2}{1-\gamma} - \frac{u^1 (u^1 - u^2)}{1-\gamma} \\ \lim_{\theta \searrow 0} \sigma_{A^1}(\theta) &= \sigma_y + \frac{u^1 - u^2}{(1-\gamma)}. \end{aligned}$$

Proposition 3.5 derives the consumption-saving decision (12) and portfolio allocation decision (13) relative to the same decisions of the large agent 2. Recall that agent's 2 choices agree in the

limit as $\theta \searrow 0$ with aggregate ones — her consumption-wealth ratio is equal to the aggregate ratio $[\xi(0)]^{-1}$ and she holds the market portfolio, $\lim_{\theta \searrow 0} \pi^2(\theta) = 1$.

This reiterates the arguments described in the introduction. When agent 1 becomes negligible, equilibrium prices have to adjust to make agent 2's choices coincide with aggregate dynamics. In particular, agent 2 has to consume the aggregate endowment, reflected by $\lim_{\theta \searrow 0} [\xi^2(\theta)]^{-1} = [\xi(0)]^{-1}$, and hold the market portfolio, reflected by $\lim_{\theta \searrow 0} \pi^2(\theta) = 1$. At these equilibrium prices, agent 1's consumption and portfolio choices deviate from the aggregate dynamics according to formulas (12) and (13). When these deviations lead to a high saving rate or a high logarithmic return on agent 1's portfolio, they can prevent her extinction. As the formulas indicate, when $u^1 = u^2$ the agents are identical and their decisions and wealth dynamics coincide.

The logic of the proof relies on showing that the current continuation value dynamics of the agent with negligible wealth is not influenced by the fact that she may become non-negligible in the future and have impact on aggregate price dynamics. Since the drift and volatility coefficients of the logarithm of the Pareto share process $\log \theta$ are bounded, then by pushing the current θ arbitrarily close to zero ($\log \theta$ arbitrarily far toward $-\infty$), one can extend the time before the presence of the agent 1 becomes noticeable from aggregate perspective (measured, e.g., by sufficiently large deviations in prices or return distributions from their homogeneous economy counterparts) arbitrarily far into the future.

The portfolio and consumption-saving decision of agent 1 as $\theta \searrow 0$ thus coincides with a 'partial equilibrium' solution where agent 1 behaves as if she lived forever as an infinitesimal agent in a homogeneous economy populated only by the large agent 2.

This implies that the survival question, whose answer only depends on the behavior at the boundaries, can be resolved by studying homogeneous economies with an infinitesimal price-taking agent. Even if the negligible agent survives with probability one and has an impact on equilibrium prices in the long run, these effects do not influence *current* prices, returns, and wealth dynamics.

3.3 Limiting relative patience and relationship to wealth growth

Importantly, the limiting discount rate $\nu^1(\theta)$ in Proposition 3.2 can be inferred from the portfolio problem of agent 1 outlined in the proof of Proposition 3.5 in equations (59)–(60), which leads to the statement of the main result of this section.

Proposition 3.6 *The expressions for the limiting behavior of the relative patience in Proposition 3.2 are*

$$\lim_{\theta^1 \searrow 0} \nu^2(\theta) - \nu^1(\theta) = \frac{\rho - \gamma}{1 - \rho} \left[(u^1 - u^2) \sigma_y + \frac{1}{2} \frac{(u^1 - u^2)^2}{1 - \gamma} \right], \quad (14)$$

$$\lim_{\theta^1 \nearrow 1} \nu^2(\theta) - \nu^1(\theta) = \frac{\rho - \gamma}{1 - \rho} \left[(u^1 - u^2) \sigma_y - \frac{1}{2} \frac{(u^1 - u^2)^2}{1 - \gamma} \right]. \quad (15)$$

Section 4 discusses which regions of the parameter space satisfy the individual survival and extinction conditions from Proposition 3.2. Notice that while Assumption A.1 imposes a restriction on the time-preference parameter β of the agents, the survival conditions do not explicitly depend on β . The survival results thus always hold with the implicit assumption that time discounting is sufficiently high.

The construction of the main survival result utilized the link between the planner's problem and the competitive equilibrium. It is therefore useful to restate the relative patience conditions in terms of conditions on the individual wealth growth rates of the two agents. An application of Itô's lemma yields $d \log A_t^n = \tilde{\mu}_{A^n}(\theta_t) dt + \sigma_{A^n}(\theta_t) dW_t$ where $\tilde{\mu}_{A^n}(\theta_t) = \mu_{A^n}(\theta_t) - \frac{1}{2}\sigma_{A^n}^2(\theta_t)$. We then obtain the following result.

Corollary 3.7 *The survival conditions in part a) of Proposition 3.2 are equivalent to:*

$$(i) \quad \lim_{\theta \searrow 0} [\tilde{\mu}_{A^1}(\theta) - \tilde{\mu}_{A^2}(\theta)] > 0, \quad (ii) \quad \lim_{\theta \nearrow 1} [\tilde{\mu}_{A^1}(\theta) - \tilde{\mu}_{A^2}(\theta)] < 0.$$

Verifying the conditions in Proposition 3.2 therefore amounts to checking that the expected growth rate of the logarithm of wealth (the drift coefficient for $d \log A_t^n$) is higher for the agent who is at the brink of extinction. In what follows, I therefore focus on the analysis of the boundary behavior. In Section 5.1, I revisit in more detail the complete dynamics of the Pareto share, discount rates, and consumption and portfolio choices.

3.4 Decomposition of wealth growth rates

The previous discussion identified the portfolio allocation and the consumption-saving decisions as the two mechanisms underlying wealth accumulation and long-run survival. It is therefore informative to decompose the growth rate of individual wealth from Corollary 3.7 into the contribution of the logarithmic return on the agent's portfolio, net of the rate of consumption rate represented by the consumption-wealth ratio,

$$d \log A_t^n = d \log R_t^n - (\xi_t^n)^{-1} dt.$$

Denoting $d \log R_t^n = \tilde{\mu}_{R^n}(\theta_t) dt + \sigma_{R^n}(\theta_t) dW_t$ where $\tilde{\mu}_{R^n}(\theta_t) \doteq \mu_{R^n}(\theta_t) - \frac{1}{2}\sigma_{R^n}^2(\theta_t)$ is the expected logarithmic return on agent's n portfolio (and $\mu_{R^n}(\theta_t)$ the expected level return), we can collect the results above to establish the following decomposition.

Proposition 3.8 *As $\theta \searrow 0$, the difference in the logarithmic wealth growth rates between the agent with negligible wealth and the large agent can be written as*

$$\lim_{\theta \searrow 0} [\tilde{\mu}_{A^1}(\theta) - \tilde{\mu}_{A^2}(\theta)] = \lim_{\theta \searrow 0} [\tilde{\mu}_{R^1}(\theta) - \tilde{\mu}_{R^2}(\theta)] - \lim_{\theta \searrow 0} [(\xi^1(\theta))^{-1} - (\xi^2(\theta))^{-1}]$$

where the difference in the expected logarithmic portfolio returns is

$$\lim_{\theta \searrow 0} [\tilde{\mu}_{R^1}(\theta) - \tilde{\mu}_{R^2}(\theta)] = \underbrace{\frac{u^1 - u^2}{(1 - \gamma) \sigma_y}}_{\text{difference in portfolios}} \underbrace{[(1 - \gamma) \sigma_y^2 - u^2 \sigma_y]}_{\text{risk premium}} - \underbrace{\frac{u^1 - u^2}{1 - \gamma} \left(\sigma_y + \frac{1}{2} \frac{u^1 - u^2}{1 - \gamma} \right)}_{\text{lognormal correction}} \quad (16)$$

and the difference in consumption rates is given by

$$\lim_{\theta \searrow 0} [(\xi^1(\theta))^{-1} - (\xi^2(\theta))^{-1}] = -\frac{1}{2} \frac{\rho}{1 - \rho} \underbrace{\left[2(u^1 - u^2) \sigma_y + \frac{(u^1 - u^2)^2}{1 - \gamma} \right]}_{\text{difference in subjective expected returns}}.$$

The proposition reveals a clear separation of the role of risk aversion and IES. The difference in the expected logarithmic portfolio returns at the boundary only depends on the relative risk aversion $1 - \gamma$, not on the parameter ρ that determines the IES. The first term represents the *risk premium channel* — the risk premium on the claim on aggregate consumption times the difference in the portfolio shares invested in the risky asset, obtained in equation (13). The risk premium itself is composed of the standard rational expectations premium $(1 - \gamma) \sigma_y^2$ and a ‘mispricing’ effect $-u^2 \sigma_y$ (if the large agent 2 is optimistic, she overprices the risky asset which leads to a lower expected return). Since survival is driven by the expected *logarithmic* return, the lognormal correction represents a penalty for choosing volatile portfolios, reflecting the *speculative volatility channel*. This penalty is the dominant force for survival when risk aversion declines to zero ($\gamma \nearrow 1$).

The difference in consumption rates consists of two components. The term in brackets is the difference between the expected portfolio return of agent 1 as perceived by agent 1, and the portfolio return of agent 2 as perceived by agent 2,

$$[\mu_{R^1}(\theta_t) + u^1 \sigma_{R^1}(\theta_t)] - [\mu_{R^2}(\theta_t) + u^2 \sigma_{R^2}(\theta_t)].$$

Here, $\mu_{R^n}(\theta_t)$ is the objective expected level return on agent’s n portfolio, and $u^n \sigma_{R^n}(\theta_t)$ is the subjective bias. It is the *subjective* expected returns (computed under Q^n , not P) that enter the formula because the consumption-saving decision of the agent depends on the expected portfolio return as perceived by herself. When $\text{IES} = 1$ ($\rho = 0$), the consumption-wealth ratios of the two agents are identical and equal to β as in the case of myopic logarithmic utility, and the consumption-saving decision plays no role in the survival outcomes. When preferences are elastic ($\text{IES} > 1$, i.e., $1 > \rho > 0$), the saving rate is an increasing function of the subjective expected portfolio return and the difference in consumption rates is therefore negatively related to the difference in subjective expected returns — vis-à-vis a high expected return, the agent with elastic preferences decides to postpone consumption and tilt the consumption profile toward the future. This helps the agent with the higher expected subjective return outsave her extinction, reflecting the *saving channel* of survival.

3.4.1 Dependence of survival results on individual parameters

Using the results from Proposition 3.6 in the inequality conditions from Proposition 3.2 reveals that the survival regions depend on the ratios of parameters u^1/σ_y and u^2/σ_y , and not on the three parameters independently. This is an important insight that shows that what matters for survival in this economy is the importance of aggregate fundamental risk embedded in σ_y relative to the willingness of the agents to generate additional volatility in their individual consumption processes through betting, reflected in the magnitude of the belief distortions u^n . Large belief distortions encourage larger speculative portfolio positions that reflect the belief differences. Aggregate risk, on the other hand, discourages additional risk taking through speculation.

For instance, if agent 2 has correct beliefs, $u^2 = 0$, then the long-run survival outcome will be the same if we fix the belief distortion u^1 and make the aggregate endowment deterministic, $\sigma_y \searrow 0$, or if we fix σ_y and make the agent's beliefs infinitely incorrect. I will revisit this aspect of the survival results in the next section.

It is also worth explaining why the survival results do not depend on the time-preference parameter β and the growth rate of the economy μ_y . Both these parameters influence individual consumption-saving decisions. Since the impact of these parameters on both agents' decisions is identical, they do not affect the difference in the rates of wealth accumulation. This would no longer be true if, for instance, the agents differed in the IES parameter.

4 Survival regions

This section analyzes the regions of the parameter space in which agents with distorted beliefs survive or dominate the economy. It turns out that all four combinations generated by the pair of inequalities in Proposition 3.2 do occur generically in nontrivial parts of the parameters space.

4.1 Asymptotic results

It is useful to start by describing the asymptotic results as either risk aversion or intertemporal elasticity of substitution moves toward extreme values, holding other parameters fixed. These extreme cases isolate the role of the individual survival channels outlined in the introduction. Section 4.2 then analyzes the interaction of these forces in the whole parameter space.

Corollary 4.1 *Holding other parameters fixed, for any given pair of beliefs u^n , $n \in \{1, 2\}$ and $\sigma_y > 0$, the survival restrictions imply the following asymptotic results.*

- (a) *As agents become risk neutral ($\gamma \nearrow 1$), each agent dominates in the long run with a strictly positive probability.*
- (b) *As risk aversion increases ($\gamma \searrow -\infty$), the agent who is relatively more optimistic about the growth rate of aggregate endowment always dominates in the long run.*

- (c) *As intertemporal elasticity of substitution increases ($\rho \nearrow 1$), the relatively more optimistic agent always survives. The relatively more pessimistic agent survives (and thus a nondegenerate long-run equilibrium exists) when risk aversion is sufficiently small.*
- (d) *As the intertemporal elasticity of substitution decreases to zero ($\rho \searrow -\infty$), a nondegenerate long-run equilibrium cannot exist.*

4.1.1 Low risk aversion and the speculative volatility channel

In order to provide the intuition underlying result (a), consider the limiting case when agents are risk neutral ($\gamma = 1$). Then the felicity function $F(C, \nu)$ in (5) is linear in C , and the agents choose to play a one-shot lottery with all their wealth. The more optimistic agent wins in states with a high realization of the next-period shock, while the other agent wins in states with a low realization. The cutoff is determined so that both agents are willing to participate (the agent with more wealth faces a higher probability of winning). After this one-shot lottery, the losing agent immediately becomes extinct, consuming zero at all subsequent dates.¹²

When the agents are not exactly risk neutral, none of the agents becomes extinct in finite time, but the same force, reflecting the *speculative volatility channel*, operates over time in the long-horizon limit. Once a sequence of unsuccessful bets reduces the wealth of one agent substantially, equilibrium prices have to adjust to make the large agent hold approximately the market portfolio. At these prices, the small agent continues to make these bets which are detrimental for her survival chances — despite the fact that she may earn a high expected level return on her portfolio, the expected logarithmic return is low due to the volatility penalty (Proposition 3.8) that comes from the high dispersion of future wealth that the agent tolerates. This leads to her extinction along a set of paths that has a strictly positive measure. Since events when either of the agents becomes sufficiently small recur with probability one, ultimately one of the agents becomes extinct with probability one, and each of the agents faces a strictly positive probability of extinction.

4.1.2 High risk aversion and the risk premium channel

In the other extreme, when agents become very risk averse ($\gamma \searrow -\infty$, result (b)), they put a high value on insuring states with low aggregate shock realizations. Since the relatively more pessimistic agent places a higher probability on these states, she is insured in equilibrium by the relatively more optimistic agent who holds a larger share of her wealth in the risky asset. The price of this insurance is the foregone risk premium in the risky asset.

¹²This result may seem puzzling but it is closely related to the exact role of the IES parameter, which captures the elasticity of substitution between current consumption and the expected risk-adjusted continuation value. When IES is finite ($\rho < 1$), then the only way how to optimally provide zero consumption in the next instant is to also provide zero continuation value in the same state, which also implies zero consumption at all subsequent dates and states (up to a set of paths of measure zero). A very similar mechanism underlies the results in Backus, Routledge, and Zin (2008). The discrete-time specification from Epstein and Zin (1989) adopted in their paper makes clear the role of IES parameter as the elasticity between current consumption and the expected risk-adjusted continuation value next period.

Expression (16) shows that as risk aversion $1 - \gamma$ increases, the risk premium grows linearly, and the pessimistic agent responds by insuring less the adverse states. This is reflected in the vanishing difference in the portfolio positions. However, the total cost of this insurance, given by the product of the risk premium and the difference in the portfolios, converges to a nonzero constant. Since the portfolio positions of the two agents converge to each other as risk aversion increases, their volatilities converge as well, and the lognormal correction vanishes. For high risk aversion levels, the *risk premium channel* dominates, and the more optimistic agent earns a strictly higher expected logarithmic return on her portfolio.

Under separable preferences, the relatively more pessimistic agent would still survive (and dominate) in this situation if her beliefs are more accurate because an increase in risk aversion implies a decrease in IES. This agent would correctly understand that she pays for her insurance of the low states with a lower expected return, and the lower IES would also motivate her to decrease current consumption rate. Under the recursive preference structure, the IES is fixed as risk aversion increases, and this saving channel does not outweigh the advantage of the relatively more optimistic agent.

4.1.3 IES and the consumption-saving decision

Result (c) highlights the role of the *saving channel*. With a high IES, agents are willing to substantially decrease their consumption rate vis-à-vis an increase in the subjective expected return on their portfolio (see also the expression for the difference in consumption rates in Proposition 3.8 which scales the difference in subjective expected returns by the term $\rho / (1 - \rho)$). Whenever an agent becomes small, she can choose a portfolio with a high subjective expected return while the market clearing mechanism forces her large counterpart to choose a portfolio that is close to the market portfolio. The high IES then gives a survival advantage to the small agent because it induces her to increase her saving rate in response to the high subjective expected return.

At the same time, risk aversion cannot be too high for this mechanism to be sufficiently strong. A high risk aversion discourages betting, and the incentives of the small agent to choose a sufficiently ‘leveraged’ portfolio with a high subjective expected return diminish.

Result (d) is a direct counterpart to (c). When preferences of the agents become inelastic ($\rho \searrow -\infty$), formulas in Proposition 3.6 imply that the survival conditions cannot hold simultaneously. Inelastic preferences imply that the agents are unwilling to substantially change the slope of their consumption profiles, and the mechanism based on differences in saving rates, which operated for high IES, is largely absent. This result again shows the critical role of the consumption-saving decision and the endogenous equilibrium price dynamics in generating equilibria in which both agents survive in the long run. Partial equilibrium models with exogenous price dynamics that do not depend on wealth shares of individual agents cannot replicate this survival mechanism.

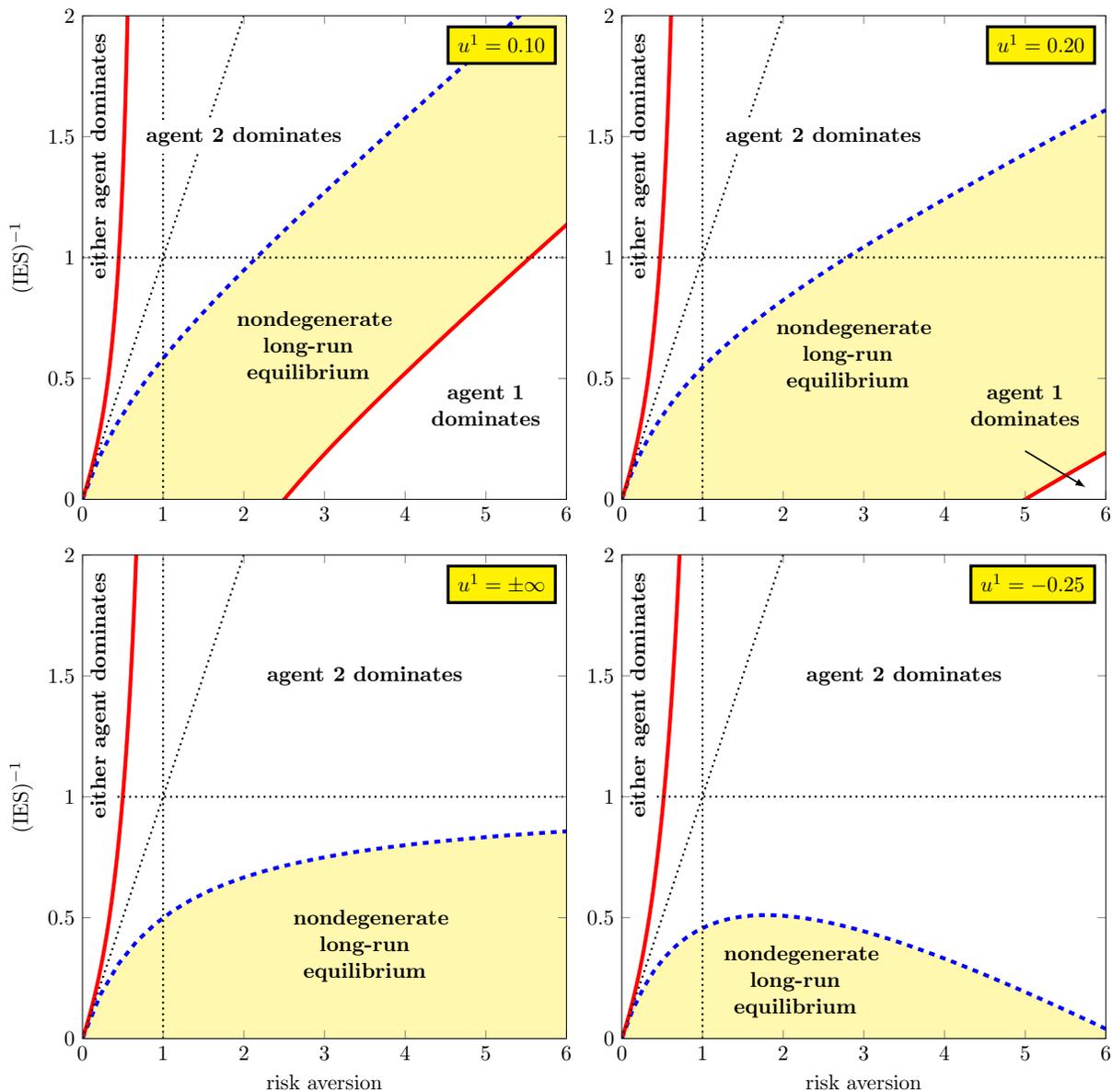


Figure 2: Survival regions for different belief distortions of agent 1 (see legend of each plot). Agent 2 always has correct beliefs, $u^2 = 0$, and the volatility of aggregate endowment is $\sigma_y = 0.02$.

4.2 Comparative statics

Survival conditions in Proposition 3.6 depend only on parameters $(\gamma, \rho, u^1/\sigma_y, u^2/\sigma_y)$. Figure 2 provides a systematic treatment of the parameter space. Each panel plots the survival results in the ‘risk aversion / inverse of IES’ plane $(1 - \gamma, 1 - \rho)$ for different levels of belief distortions. To keep the discussion focused, I concentrate on the case when agent 2 has correct beliefs, $u^2 = 0$. The online appendix considers additional cases when the beliefs’ of both agents are distorted but they are all special cases of Proposition 3.6. To get an idea about the magnitude of the belief distortions, recall that an agent with $u^1 = 0.1$ distorts the annual growth rate of aggregate endowment by $u^1\sigma_y$,

e.g., believes it to be 2.2% instead of 2% when $\sigma_y = 0.02$.

The shaded area represents the parameter combinations for which a nondegenerate long-run equilibrium exists. The blue dashed lines in the graphs depict parameter combinations for which condition (i) in Proposition 3.2 holds with equality (as $\theta \searrow 0$), while the solid red lines capture the same situation for condition (ii) (as $\theta \nearrow 1$). The results do not reveal what happens at these boundaries but the long-run outcomes for the interiors of the individual regions are completely characterized by the conditions in Proposition 3.2. The existing literature established that along the dotted diagonal, which represents the parameter combinations for separable CRRA preferences, the agent with more accurate beliefs (i.e., with a smaller $|u^n|$, in our case agent 2) dominates the economy in the long run.

4.2.1 Optimistic belief distortion

The first panel in Figure 2 starts with a moderately optimistic agent 1. The correct agent 2 dominates in the long run in the neighborhood of the dotted diagonal, extending the results for the CRRA case continuously in the parameter space. Moreover, the graph also confirms all four asymptotic results from Corollary 4.1.

At the same time, there is a nontrivial intermediate region (depicted as shaded in the graph) where both agents coexist in the long run. In this whole region, risk aversion is larger than the inverse of IES, which is a standard parametric choice in the asset pricing literature. The two boundaries in the top left panel which delimit this region are asymptotically parallel as $\gamma \searrow -\infty$ with slope $2\sigma_y / (u^1 + u^2 + 2\sigma_y)$.

As we increase the optimism of agent 1 (second panel in Figure 2), the lines delimiting the shaded region rotate clockwise. The area in which agent 2 dominates expands, reflecting an increase in inaccuracy of agent 1's beliefs, but the region in which both agents coexists in the long run never vanishes.

In fact, as $u^1 \nearrow \infty$ and agent 1 becomes infinitely optimistically biased, we obtain the third panel in Figure 2. The optimistic agent 1 never dominates the economy but there is a large set of parameter combinations for which both agents coexist in the long run. The dashed line delineating this set converges to $\text{IES} = 1$ as risk aversion increases. The shaded region includes many plausible parameterizations used in asset pricing models; for instance, much of the long-run risk literature initiated by [Bansal and Yaron \(2004\)](#) advocates IES significantly above one and risk aversion well above five.

4.2.2 Economy without aggregate risk

What matters for the result in the third panel in Figure 2 is the fact that incentives to bet, reflected in the increased belief heterogeneity, become arbitrarily large relative to aggregate risk in the economy. As we have seen, the survival results do not depend on u^1, u^2, σ_y independently but rather on the ratios $u^1/\sigma_y, u^2/\sigma_y$. An economy with $u^1/\sigma_y \rightarrow \infty$ has the feature that the magnitude of the risk premium associated with the claim on aggregate endowment, which is proportional to

σ_y^2 , becomes trivial compared to the amount of perceived mispricing that depends on the magnitude of the difference in belief distortions. The risk premium channel that benefits the more optimistic agent has a vanishing impact.

However, large differences in beliefs lead to portfolio positions that fluctuate strongly with changes in the wealth distribution. This leads to large fluctuations in perceived expected returns. When IES is larger than one, this will create a strong survival force through the saving channel.

To illuminate this mechanism, consider the case of an economy without aggregate risk, $\sigma_y = 0$ with u^1 being an arbitrary nonzero belief distortion and $u^2 = 0$. Aggregate endowment is thus deterministic but the agents can still write contracts on the realizations of the Brownian motion W , so that a nontrivial two-asset decentralization is still valid. The risky asset is a purely speculative asset that allows the agents to trade on the differences in their beliefs.

The long-run survival results for this economy are perfectly equivalent to the third panel in Figure 2. As agent 1 vanishes, agent 2 has to hold the market portfolio. Since this portfolio corresponds to a claim on the deterministic consumption stream, her stochastic discount factor for every finite horizon converges to a deterministic process and risk premia in the economy converge to zero (see Proposition 3.3 and closely related Corollary 3.4). From the perspective of agent 1, the speculative asset now offers a high perceived return and, with $\text{IES} > 1$, this translates into a higher saving rate of the negligible agent. When IES is sufficiently high, the high saving motive will always be strong enough to let the negligible agent outsave her extinction and survive in the long run.

4.2.3 Pessimistic belief distortion

The third panel in Figure 2 also represents the case when $u^1/\sigma_y \rightarrow -\infty$, i.e., the case of an infinitely pessimistic agent 1. Recall that the limit $|u^1/\sigma_y| \rightarrow \infty$ corresponds to a situation where the role of aggregate risk vanishes relative to the speculative motives generated by belief heterogeneity. In this limit, the agents are speculating on the realizations of the Brownian shock W . Because this shock is symmetric, it does not matter whether agent 1 is ‘optimistic’ and speculates on right-tail realizations of the Brownian shock or ‘pessimistic’ and speculates on left-tail realizations. This logic is most clearly visible in the case with deterministic aggregate endowment, $\sigma_y = 0$, where the survival results are the same for an arbitrary value of $u^1 \neq 0$.

What happens when the magnitude of pessimism decreases and u^1 starts moving from $-\infty$ closer to zero? The change in the survival regions is represented by a move from the third to the fourth panel of Figure 2. Maybe somewhat surprisingly, the region in which the pessimistic agent 1 survives actually *shrinks*.

Notice that the pessimistic agent invests a smaller share of her wealth into the risky asset, so she cannot benefit from the risky asset’s higher expected return through the risk premium channel. At the same time, the pessimism seems to imply that her *subjective* expected return is even lower, so that she will not improve her survival chances by choosing a higher saving rate under $\text{IES} > 1$. But since the agent is pessimistic about the return on the asset, she actually is optimistic about the

return on a short position in that asset. Observe that the last term in brackets in the consumption-wealth ratio (12), which dominates the saving decision of agent 1 when $\rho \nearrow 1$, is equal to

$$\frac{1}{2} (u^1 - u^2) \sigma_y (1 + \pi^1(0)). \quad (17)$$

If agent 1 is relatively more pessimistic, then $u^1 - u^2 < 0$, and thus $\pi^1(0) < -1$ is needed for the saving motive of agent 1 to dominate that of the large agent 2 as $\rho \nearrow 1$. While the short position in the risky asset earns a low objective expected return, a high IES can generate a sufficiently strong offsetting saving motive that will allow the pessimistic agent to outsave her extinction.

The region in the fourth graph in which the two agents coexist does not include high levels of risk aversion and shrinks for smaller belief distortions. A high level risk aversion or a lower incentive to bet caused by a smaller belief distortion will prevent the small agent from choosing a sufficiently large short position in the risky asset which is, as shown in formula (17), necessary to generate the high subjective expected return needed for the saving mechanism to operate in favor of the pessimistic agent 1.

The above discussion also explains why the described mechanism cannot lead to the long-run dominance of the pessimistic agent. As the wealth share of the pessimistic agent approaches one, she can no longer hold a short position in the risky asset, and the effect of the saving mechanism generated through the high subjective expected return disappears.

4.3 Separable preferences

The framework introduced in this paper includes as a special case the separable constant relative risk aversion preferences when $\gamma = \rho$. Yan (2008) and Kogan, Ross, Wang, and Westerfield (2011) show that in the economy presented in this paper under CRRA preferences, the agent whose beliefs are less distorted dominates in the long run under measure P . The conditions in Proposition 3.2 confirm these results as follows:

Corollary 4.2 *Under separable CRRA preferences ($\gamma = \rho$), agent n dominates in the long run under measure P if and only if $|u^n| < |u^{\sim n}|$. Agent n survives under P if and only if the inequality is non-strict. Further, agent n always survives under measure Q^n , and dominates in the long run under Q^n if and only if $u^n \neq u^{\sim n}$.¹³*

Under separable CRRA preferences, the dynamics of the Pareto share θ in (10) do not depend on the characteristics of the endowment process. Separable utility is obtained a special case of the variational utility (3)–(4) with an optimal discount rate choice $\nu^n = \beta$ where β is the time preference coefficient and the period utility function $F(C, \beta) \doteq U(C)$. The first-order condition for the planner’s problem leads to the static equation

$$\bar{\lambda}_0^1 M_t^1 U'(C_t^1) = \bar{\lambda}_0^2 M_t^2 U'(C_t^2).$$

¹³A specific situation in Corollary 4.2 arises when $u^n = -u^{\sim n}$. The proof of the corollary shows that although none of the agents becomes extinct, a nondegenerate long-run distribution for θ does not exist.

Survival analysis in the separable case thus corresponds to analyzing a sequence of state- and time-indexed static problems that are interlinked only by the initial Pareto weights $\bar{\lambda}_0^n$. If agent 1 has a constant belief distortion $u^1 \neq 0$ and agent 2 is rational, then M^1 is a strictly positive supermartingale with $\lim_{t \rightarrow \infty} M_t^1 = 0$ (P -a.s.) and $M_t^2 \equiv 1$, and thus $\lim_{t \rightarrow \infty} U'(C_t^1)/U'(C_t^2) = +\infty$ (P -a.s.). For a class of utility functions that includes the CRRA utility (the special case when $\gamma = \rho$ in this paper), this implies $\lim_{t \rightarrow \infty} \zeta_t/(1 - \zeta_t) = 0$ (P -a.s.). [Kogan, Ross, Wang, and Westerfield \(2011\)](#) analyze this case for a general class of period utility functions.

When preferences are not separable, this straightforward link breaks down because marginal utilities of future consumption also depend on continuation values and the first-order conditions involve the evolution of the endogenously determined discount rate process ν^n between 0 and t (see the stochastic discount factor [\(50\)](#)). Since these continuation values and discount rate processes are not available in closed form, they have to in general be solved for numerically. This paper avoids this issue by proving that long-run outcomes can be determined by analytically characterizing the boundary behavior of the economy.

5 Dynamics of long-run equilibria

Propositions [3.2](#) and [3.6](#) in Section [3](#) derive parametric restrictions on the survival regions. However, even if a nondegenerate long-run equilibrium exists, the question remains whether this equilibrium delivers quantitatively interesting endogenous dynamics under which each of the agents can gain a significant wealth share. The derived analytical boundary conditions cannot answer this questions, as a full solution of the model is necessary to investigate the interior of the $(0, 1)$ interval for the Pareto share θ . This section investigates numerically the equilibrium allocations and prices and their dynamics by solving the ODE [\(45\)](#) and the associated decentralization.

5.1 Survival forces in the interior of the state space

The existence of nondegenerate long-run equilibria depends on the behavior of the relative patience $\nu^2(\theta) - \nu^1(\theta)$ in the neighborhood of the boundaries. The left panel of [Figure 3](#) displays three different cases. The solid line represents the low risk aversion case in which both attracting conditions from [Proposition 3.2](#) hold and each of the agents dominates with a strictly positive probability. The dashed line corresponds to a parameterization that is close to the CRRA case when only the survival condition for the rational agent 2 is satisfied (with CRRA preferences, the relative patience would be identically zero). Finally, a case for which both survival conditions hold is shown by the dash-dotted line. In this case, a nondegenerate long-run equilibrium exists.

The right panel of [Figure 3](#) depicts the impact of relative patience on the drift coefficient of the Pareto share process. The drift vanishes at the boundaries and the boundaries are unattainable (a reflection of the Inada conditions), but sufficiently large positive (negative) slopes at the left (right) boundaries assure the existence of a nondegenerate long-run distribution of the Pareto share.

In [Section 3.4](#), we have shown that the two essential components of the survival mechanism are

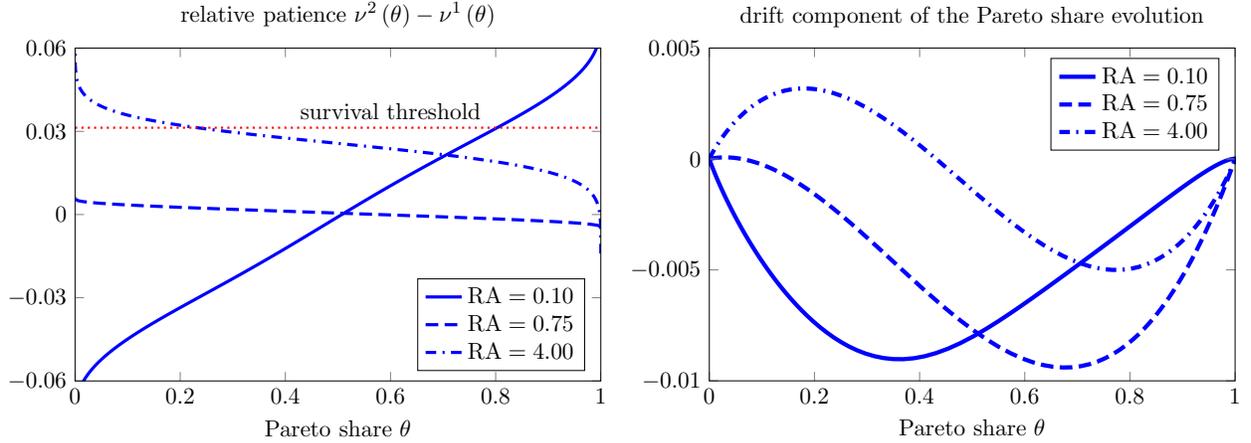


Figure 3: Relative patience $\nu^2(\theta) - \nu^1(\theta)$ (left panel) and the drift component of the Pareto share evolution $E_t[d\theta_t]/dt$ (right panel) as functions of the Pareto share θ . All models are parameterized by $u^1 = 0.25$, $u^2 = 0$, $\text{IES} = 1.5$, $\beta = 0.05$, $\mu_y = 0.02$, $\sigma_y = 0.02$, and differ in levels of risk aversion. The dotted horizontal line in the left panel represents the survival threshold $\frac{1}{2}(u^1)^2 - \frac{1}{2}(u^2)^2$ from Proposition 3.2.

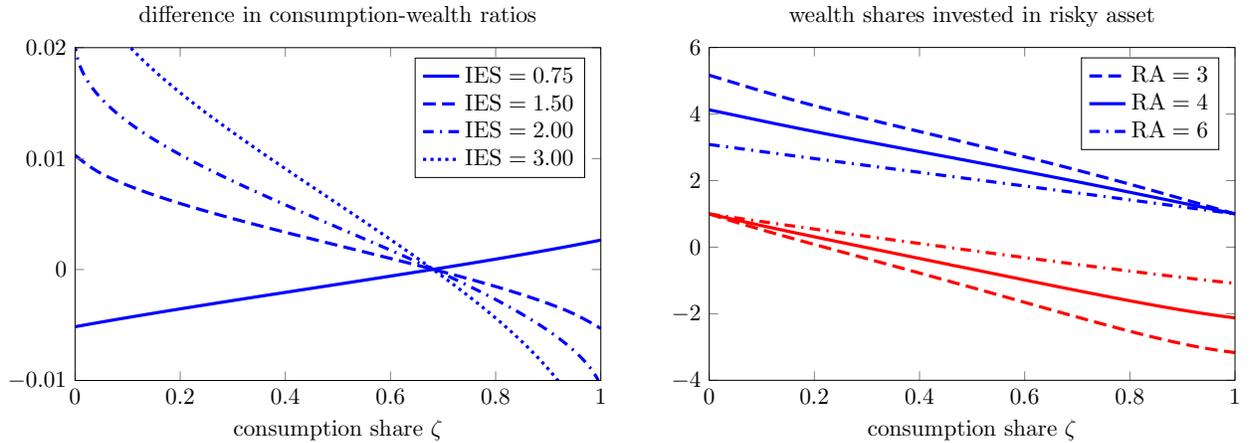


Figure 4: *Left panel*: Difference in consumption-wealth ratios $(\xi^2)^{-1} - (\xi^1)^{-1}$ as a function of the consumption share ζ of agent 1, plotted for different levels of intertemporal elasticity of substitution. The remaining parameters are $u^1 = 0.25$, $u^2 = 0$, $\text{RA} = 2$, $\beta = 0.05$, $\mu_y = 0.02$, $\sigma_y = 0.02$. *Right panel*: Wealth shares π^n of the two agents invested in the claim to aggregate endowment as functions of the consumption share ζ of agent 1, plotted for different levels of risk aversion. The remaining parameters are $u^1 = 0.25$, $u^2 = 0$, $\text{IES} = 1.5$, $\beta = 0.05$, $\mu_y = 0.02$, $\sigma_y = 0.02$, and individual curves correspond to different levels of risk aversion. Wealth share curves originating at 1 for $\zeta = 1$ ($\zeta = 0$) belong to agent 1 (agent 2).

the propensity to save and the portfolio allocation of the two agents. The left panel in Figure 4 displays the differences in the consumption-wealth ratios $[\xi^n(\theta)]^{-1}$ of the two agents, which are primarily driven by the intertemporal elasticity of substitution. For the case of $\text{IES} = 1$, the difference is exactly zero since each agent consumes a fraction β of her wealth per unit of time, and the saving channel is completely mute. Recall that a higher IES improves the survival chances of the agent who is relatively more optimistic about the return on her own wealth, as she is willing to

tilt her consumption profile more toward the future. In the graph, high levels of IES are conducive to survival of both agents — the difference in consumption rates $(\xi^2)^{-1} - (\xi^1)^{-1}$ is positive when agent 1 is negligible, and negative when agent 2 is negligible.

The portfolio allocation mechanism is depicted in the right panel of Figure 4. The share of wealth invested in the risky asset is primarily driven by the risk aversion parameter γ . The graph shows the optimistic agent's 1 wealth share π^1 invested in the risky asset in blue (upper three lines), and π^2 in red (lower three lines). As the consumption share of agent n converges to 1, her portfolio allocation $\pi^n \rightarrow 1$, reflecting the fact that the large agent's portfolio position must converge to the market portfolio.

A higher risk aversion (dash-dotted lines) limits the amount of leverage, and the portfolio positions are closer to one. This in turn limits the impact of speculative motives on market outcomes, and the role of the risk premium channel increases. Notice that some degree of speculative behavior is necessary for the survival of pessimistic agent — when risk aversion is high, a pessimistic agent does not form a large enough short stock position that would make her sufficiently optimistic about the return on her own wealth and outsave the rational agent when $\text{IES} > 1$.

5.2 Stationary distributions and evolution over time

Having solved for the equilibrium choices of the two agents, it is possible to numerically solve for the complete evolution of the economy. The top left graph of Figure 5 plots the densities $q(\zeta)$ for the stationary distribution of the consumption share ζ of the optimistic agent 1 in economies with an optimistic and a correct agent that differ in their level of risk aversion.

The graphs in Figure 2 revealed that increasing risk aversion improves the survival chances of the optimistic agent 1. The top left graph of Figure 5 provides a complementary perspective. All parameterizations in the figure generate economies where both agents survive. However, as we increase risk aversion, the distribution of the consumption share shifts in favor of the optimistic agent.

Several observations emerge. First, when both agents survive in the long run, the more incorrect agent can plausibly own and consume a substantial share of aggregate endowment. Second, the shape of the stationary densities for the consumption share indicates that long-run equilibria permit substantial variation over time in these consumption shares. Finally, the same survival channels that generate different types of long-run survival outcomes also act in favor of individual agents with each survival region.

In empirical applications, it is advantageous when the time-series of observable variables converge sufficiently quickly to their long-run distributions from any initial condition, so that data observed over finite horizons are a representative sample of the stationary distribution. For instance, Yan (2008) conducts numerical experiments under separable utility when one of the agents always vanishes, and shows that the rate of extinction can be very slow. Proposition 3.2 gives sufficient conditions for the existence of a unique stationary distribution for θ^1 but it does not say anything about the rate of convergence.

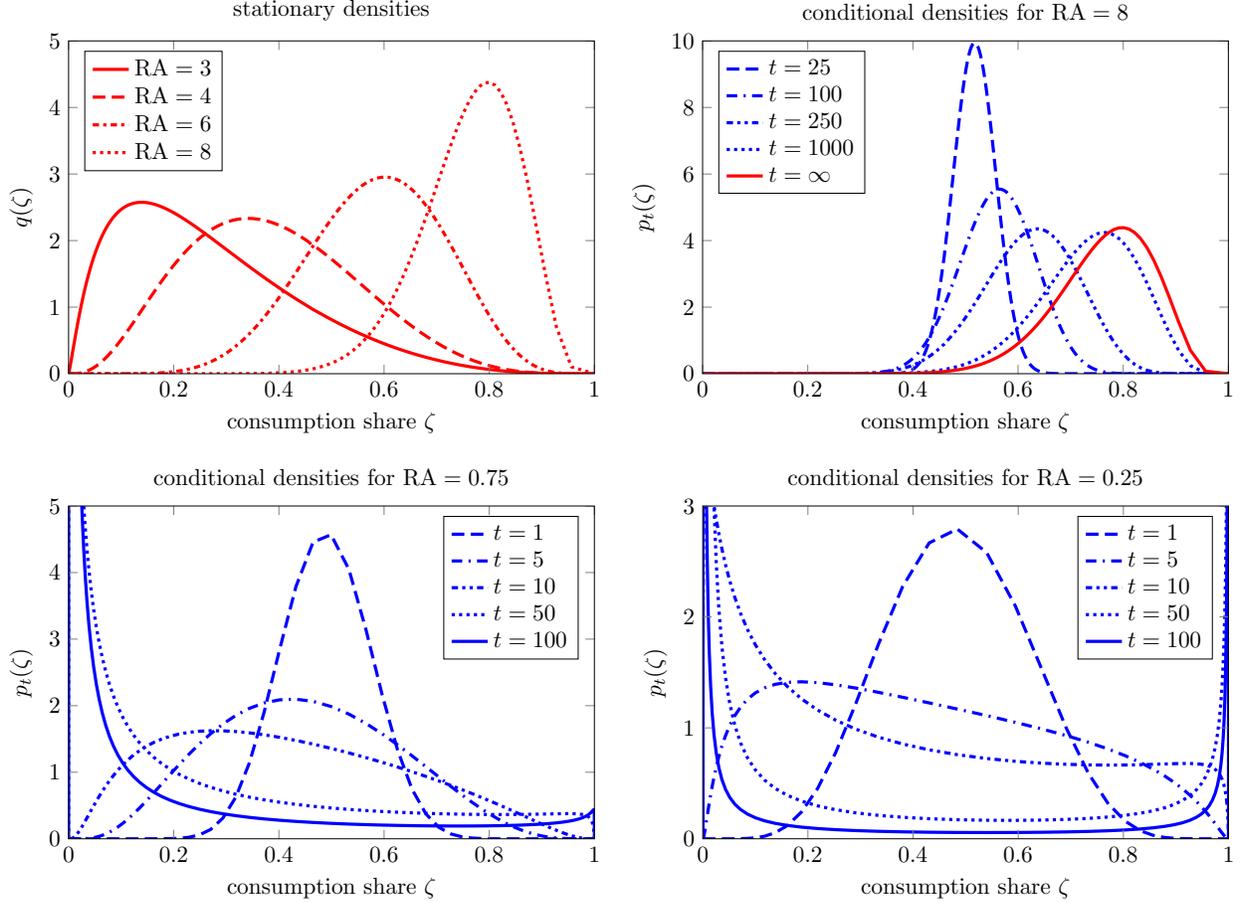


Figure 5: The top left panel depicts the stationary distributions for the consumption share ζ (θ) of the agent with optimistically distorted beliefs. All models are parameterized by $u^1 = 0.25$, $u^2 = 0$, IES = 1.5, $\beta = 0.05$, $\mu_y = 0.02$, $\sigma_y = 0.02$, and differ in levels of risk aversion, shown in the legend. The remaining three panels plot the distributions of ζ (θ_t) conditional on ζ (θ_0) = 0.5 for different time horizons t . In the top right panel (risk aversion = 8), the economy has a nondegenerate long-run distribution. In the bottom left panel (risk aversion = 0.75), the correct agent 2 dominates, and in the bottom right panel (risk aversion = 0.25), each agent dominates with a strictly positive probability.

I show in the online appendix that under the conditions from Proposition 3.2, convergence for the state variable θ occurs at an exponential rate, so that the process θ does not exhibit strong dependence properties. At the same time, the exponent in the rate calculation can still be small, and I therefore conduct a numerical simulation. The remaining three graphs in Figure 5 plot the conditional distribution of the consumption share ζ (θ_t) of the optimistic agent 1 conditional on ζ (θ_0) = 0.5 for different time horizons t . These are computed from the dynamics of the state variable θ in equation (10) by solving the associated Kolmogorov forward equation

$$\frac{\partial p_t^\theta(\theta)}{\partial t} + \frac{\partial}{\partial \theta} \left[\theta \mu_\theta(\theta) p_t^\theta(\theta) \right] - \frac{1}{2} \frac{\partial^2}{\partial \theta^2} \left[(\theta \sigma_\theta(\theta))^2 p_t^\theta(\theta) \right] = 0$$

for the conditional density $p_t^\theta(\theta)$ of θ with the initial condition $p_0^\theta(\theta) = \delta_{\theta_0}(\theta)$, where δ is the Dirac

delta function, and then transforming to obtain the conditional density for ζ

$$p_t(\zeta(\theta)) = p_t^\theta(\theta) \left[\frac{\partial \zeta}{\partial \theta}(\theta) \right]^{-1}.$$

The graphs show the evolution of the conditional distribution for three cases. In the top right graph, the conditional distribution converges to a nondegenerate long-run distribution and both agents survive. In the bottom left graph, the mass of the conditional distribution shifts to the left and agent 2 dominates. Finally, in the bottom right graph, the mass of the conditional distribution shifts out toward both boundaries, and either agent dominates with a strictly positive probability.

The speed of the evolution of the conditional distribution depends on the magnitude of the belief distortions and the level of risk aversion in the economy. When risk aversion is high, agents are not willing to engage in large bets on the realizations of the Brownian motions W , and wealth and consumption shares evolve only slowly. In the example in Figure 5, it takes roughly 2,500 periods until the density p_t is visually indistinguishable from the stationary density. As risk aversion decreases, and agents are willing to bet larger portions of their wealth, the evolution of the conditional density p_t speeds up.

While the evolution of the conditional density in Figure 5 may appear rather slow, the process can be accelerated substantially. One possible way is to increase the magnitude of the belief distortions but very large belief distortions may be rejected as empirically implausible.

A more fundamental argument relies on the appropriate interpretation of the modeled risk in this economy. In the model, the nature of risk is extremely simplistic, the agents can disagree only about the distribution of the aggregate shock. In reality, there are many other sources of aggregate or idiosyncratic risk about which the agents can disagree and write contracts on, and agents with heterogeneous beliefs would also find it optimal to introduce additional such betting devices, even if these are otherwise economically irrelevant. [Fedyk, Heyerdahl-Larsen, and Walden \(2013\)](#) show in an economy with CRRA preferences that if agents disagree about multiple sources of risk, the speed of extinction of the relatively more incorrect agent can be accelerated substantially. The same mechanism operates under recursive utility, increasing the magnitude of wealth fluctuations and the rate of convergence of $p_t(\zeta)$ to the stationary density $q(\zeta)$ when both agents survive in the long run.¹⁴ The main message arising from these considerations is that the speed of extinction or rate of convergence to the stationary distribution in stylized models with very few sources of risk should not be viewed as a strong quantitative test of the model.

¹⁴The online appendix provides an example with two imperfectly correlated Brownian motions. One concern from the perspective of the survival results may be that belief distortions about multiple sources of risk can be reinterpreted as one large belief distortion. This view is, with some qualifications, correct but the survival results show that agents can coexist in the long run even under very large belief distortions.

6 Methodology and literature overview

The modern approach in the market survival literature originates from the work of [De Long, Shleifer, Summers, and Waldmann \(1991\)](#), who study wealth accumulation in a partial equilibrium setup with exogenously specified returns and find that irrational noise traders can outgrow their rational counterparts and dominate the market. Similarly, [Blume and Easley \(1992\)](#) look at the survival problem from the vantage point of exogenously specified saving rules, albeit in a general equilibrium setting.¹⁵

Subsequent research has shown that taking into account general equilibrium effects and intertemporal optimization of agents endowed with separable preferences eliminates much of the support for survival of agents with incorrect beliefs that models with ad hoc price dynamics produce. [Sandroni \(2000\)](#) and [Blume and Easley \(2006\)](#) base their survival results on the evolution of relative entropy as a measure of disparity between subjective beliefs and the true probability distribution. In their models, aggregate endowment is bounded from above and away from zero. As a result, the local properties of the utility function are immaterial for survival. Controlling for pure time preference, the long-run fate of economic agents is determined solely by belief characteristics, and only agents whose beliefs are in a specific sense asymptotically ‘closest’ to the truth can survive.

With unbounded aggregate endowment, local properties of the utility function become an additional survival factor. Even if preferences are identical across agents, the local curvature of the utility function at low and high levels of consumption can be sufficiently different to outweigh the divergence in beliefs, and lead to the survival of agents with relatively more incorrect beliefs. [Kogan, Ross, Wang, and Westerfield \(2011\)](#) show that a sufficient condition to prevent this outcome is the boundedness of the relative risk aversion function, i.e., a condition on the preferences being uniformly ‘close’ to the homothetic CRRA case. On the other hand, in this paper, preferences are homothetic, which assures that the survival results are not driven by exogenous differences in the local properties of the utility functions.¹⁶

¹⁵Modeling of economies populated by agents endowed with heterogeneous beliefs constitutes a quickly growing branch of literature, and a thorough overview of the literature is beyond the scope of this paper. Here, I primarily focus on the intersection of this literature with the analysis of recursive nonseparable preferences. [Bhamra and Uppal \(2013\)](#) provide a more general survey that also focuses on asset pricing implications of belief and preference heterogeneity. See also the discussion of price impact by [Kogan, Ross, Wang, and Westerfield \(2011\)](#) and portfolio impact by [Cvitanic and Malamud \(2011\)](#).

I also omit the discussion of evolutionary literature which predominantly focuses on the analysis of the interaction between agents with exogenously specified portfolio rules and price dynamics. The survival mechanism in this paper critically hinges on the interaction of endogenous consumption-saving decision and portfolio allocation vis-à-vis general equilibrium prices driven by the dynamics of the wealth shares, and is thus only loosely related. See [Hommes \(2006\)](#) for a survey of the evolutionary literature, and [Evstigneev, Hens, and Schenk-Hoppé \(2006\)](#) for an analysis of portfolio rule selection.

¹⁶The survival literature also focuses on other forms of heterogeneity. [Yan \(2008\)](#) and [Muraviev \(2013\)](#) construct ‘survival indices’ that combine the contribution of belief distortions and preference parameters and show that only agents with the lowest survival index can survive.

Market incompleteness or asymmetric information may be other ways how to counteract the extinction of agents with incorrect beliefs, as long they are judiciously chosen to prevent agents to place incorrect bets, see, e.g., [Mailath and Sandroni \(2003\)](#), [Coury and Sciubba \(2012\)](#), [Cao \(2013a\)](#) or [Cogley, Sargent, and Tsyrennikov \(2013\)](#).

Importantly, survival analysis under separable preferences corresponds to analyzing a sequence of time- and state-indexed static problems that are only interlinked through the initial marginal utility of wealth, which is largely innocuous for the long-run characterization of the economy. The survival literature frequently exploits martingale methods to characterize the long-run divergence of subjective beliefs and marginal utilities of consumption.

Nonseparability of preferences breaks this straightforward link, and I therefore develop a different method that is more suitable for this environment. I analyze the survival mechanism in a two-agent, continuous-time endowment economy with complete markets and an aggregate endowment process modeled as a geometric Brownian motion. The continuous-time, Brownian information framework is not critical for the qualitative results but offers analytical tractability which allows sharp closed-form characterization of the results.

I utilize the planner's problem derived in [Dumas, Uppal, and Wang \(2000\)](#) and extend it to include heterogeneity in beliefs. The solution of the planner's problem involves endogenously determined processes that can be interpreted as stochastic Pareto weights. The analysis under separable preferences reflects the purely *intratemporal* tradeoff in the allocation of consumption vis-à-vis changes in the local curvature of the period utility function. The nonseparable nature of recursive preferences introduces an additional *intertemporal* component captured in the dynamics of the Pareto weights.

The analysis of market survival then corresponds to investigating the long-run behavior of scaled Pareto weights. I present tight sufficient conditions for the existence of nondegenerate long-run equilibria and for dominance and extinction. While the full model requires a numerical solution, I show that the behavior at the boundaries, which is essential for survival analysis, can be established analytically. I thus provide closed-form solutions for the regions of the parameter space in which the survival conditions are satisfied.

The method utilizes asymptotic properties of a differential equation for the planner's problem to characterize the asset price dynamics at the boundaries in the decentralized equilibrium. The resulting conditions from the planner's problem translate naturally into conditions on the relative logarithmic growth rates in agents' wealth.

The applicability of the derived solution method is not limited to fixed distortions. I discuss how to extend the procedure to include learning and robust preferences of [Anderson, Hansen, and Sargent \(2003\)](#). Explicit solutions of these problems are left for future work.

The approach based on the characterization of the behavior of the endogenously determined Pareto weights is closely linked to the literature on endogenous discounting, initiated by [Koopmans \(1960\)](#) and [Uzawa \(1968\)](#), and to models of heterogeneous agent economies under recursive preferences, studied by [Lucas and Stokey \(1984\)](#) and [Epstein \(1987\)](#) under certainty and by [Kan \(1995\)](#) under uncertainty. The survival conditions derived in this paper resemble a sufficient condition for the existence of a stable interior steady state in [Lucas and Stokey \(1984\)](#), called increasing marginal impatience. This condition postulates that agents discount future less as they become poorer. I show that my analysis crucially depends on a similar quantity that I call *relative patience*. The key difference lies in the determination of the two quantities. While [Lucas and Stokey](#) require

that the time preference exogenously encoded in the utility specification changes with the level of consumption, in this paper the variation in relative patience arises endogenously as a response to the equilibrium price dynamics driven by belief differences.

Anderson (2005) studies Pareto optimal allocations under heterogeneous recursive preferences in a discrete-time setup using similar methods but he does not consider survival under belief heterogeneity. Beker and Espino (2011) consider separable preferences under a richer set of beliefs. Mazoy (2005) discusses long-run consumption dynamics when agents differ in their IES. Colacito and Croce (2010) prove the existence of nondegenerate long-run equilibria in a two-good economy when agents are endowed with risk-sensitive preferences and differ in the preferences over the two goods. Guerdjikova and Sciubba (2010) study the interaction of expected-utility and smooth ambiguity averse agents. Branger, Dumitrescu, Ivanova, and Schlag (2011) analyze survival in long-run risk models with heterogeneous recursive preferences. Bhandari (2014) studies a problem where heterogeneous beliefs arise endogenously in a setup with robust preferences of Hansen and Sargent (2001).

7 Concluding remarks

Survival of agents with heterogeneous beliefs has been studied extensively under separable preferences. The main conclusion arising from the literature is a relatively robust argument in favor of the market selection hypothesis. Under complete markets and identical utility functions, a two-agent economy is dominated in the long run by the agent whose beliefs are closest to the true probability measure for a wide class of preferences and endowments. In particular, Kogan, Ross, Wang, and Westerfield (2011) show elegantly that this result holds, irrespective of the specification of the aggregate endowment process,¹⁷ as long as relative risk aversion is bounded.

This paper shows that the robust survival result is specific to the class of separable preferences. Under nonseparable recursive preferences of the Duffie–Epstein–Zin type, nondegenerate long-run equilibria in which both agents coexist arise for a broad set of plausible parameterizations when risk aversion is larger than the inverse of the intertemporal elasticity of substitution. It is equally easy to construct economies dominated by agents with relatively more incorrect beliefs.

The analysis reveals the important role played by the interaction of risk aversion with respect to intratemporal gambles that determines risk taking, and intertemporal elasticity of substitution that drives the consumption-saving decision. Critical for obtaining the survival results, and in particular the nondegenerate long-run equilibria, are the general equilibrium price effects generated by the wealth dynamics.

In particular, the paper shows the complex interaction between risk sharing and speculative motives of agents with heterogeneous beliefs, and their joint impact on the saving behavior. Speculative behavior by an agent with incorrect beliefs distorts her rationally optimal risk-return tradeoff. However, it can aid wealth accumulation if it implies a higher expected logarithmic return on the

¹⁷The survival results under separable utility thus also hold for ‘exotic’ endowment processes like the rare disaster framework in Chen, Joslin, and Tran (2012).

agent's portfolio. At the same time, a sufficiently high risk aversion is necessary to prevent excessive speculation that would lead to volatile portfolios with very low expected logarithmic returns. Finally, a higher *perceived* return implies a higher saving rate when IES is sufficiently high.

The survival results are obtained by extending the planner's problem formulation of [Dumas, Uppal, and Wang \(2000\)](#) to a setting with heterogeneous beliefs. Long-run survival of the agents is determined by the dynamics of a stochastic process for the Pareto share of one of the agents as this share becomes negligible. These dynamics can be characterized in closed form by studying the boundary behavior the solution of a Hamilton–Jacobi–Bellman equation associated with the planner's problem. I provide an existence proof of a classical solution of this equation, and of the equivalence with the planner's value function. Since this type of ODE arises in a wider class of recursive utility problems, these results can be utilized in a broader variety of economic applications.

I also provide in analytical form tight sufficient conditions that guarantee survival or extinction. These conditions can be interpreted as relative patience conditions similar to those in [Lucas and Stokey \(1984\)](#). An agent survives in the long run if her relative patience becomes sufficiently large as her wealth share vanishes. Contrary to [Lucas and Stokey \(1984\)](#), the dynamics of relative patience arises endogenously as an equilibrium outcome, and is not a direct property of agents' preferences. I also show that the survival conditions are equivalent to conditions on the limiting expected growth rates of the logarithm of individual wealth levels in a decentralized economy.

These results are obtained for a two-agent economy with an aggregate endowment process that is specified as a geometric Brownian motion, but the theoretical framework can also be utilized to derive an analog HJB equation for economies with more sophisticated Markov dynamics. In principle, the qualitative survival results should extend to a wider class of models with stable consumption growth dynamics, although the analysis of the existence of a stationary distribution for the Pareto share becomes more complicated in a multidimensional state space.

Importantly, the developed solution method is not limited to constant distortions and applies to a much wider class of belief dynamics, as well as preferences that are interpretable as deviations in beliefs. For instance, [Bhandari \(2014\)](#) uses the dynamics of Pareto weights to study a model where wealth dynamics interact with endogenous beliefs of agents concerned about model misspecification. Similarly, formulas for survival regions can be extended by incorporating heterogeneity in preferences, as in [Dumas, Uppal, and Wang \(2000\)](#).

The analysis in this paper focuses on the case of fixed belief distortions. Agents are firm believers in their probability models, and do not use new data to update their beliefs. This can be interpreted as the strongest form of incorrect beliefs, and a bias against survival of agents whose beliefs are initially incorrect. A natural question is to ask what happens when agents are allowed to learn. Learning can be incorporated into the current framework by introducing a law of motion that represents the Bayesian updating of the belief distortions u^n . These belief distortions become new state variables.

[Blume and Easley \(2006\)](#) provide a detailed analysis of the impact of Bayesian learning on survival under separable utility, and show that learning in general aids survival of agents who start with incorrect beliefs, by reducing their belief distortions. The message is much less clear

in the nonseparable preference case. For instance, Figure 2 shows that the survival region of a pessimistic agent can shrink if her belief distortion diminishes. Whether the pessimist can learn quickly enough so that her beliefs converge to rational expectations at a rate that allows survival depends on the complexity of the learning problem, as shown by Blume and Easley (2006). The limiting distribution of θ as $t \nearrow \infty$ for the case of nonseparable preferences thus remains an open question.

This paper shows that contrary to the separable preference case, long-run survival of agents with incorrect beliefs is a generic outcome under recursive preferences. The bad news for the market selection hypothesis can be interpreted as good news for models with heterogeneous agents. Models with agents who differ in preferences or beliefs often have degenerate long-run limits in which only one class of agents survives. This paper shows that coupling belief heterogeneity and recursive preferences with empirically plausible parameters leads to models in which the heterogeneity does not vanish over time.

Appendix

A Proof of Proposition 2.3

I prove the proposition through a sequence of lemmas. The proof builds on results from Fleming and Soner (2006), Pham (2009) and Strulovici and Szydlowski (2014). The framework differs, however, along important dimensions, in particular the endogenously determined discount rate and vanishing volatility at the boundaries, so that it requires a separate treatment.

In Section A.1 (Lemmas A.3 and A.6), I establish elementary properties of the value function, including the limiting values of the value function at the boundaries corresponding to one of the agents receiving a vanishing Pareto share in the planner's problem. In Section A.2, I formulate the corresponding Hamilton–Jacobi–Bellman equation.

Proving the existence and properties of the solution of this HJB equation is complicated by the fact that the volatility of the Pareto share process θ vanishes at the boundaries of the interval $\theta \in [0, 1]$. In Section A.3, I therefore formulate an auxiliary problem on the interval $[\varepsilon, 1 - \varepsilon]$. Lemma A.9 and Corollary A.10 prove the existence, uniqueness and differentiability of the solution to this problem. In Section A.4 (Corollary A.11), I extend the solution to the interval $[0, 1]$ through a limiting argument. Finally, in Section A.5 (Lemma A.13), I prove the usual verification theorem showing that the solution to the HJB equation corresponds to the value function.

The planner's problem will be well defined when the following restriction on the parameters holds.

Assumption A.1 *The parameters in the model satisfy the restrictions*

$$\beta > \max_n \rho \left(\mu_y + u^n \sigma_y - \frac{1}{2} (1 - \gamma) \sigma_y^2 \right), \quad (18)$$

$$\beta > \max_n \rho \left(\mu_y + u^{\sim n} \sigma_y - \frac{1}{2} (1 - \gamma) \sigma_y^2 \right) + \frac{\rho}{1 - \rho} \left[(u^n - u^{\sim n}) \sigma_y + \frac{1}{2} \frac{(u^n - u^{\sim n})^2}{1 - \gamma} \right] \quad (19)$$

where $\sim n$ is the index of the agent other than n .

The first restriction is sufficient for the continuation values in the homogeneous economies to be well-defined. The second restriction, which may be, depending on the parameterization, somewhat tighter, is a sufficient condition assuring that the wealth-consumption ratio is asymptotically well-behaved in the survival proofs when the agent becomes infinitesimally small. Observe that both conditions are restrictions on the time-preference parameter of the agents and can always be jointly satisfied by making the agents sufficiently impatient. Since the survival results do not depend on β , Assumption A.1 does not introduce substantial restrictions for the analysis of the problem.

Throughout the proof, I impose a restriction (Assumption A.4) on the underlying discount rate processes. Later in this appendix I verify that this assumption holds for the optimal discount rate process on every interval $[\varepsilon, 1 - \varepsilon]$. I postpone the verification of this restriction at the boundaries as $\varepsilon \searrow 0$ to Appendix B where I characterize the boundary behavior of the economy in more detail and explicitly calculate the limit.

A.1 Properties of the value function

We start with definitions and some elementary properties of the value function.

Definition A.2 The planner's control $a = (C^1, C^2, \nu^1, \nu^2)$ is admissible if $C^1 + C^2 = Y$ and, for $n \in \{1, 2\}$, the Pareto weight processes $\bar{\lambda}^n$ given by

$$\frac{d\bar{\lambda}_t^n}{\bar{\lambda}_t^n} = -\nu_t^n dt + u^n dW_t$$

have a unique strong solution and

$$V_t^n(C^n, \nu^n) \doteq E_t \left[\int_t^\infty \frac{\bar{\lambda}_s^n}{\bar{\lambda}_t^n} |F(C_s^n, \nu_s^n)| ds \right] < +\infty.$$

The set of admissible controls is denoted \mathcal{A} .

In what follows, we will use homogeneity properties and static optimization for the consumption choice to simplify the planner's control to $a = (\nu^1, \nu^2)$. To simplify notation, I will use $a \in \mathcal{A}$ also for the reduced set of admissible controls.

Lemma A.3 The value function (8) satisfies $J(\bar{\lambda}_t, Y_t) = (\bar{\lambda}_t^1 + \bar{\lambda}_t^2) Y_t^\gamma \tilde{J}(\theta_t)$ where $\tilde{J}(\theta_t)$ is a bounded function of the Pareto share $\theta_t = \bar{\lambda}_t^1 / (\bar{\lambda}_t^1 + \bar{\lambda}_t^2)$.

Proof. Define $\zeta_t = C_t^1 / Y_t$ as the consumption share of agent 1. Then (8) can be written as

$$\begin{aligned} J(\bar{\lambda}_t, Y_t) &= (\bar{\lambda}_t^1 + \bar{\lambda}_t^2) Y_t^\gamma \sup_{a \in \mathcal{A}} E_t \left[\theta_t \int_t^\infty \frac{\bar{\lambda}_s^1}{\bar{\lambda}_t^1} \left(\frac{Y_s}{Y_t} \right)^\gamma F(\zeta_s, \nu_s^1) ds \right. \\ &\quad \left. + (1 - \theta_t) \int_t^\infty \frac{\bar{\lambda}_s^2}{\bar{\lambda}_t^2} \left(\frac{Y_s}{Y_t} \right)^\gamma F(1 - \zeta_s, \nu_s^2) ds \right] \\ &\doteq (\bar{\lambda}_t^1 + \bar{\lambda}_t^2) Y_t^\gamma \tilde{J}(\theta_t) \end{aligned}$$

because the ratios $\bar{\lambda}_s^n / \bar{\lambda}_t^n$ and Y_s / Y_t do not depend on $\bar{\lambda}_t^n$ and Y_t . Here, $a = (\zeta, \nu^1, \nu^2) \in \mathcal{A}$ is an equivalent set of controls as in Definition A.2 because $C^1 = \zeta Y$ and $C^2 = (1 - \zeta) Y$. Further, because the individual value functions are increasing in consumption, we have

$$J(\bar{\lambda}_t, Y_t) = \sup_{C^1 + C^2 = Y} \bar{\lambda}_t^1 V_t^1(C^1) + \bar{\lambda}_t^2 V_t^2(C^2) \leq \bar{\lambda}_t^1 V_t^1(Y) + \bar{\lambda}_t^2 V_t^2(Y).$$

The individual value functions $V_t^n(Y)$ have a closed form solution for the iid growth process Y , given by $V_t^n(Y) = Y_t^\gamma \bar{V}^n$ where

$$\bar{V}^n = \frac{1}{\gamma} \left(\beta^{-1} \left[\beta - \rho \left(\mu_y + u^n \sigma_y - \frac{1}{2} (1 - \gamma) \sigma_y^2 \right) \right] \right)^{-\frac{\gamma}{\rho}} \quad (20)$$

with the associated optimal discount rate

$$\bar{\nu}^n = \frac{\beta}{\rho} \left(\gamma + (\rho - \gamma) (\gamma \bar{V}^n)^{-\frac{\rho}{\gamma}} \right) = \beta + (\gamma - \rho) \left(\mu_y + u^n \sigma_y - \frac{1}{2} (1 - \gamma) \sigma_y^2 \right). \quad (21)$$

Here, $V_t^n(Y)$ and $\bar{\nu}^n$ are the value function and discount rate in a homogeneous economy populated only by agent n . These objects are well-defined when the first condition of Assumption A.1 are satisfied and satisfy the same homogeneity properties as the planner's value function. Therefore, $\tilde{J}(\theta) \leq \theta \bar{V}^1 + (1 - \theta) \bar{V}^2$.

Finally, consider a suboptimal policy consisting of fixing, given an initial θ_t , the consumption shares $(\bar{\zeta}, 1 - \bar{\zeta})$ for the two agents for the whole future. Since individual consumption processes now exhibit iid

growth, the optimal choice of the discount rate will satisfy $\nu_t^n = \bar{\nu}^n$. We thus have

$$\begin{aligned} J(\bar{\lambda}_t, Y_t) &\geq \sup_{\bar{\zeta}} [\bar{\lambda}_t^1 V_t^1 (\bar{\zeta} Y_t) + \bar{\lambda}_t^2 V_t^2 ((1 - \bar{\zeta}) Y_t)] = \\ &= (\bar{\lambda}_t^1 + \bar{\lambda}_t^2) Y_t^\gamma \sup_{\bar{\zeta}} [\theta_t \bar{\zeta}^\gamma \bar{V}^1 + (1 - \theta_t) (1 - \bar{\zeta})^\gamma \bar{V}^2] \end{aligned}$$

and thus

$$\tilde{J}(\theta_t) \geq \sup_{\bar{\zeta}} \theta_t \bar{\zeta}^\gamma \bar{V}^1 + (1 - \theta_t) (1 - \bar{\zeta})^\gamma \bar{V}^2.$$

The first-order condition with respect to $\bar{\zeta}$ yields

$$\bar{\zeta}(\theta_t) = \frac{[\theta_t \gamma \bar{V}^1]^{\frac{1}{1-\gamma}}}{[\theta_t \gamma \bar{V}^1]^{\frac{1}{1-\gamma}} + [(1 - \theta_t) \gamma \bar{V}^2]^{\frac{1}{1-\gamma}}}. \quad (22)$$

Substituting this result back, we have

$$J(\bar{\lambda}_t, Y_t) \geq (\bar{\lambda}_t^1 + \bar{\lambda}_t^2) Y_t^\gamma \frac{1}{\gamma} \left[[\theta_t \gamma \bar{V}^1]^{\frac{1}{1-\gamma}} + [(1 - \theta_t) \gamma \bar{V}^2]^{\frac{1}{1-\gamma}} \right]^{1-\gamma} \doteq \quad (23)$$

$$\doteq (\bar{\lambda}_t^1 + \bar{\lambda}_t^2) Y_t^\gamma \tilde{J}(\theta_t). \quad (24)$$

which establishes the lower bound on $\tilde{J}(\theta_t)$. ■

Denote $\nu = (\nu^1, \nu^2)$. Notice that given ν , the choice of ζ only involves static optimization. Taking the first-order condition with respect to ζ , solving for ζ and substituting it back yields

$$\begin{aligned} h^0(\nu, \theta) &\doteq \max_{\zeta} \theta F(\zeta, \nu^1) + (1 - \theta) F(1 - \zeta, \nu^2) = \\ &= \frac{\beta}{\gamma} \left[\left(\theta \left(\frac{\gamma - \rho \frac{\nu^1}{\beta}}{\gamma - \rho} \right)^{1 - \frac{\gamma}{\rho}} \right)^{\frac{1}{1-\gamma}} + \left((1 - \theta) \left(\frac{\gamma - \rho \frac{\nu^2}{\beta}}{\gamma - \rho} \right)^{1 - \frac{\gamma}{\rho}} \right)^{\frac{1}{1-\gamma}} \right]^{1-\gamma}. \end{aligned}$$

Instead of $a = (\zeta(\nu), \nu^1, \nu^2)$, we will focus on the admissible controls $a = (\nu^1, \nu^2)$ and planner's indirect utility flow $h^0(\nu, \theta)$. The structure of the problem implies that the optimal Markov control of the planner is of the form $\nu_t^n = \nu^n(\theta_t)$, $n \in \{1, 2\}$.

Assumption A.4 *The discount rates $\nu_t^n = \nu^n(\theta_t)$, $n \in \{1, 2\}$ are bounded functions of θ on $[0, 1]$ that are Lipschitz continuous, and there $\exists \varepsilon > 0$ such that,*

$$\frac{\gamma - \rho \frac{\nu_t^n}{\beta}}{\gamma - \rho} > \varepsilon.$$

The results in Appendix B will show that the bounds imposed in Assumption A.4 correspond to the assumption that agents' wealth-consumption ratios are bounded and bounded away from zero. The subsequent characterization of the optimal control implies that the optimal policy will necessarily satisfy these assumptions.

Lemma A.5 *If a is admissible and satisfies Assumption A.4, then*

$$E_t \left[\int_t^\infty \frac{\bar{\lambda}_s^n}{\bar{\lambda}_t^n} \left(\frac{Y_s}{Y_t} \right)^\gamma ds \right] < +\infty, \quad n \in \{1, 2\} \quad (25)$$

and

$$\lim_{\tau \rightarrow \infty} E_t \left[\frac{\bar{\lambda}_\tau^n}{\bar{\lambda}_t^n} \left(\frac{Y_\tau}{Y_t} \right)^\gamma \right] = 0, \quad n \in \{1, 2\}. \quad (26)$$

Proof. An introspection of the function $h^0(\nu, \theta)$ implies that under Assumption A.4, this function is bounded away from zero, and thus there exists $M > 0$ such that $|h^0(\nu, \theta)| > M$. We have

$$\begin{aligned} J(\bar{\lambda}_t, Y_t) &= \sup_{(C^1, C^2, \nu^1, \nu^2)} E_t \left[\int_t^\infty [\bar{\lambda}_s^1 F(C_s^1, \nu_s^1) + \bar{\lambda}_s^2 F(C_s^2, \nu_s^2)] ds \right] = \\ &= Y_t^\gamma \sup_{(\nu^1, \nu^2)} E_t \left[\int_t^\infty (\bar{\lambda}_s^1 + \bar{\lambda}_s^2) \left(\frac{Y_s}{Y_t} \right)^\gamma h^0(\nu_s, \theta_s) ds \right]. \end{aligned}$$

Consider an arbitrary pair of processes (ν^1, ν^2) and the associated optimal consumption share ζ such that $a = (\zeta, \nu^1, \nu^2)$ is admissible. Then

$$\begin{aligned} +\infty &> E_t \left[\int_t^\infty [\bar{\lambda}_s^1 |F(C_s^1, \nu_s^1)| + \bar{\lambda}_s^2 |F(C_s^2, \nu_s^2)|] ds \right] \geq \\ &\geq Y_t^\gamma E_t \left[\int_t^\infty (\bar{\lambda}_s^1 + \bar{\lambda}_s^2) \left(\frac{Y_s}{Y_t} \right)^\gamma |h^0(\nu_s, \theta_s)| ds \right] \geq MY_t^\gamma E_t \left[\int_t^\infty (\bar{\lambda}_s^1 + \bar{\lambda}_s^2) \left(\frac{Y_s}{Y_t} \right)^\gamma ds \right] \end{aligned}$$

which proves (25). The limit in (26) is a direct consequence. ■

The following lemma characterizes the limits of $\tilde{J}(\theta_t)$ at the boundaries.

Lemma A.6 *The planner's value function $J(\bar{\lambda}_t, Y_t)$ can be continuously extended at the boundaries as $\bar{\lambda}_t^1 \searrow 0$ or $\bar{\lambda}_t^2 \searrow 0$ by the continuation values calculated for the homogeneous agent economies, i.e., for $\bar{\lambda}_t^2 > 0$*

$$J(0, \bar{\lambda}_t^2, Y_t) \doteq \lim_{\bar{\lambda}_t^1 \searrow 0} J(\bar{\lambda}_t^1, \bar{\lambda}_t^2, Y_t) = \bar{\lambda}_t^2 V_t^2(Y). \quad (27)$$

Further, the optimal choice of consumption $C_u^1(\bar{\lambda}_t^1, \bar{\lambda}_t^2, Y_t)$ for agent 1 and time $u \geq t$ satisfies

$$\forall u \geq t : \lim_{\bar{\lambda}_t^1 \searrow 0} C_u^1(\bar{\lambda}_t^1, \bar{\lambda}_t^2, Y_t) = 0 \quad P\text{-a.s.} \quad (28)$$

The case $\bar{\lambda}_t^1 \searrow 0$ is symmetric.

Notice that the convergence in (28) is not uniform in u . This is important — a vanishing Pareto weight on agent 1 leads to a vanishing consumption level for every given time $u \geq t$ but this argument does not prevent the possibility that for a given, arbitrarily small Pareto weight, the agent's consumption may recover in the future. A direct consequence of result (27) is

$$\lim_{\theta \searrow 0} \tilde{J}(\theta) = \bar{V}^1 \quad \lim_{\theta \nearrow 1} \tilde{J}(\theta) = \bar{V}^2. \quad (29)$$

Proof of Lemma A.6. Consider the case $\bar{\lambda}_t^1 \searrow 0$. Given optimal consumption streams $C^n(\bar{\lambda}_t^1, \bar{\lambda}_t^2, Y_t)$, we have

$$J(\bar{\lambda}_t^1, \bar{\lambda}_t^2, Y_t) = \alpha^1 V_t^1(C^1(\bar{\lambda}_t^1, \bar{\lambda}_t^2, Y_t)) + \alpha^2 V_t^2(C^2(\bar{\lambda}_t^1, \bar{\lambda}_t^2, Y_t)) \quad (30)$$

and since $V_t^1(C^1(\bar{\lambda}_t^1, \bar{\lambda}_t^2, Y_t))$ is bounded from above as a function of $\bar{\lambda}_t^1$ by $V_t^2(Y)$, it follows that

$$\bar{\lambda}_t^1 V_t^1(C^1(\bar{\lambda}_t^1, \bar{\lambda}_t^2, Y_t)) \xrightarrow{\bar{\lambda}_t^1 \searrow 0} \underline{V}^1 \leq 0$$

and thus $J(\bar{\lambda}_t^1, \bar{\lambda}_t^2, Y_t) \leq \lim_{\bar{\lambda}_t^1 \searrow 0} \bar{\lambda}_t^2 V_t^2 (C^2(\bar{\lambda}_t^1, \bar{\lambda}_t^2, Y_t)) \leq \bar{\lambda}_t^2 V_t^2(Y)$.

On the contrary, assume suboptimal policies $\zeta_u = \bar{\zeta}(\theta_t)$ for $u \geq t$ where $\bar{\zeta}(\theta_t)$ is given by (22). Taking the limit in (23) and noticing that $\bar{\lambda}_t^1 \searrow 0$ for a given $\bar{\lambda}_t^2 > 0$ implies $\theta_t \searrow 0$ yields

$$\lim_{\bar{\lambda}_t^1 \searrow 0} J(\bar{\lambda}_t^1, \bar{\lambda}_t^2, Y_t) \geq \bar{\lambda}_t^2 Y_t^\gamma \bar{V}^2 = \bar{\lambda}_t^2 V_t^2(Y).$$

Combining the two inequalities yields (27). The limit in (28) is a direct consequence. ■

A.2 The Hamilton–Jacobi–Bellman equation

Denoting $\bar{\lambda} = (\bar{\lambda}^1, \bar{\lambda}^2)'$ and $u = (u^1, u^2)'$, the state vector is $Z = (\bar{\lambda}', Y)'$, which suggests that the planner's problem (8) leads to the Hamilton–Jacobi–Bellman equation for $J(\bar{\lambda}, Y)$,

$$0 = \sup_{(C^1, C^2, \nu^1, \nu^2)} \sum_{n=1}^2 \bar{\lambda}^n [F(C^n, \nu^n) - J_{\bar{\lambda}^n} \nu^n] + J_y \mu_y Y + \frac{1}{2} \text{tr}(J_{zz} \Sigma), \quad (31)$$

where

$$\Sigma = \begin{pmatrix} (\text{diag}(\bar{\lambda}) u) (\text{diag}(\bar{\lambda}) u)' & (\text{diag}(\bar{\lambda}) u) \sigma_y Y \\ \sigma_y Y (\text{diag}(\bar{\lambda}) u)' & \sigma_y^2 Y^2 \end{pmatrix}$$

and $\text{diag}(\bar{\lambda})$ is a 2×2 diagonal matrix with elements of $\bar{\lambda}$ on the main diagonal. Using the conjecture $J(\bar{\lambda}, Y) = (\bar{\lambda}^1 + \bar{\lambda}^2) Y^\gamma \tilde{J}(\theta)$ reduces the problem to the ordinary differential equation for $\tilde{J}(\theta)$ given by (9) with boundary conditions $\tilde{J}(0) = \bar{V}^2$ and $\tilde{J}(1) = \bar{V}^1$, as determined by Lemma A.6.

Denote $\nu = (\nu^1, \nu^2)$. Taking the first-order condition with respect to ζ , solving for ζ and substituting it back yields

$$\begin{aligned} h^0(\nu, \theta) &\doteq \max_{\zeta} \theta F(\zeta, \nu^1) + (1 - \theta) F(1 - \zeta, \nu^2) = \\ &= \frac{\beta}{\gamma} \left[\left(\theta \left(\frac{\gamma - \rho \frac{\nu^1}{\beta}}{\gamma - \rho} \right)^{1 - \frac{\gamma}{\rho}} \right)^{\frac{1}{1-\gamma}} + \left((1 - \theta) \left(\frac{\gamma - \rho \frac{\nu^2}{\beta}}{\gamma - \rho} \right)^{1 - \frac{\gamma}{\rho}} \right)^{\frac{1}{1-\gamma}} \right]^{1-\gamma}. \end{aligned}$$

We can further define

$$\begin{aligned} h^1(\nu, \theta) &\doteq -\theta \nu^1 - (1 - \theta) \nu^2 + (\theta u^1 + (1 - \theta) u^2) \gamma \sigma_y + \gamma \mu_y + \frac{1}{2} \gamma (\gamma - 1) \sigma_y^2 \\ h^2(\nu, \theta) &\doteq \theta (1 - \theta) [\nu^2 - \nu^1 + (u^1 - u^2) \gamma \sigma_y] \\ h^3(\theta) &\doteq \frac{1}{2} \theta^2 (1 - \theta)^2 (u^1 - u^2)^2 \end{aligned}$$

Then (9) can be written as

$$0 = \sup_{\nu} h^0(\nu, \theta) + h^1(\nu, \theta) \tilde{J}'(\theta) + h^2(\nu, \theta) \tilde{J}''(\theta) + h^3(\theta) \tilde{J}'''(\theta)$$

with boundary conditions $\tilde{J}(0) = \bar{V}^2$ and $\tilde{J}(1) = \bar{V}^1$.

The goal is to show that there exists a unique twice continuously differentiable solution of this equation that corresponds to the value function. Observe that under Assumption A.4, all functions h^j are bounded and Lipschitz.

Remark A.7 The maximization over (ν^1, ν^2) in the HJB equation (31) can be solved separately. Under the optimal discount rate process ν^n for agent n ,

$$f(C^n, J_{\bar{\lambda}^n}) \doteq \sup_{\nu^n} F(C^n, \nu^n) - J_{\bar{\lambda}^n} \nu^n = \frac{\beta}{\rho} \left[(C^n)^\rho (\gamma J_{\bar{\lambda}^n})^{1-\frac{\rho}{\gamma}} - \gamma J_{\bar{\lambda}^n} \right]. \quad (32)$$

The function f is the aggregator in the stochastic differential utility representation of recursive preferences postulated by [Duffie and Epstein \(1992b\)](#). The online appendix gives more detail on this relationship. Optimal consumption share $\zeta \doteq C^1/Y$ of agent 1 is given by the first-order conditions in the consumption allocation

$$\zeta = \frac{(\bar{\lambda}^1)^{\frac{1}{1-\rho}} (\gamma J_{\bar{\lambda}^1})^{\frac{1-\rho/\gamma}{1-\rho}}}{\sum_{k=1}^2 (\bar{\lambda}^k)^{\frac{1}{1-\rho}} (\gamma J_{\bar{\lambda}^k})^{\frac{1-\rho/\gamma}{1-\rho}}} = \frac{\theta^{\frac{1}{1-\rho}} (\gamma \tilde{J}^1(\theta))^{\frac{1-\rho/\gamma}{1-\rho}}}{\theta^{\frac{1}{1-\rho}} (\gamma \tilde{J}^1(\theta))^{\frac{1-\rho/\gamma}{1-\rho}} + (1-\theta)^{\frac{1}{1-\rho}} (\gamma \tilde{J}^2(\theta))^{\frac{1-\rho/\gamma}{1-\rho}}}, \quad (33)$$

where $J_{\bar{\lambda}^n} = Y \gamma \tilde{J}^n(\theta)$ are the individual agents' continuation values under the optimal consumption allocation, with $\tilde{J}^n(\theta)$ defined as

$$\begin{aligned} \tilde{J}^1(\theta) &= \tilde{J}(\theta) + (1-\theta) \tilde{J}_\theta(\theta) \\ \tilde{J}^2(\theta) &= \tilde{J}(\theta) - \theta \tilde{J}_\theta(\theta). \end{aligned} \quad (34)$$

These are obtained through the envelope condition on the planner's value function (8).

Once the solution of the HJB equation is characterized, we will prove that it corresponds to the value function. In order to do that, the stochastic process for θ_t needs to be well defined. An application of Itô's lemma to $\theta_t = \bar{\lambda}_t^1 / (\bar{\lambda}_t^1 + \bar{\lambda}_t^2)$ yields

$$\begin{aligned} d\theta_t &= \theta_t (1-\theta_t) [\nu_t^2 - \nu_t^1 + (\theta_t u^1 + (1-\theta_t) u^2) (u^2 - u^1)] dt + \\ &\quad + \theta_t (1-\theta_t) (u^1 - u^2) dW_t. \end{aligned} \quad (35)$$

Lemma A.8 Under Assumption A.4, the stochastic differential equation (35) has a unique strong solution.

Proof. Under Assumption A.4, the drift and volatility coefficients in (35) are Lipschitz and bounded, so that a unique strong solution exists (see, e.g., [Pham \(2009\)](#), Theorem 1.3.15). ■

A.3 An auxiliary problem

Consider the following auxiliary problem with suboptimal control by the planner. Fix $\varepsilon \in (0, \frac{1}{2}]$. When $\theta_t = \bar{\lambda}_t^1 / (\bar{\lambda}_t^1 + \bar{\lambda}_t^2) \in (\varepsilon, 1-\varepsilon)$, then the planner exercises optimal control locally, given her value function. When θ_t hits the boundary ε (a symmetric argument holds for $1-\varepsilon$), the planner is restricted to (optimally) fix consumption shares of the two agents to $(\bar{\zeta}(\varepsilon), 1-\bar{\zeta}(\varepsilon))$ where $\bar{\zeta}(\varepsilon)$ is given by (22) and keep them fixed forever. Formally, the auxiliary problem for a given ε is given by

$$J^\varepsilon(\bar{\lambda}_t, Y_t) = \sup_{(C^1, C^2, \nu^1, \nu^2)} \sum_{n=1}^2 E_0 \left[\int_0^{\tau^\varepsilon} \bar{\lambda}_t^n F(C_t^n, \nu_t^n) dt + \bar{\lambda}_{\tau^\varepsilon}^n J^\varepsilon(\bar{\lambda}_{\tau^\varepsilon}, Y_{\tau^\varepsilon}) \right] \quad (36)$$

where τ^ε is a stopping time given by $\tau^\varepsilon = \min \{t : \theta_t \notin (\varepsilon, 1 - \varepsilon)\}$ and the continuation value at the stopping time is established in (22)–(24) as

$$J^\varepsilon(\bar{\lambda}_{\tau^\varepsilon}, Y_{\tau^\varepsilon}) = (\bar{\lambda}_{\tau^\varepsilon}^1 + \bar{\lambda}_{\tau^\varepsilon}^2) Y_{\tau^\varepsilon}^\gamma \bar{J}(\theta_{\tau^\varepsilon})$$

with $\bar{J}(\theta_{\tau^\varepsilon})$ given by (23) and $\theta_{\tau^\varepsilon} \in \{\varepsilon, 1 - \varepsilon\}$. It is also straightforward to verify that the value function of the auxiliary problem satisfies the homotheticity property and

$$J^\varepsilon(\bar{\lambda}_t, Y_t) = (\bar{\lambda}_{\tau^\varepsilon}^1 + \bar{\lambda}_{\tau^\varepsilon}^2) Y_t^\gamma \tilde{J}^\varepsilon(\theta_t) \quad (37)$$

where $\tilde{J}^\varepsilon(\theta_t)$ is analogous to $\tilde{J}(\theta_t)$ from Lemma A.3. The solution can be continuously extended for $\theta \in [0, 1] \setminus [\varepsilon, 1 - \varepsilon]$ through $\tilde{J}^\varepsilon(\theta) = \bar{J}(\theta)$.

Lemma A.9 *Let $\nu(\theta) = (\nu^1(\theta), \nu^2(\theta))$ be a given control that satisfies Assumption A.4. Then the boundary value problem*

$$0 = h^0(\nu, \theta) + h^1(\nu, \theta) \tilde{J}^\varepsilon(\theta) + h^2(\nu, \theta) \tilde{J}_\theta^\varepsilon(\theta) + h^3(\theta) \tilde{J}_{\theta\theta}^\varepsilon(\theta) \quad (38)$$

with boundary conditions $\tilde{J}^\varepsilon(\varepsilon) = \bar{J}(\varepsilon)$ and $\tilde{J}^\varepsilon(1 - \varepsilon) = \bar{J}(1 - \varepsilon)$ has a unique twice continuously differentiable solution on $[\varepsilon, 1 - \varepsilon]$.

Proof. The proof is a modification of the shooting algorithm argument from Strulovici and Szydlowski (2014). Consider the initial value problem given by the differential equation (38) together with initial conditions $\tilde{J}^\varepsilon(\varepsilon) = \bar{J}(\varepsilon)$ and $\tilde{J}_\theta^\varepsilon(\varepsilon) = y$. Given that the functions $h^j(\nu(\theta), \theta)$ are bounded and satisfy Lipschitz continuity, it is well known (see, e.g., the references in Strulovici and Szydlowski (2014), Appendix B) that the initial value problem has a unique, twice continuously differentiable solution that is uniformly continuous in y . The goal is to show that we can find a unique value of y such that $\tilde{J}^\varepsilon(1 - \varepsilon) = \bar{J}(1 - \varepsilon)$, and thus the boundary value problem has a unique solution.

Define $K(\theta) = \tilde{J}_\theta^\varepsilon(\theta)$ and $k^j(\theta) = -h^j(\nu(\theta), \theta) / h^3(\theta)$ for $j = 0, 1, 2$. Then (38) can be integrated to

$$\begin{aligned} K(s) &= y + \int_\varepsilon^s \left[k^0(r) + k^1(r) \tilde{J}^\varepsilon(r) + k^2(r) K(r) \right] dr \\ \tilde{J}^\varepsilon(\theta) &= \bar{J}(\varepsilon) + \int_\varepsilon^\theta K(s) ds. \end{aligned} \quad (39)$$

We are interested in the sensitivity of $\tilde{J}^\varepsilon(1 - \varepsilon)$ with respect to changes in the initial condition $\tilde{J}_\theta^\varepsilon(\varepsilon) = y$. We have

$$\begin{aligned} \frac{d}{dy} K(s) &= 1 + \int_\varepsilon^s \left[k^1(r) \frac{d}{dy} \tilde{J}^\varepsilon(r) + k^2(r) \frac{d}{dy} K(r) \right] dr = \\ &= 1 + \int_\varepsilon^s \left[k^1(r) \int_\varepsilon^r \frac{d}{dy} K(p) dp + k^2(r) \frac{d}{dy} K(r) \right] dr \\ &= 1 + \int_\varepsilon^s \left[\left(\int_r^s k^1(r') dr' + k^2(r) \right) \frac{d}{dy} K(r) \right] dr \end{aligned}$$

This is an integral form of a differential equation for $\frac{d}{dy} K(s)$ in s with $\frac{d}{dy} K(0) = 1$. Given the term $\int_r^s k^1(r') dr' + k^2(r)$ is bounded, take an $M > 0$ such that

$$\left| \int_r^s k^1(r') dr' + k^2(r) \right| < M.$$

Then

$$e^{-M(s-\varepsilon)} \leq \frac{d}{dy} K(s) \leq e^{M(s-\varepsilon)}$$

and therefore, using (39),

$$\frac{1}{M} \left[1 - e^{-M(1-2\varepsilon)} \right] \leq \frac{d}{dy} \tilde{J}^\varepsilon(1-\varepsilon) \equiv \int_\varepsilon^{1-\varepsilon} \frac{d}{dy} K(s) ds \leq \frac{1}{M} \left[e^{M(1-2\varepsilon)} - 1 \right].$$

The sensitivity of the terminal value $\tilde{J}^\varepsilon(1-\varepsilon)$ with respect to changes in the initial condition is therefore always positive, bounded and bounded away from zero. Moreover, the existence of the continuously differentiable solution for the initial value problem extends beyond $\theta = 1 - \varepsilon$. Therefore, for an arbitrary choice of the initial slope y , the terminal value $\tilde{J}^\varepsilon(1-\varepsilon)$ is finite. The lower bound on $\frac{d}{dy} \tilde{J}^\varepsilon(1-\varepsilon)$ then implies that we can always sufficiently vary y to reach an arbitrary terminal value $\tilde{J}^\varepsilon(1-\varepsilon)$. The fact that $\frac{d}{dy} \tilde{J}^\varepsilon(1-\varepsilon)$ is always positive implies that the choice of y such that the terminal value yields the boundary condition $\tilde{J}^\varepsilon(1-\varepsilon) = \bar{J}(1-\varepsilon)$ of the boundary value problem (38) is unique. This concludes the proof. ■

Corollary A.10 *The Hamilton–Jacobi–Bellman equation for the auxiliary problem,*

$$0 = \sup_\nu h^0(\nu, \theta) + h^1(\nu, \theta) \tilde{J}^\varepsilon(\theta) + h^2(\nu, \theta) \tilde{J}_\theta^\varepsilon(\theta) + h^3(\theta) \tilde{J}_{\theta\theta}^\varepsilon(\theta) \quad (40)$$

with boundary conditions $\tilde{J}^\varepsilon(\varepsilon) = \bar{J}(\varepsilon)$ and $\tilde{J}^\varepsilon(1-\varepsilon) = \bar{J}(1-\varepsilon)$ has a unique twice continuously differentiable solution on $[\varepsilon, 1-\varepsilon]$.

Proof. This is the consequence of Berge’s Maximum Theorem. The unique maximizers are given by

$$\nu^1(\theta) = \frac{\beta}{\rho} \left(\gamma + (\rho - \gamma) \left(\frac{\zeta(\theta)^\gamma}{\gamma \tilde{J}^1(\theta)} \right)^{\rho/\gamma} \right) \quad (41)$$

$$\nu^2(\theta) = \frac{\beta}{\rho} \left(\gamma + (\rho - \gamma) \left(\frac{(1-\zeta(\theta))^\gamma}{\gamma \tilde{J}^2(\theta)} \right)^{\rho/\gamma} \right), \quad (42)$$

with $\tilde{J}^n(\theta)$ defined as in (34) except for $\tilde{J}^\varepsilon(\theta)$ in place of $\tilde{J}(\theta)$. These are functions that satisfy Assumption A.4 on every interior $[\varepsilon, 1-\varepsilon]$. The limits of these formulas at the boundaries as $\varepsilon \searrow 0$ are computed in the proof of Proposition 3.6 in Appendix B. ■

A.4 Solution to the Hamilton–Jacobi–Bellman equation

Consider a sequence of auxiliary problems for $\{\varepsilon^k\}_{k=1}^\infty$ with $\varepsilon^k \searrow 0$. We want to characterize the limiting solution of this sequence. First notice that the boundary condition $\bar{J}(\varepsilon^k)$ converges to \bar{V}^2 (and $\bar{J}(1-\varepsilon^k)$ to \bar{V}^1) as $\varepsilon^k \searrow 0$. This means that the boundary conditions converges to the limiting points of the value function given by (29). We want to establish the convergence of the sequence $\tilde{J}^{\varepsilon^k}(\theta)$ to $\tilde{J}(\theta)$ on every interval $[\varepsilon, 1-\varepsilon]$, $\varepsilon > 0$.

The problem is the vanishing coefficient $h^3(\theta)$ as θ approaches 0 or 1. While the coefficients $h^j(\nu, \theta)$ are bounded for every fixed $[\varepsilon^k, 1-\varepsilon^k]$, they are not uniformly bounded across all such intervals as ε^k . However,

there is a suitable transformation of variables. Define

$$\vartheta(\theta) = \log \frac{\theta}{1-\theta} = \log \frac{\bar{\lambda}^1}{\lambda^2} \in (-\infty, +\infty)$$

and $\tilde{J}(\theta) = \hat{J}(\vartheta(\theta))$. The HJB equation (9) can be written as a differential equation for $\hat{J}(\vartheta)$ given by

$$0 = \sup_{\nu} \hat{h}^0(\nu, \vartheta) + \hat{h}^1(\nu, \vartheta) \hat{J}(\vartheta) + \hat{h}^2(\nu, \vartheta) \hat{J}_{\vartheta}(\vartheta) + \hat{h}^3(\vartheta) \hat{J}_{\vartheta\vartheta}(\vartheta) \quad (43)$$

with $\theta(\vartheta) = \exp(\vartheta) / (1 + \exp(\vartheta))$ and

$$\begin{aligned} \hat{h}^0(\nu, \vartheta) &= h^0(\nu, \theta(\vartheta)) \\ \hat{h}^1(\nu, \vartheta) &\doteq -\theta(\vartheta)\nu^1 - (1 - \theta(\vartheta))\nu^2 + (\theta(\vartheta)u^1 + (1 - \theta(\vartheta))u^2)\gamma\sigma_y + \gamma\mu_y + \frac{1}{2}\gamma(\gamma - 1)\sigma_y^2 \\ \hat{h}^2(\nu, \vartheta) &\doteq \nu^2 - \nu^1 + (u^1 - u^2)\gamma\sigma_y + \frac{1}{2}(u^1 - u^2)^2(2\theta(\vartheta) - 1) \\ \hat{h}^3(\vartheta) &\doteq \frac{1}{2}(u^1 - u^2)^2 \end{aligned}$$

The boundary conditions for the problem are given by $\lim_{\vartheta \rightarrow -\infty} \hat{J}(\vartheta) = \bar{V}^2$ and $\lim_{\vartheta \rightarrow -\infty} \hat{J}(\vartheta) = \bar{V}^1$. Under Assumption A.4, the coefficients $\hat{k}^j(\nu, \vartheta) = k^j(\nu, \vartheta) / k^3(\vartheta)$ for $j = 0, 1, 2$ are bounded for $\vartheta \in (-\infty, \infty)$.

The HJB equation (43) thus satisfies the conditions of the proof in [Strulovici and Szydlowski \(2014\)](#), Appendix B.3, that extends the solution of the HJB equation on a sequence of bounded domains to an unbounded limit. Rather than repeating the proof here, I note that the structure of (43), in particular the bounded coefficients \hat{k}^j , implies that Lemma 8 in [Strulovici and Szydlowski \(2014\)](#) is satisfied, for instance, with the function $\phi(z) = Kz$ for K sufficiently large, and that the functions \hat{J} , \hat{J}_{ϑ} , $\hat{J}_{\vartheta\vartheta}$ are bounded. An application of the Arzelà–Ascoli theorem implies that there is a uniformly convergent subsequence of solutions \hat{J}^{ε^k} on the interval $[\varepsilon, 1 - \varepsilon]$ for every $\varepsilon > 0$.

[Strulovici and Szydlowski \(2014\)](#) then use the following diagonalization argument. Start with interval $[\varepsilon^1, 1 - \varepsilon^1]$. Find the uniformly convergent subsequence of $\hat{J}^{\varepsilon^k}(\vartheta)$ on $[\varepsilon^1, 1 - \varepsilon^1]$ and denote its limit $w^1(\vartheta)$. Now take interval $[\varepsilon^2, 1 - \varepsilon^2]$ and find a subsequence of the first subsequence that converges on $[\varepsilon^2, 1 - \varepsilon^2]$. Denote the solution $w^2(\vartheta)$ and notice that $w^1(\vartheta) = w^2(\vartheta)$ for $\vartheta \in [\varepsilon^1, 1 - \varepsilon^1]$. Continue iteratively and define the limiting solution as follows: for $\vartheta \in [\varepsilon^k, 1 - \varepsilon^k] \setminus [\varepsilon^{k-1}, 1 - \varepsilon^{k-1}]$, set $\hat{J}(\vartheta) \doteq w^k(\vartheta)$.

Corollary A.11 *The limiting solution $\tilde{J}(\theta) = \hat{J}(\vartheta(\theta))$ constructed in this way exists, is twice continuously differentiable and uniquely solves the Hamilton–Jacobi–Bellman equation (9).*

The following lemma is useful as a clarification for the intuition for why the solution of the HJB equation (9) can be defined through the limit of solutions on closed subintervals.

Lemma A.12 *Let $\{\varepsilon^k\}_{k=1}^{\infty}$ satisfy $\varepsilon^k \searrow 0$. Under Assumption A.4, the sequence of stopping times $\{\tau^{\varepsilon^k}\}_{k=1}^{\infty}$ in the auxiliary problem (36) is almost surely diverging, $P\left(\tau^{\varepsilon^k} \xrightarrow[k \rightarrow \infty]{} +\infty\right) = 1$.*

Proof. We use the transformation $\vartheta(\theta) = \log \frac{\theta}{1-\theta}$. In the state space represented by ϑ , the sequence of stopping times $\{\tau^k\}_{k=1}^{\infty}$ corresponds to a sequence of first crossing times of thresholds $\pm \bar{\vartheta}^k$ as $\bar{\vartheta}^k \nearrow +\infty$. Applying Itô’s lemma to (35), we obtain

$$d\vartheta_t = \left[\nu_t^2 - \nu_t^1 + \frac{1}{2} \left((u^2)^2 - (u^1)^2 \right) \right] dt + (u^1 - u^2) dW_t.$$

This is an Itô process with bounded coefficients, for which the claim of the lemma is a standard result. ■

As we move the boundary ε^k in the auxiliary problem closer to zero, the hitting time of this boundary diverges to infinity. With discounting (under Assumption A.1 and under a uniform bound on the boundary values, their contribution to the value function for a given initial value θ_0 vanishes as $\varepsilon^k \searrow 0$.

While boundedness and the Lipschitz property stated in Assumption A.4 holds for $\nu^n(\theta)$ for every given auxiliary problem (for every fixed ε^k), it may not hold in the limit. I prove that Assumption A.4 holds also in the limit in Appendix B, by obtaining closed form solutions for these limits.

A.5 Verification theorem

The last step is to verify that the solution of the Hamilton–Jacobi–Bellman equation yields the value function. This is a standard verification argument.

Lemma A.13 *The function $J(\bar{\lambda}, Y_t) = (\bar{\lambda}_t^1 + \bar{\lambda}_t^2) Y_t^\gamma \tilde{J}(\theta_t)$, where $\tilde{J}(\theta)$ is the solution of the Hamilton–Jacobi–Bellman equation (9), coincides with the value function (8).*

Proof. Consider time $\tau \geq t$ and an admissible control $\nu = (\nu^1, \nu^2)$ with optimal choice of the consumption share $\zeta = \zeta(\nu)$. Then

$$J(\bar{\lambda}_\tau, Y_\tau) = J(\bar{\lambda}_t, Y_t) + \int_t^\tau \mu_{J,s} ds + \int_t^\tau \sigma_{J,s} ds W_s$$

where

$$\begin{aligned} \mu_{J,s} &= (\bar{\lambda}_s^1 + \bar{\lambda}_s^2) Y_s^\gamma \left\{ h^1(\nu_s, \theta_s) \tilde{J}(\theta_s) + h^2(\nu_s, \theta_s) \tilde{J}_\theta(\theta_s) + h^3(\theta_s) \tilde{J}_{\theta\theta}(\theta_s) \right\} \\ \sigma_{J,s} &= (\bar{\lambda}_s^1 + \bar{\lambda}_s^2) Y_s^\gamma \left\{ [\theta_s u^1 + (1 - \theta_s) u^2 + \gamma \sigma_y] \tilde{J}(\theta_s) + \theta_s (1 - \theta_s) (u^1 - u^2) \tilde{J}_\theta(\theta_s) \right\}. \end{aligned}$$

It follows from the discussion in Section A.4 that the terms $\tilde{J}(\theta)$ and $\theta(1 - \theta) \tilde{J}_\theta(\theta) = \widehat{J}_\vartheta(\vartheta(\theta))$ are bounded, and thus $\sigma_{J,s}$ is square integrable over $[t, \tau]$. As a consequence, the stochastic integral $\int_t^\tau \sigma_{J,s} ds W_s$ is a martingale as a function of τ , and we have

$$E_t [J(\bar{\lambda}_\tau, Y_\tau)] = J(\bar{\lambda}_t, Y_t) + E_t \left[\int_t^\tau \mu_{J,s} ds \right].$$

The limiting version of equation (40) for $\varepsilon \searrow 0$ implies that

$$E_t [J(\bar{\lambda}_\tau, Y_\tau)] \leq J(\bar{\lambda}_t, Y_t) - E_t \left[\int_t^\tau (\bar{\lambda}_s^1 + \bar{\lambda}_s^2) Y_s^\gamma h^0(\nu_s, \theta_s) ds \right]$$

with equality for the optimal control ν^n given in (41)–(42). Reorganizing, and taking the limit $\tau \rightarrow \infty$, we obtain

$$E_t \left[\int_t^\infty (\bar{\lambda}_s^1 + \bar{\lambda}_s^2) Y_s^\gamma h^0(\nu_s, \theta_s) ds \right] \leq J(\bar{\lambda}_t, Y_t) \quad (44)$$

where we utilized Lemma A.5 to show that

$$\lim_{\tau \rightarrow \infty} E_t [J(\bar{\lambda}_\tau, Y_\tau)] = \lim_{\tau \rightarrow \infty} E_t \left[(\bar{\lambda}_\tau^1 + \bar{\lambda}_\tau^2) Y_\tau^\gamma \tilde{J}(\theta_\tau) \right] = 0$$

because $\tilde{J}(\theta)$ is a bounded function. The left-hand side of (44) evaluated for the maximizers (41)–(42) is the value function, and since these maximizers are admissible, the inequality holds with equality for the value function. ■

B Characterization of the boundary behavior

In this appendix, I prove propositions from Section 3 that characterize the boundary behavior of the economy as $\theta \searrow 0$ or $\theta \nearrow 1$.

Notice that optimal choice of ν^n in (9) implies that

$$\sup_{\nu \in \mathbb{R}} F(C, \nu) - \nu V \doteq f(C, V) = \frac{\beta}{\rho} \left[C^\rho (\gamma V)^{1-\frac{\rho}{\gamma}} - \gamma V \right].$$

Substituting this expression into (9) for $F(\zeta, \tilde{J}^1)$ and $F(1-\zeta, \tilde{J}^2)$ leads to the ODE

$$\begin{aligned} 0 &= \frac{\beta}{\rho} \zeta^\rho \left(\gamma \tilde{J}^1(\theta) \right)^{1-\frac{\rho}{\gamma}} + (1-\theta) \frac{\beta}{\rho} (1-\zeta)^\rho \left(\gamma \tilde{J}^2(\theta) \right)^{1-\frac{\rho}{\gamma}} + \\ &+ \gamma \left[-\frac{\beta}{\rho} + (\theta u^1 + (1-\theta) u^2) \sigma_y + \mu_y + \frac{1}{2} (\gamma-1) \sigma_y^2 \right] \tilde{J}(\theta) \\ &+ \theta (1-\theta) (u^1 - u^2) \gamma \sigma_y \tilde{J}_\theta(\theta) + \frac{1}{2} \theta^2 (1-\theta)^2 (u^1 - u^2)^2 \tilde{J}_{\theta\theta}(\theta) \end{aligned} \quad (45)$$

where ζ is given by (33). Appendix A proofs the existence of a twice-continuously differentiable solution to this equation. The results that follow also utilize the third derivative of \tilde{J} , which can be obtained by differentiating (45).

Proof of Proposition 3.2. Given an initial condition $\theta_0 \in (0, 1)$, the process (10) lives on the open interval $(0, 1)$ with unattainable boundaries (the preferences satisfy an Inada condition at zero). For any numbers $0 < a < b < 1$, the process θ has bounded and continuous drift and volatility coefficients on (a, b) , and the volatility coefficient is bounded away from zero. It is thus sufficient to establish the appropriate boundary behavior of θ in order to make the process positive Harris recurrent (see Meyn and Tweedie (1993)). Since the process will also be φ -irreducible for the Lebesgue measure under these boundary conditions, there exists a unique stationary distribution.

Denote $\mu_\theta(\theta)$ and $\sigma_\theta(\theta)$ the drift and volatility coefficients in (10). The boundary behavior of the process θ^1 is captured by the scale measure $S : (0, 1)^2 \rightarrow \mathbb{R}$ defined as

$$s(\theta) = \exp \left\{ - \int_{\theta_0}^{\theta} \frac{2\mu_\theta(\tau)}{\sigma_\theta^2(\tau)} d\tau \right\} \quad S[\theta_l, \theta_h] = \int_{\theta_h}^{\theta_l} s(\theta) d\theta$$

for an arbitrary choice of $\theta_0 \in (0, 1)$, and the speed measure $M : (0, 1)^2 \rightarrow \mathbb{R}$

$$m(\theta) = \frac{1}{\sigma_\theta^2(\theta) s(\theta)} \quad M[\theta_l, \theta_h] = \int_{\theta_h}^{\theta_l} m(\theta) d\theta.$$

Karlin and Taylor (1981, Chapter 15) provide an extensive treatment of the boundaries.

The boundaries are nonattracting if and only if

$$\lim_{\theta_l \searrow 0} S[\theta_l, \theta_h] = \infty \quad \text{and} \quad \lim_{\theta_h \nearrow 1} S[\theta_l, \theta_h] = \infty \quad (46)$$

and this result is independent of the fixed argument that is not under the limit. With nonattracting boundaries, the stationary density will exist if the speed measure satisfies

$$\lim_{\theta_l \searrow 0} M[\theta_l, \theta_h] < \infty \quad \text{and} \quad \lim_{\theta_h \nearrow 1} M[\theta_l, \theta_h] < \infty, \quad (47)$$

again independently of the argument that is not under the limit.

In our case,

$$s(\theta) = \exp \left\{ - \int_{\theta_0}^{\theta} \frac{2(\nu^2(\tau) - \nu^1(\tau))}{\tau(1-\tau)(u^1 - u^2)^2} d\tau \right\} s_{sep}(\theta),$$

where

$$s_{sep}(\theta) = \left(\frac{1-\theta}{1-\theta_0} \right)^{-\frac{2u^1}{u^1-u^2}} \left(\frac{\theta}{\theta_0} \right)^{\frac{2u^2}{u^1-u^2}} \quad (48)$$

is the integrand of the scale function in the separable case, when $\nu^2(\theta) - \nu^1(\theta) \equiv 0$.

For the left boundary, assume that in line with condition (i), there exist $\underline{\theta} \in (0, 1)$ and $\underline{\nu} \in \mathbb{R}$ such that $\nu^2(\theta) - \nu^1(\theta) \geq \underline{\nu}$ for all $\theta \in (0, \underline{\theta})$. Taking $\theta_0 = \underline{\theta}$, the scale measure can be bounded as

$$\begin{aligned} S[\theta_l, \underline{\theta}] &\geq \int_{\theta_l}^{\underline{\theta}} \exp \left\{ - \int_{\underline{\theta}}^{\theta} \frac{2\underline{\nu}}{\tau(1-\tau)(u^1 - u^2)^2} d\tau \right\} \left(\frac{1-\theta}{1-\underline{\theta}} \right)^{-\frac{2u^1}{u^1-u^2}} \left(\frac{\theta}{\underline{\theta}} \right)^{\frac{2u^2}{u^1-u^2}} d\theta = \\ &= \int_{\theta_l}^{\underline{\theta}} \left(\frac{\theta}{\underline{\theta}} \right)^{\frac{2u^2}{u^1-u^2} - \frac{2\underline{\nu}}{(u^1-u^2)^2}} \left(\frac{1-\theta}{1-\underline{\theta}} \right)^{\frac{2\underline{\nu}}{(u^1-u^2)^2} - \frac{2u^1}{u^1-u^2}} d\theta \end{aligned}$$

The left limit in (46) thus diverges to infinity if

$$\frac{2u^2}{u^1 - u^2} - \frac{2\underline{\nu}}{(u^1 - u^2)^2} \leq -1,$$

which is satisfied when $\underline{\nu} \geq \frac{1}{2} [(u^1)^2 - (u^2)^2]$.

The argument for the right boundary is symmetric. Taking $\bar{\theta} \in (0, 1)$ and $\bar{\nu} \in \mathbb{R}$ such that $\nu^2(\theta) - \nu^1(\theta) \leq \bar{\nu}$ for all $\theta \in (\bar{\theta}, 1)$, the calculation reveals that we require $\bar{\nu} \leq \frac{1}{2} [(u^1)^2 - (u^2)^2]$.

It turns out that the bounds implied by conditions (47) are marginally tighter. Following the same bounding argument as above, sufficient conditions for (47) to hold are

$$\underline{\nu} > \frac{1}{2} [(u^1)^2 - (u^2)^2] \quad \text{and} \quad \bar{\nu} < \frac{1}{2} [(u^1)^2 - (u^2)^2]. \quad (49)$$

The construction reveals that these bounds are also the least tight bounds of this type under which the proposition holds.

It is also useful to note that the unique stationary density $q(\theta)$ is proportional to the speed density $m(\theta)$. Finally, if the limits in Proposition 3.2 do not exist, they can be replaced with appropriate limits inferior and superior.

This discussion has sorted out case (a). Conditions (i') and (ii') are sufficient conditions for the boundaries to be attracting. Lemma 6.1 in Karlin and Taylor (1981) then shows that if the 'attracting' condition is satisfied for a boundary, then θ^1 converges to this boundary on a set of paths that has a strictly positive probability. This probability is equal to one if the other boundary is non-attracting. Combining these results, we obtain statements (b), (c), and (d). ■

Proof of Corollary 4.2. Assume without loss of generality that $|u^2| \leq |u^1|$. The sufficient part is an immediate consequence of Proposition 3.2. Under separable preferences, $\nu^2 - \nu^1 \equiv 0$, and thus if $|u^2| < |u^1|$ then conditions (i') and (ii) hold, and agent 2 dominates in the long run under P .

For the necessary part, when $u^2 = u^1$, then θ is constant and both agents survive under P . When $-u^2 = u^1 = u$, then it follows from inspection of formula (48) in the proof of Proposition 3.2 that conditions

(46) are satisfied and the boundaries are non-attracting. Lemma 6.1 in [Karlin and Taylor \(1981\)](#) then implies that both agents survive under P .

Note that even though both agents survive when $-u^2 = u^1$, the speed density $m(\theta) \propto \theta^{-1} (1 - \theta)^{-1}$ is not integrable on $(0, 1)$ and thus there does not exist a finite stationary measure.

The result on survival under measure Q^n follows from the fact that the evolution of Brownian motion W under the beliefs of agent n is $dW_t = u^n dt + dW_t^n$. Since the evolution of θ completely describes the dynamics of the economy, substituting this expression into (10) and reorganizing yields the desired result. ■

Proof of Proposition 3.3. The proof of the proposition relies on showing that the dynamics of the continuation values of the two agents in the proximity of the boundaries becomes degenerate in a specific sense. From this fact, I can infer the dynamics of the stochastic discount factor implied by the consumption process of the large agent and, consequently, the equilibrium price dynamics. I state the limiting properties of the continuation values separately in [Lemmas B.1](#) and [B.2](#).

Using the construction from [Duffie and Epstein \(1992a\)](#), the stochastic discount factor process for agent n under the subjective probability measure Q^n is given by

$$S_t^2 = \exp\left(-\int_0^t \nu^2(\theta_s) ds\right) \left(\frac{Y_t}{Y_0}\right)^{\gamma-1} \left(\frac{1-\zeta(\theta_t)}{1-\zeta(\theta_0)}\right)^{\rho-1} \left(\frac{\tilde{J}^2(\theta_t)}{\tilde{J}^2(\theta_0)}\right)^{1-\frac{\rho}{\gamma}}. \quad (50)$$

[Lemma A.6](#) shows that $\lim_{\theta \searrow 0} 1 - \zeta(\theta) = 1$ and $\lim_{\theta \searrow 0} \tilde{J}^2(\theta) = \bar{V}^2$ from equation (20), implying that $\lim_{\theta \searrow 0} \nu^2(\theta) = \bar{\nu}^2$ which is given in (21). It remains to be shown that the local drift and volatility of the last two terms decline to zero as $\theta^1 \searrow 0$, which implies that the infinitesimal risk-free rate and the local price of risk converge to their counterparts from a homogeneous economy populated only by agent 2. Moreover, the price of aggregate endowment A converges as well, and so does the local return on aggregate wealth.

Homotheticity of preferences implies that individual wealth-consumption ratios are given by

$$\xi^1(\theta) = \frac{1}{\beta} \left(\frac{\gamma \tilde{J}^1(\theta)}{\zeta(\theta)^\gamma}\right)^{\frac{\rho}{\gamma}} \quad \xi^2(\theta) = \frac{1}{\beta} \left(\frac{\gamma \tilde{J}^2(\theta)}{(1-\zeta(\theta))^\gamma}\right)^{\frac{\rho}{\gamma}}. \quad (51)$$

I start by assuming that $\xi^n(\theta)$ are functions that are bounded and bounded away from zero. This, among other things, implies that the discount rate functions $\nu^n(\theta)$ in (41)–(42) are bounded and that the drift and volatility coefficients in the stochastic differential equation for θ , (10), are bounded as well. The assumption will ultimately be verified by a direct calculation of the limits of $\xi^n(\theta)$ as $\theta \searrow 0$ or $\theta \nearrow 1$. Without loss of generality, it is sufficient to focus on the case $\theta \searrow 0$. First notice some asymptotic results for the planner's continuation value $\tilde{J}(\theta)$.

Lemma B.1 *The solution of the planner's problem implies that*

$$\lim_{\theta \searrow 0} \theta \tilde{J}_\theta(\theta) = \lim_{\theta \searrow 0} (\theta)^2 \tilde{J}_{\theta\theta}(\theta) = \lim_{\theta \searrow 0} (\theta)^3 \tilde{J}_{\theta\theta\theta}(\theta) = 0.$$

Proof. [Lemma A.6](#) implies that the planner's objective function can be continuously extended at $\theta = 0$ by the continuation value for agent 2 living in a homogeneous economy. Expression (30) scaled by $(\alpha^1 + \alpha^2) \gamma^{-1} Y^\gamma$ leads to an equation in scaled continuation values

$$\tilde{J}(\theta) = \theta \tilde{J}^1(\theta) + (1 - \theta) \tilde{J}^2(\theta)$$

and the proof of Lemma A.6 yields

$$\lim_{\theta \searrow 0} \tilde{J}(\theta) = \lim_{\theta \searrow 0} \tilde{J}^2(\theta) = \bar{V}^2,$$

where \bar{V}^2 is defined in (20). Since $\tilde{J}^2(\theta) = \tilde{J}(\theta) - \theta \tilde{J}_\theta(\theta)$, then

$$\lim_{\theta \searrow 0} \theta \tilde{J}_\theta(\theta) = 0. \quad (52)$$

Further, consider the behavior of individual terms in ODE (45) as $\theta \searrow 0$. Using expression (33), the first term is proportional to

$$\begin{aligned} \theta (\zeta(\theta))^\rho \left(\tilde{J}^1(\theta) \right)^{1-\frac{\rho}{\gamma}} &= (\theta)^{\frac{1}{1-\rho}} \left(\tilde{J}^1(\theta) \right)^{\frac{1-\rho/\gamma}{1-\rho}} [K(\theta)]^{-\rho} = \\ &= \zeta(\theta) [K(\theta)]^{1-\rho}, \end{aligned}$$

where $K(\theta)$ is the denominator in the formula for the consumption share (33), and $\lim_{\theta \searrow 0} K(\theta) = (\bar{V}^2)^{\frac{1-\rho/\gamma}{1-\rho}}$, which is a finite value. Since $\lim_{\theta \searrow 0} \zeta(\theta) = 0$, the first term in (45) vanishes as $\theta \searrow 0$. The sum of the second and third term converges to

$$\frac{\beta}{\rho} (\bar{V}^2)^{1-\frac{\rho}{\gamma}} + \left(-\frac{\beta}{\rho} + \mu_y + u^2 \sigma_y + \frac{1}{2} (\gamma - 1) \sigma_y^2 \right) \bar{V}^2$$

and formula (20) implies that this expression is zero. Since the fourth term in (45) also converges to zero due to result (52), the last term in (45) must also converge to zero, or

$$\lim_{\theta \searrow 0} (\theta)^2 \tilde{J}_{\theta\theta}(\theta) = 0. \quad (53)$$

Finally, differentiate the PDE (45) by θ and multiply the equation by θ . Using comparisons with results (52)–(53), the assumption that $\zeta^1(\theta) / (\gamma \tilde{J}^1(\theta))^{1/\gamma}$ and $(1 - \zeta(\theta)) / (\gamma \tilde{J}^2(\theta))^{1/\gamma}$ are bounded and bounded away from zero and $\lim_{\theta \searrow 0} \zeta(\theta) = 0$, it is possible to determine that all terms in the new equation containing derivatives of $\tilde{J}(\theta)$ up to second order vanish as $\theta \searrow 0$. The single remaining term that contains a third derivative of $\tilde{J}(\theta)$ is multiplied by θ^3 and must necessarily converge to zero as well, and thus

$$\lim_{\theta \searrow 0} (\theta)^3 \tilde{J}_{\theta\theta\theta}(\theta) = 0.$$

■

The Markov structure of the problem implies that the evolution of the continuation values and consumption shares can be written as

$$\frac{d\tilde{J}^n(\theta_t)}{\tilde{J}^n(\theta_t)} = \mu_{\tilde{J}^n}(\theta_t) dt + \sigma_{\tilde{J}^n}(\theta_t) dW_t \quad (54)$$

$$\frac{d\zeta(\theta_t)}{\zeta(\theta_t)} = \mu_\zeta(\theta_t) dt + \sigma_\zeta(\theta_t) dW_t, \quad (55)$$

$$\frac{d(1 - \zeta(\theta_t))}{1 - \zeta(\theta_t)} = \mu_{1-\zeta}(\theta_t) dt + \sigma_{1-\zeta}(\theta_t) dW_t,$$

where the drift and volatility coefficients are functions of θ^1 , and the results from Lemma B.1 allow the characterization of their limiting behavior.

Lemma B.2 *The coefficients in equations (54)–(55) for agent 2 satisfy*

$$\lim_{\theta \searrow 0} \mu_{\tilde{J}^2}(\theta) = \lim_{\theta \searrow 0} \sigma_{\tilde{J}^2}(\theta) = \lim_{\theta \searrow 0} \mu_{1-\zeta}(\theta) = \lim_{\theta \searrow 0} \sigma_{1-\zeta}(\theta) = 0. \quad (56)$$

Proof. The result follows from an application of Itô's lemma to \tilde{J}^2 and $1 - \zeta$. Utilizing formulas (34) and (33), the coefficients will contain expressions for the value function $\tilde{J}(\theta)$ and its partial derivatives up to the third order (the third derivative can be obtained by differentiating (45)), and all the expressions can be shown to converge to zero using Lemma B.1.

Itô's lemma implies

$$\begin{aligned} d\tilde{J}^2(\theta_t) &= d\left[\tilde{J}(\theta_t) - \theta_t \tilde{J}_\theta(\theta_t)\right] = \\ &= -(\theta_t)^2 \tilde{J}_{\theta\theta}(\theta_t) \frac{d\theta_t}{\theta_t} - \frac{1}{2} \left[(\theta_t)^2 \tilde{J}_{\theta\theta}(\theta_t) + (\theta_t)^3 \tilde{J}_{\theta\theta\theta}(\theta_t) \right] \left(\frac{d\theta_t}{\theta_t} \right)^2 \end{aligned}$$

and since the drift and volatility coefficients in the dynamics of θ given by equation (10) are bounded by assumption, applying results from Lemma B.1 proves the claim about the drift and volatility coefficients of $\tilde{J}^2(\theta)$ (\tilde{J}^2 itself converges to a nonzero limit so the scaling is innocuous).

Further notice that

$$\begin{aligned} d\tilde{J}^1(\theta_t) &= d\left[\tilde{J}(\theta_t) + (1 - \theta_t) \tilde{J}_\theta(\theta_t)\right] = -(\theta_t)^2 \tilde{J}_{\theta\theta}(\theta_t) \frac{d\theta_t}{\theta_t} + \\ &\quad + \frac{1}{2} \left[(\theta_t)^2 \tilde{J}_{\theta\theta}(\theta_t) + (1 - \theta_t) (\theta_t)^2 \tilde{J}_{\theta\theta\theta}(\theta_t) \right] \left(\frac{d\theta_t}{\theta_t} \right)^2 \end{aligned} \quad (57)$$

and that

$$\frac{\zeta(\theta)}{\left(\gamma \tilde{J}^1(\theta)\right)^{\frac{1}{\gamma}}} = (\theta)^{\frac{1}{1-\rho}} \left(\tilde{J}^1(\theta)\right)^{\frac{1-\frac{1}{\gamma}}{1-\rho}} K(\theta)^{-1} \quad (58)$$

is bounded and bounded away from zero by assumption. Denote the numerators of ζ and $1 - \zeta$ in (33) as

$$Z^1(\theta) = \theta^{\frac{1}{1-\rho}} \left(\gamma \tilde{J}^1(\theta)\right)^{\frac{1-\frac{\rho}{\gamma}}{1-\rho}} \quad Z^2(\theta) = (1 - \theta)^{\frac{1}{1-\rho}} \left(\gamma \tilde{J}^2(\theta)\right)^{\frac{1-\frac{\rho}{\gamma}}{1-\rho}}.$$

Then $1 - \zeta = Z^2 / (Z^1 + Z^2)$ and, omitting arguments,

$$\begin{aligned} dZ^1 &= \frac{1}{1-\rho} Z^1 \frac{d\theta}{\theta} + \frac{1-\frac{\rho}{\gamma}}{1-\rho} Z^1 \frac{d\tilde{J}^1}{\tilde{J}^1} + \frac{1}{2} \frac{\rho}{(1-\rho)^2} Z^1 \left(\frac{d\theta}{\theta} \right)^2 + \\ &\quad + \frac{1}{2} \frac{\left(\rho - \frac{\rho}{\gamma}\right) \left(1 - \frac{\rho}{\gamma}\right)}{(1-\rho)^2} Z^1 \left(\frac{d\tilde{J}^1}{\tilde{J}^1} \right)^2 + \frac{1-\frac{\rho}{\gamma}}{(1-\rho)^2} Z^1 \frac{d\theta}{\theta} \frac{d\tilde{J}^1}{\tilde{J}^1} \\ dZ^2 &= -\frac{1}{1-\rho} Z^2 \frac{\theta}{1-\theta} \frac{d\theta}{\theta} + \frac{1-\frac{\rho}{\gamma}}{1-\rho} Z^2 \frac{d\tilde{J}^2}{\tilde{J}^2} + \frac{1}{2} \frac{\rho}{(1-\rho)^2} Z^2 \left(\frac{\theta}{1-\theta} \right)^2 \left(\frac{d\theta}{\theta} \right)^2 + \\ &\quad + \frac{1}{2} \frac{\left(\rho - \frac{\rho}{\gamma}\right) \left(1 - \frac{\rho}{\gamma}\right)}{(1-\rho)^2} Z^2 \left(\frac{d\tilde{J}^2}{\tilde{J}^2} \right)^2 - \frac{1-\frac{\rho}{\gamma}}{(1-\rho)^2} Z^2 \frac{\theta}{1-\theta} \frac{d\theta}{\theta} \frac{d\tilde{J}^2}{\tilde{J}^2}. \end{aligned}$$

Since the drift and volatility coefficients of $d\tilde{J}^2/\tilde{J}^2$ vanish as $\theta \searrow 0$, and $\lim_{\theta \searrow 0} Z^2(\theta) = (\gamma \bar{V}^2)^{\frac{1-\frac{\rho}{\gamma}}{1-\rho}}$,

the drift and volatility coefficients in the equation for dZ^2 vanish. In the equation for dZ^1 , it remains to determine the behavior of terms containing $d\tilde{J}^1$ (the remaining contributions to drift and volatility terms converge to zero because $\lim_{\theta \searrow 0} Z^1(\theta) = 0$):

$$\frac{Z^1}{\tilde{J}^1} = \theta \left[(\theta)^{\frac{1}{1-\rho}} \left(\gamma \tilde{J}^1 \right)^{\frac{1-\frac{1}{\gamma}}{1-\rho}} \right]^\rho,$$

where the term in brackets is bounded and bounded away from zero by utilizing (58). Using the first θ to multiply the coefficients in $d\tilde{J}^1$ in formula (57), we conclude that the coefficients in $Z^1 d\tilde{J}^1 / \tilde{J}^1$ vanish as $\theta \searrow 0$. Finally, the drift term arising from $(d\tilde{J}^1)^2$ vanishes, and

$$Z^1 \left(\frac{d\tilde{J}^1}{\tilde{J}^1} \right)^2 = \frac{(\theta)^5 \left(\tilde{J}_{\theta\theta} \right)^2}{\tilde{J} + (1-\theta) \tilde{J}_\theta} \left[(\theta)^{\frac{1}{1-\rho}} \left(\gamma \tilde{J}^1 \right)^{\frac{1-\frac{1}{\gamma}}{1-\rho}} \right]^\rho \left(\frac{d\theta}{\theta} \right)^2.$$

Here, the last term has bounded drift, the second last term is bounded, and the first term converges to zero as $\theta \searrow 0$, which can be shown by using the l'Hôpital's rule (the numerator converges to zero and the denominator to zero or $-\infty$, depending on the sign of γ):

$$\lim_{\theta \searrow 0} \frac{(\theta)^5 \left(\tilde{J}_{\theta\theta} \right)^2}{\tilde{J} + (1-\theta) \tilde{J}_\theta} = \lim_{\theta \searrow 0} \frac{5(\theta)^4 \tilde{J}_{\theta\theta} + 2(\theta)^5 \tilde{J}_{\theta\theta\theta}}{1-\theta} = 0.$$

Thus all terms in the drift and volatility coefficients of dZ^1 vanish.

Applying Itô's lemma to $1 - \zeta$ yields

$$\begin{aligned} d(1 - \zeta) &= \frac{1}{Z^1 + Z^2} dZ^2 - \frac{Z^2}{(Z^1 + Z^2)^2} (dZ^1 + dZ^2) + \\ &+ \frac{Z^2}{(Z^1 + Z^2)^3} (dZ^1 + dZ^2)^2 - \frac{1}{(Z^1 + Z^2)^2} dZ^2 (dZ^1 + dZ^2) \end{aligned}$$

and the results on the behavior of dZ^1 and dZ^2 as $\theta \searrow 0$ lead to the desired conclusion about the convergence of drift and volatility coefficients of $d(1 - \zeta)$. ■

We can now finally proceed with the proof of Proposition 3.3. Convergence of the risk-free interest rate follows from the direct calculation of

$$r(0) = \lim_{t \searrow 0} -\frac{1}{t} \log E [M_t^2 S_t^2(0) | \mathcal{F}_0]$$

where $S_t^2(0)$ is the limiting stochastic discount factor corresponding to the one prevailing in a homogeneous economy populated only by agent 2. Lemma B.2 shows that the local behavior of S_t^2 converges to $S_t^2(0)$ as $\theta_0 \searrow 0$. Similarly, convergence of the aggregate wealth-consumption ratio follows from

$$\xi(\theta) = \xi^1(\theta) \zeta(\theta) + \xi^2(\theta) (1 - \zeta(\theta)).$$

Since $\xi^n(\theta)$ are bounded and $\zeta(\theta)$ converges to zero, we have

$$\lim_{\theta \searrow 0} \xi(\theta) = \lim_{\theta \searrow 0} \xi^2(\theta) = \frac{1}{\beta} (\gamma \bar{V}^2)^\rho,$$

where \bar{V}^2 is given by (20).

In order to obtain the convergence of the infinitesimal return, observe that

$$\xi^1(\theta) \zeta(\theta) = \beta^{-1} \theta \left(\gamma \tilde{J}^1(\theta) \right) [Z^1(\theta) + Z^2(\theta)]^{\rho-1}$$

and

$$d \left[\theta \gamma \tilde{J}^1(\theta) \right] = \theta \gamma \tilde{J}^1(\theta) \frac{d\theta}{\theta} + \theta \gamma d\tilde{J}^1(\theta) + \theta \gamma d\tilde{J}^1(\theta) \frac{d\theta}{\theta}.$$

The drift and volatility coefficients of the first term on the right-hand side vanish as $\theta \searrow 0$ by the proof of Lemma B.1, and the coefficients of the other two terms vanish by combining the results in that Lemma with equation (57). Further,

$$\begin{aligned} d \left\{ [Z^1 + Z^2]^{\rho-1} \right\} &= (\rho-1) [Z^1(\theta^1) + Z^2(\theta^1)]^{\rho-2} (dZ^1 + dZ^2) + \\ &\quad + \frac{1}{2} (\rho-2)(\rho-1) [Z^1(\theta^1) + Z^2(\theta^1)]^{\rho-3} (dZ^1 + dZ^2)^2 \end{aligned}$$

and since dZ^1 and dZ^2 have vanishing coefficients by the proof of Lemma B.2 and the remaining terms are bounded, we obtain that $d[\xi^1(\theta) \zeta(\theta)]$ has vanishing drift and volatility coefficients as $\theta \searrow 0$. The same argument holds for $d[\xi^2(\theta)(1-\zeta(\theta))]$, and thus $d\xi(\theta)$ has vanishing coefficients as well. Therefore all but the first term in

$$dA_t = d[\xi(\theta_t) Y_t] = A_t \frac{dY_t}{Y_t} + Y_t d\xi(\theta_t) + d\xi(\theta_t) dY_t$$

have coefficients that decline to zero as $\theta_t \searrow 0$, which proves the result. ■

Proof of Proposition 3.4. The evolution of θ given by equation (10) implies that for every fixed $t \geq 0$

$$\theta_0 \searrow 0 \implies \theta_t \rightarrow 0, P\text{-a.s.}$$

and thus also $1-\zeta(\theta_t) \rightarrow 1$ and $\tilde{J}^2(\theta_t) \rightarrow \bar{V}^2$, P -a.s.¹⁸ The last two terms in the expression for the stochastic discount factor, S_t^2 , equation (50), converge to one, P -a.s., and since $\nu^2(\theta_s^1)$, $0 \leq s \leq t$ also converges to $\nu^2(0)$ and is bounded, we have $S_t^2 \xrightarrow{P} S_t^2(0)$. Consider a family of random variables $M_t^2 S_t^2(\theta_0^1)$ indexed by the initial Pareto share θ_0^1 . Since this family is uniformly integrable, then convergence in probability implies convergence in mean, and we obtain the convergence result for bond prices

$$E [M_t^2 S_t^2(\theta_0) | \mathcal{F}_0] \xrightarrow{\theta_0 \searrow 0} E [M_t^2 S_t^2(0) | \mathcal{F}_0].$$

The same argument holds for $M_t^2 S_t^2(\theta_0) Y_t$, which yields the result for the price of individual cash flows from the aggregate endowment. ■

¹⁸This result becomes more transparent if we consider $1-\zeta$ and \tilde{J}^2 as functions of $\log \theta$. The dynamics of $\log \theta$

$$\begin{aligned} d \log \theta_t &= (1-\theta_t) \left[\nu^2(\theta_t) - \nu^1(\theta_t) + \frac{1}{2} \left((u^2)^2 - (u^1)^2 \right) - \frac{1}{2} \theta_t (u^1 - u^2)^2 \right] dt + \\ &\quad + (1-\theta_t) (u^1 - u^2) dW_t \end{aligned}$$

has bounded drift and volatility coefficients and thus for $\forall \varepsilon > 0, \forall k > 0$, it is possible to achieve

$$P[\theta_t < k] = P[\log \theta_t < \log k] > 1 - \varepsilon$$

by setting $\log \theta_0$ sufficiently low.

Proof of Proposition 3.5. Agent 1, whose wealth A^1 is close to zero, solves

$$\bar{\lambda}_t^1 V_t^1 = \max_{C^1, \pi^1, \nu^1} E_t \left[\int_t^\infty \bar{\lambda}_s^1 F(C_s^1, \nu_s^1) ds \right] \quad (59)$$

subject to (7) and the budget constraint,

$$\begin{aligned} \frac{dA_t^1}{A_t^1} &= \left[r(\theta_t^1) + \pi_t^1 \left([\xi(\theta_t)]^{-1} + \mu_A(\theta_t) - r(\theta_t) \right) - \frac{C_t^1}{A_t^1} \right] dt + \pi_t^1 \sigma_A(\theta_t) dW_t = \\ &= \mu_{A^1}(\theta_t) dt + \sigma_{A^1}(\theta_t) dW_t \end{aligned} \quad (60)$$

where π^1 is the portfolio share invested in the risky asset. The local behavior of returns on the risk-free bond $r(\theta)$ and risky asset (11) as $\theta \searrow 0$ is known from Proposition 3.3.

The homogeneity of the problem (59–60) motivates the guess

$$V_t^1 = (A_t^1)^\gamma \hat{V}^1(\theta_t). \quad (61)$$

The drift and volatility coefficients depend explicitly on θ because A^1 and θ are linked through

$$A_t^1 = Y_t \zeta(\theta_t) \beta^{-\frac{1}{1-\rho}} \left[\gamma \hat{V}^1(\theta_t) \right]^{\frac{\rho}{\gamma(1-\rho)}}. \quad (62)$$

Recall that we are interested in the characterization of the limiting solution as $\theta \searrow 0$. The associated HJB equation leads to a second-order ODE (omitting dependence on θ)

$$\begin{aligned} 0 &= \max_{C^1, \pi^1, \nu^1} \frac{1}{\rho} \beta^{\frac{1}{1-\rho}} \left(\gamma \hat{V}^1 \right)^{1-\frac{\rho}{\gamma(1-\rho)}} + \hat{V}^1 \gamma \left(-\frac{\beta}{\rho} + \mu_{A^1} + u^1 \sigma_{A^1} - \frac{1}{2} (1-\gamma) (\sigma_{A^1})^2 \right) + \\ &\quad + \hat{V}_\theta^1 \theta (\mu_\theta + u^1 \sigma_\theta + \gamma \sigma_\theta \sigma_{A^1}) + \hat{V}_{\theta\theta}^1 (\theta)^2 \frac{1}{2} (\sigma_\theta)^2, \end{aligned} \quad (63)$$

which yields the first-order conditions on C_t^1 and π_t^1 :

$$\begin{aligned} \frac{C_t^1}{A_t^1} &= \beta^{\frac{1}{1-\rho}} \left(\gamma \hat{V}^1(\theta_t) \right)^{-\frac{\rho}{\gamma(1-\rho)}} \\ \pi_t^1 &= \frac{[\xi(\theta_t)]^{-1} + \mu_A(\theta_t) + u_1 \sigma_A(\theta_t) - r(\theta_t) + \frac{\theta \hat{V}_\theta^1(\theta_t)}{\hat{V}^1(\theta_t)} \sigma_\theta(\theta_t) \sigma_{A^1}(\theta_t)}{(1-\gamma) (\sigma_A(\theta_t))^2}, \end{aligned} \quad (64)$$

where μ_{A^1} and σ_{A^1} are the drift and volatility coefficients on the right-hand side of (60), and μ_θ and σ_θ are the coefficients associated with the evolution of $d\theta_t/\theta_t$. Notice that the portfolio choice π^1 almost corresponds to the standard Merton (1971) result, except the last term in the numerator which explicitly takes into account the covariance between agent's 1 wealth and the evolution in the state variable θ imposed by (62).

The solution of this equation determines the consumption-wealth ratio of agent 1 and, consequently, the evolution of her wealth. While a closed-form solution of this equation is not available, it is again possible to characterize the asymptotic behavior as $\theta \searrow 0$, established in Lemma B.4. For the proof of that lemma, the following result, the following result will be useful:

Lemma B.3 *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be differentiable with a monotone first derivative in a neighborhood of $-\infty$ and have a finite limit $\lim_{x \rightarrow -\infty} f(x)$. Then $\lim_{x \rightarrow -\infty} f'(x) = 0$.*

Lemma B.4 *The following results hold:*

$$\lim_{\theta \searrow 0} \theta \hat{V}_\theta^1(\theta) = \lim_{\theta \searrow 0} (\theta)^2 \hat{V}_{\theta\theta}^1(\theta) = 0.$$

Proof of Lemma B.4. Transformation (61) together with the previously used $V_t^1 = Y^\gamma \tilde{J}^1(\theta_t)$ imply that

$$\hat{V}^1(\theta) = \beta^\gamma \left(\frac{\gamma \tilde{J}^1(\theta^1)}{\zeta(\theta)^\gamma} \right)^{1-\rho}. \quad (65)$$

Think for a moment of \hat{V}^1 as a function of $\log \theta$, where we are interested in the limiting behavior as $\log \theta \rightarrow -\infty$. We have

$$\theta \hat{V}_\theta^1 = \hat{V}_{\log \theta}^1 \quad \text{and} \quad (\theta)^2 \hat{V}_{\theta\theta}^1 = \hat{V}_{(\log \theta)^2}^1 - \hat{V}_{\log \theta}^1. \quad (66)$$

Differentiating repeatedly expression (65) and exploiting the local behavior of $\tilde{J}^1(\theta)$ as $\theta \searrow 0$, we conclude that the assumptions of Lemma B.3 hold, and thus both expressions in (66) converge to zero as $\theta \searrow 0$. ■

These results are similar to those in Lemma B.1. They imply that the derivative terms in the ODE (63) vanish as $\theta \searrow 0$, and we obtain the limit for $\hat{V}^1(\theta)$ and the evolution of A^1 in closed form.

We have thus pinned down the behavior of the last term in the numerator of the portfolio share π^1 in equation (64). This term explicitly takes into account agent 1's knowledge about the impact of her portfolio decision on equilibrium prices. Since this term vanishes as $\theta \searrow 0$, the agent understands that asymptotically the portfolio decisions made by agents of her type will not have any impact on local equilibrium price dynamics, and thus behaves as if she resided in an economy populated only by agent 2.

We can now continue with the proof of Proposition 3.5. Utilizing Lemma B.4 to deduce which terms in ODE (63) vanish and Proposition 3.3 to determine the limiting values of the remaining coefficients, we obtain

$$\begin{aligned} \lim_{\theta \searrow 0} \beta^{\frac{1}{1-\rho}} \left(\hat{V}^1(\theta) \right)^{-\frac{\rho}{\gamma(1-\rho)}} &= \lim_{\theta \searrow 0} [\xi^1(\theta)]^{-1} = \beta - \rho \left(\mu_y + u^2 \sigma_y - \frac{1}{2} (1-\gamma) (\sigma_y)^2 \right) - \\ &\quad - \frac{\rho}{1-\rho} \left[(u^1 - u^2) \sigma_y + \frac{1}{2} \frac{(u^1 - u^2)^2}{1-\gamma} \right], \end{aligned}$$

which is the limiting consumption-wealth ratio for agent 1. The formulas for the wealth share invested in the claim on aggregate consumption and the coefficients of the wealth process are obtained by plugging in the previous results into expressions (60) and (64). ■

Proof of Proposition 3.6. Given convergence to the homogeneous economy counterpart, the expression for $\lim_{\theta \searrow 0} \nu^2(\theta)$ is given by equation (21). Utilizing the formula for the wealth-consumption ratio (51) and the result from Lemma 3.5 then yields

$$\begin{aligned} \lim_{\theta \searrow 0} \nu^1(\theta) &= \lim_{\theta \searrow 0} \beta^{\frac{\gamma}{\rho}} + (\rho - \gamma) [\xi^1(\theta)]^{-1} = \beta + (\gamma - \rho) \left(\mu_y + u^2 \sigma_y - \frac{1}{2} (1-\gamma) \sigma_y^2 \right) + \\ &\quad + \frac{\gamma - \rho}{1-\rho} \left[(u^1 - u^2) \sigma_y + \frac{1}{2} \frac{(u^1 - u^2)^2}{1-\gamma} \right]. \end{aligned}$$

The first two terms in the last expression are equal to the limit for $\nu^2(\theta)$, which yields the result for the difference of the discount rates. The expression for part (ii) is obtained by symmetry. ■

It remains for me to verify that the assumption about the boundedness of wealth consumption ratios indeed holds.

Corollary B.5 *Under parameter restrictions in Assumption A.1, the wealth-consumption ratios are bounded and bounded away from zero.*

Proof of Corollary B.5. The critical point is the limits for the consumption-wealth ratios as the Pareto share of one of the agents becomes small. Since the large agent's consumption-wealth ratio converges to that in a homogeneous economy, the relevant parameter restriction is the same as restriction (18) in Assumption A.1. The consumption-wealth ratio of the small agent is given in expression (51), and restriction (19) in Assumption A.1 assures that this quantity is strictly positive, and the wealth-consumption ratio finite.

■

Proof of Corollary 3.7. Utilize results in Proposition 3.5 and the fact that $\lim_{\theta \searrow 0} \mu_{A^2}(\theta) = \mu_y$ and $\lim_{\theta \searrow 0} \sigma_{A^2}(\theta) = \sigma_y$, then form the differences in the limiting expected logarithmic growth rates, and compare them to inequalities in Proposition 3.2. ■

Proof of Proposition 3.8. The difference in expected logarithmic returns is obtained by computing the limiting behavior of

$$(\pi^1(\theta) - \pi^2(\theta)) \left[[\xi(\theta)]^{-1} + \mu_A(\theta) - r(\theta) \right] - \frac{1}{2} \left[(\pi^1(\theta))^2 - (\pi^2(\theta))^2 \right] (\sigma_A(\theta))^2,$$

utilizing the results for $\theta \searrow 0$ from Propositions 3.3 and 3.5. The first term above is the difference in the risk premium associated with the two portfolios, and the second term is the lognormal correction. The same propositions also contain the results for the consumption-wealth ratios of the two agents. ■

Proof of Corollary 4.1. The results are obtained by taking limits of the expressions in Proposition 3.6.

■

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