Forward-Looking Behavior Revisited: A Foundation of Time Inconsistency

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April 2, 2014

Abstract

Regardless of its interpretation, the standard exponentially-discounted-utility model implies myopically forward-looking behavior: Preferences at a given period are completely determined by consumption in that period and preferences at the next period. This paper axiomatizes preferences that capture fully forward-looking behavior, delivering a new class of utility representations which includes quasi-hyperbolic (β-δ) discounting as a particular case. These representations rationalize phenomena left unexplained not only by the standard model, but also by the β-δ discounting model. Time inconsistency, present bias, and other phenomena are necessary, logical consequences of fully forward-looking behavior and do not require time-varying preferences or psychological considerations absent from the standard model. The approach also delivers tractable, Bellman-type equations characterizing optimal consumption streams, as well as new insights for the welfare analysis of time-inconsistent agents.

Keywords: time inconsistency, forward-looking behavior, hyperbolic discounting, beta-delta discounting, anticipations, welfare criterion.
JEL Classification: D01, D60, D90

*The authors are grateful to Nageeb S. Ali, Nabil Al-Najjar, Eddie Dekel, Bart Lipman, Peter Klibanoff, Mark Machina, Paul Milgrom, David G. Pearce, Todd Sarver, Marciano Siniscalchi, and Asher Wolinsky, as well as seminar participants at UC San Diego, Stanford, Université de Montréal for useful comments. An early version of this project was developed by Strulovici and presented at Northwestern University and the 2011 SAET conference in Faro, Portugal under the title "Forward-Looking Behavior, Well-Being Representation, and Time Inconsistency." Strulovici is grateful for financial support from the National Science Foundation and a Sloan Research Fellowship.

1 Introduction

Except in few pathological cases, decision makers (DMs, hereafter) are forward-looking: Facing a decision that entails later consequences, a DM takes into account his future preferences when making the decision. Such forward-looking behavior shapes many intertemporal economic choices, making it essential to understand how, in our models, DMs incorporate their future preferences into current decisions.

This paper proposes a general theory of intertemporal choice by a DM who is 'fully' forward-looking, in the following sense: DM's current preference over consumption streams directly depends on his preferences over streams at all future dates.[1] The theory departs from the standard exponentially-discounted-utility (EDU) paradigm (Samuelson (1937) and Koopmans (1960, 1964)), and axiomatizes a new class of models that are consistent with several phenomena perceived as anomalies under EDU. In particular, it provides a new, rational foundation for well-known models of time-inconsistent behavior, without invoking time-varying preferences, stochastic outcomes, and psychological or normative considerations absent from EDU.

The departure from EDU is simple, and has a natural motivation. Introducing EDU, Samuelson (1937) conceived a forward-looking DM who ranks consumption streams $c = (c_0, c_1, \ldots)$ by considering the utility $u(c_t)$, "regarded as a time flow," at each t and then computing "the sum of all future utilities, they being reduced to comparable magnitude by [...] some simple regular" form of discounting. That is, DM maximizes

$$U(c) = u(c_0) + \delta u(c_1) + \delta^2 u(c_2) + \cdots .$$

However, Samuelson viewed it as "completely arbitrary to assume that [DM] behaves so as to maximize" an objective of this form. In particular, he was uncomfortable with "the assumption that at every instant of time [DM's] satisfaction depends only upon the consumption at that time." One problem with interpreting $u(c_t)$ at $t \geq 1$ as representing DM's "satisfaction" (i.e., well-being) at t is the following. If at t DM continues to be forward-looking and to care about future consumption, his time-t well-being should depend on $(c_t, c_{t+1}, \ldots)$.

This suggests a second, perhaps more coherent, interpretation, which is to think of $U(c_t, c_{t+1}, \ldots)$—representing DM's preference over consumption streams at t—as his

[1] Since the goal here is to study forward-looking behavior, the paper focuses on the case in which current preference depends on future preferences. Of course, it is natural to consider the case in which current preference may also depend on past preferences. But this is beyond the scope of the present paper.

time-t well-being. Following this view, EDU is equivalent to

$$U(c_0, c_1, \dots) = u(c_0) + \delta U(c_1, c_2, \dots).$$

With this interpretation, a DM who behaves according to EDU does not incorporate his well-being beyond tomorrow (i.e., at $t \geq 2$) into today's decisions. In other words, DM's time-0 preference over consumption streams cannot depend directly on his time-t preference over streams, for any $t \geq 2$. One might look for a third, more compelling interpretation of EDU that circumvents this irrelevance of future preferences. But such an interpretation does not exist: That property holds *regardless of the interpretation,* as it is a direct consequence of the mathematical form taken by EDU. To see why this irrelevance property limits the extent to which EDU can capture forward-looking behavior, consider two streams, c and c', starting at time 1. Suppose that c provides a better consumption than c' for *all* $t > 1$, but that c'_1 is so much better than c_1 that, if asked at time 1, DM would be indifferent between c and c'. Now suppose DM must choose at time 0 between c or c'. Under EDU, he should be exactly indifferent. It seems plausible, however, that at least some DMs may strictly prefer c, taking into account that it systematically dominates c' after time 1.

Such preferences are captured by the more general representation

$$U(c) = V(c_0, U_1, U_2, \dots), \tag{1}$$

where $U_t = U(c_t, c_{t+1}, \dots)$ represents DM's preferences at t. We will sometimes interpret U_t as DM's *well-being* in period t. This well-being interpretation of the total utility, while somewhat restrictive, is helpful to convey the intuition for this paper's results.[2] Representation (1) highlights the two conceptual premises of the paper. First, future well-being, not future instantaneous utility, determines today's well-being (together with today's consumption). This implies, in particular, that DM is insensitive to a shuffling of consumption across future dates, as long as this does not change his well-being in future periods. The second premise is that today's consumption is inherently different from (future) well-being, a more general and comprehensive entity. Such a distinction links this paper with Böhm-Bawerk's (1890) and Fisher's (1930) early work on intertemporal choice.

Despite preferences being invariant over time—a hypothesis maintained throughout—fully forward-looking behavior turns out to be incompatible with time consistency. This latter property requires that, if a course of action is preferable according to tomorrow's

[2]This interpretation may also suggest a connection with the literature on intergenerational altruism. As explained in Section 2, a connection exists only at a superficial, formal level. Conceptually, however, this paper is unrelated to that literature.

preference, then it remains preferable, for tomorrow, according to today's preference.[3] DM is time consistent if and only if V in (1) depends only on its first two arguments, i.e., if he cares about his future preferences only through his preference at the next period. Moreover, fully forward-looking behavior implies a specific form of time inconsistency: present bias, namely a tendency to pursue immediate gratification. This result may seem paradoxical, for a fully forward-looking DM cares *more* about his future preferences than in standard models. This behavior has a simple explanation which, while resolving the paradox, differs significantly from one's usual understanding of present bias. Suppose that a fully forward-looking DM is indifferent, at time 0, between two continuation streams c and c' that start at time 1, with $c_1 > c'_1$, $c_2 < c'_2$, and $c_t = c'_t$ for $t > 2$. Suppose also that, everything else equal, DM likes higher consumption better. So c' must provide a higher well-being than c at time 2. *Because* this higher well-being at 2 is directly taken into account at 0, for DM to be indifferent, it *must be* that c provides a higher well-being than c' at time 1. Now recall that preferences are time invariant. It follow that, if DM had to choose between c and c' at time 0, he would strictly prefer c, which has higher immediate consumption, and hence exhibits present bias.[4]

The general representation (1) hinges on the following, simple axiom. Given two streams with equal initial consumption, if DM is indifferent between their continuations starting at any future date (i.e., ${}_tc = (c_t, c_{t+1}, \ldots)$), then he cannot prefer either stream—for example, because they allocate consumption differently over time. This ensures that DM's current preference depends on future consumption only through future preferences. To obtain a more tractable model of fully forward-looking behavior, we explore the consequences of imposing time separability and stationary dependence on future *preferences/well-being*, in contrast to future *consumption/instantaneous utility*. These additional axioms lead to a new class of utility representations, which takes the form

$$U(c) = u(c_0) + \sum_{t=1}^{\infty} \alpha^t G(U({}_tc)), \tag{2}$$

where the function G captures how DM values, at time 0, his future well-being. Separability in immediate consumption and future well-being follows from an adaptation of Debreu's (1960) separability axiom. Regarding stationarity, in general, it says that *ex ante* DM does not change how he ranks options, if all options are simply postponed by one period. Of course, for this property to be reasonable, the time shift should not

[3] Time consistency is conceptually different from stationarity of a preference (Koopmans (1960)), even though these properties are often viewed as synonyms (see Section 8).

[4] As explained in the paper, the source of time inconsistency here differs from usual rationales that view DM as overweighing the present vs. the future (Laibson (1997)), or imply that he is prone to temptations (Gul and Pesendorfer (2001)), or view his preferences as changing over time (Strotz (1955)).

change the nature of the considered options. More concretely, postponing consumption streams that start at a future t does not change their nature, for it involves shifting future well-being. Instead, postponing consumption streams that start today changes their nature, for it transforms immediate consumption into future well-being. Therefore, this paper's stationarity axiom refers only to future well-being; it also differs substantively from Koopmans's (1960) stationarity as well as from Olea and Strzalecki's (2013) quasi-stationarity (see Sections 2 and 8).

The behavioral implications of representation (2) crucially depend on the properties of G. The paper first characterizes properties of G (within the class of H-continuous functions, a concept that we introduce to deal with infinite horizon problems) that ensure existence and uniqueness of a solution U, given α and u, and thus help to choose G in applications. These properties also imply that a fully forward-looking DM always exhibits impatience, namely a general aversion to delaying consumption at any time—akin to positive discounting under EDU. The paper then shows how to apply representation (2) to study, for example, consumption-saving problems using a generalized, Bellman-type equation. It also derives an explicit formula for DM's discount factor between 0 and t. For general G, this factor will depend (through well-being) not only on consumption at t—as empirically observed (see Frederick et al. (2002))—but also on the entire infinite consumption stream—as suggested by Fisher (1930). Finally, representation (2) can explain other phenomena that appear as anomalies under EDU:[5] consumption interdependences across periods, higher discounting of gains than of losses ('sign effect'), and a desire to advance dreadful events and to postpone delightful ones.

A particularly interesting case of representation (2) arises when G is linear. In this case, the discount factor between 0 and t takes the renown form $\beta\delta^t$—i.e., quasi-hyperbolic discounting (Phelps and Pollak (1968); Laibson (1997))—where β and δ are simple functions of α and G's constant slope. Linearity of G is characterized by one additional axiom. Even if DM's preference is separable in immediate consumption and future well-being, consumption trade-offs between two periods can depend on well-being—hence consumption—in intermediate as well as future periods. This interdependence, though consistent with some evidence, is clearly absent in the β-δ model. So the additional axiom simply rules it out.

The resulting axiomatic foundation of β-δ discounting sheds light on its main features. First, the stark, apparently ad-hoc, distinction between short- and long-run discounting of instantaneous utility arises from the conceptual distinction between immediate con-

[5]For evidence of these phenomena, see, e.g., Frederick et al. (2002).

sumption and future well-being, combined with stationarity in future well-being. Second, the time separability in instantaneous utility follows from separability in immediate consumption and future well-being, combined with the impossibility that intertemporal consumption trade-offs depend on intermediate and future well-being. This surprising, yet natural, axiomatization of β-δ discounting complements the existing ones (see Section 2).

Finally, this paper suggests a different angle from which we can think about welfare in contexts with time-inconsistent preferences. An immediate consequence of the paper's results is to weaken the case for paternalistic interventions, which usually views time inconsistency as a form of bounded rationality. Since a fully forward-looking DM already takes into account his well-being (i.e., preferences) in all future periods, a 'libertarian' planner may reasonably use his ex-ante preference to assess welfare.[6] As Rubinstein (2003) notes, in the case of β-δ discounting, the literature has usually assumed that welfare should be assessed using the associated EDU with discount factor δ. The present paper *derives* such a welfare criterion from representation (2) with linear G. That is, the EDU criterion based on DM's revealed δ is equivalent to a criterion that weighs DM's well-being at t using the revealed discount factor α^t in (2). This criterion, of course, implies that the planner is time consistent.

2 Related Literature

Several empirical studies have criticized EDU, and several papers have addressed its flaws in various ways.[7] For instance, Loewenstein (1987), Caplin and Leahy (2001), and Kőszegi (2010) add to EDU a dimension called anticipation utility. That is, besides the usual instantaneous utility at date t, consumption at t also generates a distinct utility at all $t' < t$. Such a utility captures the effects on DM's overall happiness of feelings, like excitement or anxiety, that he harbors today when thinking about future events. This is different from anticipating, in the sense of imagining, the future feelings arising when those events occur.[8]

Therefore, anticipation utility differs from fully forward-looking behavior, which simply means that DM cares about the future per se—as in EDU—and correctly predicts that he continues to do so in the future. To see this, consider a simple two-period example.

[6] Note that such a DM is also never subject to money-pump schemes, for instance.

[7] See also Epstein (1983), Epstein and Hynes (1983).

[8] In Böhm-Bawerk's (1890) words, "We must distinguish between two fundamentally distinct things [...]. It is one thing to represent to ourselves, or imagine, a future pleasure or future pain [...]. It is quite another thing to experience, in this imagination itself, [...] an actual present pleasure of anticipation." (Book V, Ch. 1, p. 239).

In period 2, DM consumes a concert and, while there, a drink. His period-2 utility is $u(s,d)$, which is strictly increasing in the seat's proximity to the stage, s, and the drink's quality, d. In period 1, DM derives anticipation utility in the form of excitement from thinking about being at the concert, but no anticipation utility from the drink. Let DM's excitement utility be $e(s)$, which is strictly increasing. In period 1, DM has \$100 that he can use to buy a concert ticket; whatever is left will go for the drink. In period 1, DM maximizes $e(s) + u(s,d)$, whereas in period 2 he would maximize $u(s,d)$. So, for any d, DM's marginal utility from s is higher in period 1. He is therefore willing to spend more for a ticket in period 1 than in period 2; i.e., he is time inconsistent. By contrast, with only two periods, a fully forward-looking DM simply maximizes $u(s,d)$ in both periods and consequently is time consistent.

Although conceptually different, anticipation-utility models can also lead to time inconsistency and can accommodate some anomalies of EDU addressed in the present paper. This paper, however, does so relying on the notion of forward-looking behavior and therefore complements the explanations based on anticipation utility.

Phelps and Pollak (1968) introduced β-δ discounting, so as to analyze economies populated by non-overlapping generations that are 'imperfectly altruistic.' That is, this paper views as more plausible that the current generation cares significantly more about itself than about any future generation—composed, after all, of unborn strangers. In this view, the β-δ formula, though simple, has a natural justification. Laibson (1997) makes a significant conceptual leap, by applying the same formula to individual decision-making. He justifies using it based on its ability "to capture the qualitative properties" of hyperbolic discounting, which has received substantial empirical support. Nonetheless, with a single DM, it is more difficult to justify why he cares significantly more about his immediate consumption than *his own* future consumption in a uniform way.

Several papers have provided axiomatic justifications for the β-δ model in settings with a single DM (see, e.g., Hayashi (2003); Olea and Strzalecki (2013)). These axiomatizations differ from that of the present paper as follows. First, they continue to view DM as caring about instantaneous utilities. Within this framework, they replace Koopmans's (1960) stationarity—clearly violated by β-δ discounting—with quasi-stationarity, namely stationarity from the second period onward. Assuming quasi-stationarity, however, does not address the conceptual issue posed by Laibson's model. Second, to obtain the β-δ representation, they need to ensure that current and future instantaneous utilities are cardinally equivalent. Olea and Strzalecki's axioms permit useful experiments to identify and measure β and δ, but they seem difficult to interpret. In contrast, the present paper views DM as caring about immediate consumption and future well-being (i.e., preferences). As a result, its notion of stationarity—involving only well-being—has a more natural interpretation and, as noted, can offer a rationale for the β-δ formula in Laibson's setting. The paper also shows that this formula is tightly linked with an intuitive property, namely that intertemporal consumption trade-offs do not depend on intermediate and future well-being.

As noted, this paper is related to the literature on intergenerational altruism at a superficial, formal level, but it differs at a deeper, conceptual level. That literature considers a collection of individuals, not a single DM. Clearly, a DM's time preference and his altruism towards other individuals are two distinct and orthogonal aspects of behavior. More specifically, Saez-Marti and Weibull (2005) derive the mathematical equivalence between the β-δ formula and the representation in (2) with linear G. But they do not provide an axiomatic foundation of either representation and, hence, do not address the conceptual issues mentioned before. Pearce's (2008) work on nonpaternalistic sympathy considers a cake-eating model with finitely many generations; each generation's well-being depends on its consumption as well as the well-being of all other or only future generations. That paper focuses on equilibrium analysis, proving general inefficiency results, and takes utility functions as given. In a similar vein, Bergstrom (1999) studies the systems of utility functions that include altruism towards others, focusing on the infinite regress that it may generate. Finally, in his study of hedonistic altruism and welfare, Ray (2014) examines welfare criteria that aggregate the well-being of altruistic, time-consistent, generations and that are formally similar to those in Section 6.[9]

Finally, a literature in philosophy studies the normative question of how an individual's different 'selves,' corresponding to different times, should be weighted in a given decision problem (Parfit [1971, 1976, 1982]). Similarly, economists have studied normative criteria for intergenerational problems (see, e.g., for catastrophe risk). These normative issues, while interesting, are orthogonal to the question considered here.

3 Preliminaries

This paper considers how a decision maker (DM) chooses consumption streams. At each time t, the set of feasible consumption levels is X, a connected, separable, metric space. Time is discrete with infinite horizon: $T = \{0, 1, 2, \ldots\}$. For $t \geq 0$, the set of

[9] Jackson and Yariv (2011) study the problem of how to aggregate preferences of heterogeneous, *time-consistent* individuals in collective dynamic decisions and show that natural procedures lead to time inconsistency.

consumption streams starting at time t is ${}_tC \subseteq X^T$—elements of this set will be denoted by ${}_tc = (c_t, c_{t+1}, \ldots)$. For each t, consider the preference relation $\succ^t$ with domain ${}_tC$, which represents DM's choices over consumption streams at t.[10] In principle, ${}_tC$ and $\succ^t$ could vary over time. In this paper, however, they are time invariant.

Assumption 1 (Time Invariance)**.** *For all $t \geq 0$, ${}_tC = C$ and $\succ^t = \succ$.*

The set C is endowed with the sup-norm: $\|c - c'\|_C = \sup_t d(c_t, c'_t)$ where d is the metric on X.

A basic premise of this paper is that $\succ$ has a utility representation. As shown in Appendix B, this property follows from standard axioms. To simplify exposition, it is stated here as an assumption.

Assumption 2 (Utility Representation)**.** *There is a continuous function $U : C \to \mathbb{R}$ such that $c \succ c'$ if and only if $U(c) > U(c')$. Moreover, U is nonconstant in the first and some other argument.*

Remark 1. In the rest of the paper, $U({}_tc)$ will be interpreted as DM's *well-being* (i.e., total utility) generated by consumption stream ${}_tc$. This is done only for the sake of conveying intuitions more easily,

Since this paper is interested in forward-looking behavior, by assumption, well-being at t depends on consumption after t. It is also natural that well-being depends on immediate consumption. For future reference, let $\mathcal{U}$ be the range of U. Note that $\mathcal{U}$ is an interval because U is continuous and non-constant and X is connected. Since DM's preferences are time invariant (Assumption 1), for simplicity, hereafter the paper takes the perspective of $t = 0$.

4 Forward-looking Behavior and Time Inconsistency

4.1 'Well-Being' Preference Representation

This paper considers a DM who, at each t, is forward-looking in the sense that he cares about his future preferences, represented by well-being U. That is, it focuses on $\succ$ that has the following recursive representation.

[10] This paper continues to assume that, at each t, preferences do not depend on past consumption. Relaxing this assumption is beyond the paper's scope.

Definition 1 (Well-Being Representation)**.** Preference $\succ$ has a well-being representation if and only if

$$U(c) = V(c_0, U(_1c), U(_2c), \ldots) \tag{3}$$

for some function $V : X \times \mathcal{U}^{\mathbb{N}} \to \mathbb{R}$ that is nonconstant in c_0 and $U(_tc)$ for some $t > 0$.[11]

So, how DM ranks consumption streams c and c' at 0 depends *only* on the immediate consumption levels c_0 and c'_0, and on how he ranks continuation streams $_tc$ and $_tc'$ for at least some future period t. In other words, DM well-being from any c depends *only* on his immediate consumption c_0 and, for at least some future t, on his per-period well-being from $_tc$. Well-being is therefore conceptually different from the immediate utility from a single consumption event: Well-being captures an aggregate of sensorial pleasure from immediate consumption as well as purely mental satisfaction (or dissatisfaction) from future well-being. While such an aggregation seems difficult to make, it is somehow performed by any forward-looking DM who must choose current and future consumption in a dynamic optimization problem. The definition is recursive, as one should expect: For a forward-looking DM, well-being today involves well-being in the future. Finally, note that, at this stage, V may be strictly decreasing in $U(_tc)$ for some $t > 0$.

The key axiom to obtain representation (3) is the following. It captures the idea that consumption streams starting tomorrow affect well-being today, only through the well-being that they generate at each future time.

Axiom 1. *If $_tc \sim {_tc'}$ for all $t > 0$, then $(c_0, {_1c}) \sim (c_0, {_1c'})$.*

Axiom 1 rules out the possibility that DM prefers stream c over c' because, even though they generate the same stream of immediate consumption and future well-being, they allocate future consumption differently over time.

Theorem 1. *Axiom 1 holds if and only if $\succ$ has a well-being representation.*

Proof. Let $f_0(c) = c_0$ and, for $t > 0$, $f_t(c) = U(_tc)$. Also, let $f(c) = (f_0, f_1, f_2, \ldots)$ and

$$\mathcal{F} = \{f(c) : c \in C\}. \tag{4}$$

($\Rightarrow$) First, define equivalence classes on C as follows. Say that c is equivalent to c' if $f_t(c) = f_t(c')$ for all $t \geq 0$.[12] Let C^* be the set of equivalence classes of C, and let

[11]Of course, one can allow DM's preferences to differ across time, yet obtain for each $\succ^t$ a well-being representation. In this case, U and V will depend on t.

[12]In general, there may be several consumption streams in an equivalence class. For example, suppose that $U(c) = c_0 + c_1 + c_2 + c_3$, and let $c = (1, 1, -1, -1, 1, 1, -1, -1, \ldots)$ and $c' = (1, -1, -1, 1, 1, -1, -1, 1, 1, \ldots)$.

the function U^* be defined by U on C^*. Then, the function $f^* : C^* \to \mathcal{F}$, defined by $f^*(c^*) = f(c)$ for c in the equivalence class c^*, is by construction one-to-one and onto; so let $(f^*)^{-1}$ denote its inverse. Finally, for any $f \in \mathcal{F}$, define

$$V(f) = U^*((f^*)^{-1}(f)).$$

By Axiom 1, V is a well-defined function, and $V(f(c)) = U(c)$ for every c. By Assumption 2, V is nonconstant in f_0 and f_t for some $t > 0$.

($\Leftarrow$) Suppose that $V : \mathcal{F} \to \mathbb{R}$ is a continuous function such that $V(f(c)) = U(c)$. Then, it is immediate to see that the implied preference satisfies Axioms 1.

□

Note that Axiom 1 is weak: It requires that DM be indifferent at 0 between two consumption streams, only if he is indifferent between their truncations at *all* future dates, not just next one. Clearly, if at 0 DM cares only about his well-being at 1—as under EDU—Axiom 1 holds. As clarified in Section 8, by allowing current preference to depend on future preference in a richer way, Axiom 1 represents a key departure of this paper from the set of axioms characterizing EDU (Koopmans [1960, 1964]).

4.2 Time (In)consistency

Among the properties of intertemporal choice, time consistency is perhaps the most prominent and studied one. Therefore, this paper first considers what implications time consistency has on the extent to which DM is forward-looking—that is, on how V depends on future well-being. Surprisingly, there is a tension between being forward-looking and being time consistent.

In the present setup, the idea that choices are consistent over time is captured as follows (see, e.g., Siniscalchi (2011)).

Definition 2 (Time Consistency)**.** If ${}_1c \sim^1 {}_1c'$, then $(c_0, {}_1c) \sim^0 (c_0, {}_1c')$. If ${}_1c \succ^1 {}_1c'$, then $(c_0, {}_1c) \succ^0 (c_0, {}_1c')$.[13]

To understand this definition, suppose that at 1 DM will have to choose whether to buy or not a ticket to go to Hawaii at $t > 1$. Does he rank these two options in the same way at 1 and at 0? Importantly, these rankings are those revealed by DM's choices at 1 and at 0—which, by Assumption 1, are generated by the same preference. Definition 2 first

[13] Definition 2 looks similar to the stationarity axiom in Koopmans [1960, 1964]. However, time consistency and stationarity are conceptually very different (see Section 8).

requires that, if at 1 DM is indifferent between buying the ticket or not, then at 0 he should also be indifferent between buying or not at 1. Similarly, if at 1 DM prefers not to buy the ticket, then at 0 he should also prefer not to buy it at 1.[14]

Time consistency has strong implications on the degree of forward-looking behavior. Indeed, DM's preference is time consistent if and only if he cares about immediate consumption and (positively) about his well-being in the next period only.[15]

Proposition 1. *Preference $\succ$ satisfies time consistency if and only if* $V(c_0, U({}_1c), U({}_2c), \ldots) = V(c_0, U({}_1c))$, *for all* $c \in C$, *and* V *is strictly increasing in its second argument.*

To see the intuition, suppose that ${}_1c$ corresponds to buying a ticket to go to Hawaii at $t > 1$, and that, at 1, DM slightly prefers not to buy the ticket. Now imagine that, at 0, DM can choose whether to buy the ticket at 1. Then, at 0, if DM cares directly about his well-being beyond 1—i.e., when he will be at Hawaii—he should strictly prefer to buy the ticket at 1. Intuitively, from the perspective of 1, the negative effect on immediate well-being of spending money for the ticket just offsets the positive effect of higher period-t well-being. But, at 0, since DM cares *directly* about well-being beyond 1, he weighs more the positive effect of the Hawaii trip, thus strictly preferring to buy the ticket at 1. More generally, this mechanism can lead to situations in which, at 1, DM wants to review plans he made at 0.

Proposition 1 has several implications. First, a DM who cares about his well-being beyond the immediate future must be time inconsistent. So this paper identifies a cause of time inconsistency in DM's caring, at 0, about his well-being beyond 1. Moreover, if we deem natural that a DM should care about his future in this way, then based on Proposition 1, we must conclude that time inconsistency should be the rule, rather than the exception. The second implication is that DM's preference is time consistent if and only if, at each t, it has a specific, well-known recursive representation: DM's utility from a consumption stream, $U(c)$, depends only on immediate consumption, c_0, and continuation utility $U({}_1c)$. Moreover, at 0, DM must always be better off if his continuation utility is higher. Of course, in this case, at 1 DM never wants to review a consumption plan chosen at 0, since it was already maximizing his period-1 well-being.

As noted, EDU satisfies

$$U(c) = u(c_0) + \delta U({}_1c) = V(c_0, U({}_1c)).$$

[14] One could consider a weaker version of the second part of Definition 2: If ${}_1c \succ^1 {}_1c'$, then $(c_0, {}_1c) \succsim^0 (c_0, {}_1c')$. This version, however, would be at odds with time consistency, unless DM is fully myopic.

[15] All omitted proofs are in Appendix B.

In this paper, if the preference at t depends only on immediate consumption and the preference at $t+1$, then DM is called 'myopically forward-looking.'[16] On the other hand, a DM is called 'fully forward-looking' if, at all t, his preference also depends directly on his preferences at all $s > t$.[17]

4.3 Present Bias

In general, time inconsistency can manifest itself in different forms. So a natural question is whether fully forward-looking behavior implies specific forms of time inconsistency. We already know from Assumption 1 that, here, time inconsistency cannot arise because DM's preference changes over time. For example, it cannot take the form that at 0 DM always prefers higher consumption for period 1, but at 1 he always prefers lower consumption for period 1. Indeed, forward-looking behavior implies a form of time inconsistency that involves changes in intertemporal trade-offs depending on whether these occur in the future or in the present. In short, fully forward-looking behavior implies present-bias, a tendency to pursue immediate gratification.

Definition 3 (Present Bias)**.** Let x, y, w, $h \in X$ and $c' \in C$. Suppose $(x, c) \succ (y, c)$ for all c and $(z_0, \ldots, z_t, x, w, c') \sim (z_0, \ldots, z_t, y, h, c')$ for some $t \geq 0$, then $(x, w, c') \succ (y, h, c')$.

Intuitively, this definition says the following. Suppose that, fixing consumption in all other periods, DM always strictly prefers consumption x to y. Also, suppose there is a consumption h (e.g., a trip to Hawaii) that can make DM indifferent at 0 between getting x or y at some future t, provided y is 'compensated' with h at $t+1$. Then, given the same choice at 0, a present-biased DM continues to strictly prefers x to y, thus pursuing immediate gratification.

Proposition 2. *If $V(c_0, U({}_1c), U({}_2c), \ldots)$ is strictly increasing in $U({}_tc)$ for all $t > 0$,*[18] *then it implies present bias.*

Though perhaps surprising, this result has a simple intuition. Suppose V is strictly increasing in well-being at all future dates. Also, suppose that y followed by h corresponds to buying a ticket to go to Hawaii next period, whereas x followed by w corresponds to not buying it. Thinking at 0 about whether to buy the ticket at t, a fully forward-looking DM takes into account how the trip affects his period-0 well-being both directly through

[16] Another interpretation is that such a DM fails to realize that, in the future, he will continue to care about future well-being.

[17] Of course, one could also consider the case in which V depends on $U({}_tc)$ up to some finite $\bar{t} > 1$.

[18] Section 5.1 will introduce an axiom that captures this property (Axiom 5).

his well-being at $t+1$ and indirectly through his well-being at t and before. Instead, facing the same choice at 0, DM takes into account only how the trip directly affects his period-0 well-being through his period-1 well-being. Thus, at 0, although the enjoyment of the trip at $t+1$ and, indirectly, in all previous periods is enough to compensate the ticket price at t, the direct enjoyment of the trip next period only is not enough. So, at 0, DM prefers not to buy the ticket; this gives him higher immediate gratification.

5 Refining the Well-being Representation

5.1 Time Separability and Well-being Stationarity

This section refines the general representation V in (3), by considering preferences that satisfy some form of time separability and stationarity. The literature has studied these properties in relation to per-period utility from consumption events. The goal here is to explore their consequences when applied to per-period well-being from consumption streams.

The first two axioms imply that $\succ$ is time separable—that is, separable in immediate consumption and future well-being, as well as across future well-being. Let Π consist of all unions of subsets of $\{\{1\},\{2\},\{3,4,\ldots\}\}$.

Axiom 2 (Immediate-Consumption and Well-Being Separability)**.** *Fix any $\pi \in \Pi$. If c, $\hat{c}$, c', $\hat{c}' \in C$ satisfy*
(i) ${}_tc \sim {}_t\hat{c}$ and ${}_tc' \sim {}_t\hat{c}'$ for all $t \in \pi$,
(ii) ${}_tc \sim {}_tc'$ and ${}_t\hat{c} \sim {}_t\hat{c}'$ for all $t \in T \setminus \pi$,
(iii) either $c_0 = c'_0$ and $\hat{c}_0 = \hat{c}'_0$, or $c_0 = \hat{c}_0$ and $c'_0 = \hat{c}'_0$,
then $c \succ c'$ if and only if $\hat{c} \succ \hat{c}'$.

Intuitively, Axiom 2 says the following. Consider $\pi = \{1\}$, for example. Suppose c yields higher well-being than c' at 1, but otherwise immediate consumption and future well-being are the same under c and c'. Suppose DM prefers c to c'. Then he should also prefer $\hat{c}$ to $\hat{c}'$, if at 1 these streams yield the same well-being as c and c' respectively, but otherwise immediate consumption and future well-being are again equal under $\hat{c}$ and $\hat{c}'$. Axiom 2 is inspired by Debreu's (1960) and Koopmans's (1960) separability axioms. It differs to the extent that it requires that certain consumption streams be indifferent, rather than that certain consumption events be equal. This is because we want separability in well-being, which can be the same across streams allocating consumption differently over time.

Axiom 3 is of technical nature; it ensures that $\succ$ does depend on well-being at period 1, 2, and 3 (Debreu's essentiality condition).

Axiom 3 (Essentiality)**.** *There are* x, x', y, $y' \in X$ *and* c, $c' \in C$ *such that* $(z, x, \hat{c}) \succ (z, x', \hat{c})$, $(z', z'', y, c'') \succ (z', z'', y', c'')$, *and* $(w, w', w'', c) \succ (w, w', w'', c')$ *for some* z, z', z'', w, w', $w'' \in X$, *and* $\hat{c}, c'' \in C$.

Axiom 4, also inspired by Koopmans (1960), captures the idea that $\succ$ is stationary. Intuitively, stationarity means that, at 0, DM does not change how he ranks consumption events, simply because they are postponed to a subsequent date. Of course, requiring this property is reasonable only if postponing an event does not change its nature. In this paper, however, instantaneous consumption and future well-being are conceptually different. So Axiom 4 requires stationarity only with respect to future well-being.

Axiom 4 (Well-Being Stationarity)**.** *If* c, $c' \in C$ *satisfy* $c_0 = c'_0$ *and* ${}_1c \sim {}_1c'$, *then*

$$(c_0, {}_2c) \succsim (c'_0, {}_2c') \Leftrightarrow c \succsim c'.$$

Intuitively, Axiom 4 says the following. Suppose c and c' differ only in well-being from period 2 onward. Then, if we drop consumption at 1 and shift both continuation streams back one period, DM should rank the new streams as he ranked c and c'. Section 8 compares Axiom 4 with Koopmans's [1960] stationarity and Olea and Strzalecki's [2013] quasi-stationarity.

Finally, Axiom 5 captures the natural case in which a fully forward-looking DM is better off at 0 when his future well-being improves.

Axiom 5 (Well-being Monotonicity)**.** *If* c, $c' \in C$ *satisfy* $c_0 = c'_0$ *and* ${}_tc \succsim {}_tc'$ *for all* $t > 0$, *then* $c \succsim c'$. *Moreover, if* ${}_tc \succ {}_tc'$ *for some* $t > 0$, *then* $c \succ c'$.

Theorem 2 (Additive Well-Being Representation)**.** *Axioms 1-4 hold if and only if the function* U *may be chosen so that*

$$U(c) = u(c_0) + \sum_{t=1}^{\infty} \alpha^t G(U({}_tc)). \tag{5}$$

where $u : X \to \mathbb{R}$ *and* $G : \mathcal{U} \to \mathbb{R}$ *are continuous, nonconstant functions and* $\alpha \in (0, 1)$. *Moreover, if* $\hat{U}$, $\hat{u}$, $\hat{\alpha}$, *and* $\hat{G}$ *represent the same* $\succ$ *as in (5), then* $\hat{\alpha} = \alpha$, *and there exist* $a > 0$ *and* $b \in \mathbb{R}$ *such that* $\hat{U}(c) = aU(c) + b$, $\hat{u}(x) = au(x) + b$, *and* $\hat{G}(\hat{U}) = aG(\frac{\hat{U}-b}{a})$ *for all* c, x, *and* $\hat{U}$. *Finally, Axiom 5 holds if and only if* G *is strictly increasing.*

According to Theorem 2, DM derives an instantaneous utility u from immediate consumption, as usual, and a per-period utility G from future well-being. Moreover, DM discounts utility from future well-being exponentially.

One might wonder whether the expression in (5) is always well-defined for any function G. The cumulative effects of future well-being on preceding well-being may inevitably lead the summation in (5) to diverge to infinity. However, by Assumption 2—i.e., by Axioms 7-10 in Appendix 11.1—U is a nonconstant representation of $\succ$ with values in the interval $\mathcal{U} \subset \mathbb{R}$. Therefore, there always exist consumption streams c such that $U(c)$ is bounded. This implies joint restrictions on α and G.

Proposition 3 considers the natural case of increasing G, identifying a sufficient (and almost necessary) condition for (5) to be well-defined, as well as other properties of G.

Definition 4. A function $U : C \to \mathbb{R}$ is H-continuous if, for every $\varepsilon > 0$, there exists a time $T(\varepsilon)$ such that the following holds: If $c, \tilde{c} \in C$ satisfy $c_t = \tilde{c}_t$ for $t \leq T(\varepsilon)$, then $|U(c) - U(\tilde{c})| < \varepsilon$.

Proposition 3.
(i) Under axioms 1-5, in representation (5) G is bounded, U is H-continuous, and for U', $U \in \mathcal{U}$

$$\frac{1-\alpha}{\alpha} |U' - U| > |G(U') - G(U)| .$$

(ii) Suppose G is strictly increasing, bounded, and K-Lipschitz continuous with $K < \frac{1-\alpha}{\alpha}$, i.e., for all U', $U \in \mathcal{U}$

$$K |U' - U| \geq |G(U') - G(U)| .$$

Then, there is a unique H-continuous function $U : C \to \mathbb{R}$ that solves (5).

This result helps to choose G appropriately in applications. Note that boundedness of G implies that future well-being can have only a limited impact on current well-being: Intuitively, DM cannot become infinitely happy or unhappy just from imagining his future well-being. Instead, no such bound applies to the immediate instantaneous utility.

Proposition 3 also implies that a fully forward-looking DM is always impatient—even if his future well-being at each date depends on subsequent well-being and he correctly anticipates this.

Definition 5 (Impatience)**.** Let x, $y \in X$ be such that $(x, c) \succ (y, c)$ for all $c \in C$. Preference $\succ$ exhibits impatience if, for any $t > 0$, $c^x \succ c^y$ where $c_0^x = x$, $c_t^x = y$, $c_0^y = y$, $c_t^y = x$, and $c_s^x = c_s^y$ otherwise.

Note that impatience differs from present bias (Definition 3). Impatience refers to the trade-off between achieving higher satisfaction at earlier rather than later periods; present bias refers to how this trade-off changes when the earlier period occurs in the present rather than in the future.

Corollary 1. *If axioms 1-5 hold, then $\succ$ exhibits impatience.*

Finally, Theorem 2 suggests that, given α and G, DM's preference is entirely driven by the instantaneous utility u. Corollary (2) formalizes this point.

Corollary 2 (u-Representation)**.** *Given representation* (5) *and H-continuity, there is a continuous nonconstant function $\hat{U}$ such that, for all $c \in C$,*

$$U(c) = \hat{U}(u(c_0), u(c_1), \ldots).$$

5.2 A Bellman-type Equation for Dynamic Choice Problems

To see how to work with representation (5), consider the following consumption-saving problem. For expositional simplicity, formulate it as a cake-eating problem—it should be clear that the method described here can be generalized to other Markovian decision problems. At 0, DM must commit to a stream $(c_0, c_1, \ldots) \in \mathbb{R}_+^{\mathbb{N}}$ subject to the constraint $\sum_{t \geq 0} c_t \leq b$, where b is the cake size. Also, let $C(b)$ be the set of all nonnegative consumption streams satisfying this constraint. Based on representation (5), the optimal utility is given by

$$U^*(b) = \sup_{c_0 \leq b} \{u(c_0) + \alpha W(b - c_0)\}, \tag{6}$$

where

$$W(b') = \sup_{c' \in C(b')} \sum_{t=0}^{\infty} \alpha^t G(U({}_t c')).$$

If we can solve for W, we can then easily determine the optimal consumption plan. Note that, for any $b \geq 0$, we can re-express $W(b)$ as

$$W(b) = \sup_{c_0 \leq b} \left\{ \sup_{c' \in C(b - c_0)} \left\{ G\left(u(c_0) + \alpha \sum_{t=0}^{\infty} \alpha^t G(U({}_t c')) \right) + \alpha \sum_{t=0}^{\infty} \alpha^t G(U({}_t c')) \right\} \right\}.$$

With increasing G, this yields the following Bellman-type equation for W:

$$W(b) = \sup_{c_0 \leq b} \{G(u(c_0) + \alpha W(b - c_0)) + \alpha W(b - c_0)\} \tag{7}$$

Given W, the maximization in (6) determines the optimal c_0 and that in (7) determines c_t for all $t > 0$.

Equation (7) differs from usual Bellman equations mainly because the instantaneous-utility term is inside the function G. Indeed, it reduces to a standard equation if G is linear. However, under minor regularity conditions on G, equation (7) has a well-defined solution W. To see this, define the operator $\mathcal{J}$ on the set $B(\mathbb{R}_+)$ of bounded real-valued functions of $\mathbb{R}_+$ by

$$\mathcal{J}(W)(b) = \sup_{c_0 \leq b} \{G(u(c_0) + \alpha W(b - c_0)) + \alpha W(b - c_0)\}.$$

Then, if G is bounded and K-Lipschitz continuous with $K < (1-\alpha)/\alpha$, it is easy to show that $\mathcal{J}$ is a contraction and therefore has a unique fixed point. So equation (7) uniquely defines W. It is straightforward to approximate numerically this fixed-point, which is just a univariate function, and the rate of convergence of numerical schemes is given as a function of the Lipschitz constant of G.

When DM cannot commit at 0, time inconsistency leads to an equilibrium problem, in which DM chooses c_t at each t. Existence and properties of Markovian equilibria in a similar setting—the 'buffer-stock model,' which includes stochastic shocks to the state (b here)—have been studied by Ray (1987), Bernheim and Ray (1989), Harris and Laibson (2001), and Quah and Strulovici (2013). Bernheim and Ray study a set of utility functions that includes those in Theorem 2, so their equilibrium analysis applies to the preferences studied here.

5.3 Intertemporal Rate of Utility Substitution

The representation in Theorem 2 has interesting implications regarding how DM trades off consumption across periods. By Axiom 2, separability holds between immediate consumption and future well-being, as well as across future well-being. Nonetheless, the trade-off between consumption at 0 and $t > 0$ can depend on well-being—and hence consumption—between 0 and t and after t. This is because well-being at t affects well-being at all $s < t$ and depends on well-being after t.

To see this, consider DM's discount factor between 0 and t. Of course, consumption trade-offs between 0 and t also depend on the curvature of u. To bypass this dependence, we can rely on Corollary 2 and, given stream c, define $u_s = u(c_s)$ and the discount factor as

$$d(t, c) = \frac{\partial U(c)/\partial u_t}{\partial U(c)/\partial u_0}. \tag{8}$$

That is, $d(t, c)$ is the rate at which DM substitutes instantaneous utility between 0 and t. Note that, under EDU, $d(t, c) = \delta^t$. In general, for $d(t, c)$ to be well defined, the derivatives in (8) must exist. This is always the case when G is differentiable.[19]

Proposition 4. *Suppose G in representation (5) is differentiable. Then,* $d(1, c) =$

[19] Note that, when increasing, G is already differentiable at almost every point in $\mathcal{U}$.

$\alpha G'(U({}_1c))$ *and, for* $t > 1$,

$$d(t,c) = \alpha^t G'(U({}_tc)) \left[1 + \sum_{\tau=1}^{t-1} G'(U({}_{t-\tau}c)) \prod_{s=1}^{\tau-1} (1 + G'(U({}_{t-s}c))) \right],$$

where $\prod_{s=1}^{\tau-1}(1 + G'(U({}_{t-s}c))) \equiv 1$ if $\tau = 1$.

This formula has a simple explanation. Suppose c_t changes so that $u(c_t)$ rises by a small amount. This has two effects: (1) well-being rises at t, which explains the term $G'(U({}_tc))$; consequently, (2) well-being rises for all τ between period 1 and t, which explains the summation. Moreover, the rise in $U({}_tc)$ affects $U({}_{t-\tau}c)$ through all well-beings between $t - \tau$ and t, which explains the product.

By Proposition 4, for general G, the discount factor $d(t,c)$ depends on consumption at t as well as on well-being before and after t—hence it depends on the entire stream c (see Section 5.5 for details). The only case in which $d(t,c)$ is independent of c is when G is linear. Surprisingly, in this case, the discount factor takes a very well-known form.

Corollary 3. *Suppose* $G(U) = \gamma U$ *with* $0 < \gamma < \frac{1-\alpha}{\alpha}$. *Then, for all* $t > 0$,

$$d(t,c) = \beta\delta^t,$$

where $\beta = \frac{\gamma}{1+\gamma}$ *and* $\delta = (1+\gamma)\alpha < 1$.

Proof. By Proposition 4, the result is immediate for $t = 1, 2$. For $t > 2$,

$$d(t,c) = \alpha^t \gamma \left[1 + \gamma \sum_{\tau=1}^{t-1} (1+\gamma)^{\tau-1} \right] = \alpha^t \gamma (1+\gamma)^{t-1}.$$

□

5.4 Quasi-hyperbolic Discounting of Instantaneous Utility

A natural question is which properties of DM's preference correspond to G being linear and hence to quasi-hyperbolic discounting. As noted, unless G is linear, the trade-off between utility from consumption at 0 and at $t > 0$ depends on well-being between 0 and t, and after t. This observation suggests Axiom 6.

Axiom 6 (Trade-off Independence).
(i) $(c_0, c_1, {}_2c) \succ (c_0', c_1', {}_2c)$ *if and only if* $(c_0, c_1, {}_2c') \succ (c_0', c_1', {}_2c')$;
(ii) $(c_0, c_1, {}_2c) \succ (c_0', c_1, {}_2c')$ *if and only if* $(c_0, c_1', {}_2c) \succ (c_0', c_1', {}_2c')$.

Intuitively, condition (i) says that the trade-off between consumption at 0 and at 1 is independent of the continuation stream ${}_2c$ and hence of future well-being. Condition

(ii) says that the trade-off between consumption at 0 and *after* 1 is independent of consumption at 1 and hence of well-being at 1.

Theorem 3 ('Vividness' Well-being Representation)**.** *Axiom 1-6 hold if and only if the function U may be chosen do that*

$$U(c) = u(c_0) + \sum_{t=1}^{\infty} \alpha^t \gamma U({}_t c), \tag{9}$$

where $\alpha \in (0, 1)$, $\gamma \in (0, \frac{1-\alpha}{\alpha})$, and $u : X \to \mathbb{R}$ is a continuous nonconstant function.

Corollary 4 (Quasi-Hyperbolic Discounting)**.** *Axiom 1-6 hold if and only if there are β, $\delta \in (0, 1)$ and a continuous nonconstant function u such that*

$$U(c) = u(c_0) + \beta \sum_{t=1}^{\infty} \delta^t u(c_t).$$

Proof. By Theorem 3, for all t, $U(c)$ is a strictly increasing, linear function of $U({}_t c)$, which is in turn a strictly increasing, linear function of $u(c_t)$. Hence, there is a function $\kappa(t) : T \setminus \{0\} \to \mathbb{R}_{++}$ such that

$$U(c) = u(c_0) + \sum_{t=1}^{\infty} \kappa(t) u(c_t).$$

Clearly, for all $t > 0$, $\kappa(t) = d(t, c)$ defined in (8). Corollary 3 implies the result. □

This result allows us to understand β-δ discounting of instantaneous utility in terms of properties of preferences that depend on immediate consumption and future well-being (i.e., preferences). First, the stark, apparently ad-hoc, difference between discounting of future instantaneous utility from today's perspective and from any future period's perspective comes from the conceptual difference between immediate consumption and future well-being. Second, additive time separability in instantaneous utility requires separability between immediate consumption and future well-being as well as across future well-being, but also that the consumption trade-off between any two periods be unaffected by intermediate as well as future well-being.

This paper offers a foundation of quasi-hyperbolic discounting based on forward-looking behavior of a DM who takes account of his well-being beyond the immediate future. This explanation differs from the usual one based on myopia, which says that a DM would discount future consumption quasi-hyperbolically because he cares disproportionately about the present against any future period—if anything, fully forward-looking behavior captures the opposite. Moreover, Corollary 4 tightly links the degree of present bias, β, with DM's ex-ante marginal utility from future well-being, γ. For example, exponential discounting of well-being ($\gamma = 1$) corresponds to $\beta = \frac{1}{2}$. We can interpret the parameter γ as measuring the degree to which DM finds future outcomes 'imaginable' or 'vivid.' This vividness interpretation relates to Böhm-Bawerk's (1890) and Fisher's (1930) idea that DM's current utility depends on immediate consumption as well as on his ability to imagine or foresee his future 'wants.' This essential difference between immediate consumption and future well-being results in an asymmetric treatment of current and future instantaneous utilities from consumption. Such an asymmetry is otherwise difficult to justify, for it involves entities that are identical in nature.

Corollary 4 has several implications. First, consistent with Proposition 2, it implies that β is always strictly less than one. Second, it can give an explanation for why present bias may weaken as the period length shortens. If each period represents a shorter time horizon—e.g., a week rather than a month—current instantaneous utility $u(c_0)$ should play a smaller role in determining current well-being; that is, γ should be larger in (9). Consequently, β should get closer to one. The third implication is about long-run discount factors and how increasing vividness of future well-being can mitigate present bias.[20]

Corollary 5. *For any degree of present bias β, a fully forward-looking DM discounts instantaneous utility using a long-run factor δ that is strictly higher than the factor α used to discount well-being. Moreover, for any α, increasing vividness γ mitigates present bias and improves the long-run factor δ.*

For example, using Laibson et al.'s (2007) estimates of $\beta = 0.7$ and $\delta = 0.95$, we get $\gamma = 2.33$ and $\alpha = 0.285$. So well-being discounting is actually much steeper than one might think by looking only at consumption discounting.

5.5 Beyond Quasi-hyperbolic Discounting

Representation (9) is particularly appealing, for it corresponds to a widely used and tractable model of intertemporal choice. Although quasi-hyperbolic discounting captures well-documented behavioral phenomena like present bias, it cannot capture other phenomena that appear as anomalies through the lenses of EDU. Instead, representation (5) can capture and, most importantly, can offer an explanation for some of these anomalies.

First of all, some people discount future consumption at a rate that may depend on the level of consumption itself (for evidence, see Frederick et al. (2002)). By Proposition 4,

[20] Vividness of the future well-being implied by today's decisions may be influenced with specific ad campaigns. For example, consider the dramatic pictures and reminders printed on cigarette packs. It is hard to believe that such packaging is just meant to inform unaware customers of the consequences of smoking.

this phenomenon would occur because how much DM discounts instantaneous utility from c_t depends on how much his ex-ante utility from well-being at t responds to changes in c_t, which in turn may vary with c_t itself. As noted, more generally, $d(t, c)$ can depend on the entire stream c. This may capture Fisher's (1930) idea that "[an individual's] degree of impatience for, say, $100 worth of this year's income over $100 worth of next year income depends upon the entire character of his [...] income stream pictured as beginning today and extending into the indefinite future." (Ch. 4, §5)

Formally, how $d(t, c)$ varies with c ultimately depends on the properties of G. To state the next result, for any c, $c' \in C$, let $c \geq^* c'$ if and only if $u(c_t) \geq u(c'_t)$ for all $t \geq 0$.

Corollary 6. *Let $d(t, c)$ be as in Proposition 4. For any $t > 0$, $c \geq^* c'$ implies $d(t, c) \leq d(t, c')$ if and only if G' is decreasing. Conversely, $c \geq^* c'$ implies $d(t, c) \geq d(t, c')$ if and only if G' is increasing.*

This result shows a tight link between behavior and a property of G, which in principle can be empirically tested.

Some evidence also suggests that intertemporal choices exhibit consumption interdependences across dates, in contrast with both EDU and the β-δ model (again, see Frederick et al. (2002)). Clearly, a general representation like that in Assumption 2 allows for such interdependences, but offers no insight on their origin. Representation (5), instead, identifies a specific source of interdependences. Although DM treats each period separately with regard to well-being, how he evaluates consumption at a future t can depend on well-being, and hence consumption, before and after t. So, this can explain forward- and backward-looking interdependences that arise in the future.[21]

Such interdependences can create preferences for spreading consumption over time, in a way that is consistent with some evidence. For example, Frederick et al. (2002) discuss the following experiment (p. 364). There are five periods and two opportunities, A and B, to dine at a fancy restaurant. Subjects are asked to rank consumption streams in two scenarios. In scenario 1, one stream features A in period 0 and the other features A in period 2; in all other periods both streams involve dining at home. Scenario 2 coincides with scenario 1, except that both streams feature B in period 4. Presumably, dining at a fancy restaurant yields higher instantaneous utility than dining at home. According to the experiment, in scenario 1, most of the subjects prefers A in period 2. But in scenario 2 a significant fraction of them prefers A in period 0. This change is consistent with a concave G, perhaps the most natural case—concavity of G simply means that

[21]Such interdependences differ from other backward-looking interdependences, like habit formation, which this paper cannot capture.

the marginal value at 0 of increasing well-being at any $t > 0$ decreases in its level. In scenario 1, a DM may prefer to delay A, as the improvement in well-being both at 2 and at 1 may offset discounting.[22] But, when well-being at 4 is higher because of B, well-being at 2 is also higher. Consequently, the benefit of improving period-2 well-being with A is lower than when B is absent; hence DM prefers enjoying A right away.

Finally, representation (5) can explain other empirical findings summarized by Frederick et al. (2002). The *sign effect*, for example, means that gains are discounted more than losses. When G is concave, increasing consumption at some period t reduces the discount factor between 0 and t. Hence, from the point of view of EDU, it appears as if the discount rate increases. The discount factor captures an indifference point in the consumption trade-off between 0 and t. If consumption at t is higher, this indifference occurs at a lower point, for the marginal utility of future well-being at t is lower—in a sense, improving future well-being matters less.

6 Welfare Criteria and Normative Analysis

Models that allow for time-inconsistent preferences pose serious conceptual problems when defining welfare criteria and addressing policy questions. Discussing hyperbolic discounting, Rubinstein (2003) notes,

> "Policy questions were freely discussed in these models even though welfare assessment is particularly tricky in the presence of time inconsistency. The literature often assumed, though with some hesitation, that the welfare criterion is the utility function with stationary discounting rate δ (which is independent of β)." (p. 1208)

Another, perhaps more fundamental, issue is whether the existence of time-inconsistent DMs justifies some form of paternalistic interventions. An immediate consequence of the present paper is to weaken the case for such interventions. If time inconsistency were the result of some bounded rationality, one may be tempted to argue that time-inconsistent DMs may benefit from paternalistic interventions. This argument, however, is not valid if time inconsistency is the result of fully forward-looking behavior. In this case, there seems to be no reason why a planner should use welfare criteria other than DM's ex-ante preference. If at 0 DM is already fully taking account of his future preferences over continuation streams, then the planner may simply adopt a 'libertarian' stance and use as a welfare criterion $W^L(c) = U(c)$.

[22]In infinite-horizon settings, by Corollary 1, DM never wants to delay delightful events. However, a finite-horizon version of representation (5) can generate such a preference (see Appendix A).

From the point of view of this paper, a libertarian criterion seems even more appropriate for time-inconsistent DMs than for time-consistent ones. Indeed, under EDU, welfare is usually and uncontroversially measured as the sum of instantaneous utilities, discounted at DM's subjective rate. That model implies, however, that DM's current preference takes into account his preferences only in the next period, but not in future periods. In contrast, the β-δ model implies that DM's current preference takes into account, albeit in a simple way, his preferences in all future periods—of course, this also holds with the more general representations in the paper. In this view, the ex-ante preference implied by the β-δ model may appear as a more reasonable welfare criterion than that implied by EDU. For example, consider two policies, A and B, inducing streams c^A and c^B such that ${}_1c^A \sim^1 {}_1c^B$ but ${}_tc^A \succ^t {}_tc^B$ for all $t > 1$. The criterion based on EDU implies that, at 0, A is as desirable as B. Instead, the criterion based on the β-δ model implies that, at 0, A is strictly more desirable than B. Thus, the second criterion favors what we may call long-run sustainability and may therefore seem more appealing.

Using the well-being representation corresponding to the β-δ model (Theorem 3), we can also *derive* the welfare criterion that the literature has so far assumed for this model. Recall that the ex-ante choices of a β-δ DM reveal the corresponding parameters α and γ. In particular, we may interpret the factor α^t as capturing DM's assessment of how likely it is that a policy under scrutiny will continue to matter for him at t. This assessment may combine subjective aspects as well as objective information, which DM may know better than the planner. Therefore, the planner may use the weights α^t to aggregate DM's well-being across periods. Surprisingly, doing so is equivalent to aggregating DM's instantaneous utilities using the familiar weights δ^t.

Proposition 5. *Suppose $U(c)$ can be represented as in Theorem 3 and Corollary 4, with corresponding parameters (u, α, γ) and (u, β, δ). Let $w : T \to \mathbb{R}_+$ and define $W(c) = \sum_{t=0}^{\infty} w(t)U({}_tc)$. Then,*

$$W(c) = \sum_{t=0}^{\infty} \delta^t u(c_t)$$

if and only if $w(t) = \alpha^t$.

This result has another remarkable and convenient implication. Under a natural, focal specification of welfare weights for a time-inconsistent agent—revealed by his ex-ante choices—we recover a social planner that is time consistent.

Since time consistency may reveal that a DM is not fully forward-looking, a natural question is then whether the planner should weigh more DM's well-being beyond the immediate future than does DM himself and, if so, by how much. In the case of EDU, i.e., $U({}_tc) = \sum_{t=0}^{\infty} \delta^t u(c_t)$, one might rely on δ to aggregate DM's well-being across periods using the criterion $\hat{W}(c) = \sum_{t=0}^{\infty} \delta^t U({}_tc)$. Doing so, however, makes the planner time inconsistent. Indeed, one can show that $\hat{W}(c) = u(c_0) + \sum_{t=1}^{\infty} \delta^t (1+t) u(c_t)$.[23]

To summarize, in relation to the β-δ model discussed by Rubinstein, this paper shows that the arguably most natural, welfare criteria are precisely those that have been used in practice. The paper also sheds light on their respective implications on how the planner weighs DM's well-being across periods.

7 General Discounting: Well-being vs. Instantaneous Utility

Discounting is a standard feature in dynamic economic models. This section investigates the general relationship between how DM discounts future instantaneous utility—e.g., in the case of hyperbolic discounting—and how he cares about future well-being.

In general, a discount function captures the weight that, at t, DM assigns to his instantaneous utility at $s > t$. For every $t \geq 0$ and $s > t$, let this weight be $d(t,s) \in (0,1)$. Also, suppose DM's preference can be represented by

$$U_t({}_tc) = u(c_t) + \sum_{s>t} d(t,s) u(c_s) \tag{10}$$

for some function $u : X \to \mathbb{R}$.[24] The questions here are whether we can always find a weight function q such that

$$U_t({}_tc) = u(c_t) + \sum_{s>t} q(t,s) U_s({}_sc), \tag{11}$$

and how q relates to d. To state the answer, for $s > t$, let

$$\mathcal{T}(t,s) = \{\mathbf{t} = (\tau_0, \ldots, \tau_n) \mid 1 \leq n \leq |s-t|, \tau_0 = t, \tau_n = s, \tau_i < \tau_{i+1}\}, \tag{12}$$

and for $s > t+1$, let

$$\hat{\mathcal{T}}(t,s) = \{\mathbf{t} = (\tau_0, \ldots, \tau_n) \mid 2 \leq n \leq |s-t|, \tau_0 = t, \tau_n = s, \tau_i < \tau_{i+1}\}. \tag{13}$$

In words, $\mathcal{T}(t,s)$ is the set of all increasing vectors starting at t and ending at s; $\hat{\mathcal{T}}(t,s)$ is the set of all such vectors with at least one intermediate date.

Proposition 6.

[23] At first glance, EDU appears as the limit of the β-δ model as $\beta \to 1$. However, by Theorem 3 and Corollary 3, in the limit the corresponding α and γ must take extreme, implausible values. Therefore, it seems more natural to think of time-consistent and time-inconsistent DMs as having radically different preferences.

[24] This section allows for the possibility that preferences are not time invariant (Assumption 1).

(i) If $U_t({}_tc)$ satisfies (10) *for some discount function d, then it also satisfies* (11) *for some weight function q. In particular, for all $t \geq 0$ and $s > t+1$, $q(t,t+1) = d(t,t+1)$ and*

$$q(t,s) = d(t,s) + \sum_{\mathbf{t} \in \hat{\mathcal{T}}(t,s)} (-1)^{|\mathbf{t}|} \prod_{n=1}^{|\mathbf{t}|-1} d(\tau_{n-1}, \tau_n). \tag{14}$$

(ii) If $U_t({}_tc)$ satisfies (11) *for some non-negative weight function q, then it also satisfies* (10) *for some discount function d. Moreover, for all $t \geq 0$ and $s > t$,*

$$d(t,s) = \sum_{\mathbf{t} \in \mathcal{T}(t,s)} \prod_{n=1}^{|\mathbf{t}|-1} q(\tau_{n-1}, \tau_n). \tag{15}$$

The intuition behind Proposition 6 is simple. When DM is fully forward-looking (i.e., $q(t,s) > 0$ for all t, s), $d(t,s)$ takes account of all channels through which consumption at s affects well-being at t (see (15)). Such channels are as many as the ways of reaching s from t in j jumps, for $j \leq s - t$. For example, consumption at $t+2$ affects well-being at t via two channels: a direct one, for at t DM cares about well-being at $t+2$; and an indirect one, for at t DM cares about well-being at $t+1$, which in turn depends on well-being at $t+2$. As a result, independently of how DM weighs future well-being, he weighs instantaneous utility at least two periods in the future strictly more than his well-being at that period.

Corollary 7. *Suppose that d and q are the instantaneous-utility discount function and the well-being weight function corresponding to the same $\succ$. Then, $d(t,s) > q(t,s)$ whenever $s - t \geq 2$.*[25]

Using Proposition 6, we can also derive which properties of the general discount function d are equivalent to myopically forward-looking behavior, namely $q(t,s) = 0$ for $s > t+1$. Note that, by Proposition 6, in all cases $d(t,t+1) = q(t,t+1)$.

Corollary 8. *For all $t \geq 0$, $U_t({}_tc) = u(c_t) + d(t,t+1)U_{t+1}({}_{t+1}c)$ if and only if, for $s > t$,*

$$d(t,s) = \prod_{j=0}^{s-t-1} d(t+j, t+j+1). \tag{16}$$

Corollary 8 has two implications. First, if d depends only on the time distance $s - t$, then of course we must have exponential discounting: Letting $\delta = d(1)$, by (16) we get $d(s,t) = \delta^{s-t}$. We can therefore interpret any form of discounting that depends only on $s - t$ and is not exponential—like hyperbolic discounting—as implying that,

[25]Using a recursive formulation, Saez-Marti and Weibull (2005) Saez-Marti and Weibull (2005) make a similar but weaker observation, namely that $d(t,s) \geq q(t,s)$ for all t and s.

when comparing consumption streams, DM cares directly about well-being beyond the immediate future—i.e., he is not myopically forward-looking. The second implication concerns how future consumption affects current well-being. By (16), $d(t, s)$ takes into account only the direct channel through which consumption at s affects well-being at t.

As an illustration of the relationship between d and q, consider the well-known and empirically supported case of hyperbolic discounting (see Frederick et al. (2002) and references therein). In its most general version, hyperbolic discounting takes the form

$$d(t, s) = d^h(s - t) = [1 + k(s - t)]^{-\frac{p}{k}} \tag{17}$$

with k, $p > 0$ (see Loewenstein and Prelec (1992)). Unfortunately, it is hard to derive in closed form the well-being weight function q^h corresponding to d^h.[26] However, using Proposition 6, we can simulate q^h for different values of k and p. This gives us an idea of how q^h qualitatively relates to d^h.

Figure 1 represents d^h, q^h, and the standard exponential discount function for $\delta = 0.9$ (curve ED). In all three panels, given k, the parameter p is set so that $d^h(\tau) = \delta^\tau$ at $\tau = 10$ (this period is arbitrary). Recall that hyperbolic discounting is characterized by declining discount rates, as highlighted by panel (a) and (b). Also, recall that, given p, $d^h(\tau)$ converges to δ^τ for every τ as $k \to 0$—this pattern is clear going from panel (a) to (c). All three panels illustrate the point of Corollary 7: Starting in period 2, q^h is strictly below d^h. Moreover, they illustrate that, when a hyperbolically-discounting time-inconsistent DM becomes more similar to an exponentially-discounting time-consistent DM, he discounts more his well-being from period 2 onwards. Indeed, for these periods, q^h moves closer to zero as d^h moves closer to ED. This is, of course, an illustration of Propositions 1 and Corollary 8: DM can be time consistent if and only if he does not care about well-being beyond period 1, i.e., $q^h(\tau) = 0$ for $\tau > 1$.

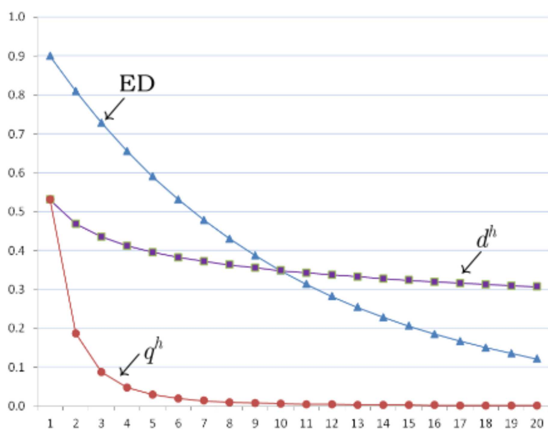
8 Discussion: Relation with Other Stationarity Notions

This section briefly compares well-being stationarity (Axiom 4) with Koopmans's [1960, 1964] stationarity and Olea and Strzalecki's [2013] quasi-stationarity.

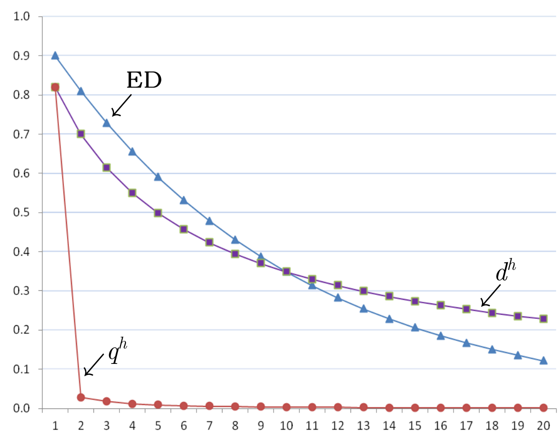
In his general analysis of impatience, Koopmans considers a single preference, $\succ^0$, and calls it stationary if it satisfies the following property: If ${}_1c \sim^0 {}_1c'$, then $(c_0, {}_1c) \sim^0 (c_0, {}_1c')$; and if ${}_1c \succ^0 {}_1c'$, then $(c_0, {}_1c) \succ^0 (c_0, {}_1c')$. This property looks similar to time consistency

[26] Proposition 2 in Saez-Marti and Weibull (2005) implies that, if d^h satisfies (17), then $q^h \geq 0$.

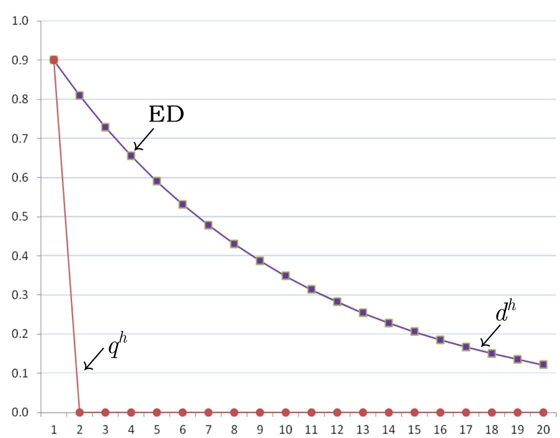
Figure 1: Hyperbolic-Discounting Simulation



(a) $k = 30$



(b) $k = 0.3$



(c) $k = 0.0001$

(Definition 2), but is conceptually very different.[27] Indeed, Koopmans writes:

> "[Stationarity] does *not* imply that, after one period has elapsed, the ordering then applicable to the 'then' future will be the same as that now applicable to the 'present' future. All postulates are concerned with only one ordering, namely that guiding decisions to be taken in the present. Any question of change or consistency of preferences as the time of choice changes is therefore extraneous to the present study." (Koopmans [1964], p. 85, emphasis in the original)

The point can be illustrated with two simple examples.

Example 1 (Stationarity $\not\Rightarrow$ Time Consistency): For $t \geq 0$, $\succ^t$ is represented by

$$U_t({}_tc) = u(c_t) + \sum_{s>t}^{\infty} \delta_t^{s-t} u(c_s) = u(c_t) + \delta_t U_t({}_{t+1}c),$$

where $u : X \to \mathbb{R}$ is continuous and strictly increasing. Moreover, $\delta_t \in (0,1)$ and $\delta_t > \delta_{t+1}$ for $t \geq 0$. Then, each $\succ^t$ satisfies stationarity, but $\{\succ^t\}_{t\geq 0}$ violates time consistency.

Example 2 (Time Consistency $\not\Rightarrow$ Stationarity): For $t \geq 0$, $\succ^t$ is represented by

$$U_t({}_tc) = u(c_t) + \phi_t \sum_{s>t}^{\infty} \delta^{s-t} u(c_s) = u(c_t) + \phi_t \delta U_{t+1}({}_{t+1}c),$$

where $\delta \in (0,1)$, $\phi_0 \in (0,1)$, and $\phi_t = 1$ for $t > 0$; $u(\cdot)$ is as before. Then $\{\succ^t\}_{t\geq 0}$ satisfies time consistency, but $\succ^0$ violates stationarity.

Using another axiom—i.e., separability of $\succ^0$ in c_0 and ${}_1c$—Koopmans obtains a recursive representation of the form $U(c) = \hat{V}(u(c_0), U({}_1c))$, where $\hat{V}$ is strictly increasing in each argument. Note that separability implies that $U(c)$ depends on ${}_1c$ only through a 'perspective utility' $U_1({}_1c)$, which equals $U({}_1c)$ by stationarity. With further axioms, Koopmans shows that there exists a special form of $\hat{V}$ which corresponds to EDU. The previous remarks and Proposition 1 remind us that EDU features time consistency not because it discounts future instantaneous utility at a constant rate—a cardinal property of a specific $\hat{V}$—but because today's well-being depends only on tomorrow's well-being—an ordinal property of $\hat{V}$.[28]

If we view DM as evaluating consumption streams based only on the instantaneous utility generated in each period—so that consumption at 0 and at 1 are conceptually equivalent—then Koopmans's stationarity seems reasonable. If we delay two consumption streams from 0 to 1, replacing consumption at 0 with the same level in both cases, why should DM, at 0, rank such new streams differently from the original ones? Note that,

[27] Of course, if Assumption 1 holds, time consistency and stationarity are mathematically equivalent.

[28] A property of a representation is ordinal if it is invariant to strictly increasing monotonic transformations.

using the recursive representation $\hat{V}$, in Koopmans stationarity holds between any two periods.

Olea and Strzalecki's paper allows for violations of stationarity, but only between period 0 and 1—the resulting property is called quasi-stationarity. However, their paper continues to view DM as evaluating streams based only on instantaneous utilities. In this view, quasi-stationarity seems more difficult to justify. If DM views consumption in the same way in all periods, why should stationarity hold between tomorrow and the day after, but not between today and tomorrow? This issue does not arise with well-being stationarity (Axiom 4), for in the present model tomorrow's well-being is equivalent to well-being thereafter, but differs conceptually from today's consumption.

These remarks clarify in which way the present paper departs from other intertemporal utility models. It does not relax stationarity directly—for example, as Hayashi (2003) and Olea and Strzalecki (2013) do. Instead, it takes a conceptually different view of what determines preferences over consumption streams. The starting point is to allow forward-looking behavior to extend beyond the immediate future. By Proposition 1, doing so requires abandoning time consistency, which by time invariance (Assumption 1) happens to be formally equivalent to stationarity.

9 Conclusion

This paper develops a general theory of fully forward-looking behavior in intertemporal choice problems. Fully forward-looking preferences—which in each period depend on the preferences in all subsequent periods—can explain time inconsistency in the form of present bias. Such preferences have tractable representations which can capture multiple phenomena relegated to the status of anomalies under the exponentially-discounted-utility paradigm. Despite recursively depending on itself in all future periods, any fully forward-looking preference exhibits impatience. As Koopmans's [1960] axiomatization of EDU implies a strictly positive discount rate and, hence, well-defined preferences over an infinite horizon, this paper's axiomatization implies that future preferences are also discounted positively and, under its stationarity axiom, that variations in future well-being (under an intuitive interpretation of the model) have limited effects on current well-being. This implies, for instance, that fully forward-looking agents cannot become infinitely happy by anticipating future pleasurable events. These features, among other things, distinguish this theory from previous ad-hoc models of anticipations, in which parameter restrictions ensuring that the model was well-defined were left unjustified.

Finally, a specific form of fully forward-looking preferences, axiomatized in the paper, corresponds to the renowned quasi-hyperbolic discounting model.

This theory allows us to conduct rigorous welfare analysis with time-inconsistent agents of the type studied here. Paternalistic interventions seem difficult to justify, since fully forward-looking agents take into account their well-being in all future periods. For instance, these agents are never victim of money-pump schemes, for they fully anticipate how such schemes would affect their future well-being. These agents may, however, demand commitment and perfectly predict that, in the future, they will want to walk away from it. Any justification of policies helping time-inconsistent agents honor their commitments ultimately hinges on how we measure their welfare. Welfare criteria that provide such a justification have been usually assumed in the literature. The paper shows that we can derive such criteria from representations of fully forward-looking preferences.

Finally, this paper contributes to the discussion about which behaviors can be classified as resulting from 'bounded rationality.' Indeed, it shows that a form of behavior—time inconsistency—that may seem to belong to this category can be explained by a nonstandard, yet plausible, model of fully rational decision making.

10 Appendix A: Anticipating Dreadful Events and Delaying Delightful Ones.

This section briefly illustrates that, in a finite-horizon setting, a well-being representation similar to that in Theorem 2 can accommodate a desire to anticipate dreadful events and to delay delightful ones. Let $T = \{0, 1, \ldots, \bar{t}\}$ with $\bar{t} < \infty$, $\overline{C} = X^{\bar{t}+1}$, and $\alpha \in (0, 1)$, u, and G as in Theorem 2. We can then recursively define a well-being representation as follows. For any $c \in \overline{C}$, let $U_{\bar{t}}(c_{\bar{t}}) = u(c_{\bar{t}})$ and, for $t < \bar{t}$, let

$$U_t({}_tc) = u(c_t) + \sum_{\tau=1}^{\bar{t}-t} \alpha^\tau G(U_{t+\tau}({}_{t+\tau}c)). \tag{18}$$

Since U_t is constructed by backward induction for each t, it is well defined even if α and G do not satisfy the restrictions in Proposition 3. Hereafter, suppose G is strictly increasing. It is also easy to see—adapting Proposition 4 and Corollary 3—that, if $G(U) = \gamma U$ with $\gamma > 0$, then

$$U_t({}_tc) = u(c_t) + \sum_{\tau=1}^{\bar{t}-t} \beta\delta^\tau u(c_{t+\tau}),$$

where $\beta = \frac{\gamma}{1+\gamma}$ and $\delta = \alpha(1+\gamma)$. Note that, for any $\gamma > 0$, $\beta < 1$; yet, it is possible that $\delta > 1$.

Consider now the effect of delaying an event that involves changing consumption from y to x. Let $\overline{u} = u(x) - u(y)$—e.g., for a dreadful event $\overline{u} > 0$. If DM delays this event from 0 to t, his instantaneous utility changes by $\overline{u}$ at 0 and by $-\overline{u}$ at t. Consequently, his well-being changes at t and, indirectly, at all τ between 1 and t. Formally, we can compute the overall effect of an event delay, which changes instantaneous utility by $\overline{u}$, as follows. For $\tau = 1\ldots, t$, define ΔG_τ recursively as

$$\Delta G_t = G(U_t({}_t c) - \overline{u}) - G(U_t({}_t c))$$

$$\Delta G_{t-s} = G\left(U_{t-s}({}_{t-s}c) + \sum_{k=1}^{s} \alpha^k \Delta G_{t-s+k}\right) - G(U_{t-s}({}_{t-s}c)).$$

The overall effect of the delay until t on well-being at 0 is

$$\theta_t(\overline{u}) = \overline{u} + \Gamma_t(\overline{u}),$$

where

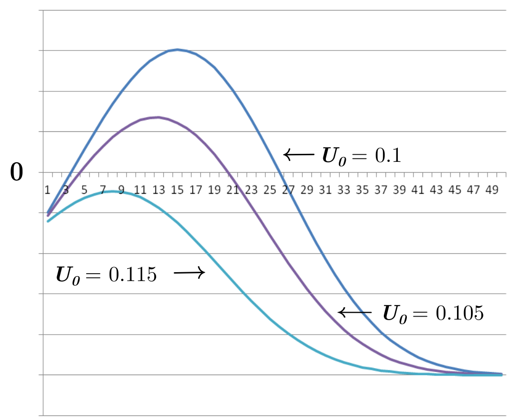
$$\Gamma_t \equiv \sum_{s=0}^{t-1} \alpha^{t-s} \Delta G_{t-s}.$$

So, DM wants to delay an event from 0 to t if and only if $\theta_t(\overline{u}) > 0$. In this case, he also wants to choose t so as to maximize $\theta_t(\overline{u})$.

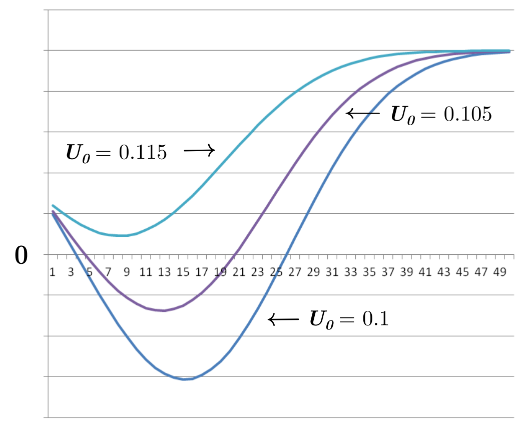
Despite DM discounts future utility from well-being ($\alpha < 1$), he may prefer to delay delightful events or to anticipate dreadful ones. Note that, in general, Γ_t depends on the stream $\{U_\tau\}_{\tau=1}^t$ of well-being that DM expects up to period t—unless G is linear or U_τ is constant and $\overline{u}$ is infinitesimal. A complete investigation of this dependence is beyond the scope of this section, which therefore focuses on a simulation. Suppose that $G(U) = -e^{-gU}$ with $g > 0$ and $U_\tau = U_0(1+r)^\tau$ for some $r \in [0,1]$ and $U_0 > 0$. Figure 2 reports $\theta_t(\overline{u})$ for different specifications of the parameters and $\overline{u}$.

Panel (a) considers a delightful event ($\overline{u} < 0$) and different well-being streams $\{U_\tau\}_{\tau=1}^t$—note that the higher U_0, the higher the entire stream. When DM faces a relatively low well-being stream ($U_0 = 0.1$), he prefers to delay the event to some period in the near future, *beyond* period 1. This is because, for low U_τ, G responds enough to improvements in well-being, so that the cumulative effects of the benefit $-\overline{u}$ at some period in the near future can prevail over discounting. However, when facing higher streams, DM prefers to delay the event to a closer period ($U_0 = 0.105$) or not to delay it at all ($U_0 = 0.115$). This is because, for higher U_τ, the concavity of G makes it less responsive to the benefit $-\overline{u}$. Finally, in all cases, eventually discounting dominates, so that DM does not want to delay

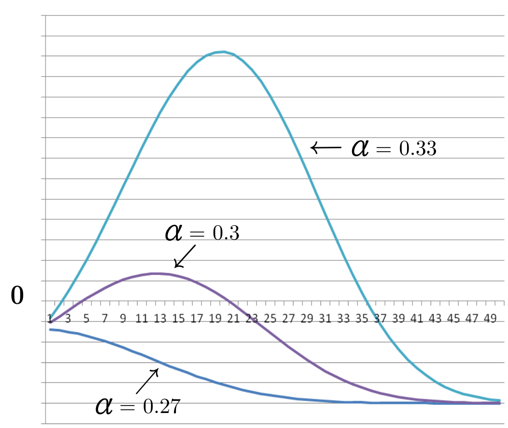
Figure 2: Anticipating and Delaying Consumption Events



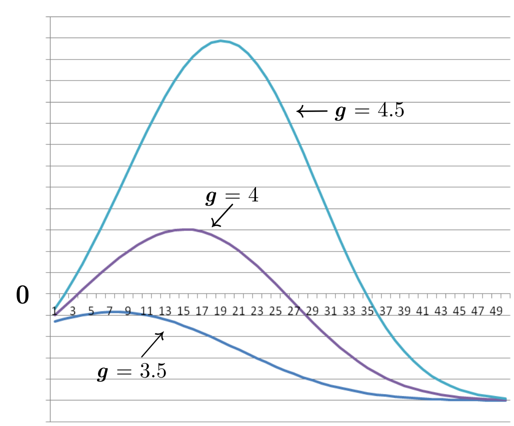
(a) $\overline{u} = -0.0001$, $g = 4$, $\alpha = 0.3$, $r = 0.02$



(b) $\overline{u} = 0.0001$, $g = 4$, $\alpha = 0.3$, $r = 0.02$



(c) $U_0 = 0.1$, $\overline{u} = -0.0001$, $g = 4$, $r = 0.02$



(d) $U_0 = 0.1$, $\overline{u} = 0.0001$, $\alpha = 0.3$, $r = 0.02$

a delightful event indefinitely.[29]

Panel (b) considers a dreadful event ($\overline{u} > 0$) and the same well-being streams as in panel (a). If possible, DM would always want to delay the event indefinitely.[30] This is intuitive: Eventually discounting dominates, killing any effect of future well-being losses on current well-being. However, in the short run, DM prefer to delay the event to period 1, than to any period right after 1. This non-monotonic preference arises because, in the short run, G's sensitivity to well-being losses due to $-\overline{u}$ dominates on discounting. Therefore, anticipating $-\overline{u}$ from, say, period 3 to period 1 allows DM to avoid its cumulative effects in periods 1 and 2.

Panel (c) focuses on the effect of discounting on DM's desire to delay delightful events—the case of dreadful events is similar. DM wants to delay as long as discounting is not too strong. This is intuitive: DM is willing to delay as long as his current well-being can benefit from the cumulative effect, on intermediate well-being, of increasing well-being in a future period by $-\overline{u}$. As α decreases, DM discounts this cumulative benefit more and consequently is less willing to forgo enjoying the event right away.

Finally, panel (d) focuses on the effect of G's sensitivity on DM desire to delay delightful events—again, the case of dreadful events is similar. DM wants to delay as long as G is sensitive enough to the benefit $-\overline{u}$ on future well-being. Note that g measures the elasticity of G normalized by U. Hence, as g increases, the benefits of increasing well-being in the near future by $-\overline{u}$ are more likely to dominate discounting, thus making DM delay.

11 Appendix B: Omitted Proofs

11.1 Existence of a Utility Representation

The following axioms are standard.

Axiom 7 (Weak Order). *$\succ$ is complete and transitive.*

Axiom 8 (Continuity). *For all $c \in C$, the sets $\{c' \in C : c' \prec c\}$ and $\{c' \in C : c' \succ c\}$ are open.*

Axiom 9 (Constant-Flow Dominance). *For all $c \in C$, there are constant sequences $\overline{c}$ and $\underline{c}$ such that $\underline{c} \precsim c \precsim \overline{c}$.*

[29]DM would want to delay indefinitely if G were linear and $\delta = \alpha(1+\gamma) > 1$, which seems unreasonable.

[30]DM would never want to delay indefinitely if G were linear and $\delta = \alpha(1+\gamma) > 1$, which again seems unreasonable.

These axioms lead to the following standard result, which builds on Diamond (1965).

Theorem 4. *Under Axioms 7-9, there is a continuous function* $U : C \to \mathbb{R}$ *such that* $c \succ c'$ *if and only if* $U(c) > U(c')$.

Proof. The proof follows and generalizes that of Diamond (1965), and is based on the following lemma by Debreu (1954).

Lemma 1. *Let* C *be a completely ordered set and* $Z = (z_0, z_1, ...)$ *be a countable subset of* C. *If for every* $c, c' \in C$ *such that* $c \prec c'$, *there is* $z \in Z$ *such that* $c \preceq z \preceq c'$, *then there exists on* C *a real, order-preserving function, continuous in any natural topology.*[31]

Lemma 2. *For any* $c \in C$, *there is a constant stream* c^* *such that* $c \sim c^*$.

Proof. Let D be the set of constant streams and, for any fixed $c \in C$, let $A = \{d \in D : d \precsim c\}$ and $B = \{d \in D : d \succsim c\}$. By Axiom 7, $A \cup B = D$; by Axiom 8, A and B are closed; by Axiom 9, A and B are nonempty. Moreover, D is connected. Indeed, for any continuous function $\phi : D \to \{0, 1\}$, the function $\bar{\phi} : X \to \{0, 1\}$ defined by $\bar{\phi}(x) = \phi(x, x, \ldots)$ is also continuous. Connectedness of X implies that $\bar{\phi}$ is constant and, hence, that ϕ is constant, showing connectedness of D. This implies that $A \cap B \neq \varnothing$.

□

To conclude the proof of Theorem 4, let Z_0 denote a countable dense subset of X, which exists since X is separable, and let Z denote the subset C consisting of constant sequences whose elements belong to Z_0. Lemma 2 implies that Z satisfies the hypothesis of Lemma 1, which yields the result.

□

The next axiom ensures that U is nonconstant in the first and some other argument.

Axiom 10 (Non triviality)**.** *There are* $x, x', \hat{x} \in X$ *and* $c, c', \hat{c} \in C$, *such that*

$$(x, \hat{c}) \succ (x', \hat{c}) \text{ and } (\hat{x}, c) \succ (\hat{x}, c').$$

11.2 Proof of Proposition 1

Suppose that $V(c_0, U({}_1c), U({}_2c), \ldots) = V(c_0, U({}_1c))$ for all $c \in C$ and V is strictly increasing in $U({}_1c)$. By Assumption 1, if ${}_1c \sim^1 {}_1c'$, then[32] $U({}_1c) = U({}_1c')$ and, since V is a function,

[31] A natural topology is one under which Axiom 8 holds for that topology.

[32] This step would be meaningless if $U({}_1c)$ represented how DM evaluates consumption streams starting at 1 from the perspective of 0, but not necessarily how he evaluates such streams from the perspective of 1. This observation applies to the rest of the proof.

$V(c_0, U({}_1c)) = V(c_0, U({}_1c'))$; hence $(c_0, {}_1c) \sim^0 (c_0, {}_1c')$. If ${}_1c \succ^1 {}_1c'$, then $U({}_1c) > U({}_1c')$ and, since V is strictly increasing in its second argument, $V(c_0, U({}_1c)) > V(c_0, U({}_1c'))$; hence $(c_0, {}_1c) \succ^0 (c_0, {}_1c')$.

Suppose ${}_1c \sim^1 {}_1c'$ implies $(c_0, {}_1c) \sim^0 (c_0, {}_1c')$. Then, for any $(U({}_1c), U({}_2c), \ldots)$ and $(U({}_1c'), U({}_2c'), \ldots)$ such that $U({}_1c) = U({}_1c')$,

$$V(c_0, U({}_1c), U({}_2c), \ldots) = V(c_0, U({}_1c'), U({}_2c'), \ldots).$$

So V can depend only on its first two arguments. Suppose ${}_1c \succ^1 {}_1c'$ implies $(c_0, {}_1c) \succ^0 (c_0, {}_1c')$. Then, $U({}_1c) > U({}_1c')$. Moreover, it must be that $V(c_0, U({}_1c)) > V(c_0, U({}_1c'))$; that is, V must be strictly increasing in its second argument.

11.3 Proof of Proposition 2

Let $U(c) = V(c_0, U({}_1c), U({}_2c), \ldots)$ where V is strictly increasing in $U({}_tc)$ for all $t > 0$. By definition, $(x, c) \succ (y, c)$ means that $U(x, c) > U(y, c)$. Hence, for all $0 \leq s \leq t$,

$$U({}_sz_t, x, c) > U({}_sz_t, y, c),$$

where, for $s < t$, ${}_sz_t = (z_s, \ldots, z_t)$ and ${}_tz_t = z_t$. This follows by induction. For $s = t$,

$$\begin{aligned} U({}_tz_t, x, c) &= V({}_tz_t, U(x, c), U(c), \ldots) \\ &> V({}_tz_t, U(y, c), U(c), \ldots) = U({}_tz_t, y, c). \end{aligned}$$

Now suppose that the claim holds for $r + 1 \leq s \leq t$, with $0 \leq r < t$. Then

$$\begin{aligned} U({}_rz_t, x, c) &= V(z_r, U({}_{r+1}z_t, x, c), \ldots, U({}_tz_t, x, c), U(x, c), \ldots) \\ &> V(z_r, U({}_{r+1}z_t, y, c), \ldots, U({}_tz_t, y, c), U(y, c), \ldots) \ = \ U({}_rz_t, y, c). \end{aligned}$$

Again, by definition $({}_0z_t, x, w, c') \sim ({}_0z_t, y, h, c')$ means that

$$\begin{aligned} &V(z_0, U({}_1z_t, x, w, c'), \ldots, U({}_tz_t, x, w, c'), U(x, w, c'), U(c'), \ldots) \\ &= V(z_0, U({}_1z_t, y, h, c'), \ldots, U({}_tz_t, y, h, c'), U(y, h, c'), U(h, c'), \ldots). \end{aligned}$$

Since $U({}_0z_t, x, c) > U({}_0z_t, y, c)$ for all c,

$$\begin{aligned} &V(z_0, U({}_1z_t, y, w, c'), \ldots, U({}_tz_t, y, w, c'), U(y, w, c'), U(w, c'), \ldots) \\ &< V(z_0, U({}_1z_t, y, h, c'), \ldots, U(z_t, y, h, c'), U(y, h, c'), U(h, c'), \ldots). \end{aligned}$$

This implies that $U(h, c') > U(w, c')$. Otherwise, $U(y, h, c') \leq U(y, w, c')$ and, by induction, $U({}_sz_t, y, h, c') \leq U({}_sz_t, y, w, c')$ for all $0 \leq s \leq t$, which is a contradiction.

Finally, it must be that $U(x, w, c') > U(y, h, c')$. Otherwise, again by induction, for all $0 \leq s \leq t$

$$U({}_sz_t, y, h, c') > U({}_sz_t, x, w, c'),$$

which contradicts $({}_0z_t, x, w, c') \sim ({}_0z_t, y, h, c')$.

11.4 Proof of Theorem 2

This proof adapts arguments in Debreu (1960) and Koopmans [1960, 1964] to the present environment. It is convenient to work in terms of the streams of immediate consumption and future well-being f, defined in (4), and the binary relation $\succ^*$ on $\mathcal{F}$ induced by the function $V : \mathcal{F} \to \mathbb{R}$ in the proof of Theorem 1.

Let Π' consist of all unions of subsets of $\{\{0\}, \{1\}, \{2\}, \{3, 4, \ldots\}\}$.

Lemma 3. *Axiom 2 implies that $\succ^*$ satisfies the following property. For any f, $f' \in \mathcal{F}$ and $\pi \in \Pi'$,*

$$(f_\pi, f_{\pi^c}) \succ^* (f'_\pi, f_{\pi^c}) \iff (f_\pi, f'_{\pi^c}) \succ^* (f'_\pi, f'_{\pi^c}),$$

where $\pi^c = T \setminus \pi$. By Axiom 3, $\succ^*$ depends on f_0, f_1, f_2, and ${}_3f$.

Proof. Recall that ${}_tc \sim {}_tc'$ implies $U({}_tc) = U({}_tc')$, which is equivalent to $f_t = f'_t$. Then, by Axiom 2, for any $\pi \in \Pi'$

$$V(f_\pi, f_{\pi^c}) > V(f'_\pi, f_{\pi^c}) \iff V(f_\pi, f'_{\pi^c}) > V(f'_\pi, f'_{\pi^c}).$$

□

By Debreu (1960), there exist then continuous nonconstant functions $\overline{V}$, $\hat{u}$, a, b, and d such that

$$\overline{V}(f) = \hat{u}(f_0) + a(f_1) + b(f_2) + d({}_3f) \tag{19}$$

and

$$f \succ^* f' \iff \overline{V}(f) > \overline{V}(f').$$

By Lemma 3 with $\pi = \{0\}$, Axiom 10, and Koopmans's [1960] argument, V can be expressed as

$$V(f) = W(v(f_0), A({}_1f)) \tag{20}$$

for some continuous, nonconstant functions W, v, and A, where W is strictly increasing. Similarly, by Lemma 3 with $\pi = \{1\}$, Axiom 10, and Koopmans's [1960] argument, V can be expressed as

$$V(f) = \overline{W}(v(f_0), \overline{A}(\overline{G}(f_1), B({}_2f))),$$

for some continuous, nonconstant functions $\overline{W}$, $\overline{A}$, $\overline{G}$, and B, where $\overline{W}$ and $\overline{A}$ are strictly increasing. Now use Axiom 4 to obtain, as shown by Koopmans [1960], that A in (20) and B in (21) are homeomorphic and therefore B can be taken to equal A by a simple modification of the function $\overline{A}$. This leads to

$$V(f) = \hat{W}(v(f_0), \hat{A}(\hat{G}(f_1), A({}_2f))). \tag{21}$$

According to (20), for every $v(f_0)$, $\succ^*$ depends on ${}_1f$ only through $A({}_1f)$. Therefore, for all ${}_1f$,

$$A({}_1f) = \phi_1(a(f_1) + h({}_2f)), \tag{22}$$

for some strictly increasing and continuous function ϕ_1, where $h({}_2f) = b(f_2) + d({}_3f)$.

According to (21), for every $v(f_0)$ and $\hat{G}(f_1)$, $\succ^*$ depends on ${}_2f$ only through $A({}_2f)$. Therefore, for all ${}_2f$,

$$A({}_2f) = \phi_2(b(f_2) + d({}_3f)), \tag{23}$$

for some strictly increasing and continuous function ϕ_2.

According to (21), for every $v(f_0)$ and $A({}_2f)$, $\succ^*$ depends on f_1 only through $\hat{G}(f_1)$. Therefore,

$$a(f_1) \equiv G(f_1) = \phi_3(\hat{G}(f_1)),$$

for some strictly increasing and continuous function ϕ_3.

Now comparing (22) and (23) implies that, for all f,

$$a(f_2) + h({}_3f) = \phi(b(f_2) + d({}_3f)),$$

where ϕ is some strictly increasing continuous function.

Lemma. *ϕ is affine.*

Proof. Let $x = f_2 \in \mathcal{X}$ and $y = {}_3f \in \mathcal{Y}$. We have

$$a(x) + h(y) = \phi(b(x) + d(y)),$$

where ϕ is increasing and continuous. Note that, since b, d, and U are continuous and nonconstant and X is connected, without loss of generality $I = \{b(x) + d(y) : x \in \mathcal{X},\ y \in \mathcal{Y}\}$ is a connected, nonempty interval. Choose $x_0 \in \mathcal{X}$ and $y_0 \in \mathcal{Y}$ arbitrarily, and define $\overline{a}(x) =$

$a(x) - a(x_0)$, $\overline{h}(y) = h(y) - h(y_0)$, and $\overline{b}(\cdot)$ and $\overline{d}(\cdot)$ similarly. So

$$\begin{aligned}\overline{a}(x) + \overline{h}(y) &= \phi(\overline{b}(x) + \overline{d}(y) + b(x_0) + d(y_0)) - \phi(b(x_0) + d(y_0)) \\ &\equiv \overline{\phi}(\overline{b}(x) + \overline{d}(y)).\end{aligned}$$

Note that $\overline{\phi}$ is continuous on the connected nonempty interval $\overline{I} = I - b(x_0) - d(y_0)$, which contains 0, and that $\overline{a}(x) = \overline{\phi}(\overline{b}(x))$ and $\overline{h}(y) = \overline{\phi}(\overline{d}(y))$. So,

$$\overline{\phi}(\tilde{b} + \tilde{d}) = \overline{\phi}(\tilde{b}) + \overline{\phi}(\tilde{d}) \tag{24}$$

for all $\tilde{b} \in I_b = \{\overline{b}(x) : x \in \mathcal{X}\}$ and $\tilde{d} \in I_d = \{\overline{d}(y) : y \in \mathcal{Y}\}$.

Using (24), we can now show that $\overline{\phi}$ is linear, thus ϕ is affine. First, note that since $\overline{\phi}(0) = 0$, $\overline{\phi}(z) = -\overline{\phi}(-z)$; so we can focus on the positive part, $\overline{I}_+$, or the negative part, $\overline{I}_-$, of $\overline{I}$. Suppose, without loss of generality, that $\overline{I}_+ \neq \varnothing$. Consider any $b > b' > 0$ in $\overline{I}_+$ such that b and b' are rational. Then, by (24), $\overline{\phi}(b) = m\overline{\phi}(\frac{1}{n})$ and $\overline{\phi}(b') = m'\overline{\phi}(\frac{1}{n'})$ for m, m', n, $n' \in \mathbb{N}$. Since $\overline{\phi}(\frac{1}{n})n = \overline{\phi}(1) = \overline{\phi}(\frac{1}{n'})n'$, it follows that $\overline{\phi}(b) = \frac{b}{b'}\overline{\phi}(b')$. Since rationals are dense in $\overline{I}_+$ and $\overline{\phi}$ is continuous, $\overline{\phi}(b) = \frac{b}{b'}\overline{\phi}(b')$ holds for all $b, b' \in \overline{I}_+$, which implies linearity.

□

Since ϕ must be increasing, there exists $\alpha > 0$ such that $b(f_2) = \alpha a(f_2)$ and

$$d({}_3f) = \alpha h({}_3f) = \alpha(b(f_3) + d({}_4f)). \tag{25}$$

It follows that

$$\overline{V}(f) = \hat{u}(f_0) + G(f_1) + \alpha G(f_2) + d({}_3f).$$

Now restrict attention to streams f that are constant after $t = 3$—which correspond to consumption streams that are constant after $t = 3$. For such streams, $d({}_3f) = \hat{d}(f_3)$. By Axiom 4 and $\alpha > 0$,

$$G(f_3) \geq G(f_3') \iff \hat{d}(f_3) \geq \hat{d}(f_3').$$

So $\hat{d}(\cdot) = \varphi(G(\cdot))$ for some strictly increasing and continuous function φ. Again by Axiom 4,

$$\alpha G(f_2) + \varphi(G(f_3)) \geq \alpha G(f_2') + \varphi(G(f_3'))$$

if and only if

$$G(f_2) + \alpha G(f_3) + \varphi(G(f_3)) \geq G(f_2) + \alpha G(f_3) + \varphi(G(f_3)).$$

So

$$\alpha G(f_2) + \varphi(G(f_3)) = \alpha(G(f_2) + \alpha G(f_3) + \varphi(G(f_3))) + k,$$

which implies $\varphi(G)(1-\alpha) = \alpha^2 G + k$. Since φ must be strictly increasing, it follows that $\alpha < 1$.

Finally, by iteratively applying (25) and relying on $\alpha \in (0,1)$, we get that $\succ^*$ can be represented by

$$V^*(f) = u(f_0) + \sum_{t=1}^{\infty} \alpha^t G(f_t).$$

To conclude, we choose, as the utility $\hat{U}$ representing preference $\succ$ over C, the function defined by

$$\hat{U}(c) = V^*(c_0, U({}_1c), \ldots).$$

The functions $\hat{U}$ and U are strict increasing transformations of one another. Letting $\hat{G}$ denote the function of $\hat{U}$ such that $\hat{G}(\hat{U}(c)) = G(U(c))$ for all c, we obtain the representation formula (5). For the uniqueness part, note that the additive form of $\hat{U}$ is unique up to affine transformations, i.e., $\tilde{U} = a\hat{U} + b$ for $a > 0$ and $b \in \mathbb{R}$. So,

$$\begin{aligned}\tilde{U}(c) &= au(c_0) + b + \sum_{t=1}^{\infty} \alpha^t a G(\hat{U}({}_tc)) \\ &= au(c_0) + b + \sum_{t=1}^{\infty} \alpha^t a G\left(\frac{\tilde{U}({}_tc) - b}{a}\right).\end{aligned}$$

Finally, it is easy to see that Axiom 5 holds if and only if G is strictly increasing—using Lemma 2.

11.5 Proof of Proposition 3

(i) To show that G is bounded on $\mathcal{U}$, recall from Axiom 9 (Constant-flow Dominance) that $U(c)$ is finite for all $c \in C$. Suppose, by contradiction, that G is unbounded on $\mathcal{U}$. Then, for each n, there must be a constant stream c^n with utility U^n such that $G^n \equiv G(U^n) \geq n$, and $G^n > G^m$ for $n > m$. Moreover, by Axioms 5, a stream c that equals c^n for the first k periods and c^m forever after, for $m > n$, must satisfy $G(U(c)) \geq n$. This is because $U({}_tc) > U({}_tc^n)$ for $t > k$ and hence $U({}_kc) > U({}_kc^n)$; then, by induction, $U({}_tc) > U({}_tc^n)$ for $0 \leq t < k$. Now construct a new stream as follows: for some large M, start with (the consumption defining) c^M for the first 10 periods, then $c^{M^{10}/\alpha^{10}}$ for the next 10 periods, then $c^{M^{20}/\alpha^{20}}$ for the next 10 periods, and so on. By construction, $U(c)$ must exceed the sum of $\alpha^t G(U({}_tc))$ over all t's that are multiples of 10 (this is because the remaining terms are nonnegative). But, since by construction $\alpha^t G(U({}_tc)) \geq M^t$, that sum diverges to infinity and $U(c)$ must be infinite, violating Axiom 9.

To show that U is H-continuous, note that for any $c, \tilde{c} \in C$

$$|U(c) - U(\tilde{c})| = \left| u(c_0) - u(\tilde{c}_0) + \sum_{t=1}^{\infty} \alpha^t \left[G(U({}_tc)) - G(U({}_t\tilde{c})) \right] \right|$$

$$\leq \sum_{t=1}^{T} \alpha^t \left| G(U({}_tc)) - G(U({}_t\tilde{c})) \right| + \alpha^T \frac{\alpha 2\overline{G}}{1-\alpha},$$

where $\bar{G} = \sup_{\tilde{U} \in \mathcal{U}} |G(\tilde{U})|$. So, for any $\varepsilon > 0$, choose $T(\varepsilon)$ so that $\alpha^{T(\varepsilon)} \frac{\alpha 2\overline{G}}{1-\alpha} < \varepsilon$.

Finally, take $U', U \in \mathcal{U}$. By definition, there are $c', c \in C$ such that $U(c') = U'$ and $U(c) = U$. By Lemma 2, we can take $c' = (x, x, \ldots)$ and $c = (y, y, \ldots)$ for some $x, y \in X$. Suppose $u(x) > u(y)$. Then, $U(x, y, \ldots) > U(y, y, \ldots)$ and for all finite $t > 1$, by induction, $U(c_0, \ldots, c_t, y, \ldots) > U(x, y, \ldots)$, where $c_\tau = x$ for $0 < \tau \leq t$. Since U is H-continuous, $U(x, \ldots) > U(y, \ldots)$. By representation (5),

$$U(x) - \frac{\alpha}{1-\alpha} G(U(x)) > U(y) - \frac{\alpha}{1-\alpha} G(U(y)).$$

Rearranging, we get that for any $U' > U$ in $\mathcal{U}$

$$\frac{1-\alpha}{\alpha}(U' - U) > (G(U') - G(U)).$$

(ii) Let $C(M)$ be the set of consumption streams such that $|u(c_t)| \leq M$ for all t, and $B(M)$ be the space of bounded real-valued functions with domain $C(M)$. Endowed with the sup norm $\|U\|_\infty = \sup_{c \in C(M)} |U(c)|$, $B(M)$ is a complete metric space. Let $\mathcal{J}$ be the operator on $B(M)$ defined by

$$\mathcal{J}(U)(c) = u(c_0) + \sum_{t=1}^{\infty} \alpha^t G(U({}_tc)).$$

By construction, $\mathcal{J}(U)$ is bounded over $C(M)$, as u is bounded by M and U is bounded over $C(M)$. Moreover, since G is K-Lipschitz continuous with $K < (1-\alpha)/\alpha$, $\mathcal{J}$ must be a contraction, as is easily checked. So, $\mathcal{J}$ has a unique fixed point; call it U_M. As M increases, the domain of U_M increases. However, for any M, N, uniqueness of the fixed point guarantees that U_M and U_N coincide on the intersection of their domains. Thus, we obtain a unique solution U^* to (5) over $C(B) = \cup_M C(M)$.

Let $\mathcal{H}$ be the set of H-continuous functions. To verify that U^* is H-continuous, it suffices to show that (a) $\mathcal{J}$ maps $\mathcal{H}$ onto itself, and (b) $\mathcal{H}$ is closed under the sup norm. Indeed, this will guarantee that $\mathcal{J}$'s fixed-point belongs to $\mathcal{H}$. To show (a), take any $U \in \mathcal{H}$ and $\varepsilon > 0$. Since $\alpha < 1$ and G is bounded, there is $T > 0$ such that $\frac{\alpha^T 2\bar{G}}{1-\alpha} < \varepsilon/2$, where $\bar{G} = \sup_{\tilde{U} \in \mathcal{U}} |G(\tilde{U})|$. Moreover, since $U \in \mathcal{H}$, there exists N such $|U(c) - U(\tilde{c})| < \varepsilon/2$ whenever $c_t = \tilde{c}_t$ for all $t \leq N$. For any c and $\tilde{c}$,

$$\begin{aligned} |\mathcal{J}(U)(c) - \mathcal{J}(U)(\tilde{c})| &\leq \left| \sum_{t=1}^{\infty} \alpha^t \left[G(U({}_tc)) - G(U({}_t\tilde{c})) \right] \right| \\ &\leq K \sum_{t=1}^{T-1} \alpha^t |U({}_tc) - U({}_t\tilde{c})| + \alpha^T \frac{2\bar{G}}{1-\alpha}. \end{aligned}$$

where K is the Lipschitz constant of G. The first term is less than $\frac{K\alpha}{(1-\alpha)} \max_{t \leq T-1} |U({}_tc) - U({}_t\tilde{c})|$. Now suppose that $c_t = \tilde{c}_t$ for all $t \leq N' = N + T$. This implies that $({}_tc)_{t'} = ({}_t\tilde{c})_{t'}$ for all $t \leq T$ and $t' \leq N$, because ${}_tc$ is truncating at most T elements of c, and c and $\tilde{c}$ were identical up to time $T+N$, by construction. By definition of N, we have $|U({}_tc) - U({}_t\tilde{c})| < \varepsilon/2$ for all $t \leq T$ and, hence, $|\mathcal{J}(U)(c) - \mathcal{J}(U)(\tilde{c})| < \varepsilon$. Setting $T(\varepsilon) = N'$ shows that $\mathcal{J}(U)$ satisfies H-continuity. To prove (b), consider a sequence $\{U^m\}$ in $\mathcal{H}$ that converges to some limit U in the sup norm. Now fix $\varepsilon > 0$. There is m such that $\|U^m - U\|_\infty < \varepsilon/3$. Since $U^m \in \mathcal{H}$, there is N such that $|U^m(c) - U^m(\tilde{c})| < \varepsilon/3$ whenever $c_t = \tilde{c}_t$ for all $t \leq N$. Thus, for such $c, \tilde{c}$,

$$|U(c) - U(\tilde{c})| \leq |U(c) - U^m(c)| + |U^m(c) - U^m(\tilde{c})| + |U^m(\tilde{c}) - U(\tilde{c})| < \varepsilon,$$

which shows that $U \in \mathcal{H}$.

To extend the definition of U^* from $C(B)$ to C, for any $c \in C \setminus C(B)$, consider any sequence $\{c^n\}$ in $C(B)$ such that $c^n_t = c_t$ for all $t \leq n$, and let $U^*(c) = \lim_{n \to +\infty} U^*(c^n)$. This limit is well-defined and independent of the chosen sequence. To see this, note that, for any such sequence $\{c^n\}$ and any $\varepsilon > 0$, H-continuity of U^* implies that there is T such that $|U^*(c) - U^*(\tilde{c})| < \varepsilon$ whenever $c_t = \tilde{c}_t$ for all $t \leq T$. Hence, $|U^*(c^n) - U^*(c^m)| < \varepsilon$ for all $n, m \geq T$, since the consumption levels of c^n and c^m coincide up to $\min\{n, m\}$. So, $\{U^*(c^n)\}$ forms a Cauchy sequence in $\mathbb{R}$ and thus converges. Moreover, the limit is independent of the chosen sequence, as for any $\varepsilon > 0$, $|U^*(c^n) - U^*(\tilde{c}^n)| < \varepsilon$ for n large enough and sequences $\{c^n\}$ and $\{\tilde{c}^n\}$ of the type constructed above.

The limit U thus defined satisfies representation (5). Since U^* is a fixed point of $\mathcal{J}$ on $C(B)$ and c^n belongs to $C(B)$, for each n

$$U^*(c^n) = u(c^n_0) + \sum_{t=1}^{\infty} \alpha^t G(U^*({}_tc^n))$$

The left-hand side converges to $U^*(c)$. Moreover, for each t, $U^*({}_tc^n)$ converges to $U^*({}_tc)$ (which is similarly well defined). Since G is continuous, $G(U^*({}_tc^n))$ converges to $G(U^*({}_tc^n))$ for each t. Since $\alpha < 1$ and G is bounded, by the dominated convergence theorem, the right-hand side converges to $u(c_0) + \sum_{t=1}^{\infty} \alpha^t G(U^*({}_tc))$, which proves that (5) holds for all $c \in C$.

Finally, there is a unique H-continuous extension of U^* from $C(B)$ to C that solves (5). To see this, let U be any H-continuous solution to (5). Since U is a fixed point of $\mathcal{J}$ and the fixed point is unique on $C(B)$, U must coincide with U^* on $C(B)$. Take any $c \in C \setminus C(B)$ and $\varepsilon > 0$. By H-continuity of U and U^*, both $|U(c) - U(\tilde{c})|$ and $|U^*(c) - U^*(\tilde{c})|$ are less than $\varepsilon/2$ for some $\tilde{c} \in C(B)$ equal to c for all t up to a large N. Since U and U^* must be equal at $\tilde{c}$, $|U(c) - U^*(c)| < \varepsilon$. Since ε was arbitrary, $U(c) = U^*(c)$ for all c, establishing uniqueness.

11.6 Proof of Corollary 1

By Theorem 2, $\succ$ can be represented by

$$U(c) = u(c_0) + \sum_{t=1}^{\infty} \alpha^t G(U({}_tc)).$$

Since $(x, c) \succ (y, c)$, $u(x) = u(y) + \overline{u}$ for some $\overline{u} > 0$. Hence, for any $t > 0$, $U(c^x) - U(c^y)$ equals $\overline{u} - \sum_{s=1}^{t} \alpha^s \Delta G_s$, where ΔG_s is defined recursively as follows: for $s = t$,

$$\Delta G_t = G(U({}_tc^y)) - G(U({}_tc^y) - \overline{u}),$$

otherwise

$$\Delta G_s = G(U_s({}_sc^y)) - G\left(U_s({}_sc^y) - \sum_{k=1}^{t-s} \alpha^k \Delta G_{s+k}\right).$$

By Proposition 3, $\Delta G_t < \frac{1-\alpha}{\alpha}\overline{u}$ and

$$\begin{aligned}\Delta G_{t-1} &= G(U_{t-1}({}_{t-1}c^y)) - G\left(U_{t-1}({}_{t-1}c^y) - \alpha\Delta G_t\right) \\ &< (1-\alpha)\Delta G_t < \frac{(1-\alpha)^2}{\alpha}\overline{u}.\end{aligned}$$

Now, suppose that, for all k such that $s < k \leq t-1$, $\Delta G_k < \frac{(1-\alpha)^2}{\alpha}\overline{u}$. It follows that

$$\begin{aligned}\Delta G_s &< \frac{1-\alpha}{\alpha}\left[\sum_{\tau=1}^{t-s} \alpha^\tau \Delta G_{s+\tau}\right] \\ &< \frac{1-\alpha}{\alpha}\left[\sum_{\tau=1}^{t-s-1} \alpha^\tau \frac{(1-\alpha)^2}{\alpha} + \alpha^{t-s}\frac{(1-\alpha)}{\alpha}\right]\overline{u} \\ &= \frac{(1-\alpha)^2}{\alpha}\left[\sum_{\tau=0}^{t-s-2} \alpha^\tau (1-\alpha) + \alpha^{t-s-1}\right]\overline{u} \\ &= \frac{(1-\alpha)^2}{\alpha}\overline{u}.\end{aligned}$$

Therefore,

$$\sum_{s=1}^{t} \alpha^s \Delta G_s < \overline{u}\left[\alpha^t \frac{1-\alpha}{\alpha} + \sum_{s=1}^{t-1} \alpha^s \frac{(1-\alpha)^2}{\alpha}\right] = \overline{u}(1-\alpha).$$

We conclude that $U(c^x) - U(c^y) > \alpha\overline{u} > 0$.

11.7 Proof of Corollary 2

By representation (5), U clearly depends on c_0 only through $u_0 = u(c_0)$. This implies that $U({}_1c)$—and hence also $U(c)$ (from (5))—depends on c_1 only through $u_1 = u(c_1)$. By induction, $U(c)$ depends on $(c_0, \ldots, c_t)$ only through $(u_0, \ldots, u_t)$, for each t. There remains to establish the result at infinity: If c and $\tilde{c}$ are two streams such that $u(c_t) = u(\tilde{c}_t)$ for all t, we need to show that $U(c) = U(\tilde{c})$. From the previous step, assume without loss of generality that $c_t = \tilde{c}_t$ for all $t \leq T$, where T is any large, finite constant. Since U is H-continuous, we can choose T so that $|U(c') - U(\tilde{c}')| < \varepsilon$ for all $c', \tilde{c}'$ that coincide up to T. Since c and $\tilde{c}$ satisfy this property, $|U(c) - U(\tilde{c})| < \varepsilon$, and since ε was arbitrary, $U(c) = U(\tilde{c})$. This shows that the sequence $\{u_t = u(c_t)\}_{t=0}^{\infty}$ of period-utility levels entirely determines the value of $U(c)$, proving the result.

11.8 Proof of Proposition 4

Consider representation (5) in Theorem 2. For every $c \in C$, we have sequences $\{u_s\}_{s=0}^{\infty}$ and $\{U_s\}_{s=0}^{\infty}$, where $u_s = u(c_s)$ and $U_s = U({}_sc)$. Since u is continuous and X is connected, the range of u is a connected interval $I_u \subset \mathbb{R}$. Recall that the range of U is also a connected interval $\mathcal{U} \subset \mathbb{R}$. Using this notation,

$$d(t, c) = \frac{\partial U_0 / \partial u_t}{\partial U_0 / \partial u_0}.$$

Note that $\frac{\partial U_s}{\partial u_s} = 1$ for all $s \geq 0$. Since G is differentiable, we have

$$\frac{\partial U_0}{\partial u_t} = \sum_{\tau=0}^{t-1} \alpha^{t-\tau} G'(U_{t-\tau}) \frac{\partial U_{t-\tau}}{\partial u_t}.$$

More generally, for $1 \leq \tau \leq t$,

$$\frac{\partial U_{t-\tau}}{\partial u_t} = \sum_{s=0}^{\tau-1} \alpha^{\tau-s} G'(U_{t-s}) \frac{\partial U_{t-s}}{\partial u_t}.$$

So, for $\tau = 1$, $\frac{\partial U_{t-1}}{\partial u_t} = \alpha G'(U_t)$. More generally, for $2 \leq \tau \leq t$,

$$\begin{aligned}
\frac{\partial U_{t-\tau}}{\partial u_t} &= \alpha \sum_{s=0}^{(\tau-1)-1} \alpha^{(\tau-1)-s} G'(U_{t-s}) \frac{\partial U_{t-s}}{\partial u_t} + \alpha G'(U_{t-(\tau-1)}) \frac{\partial U_{t-(\tau-1)}}{\partial u_t} \\
&= \frac{\partial U_{t-(\tau-1)}}{\partial u_t} \alpha (1 + G'(U_{t-(\tau-1)})).
\end{aligned}$$

So,

$$\frac{\partial U_{t-\tau}}{\partial u_t} = \alpha^{\tau} G'(U_t) \prod_{s=1}^{\tau-1} (1 + G'(U_{t-s})).$$

Let $\prod_{s=1}^{\tau-1}(1+G'(U_{t-s}))=1$ if $\tau=1$. Then,

$$\begin{aligned}\frac{\partial U_0}{\partial u_t} &= \alpha^t G'(U_t)+G'(U_t)\sum_{\tau=1}^{t-1}\alpha^t G'(U_{t-\tau})\prod_{s=1}^{\tau-1}(1+G'(U_{t-s})) \\ &= \alpha^t G'(U_t)\left[1+\sum_{\tau=1}^{t-1}G'(U_{t-\tau})\prod_{s=1}^{\tau-1}(1+G'(U_{t-s}))\right].\end{aligned}$$

11.9 Proof of Theorem 3

Using Axiom 6 and Theorem 2, we also have

$$(c_0, c_1, {}_2c) \succ (c_0, c_1', {}_2c) \Leftrightarrow (\hat{c}_0, c_1, {}_2c') \succ (\hat{c}_0, c_1', {}_2c') \tag{26}$$

$$(c_0, c_1, {}_2c) \succ (c_0, c_1, {}_2c') \Leftrightarrow (\hat{c}_0, c_1', {}_2c) \succ (\hat{c}_0, c_1', {}_2c') \tag{27}$$

$$(c_0, c_1, {}_2c) \succ (c_0', c_1, {}_2c) \Leftrightarrow (c_0, c_1', {}_2c') \succ (c_0', c_1', {}_2c') \tag{28}$$

$$(c_0, c_1, {}_2c) \succ (c_0, c_1', {}_2c') \Leftrightarrow (\hat{c}_0, c_1, {}_2c) \succ (\hat{c}_0, c_1', {}_2c') \tag{29}$$

By Debreu's [1960], conditions (26)-(29) and (i)-(ii) in Axiom 6 imply that $\succ$ can be represented by

$$w_0(c_0)+w_1(c_1)+w_2({}_2c),$$

for some continuous and nonconstant functions w_0, w_1, and w_2. By Theorem 2, $\succ$ is also represented by

$$u(c_0)+\alpha G(u(c_1)+g({}_2c))+\alpha g({}_2c),$$

where $g({}_2c)=\sum_{t=2}^{\infty}\alpha^{t-1}G(U({}_tc))$. It follows that

$$u(c_0)+\alpha G(u(c_1)+g({}_2c))+\alpha g({}_2c)=\xi\left[w_0(c_0)+w_1(c_1)+w_2({}_2c)\right]+\chi,$$

where $\xi>0$ and $\chi\in\mathbb{R}$. This implies that

$$\alpha G(u(c_1)+g({}_2c))+\alpha g({}_2c)=\xi\left[w_1(c_1)+w_2({}_2c)\right],$$

and therefore G must be affine. Since G must be increasing, without loss of generality let $G(U) = \gamma U$ with $\gamma>0$. Finally, by Proposition 3, $\gamma<\frac{1-\alpha}{\alpha}$.

11.10 Proof of Corollary 6

For every $c\in C$, consider the sequence $\{U_s\}_{s=0}$ in the proof of Proposition 4. By Axiom 5, $c\geq^* c'$ implies $U_s\geq U_s'$ for all $s\geq 0$. It is immediate that, if G' is increasing (decreasing), then $d(t,c)\geq(\leq)\ d(t,c')$ for all $t>0$. On the other hand, suppose G' is not increasing, i.e., there is

$U > U'$ in $\mathcal{U}$ such that $G'(U) < G'(U')$—the other case is similar. By definition and Lemma 2, $U = U(c)$ and $U' = U(c')$ for some constant streams c and c'. By Axiom 5, $c \geq^* c'$. However, for all $t > 0$, $d(t, c) < d(t, c')$.

11.11 Proof of Proposition 6

By assumption, for all t,

$$U({}_tc) = u(c_t) + \sum_{\tau=t+1}^{\infty} \beta\delta^{\tau-t} u(c_\tau),$$

where $0 < \beta = \frac{\gamma}{1+\gamma} < 1$, $0 < \delta = (1+\gamma)\alpha < 1$, $0 < \alpha < 1$.

($\Leftarrow$) Suppose that $W(c) = \sum_{t=0}^{\infty} \alpha^t U({}_tc)$. Now substituting the expression for U, we get

$$\begin{aligned} W(c) &= u(c_0) + \sum_{t=1}^{\infty} [\beta\delta^t + \alpha^t] u(c_t) + \sum_{t=1}^{\infty} \left(\frac{\alpha}{\delta}\right)^t \sum_{\tau=t+1}^{\infty} \beta\delta^\tau u(c_\tau) \\ &= u(c_0) + \sum_{t=1}^{\infty} [\beta\delta^t + \alpha^t] u(c_t) + \sum_{t=2}^{\infty} \beta\delta^t \left(\sum_{\tau=1}^{t-1} \left(\frac{\alpha}{\delta}\right)^\tau \right) u(c_t) \\ &= u(c_0) + \sum_{t=1}^{\infty} u(c_t) \left[\alpha^t + \beta\delta^t \frac{1 - \left(\frac{\alpha}{\delta}\right)^t}{1 - \frac{\alpha}{\delta}} \right]. \end{aligned}$$

Now, the formulas for β and δ imply

$$1 - \frac{\alpha}{\delta} = \frac{\gamma}{1+\gamma} = \beta.$$

Therefore,

$$W(c) = u(c_0) + \sum_{t=1}^{\infty} \delta^t u(c_t).$$

($\Rightarrow$) Using the expression for U, we get

$$\sum_{t=0}^{\infty} w(t) U({}_tc) = w(0) u(c_0) + \sum_{t=1}^{\infty} u(c_t) \left[w(t) + \beta\delta^t \left(\sum_{\tau=0}^{\infty} \frac{w(\tau)}{\delta^\tau} \right) \right].$$

By assumption, $\sum_{t=0}^{\infty} w(t) U({}_tc) = \sum_{t=0}^{\infty} \delta^t u(c_t)$. Therefore, the coefficients of $u(c_t)$ must match for all t. For $t = 0$, $w(0) = 1$. Then, for $t = 1$,

$$w(1) = (1-\beta)\delta = \alpha.$$

Now suppose $w(t) = \alpha^t$ for all $t = 0, \ldots, \tau$. Then,

$$w(\tau+1) = \delta^{\tau+1} - \beta\delta^{\tau+1}\frac{1-\frac{\alpha^{T+1}}{\delta^{\tau+1}}}{1-\frac{\alpha}{\delta}} = \alpha^{\tau+1}.$$

Hence, by induction, $w(t) = \alpha^t$ for all t.

11.12 Proof of Proposition 6

(Part i) Suppose $\succ$ can be represented by $U_t({}_tc) = u(c_t) + \sum_{s>t} d(t,s)u(c_s)$. We want to show that there is an alternative representation given by

$$U_t({}_tc) = u(c_t) + \sum_{s>t} q(t,s)U_s({}_sc),$$

for some function q. If this is true, then for all $t \geq 0$,

$$u(c_t) = U_t({}_tc) - \sum_{s>t} q(t,s)U_s({}_sc)$$

and

$$\begin{aligned} U_t({}_tc) &= u(c_t) + \sum_{s>t} d(t,s)\left[U_s({}_sc) - \sum_{r>s} q(s,r)U_r({}_rc)\right] \\ &= u(c_t) + d(t,t+1)U_{t+1}({}_{t+1}c) + \sum_{s>t+1} U_s({}_sc)\left[d(t,s) - \sum_{t<r\leq s-1} q(r,s)d(t,r)\right] \end{aligned}$$

So, for all $t \geq 0$ and $s > t+1$, $q(t,t+1) = d(t,t+1)$ and

$$q(t,s) = d(t,s) - \sum_{t<r\leq s-1} q(r,s)d(t,r). \tag{30}$$

Define $\hat{\mathcal{T}}(t,s)$ as in (13). For $s = t+2$, (30) becomes

$$\begin{aligned} q(t,t+2) &= d(t,t+2) - d(t,t+1)q(t+1,t+2) \\ &= d(t,t+2) - d(t,t+1)d(t+1,t+2). \end{aligned}$$

So (14) holds for all $t \geq 0$ and $s = t+2$, since $\hat{\mathcal{T}}(t,t+2) = \{(t,t+1,t+2)\}$. Now suppose that (14) holds for all $t \geq 0$ and $s = t+k$ with $2 \leq k \leq n-1$. Then, by (30), for $s' = t+n$

$$q(t,s') = d(t,s') - d(t,s'-1)d(s'-1,s')$$

$$
\begin{aligned}
&- \sum_{t<r\leq s'-2} d(t,r)\left[d(r,s') + \sum_{\mathbf{t}\in\hat{\mathcal{T}}(r,s')} (-1)^{|\mathbf{t}|}\prod_{j=1}^{|\mathbf{t}|-1} d(\tau_{j-1},\tau_j)\right] \\
= \;& d(t,s') - \sum_{t<r\leq s'-1} d(r,s')d(t,r) + \\
&- \sum_{t<r\leq s'-2}\sum_{\mathbf{t}\in\hat{\mathcal{T}}(r,s')} (-1)^{|\mathbf{t}|}\prod_{j=1}^{|\mathbf{t}|-1} d(\tau_{j-1},\tau_j)d(t,r) \\
= \;& d(t,s') + \sum_{\mathbf{t}\in\hat{\mathcal{T}}(t,s')} (-1)^{|\mathbf{t}|}\prod_{j=1}^{|\mathbf{t}|-1} d(\tau_{j-1},\tau_j).
\end{aligned}
$$

The result follows by induction.

(Part ii) Suppose $\succ$ can be represented by $U_t({}_tc) = u(c_t) + \sum_{s>t} q(t,s)U_s({}_sc)$. We want to show that there is an alternative representation given by

$$
U_t({}_tc) = u(c_t) + \sum_{s>t} d(t,s)u(c_s)
$$

for some discount function d. If this is true, then (letting $d(t,t) = 1$ for all t)

$$
\begin{aligned}
U_t({}_tc) &= u(c_t) + \sum_{s>t} q(t,s)U_s({}_sc) \\
&= u(c_t) + \sum_{s>t} q(t,s)\left[\sum_{r\geq s} d(s,r)u(c_r)\right] \\
&= u(c_t) + \sum_{s>t}\left[\sum_{t<r\leq s} q(t,r)d(r,s)\right]u(c_s).
\end{aligned}
$$

So, for all $t \geq 0$ and $s > t$,

$$
d(t,s) = \sum_{t<r\leq s} q(t,r)d(r,s). \tag{31}
$$

Therefore, $d(t,t+1) = q(t,t+1)$ for all $t \geq 0$. Now define $\mathcal{T}(t,s)$ as in (12). Suppose that (15) holds for all $t \geq 0$ and $s = t+k$ with $1 \leq k \leq n-1$. Then, by (31)

$$
\begin{aligned}
d(t,t+n) &= \sum_{t<r\leq t+n} q(t,r)d(r,s) \\
&= q(t,t+n) + \sum_{t<r\leq t+n-1}\left[\sum_{\mathbf{t}\in\mathcal{T}(r,t+n)}\prod_{j=1}^{|\mathbf{t}|-1} q(\tau_{j-1},\tau_j)q(t,r)\right] \\
&= q(t,t+n) + \sum_{\mathbf{t}\in\mathcal{T}(t,t+n)\setminus\{(t,t+n)\}}\prod_{j=1}^{|\mathbf{t}|-1} q(\tau_{j-1},\tau_j)q(t,r).
\end{aligned}
$$

The result follows by induction.

11.13 Proof of Corollary 8

($\Rightarrow$) The claim follows immediately by iteratively substituting the expression of $U_{t+1}({}_{t+1}c)$.

($\Leftarrow$) The claim follows using condition (30) in the proof of Proposition (6). For any $t \geq 0$, if $s = t + 2$, then

$$q(t, t+2) = d(t, t+2) - d(t, t+1)q(t+1, t+2) = 0,$$

using (16) for $d(t, t+2)$ and $q(t+1.t+2) = d(t+1, t+2)$. Now take $s = t+n$, for $n > 2$. Suppose $q(t, t+k) = 0$ for all $k = 2, \ldots, n-1$ and all $t \geq 0$. Then,

$$q(t, t+n) = d(t, t+n) - d(t, t+n-1)q(t+n-1, t+n) = 0,$$

again using (16) for $d(t, t+n)$ and $d(t, t+n-1)$ and $q(t+n-1, t+n) = d(t+n-1, t+n)$.

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