

Nonparametric Welfare Analysis for Discrete Choice

Debopam Bhattacharya*

University of Oxford

September 26, 2014.

Abstract

We consider empirical measurement of exact equivalent/compensating variation resulting from price-change of a discrete good using individual-level data, when there is unobserved heterogeneity in preferences. We show that for binary and multinomial choice, the marginal distributions of EV/CV are nonparametrically point-identified solely from the conditional choice-probabilities, under extremely general preference-distributions. These results hold even when the distribution/dimension of unobserved heterogeneity are neither specified, nor identified and utilities are neither quasi-linear nor parametrically specified. Welfare-distributions can be expressed as closed-form functionals of choice-probabilities, thus enabling easy computation in applications. Average EV for price-rise equals the change in average consumer-surplus and is smaller than average CV for a normal good. Point-identification fails for ordered choice if the unit-price is identical for all alternatives, thereby providing a connection to Hausman-Newey's (2013) partial identification results for the limiting case of *continuous* choice.

1 Introduction

This paper concerns the empirical measurement of money-metric welfare in regard to goods which are consumed or, more generally, decisions which are made, in discrete form. The specific focus is on price variations brought about by taxes and subsidies which affect consumer utility and can give rise to deadweight loss. Examples include, inter alia, the effects of taxing unemployment-benefits on exiting unemployment, of price changes on choice of the mode of transportation and of lowering property taxes on home-ownership. The setting is where the researcher observes realizations of

*I am grateful to Richard Blundell, Martin Browning, Ian Crawford, Eric Gautier, Michael Keane, Arthur Lewbel and Whitney Newey for helpful feedback and encouragement. All errors are mine.

the discrete decision at the individual-level from micro-datasets which also record the individual's characteristics, including income, and prices faced by her in regard to the discrete decision.¹ The goal is to estimate exact – rather than approximate – impact on individual welfare, measured in terms of income compensation, of a change in price brought about by taxes or subsidies and the associated deadweight loss. The analysis incorporates unobserved, individual heterogeneity in utility functions and focuses on recovering the distribution of the impact of price change on individual welfare arising from such heterogeneity without restricting the nature of heterogeneity or specifying functional form of utilities.

Overview of results: Our key results for multinomial choice (binary choice being a special case) are: (i) the marginal distribution and, consequently, the average of compensating variation (CV) and equivalent variation (EV) corresponding to a price change are nonparametrically point-identified solely from choice probabilities, (ii) for a price rise, the average EV is identical to the change in average consumer surplus even if utility is not quasi-linear in income, (iii) the average CV exceeds the average EV if the good is normal and (iv) the above conclusions hold even if the dimension/distribution of heterogeneity are *not* identified. These results are fully nonparametric in the sense that they do not require one to specify the dimension or distribution of heterogeneity and the functional form of utilities – thus allowing for extremely general preference distributions in the population. A practical advantage of our results is that they express welfare distributions as simple, closed-form functionals of the choice probabilities and thus can be easily computed in applications. Finally, we show that nonparametric point-identification of EV/CV distributions fails in the case of *ordered* choice with three or more alternatives if the unit price is identical across all alternatives. To our knowledge, these constitute the first set of results in the econometric literature on the nonparametric identification of exact welfare distributions for discrete choice.

Related literature: A large literature exists in econometrics on estimation of demand from individual level consumption data. Indeed, an important use of such demand estimates is the calculation of welfare effects of price change arising from taxes and subsidies.² For demand of a good that is consumed in continuous quantities, such as gasoline, Hausman, 1981, in a seminal paper, formulated the nonparametric identification and parametric estimation of exact welfare effects of a price change. Vartia, 1983 provided an alternative computational approach to the same problem.

¹This is in contrast to demand analysis using market-level data, popular in empirical Industrial Organization.

²See, for instance, Lewbel, 2001 for a discussion of individual heterogeneity for demand analysis and welfare measurement.

Hausman and Newey, 1995, extended these analyses by formulating semiparametric estimation of the welfare effects and developing the corresponding theory of statistical inference. Their methods cannot be directly used in discrete choice settings where the effect of a price change on individual utilities depends in a fundamental way on the discreteness of choice possibilities as well as on general individual heterogeneity. Specifically, in discrete choice scenarios, corner solutions are generic and this makes it difficult to recover the compensation functions using the differential-equation based approach of the above papers.

In the discrete choice setting, Domencich and McFadden, 1975 (DM75, henceforth) made the strong assumption that utility is quasi-linear, i.e., additively separable in income. Under this simplifying but restrictive assumption, Marshallian and Hicksian welfare measures are identical and average welfare effects of price change can be expressed as the integral of the ordinary Marshallian choice-probabilities (c.f. DM75, pages 94-99). In a highly influential subsequent paper, Small and Rosen, 1981 (SR81, henceforth) investigated the measurement of welfare effects of price and quality change for discrete choice. In their empirical formulation, SR81 introduced additive scalar heterogeneity in utility functions but assumed that the discrete good is sufficiently unimportant to the consumer so that income effects from price or quality changes are negligible (c.f., SR81, page 124, assumptions a and b) – thereby equating Marshallian and Hicksian welfare measures. More recently, Herriges and Kling, 1999 (HK99) and Dagsvik and Karlstrom, 2005 (DK05), in their analysis of the same problem, allowed utility to be nonlinear in income and incorporated unobservables in utility but assumed that these unobservables have both a known dimension and follow a known parametric distribution for identifying and estimating the distribution of welfare effects of price changes. The HK99 and DK05 analysis also require the functional forms of utilities to be known up to finite dimensional parameters which are either non-stochastic, or stochastic with fully known distributions and with the heterogeneity entering the utility function in a known way. Apart from the usual concerns about mis-specification, these parametric approaches do not clarify whether the identification arises from the functional form assumptions or is more fundamental in the sense that the choice probabilities contain all the identification-relevant information, with parametric computations being simply a convenient approximation. Indeed, we will show below that our nonparametric point-identification results which hold for *unordered* multinomial choice fail for *ordered* choice – a fundamental difference which would not be apparent if one were to focus only on parametric models.

In contrast to the works cited above, the purpose of the present paper is to establish *non-*

parametric point-identification of the distribution of welfare effects of price change in a discrete choice setting, incorporating unobservable heterogeneity in the utility function, and assuming no knowledge of the dimension and thus of the distribution of these unobservables. Indeed, in many applications, it is important to allow for multiple sources of unobserved heterogeneity and it is easy to see that restricting the dimension of heterogeneity can place arbitrary restrictions on the variation of individual preferences in the population. For instance, consider the canonical non-parametric binary choice equation $q = 1 \{\beta(p, y) + \varepsilon > 0\}$, where $\beta(p, y)$ is an unknown function of price p and income y , and ε is a scalar additive heterogeneity with an unknown distribution.³ This model implies that if at a specific price and income (p, y) an individual i prefers to buy while individual j does not (implying $\varepsilon_i > \varepsilon_j$), then at no price-income combination (p', y') , can we have a situation where individual i prefers not to buy while individual j prefers to buy – an extremely strong (and untestable) restriction. This restriction is also implied by the more general model $q = 1 \{\beta(p, y, \varepsilon) > 0\}$, where ε is a scalar and $\beta(p, y, \cdot)$ is strictly increasing. Thus allowing for heterogeneity of unrestricted dimension immediately extends the scope of the results to a much larger set of preference profiles.

The *continuous* consumption analog of the present paper is Hausman and Newey, 2013 (HN13). They consider unobserved individual heterogeneity of unspecified dimension in utility functions and focus on recovering average equivalent variation resulting from price change of a *continuous* consumption good, based on demand data. HN13 show that (a) the dimension of heterogeneity is not identified from observed demand and (b) if one allows for heterogeneity of unspecified dimension, then one cannot point-identify average welfare but may obtain bounds on it.⁴ Kitamura and Stoye, 2013 (see also Hoderlein and Stoye, 2013) have also worked under heterogeneity of unspecified dimension and provided tests of consumer rationality in that framework, based on results of McFadden and Richter, 1991. We are not aware of any other work which performs demand analysis at this level of generality regarding both individual heterogeneity and the form of utility functions.⁵ For general binary choice, Ichimura and Thompson, 1998 (IT98) and Gautier and Kitamura, 2012 (GK12) considered random coefficient models where the dimension of heterogeneity is specified to

³See Matzkin (1992) for a nonparametric and and Manski (1975) for a semiparametric analysis of this model.

⁴The HN95 and HN13 approach require solving a differential equation. Such approaches are not useful in the discrete choice case where corner solutions to consumer optimization are generic. Our approach instead works directly from definitions of Hicksian welfare.

⁵Hoderlein and Vanhems 2011, Blundell et al 2012, Blundell et al, 2013, Lewbel and Pendakur, 2013 among others discuss demand and welfare estimation for continuous choice under restricted heterogeneity.

be equal to the number of regressors (plus one for the random intercept) and enter the individual outcome equation in a special way, viz., as scalar multipliers – one attached to each regressor. These authors provide conditions – including large support requirements for regressors – under which the distribution of these random coefficients can be nonparametrically identified up to scale normalization. Such support requirements may be restrictive in specific demand applications. Besides, estimation of the heterogeneity distribution is complicated owing to ill-posed inverse problems. The present paper shows that for the purpose of welfare analysis, this exercise is not necessary and that even the dimension of heterogeneity does not need to be specified in advance.⁶ This finding appears to be of significant practical importance because in demand applications, a key motivation for recovering the distribution of consumer heterogeneity is the calculation of welfare distributions resulting from price change.

The rest of the paper is organized as follows. In section 2, we analyze the leading case of binary choice; in section 3 we show that the results for the binary case generalize to multinomial choice and in section 4, we discuss ordered choice. All proofs are collected in an appendix where we also provide an example where CV and EV distributions are point-identified even when the dimension and distribution of unobserved heterogeneity are not.

2 Binary Choice

2.1 Set-up

Consider an individual with income y , who faces the choice between buying and not buying a binary good which costs p . Let a represent the quantity of numeraire which the individual consumes in addition to the binary good. Suppose that the utility realized by the individual is given by

$$\begin{cases} U_1(a, \eta), & \text{if choose 1,} \\ U_0(a, \eta), & \text{if choose 0,} \end{cases} \quad (1)$$

Here η is a possibly vector-valued, individual-specific taste-variable *of unknown dimension*, unobserved by the econometrician, which enter the utility functions in any arbitrary way. Income and prices faced by the individual are observed by the econometrician. In addition, he may observe a set of covariates. The latter will be suppressed in the exposition below for notational clarity, i.e., the entire analysis should be thought of as implicitly conditioned on these observed covariates.

⁶Of course, the IT98 and GK13 are general mathematical results which are applicable to any random coefficient model of binary outcome, not necessarily related to utility-based welfare analysis.

Given total income y , the budget constraint is $a + pq = y$ where $q \in \{0, 1\}$ represents the binary choice. Replacing the budget constraint in (1), the consumer's realized utility is given by

$$\begin{cases} U_1(y - p, \eta), & \text{if choose 1,} \\ U_0(y, \eta), & \text{if choose 0.} \end{cases}$$

A consumer of type η , income y and facing price p chooses option 1 (buy the good) iff $U_1(y - p, \eta) > U_0(y, \eta)$.⁷

Define the structural choice probability at hypothetical price and income (p, y) as

$$\begin{aligned} \bar{q}(p, y) &= \Pr \{U_1(y - p, \eta) > U_0(y, \eta)\} \\ &\equiv \int 1 \{U_1(y - p, \eta) > U_0(y, \eta)\} dF(\eta), \end{aligned} \quad (2)$$

where $F(\cdot)$ denotes the marginal distribution of η . This is akin to the "average structural function" defined in Blundell and Powell, 2003. Our identification results in this paper are concerned with expressing the marginal distributions of welfare in terms of $\bar{q}(\cdot, \cdot)$.

When observed realizations of price and income are jointly independent of preference heterogeneity η (conditional on other covariates), one can obtain $\bar{q}(p, y)$ by a nonparametric regression of the individual's decision to buy on price and income – i.e., the Marshallian choice probabilities. Independence of preference heterogeneity and budget sets has been assumed in all pre-existing research on welfare estimation for discrete choice, including Domencich and McFadden, 1975, Small and Rosen, 1981 and Dagsvik and Karlstrom, 2005. It is a maintained assumption in random coefficient models, c.f., Ichimura and Thompson, 1998 and Gautier and Kitamura, 2012, which are more restrictive (c.f., footnote 10) in regard to the form and dimension of heterogeneity than our set-up. Conditional independence is also assumed in Hausman and Newey, 2013 in the context of continuous choice with unrestricted heterogeneity. If η is correlated with price or income in the data, then $\bar{q}(\cdot, \cdot)$ can be recovered using control functions (Blundell and Powell, 2003), which satisfy that η is independent of price and income, conditional on the control function.⁸

⁷Our set-up corresponds to data from a single cross-section and the analysis is static. This is in line with essentially the entire econometric literature on empirical welfare analysis, including all of the papers cited above. However, the present set-up is also applicable to the dynamic context if one interprets $U_1(\cdot, \cdot)$ and $U_0(\cdot, \cdot)$ as the expected discounted stream of future utilities resulting from choice of alternative 1 and alternative 0 in the current period. I am grateful to a referee for suggesting this point.

⁸To be clear, our identification results in this paper, viz., propositions 1-5 below, concern identifiability of the relationship between welfare distributions and the structural choice probabilities $\bar{q}(\cdot, \cdot)$. These results hold no matter whether price/income are endogenous or exogenous. The impact of endogeneity is that it affects how one could consistently estimate $\bar{q}(\cdot, \cdot)$. See also footnote 13.

Now, we impose the following assumption on the utility functions – our only substantive assumption in this paper:

Assumption 1 *Suppose that for each η , $U_0(a, \eta)$ and $U_1(a, \eta)$ are continuous and strictly increasing in a . Let $U_1^{-1}(b, \eta)$ denotes the unique solution x to the equation $U_1(x, \eta) = b$ and $U_0^{-1}(b, \eta)$ denotes the unique solution in x to the equation $U_0(x, \eta) = b$.*

The above specification is far more general than the additive, scalar heterogeneity structure used in DM75 and SR81 where utility is given by

$$\begin{cases} U_1(y - p) + \varepsilon_1, & \text{if choose 1,} \\ U_0(y) + \varepsilon_0, & \text{if choose 0,} \end{cases}$$

where the functional forms of $U_1(\cdot, \cdot)$ and $U_0(\cdot, \cdot)$ are *known* (up to estimable finite dimensional parameters), ε_1 and ε_0 are scalar random variables, independent of price and income and have a known distribution. DK05 (c.f. section 5 of their paper) consider the mixed multinomial logit-type structure

$$\begin{cases} U_1(y - p, \beta) + \varepsilon_1, & \text{if choose 1,} \\ U_0(y, \beta) + \varepsilon_0, & \text{if choose 0,} \end{cases} \quad (3)$$

where utilities are of *known* functional form and smooth in parameters; the random coefficients β , which enter the utilities in a known way, are independent of scalar-valued additive errors $\varepsilon_1, \varepsilon_0$, and $(\beta, \varepsilon_1, \varepsilon_0)$ have a fully *known* joint probability distribution independent of (p, y) . DK05 derive expressions for the distribution of Hicksian welfare measures in terms of these known heterogeneity distributions and known utility functions.⁹ Indeed, if (i) the dimension and distribution of unobserved heterogeneity are separately identified from choice probabilities, (ii) functional forms of

⁹For example, DK05 page 63, after the proof of theorem 2, clarify that “...one can calculate the Hicksian choice probabilities readily, **provided the cumulative distribution $F^B(\cdot)$ is known** since only a one-dimensional integral is involved in the formula for $P_B^h(j, w, u)$.” When the dimension of heterogeneity and/or its distribution $F^B(\cdot)$ are *unknown*, the theorems and corollaries of DK05 do not provide identification results; they express the objects of interest – written on the LHS in equation (12), theorem 2, corollary 2 etc. – in terms of the model primitives, viz., the individual utility functions and the distribution of heterogeneity (and its partial derivatives) which appear on the RHS. They do not establish that the RHS expressions are nonparametrically identified from observed choice data without knowledge of the heterogeneity distribution. This may be contrasted with equations (5) and (9) of the present paper, where the LHS are the object of interest and the RHS are choice probabilities which are nonparametric functions of the data.

utilities are known and (iii) the key unobservables (denoted by ε s) enter as additive scalar errors in utility functions and all other unobservables (denoted by β) are either non-existent or enter utilities in a known way and are independent of the ε s, then one can use the expressions in DK05 to calculate the distribution of Hicksian welfare measures. These conditions are restrictive and somewhat arbitrary. In the appendix of the present paper, we provide an example of binary choice where the dimension and thus distribution of heterogeneity are *not* identified and thus the DK05 results cannot be applied, and yet welfare distributions are nonparametrically point-identified using our results because they are based solely on the choice probability functions.

The limitations of a fully parametric approach are brought out more clearly in the context of ordered choice, discussed below in section 4. In this case, making parametric assumptions would enable one to "calculate" the distribution functions of CV/EV exactly (c.f., remark 8 below). However, we show in section 4 that for ordered choice, the distributions of CV/EV generically *fail* to be nonparametrically point-identified; thus the parametric calculations in this case would correspond to identification obtained solely via functional form assumptions. This important difference between the ordered and unordered case would not be apparent if one were only using parametrically specified utility functions and heterogeneity distributions.

In contrast, our identification results require only that utilities are continuous and strictly increasing in the numeraire (assumption 1) and do not require one to specify the functional form of the utilities (including how unobserved heterogeneity enter utilities), to impose differentiability, to specify the dimension and distribution of unobserved heterogeneity or to assume arbitrary independence conditions among different components of heterogeneity, as in model (3) above.¹⁰

¹⁰Our set-up is also more general than a pure random coefficient model, which postulates $q = 1 \{-p + \varepsilon_1 y + \varepsilon_0 > 0\}$, where $(\varepsilon_0, \varepsilon_1)$ is a 2-dimensional random vector independent of (p, y) . For example, the random coefficient model implies that for fixed p , if an individual prefers to buy at income y and also at income $y' > y$, then she must also prefer to buy at any intermediate income $\lambda y + (1 - \lambda) y'$ for $\lambda \in (0, 1)$. This is because

$$\begin{aligned} & -p + \varepsilon_1 (\lambda y + (1 - \lambda) y') + \varepsilon_0 \\ = & \lambda \{-p + \varepsilon_1 y + \varepsilon_0\} + (1 - \lambda) \{-p + \varepsilon_1 y' + \varepsilon_0\}, \end{aligned}$$

so that if each term on the RHS is positive, then so is the LHS. This is not imposed by our set-up because

$$U_1(y - p, \eta) > U_0(y, \eta) \quad \text{and} \quad U_1(y' - p, \eta) > U_0(y', \eta)$$

need **not** imply that

$$U_1(\lambda y + (1 - \lambda) y' - p, \eta) > U_0(\lambda y + (1 - \lambda) y', \eta).$$

We now turn to welfare analysis within our set-up.

2.2 Equivalent Variation

First, consider equivalent variation as a measure of welfare. We will evaluate welfare change for a ceteris paribus price increase from p_0 to p_1 . That is, for fixed y , we will calculate the amount of income to be subtracted from an η type individual with income y and facing prices p_0 in order that her (maximized) utility in this situation equals that when she were facing prices p_1 where $p_1 > p_0$. For lack of a better word, we will call this "compensation" but with the understanding that it is an amount of money being taken away.

Note that by definition, the EV, denoted by $S^{EV}(y, p_0, p_1, \eta)$, will satisfy

$$\begin{aligned} & \max \{U_0(y - S^{EV}(y, p_0, p_1, \eta), \eta), U_1(y - S^{EV}(y, p_0, p_1, \eta) - p_0, \eta)\} \\ = & \max \{U_0(y, \eta), U_1(y - p_1, \eta)\}. \end{aligned} \quad (4)$$

We will demonstrate that the expressions for $S^{EV}(y, p_0, p_1, \eta)$ take the following forms for different ranges of η , for given y, p_1 and p_0 .

Proposition 1 *Suppose Assumption 1 holds. Then (i) if $U_1(y - p_0, \eta) \leq U_0(y, \eta)$, then $S^{EV}(y, p_0, p_1, \eta) = 0$; (ii) if $U_1(y - p_1, \eta) \leq U_0(y, \eta) < U_1(y - p_0, \eta)$, then $S^{EV}(y, p_0, p_1, \eta) = y - p_0 - U_1^{-1}(U_0(y, \eta), \eta)$; (iii) if $U_1(y - p_1, \eta) > U_0(y, \eta)$, then $S^{EV}(y, p_0, p_1, \eta) = p_1 - p_0$.*

(See appendix for a proof.)

These three cases correspond respectively to η s who do not buy at the lower price, those who switch from buying at lower price to not buying at the higher price and those who buy at both low and high price. While the zero EV is obvious for the first group, the other cases are not entirely obvious because one needs to understand how buying is affected when income is deducted from a situation of low price.

The above proposition is an intermediate result which leads us to the first identification result, where we provide a closed-form expression for the marginal distribution of EV, induced by the distribution of the heterogeneity η .

Proposition 2 *Suppose Assumption 1 holds. Consider a price rise from p_0 to p_1 . Then the*

equivalent variation evaluated at hypothetical income y has marginal distribution given by

$$\Pr \{S^{EV}(y, p_0, p_1, \eta) \leq a\} = \begin{cases} 0, & \text{if } a < 0, \\ 1 - \bar{q}(p_0, y) & \text{if } a = 0, \\ 1 - \bar{q}(a + p_0, y), & \text{if } 0 < a < p_1 - p_0, \\ 1, & \text{if } a \geq p_1 - p_0, \end{cases} \quad (5)$$

where $\bar{q}(\cdot, \cdot)$ is defined above in (2).

(See appendix for proof)

Interpretation: An intuitive interpretation of the result is as follows. Consider an η type individual whose reservation price for the binary good at income y is $p_0 + t(y, \eta)$ where $0 < t(y, \eta) < p_1 - p_0$. This means that she is indifferent between buying and not buying at price $p_0 + t(y, \eta)$ when she has income y so that

$$U_0(y, \eta) = U_1(y - (p_0 + t(y, \eta)), \eta). \quad (6)$$

At any higher price, the RHS is smaller and she does not buy; at any lower price, the RHS is larger and she buys. But since $y - (p_0 + t(y, \eta)) = (y - t(y, \eta)) - p_0$, we get that

$$U_0(y, \eta) = U_1((y - t(y, \eta)) - p_0, \eta),$$

which means that if we take away an amount of the numeraire equal to $t(y, \eta)$, then she would reach the same level of utility from buying at price p_0 as she would when not buying. Recall that since $t(y, \eta) < p_1 - p_0$, she was not buying at the higher price p_1 and getting utility $U_0(y, \eta)$, which is precisely the reference utility for calculation of her EV. The previous display therefore implies that the EV for a price increase from p_0 to p_1 is $t(y, \eta)$ for this consumer. That is, the EV equals the difference between the reservation price and the initial lower price p_0 . Therefore, the probability that EV is less than a equals the proportion of individuals with reservation price less than $p_0 + a$ which, by definition, equals the fraction of individuals who do not buy at prices higher than $p_0 + a$, and is thus given by $1 - \bar{q}(p_0 + a, y)$.

Having obtained the distribution of the EV, we now aggregate these functions w.r.t. the marginal distribution of heterogeneity to obtain the average EV.

Corollary 1 *Suppose Assumption 1 holds. Then for a price increase from p_0 to p_1 , the average*

equivalent variation evaluated at income y is given by

$$\mu^{EV}(y, p_0, p_1) = \int_{p_0}^{p_1} \bar{q}(p, y) dp, \quad (7)$$

where $\bar{q}(p, y)$ is defined in (2). (see appendix for proof)

Remark 1 Since $\bar{q}(p, y)$ is simply the average Marshallian choice probability observed in the data, the above conclusion shows that the change in Marshallian consumer surplus and the average Hicksian equivalent variations are equal. This result obtains although the utility functions were not specified to be quasi-linear. Indeed, for quasi-linear utility, **both CV and EV** are equal to the change in Marshallian consumer surplus which will not (necessarily) be the case here, as we shall see below.

2.3 Compensating Variation

Now consider measurement of compensating variation. Fix a hypothetical level of income y and consider a price rise from an initial value p_0 to p_1 . The compensating variation $S^{CV}(y, p_0, p_1, \eta)$ measures the income S to be given to an individual of type η at income y and facing price $p_1 > p_0$, so that their maximized utility in this situation is equal to their maximized utility when prices were p_0 and income was y , i.e.,

$$\begin{aligned} & \max \{U_0(y + S, \eta), U_1(y + S - p_1, \eta)\} \\ &= \max \{U_0(y, \eta), U_1(y - p_0, \eta)\}. \end{aligned} \quad (8)$$

We will demonstrate that the expressions for $S^{CV}(y, p_0, p_1, \eta)$ take the following form.

Proposition 3 *Suppose Assumption 1 holds. Then, (i) if $U_1(y - p_0, \eta) \leq U_0(y, \eta)$, then $S^{CV}(y, p_0, p_1, \eta) = 0$; (ii) if $U_0(y, \eta) < U_1(y - p_0, \eta) \leq U_0(y + p_1 - p_0, \eta)$, then $S^{CV}(y, p_0, p_1, \eta) = U_0^{-1}(U_1(y - p_0, \eta), \eta) - y$; (iii) if $U_1(y - p_0, \eta) > U_0(y + p_1 - p_0, \eta)$, then $S^{CV}(y, p_0, p_1, \eta) = p_1 - p_0$ (see appendix for proof).*

These three cases correspond respectively to η s who (i) do not buy at the lower price, (ii) those who switch from buying at lower price to not buying at the higher price *but when compensated by the amount of price change would prefer not to buy* and (iii) switchers who when compensated by the amount of price change would prefer to buy as well as those who buy at both low and high price. While the zero CV is obvious for the first group, the other cases are not entirely obvious

because one needs to understand how buying is affected when income is raised from a situation of high price.

The above proposition, like proposition 1, is also an intermediate result which helps us reach our second identification result, where we establish that the distribution of CV, induced by the distribution of the heterogeneity η can also be expressed solely in terms of the choice probability functional $\bar{q}(\cdot, \cdot)$.

Proposition 4 *Suppose Assumption 1 holds. Consider a price rise from p_0 to p_1 . Then the compensating variation evaluated at hypothetical income y has marginal distribution given by*

$$\Pr \{S^{CV}(y, p_0, p_1, \eta) \leq a\} = \begin{cases} 0, & \text{if } a < 0, \\ 1 - \bar{q}(p_0, y) & \text{if } a = 0, \\ 1 - \bar{q}(p_0 + a, y + a), & \text{if } 0 < a < p_1 - p_0, \\ 1, & \text{if } a \geq p_1 - p_0, \end{cases} \quad (9)$$

where $\bar{q}(p, y)$ is defined in (2) (see appendix for proof).

Interpretation: An intuitive interpretation of the result is as follows. Consider an η type individual, who faces initial price p_0 and has income y . Now consider a situation where price goes up to $p_0 + a$ where $0 \leq a < p_1 - p_0$ and her income is compensated by the amount of price-rise a . Suppose that in this new situation, her utility from not buying exceeds her utility from buying, i.e.,

$$U_0(y + a, \eta) > U_1((y + a) - (p_0 + a), \eta) = U_1(y - p_0, \eta).$$

Since $U_0(\cdot, \eta)$ is strictly increasing, it is also true that $U_0(y + a, \eta) > U_0(y, \eta)$. Putting the two together,

$$U_0(y + a, \eta) > \max\{U_1(y - p_0, \eta), U_0(y, \eta)\}.$$

For CV, the RHS utility is the reference utility and therefore any compensation exceeding a will lead such a person to not buy and enjoy a utility level exceeding the reference utility level $\max\{U_1(y - p_0, \eta), U_0(y, \eta)\}$. Thus for such a person, the CV must not exceed a . Such individuals can be identified in the data as those who do not buy when they have income $y + a$ and price is $p_0 + a$ and thus

$$\Pr(S^{CV}(y, p_0, p_1, \eta) \leq a) = 1 - \bar{q}(p_0 + a, y + a).$$

In particular, the CV must not exceed a for those who had switched initially from buying to not buying due to the price-rise from p_0 to $p_0 + a$ but who would nonetheless, upon getting compensated income $y + a$, strictly prefer to not buy and instead leave with the extra money when price is $p_0 + a$. These individuals can be made as happy as they originally were by paying a compensation strictly less than a and letting them leave with this extra money without buying.

Corollary 2 *Suppose Assumption 1 holds. Then for a price increase from p_0 to p_1 , the average compensating variation evaluated at income y is given by*

$$\mu^{CV}(y, p_0, p_1) = \int_{p_0}^{p_1} \bar{q}(p, y + p - p_0) dp. \quad (10)$$

(see appendix for proof).¹¹

The following remarks are in order.

Remark 2 *If the binary good is normal, then $\bar{q}(p, y) \leq \bar{q}(p, y + p - p_0)$ for fixed y and for all $p \geq p_0$. Hence, it follows from the expressions for CV and EV in (7) and (10) that for a price increase from p_0 to p_1 , $\mu^{EV}(y, p_0, p_1) \leq \mu^{CV}(y, p_0, p_1)$.*

Remark 3 *It is important to note that the proofs of the four propositions above do not rely on whether the distribution of unobserved heterogeneity η is known or identified. In the appendix, we provide an example where the distribution and even the dimension of heterogeneity are not identified and yet the conditions of proposition 2 and 4 are satisfied, so that the distribution of welfare measures are point-identified.*

Remark 4 *For a per unit tax of τ , the average deadweight loss is given by*

$$\begin{aligned} DWL^{tax}(EV) &= \int_{p_0}^{p_0(1+\tau)} \bar{q}(p, y) dp - \tau p_0 \times \bar{q}(p_0(1+\tau), y), \\ DWL^{tax}(CV) &= \int_{p_0}^{p_0(1+\tau)} \bar{q}(p, y + p - p_0) dp - \tau p_0 \times \bar{q}(p_0(1+\tau), y). \end{aligned}$$

Remark 5 *When a subsidy reduces prices from p_1 to p_0 , the labelling of EV and CV reverses and we get that*

$$\mu^{CV}(p_0, p_1, y) = \int_{p_0}^{p_1} \bar{q}(p, y) dp, \quad \mu^{EV}(p_0, p_1, y) = \int_{p_0}^{p_1} \bar{q}(p, y + p - p_0) dp.$$

¹¹Our expressions (7) and (10) may be contrasted with the claim in Maler, 1974, page 139 that mean CV and EV are both identical to the change in aggregate Marshallian consumer surplus. This claim would be true if and only if individual demand for the discrete good is income-inelastic. Otherwise, mean EV would still equal mean change in CS but not mean CV.

Remark 6 In order for the C.D.F. of CV to be weakly increasing, one needs to check whether for all $a > 0$ and for all y , $\bar{q}(p_0 + a, y + a) \leq \bar{q}(p_0, y)$. Similarly, for C.D.F. of EV to be weakly increasing, one needs to check that for all $a > 0$, $\bar{q}(p_0 + a, y) \leq \bar{q}(p_0, y)$, for all y . Note that according to the random utility model considered here,

$$\begin{aligned}
& \bar{q}(p_0, y) - \bar{q}(p_0 + a, y + a) \\
= & \Pr \{U_0(y, \eta) - U_1(y - p_0, \eta) \leq 0\} - \Pr \{0 > U_0(y + a, \eta) - U_1(y + a - (p_0 + a), \eta)\} \\
= & \Pr \{U_0(y, \eta) - U_1(y - p_0, \eta) \leq 0\} - \Pr \{U_0(y + a, \eta) - U_1(y - p_0, \eta) < 0\} \\
\equiv & \int [1 \{U_0(y, \eta) - U_1(y - p_0, \eta) \leq 0\} - 1 \{U_0(y + a, \eta) - U_1(y - p_0, \eta) < 0\}] dF(\eta),
\end{aligned}$$

which is non-negative since $U_0(\cdot, \eta)$ is assumed to be strictly increasing and $a > 0$. Similarly,

$$\begin{aligned}
& \bar{q}(p_0, y) - \bar{q}(p_0 + a, y) \\
= & \Pr \{U_0(y, \eta) < U_1(y - p_0, \eta)\} - \Pr \{U_0(y, \eta) < U_1(y - (p_0 + a), \eta)\} \\
\equiv & \int [1 \{U_0(y, \eta) < U_1(y - p_0, \eta)\} - 1 \{U_0(y, \eta) < U_1(y - (p_0 + a), \eta)\}] dF(\eta),
\end{aligned}$$

which is non-negative since $U_1(\cdot, \eta)$ is assumed to be strictly increasing and $a > 0$. One may test these conditions after estimating an unrestricted $\bar{q}(\cdot, \cdot)$ or impose these restrictions during estimation. The first condition, viz., $\bar{q}(p_0 + a, y + a) \leq \bar{q}(p_0, y)$ can also be interpreted as a stochastic revealed preference (c.f., McFadden and Richter, 1991) type inequality for binary choice.¹²

Remark 7 When income is endogenous, there is a distinction between (i) the marginal distribution of welfare evaluated at hypothetical income y (the parameter of interest in the present paper) and (ii) the conditional distribution of welfare evaluated at hypothetical income y for the subpopulation

¹²When facing price $p_0 + a$ with income $y + a$, $a > 0$, a consumer's budget-set is the pair $\{(0, y + a), (1, y - p_0)\}$, where the first entry is the binary choice and the second is the quantity of numeraire. Consider those η s who choose the bundle $(1, y - p_0)$ and thus for whom $(1, y - p_0)$ is directly revealed preferred to $(0, y + a)$. If "more numeraire is better", then $(0, y + a)$ must be preferred by all consumers over $(0, y)$. Therefore for the above η s, $(1, y - p_0)$ is revealed preferred to $(0, y)$. By a standard RP axiom, it follows that in a situation where the budget-set is the pair $\{(0, y), (1, y - p_0)\}$, these η s must not choose $(0, y)$. Thus the set

$$\begin{aligned}
& \{\eta : (1, y - p_0) \succ (0, y + a)\} \\
\subseteq & \{\eta : (1, y - p_0) \succ (0, y)\},
\end{aligned}$$

implying $\bar{q}(p_0 + a, y + a) \leq \bar{q}(p_0, y)$.

whose value of realized income is y' . For EV, say, these are given respectively by

$$\begin{aligned}\pi(a, y) &\stackrel{def}{=} \int 1\{EV(p_0, p_1, y, \eta) \leq a\} dF(\eta), \\ \pi^c(a, y, y') &\stackrel{def}{=} \int 1\{EV(p_0, p_1, y, \eta) \leq a\} dF_{\eta|Y}(\eta|y'),\end{aligned}$$

where $F(\cdot)$ denotes the marginal distribution of η and $F_{\eta|Y}(\cdot|y')$ denotes the conditional distribution of η for the sub-population whose current income is y' . In a treatment effect context, $\pi(a, y)$ is analogous to the average treatment effect and $\pi^c(a, y, y')$ is analogous to the average effect of treatment on the treated. In this paper, we focus on the former parameter. Of course, if income is exogenous conditional on observed covariates (as assumed in DM75, HK99, DK05), then the two parameters coincide.

3 Multinomial Choice

We now extend the above results for binary outcomes to multinomial choice. Assume that a consumer with income y and taste η (again, possibly vector-valued and of unknown dimension) faces a mutually exclusive set of alternatives with alternative-specific prices – the classic example being choice of the mode of transportation (e.g., bus, train, walk etc.). The consumer can pick only one among the various alternatives. Let the set of alternatives be denoted by $\{0, 1, \dots, J\}$ with p_j denoting the price of the j th alternative for $j = 1, \dots, J$ and the 0th alternative denoting not choosing any of the J alternatives. As before, assume that utility from choosing alternative 0 is $U_0(y, \eta)$ and choosing alternative j produces utility $U_j(y - p_j, \eta)$.

Let $p_{-1} = (p_2, p_3, \dots, p_J)$, and let $\bar{q}_1(t, p_{-1}, y)$ denotes (analogous to (2)) the structural choice probability of choosing alternative 1 when its price is t , prices of the other alternatives are held fixed at p_{-1} and income is fixed at y , i.e.,

$$\begin{aligned}\bar{q}_1(t, p_{-1}, y) &\stackrel{def}{=} \int 1\{U_1(y - t, \eta) \geq \max\{U_0(y, \eta), U_2(y - p_2, \eta), \dots, U_J(y - p_J, \eta)\}\} dF(\eta),\end{aligned}\quad (11)$$

where $F(\cdot)$ denotes the marginal distribution of η .

Assumption 2 $U_j(\cdot, \eta)$ is continuous and strictly increasing for each η , for $j = 0, 1, \dots, J$.

Consider a price increase for alternative 1 from p_{10} to p_{11} , with the prices of all other alternatives

held fixed at p_{-1} and income fixed at y . Define CV to be the solution S to the equation

$$\begin{aligned} & \max \{U_0(y + S, \eta), U_1(y + S - p_{11}, \eta), U_2(y + S - p_2, \eta), \dots, U_J(y + S - p_J, \eta)\} \\ = & \max \{U_0(y, \eta), U_1(y - p_{10}, \eta), U_2(y - p_2, \eta), \dots, U_J(y - p_J, \eta)\}. \end{aligned}$$

Similarly, define EV to be the solution S to the equation

$$\begin{aligned} & \max \{U_0(y - S, \eta), U_1(y - S - p_{10}, \eta), U_2(y - S - p_2, \eta), \dots, U_J(y - S - p_J, \eta)\} \\ = & \max \{U_0(y, \eta), U_1(y - p_{11}, \eta), U_2(y - p_2, \eta), \dots, U_J(y - p_J, \eta)\}. \end{aligned}$$

Then we have the following result.

Proposition 5 *Suppose assumption 2 holds. Consider a price increase for alternative 1 from p_{10} to p_{11} , with the prices of all other alternatives held fixed at p_{-1} . Then EV and CV evaluated at hypothetical income y have marginal distributions given by*

$$\Pr [EV(y, p_{10}, p_{11}, p_{-1}, \eta) \leq r] = \begin{cases} 0, & \text{if } r < 0, \\ 1 - \bar{q}_1(p_{10}, p_{-1}, y) & \text{if } r = 0, \\ 1 - \bar{q}_1(p_{10} + r, p_{-1}, y) & \text{if } 0 < r < p_{11} - p_{10}, \\ 1, & \text{if } r \geq p_{11} - p_{10}, \end{cases} \quad (12)$$

and

$$\Pr [CV(y, p_{10}, p_{11}, p_{-1}, \eta) \leq r] = \begin{cases} 0, & \text{if } r < 0, \\ 1 - \bar{q}_1(p_{10}, p_{-1}, y) & \text{if } r = 0, \\ 1 - \bar{q}_1(p_{10} + r, p_{-1}, y + r) & \text{if } 0 < r < p_{11} - p_{10}, \\ 1, & \text{if } r \geq p_{11} - p_{10}, \end{cases} \quad (13)$$

with $\bar{q}_1(\cdot, \cdot, \cdot)$ defined in (11).

(Proof in appendix)

Corollary 3 *Under assumption 2, the expected values of EV and CV evaluated at income y are given respectively by*

$$E(EV) = \int_{p_{10}}^{p_{11}} \bar{q}_1(r, p_{-1}, y) dr, \quad (14)$$

$$E(CV) = \int_{p_{10}}^{p_{11}} \bar{q}_1(r, p_{-1}, y + r - p_{10}) dr. \quad (15)$$

(Proof exactly analogous to corollaries 1 and 2).

Remark 8 *Even when one is willing to make parametric assumptions (e.g., multinomial logit), proposition 5 and corollary 3 can be used directly to compute welfare distributions from choice-probabilities without going back to computing expenditure functions etc., as in HK99 and DK05. For example, for a multinomial logit specification with alternatives $\{0, 1, 2, \dots, J\}$, the conditional choice probability for alternative 1 is given by*

$$\bar{q}_1(p, p_{-1}, y) = \frac{\exp(\beta_{01} + \beta_{11}p + \beta'_{-11}p_{-1} + \beta_{Y1}y)}{1 + \sum_{j \neq 0} \exp(\beta_{0j} + \beta_{1j}p + \beta'_{-1j}p_{-1} + \beta_{Yj}y)}$$

where β_{0j} , β_{1j} , β_{-1j} and β_{Yj} denote respectively the intercept, coefficients on p_j and p_{-j} and the income coefficient for the j th alternative, estimable via Maximum Likelihood. Accordingly,

$$\begin{aligned} & E(CV) \\ &= \int_{p_{10}}^{p_{11}} \bar{q}_1(r, p_{-1}, y + r - p_{10}) dr \\ &= \int_{p_{10}}^{p_{11}} \left\{ \frac{\exp(\beta_{01} + \beta_{11}r + \beta'_{-11}p_{-1} + \beta_{Y1}(y + r - p_{10}))}{1 + \sum_{j \neq 0} \exp(\beta_{0j} + \beta_{1j}r + \beta'_{-1j}p_{-1} + \beta_{Yj}(y + r - p_{10}))} \right\} dr. \end{aligned} \quad (16)$$

Estimation: The present paper is mainly concerned with identification; a full-scale treatment of estimation and inference is being investigated in a separate paper.¹³ We simply note here that once a nonparametric estimate of $\bar{q}_1(p, p_{-1}, y)$, is obtained, say, using kernels or series methods, one can estimate the distribution functions of EV/CV by direct plug-in into equations (12) and (13). For inference on expected welfare (14) and (15), obtained by integrating the $\bar{q}_1(p, p_{-1}, y)$ functions over p holding y fixed, one can use distributional results for partial means (c.f., Newey, 1994, Lee, 2014). For a parametric specification of choice probabilities, e.g. multinomial logit, one first obtains the logit coefficients and then obtains an estimate of expected welfare which, as is apparent from (16), is a smooth functional of these coefficients. Accordingly, one can employ the bootstrap to conduct inference on mean welfare, whose justification follows via the delta method.

4 Ordered Choice

The multinomial choice scenario of the previous section may be contrasted with a situation of ordered choice, i.e., where a good can be bought in discrete units of 0, 1, 2, etc. *and the per unit*

¹³In Bhattacharya and Lee, 2014, we are developing large-sample inference theory for the endogenous income case, using recent results of Lee (2014). We are doing this for both the unconditional distribution of welfare (considered above) and the distribution of welfare conditional on current income, analogous to average treatment effect and average effect of treatment on the treated, respectively. These two parameters coincide when income is exogenous.

price is the same, no matter how many units are bought. Then an increase in the unit price will change the price of all choices (viz., buy 1 unit, buy 2 units etc.) *simultaneously*, making this case different from the multinomial case considered above. Continuous choice, considered in Hausman and Newey, 2013, can be viewed as a limiting case of ordered choice. We will demonstrate that for ordered choice, the distributions of welfare changes are *not* point-identified in general. However, if the per unit price is allowed to be *different* depending on how many units are bought (e.g., a discount is provided for larger purchases), then it becomes possible to change, say, the unit price of buying 1 unit while holding the unit price of other alternatives (e.g., buying 2 units, 3 units etc.) fixed. This case will then reduce to the multinomial case above and one would end up with a point-identification result. To take a simple example, consider two scenarios. In scenario A, one chooses between 1 banana, 2 bananas and no banana where the price per banana is p . In this case, the distributions of CV/EV corresponding to changing the price p will not be point-identified. In scenario B, one still chooses between 1 banana, 2 bananas and no banana but the price is p per banana if one buys a single banana but it is p_2 per banana if one buys 2 bananas. In this second scenario, the distribution of CV/EV arising from changing p while holding p_2 fixed (or vice versa) are point-identified since one may simply view this as a multinomial choice problem with 3 alternatives – where the "price" of alternative 1 (buy no banana) is zero, that of alternative 2 (buy 1 banana) is p and that of alternative 3 (buy 2 bananas) is $2p_2$.

Scenario A: To see the failure of point-identification in scenario A explicitly, let $U_0(y, \eta)$, $U_1(y - p, \eta)$ and $U_2(y - 2p, \eta)$ denote the utility from buying 0, 1 or 2 bananas respectively, where p denotes the (uniform) unit price per banana. Assume that for each $j = \{0, 1, 2\}$, $U_j(\cdot, \eta)$ is strictly increasing with probability 1. From the conditional choice probabilities of alternatives 0, 1 and 2 on observed budget sets (p^*, y^*) , we can identify the probabilities of the following sets

$$\begin{aligned}
A_0(p^*, y^*) &= \{\eta : U_0(y^*, \eta) \geq \max\{U_1(y^* - p^*, \eta), U_2(y^* - 2p^*, \eta)\}\}, \\
A_1(p^*, y^*) &= \{\eta : U_1(y^* - p^*, \eta) \geq \max\{U_0(y^*, \eta), U_2(y^* - 2p^*, \eta)\}\}, \\
A_2(p^*, y^*) &= \{\eta : U_2(y^* - 2p^*, \eta) \geq \max\{U_0(y^*, \eta), U_1(y^* - p^*, \eta)\}\}, \tag{17}
\end{aligned}$$

for different values of (p^*, y^*) . It is important to note in each line of the previous display the utilities for the 3 alternatives being compared are evaluated at the same (p^*, y^*) . The choice probabilities cannot point-identify, for example, the probability of a set like

$$\{\eta : U_1(y^* - p^*, \eta) \geq \max\{U_0(y^*, \eta), U_2(y^* - 2\tilde{p}, \eta)\}\},$$

where $\tilde{p} \neq p^*$, because no individual in the data faces two different unit prices for alternative 1 and alternative 2 in scenario A. In other words, no consumer in the population may be observed to make a choice among the three bundles $(0, y^*)$, $(1, y^* - p^*)$ and $(2, y - 2\tilde{p})$ if $\tilde{p} \neq p^*$.

Now, suppose the per unit price p changes from p_0 to p_1 where $p_1 > p_0$. Then the resulting CV is the solution S to the equation

$$\begin{aligned} & \max \{U_0(y, \eta), U_1(y - p_0, \eta), U_2(y - 2p_0, \eta)\} \\ = & \max \{U_0(y + S, \eta), U_1(y + S - p_1, \eta), U_2(y + S - 2p_1, \eta)\}. \end{aligned} \quad (18)$$

Consider the probability that $S = p_1 - p_0$. If the utility differences are continuously distributed, then the only situation where $S = p_1 - p_0$ is where the maximum on the LHS of (18) is $U_1(y - p_0, \eta)$ and that on the RHS is $U_1(y + S - p_1, \eta)$.¹⁴ Thus

$$\begin{aligned} & \Pr[S = p_1 - p_0] \\ = & \Pr \left[\begin{array}{l} U_1(y - p_0, \eta) \geq \max \{U_0(y, \eta), U_2(y - 2p_0, \eta)\}, \\ U_1(y + p_1 - p_0 - p_1, \eta) \geq \max \{U_0(y + p_1 - p_0, \eta), U_2(y + p_1 - p_0 - 2p_1, \eta)\} \end{array} \right] \\ = & \Pr \left[\begin{array}{l} U_1(y - p_0, \eta) \geq \max \{U_0(y, \eta), U_2(y - 2p_0, \eta)\}, \\ U_1(y - p_0, \eta) \geq \max \{U_0(y + p_1 - p_0, \eta), U_2(y - p_0 - p_1, \eta)\} \end{array} \right] \\ = & \Pr[U_1(y - p_0, \eta) \geq \max \{U_0(y + p_1 - p_0, \eta), U_2(y - 2p_0, \eta)\}], \end{aligned} \quad (19)$$

by strict monotonicity of $U_0(\cdot, \eta)$ and $U_2(\cdot, \eta)$. For standard parametric models (e.g., $U_j(y, \eta) = \alpha_j \ln(y) + \eta_j$ with η_j scalar and normally distributed), the probability in (19) is positive. However, it is clear from (19) that the probability that $S = p_1 - p_0$ equals the probability of choosing the bundle $(1, y - p_0)$ over the bundles $(0, y + p_1 - p_0)$ and $(2, y - 2p_0)$. The latter probability can be nonparametrically point-identified if and only if some consumers in the population face the choice

¹⁴For example, if instead the first term is the maximum for the LHS of (18) and the second term is the maximum for the RHS of (18) with $S = p_1 - p_0$, then

$$U_0(y, \eta) = U_1 \left(y + \overbrace{p_1 - p_0}^S - p_1, \eta \right) = U_1(y - p_0, \eta)$$

which will have zero probability if $U_1(y - p_0, \eta) - U_0(y, \eta)$ is continuously distributed, e.g., for $j = 0, 1, 2$, $U_j(y - p_j, \eta) = \alpha_j \ln(y - p_j) + \eta_j$ with $\{\eta_0, \eta_1, \eta_2\}$ having extreme value or joint normal distribution.

among these three bundles, which can happen if and only if for some (p^*, y^*) , we have that

$$\begin{aligned} y^* - p^* &= y - p_0 \\ y^* &= y + p_1 - p_0 \\ y^* - 2p^* &= y - 2p_0. \end{aligned}$$

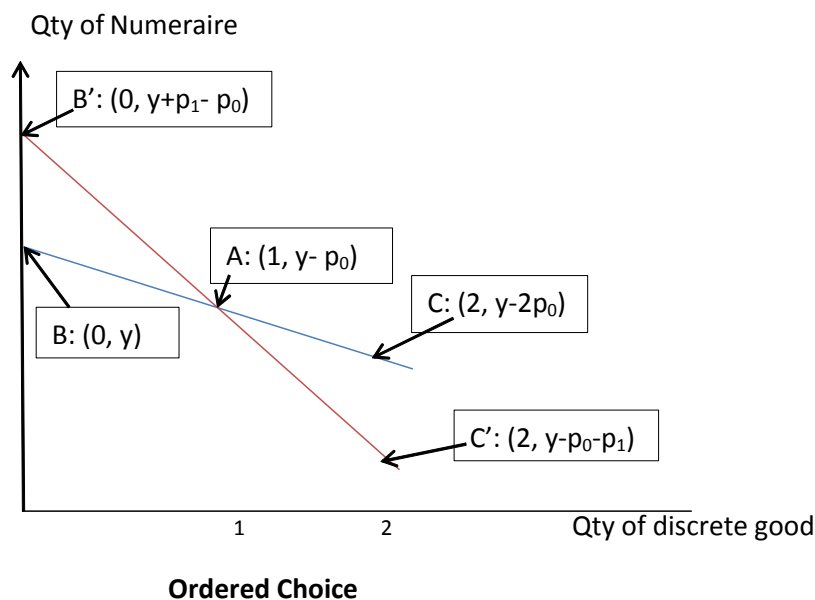
Replacing the second equation in the first yields $p^* = y + p_1 - p_0 - y + p_0 = p_1$ and replacing this in the first equation yields $y^* = y + p_1 - p_0$. But $y^* = y + p_1 - p_0$ and $p^* = p_1$ does not satisfy the third equation. Thus, there is no (p^*, y^*) which satisfies all three equations simultaneously. In other words, there cannot be any consumer in the population who may be observed to make a choice among the bundles $(0, y + p_1 - p_0)$, $(1, y - p_0)$ and $(2, y - 2p_0)$, so that we cannot identify the probability of choosing the bundle $(1, y - p_0)$ over the bundles $(0, y + p_1 - p_0)$ and $(2, y - 2p_0)$ which is the probability of $S = p_1 - p_0$. Thus the probability that $S = p_1 - p_0$ is positive but it is nonparametrically *unidentified*; so the distribution of the CV cannot be nonparametrically point-identified.

Scenario B: Now, consider scenario B. Indeed, if the *per unit* price when 2 units are consumed is fixed at p_2 and the unit price for consuming 1 unit rises from p_0 to p_1 , then

$$\begin{aligned} & \Pr [S = p_1 - p_0] \\ = & \Pr \left[\begin{array}{l} U_1 (y - p_0, \eta) \geq \max \{U_0 (y, \eta), U_2 (y - 2p_2, \eta)\}, \\ U_1 (y + p_1 - p_0 - p_1, \eta) \geq \max \{U_0 (y + p_1 - p_0, \eta), U_2 (y + p_1 - p_0 - 2p_2, \eta)\} \end{array} \right] \\ = & \Pr \left[\begin{array}{l} U_1 (y - p_0, \eta) \geq \max \{U_0 (y, \eta), U_2 (y - 2p_2, \eta)\}, \\ U_1 (y - p_0, \eta) \geq \max \{U_0 (y + p_1 - p_0, \eta), U_2 (y + p_1 - p_0 - 2p_2, \eta)\} \end{array} \right] \\ = & \Pr [U_1 (y - p_0, \eta) \geq \max \{U_0 (y + p_1 - p_0, \eta), U_2 (y + p_1 - p_0 - 2p_2, \eta)\}] \\ = & \Pr [U_1 ((y + p_1 - p_0) - p_1, \eta) \geq \max \{U_0 (y + p_1 - p_0, \eta), U_2 (y + p_1 - p_0 - 2p_2, \eta)\}] \\ = & \bar{q}_1 (p_1, p_2, y + p_1 - p_0), \end{aligned}$$

which is point-identified from the observed choice probability of alternative 1 at price p_1 , income $y + p_1 - p_0$ and unit price of alternative 2 (i.e., of buying 2 units) fixed at p_2 .

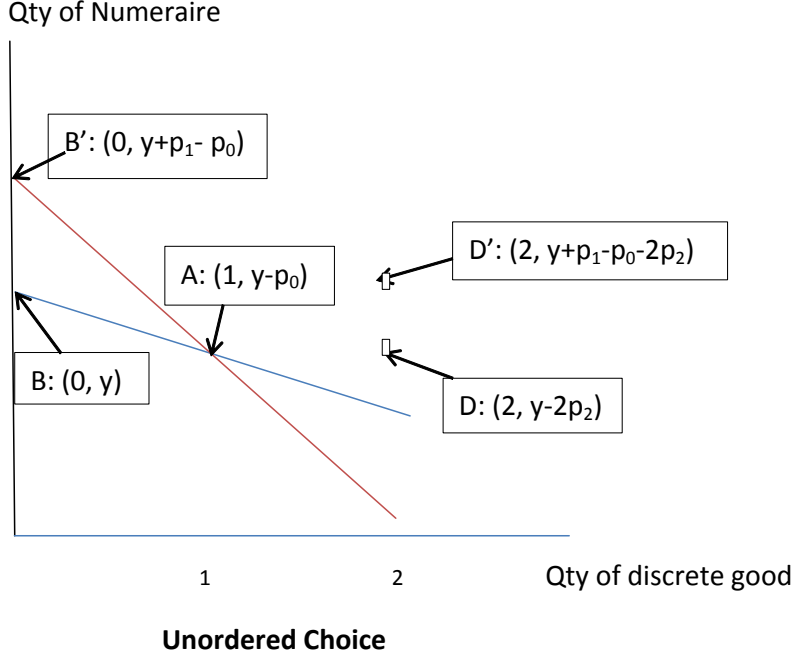
Graphical Illustration: The above discussion is graphically depicted in the following figures. First, considered ordered choice, depicted in the figure below.



At price p_0 and income y , individuals choose between the bundles $B = (0, y)$, $A = (1, y - p_0)$ and $C = (2, y - 2p_0)$ and at price p_1 and compensated income $y + p_1 - p_0$, they choose between the bundles $B' = (0, y + p_1 - p_0)$, $A = (1, y + p_1 - p_0 - p_1) = (1, y - p_0)$ and $C' = (2, y + p_1 - p_0 - 2p_1) = (2, y - p_1 - p_0)$. By monotonicity of utility, for every η , $B' \succ B$ and $C \succ C'$. Therefore,

$$\begin{aligned} \Pr(S = p_1 - p_0) &= \Pr[\{A \succ B, A \succ C\} \cap \{A \succ B', A \succ C'\}] \\ &= \Pr(A \succ C, A \succ B'). \end{aligned}$$

But in the ordered choice case, we observe choice behavior across bundles which lie on the same straight line (e.g., (B, A, C) or (B', A, C')). Since (B', A, C) do not lie on a straight line, we cannot point-identify the probability $\Pr(A \succ C, A \succ B')$. Now consider unordered choice.



With price of alternative 2 fixed at p_2 , we observe choice behavior across bundles of the form

$$C(z, p, p_2) := \{(0, z), (1, z - p), (2, z - 2p_2)\}$$

for every combination of (z, p) . Now,

$$\begin{aligned} \Pr(S = p_1 - p_0) &= \Pr[\{A \succ B, A \succ D\} \cap \{A \succ B', A \succ D'\}] \\ &= \Pr(A \succ D', A \succ B'), \end{aligned} \quad (20)$$

since $D' \succ D$ by monotonicity. Therefore, we can compute the probability (20) as the probability of choosing $(1, z - p)$ from $C(z, p, p_2)$, with $z = y + p_1 - p_0$ and $p = p_1$.

Remark 9 *It is interesting to note that if the functional forms of the utility functions were parametrically specified and the dimension and distribution of heterogeneity were assumed to be known (as in HK99 or DK05), then the probability in (19) would appear to be point-identified. For example, if one assumes that for $j = 0, 1, 2$, $U_j(y, \eta) = \alpha_j \ln(y) + \eta_j$ with $\{\eta_0, \eta_1, \eta_2\}$ distributed as IID extreme valued, then (19) reduces to*

$$\begin{aligned} &\Pr[U_1(y - p_0, \eta) \geq \max\{U_0(y + p_1 - p_0, \eta), U_2(y - 2p_0, \eta)\}] \\ &= \Pr[\alpha_1 \ln(y - p_0) + \eta_1 \geq \max\{\alpha_0 \ln(y + p_1 - p_0) + \eta_0, \alpha_2 \ln(y - 2p_0) + \eta_2\}] \\ &= \frac{\exp(\alpha_1 \ln(y - p_0))}{\exp(\alpha_0 \ln(y + p_1 - p_0)) + \exp(\alpha_1 \ln(y - p_0)) + \exp(\alpha_2 \ln(y - 2p_0))}, \end{aligned}$$

which can be readily "calculated" using ML estimates of the α s. However, the identification result underlying this calculation is artificial in that it is driven entirely by functional form assumptions. In other words, nonparametric point-identifiability of welfare distributions in the case of unordered choice and its failure in the ordered case is a fundamental difference which would not be apparent if one only considered parametric models.

5 Summary and Conclusion

The key insight of this paper is that for binary and multinomial choice, the choice probabilities alone contain all the relevant information for *nonparametric* point-identification of exact money-metric welfare distributions. These identification results are valid under essentially unrestricted forms of unobserved heterogeneity and utility functions, and continue to hold even if the dimension – and therefore the distribution – of unobserved heterogeneity are neither specified, nor identified. Interestingly, point-identification fails in the case of ordered choice with three or more alternatives if the unit price is required to be the same no matter how many units are bought. These results complement Hausman and Newey's (2013) recent finding that for price change of a *continuous* good, averages of money-metric welfare measures are only set-identified under unrestricted heterogeneity.

On the practical side, the distributions of welfare are expressed as simple closed-form transformations of the choice probabilities, enabling easy computation and inference. Even if one approximates the choice probabilities by a parametric model (e.g., multinomial logit), these closed-form expressions can be used to compute welfare-distributions directly from these choice probabilities without requiring one to compute expenditure functions by reverting to utility functions and heterogeneity distributions.

To our knowledge, the present paper derives the first set of results on the nonparametric identification of CV/EV distributions for discrete choice and does so for essentially unrestricted preference distributions. Consequently, it significantly advances the existing literature which (a) either assumed away the key identification problem by assuming negligible income effects, or (b) based welfare calculations on parametrically specified – and, consequently, potentially misspecified – utility functions and heterogeneity distributions without recognizing that the conditional choice probabilities themselves contain all the relevant information for exact welfare analysis under essentially unrestricted preference distributions.

References

- [1] Bhattacharya, D. and Y. Lee (2014): Nonparametric Inference for Welfare Distributions in Discrete Choice Models, in progress.
- [2] Blundell, R. and J. Powell (2003) "Endogeneity in Nonparametric and Semiparametric Regression Models," in *Advances in Economics and Econometrics, Theory and Applications, Eighth World Congress, Volume II*, edited by M. Dewatripont, L.P. Hansen, and S.J. Turnovsky, Cambridge University Press, Cambridge, U.K.
- [3] Blundell, R., J. Horowitz and Perry, M. (2012): Measuring the price responsiveness of gasoline demand: Economic shape restrictions and nonparametric demand estimation, *Quantitative Economics*, Volume 3, Issue 1, pp.29-51.
- [4] Blundell, R., D. Kristensen and R. Matzkin (2013): "Bounding Quantile Demand Functions Using Revealed Preference Inequalities", forthcoming, *Journal of Econometrics*.
- [5] Dagsvik, J. and A. Karlstrom (2005): Compensating Variation and Hicksian Choice Probabilities in Random Utility Models that are Nonlinear in Income, *Review of Economic Studies* (2005) 72 (1): pp. 57-76.
- [6] Domencich, T. and D. McFadden (1975): *Urban Travel Demand - A Behavioral Analysis*, North-Holland, Oxford.
- [7] Hausman, J. (1981): Exact Consumer's Surplus and Deadweight Loss, *The American Economic Review*, Vol. 71, No. 4, pp. 662-676.
- [8] Hausman, J. and Newey, W. (1995): Nonparametric Estimation of Exact Consumers Surplus and Deadweight Loss, *Econometrica*, Vol. 63, No. 6, pp. 1445-1476.
- [9] Hausman, J. and Newey, W. (2013): Individual Heterogeneity and Average Welfare, CEMMAP working paper, 34/13.
- [10] Herridges, J and C. Kling, 1999: Nonlinear Income Effects in Random Utility Models," *The Review of Economics and Statistics*, Vol. 81, No. 1, 1999, pp. 62-72.
- [11] Hoderlein S. and A. Vanhems (2011): Welfare analysis using nonseparable models", CEMMAP Discussion Paper CWP01/11.

- [12] Ichimura, H., and Thompson, T. S. (1998): "Maximum Likelihood Estimation of a Binary Choice Model with Random Coefficients of Unknown Distribution". *Journal of Econometrics*, 86, 269-295.
- [13] Karr, A.F. (1993). *Probability*. Springer-Verlag.
- [14] Kitamura, Y. and J. Stoye (2013): *Nonparametric Analysis of Random Utility Models: Testing*, mimeo. Yale University.
- [15] Kitamura, Y. and Gautier, E (2013): *Nonparametric Estimation in Random Coefficients Binary Choice Models*, *Econometrica*, Volume 81, Issue 2, pages 581–607, March 2013.
- [16] Lee, Y. (2014). *Partial mean processes with generated regressors: Continuous treatment effects and nonseparable models*. Working paper, University of Oxford.
- [17] Lewbel, A. (2001): *Demand Systems with and without Errors*, *The American Economic Review*, Vol. 91, No. 3, Jun., 2001.
- [18] Lewbel, A. and K. Pendakur (2013): *Unobserved Preference Heterogeneity in Demand Using Generalized Random Coefficients*, mimeo. Boston College and Simon Fraser University.
- [19] Manski, C. (1975): *Maximum Score Estimation of the Stochastic Utility Model of Choice*", *Journal of Econometrics*, 3, pp 205-228.
- [20] Matzkin, R.L. (1992): *Nonparametric and Distribution-Free Estimation of the Binary Threshold Crossing and The Binary Choice Models*, *Econometrica*, Vol. 60, No. 2, pp. 239-270.
- [21] McFadden, D. (2005): *"Revealed Stochastic Preference: A Synthesis"*, mimeo. University of California, Berkeley.
- [22] McFadden, D.; K. Richter (1991) *"Stochastic Rationality and Revealed Stochastic Preference,"* in J. Chipman, D. McFadden, K. Richter (eds) *Preferences, Uncertainty, and Rationality*, Westview Press, 161-186.
- [23] Newey, W. (1994). *"Kernel Estimation of Partial Means and a General Variance Estimator,"* *Econometric Theory*, Vol 10, 233-253.
- [24] Small, K. and Rosen, H. (1981): *Applied Welfare Economics with Discrete Choice Models*, *Econometrica*, Vol. 49, No. 1, pp. 105- 130.

[25] Vartia, Y. (1983): Efficient Methods of Measuring Welfare Change and Compensated Income in Terms of Ordinary Demand Functions, *Econometrica*, Vol. 51, No. 1, pp. 79-98.

Appendix

A. Proof of proposition 1:

We denote the individual level EV $S^{EV}(y, p_0, p_1, \eta)$ by simply $S(y, \eta)$ to avoid cumbersome notation in the proof. Also for ease of reference, we rewrite condition (4) again:

$$\begin{aligned} & \max \{U_0(y - S(y, \eta), \eta), U_1(y - S(y, \eta) - p_0, \eta)\} \\ = & \max \{U_0(y, \eta), U_1(y - p_1, \eta)\}. \end{aligned}$$

Proof. By monotonicity of $U_1(\cdot, \eta)$ and $U_0(\cdot, \eta)$, we must have that $S(y, \eta) \geq 0$ in order for (4) to hold.

Now, in **case (i)**, $0 \leq U_0(y, \eta) - U_1(y - p_0, \eta)$. So if $S(y, \eta) > 0$, then

$$\begin{aligned} & \max \{U_0(y - S(y, \eta), \eta), U_1(y - S(y, \eta) - p_0, \eta)\} \\ < & \max \{U_0(y, \eta), U_1(y - p_0, \eta)\} \\ = & U_0(y, \eta) \\ \leq & \max \{U_0(y, \eta), U_1(y - p_1, \eta)\}, \end{aligned}$$

contradicting (4). This implies that $S(y, \eta) = 0$.

Now, consider **case (ii)**. Given the restriction $0 \leq U_0(y, \eta) - U_1(y - p_1, \eta)$, the RHS of (4) is $U_0(y, \eta)$. Therefore from (4), $S(y, \eta)$ must satisfy

$$U_0(y, \eta) = \max \{U_1(y - S(y, \eta) - p_0, \eta), U_0(y - S(y, \eta), \eta)\}. \quad (21)$$

Now, equation (21) is equivalent to

$$U_0(y, \eta) = U_1(y - S(y, \eta) - p_0, \eta). \quad (22)$$

To see why, suppose the first term on the RHS of (21) is smaller than the second, i.e.,

$$U_1(y - S(y, \eta) - p_0, \eta) < U_0(y - S(y, \eta), \eta). \quad (23)$$

Then equation (21) implies

$$U_0(y, \eta) = U_0(y - S(y, \eta), \eta)$$

implying $S(y, \eta) = 0$ by strict monotonicity of $U_0(\cdot, \eta)$. But then, the first term on the RHS of equation (21), viz., $U_1(y - S(y, \eta) - p_0, \eta)$ equals $U_1(y - p_0, \eta)$ while the second term equals $U_0(y, \eta)$; but because we know that $U_1(y - p_0, \eta) \geq U_0(y, \eta)$ (we are in case (ii) of the proposition), the inequality (23) is violated. Therefore, we must have that

$$U_1(y - S(y, \eta) - p_0, \eta) \geq U_0(y - S(y, \eta), \eta),$$

so that the maximum on the RHS of equation (21) must equal $U_1(y - S(y, \eta) - p_0, \eta)$, whence the conclusion (22) follows.

From (22), using monotonicity of $U_1(\cdot, \eta)$, we have that

$$S(y, \eta) = y - p_0 - U_1^{-1}(U_0(y, \eta), \eta).$$

Finally, consider **case (iii)**: Given the restriction, $0 > U_0(y, \eta) - U_1(y - p_1, \eta)$, the RHS of (4) is $U_1(y - p_1, \eta)$. Now, suppose the LHS of (4) is $U_0(y - S(y, \eta), \eta)$. But since $S(y, \eta) \geq 0$, we must have that

$$\begin{aligned} U_0(y - S(y, \eta), \eta) &\leq U_0(y, \eta) < U_1(y - p_1, \eta) \\ &= U_0(y - S(y, \eta), \eta), \text{ by (4),} \end{aligned}$$

a contradiction. Therefore, the LHS of (4) must be $U_1(y - S(y, \eta) - p_0, \eta)$ and therefore by (4),

$$U_1(y - S(y, \eta) - p_0, \eta) = U_1(y - p_1, \eta),$$

whence, by strict monotonicity of $U_1(\cdot, \eta)$, we get that $S(y, \eta) = p_1 - p_0$. ■

B. Proof of proposition 2:

Proof. The compensation must be non-negative and no larger than $p_1 - p_0$; otherwise (4) will be violated. EV is zero for those not purchasing at p_0 and hence EV has a point mass equal to the probability of no purchase at p_0 , which is given by $1 - \bar{q}(p_0, y)$. So the only nontrivial step is for $0 < a < p_1 - p_0$. This case corresponds to case (ii) of proposition 1. Accordingly, for $0 < a < p_1 - p_0$,

the probability of compensation not exceeding a is given by

$$\begin{aligned}
& \Pr(S(y, \eta) = 0, 0 \leq U_0(y, \eta) - U_1(y - p_0, \eta)) \\
& + \Pr\left(\begin{array}{c} y - p_0 - U_1^{-1}(U_0(y, \eta), \eta) \leq a, \\ U_0(y, \eta) - U_1(y - p_0, \eta) < 0 \leq U_0(y, \eta) - U_1(y - p_1, \eta) \end{array}\right) \\
= & 1 - \bar{q}(p_0, y) \\
& + \Pr\left(\begin{array}{c} 0 \leq U_0(y, \eta) - U_1(y - p_0 - a, \eta), \\ U_0(y, \eta) - U_1(y - p_0, \eta) < 0 \leq U_0(y, \eta) - U_1(y - p_1, \eta) \end{array}\right) \\
= & 1 - \bar{q}(p_0, y) \\
& + \Pr(U_0(y, \eta) - U_1(y - p_0, \eta) < 0 \leq U_0(y, \eta) - U_1(y - p_0 - a, \eta)), \text{ since } a < p_1 - p_0 \\
= & 1 - \bar{q}(p_0, y) \\
& + 1 - \bar{q}(p_0 + a, y) - (1 - \bar{q}(p_0, y)) \\
= & 1 - \bar{q}(p_0 + a, y).
\end{aligned}$$

This finishes the proof. ■

Proof of Corollary 1

We use the well-known result (c.f., Karr, 1993, p. 113) that for a positive random variable, X with C.D.F. $F_X(\cdot)$, the expectation is given by

$$E(X) = \int_0^\infty (1 - F_X(a)) da. \quad (24)$$

From (5), it is clear that EV is a positive random variable with support $[0, p_1 - p_0]$. Let $F_{EV(y, p_0, p_1)}(\cdot)$ denote the cumulative distribution function of EV obtained from (5). Consequently, expected EV is given by

$$\begin{aligned}
\mu^{EV}(y, p_0, p_1) &= \int_0^{p_1 - p_0} (1 - F_{EV(y, p_0, p_1)}(a)) da \\
&= \int_0^{p_1 - p_0} \bar{q}(a + p_0, y) da \\
&= \int_{p_0}^{p_1} \bar{q}(z, y) dz, \text{ by change of variables } z = a + p_0.
\end{aligned}$$

Thus we get (7).

C. Proof of proposition 3

We denote the individual level CV $S^{CV}(y, p_0, p_1, \eta)$ by simply $S(y, \eta)$ to avoid cumbersome notation in the proof. Also for ease of reference, we rewrite condition (8) again:

$$\begin{aligned} & \max \{U_0(y + S(y, \eta), \eta), U_1(y + S(y, \eta) - p_1, \eta)\} \\ &= \max \{U_0(y, \eta), U_1(y - p_0, \eta)\}. \end{aligned}$$

Proof. First observe that by (8), we must have that $S(y, \eta) \geq 0$. Otherwise, the LHS of (8) must be strictly smaller than the RHS, by the monotonicity of $U_0(\cdot)$ and $U_1(\cdot)$. Now consider the following cases.

Case (i): $0 \leq U_0(y, \eta) - U_1(y - p_0, \eta)$

Since $0 \leq U_0(y, \eta) - U_1(y - p_0, \eta)$, then the RHS of (8) is $U_0(y, \eta)$. If $S(y, \eta) > 0$, then the first term of LHS of (8) must be strictly larger than $U_0(y, \eta)$, by strict monotonicity of $U_0(\cdot, \eta)$. This would imply that the LHS of (8) must be strictly larger than $U_0(y, \eta)$ – a contradiction. Therefore, in this case, we must have $S(y, \eta) = 0$. Intuitively, this means that those η who were not buying at the initial price p_0 do not need to be compensated.

Now, suppose case (i) does not hold, so that RHS maximum is in fact $U_1(y - p_0, \eta)$, i.e., $U_0(y, \eta) - U_1(y - p_0, \eta) < 0$. This corresponds to those η 's who buy the good at price p_0 . Now, there are two possibilities regarding which term is the maximum in the LHS of (8) – case (ii) corresponds to when the maximum is the first term and case (iii) to when the maximum is the second term.

Case (ii): Accordingly, first assume that the LHS maximum is $U_0(y + S(y, \eta), \eta)$. Then $S(y, \eta)$ must satisfy

$$\begin{aligned} U_0(y + S(y, \eta), \eta) &= U_1(y - p_0, \eta) \\ \implies S(y, \eta) &= U_0^{-1}(U_1(y - p_0, \eta), \eta) - y. \end{aligned} \tag{25}$$

In order for this to simultaneously satisfy that the LHS maximum of (8) is $U_0(y + S(y, \eta), \eta)$, we need

$$\begin{aligned} U_0(y + S(y, \eta), \eta) &\geq U_1(y + S(y, \eta) - p_1, \eta) \\ \text{substituting } \underline{\underline{S(y, \eta)}} \text{ from (25)} \quad U_1(y - p_0, \eta) &\geq U_1(y + S(y, \eta) - p_1, \eta) \\ &\implies y - p_0 \geq y + S(y, \eta) - p_1 \\ &\implies p_1 - p_0 \geq S(y, \eta) = U_0^{-1}(U_1(y - p_0, \eta), \eta) - y \\ &\implies U_0(y + p_1 - p_0, \eta) \geq U_1(y - p_0, \eta) \\ &\implies 0 \leq U_0(y + p_1 - p_0, \eta) - U_1(y - p_0, \eta). \end{aligned}$$

Thus we arrive at the conclusion of case (ii), viz.,

$$\begin{aligned} U_0(y) - U_1(y - p_0) &< 0 \leq U_0(y + p_1 - p_0, \eta) - U_1(y - p_0, \eta) \text{ and} \\ S(y, \eta) &= U_0^{-1}(U_1(y - p_0, \eta), \eta) - y. \end{aligned}$$

Case (iii): Finally, consider the remaining case where the maximum of the LHS of (8) is $U_1(y + S(y, \eta) - p_1, \eta)$, whence we have

$$\begin{aligned} U_1(y + S(y, \eta) - p_1, \eta) &= U_1(y - p_0, \eta) \\ \Rightarrow y + S(y, \eta) - p_1 &= y - p_0 \\ \Rightarrow S(y, \eta) &= p_1 - p_0. \end{aligned}$$

Replacing $S(y, \eta)$ in the LHS of (8), in order to be consistent with our assumption that the LHS maximum is $U_1(y + S(y, \eta) - p_1, \eta)$, we must have that

$$\begin{aligned} U_1(y + S(y, \eta) - p_1, \eta) &\geq U_0(y + S(y, \eta), \eta) \\ \Leftrightarrow U_1(y - p_0, \eta) &\geq U_0(y + p_1 - p_0, \eta) \end{aligned}$$

which is precisely case (iii) of the proposition, viz., $U_1(y - p_0, \eta) \geq U_0(y + p_1 - p_0, \eta)$. ■

D: Proof of proposition 4

Proof. First recall from (2) that for any $a > 0$, we have that

$$\begin{aligned} \bar{q}(p_0, y) &= \Pr(0 > U_0(y, \eta) - U_1(y - p_0, \eta)), \\ \bar{q}(a + p_0, y + a) &= \Pr(0 > U_0(y + a, \eta) - U_1(y - p_0, \eta)), \end{aligned}$$

where the probabilities are computed w.r.t. the marginal distribution of η .

Now, from proposition 3 and its proof, it is clear that for any η , the compensation is non-negative and it equals zero for those not buying at price p_0 . Hence the CV has no mass below 0 and a point mass of $1 - \bar{q}(p_0, y)$ at 0. Also, it is clear that the compensation cannot exceed $p_1 - p_0$ for any η . Otherwise, (8) will be violated. Therefore the C.D.F. of CV must reach 1 at $p_1 - p_0$. So the only nontrivial case is $0 < a < p_1 - p_0$. This corresponds to case (ii) of proposition 3. Accordingly, for $0 < a < p_1 - p_0$, the probability of the compensation being no larger than a is

given by

$$\begin{aligned}
& \Pr(S(y, \eta) = 0, U_0(y, \eta) - U_1(y - p_0, \eta) \geq 0) \\
& + \Pr \left\{ \begin{array}{l} U_0^{-1}(U_1(y - p_0, \eta), \eta) - y < a, \\ U_0(y, \eta) - U_1(y - p_0, \eta) < 0 \leq U_0(y + p_1 - p_0, \eta) - U_1(y - p_0, \eta) \end{array} \right\} \\
= & 1 - \bar{q}(p_0, y) \\
& + \Pr \left\{ \begin{array}{l} U_0^{-1}(U_1(y - p_0, \eta), \eta) - y < a, \\ U_0(y, \eta) - U_1(y - p_0, \eta) < 0 \leq U_0(y + p_1 - p_0, \eta) - U_1(y - p_0, \eta) \end{array} \right\}. \quad (26)
\end{aligned}$$

Now,

$$\begin{aligned}
& \Pr \left(\begin{array}{l} U_0^{-1}(U_1(y - p_0, \eta), \eta) - y < a, \\ U_0(y, \eta) - U_1(y - p_0, \eta) < 0 \leq U_0(y + p_1 - p_0, \eta) - U_1(y - p_0, \eta) \end{array} \right) \\
= & \Pr \left(\begin{array}{l} 0 < U_0(y + a, \eta) - U_1(y - p_0, \eta), \\ U_0(y, \eta) - U_1(y - p_0, \eta) < 0 \leq U_0(y + p_1 - p_0, \eta) - U_1(y - p_0, \eta) \end{array} \right) \\
= & \Pr(U_0(y, \eta) - U_1(y - p_0, \eta) < 0 \leq U_0(y + a, \eta) - U_1(y - p_0, \eta)), \text{ since } a < p_1 - p_0 \\
= & \Pr(0 \leq U_0(y + a, \eta) - U_1(y - p_0, \eta)) - \Pr \left(\begin{array}{l} U_0(y, \eta) - U_1(y - p_0, \eta) \geq 0, \\ U_0(y + a, \eta) - U_1(y - p_0, \eta) \geq 0 \end{array} \right) \\
= & \Pr(0 < U_0(y + a, \eta) - U_1(y - p_0, \eta)) - \Pr(0 \leq U_0(y, \eta) - U_1(y - p_0, \eta)) \text{ since } a \geq 0 \\
= & \Pr(0 < U_0(y + a, \eta) - U_1(y + a - (a + p_0), \eta)) - \Pr(0 \leq U_0(y, \eta) - U_1(y - p_0, \eta)) \\
= & \Pr(0 > U_0(y, \eta) - U_1(y - p_0, \eta)) - \Pr(0 > U_0(y + a, \eta) - U_1(y + a - (a + p_0), \eta)) \\
= & \bar{q}(p_0, y) - \bar{q}(a + p_0, y + a).
\end{aligned}$$

Substituting in (26), we get that for $0 < a < p_1 - p_0$,

$$\begin{aligned}
& \Pr\{S(y, \eta) \leq a\} \\
= & 1 - \bar{q}(p_0, y) + \bar{q}(p_0, y) - \bar{q}(a + p_0, y + a) \\
= & 1 - \bar{q}(a + p_0, y + a),
\end{aligned}$$

as desired. ■

E: Proof of Corollary 2

Proof. Let $F_{CV(y, p_0, p_1)}(\cdot)$ denote the cumulative distribution function of EV obtained from (9).

Using (24) and the c.d.f. of CV in (9), we get that expected CV is given by

$$\begin{aligned}
& \mu^{CV}(y, p_0, p_1) \\
&= \int_0^{p_1 - p_0} (1 - F_{CV(y, p_0, p_1)}(a)) da \\
&= \int_0^{p_1 - p_0} \bar{q}(a + p_0, y + a) da \\
&= \int_{p_0}^{p_1} \bar{q}(z, y + z - p_0) dz, \text{ substituting } z = a + p_0.
\end{aligned}$$

■

F. Proof of proposition 5

We provide the proof for CV; the proof of EV is exactly analogous. Recall the notation $p_{-1} = (p_2, p_3, \dots, p_J)$, and $\bar{q}_1(t, p_{-1}, y)$ the structural choice probability of alternative 1 when its price is t , prices of the other alternatives are held fixed at p_{-1} and income is y .

Proof. The compensating variation is the solution S to the equation

$$\begin{aligned}
& \max \{U_0(y + S, \eta), U_1(y + S - p_{11}, \eta), U_2(y + S - p_2, \eta), \dots, U_J(y + S - p_J, \eta)\} \\
&= \max \{U_0(y, \eta), U_1(y - p_{10}, \eta), U_2(y - p_2, \eta), \dots, U_J(y - p_J, \eta)\}.
\end{aligned}$$

Since $\max \{a, b, c\} = \max \{a, \max \{b, c\}\}$, the previous display can be rewritten as

$$\begin{aligned}
& \max \{U^*(y + S, p_{-1}, \eta), U_1(y + S - p_{11}, \eta)\} \\
&= \max \{U^*(y, p_{-1}, \eta), U_1(y - p_{10}, \eta)\}, \tag{27}
\end{aligned}$$

where

$$U^*(y, p_{-1}, \eta) \equiv \max \{U_0(y, \eta), U_2(y - p_2, \eta), \dots, U_J(y - p_J, \eta)\}.$$

Equation (27) is exactly analogous to (8) with $U_0(y, \eta)$ replaced by $U^*(y, p_{-1}, \eta)$.

Assumption 2 implies that $U^*(\cdot, p_{-1}, \eta)$ is strictly increasing and continuous for each η . Let $U^{*-1}(a, p_{-1}, \eta)$ denote the unique solution for x in the equation $U^*(x, p_{-1}, \eta) = a$. It is clear from (27) that given the monotonicity of $U_1(\cdot, \eta)$ and $U^*(\cdot, p_{-1}, \eta)$, the CV must lie in $[0, p_{11} - p_{10}]$ with probability 1.

Now, it can be seen that all our results from the binary case carry over with the utility $U_0(y, \eta)$ replaced by $U^*(y, p_{-1}, \eta)$, since the latter does not involve the price of alternative 1. Following

through the proof of proposition 3 with $U_0(y, \eta)$ replaced by $U^*(y, p_{-1}, \eta)$ yields

$$CV = \begin{cases} 0 & \text{if } U_1(y - p_{10}, \eta) < U^*(y, p_{-1}, \eta) \\ U^{*-1}(U_1(y - p_{10}, \eta), p_{-1}, \eta) - y, & \text{if } \begin{pmatrix} U^*(y, p_{-1}, \eta) \leq U_1(y - p_{10}, \eta) \\ < U_0(y + p_{11} - p_{10}, \eta) \end{pmatrix} \\ p_{11} - p_{10} & \text{if } U^*(y + p_{11} - p_{10}, p_{-1}, \eta) < U_1(y - p_{10}, \eta) \end{cases}.$$

Now,

$$\begin{aligned} \Pr(CV = 0) &= \Pr(U_1(y - p_{10}, \eta) \leq U^*(y, p_{-1}, \eta)) \\ &= 1 - \bar{q}_1(p_{10}, p_{-1}, y). \end{aligned} \quad (28)$$

For $0 < r < p_{11} - p_{10}$,

$$\begin{aligned} &\Pr(U^{*-1}(U_1(y - p_{10}, \eta), p_{-1}, \eta) - y < r) \\ &= \Pr(U_1(y - p_{10}, \eta) \leq U^*(y + r, p_{-1}, \eta)) \\ &= \Pr(U_1(y + r - (p_{10} + r), \eta) \leq U^*(y + r, p_{-1}, \eta)) \\ &= 1 - \bar{q}_1(p_{10} + r, p_{-1}, y + r). \end{aligned} \quad (29)$$

Finally,

$$\begin{aligned} &\Pr(CV = p_{11} - p_{10}) \\ &= \Pr(U^*(y + p_{11} - p_{10}, p_{-1}, \eta) < U_1(y - p_{10}, \eta)) \\ &= \Pr(U^*(y + p_{11} - p_{10}, p_{-1}, \eta) < U_1((y + p_{11} - p_{10}) - p_{11}, \eta)) \\ &= \bar{q}_1(p_{11}, p_{-1}, y + p_{11} - p_{10}). \end{aligned} \quad (30)$$

Aggregating from (28), (29) and (30), we get the C.D.F. stated in proposition 5. ■

G: Binary choice example where heterogeneity distribution is not point-identified but welfare distributions are:

Suppose $\eta \equiv (\eta_1, \eta_0)$ is jointly independent of (p, y) and $\eta_1 \perp \eta_0$. Assume that the support of price distribution in the data is contained in $[0, p_H]$ and income is bounded below by y_L with $y_L > p_H > 0$. Let

$$U_1(y - p, \eta) = y - p + \eta_1, \quad U_0(y, \eta) = (1 - \eta_0)y,$$

where η_0 is distributed uniform $(0, 1)$ and the support of η_1 – denoted by T – is contained in $(p_H - y_L, 0)$. Denote the c.d.f. of η_1 by $G(\cdot)$. An individual of type (y, η) and facing price p buys the good if and only if $y - p + \eta_1 > (1 - \eta_0)y$.

Thus for any fixed $\eta = (\eta_1, \eta_0)$ in the support, the utility functions are continuous and strictly increasing in income. Thus proposition 2 and 4 apply and imply that the distribution of EV and CV arising from a price change are point-identified.

Now, consider the choice probability in this model. Since $p_H - y_L \leq \eta_1 \leq 0$ wp1, it follows that for any p, y in the support of the data, we must have that $p - y < \eta_1 < p$, or

$$0 < \frac{p - \eta_1}{y} < 1, \text{ w.p. } 1. \quad (31)$$

Therefore, choice probability of alternative 1 at price p and income y is given by

$$\begin{aligned} \bar{q}(p, y) &= \Pr \{y - p + \eta_1 > (1 - \eta_2)y\} \\ &= \Pr \{\eta_2 y + \eta_1 > p\} \\ &= \Pr \left\{ \eta_2 > \frac{p - \eta_1}{y} \right\} \text{ since } y > 0 \\ &= \int_T \left(1 - \frac{p - \eta_1}{y} \right) dG(\eta_1), \text{ by } \eta_1 \perp \eta_2, \text{ inequality (31) and } \eta_2 \sim U(0, 1) \\ &= \left(1 - \frac{p}{y} \right) + \frac{1}{y} \int_T \eta_1 dG(\eta_1) \\ &= \left(1 - \frac{p}{y} \right) + \frac{1}{y} E(\eta_1). \end{aligned}$$

Thus, the choice probability $\bar{q}(p, y)$ depends on the distribution of η_1 only through its expectation. Therefore, all distributions for η_1 with support contained in $(p_H - y_L, 0)$ and having the same expectation will give rise to the same choice probability *for each value of p and y* , implying that the distribution of η_1 cannot be identified from the choice probabilities alone. In particular, $\eta_1 \sim \text{Uniform}[p_H - y_L, 0]$ and $\eta_1 = \frac{p_H - y_L}{2}$ with probability 1 will both produce identical $\bar{q}(p, y)$ *for all* (p, y) in the support of price and income. This implies that the dimension of heterogeneity is also not identified ($\dim(\eta_1, \eta_0)$ is 1 when η_1 is degenerate and 2 when η_1 is uniform). Yet the distribution of EV and CV are point-identified from $\bar{q}(\cdot, \cdot)$ as implied by propositions 2 and 4 of my paper.

The above example demonstrates that identifiability of the heterogeneity distribution, or even correct specification of its dimension are not a requirement for identifiability of welfare distributions.