EVALUATING STRATEGIC FORECASTERS

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ABSTRACT: Motivated by the question of how one should evaluate professional election forecasters, we study a novel dynamic mechanism design problem without transfers. A principal who wishes to hire only high quality forecasters is faced with an agent of unknown quality. The agent privately observes signals about a publicly observable future event, and may strategically misrepresent information to inflate the principal's perception of his quality. We show that the optimal deterministic mechanism is simple and easy to implement in practice: it evaluates a single, optimally timed prediction. We study the generality of this result and its robustness to randomization and noncommitment.

KEYWORDS: dynamic mechanism design, mechanism design without transfers, forecasting, learning, election predictions.

JEL CLASSIFICATION: D82, D83, D86.

> *"A foolish consistency is the hobgoblin of little minds, adored by little statesmen and philosophers and divines."*
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> —Ralph Waldo Emerson

Forecasting is an important industry whose experts' services are utilized in a variety of different fields, including politics, sports, meteorology, banking, finance, and economics. Forecasters differ based on the quality of their predictions which, in turn, is determined by the accuracy of their information and their ability to process it. The career prospects of an expert depend on public perceptions of his ability, and hence a strategic forecaster may make predictions designed to inflate those perceptions. In this paper, we study the dynamic mechanism design problem of a principal who uses an expert's predictions to determine whether that expert is worth hiring. In a nutshell, we are interested in determining the optimal method of screening strategic forecasters.

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For example, consider a governor or senator who is contemplating a presidential run in the next electoral cycle. She would like to hire a professional election forecaster to help her accurately determine the viability of her future candidacy. To evaluate the forecaster's ability, she observes his predictions at various points in the current electoral cycle; she also eventually observes the current electoral outcome. What is the best way for her to determine whether the forecaster is worth hiring? The important factors the politician needs to incorporate into her hiring decision are that (i) the forecaster's information and the election outcome are both noisy signals of the underlying preferences of the electorate; (ii) the forecaster learns about those preferences more precisely as the election nears; and (iii) his predictions are strategically chosen to make himself appear to be of higher quality as he anticipates the career implications of his performance.

We develop a novel dynamic model to study these issues. In its simplest symmetric and binary form (on which the bulk of the paper focuses), the framework can be described as follows. There is a persistent, unknown state of the world governing the data-generating process. This unobserved state takes one of two values with equal probability. A binary public outcome, which is a noisy signal of the underlying state, occurs at time $T + 1$. Leading up to that outcome, the agent (forecaster) privately learns about the state (and therefore the expected outcome) via a sequence of T noisy signals. These binary signals are correlated with the state but are otherwise conditionally independent and identically distributed. The agent is equally likely to be either a "good" or a "bad" type, where a good type observes more precise information. At each point of time, the agent strategically reports his signal. After the outcome has been realized, the principal decides whether or not to hire the agent based on a mechanism that is announced (and committed to) at the beginning of the game. A mechanism in this context is a deterministic mapping from the history of reported signals and the eventual outcome to a hiring decision. Both parties care only about this hiring decision, as both the underlying state and the agent's signals are payoff irrelevant. Their incentives diverge, however: the principal only wants to hire the good type, while the agent always wants to be hired, regardless of his private type.

The critical modeling assumptions of our environment are supported by the disparate literatures that study forecasting in psychology, statistics, economics, and finance. Our underlying information structure—an unknown data generating process that the forecaster learns over time—is a standard (albeit simplified) feature of statistical models of forecasting (an up-to-date survey is Elliott and Timmermann (2016); recent empirical evidence on learning by professional forecasters can be found in Lahiri and Sheng (2008) and the papers that follow). Psychologists have shown that experts differ in their forecasting abilities and that better forecasters are consistently more accurate (see, for the instance, the work described in Tetlock (2005) and Tetlock and Gardner (2015)). Trueman (1994), Ottaviani and Sørensen (2006c), and others have argued that experts who differ by ability choose their forecasts with the intention of influencing clients' assessments of that ability. At a high level, the key departure of this paper from this latter economics literature is that we incorporate a strategic principal (as opposed to a passive market) who optimally chooses her method of evaluating such strategic forecasters.

To understand the role played by incentives in this environment, it is instructive to examine the benchmark case where the principal does not know the agent's type but can observe his signals. Here, the principal can screen the agent—even in the absence of a public outcome—by using the *variance* of the observed signals: since the good type receives more precise information, he is more likely to receive signal profiles with a large proportion of identical signals (or, equivalently, profiles with lower signal variance). The public outcome provides another means of screening as, in addition to decreased variance, the good type is more likely to receive signal profiles in which a large fraction of signals match the outcome. Hence, the principal's optimal hiring decision in this benchmark takes the form of two thresholds: one for *accuracy* and one for *consistency* (Theorem 1). Here, the agent is hired either when he receives sufficiently many signals matching the realized outcome (more than the accuracy threshold) or sufficiently many signals mismatching the realized outcome (more than the consistency threshold).[1] Hence, when the agent cannot strategically report his signals, the principal screens using *both* the accuracy and the consistency of the agent's information. Note that, given a profile of received signals, the order in which the signals arrive plays no role as they are generated from a conditionally i.i.d. process.

An immediate and important economic insight is that when the agent is free to report his signals strategically, the optimal mechanism does not screen using consistency. The reason is quite intuitive: it is always possible for the agent to report consistent signals regardless of the actual information he receives. Instead, we show that it is optimal for the principal to screen using a combination of the *accuracy* of the agent's signals along with the *order* in which they arrive. Specifically, our main result shows that the optimal deterministic mechanism takes the very simple form of a *prediction mechanism*: the principal optimally chooses a time period $\overline{T} \leq T$ to solicit a *single* prediction of the final outcome, and the agent is hired if, and only if, that prediction matches that outcome (Theorem 3).[2] The principal utilizes the order in which the signals arrive by ignoring information that arrives after $\overline{T}$.

This result has a number of features that are worth emphasizing. It may be surprising to some readers (perhaps in light of the "testing experts" literature we discuss below) that screening is possible at all in this strategic environment, especially when the principal's only screening instrument is a coarse hiring decision. Unlike the benchmark case, screening a strategic agent is not possible in the absence of a public outcome as the bad type is free to follow any reporting strategy. However, with a public outcome, screening becomes possible: since the good type receives more precise information, his prediction (if truthful) of the outcome in any period is more likely to be correct than the bad type's. Thus, a hiring rule where the agent is picked if, and only if, his prediction in a given period turns out to be accurate is more likely (compared to the initial belief) to result in the hiring of the good type. Moreover, focusing on a single period's prediction (and ignoring the agent's reports at all other periods) also ensures that it is optimal for the strategic agent to sincerely predict the outcome he believes to be more likely.

[1] The most transparent demonstration of why two cutoffs are optimal (as opposed to only one for accuracy) is the corner case where the good type's signals are perfectly informative while the bad type's are completely uninformative. Here, any variance in the signals immediately reveals the agent to be the bad type, regardless of the fraction that are correct. The likelihood that the bad type receives the same signal over and over again is thus sufficiently small to ensure that the principal is happy to hire the agent when his signal profile is perfectly consistent but does not match the outcome.

[2] It is worth stressing that this mechanism is optimal within the full class of deterministic direct revelation mechanisms that, in addition to the signals, also ask the agent to report his initial private type. Since such type reporting is not observed in practice and—as Lemma 1 shows—there is no loss of generality in dispensing with it, we deliberately focus on mechanisms that do not solicit this information. As we discuss later in Section 6, our results also generalize to the case where the agent has no initial private information.

As mentioned above, the optimal mechanism uses the order of signals for screening by discarding information that the agent receives after period $\overline{T}$. To see why doing so might help the principal, suppose she instead always chooses to solicit predictions at the end of period T after the agent has acquired all possible information. When T is large, both types of the agent learn the underlying state with high probability, which makes screening by predictions ineffective. Instead, the principal can choose to screen at an intermediate time period when the learning advantage for the good type (from receiving more precise information) is at its highest. An insight from the main result is that the principal is unable to improve screening in a deterministic mechanism (over and above soliciting a prediction) by using any information that arrives after period $\overline{T}$.

A strength of the optimal mechanism is that is very easy to describe and implement in practice. Moreover, we show that the same optimal outcome can be achieved even *without commitment* (Theorem 4), thereby making our results applicable in settings where the principal has little or no commitment power. This is another novel aspect of our framework as it is quite unusual for commitment power to not benefit the principal in a dynamic mechanism design environment.

In Section 6, we discuss the generality of the main insight driving our result by showing that it also applies to very general environments (Theorem 7). We show that the key assumption we need for the optimality of prediction mechanisms is that the public outcome is binary. As long as this assumption holds, prediction mechanisms remain optimal even if the agent's type is drawn from a general space and the information he receives is generated from a general time-varying signal process. Additionally, even in this general environment, commitment is not required to implement the optimal mechanism. The simplicity of optimal mechanisms in so general a setting opens the door to further research on even richer models which have strategic forecasting as a component (and we discuss a few avenues for future research in our concluding remarks).

Finally, while our focus on deterministic hiring rules is driven by their suitability for our motivating applications (as it is well known that committment to stochastic policies can be extremely difficult in practice), randomization plays an interesting theoretical role in our environment.[3] This is most easily demonstrated in the optimal *stochastic* mechanism for the special case of $T = 3$ periods (Theorem 5). Here, we show how the principal fine-tunes her screening by hiring the agent with different (strictly positive) probabilities that depend on the order of signal arrivals in addition to the overall composition of the signal profile. Finally, we show that a sufficient condition for the optimality of randomization is that the time horizon is long enough (Theorem 6).

Related Literature. Expert forecasting is an important industry and the input of forecasters is often solicited for numerous decisions made by firms and policy makers alike. While the statistical work on evaluating forecasting models is well developed (see, for instance, the aforementioned survey Elliott and Timmermann (2016)), there is relatively less research examining the incentives of strategic experts and how these incentives influence their forecasts (a recent survey of this work is Marinovic, Ottaviani, and Sørensen (2013)). The theoretical work in this latter literature (see, for instance, Ottaviani and Sørensen (2006a), Ottaviani and Sørensen (2006c)) differs in that forecasters are evaluated by a rational, but otherwise passive, market and that the environment is static.

[3]Enforcement is a standard concern with stochastic mechanisms as the randomization device employed by the principal must also be verifiable in practice; see, for instance, Laffont and Martimort (2009, p. 67).

This paper differs in that we consider a dynamic environment in which a strategic principal can alter the incentives of the forecaster by choosing her evaluation criterion.

The literature on testing experts (starting with Foster and Vohra (1998); a recent survey is Olszewski (2015)) shares a similar motivation. Here an abstract dynamic environment is considered and the focus is on determining the existence of a test which (i) cannot be passed by a strategic forecaster without knowledge of the true data generating process and (ii) can be passed almost surely when the forecaster knows the process. Both our model and overall objective differ in that we allow the agent to be imperfectly informed about the data generating process and that the principal's goal is to design a mechanism that maximally separates the good forecaster from the bad, even if that screening is imperfect.

Since we consider a setting where a principal can commit to her hiring policy (based on sequential information received from the agent), our results are related to those in the literature on dynamic mechanism design. The binary private signals in our simplified model is a key feature of Battaglini (2005) and Boleslavsky and Said (2013), of which the latter also features private information about the signal process. These papers differ not only in their reliance on transfers but also in terms of the payoffs, the structure of the stochastic process governing signal evolution, and (as a result) the applications to which their models apply. Though it also differs along these latter dimensions, Guo and Hörner (2015) is more closely related as it also examines a dynamic mechanism design problem in a binary environment without transfers. In another strand of this literature, Aghion and Jackson (2016) show that tenure schemes can provide incentives for an agent to take actions that reveal his competence. However, their setting yields distinct economic insights as (among other differences) they rely on having multiple opportunities to learn about the agent's competence as well as on principal preferences that depend on the agent's actions instead of his underlying type. We will further discuss the relation of our results to the dynamic mechanism design literature in more detail in Section 7.1.

Finally, since we also examine the dynamic cheap talk setting where the principal cannot commit, this paper is related to the literature studying how an agent with a privately known type builds reputation via dynamic communication. The key difference between our setting and this literature (in addition to the different applications modeled and the fact that we also characterize the full commitment optimum) is that our principal dynamically screens across types with differential rates of learning of a fixed underlying state. This is in contrast with Ottaviani and Sørensen (2006b) and Li (2007), where the agent is evaluated by a competitive market and so his payoff is simply the posterior belief about his type. Alternatively, Morris (2001) considers a repeated, two-period setting where a principal makes a decision in each period based on the agent's report. While both the principal and the agent in his setting have very different preferences from ours, an important distinction is that our principal makes a single decision after cheap talk has ended.[4]

1. Model

We consider a T-period, discrete time, finite horizon framework in which a principal determines whether or not to hire an agent who is an expert forecaster. To make the main insights transparent,

[4]This aspect is also reminiscent of Krishna and Morgan (2004) (and the papers that follow), where an additional long communication protocol is added to the canonical model of Crawford and Sobel (1982).

we define a simplified, symmetric version of the model on which the majority of the paper focuses. We discuss the full generality of the results in Section 6.

1.1. *The Environment*

State: The forecaster is being judged on his ability to learn about an unknown state of the world ω. This state, which governs the data-generating process, is equally likely to be either high (H) or low (L), so the commonly known prior distribution of states is $\Pr(\omega = H) = \Pr(\omega = L) = \frac{1}{2}$.

Agent's private information: There is a single forecaster whose privately known type (his forecasting ability) θ can either be good (g) or bad (b) with equal likelihood; thus, the commonly known prior distribution of ability is $\Pr(\theta = g) = \Pr(\theta = b) = \frac{1}{2}$.[5]

In each period $t = 1, \ldots, T$, the forecaster privately observes a binary signal $s_t \in \{h, l\}$ about the unknown state ω. The accuracy of these signals (that is, the probability that each signal "matches" the true state) is

$$\alpha_\theta := \Pr(s_t = h|\omega = H, \theta) = \Pr(s_t = l|\omega = L, \theta).$$

We assume that $\frac{1}{2} < \alpha_b < \alpha_g < 1$, so that the type-$g$ agent's signals are more precise than the type-b agent's.[6] We write $s^t := (s_1, \ldots, s_t)$ to denote a sequence of t signals.

Outcome: At the end of period T, a publicly observed binary outcome $r \in \{h, l\}$ is realized. This outcome is correlated with the true state ω; we denote by $\gamma \in \left[\frac{1}{2}, 1\right]$ the probability with which the outcome r "matches" the true state ω, so

$$\gamma := \Pr(r = h|\omega = H) = \Pr(r = l|\omega = L).$$

The corner case where $\gamma = 1$ corresponds to situations where the public outcome fully reveals the underlying state, while $\gamma < 1$ reflects environments where that outcome is only a noisy signal.[7]

1.2. *The Game*

In each period t, the agent strategically reports his signal $\tilde{s}_t \in \{h, l\}$, possibly as the realization of a mixed strategy (we will discuss implementations where the agent makes predictions instead of reporting signals in Section 3.3). Our main focus will be on the case where the principal has full commitment, but we will also examine what happens in the absence of commitment power.

Histories: At the beginning of any period t, $h_t^A = (s^t, \tilde{s}^{t-1})$ denotes the agent's private history. This contains the t privately observed signals s^t and the $t-1$ reports $\tilde{s}^{t-1}$ made prior to period t. We use $\mathcal{H}^A = \bigcup_{t=1}^T \left(\{h, l\}^t \times \{h, l\}^{t-1}\right)$ to denote the set of all histories for the agent.

The relevant history for the principal $h^P = (\tilde{s}^T, r)$ at which she makes a hiring decision contains the entire sequence reports made by the agent in all T periods and the final outcome. We use $\mathcal{H}^P = \{h, l\}^{T+1}$ to denote the set of all such public histories.

[5] Note that we do not require symmetry in either the state or type distributions for any of the results in the binary model. This assumption merely allows us to simplify the notation and shorten the proofs without compromising our main economic insights.

[6] We exclude the corner cases $\alpha_g = 1$ and $\alpha_b = \frac{1}{2}$ to simplify our exposition, though our results continue to hold.

[7] Election outcomes are often affected by last-minute events uncorrelated with the electorate's underlying preferences (unexpected news or election-day weather, for instance, might affect turnout). Similarly, unanticipated in-game injuries often lead to upsets of the "better" team in a sporting event.

Agent's strategy: The type-θ agent's strategy $\sigma^\theta : \mathcal{H}^A \to \Delta\{h,l\}$ determines the distribution of signal reports at each history. We will use the signal subscript $\sigma_s^\theta(h_t^A)$ to denote the probability that the agent reports signal $s \in \{h,l\}$.

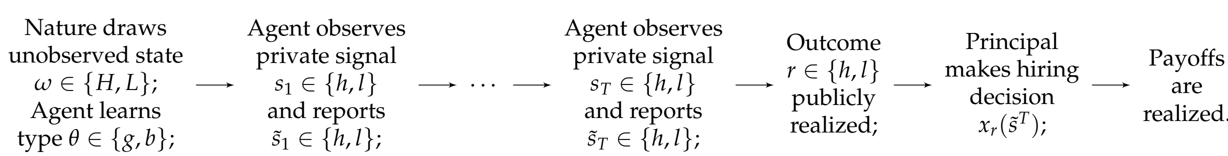
Principal's strategy: We denote by $x_r(\tilde{s}^T) \in \{0,1\}$ the principal's strategy at history $(\tilde{s}^T, r) \in \mathcal{H}^P$. It determines the probability with which she hires the agent as a function of the T reported signals $\tilde{s}^T$ and the outcome r. We focus on deterministic hiring decisions for the principal as we feel that this is the more natural modeling assumption for the applications we consider. In particular, as mentioned in the introduction, it is hard in practice for a principal to commit to a stochastic hiring policy. That said, randomization plays an interesting theoretical role in our model that we discuss in Section 5.

When the principal has commitment power, we sometimes refer to x as a mechanism, although it does *not* correspond to a direct revelation mechanism (since x does not condition on the agent's private type θ). We restrict attention to this game as it more closely mirrors the applications of our model (forecasters do not typically report their types in practice); note, however, that this restriction is without loss of generality (as we show in Lemma 1).

Payoffs: The payoffs only depend on the agent's type and the hiring decision. The principal receives a normalized payoff of 1 if she hires a good ($\theta = g$) forecaster, a normalized payoff of -1 if she hires a bad ($\theta = b$) forecaster, and a payoff of 0 otherwise. Essentially, these payoffs capture the principal's preference to maximize the probability of hiring the good type while simultaneously minimizing the probability of hiring the bad type.

The agent's preferences are type-independent: both types want to maximize the probability with which they are hired. To capture this, we assume that the agent receives a normalized payoff of 1 if she is hired and a payoff of 0 if not.

Timing: For easy reference, the following flow chart summarizes the game.



2. Benchmark: Publicly Observed Signals

Before we analyze the game, we consider a simple benchmark in which the agent's signals are publicly observed. Here, the agent is passive and the only private information is his initial type. This benchmark helps highlight the issues inherent in trying to attain this "first-best" payoff for the principal when the agent must be incentivized to truthfully reveal his private signals.

A consequence of the payoff structure is that the principal's ex-ante expected payoff from any hiring decision x can be written as

$$\Pi := \sum_{r \in \{h,l\}} \sum_{s^T \in \{h,l\}^T} \Pr(r, s^T) \left[\Pr(\theta = g | r, s^T) - \Pr(\theta = b | r, s^T) \right] x_r(s^T)$$

$$= \frac{1}{2} \sum_{r \in \{h,l\}} \sum_{s^T \in \{h,l\}^T} \Big[\Pr(r, s^T | \theta = g) - \Pr(r, s^T | \theta = b) \Big] \, x_r(s^T),$$

which is the difference in the expected probabilities that the g and b types of the agent are hired.[8] Therefore, the optimal hiring decision in this benchmark is given by

$$x_r^{FB}(s^T) = \begin{cases} 1 & \text{if } \Pr(r, s^T | \theta = g) \geq \Pr(r, s^T | \theta = b), \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Observe that the principal cannot benefit from randomizing her hiring decision in this first-best benchmark. In addition, the probabilities that determine the first-best hiring policy can be readily expressed in terms of the model primitives. In particular, since signals are conditionally i.i.d., only their frequencies (and not the specific order in which signals arrive) play a role. With this in mind, the probability of type θ observing a signal profile s^T in which n signals that match the outcome is

$$\Pr\left(\sum_{t=1}^{T} \mathbb{1}_r(s_t) = n \,\middle|\, \theta \right) = \binom{T}{n} \beta_{n,T,\theta}, \text{ where } \beta_{n,T,\theta} := \gamma \alpha_\theta^n (1-\alpha_\theta)^{T-n} + (1-\gamma) \alpha_\theta^{T-n} (1-\alpha_\theta)^n$$

and we define $\mathbb{1}_r(s_t)$ to be the indicator function that takes the value 1 if the period t signal matches the outcome ($s_t = r$) and 0 otherwise.

The first term of $\beta_{n,T,\theta}$ corresponds to the cases where the outcome matches the underlying state (that is, $(r, \omega) \in \{(h, H), (l, L)\}$), while the second term corresponds to the complementary cases where the outcome does not match the state (that is, $(r, \omega) \in \{(h, L), (l, H)\}$). Since $\Pr(r, s^T)$ is constant across all signal profiles s^T with the same number of signals matching the outcome r, the first-best is then easy to state: hire an agent who receives exactly n signals that match the outcome if, and only if, the agent is more likely to be of type g than type b, so that

$$\Delta_{n,T} := \beta_{n,T,g} - \beta_{n,T,b} \geq 0.$$

To make the incentive issues in implementing x^{FB} explicit, we now provide a qualitative characterization of the first-best hiring policy in (1).

THEOREM 1. *In the benchmark with publicly observable signals, the first-best hiring policy x^{FB} can be characterized by two cutoffs $\overline{n}$ and $\underline{n}$ with $T/2 < \overline{n} \leq T$ and $\underline{n} \leq T - \overline{n}$ such that*

$$x_r^{FB}(s^T) = \begin{cases} 1 & \text{if } \sum_{t=1}^{T} \mathbb{1}_r(s_t) \geq \overline{n} \text{ or } \sum_{t=1}^{T} \mathbb{1}_r(s_t) \leq \underline{n}, \\ 0 & \text{otherwise.} \end{cases}$$

In words, Theorem 1 states that there are cutoffs $\overline{n}$ and $\underline{n}$ such that the agent is hired whenever he receives at least $\overline{n}$ signals that match the outcome or at least $\underline{n}$ signals that do not. We refer to $\overline{n}$ as threshold for *accuracy* and $\underline{n}$ as the threshold for *consistency*.

For some intuition on the role played by consistency, consider the case where the outcome is very uninformative about the true underlying state (that is, $\gamma \approx 1/2$). Here the outcome provides very little information with which to evaluate the forecaster; instead, the principal can exploit the fact that the good type is more likely to get similar signals since she receives more precise information. Of course, the good type's signals are also more likely to match the outcome, thereby

[8] This follows from the observation that $\Pr(\theta | r, s^T) \Pr(r, s^T) = \Pr(r, s^T, \theta) = \frac{1}{2} \Pr(r, s^T | \theta)$.

explaining why the threshold for consistency is higher than that for accuracy ($\underline{n} \leq T - \overline{n}$). At the other extreme, when the outcome is almost perfectly informative about the true underlying state (so that $\gamma \approx 1$), many incorrect signals are an even stronger indication that the agent is the bad type, and so consistency ceases to be a useful screening device.[9]

Observe that an immediate consequence of Theorem 1 is that strategic behavior by the agent will preclude the implementation of the first-best policy whenever $\underline{n} \geq 0$. This is because the agent always gets hired for sure when she reports the same signal in all periods (regardless of whether it matches the outcome) and such consistency can—and will—always be mimicked. As a result, in this case, the principal will achieve no separation whatsoever (both types will be hired with probability 1) using the first-best policy x^{FB} when faced with a strategic agent. Nontrivial screening is, however, always possible using the simple class of mechanisms (that are easy to implement in practice) that we present in the next section.

3. The Optimal Mechanism with Commitment

In this section, we consider the case where the principal can commit in advance to the mechanism x. We begin by describing a simple class of mechanisms via which screening can always be achieved. As we will argue, the optimal mechanism also belongs to this class.

3.1. *Prediction Mechanisms*

A hiring policy x is a *period-t prediction mechanism* if the agent is hired whenever the outcome r matches the state viewed as more likely by the agent's period-t posterior beliefs (which are, of course, based on his signals s^t). Put differently, the agent is asked to predict the final outcome at period t, and is hired whenever this prediction matched the outcome.

Formally, a period-t prediction mechanism can be implemented as a function of the reported signals as follows

$$x_r(\tilde{s}^T) = \begin{cases} 1 & \text{if } \sum_{t' \leq t} \mathbb{1}_r(\tilde{s}_{t'}) > t/2, \\ 1 & \text{if } \sum_{t' \leq t} \mathbb{1}_r(\tilde{s}_{t'}) = t/2 \text{ and } \tilde{s}_t = r, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

In words, this mechanism hires the agent for sure when a strict majority of his reported signals up to t match the outcome. When the agent reports an equal number of h and l signals through period t, then his period-t report determines the hiring decision. Effectively, this is a tie breaking rule since the agent's belief assigns equal likelihood to both states.[10]

It is straightforward to argue that truthtelling is optimal for *both* types g and b in response to the mechanism defined in (2). To see this, note that the agent does not have an incentive to misreport before period t even if he could pick a reporting strategy $\tilde{s}^t$ after observing all t signals s^t (instead of having to report them sequentially), as the majority of the signals corresponds to the outcome

[9] As either type's precision α_θ rises and mistakes become less likely, the accuracy threshold $\overline{n}$ becomes less forgiving of incorrect signals (that is, $\overline{n}$ increases). Meanwhile, the consistency threshold $\underline{n}$ may be nonmonotone in α_θ: increasing the quality of an agent's information decreases the variance of his signals while increasing the accuracy, and these two effects have countervailing impacts on the likelihood of signals consistently contradicting the public outcome.

[10] Note that, when $\sum_{t' \leq t} \mathbb{1}_r(\tilde{s}_{t'}) = t/2$, we can use other tie breaking rules to implement the same outcome. For instance, we can choose an arbitrary period $t' \leq t$ and hire the agent whenever $\tilde{s}_{t'} = r$.

that is more likely to arise. Additionally, since the signals reported after period t do not affect the hiring decision, it is trivially optimal to report them truthfully. Finally, since the good type is always more likely to observe a majority of signals corresponding to the underlying state, he will be hired with greater probability than the bad type, and hence this mechanism always achieves nontrivial screening.

The principal can optimize within this class of mechanisms by choosing the period in which she solicits a prediction. The next result shows that it may not be optimal for the principal to wait until the final period, but instead should limit the information observed by the agent.

THEOREM 2. *There exists a $\overline{T} \geq 1$ such that the principal's payoff from a period-t prediction mechanism is increasing in t for all $t \leq \overline{T}$ and decreasing in t for all $t \geq \overline{T}$.*

The intuition for the nonmonotonicity of payoffs in t is simple. As t grows larger, both types learn about the underlying state more precisely. But in the limit as t becomes arbitrarily large, both types learn the state perfectly and thus make the same prediction. As a result, screening becomes less effective, and for sufficiently long time horizons T, the principal prefers to solicit a prediction at an intermediate time when the learning advantage of the type-g agent over his type-b counterpart is at its highest. In what follows, we will use $\overline{T}$ to denote the optimal period for the principal to solicit the prediction.[11]

3.2. *The Optimal Mechanism*

In this section, we will argue that the optimal mechanism is a prediction mechanism. To begin, it is worth reiterating that the class of mechanisms we consider (functions of the reported signals alone) is a strict subset of the set of direct revelation mechanisms. This is because the mechanisms x do not condition on the agent's initial private type. While a restriction to such mechanisms can be justified by appealing to their realism, we now argue that this restriction is also without loss of generality: the principal can achieve the same payoff by maximizing over the class of (indirect) mechanisms x as she can from the optimal direct revelation mechanism.

A *direct revelation mechanism* $\chi_r(\theta, s^T) \in \{0, 1\}$ (or direct mechanism for short) determines the probability that the agent is hired as a function of his reported initial type θ, his profile of reported signals s^T, and the final outcome r. Note that the revelation principle applies in this environment, so it is without loss to consider the message space $\{g, b\} \times \{h, l\}^T$.[12] The next result states that the principal cannot attain a higher payoff from using this larger class of direct mechanisms.

LEMMA 1. *There is an optimal direct mechanism that does not depend on the reported type. Specifically, for any incentive compatible direct mechanism χ, there is an indirect mechanism x with the following properties:*

(1) *the principal's payoff from x is (weakly) higher than her payoff from χ;*

[11]It is straightforward to show that $\overline{T}$ is decreasing in both α_g and α_b. Note that as α_g grows larger, the type-g agent's information becomes more precise quickly and his relative advantage over type b peaks sooner; meanwhile, as α_b grows larger, the type-b agent is able to "catch up" quickly to the type-g agent.

[12]Strausz (2003) shows that the revelation principle does not always apply when the principal is restricted to deterministic mechanisms. However, it does apply in single-agent settings such as ours; for reference, we present a formal statement and proof in our supplementary Appendix B.1.

(2) *the type-g agent has an incentive to report his signals truthfully; and*
(3) *the type-b agent reports his signals optimally.*

The intuition for this lemma is transparent. Fix any incentive compatible direct mechanism χ that depends on the agent's reported type. Incentive compatibility implies that it is optimal for the agent to report his initial type truthfully; in particular, the type-b agent receives a lower payoff from initially misreporting his type as g and then optimally reporting his signals. So consider the indirect mechanism $x_r(\cdot) := \chi_r(g, \cdot)$. By definition, it is optimal for the good type to report his signals truthfully (property (2)). Additionally, incentive compatibility of χ ensures that the bad type's payoff is lower from x than from χ. Since the principal's payoff is increasing in type g's payoff and decreasing in type b's, this indirect mechanism will be no worse for her (property (1)). Finally, note that since x is not a direct mechanism, we cannot a priori restrict attention to mechanisms where *both* types report their signals truthfully (made explicit by property (3)).

Henceforth, when we refer to an *optimal* mechanism x, this will correspond to a mechanism that yields the highest payoff that the principal can achieve in the *full space* of direct mechanisms χ. The next theorem characterizes the optimal mechanism.

THEOREM 3. *Let $\widetilde{T} := \min\{T, \overline{T}\}$. A period-$\widetilde{T}$ prediction mechanism is an optimal mechanism.*

There are several aspects of the above result that are worth emphasizing. First, the optimal mechanism takes a very simple form that is easy to implement in practice, as it is both easy to time when predictions are solicited and to institute a hiring policy that depends on the accuracy of the predictions. Second, observe that the optimal mechanism has the property that truthtelling is optimal for *both* types of the agent. This property finds support from the empirical evidence on anonymous analyst surveys. In their handbook chapter, Marinovic, Ottaviani, and Sørensen (2013) point out that, "According to industry experts, forecasters often seem to submit to the anonymous surveys the same forecasts they have already prepared for public (i.e. nonanonymous) release." This is suggestive evidence for the fact that strategic forecasters in the real world predict truthfully as they have no reason to lie in anonymous surveys.

Third, while the optimal period at which to solicit the prediction $\widetilde{T}$ depends on the underlying parameters, it does not do so in a fine-grained way. Put differently, the optimality of the period-$\widetilde{T}$ prediction mechanism will be robust to "small" inaccuracies in the principal's beliefs about the underlying parameters. Finally, while we will show the full generality of the insight driving the above result (in Section 6), it is worth mentioning that it is easy to incorporate asymmetries (in the prior belief regarding the state, the agent's type distribution, and the principal's payoffs from hiring the good or the bad type) within this simplified version of the model. The only change to the above result is that a trivial decision (to always or never hire the agent) may become optimal under some model parameters.

It might seem surprising that the optimal mechanism does not involve more elaborate screening. The reason is that the principal has limited instruments at her disposal and, as a result, incentive compatibility significantly restricts the set of mechanisms the principal can utilize. The characterization of the set of mechanisms that induce the good type-g to report truthfully is the crucial step in the proof of Theorem 3 and is described in the following lemma.

LEMMA 2. *A mechanism x induces truthful signal reporting from the type-g agent if, and only if, it is one of the following mechanisms:*

(1) *a trivial mechanism: the principal's hiring decision does not depend on the agent's reports, so that $x_r(s^T) = x_r(\hat{s}^T)$ for $r = h, l$ and all $s^T, \hat{s}^T \in \{h, l\}^T$; or*
(2) *a period-t prediction mechanism for some $1 \leq t \leq T$.*

Consequently, a mechanism that induces truthful reporting by the type-g agent also induces truthful reporting by the type-b agent.

This lemma shows that incentive compatibility for type g (which, by Lemma 1, is a property of an optimal mechanism) implies that the only nontrivial mechanisms at the principal's disposal are prediction mechanisms. Combined with Theorem 2's payoff single-peakedness result, this implies Theorem 3. As we will argue further in Section 6, the insight in this lemma is remarkably general, applying immediately to substantially generalized versions of the model.

The following example is useful to develop intuition for the lemma. Suppose that a type-g incentive compatible mechanism x is such that, at some period-T history, the hiring decision is a nontrivial function of the final period-T report s_T and the outcome r; that is, there is a sequence of signals s^{T-1} such that the hiring decisions $(x_h(s^{T-1}, h), x_l(s^{T-1}, h))$ after a report $\tilde{s}_T = h$ differ from the hiring decisions $(x_h(s^{T-1}, l), x_l(s^{T-1}, l))$ after a report $\tilde{s}_T = l$. Since the hiring rule is deterministic, this is only possible when $(x_h(s^{T-1}, \tilde{s}_T), x_l(s^{T-1}, \tilde{s}_T))$ equals either $(1, 0)$ or $(0, 1)$.[13] Incentive compatibility implies that, if $(x_h(s^{T-1}, s_T), x_l(s^{T-1}, s_T)) = (1, 0)$, then the agent must believe the outcome $r = h$ is more likely, as he could instead report $s'_T \neq s_T$ and face $(x_h(s^{T-1}, s'_T), x_l(s^{T-1}, s'_T)) = (0, 1)$. However, this essentially implies that the agent is hired if, and only if, the outcome he believes to be more likely is realized; in other words, the mechanism is effectively soliciting a prediction at this history and then hiring based on its accuracy. The proof of Lemma 2 generalizes this argument to all histories.

3.3. *Alternative Implementations and Interpretations of the Optimal Mechanism*

In this section, we revisit our leading example (a political candidate seeking to hire a forecaster) to discuss alternate ways in which the optimal mechanism can be implemented. This discussion also allows us to demonstrate the flexibility of our framework to capture the various different forms that political forecasts often take. The period-t prediction mechanism as defined in (2) captures the case where the agent reports his signal in each period: this can be interpreted as a pollster sequentially releasing the predicted outcomes from each poll he conducts. (As we will argue in Section 6, the model can be generalized to allow for signals with a continuous support in which case this will correspond to releasing the poll results as a percentage instead of a prediction.) Here, the prediction from a period-t poll corresponds to his period-t signal and not the cumulative information he has acquired.

[13] If $x_h(\hat{s}^T) = x_l(\hat{s}^T) = 1$ for some sequence $\hat{s}^T$, then the agent can guarantee he is hired with probability 1 by *always* reporting $\hat{s}^T$, regardless of his true signals. Since this potential deviation remains unused (as the type-g agent is willing to report truthfully), the principal must therefore always (trivially) hire the agent. An analogous argument applies if there is some $\hat{s}^T$ with $x_h(\hat{s}^T) = x_l(\hat{s}^T) = 0$.

Alternatively, political punditry often takes the form of an expert predicting who he thinks is more likely to win in each period after aggregating all his past information. In this case, each report $\tilde{s}_t$ can be interpreted as a prediction of the final outcome and not a signal report. When the agent reports this way, the period-t prediction mechanism simply becomes

$$x_r(\tilde{s}^T) = \begin{cases} 1 & \text{if } \tilde{s}_t = r, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

In words, this mechanism asks the agent to predict the final outcome at each period and hires the agent if and only if the period-t prediction matches the outcome (all other reports are ignored). A strategic agent facing this mechanism will predict at period t whichever outcome he believes is more likely to eventually arise. This will simply be the outcome for which the agent has received more signals up to period t (and he will be indifferent if he has received an equal number of h and l signals). This implementation clearly achieves the identical payoff to the principal as that in (2).

Finally, our framework is flexible enough to allow for the forecaster to make predictions on the odds of the likely winner (in the form of a percentage). Such predictions are made by forecasters like The Upshot of the New York Times or FiveThirtyEight by Nate Silver who aggregate information from various polls (and their type determines the accuracy of this aggregation). To model this, we can simply alter the message space so that the agent is asked to make a percentage prediction (of course, the revelation principle implies that enlarging the message space in this way does not alter the optimal mechanism). Here the period-t prediction mechanism can be implemented by hiring the agent if and only if the outcome he predicts is more likely (in a percentage sense) in period t ends up occurring.

4. The Optimal Mechanism without Commitment

In this section, we derive the equilibrium that maximizes the principal's payoff when she cannot commit to her hiring policy x. Of course, the principal is always weakly better off with commitment power as she can always choose to commit to whatever strategy she can play in its absence. We show that the principal can achieve the same payoff as in Theorem 3 even when she does not have commitment power. We view this as further support for the optimal mechanism in Theorem 3 as, in practice, the level of commitment possessed by principals may vary.

In the absence of commitment, our setting constitutes a dynamic cheap talk game. Here, the agent (the sender) can costlessly make either report in every period t. The reports $\tilde{s}^T$ themselves are not payoff relevant; instead, their only purpose is to inform the principal's (the receiver) decision. The principal's payoff-relevant information is, instead, the agent's private type, and (as in the standard cheap-talk setting) the principal and agent have divergent preferences over the former's action choice as a function of this type. Finally, rather than consider alternative message spaces or games, we will directly show that the principal can achieve the same payoff both with and without commitment power.

Theorem 4. *There is a sequential equilibrium of the game without commitment that yields the principal the same payoff as a period-t prediction mechanism. In particular, this implies that the principal can achieve the same payoff as in the optimal mechanism with commitment.*

Due to the simple structure of the optimal mechanism under full commitment, this result is remarkably straightforward to show. We now describe equilibrium strategies that replicate the outcome of a period-t prediction mechanism. The principal's strategy is to ignore all reports of the agent except that in period t, and she hires the agent if, and only if, his period-t report matches the outcome. In response, both types of the agent babble in all periods except period t, where they report the signal corresponding to the outcome that they consider more likely to arise.

Formally, the principal's strategy is

$$x_r(s^T) = \begin{cases} 1 & \text{if } s_t = r, \\ 0 & \text{otherwise.} \end{cases}$$

For any $t' \neq t$, the agent's strategy is

$$\sigma_h^\theta(h_{t'}^A) = 1 - \sigma_l^\theta(h_{t'}^A) = \frac{1}{2}$$

for all period-t' agent histories; that is, he mixes both reports with equal probability. For a period t history $h_t^A = (s^t, \tilde{s}^{t-1})$ (recalling that s^t denotes the t observed signals and $\tilde{s}^{t-1}$ denotes the $t-1$ reports made prior to period t), the agent's strategy is

$$\sigma_h^\theta(s^t, \tilde{s}^{t-1}) = 1 - \sigma_l^\theta(s^t, \tilde{s}^{t-1}) = \begin{cases} 1 & \text{if } \sum_{t' \leq t} \mathbb{1}_h(s_{t'}) > t/2, \\ \frac{1}{2} & \text{if } \sum_{t' \leq t} \mathbb{1}_h(s_{t'}) = t/2, \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to see that these strategies constitute an equilibrium. Since the principal ignores the reports at all period except t, the agent is indifferent at all such histories; in particular, babbling is therefore a best response. In addition, he is hired only if his period-t report matches the outcome, so it is a best response for him to report whichever signal he has seen more often (and is indifferent if he has seen an equal number of h and l signals). Conversely, since the agent is babbling at all periods except t, it is a best response for the principal to ignore these reports. Finally, since the type-g agent is more likely to correctly predict the outcome, it is optimal for the principal to hire the agent when his period-t report matches the outcome.

Note that all possible signal reports are on-path in the agent's strategy above. Thus, as in the canonical cheap talk setting of Crawford and Sobel (1982), standard refinements have no bite as there is no need to discipline off-path behavior. Hence, in particular, the constructed equilibrium above is a sequential equilibrium. To the best of our knowledge, there are no accepted refinements of dynamic cheap talk games and, moreover, since our setting is quite different from the canonical setting, it is not clear how to extend the refinements designed specifically for the static Crawford and Sobel (1982) environment (most notably Chen, Kartik, and Sobel (2008)). The design of such refinements for dynamic cheap talk games is an important topic of research but is beyond the scope of this paper.

5. Stochastic Mechanisms

In this section, we describe how the principal can utilize randomization to fine-tune screening. Formally, the principal's strategy, which we refer to as a *stochastic mechanism* when she has commitment, now has the entire unit interval as its range. We will use the same notation as before: $x_r(\tilde{s}^T) \in [0,1]$ denotes the probability with which she hires the agent as a function of the T reported signals $\tilde{s}^T$ and the outcome r. For brevity, we will sometimes drop the additional "stochastic" qualifier in this section when it is clear that we are referring to a stochastic mechanism.

The optimal mechanism is difficult to derive for arbitrary time horizons T. This is primarily because the set of incentive compatible stochastic mechanisms is much larger and harder to characterize than in the deterministic mechanism case. Similar issues are also encountered in dynamic mechanism design environments with transfers (hence the restriction to deterministic mechanisms in Courty and Li (2000) or Krähmer and Strausz (2011), for instance). The main aim of this section is to show that the screening is more subtle with randomization for which the restriction to the special case of $T = 3$ suffices. That said, we also provide a simple sufficient condition in Section 5.2 for when randomization is a feature of the optimal stochastic mechanism (Theorem 6).

5.1. *The Role of Randomization when $T = 3$*

In this subsection, we describe the optimal stochastic mechanism for the $T = 3$ period case and discuss its qualitative properties. This special case is convenient to highlight the role played by randomization as the optimal stochastic mechanism can be characterized and is easy to describe.

We begin by describing the first-best mechanism x^{FB} for the case where the agent's signals (but not his initial type θ) are also observed by the principal. The following characterization of the set of possible accuracy and consistency thresholds (corresponding to Theorem 1) that can arise in x^{FB} is instructive as a point of contrast with the optimal mechanism.

LEMMA 3. *Suppose $T = 3$. Then the first-best mechanism x^{FB} is one of the following:*

(1) *hire the agent if, and only if, all three of his signals are accurate ($\overline{n} = 3$, $\underline{n} = -1$);*
(2) *hire the agent if, and only if, all three of his signals are consistent ($\overline{n} = 3$, $\underline{n} = 0$); or*
(3) *hire the agent if, and only if, a majority of his signals match the outcome ($\overline{n} = 2$, $\underline{n} = -1$).*

Observe that the first-best mechanism in case (3) is simply a period-3 prediction mechanism: the agent will predict the outcome corresponding to the majority of his signals. This is the case when $\Delta_{2,3} \geq 0$ and the type-g agent is more likely to observe exactly two matches than the type-b agent; when this is the case, the first-best payoff is achievable. When this is not the case, however, the first-best payoffs corresponding to cases (1) and (2) cannot be achieved as a strategic agent can easily feign consistency by simply "cascading" on his first signal. In such circumstances, the optimal mechanism (characterized in the following theorem) is distorted away from the first-best.

THEOREM 5. *Suppose $T = 3$. When $\Delta_{2,3} \geq 0$, the period-3 prediction mechanism is an optimal stochastic mechanism. Conversely, when $\Delta_{2,3} < 0$, the optimal stochastic mechanism is given by*

$$x_r(s^3) = \begin{cases} 1 & \text{if } s_1 = s_2 = r, \\ \frac{1}{2(\gamma\alpha_b + (1-\gamma)(1-\alpha_b))} & \text{if } s_1 \neq s_2 \text{ and } s_3 = r, \\ 0 & \text{otherwise.} \end{cases}$$

Faced with this mechanism, it is optimal for both types of the agent to truthfully report.

The optimal stochastic mechanism when $T = 3$ hires the agent only if a majority of his reported signals match the outcome. Moreover, when the type-b agent is more likely to match exactly two of three signals than the type-g agent (that is, when $\Delta_{2,3} < 0$), the *order* of reported signals influences the hiring decision. Specifically, the optimal mechanism rewards early accuracy: in profiles where exactly two of the three reports match the outcome, the agent is hired with higher probability when the first two reports are correct than when one of them mismatches.

Intuitively, when $\Delta_{2,3} < 0$, the principal would prefer not to hire the agent at histories where he truthfully reports only two signals matching the outcome (as such profiles are more likely for the type-b agent). But as we have seen, deterministic mechanisms compel the principal to hire the agent at such profiles whereas, when the principal can randomize, she can reduce the hiring probability at such profiles without violating incentive compatibility.

To better understand how randomization permits such a reduction, it is helpful to reinterpret the hiring rule in Theorem 5 as an option mechanism: in the second period, the agent is offered the opportunity to make a prediction immediately or to delay his prediction to period three. A correct prediction in period two is rewarded by hiring the agent for sure, while a correct prediction in period three is rewarded by hiring the agent with a probability strictly less than one. (An incorrect forecaster is never hired, regardless of the timing of his prediction.) Faced with this option, an agent who has observed two identical signals will always make a prediction in period two—no matter what he observes in the third period, the agent's prediction will remain unchanged but his probability of being hired is lower. However, an agent who has observed two mismatched signals is uncertain about the underlying state, and therefore benefits from delaying his prediction by a period. Indeed, the reduced probability of being hired in period three after mixed signals is chosen precisely to ensure that the type-b agent is indifferent about delaying his prediction, while type g's better information gives him a strict incentive to wait for an additional signal.[14] Of course, since the type-g agent is more likely to observe two matching signals in the first two periods, he is correspondingly more likely to make an early prediction (with a larger hiring probability), compounding his pure informational advantage over the type-b agent.

It is instructive to briefly contrast the proof strategy for Theorem 5 with that for Theorem 3. First, observe that Lemma 1 also applies to stochastic mechanisms, so it is without loss to consider stochastic mechanisms where the type-g agent reports truthfully while type b is allowed to optimally misreport. Unlike with deterministic mechanisms, where Lemma 2 showed that incentive compatibility for type g implies incentive compatibility for type b, incentive constraints in a stochastic mechanism may be less restrictive. In particular, there are stochastic mechanisms where truthful reporting of signals is incentive compatible for type g but not for type b. Therefore, it is difficult to formulate a tractable version of the principal's optimization problem. Our proof instead relies on an auxiliary problem that is both easier to solve and yields the principal a greater payoff; we then show that resulting solution is in fact feasible in the original problem.

[14]Note that the randomization necessary to generate this indifference for the type-b agent relies on the informativeness γ of the public outcome. This is in contrast to the case of deterministic mechanisms where, as long as the public outcome is equally accurate in both states of the world, the optimal prediction mechanism does not depend on γ.

5.2. *When is Randomization Optimal?*

We now provide a simple sufficient condition for the optimality of randomization.

THEOREM 6. *The principal's payoff from the optimal stochastic mechanism is strictly higher than that from the optimal (deterministic) mechanism when $T > \overline{T} + 1$.*

This result states that the principal strictly benefits from using randomization for sufficiently long time horizons. Intuitively, recall that the optimal (deterministic) mechanism for $T > \overline{T}$ is a period-$\overline{T}$ prediction mechanism: in this mechanism, the principal ignores reports after $\overline{T}$. In the optimal stochastic mechanism for $T = 3$, when the agent in period 2 has conflicting signals (and therefore thinks both outcomes are equally likely), the principal can fine-tune screening by lowering the hiring probability (which is beneficial since type b's higher signal variance implies he is more likely to receive an equal number of h and l signals) without destroying incentive compatibility. The principal can similarly lower the hiring probability at profiles in which the agent reports the same number of h and l signals (or in which the difference between h and l signals is one) by conditioning the mechanism on reports after period $\overline{T}$. The agent prefers such a mechanism as it allows him the chance to better learn the underlying state.

It is hard to fully characterize the optimal stochastic mechanism for $T > 3$ since incentive compatibility for type g alone is no longer sufficient to pin down type b's reporting strategy. As a result, the derivation of the optimal mechanism must account for optimal misreporting, which makes the problem intractable. In the $T = 3$ period case, it is possible to identify and individually account for histories at which the type-b agent might have an incentive to misreport; when $T > 3$, however, the set of such histories becomes large and this approach is no longer feasible.

6. A MORE GENERAL MODEL

In this section, we show the generality of our main insight that prediction mechanisms are the optimal way to screen strategic forecasters. As we will argue, the critical assumption driving our result is that the outcome r that is being predicted is binary; every other assumption can be substantially generalized. We describe the key components of the model in their full generality below and deliberately overload the notation to make the generalization of each assumption explicit. The timing of the model remains unchanged.

State: There is an unknown underlying state ω drawn from an arbitrary (not necessarily binary) set Ω that drives the data-generating process. ω is distributed according to a commonly known probability measure $p_0 \in \Delta(\Omega)$.

Agent's private information: The agent's type θ is drawn from an arbitrary (again, not necessarily binary) set Θ. The commonly known prior distribution of θ is given by $\mu_0 \in \Delta(\Theta)$.

In this general setting, we allow for the possibility that the agent does not perfectly observe his initial type, but instead learns about his forecasting ability over time.[15] We model this by adding

[15] In discussing the important directions for future research on strategic forecasters, Marinovic, Ottaviani, and Sørensen (2013) state that a "key challenge lies in finding a tractable and sufficiently general multi-period environment with learning about the precision as well as about the state." Our general model takes a step in this direction.

an additional signal: formally, in period 0, the forecaster observes a single private signal $\lambda \in \Lambda$, where the set Λ is arbitrary. This signal λ is drawn from a (commonly known) measure $\mu_\theta \in \Delta(\Lambda)$ that may vary by type θ. Thus, the case of a perfectly informed agent corresponds to $\Lambda = \Theta$ and $\mu_\theta(\{\theta\}) = 1$ for all $\theta \in \Theta$. On the other hand, the case where $\mu_\theta = \mu_{\theta'}$ for all $\theta, \theta' \in \Theta$, so the distribution of λ does not vary by type, corresponds to a "signal jamming" version of our model (similar to the career concerns literature following Holmstrom (1999)) where both the principal and the agent start with the same information.

In each period, $t = 1, \ldots, T$, the forecaster privately observes a noisy but informative signal s_t, drawn from an arbitrary signal space S_t, about the unknown state ω. Signals are conditionally independent given the underlying state ω and the agent's type θ, and s_t is drawn from a distribution $\alpha_{\omega,\theta,t} \in \Delta(S_t)$. Note that both the signal spaces and distributions may vary over time.

Outcome: As in the simplified model, a binary outcome $r \in \{h, l\}$ is publicly realized at the end of period T. We denote by $\gamma_\omega \in [0, 1]$ the probability of outcome h arising when the true state is ω. Note that the outcome remains a potentially noisy signal of the state, but the joint distribution is not restricted in any way.

Payoffs: The agent of type θ now receives a payoff $u_\theta > 0$ from being hired, and a normalized payoff of 0 if he is not. Note that this does not change the agent's incentives compared to the simplified model in Section 1 as his objective is still to maximize the probability of being hired.

Finally, the principal's payoff can also be made type-dependent: she receives a payoff $\pi_\theta \in \mathbb{R}$ if she hires an agent of type θ, and a normalized payoff of 0 from not hiring the agent.

Appropriate definitions of strategies and mechanisms generalize to this richer environment in the obvious way. The next result shows that prediction mechanisms remain optimal even in this very general environment. Note that the definition of a prediction mechanism as a function of reported signals will differ from that in (2) as the environment is no longer symmetric and the signal space is not binary. Instead, we make use of the alternative definition in (3).

THEOREM 7. *In the general model, one of the following mechanisms is optimal:*

(1) *a trivial mechanism: the principal's hiring decision does not depend on the agent's reports; or*
(2) *a period-t prediction mechanism for appropriately chosen t.*

Additionally, the principal can implement the same outcome and thereby achieve the same payoff in a sequential equilibrium of the game without commitment.

As in the case of simplified model (Theorem 3), prediction mechanisms are optimal within the full class of direct revelation mechanisms. As we allow for more than two types, it is no longer possible to directly argue (as in Lemma 1) that the principal cannot benefit by asking the agent to report his type in a direct revelation mechanism. Instead, our proof characterizes incentive compatible direct mechanisms in this general setting. Effectively, we show that the only nontrivial incentive compatible direct mechanisms are prediction mechanisms and thus, finding the optimal mechanism only involves choosing the time at which to solicit the prediction. Of course, additional structure is necessary to fully characterize the optimal prediction period as a function of the time horizon T.

Essential to Theorem 7's characterization of incentive compatibility is the assumption that the publicly observable outcome—that is, the information available to the principal when evaluating a forecast—is binary. Enriching the set of possible outcomes yields the principal a substantially more complex set of instruments: agents could, for instance, be asked to make predictions about nested partitions of possible outcomes. This increased dimensionality of the set of possible mechanisms precludes a characterization of the principal's optimal mechanism.

On the other hand, the result does not require us to take a stand on the relationship between the principal's payoff and the information of the agent. For instance, we do not need to assume that "good" types (for which $v(\theta) > 0$) receive better information than "bad" types (for which $v(\theta) < 0$). Of course, some additional structure (like that we impose in our simplified model) is desirable to capture specific applications.

The critical assumptions of the general model in this section are supported by our leading application. The richness of the time-varying signal space captures the myriad different sources of information that are available to political pundits. Importantly, this application satisfies the key driving assumption of our model: political predictions are always about the eventual winner which is a binary outcome in the (effectively) two-party US political system. As we argued in Section 3.3, our model is flexible enough to capture the variety of different forms that political predictions come in. It is worth noting that the an election result taken as the difference in vote counts can be considered as a continuous outcome variable; however, to the best of our knowledge, political forecasters always predict, and are judged on, an election's winners and not their win margins.

7. Discussion

In this section, we address a few important structural assumptions of the model. For ease of exposition, the discussion will employ the simplified setup of Section 1.

7.1. *The Role of Sequential Reporting*

In the canonical dynamic mechanism design environment with transfers (see, for instance, Courty and Li (2000) or Pavan, Segal, and Toikka (2014)), the fact that the agent receives his private information sequentially plays an important role for the tractability of the model. Because the agent has single dimensional private information at the time of contracting, incentive compatibility is easier to characterize than in static, multidimensional mechanism design environments where the agent has acquired all his private information before contracting (Eső and Szentes (2017) demonstrate the generality of this technique). An important underlying economic insight is that the principal benefits from being able to contract with an agent when her informational disadvantage is at its lowest as the agent has not acquired the entirety of his private information.

We now isolate the role played by sequential reporting in our model by drawing a contrast with the optimal stochastic mechanism in the static multidimensional version of our environment. In the static game, the agent's strategy σ^θ is defined as follows: he first observes his T signals s^T and his strategy $\sigma^\theta(s^T) \in \Delta(\{h,l\}^T)$ determines the distribution over T-vectors $\tilde{s}^T \in \{h,l\}^T$ of signal reports. As before, the principal's strategy $x(\tilde{s}^T, r) \in [0,1]$ depends on the vector of reported signals and the outcome but observe that we also allow the principal to randomize. We refer to the principal's strategy when she can commit as a *static stochastic mechanism*.

THEOREM 8. *The optimal static stochastic mechanism yields the principal the same payoff as that from a period-T prediction mechanism in the dynamic game with sequential reporting.*[16]

The optimal static mechanism is equivalent to the period-T prediction mechanism in the dynamic environment. There are two aspects of Theorem 8 that are worth highlighting. The first is that, with long time horizons $T \geq \overline{T}$, the principal cannot prevent the agent from using the information he receives after period $\overline{T}$. This is in contrast with Theorem 3 where the principal chooses to ignore reports after $\overline{T}$. Recall also, that for the case of deterministic hiring policies, the principal does not even need commitment to maximize her payoff (Theorem 4). Thus, in our main setting of interest (deterministic mechanisms), the dynamics of agent learning plays a greater role than commitment.

Secondly, observe that it is not optimal for the principal to employ randomization in the static setting. This is in contrast with the optimality of randomization when the time horizon is long (Theorem 6). This latter aspect is also a feature of the sequential screening setting of Courty and Li (2000). There too, the principal may employ randomization with dynamic reporting but will not if restricted to using a static mechanism after the agent has acquired all his private information. This similarity is captured by the Myersonian approach that we take in the proof of Theorem 8.

To summarize, simple mechanisms are optimal in our model unless the principal can randomize *and* the environment is dynamic. Put differently, *both* these aspects must be present simultaneously in order for the optimal mechanism to take a form more complex than a prediction mechanism.

7.2. *Transfers*

While our setting without transfers is appropriate for the applications we have in mind, it is natural to explore the theoretical implications of permitting them. We begin by discussing the optimal *direct* mechanism but, as we will argue below, it is also possible to implement this direct mechanism by with an indirect mechanisms that does not condition on the agent's type. A *direct mechanism with transfers* consists of two functions

$$\chi_r(\theta, s^T) \in \{0,1\} \text{ and } \tau_r(\theta, s^T) \in \mathbb{R},$$

where χ is (as before) the hiring decision and τ is a transfer that also depends on the reported type, signals, and outcome. Both types receive (arbitrary) strictly positive utility from being hired and zero utility if they are not.

Since the signal distributions of both types are correlated, we can use the insight of Crémer and McLean (1988) to induce the agent to reveal his type with a zero expected transfer, thereby

[16] The proof of this result can be found in our supplementary Appendix B.2.

ensuring that the principal only hires the type-g agent.[17] To see this, consider the mechanism

$$\chi_r(\theta, s^T) = \begin{cases} 1 & \text{if } \theta = g, \\ 0 & \text{if } \theta = b, \end{cases} \text{ and } \tau_r(\theta, s^T) = \begin{cases} \kappa\overline{\beta} & \text{if } \theta = g \text{ and } s_1 = r, \\ \kappa\underline{\beta} & \text{if } \theta = g \text{ and } s_1 \neq r, \\ 0 & \text{if } \theta = b, \end{cases}$$

where $\overline{\beta}, \underline{\beta}, \kappa > 0$ and

$$[\alpha_g \gamma + (1-\alpha_g)(1-\gamma)]\kappa\overline{\beta} - [\alpha_g(1-\gamma) + (1-\alpha_g)\gamma]\kappa\underline{\beta} = 0.$$

In words, this mechanism hires the agent only if he reports type g, and the transfer depends on the reported period-1 signal. Moreover, this transfer is such that, if the agent reports type g, he receives $\kappa\overline{\beta}$ if the first signal matches the outcome, makes a payment of $\kappa\underline{\beta}$ to the principal if it does not, and has an expected payment from reporting truthfully (for a type-g agent) of 0.

Now observe that reporting truthfully is optimal for type g. He has no incentive to report his initial type as b, as truthful reporting leads to him being hired with an expected transfer of zero. Moreover, he has no incentive to misreport his period-1 signal s_1 as he receives a positive transfer when the signal matches the outcome (and a negative transfer when it does not). Type b receives zero utility from truthful reporting. If, instead, the type-b agent misreports his type as g, he will then find it optimal to report his period-1 signal s_1 truthfully (for the same reason as type-g does). However, type b will now have to make a strictly positive expected payment to the principal (as $\alpha_b < \alpha_g$). κ can always be chosen to be large enough so that this payment will be greater than the utility that type b gets from being hired. Thus, this mechanism achieves the best possible outcome for the principal.

Finally note that we can implement a similar outcome using the class of mechanisms that does not depend on the explicit announcement of the type (due to the presence of transfers, Lemma 1 does not apply here). The principal can always use the first signal report to proxy for the type announcement (for instance, interpreting $s_1 = h$ as an announcement that the agent is type-g and vice versa). Having solicited this information, the principal can choose her hiring rule and construct similar transfer lotteries as above using the signal reports s_t from periods $t > 1$ to ensure that only type-g is hired at a zero expected transfer.

8. Concluding Remarks

In this paper, we introduce the problem of evaluating a strategic forecaster based on the dynamics of the predictions he makes about an upcoming event. In doing so, we bring two novel aspects to the study of evaluating forecasters that differ from the existing literature in economics and psychology: prediction dynamics and mechanism design by the evaluator. In a very general setting, we derive the optimal deterministic dynamic mechanism for the principal and show that it takes a very simple and easy to implement form. The simplicity of the optimal mechanism combined with the fact that commitment is not necessary to implement it implies that it can serve as a simple guideline for hiring forecasters.

[17] Olszewski and Pęski (2011) apply a general version of this insight to the "testing experts" problem discussed earlier.

The tractability of our setting opens the door to future research on more complex forecasting environments. There at least two obvious generalizations of the model that merit further study. The first is to enrich the outcome space. As we mentioned earlier, forecasters predict a variety of different events many of which need not be binary (for instance, predictions about economic variables). A second natural generalization would be the optimal contest design for multiple forecasters. We hope to investigate these intriguing questions in future research.

APPENDIX A. OMITTED PROOFS

PROOF OF THEOREM 1. In the first-best, the principal observes the agent's signals s^T, but not the agent's type. Therefore, the first-best optimal mechanism must solve

$$\max_{x_h(\cdot),x_l(\cdot)} \left\{ \sum_{r\in\{h,l\}} \sum_{s^T\in\{h,l\}^T} \left[\frac{1}{2}\Pr(r,s^T|\theta=g) - \frac{1}{2}\Pr(r,s^T|\theta=b) \right] x_r(s^T) \right\}.$$

Note, however, that $\Pr(r,s^T|\theta)$ is constant across all s^T with the same number of matching signals $s_t = r$, regardless of the order of those signals. Therefore, with slight abuse of notation, we can write any solution $x_r(s^T)$ to the principal's problem as $x(n)$, where $n = \sum_t \mathbb{1}_r(s_t)$.

Therefore, with slight abuse of the notation from the main text, we write

$$\beta_{n,\theta,\gamma} := \gamma\alpha_\theta^n(1-\alpha_\theta)^{T-n} + (1-\gamma)\alpha_\theta^{T-n}(1-\alpha_\theta)^n \text{ and } \Delta_{n,\gamma} := \beta_{n,g,\gamma} - \beta_{n,b,\gamma}$$

for all $n \in [0,T]$. (Recall that $\binom{T}{n}\beta_{n,\theta,\gamma}$ is the probability that exactly n of the agent's T signals with precision α_θ match the precision-γ realized outcome.) We can then write the principal's observable-signal problem as

$$\max_{x(\cdot)} \left\{ \frac{1}{2}\sum_{n=0}^{T} \binom{T}{n} \Delta_{n,\gamma} x(n) \right\}.$$

It is trivial to see that the solution of this linear program depends entirely on the signs of the $\Delta_{n,\gamma}$ coefficients: we have $x^{FB}(n) = 1$ if $\Delta_{n,\gamma} > 0$, and $x^{FB}(n) = 0$ if $\Delta_{n,\gamma} < 0$.

CLAIM. *Suppose $\Delta_{n,\gamma} \geq 0$. Then $\frac{\partial^2}{\partial n^2}\Delta_{n,\gamma} > 0$.*

PROOF OF CLAIM. Note first that

$$\frac{\partial}{\partial n}\Delta_{n,\gamma} = \left[\ln\left(\frac{\alpha}{1-\alpha}\right) \left(\gamma\alpha^n(1-\alpha)^{T-n} - (1-\gamma)\alpha^{T-n}(1-\alpha)^n \right) \right]_{\alpha_b}^{\alpha_g},$$

implying that

$$\begin{aligned}
\frac{\partial^2}{\partial n^2}\Delta_{n,\gamma} &= \left[\ln^2\left(\frac{\alpha}{1-\alpha}\right) \left(\gamma\alpha^n(1-\alpha)^{T-n} + (1-\gamma)\alpha^{T-n}(1-\alpha)^n \right) \right]_{\alpha_b}^{\alpha_g} \\
&= \left[\ln^2\left(\frac{\alpha}{1-\alpha}\right) \right]_{\alpha_b}^{\alpha_g} \left(\gamma\alpha_g^n(1-\alpha_g)^{T-n} + (1-\gamma)\alpha_g^{T-n}(1-\alpha_g)^n \right) \\
&\quad + \ln^2\left(\frac{\alpha_b}{1-\alpha_b}\right) \left[\gamma\alpha^n(1-\alpha)^{T-n} + (1-\gamma)\alpha^{T-n}(1-\alpha)^n \right]_{\alpha_b}^{\alpha_g} \\
&= \left[\ln^2\left(\frac{\alpha}{1-\alpha}\right) \right]_{\alpha_b}^{\alpha_g} \beta_{n,g,\gamma} + \ln^2\left(\frac{\alpha_b}{1-\alpha_b}\right) \Delta_{n,\gamma}.
\end{aligned}$$

Since $\ln\left(\frac{\alpha}{1-\alpha}\right)$ is strictly positive and increasing on $(\frac{1}{2},1)$ and $\beta_{n,g,\gamma} > 0$, the assumption that $\Delta_{n,\gamma} \geq 0$ implies that the expression above is strictly positive. ◊

Thus, $\Delta_{n,\gamma}$ is strictly convex on a neighborhood of any $m \in [0,T]$ at which $\Delta_{m,\gamma} \geq 0$. Therefore, if there exists some $\underline{n} \in [0,T]$ with $\Delta_{\underline{n},\gamma} = 0$ and $\frac{\partial}{\partial n}\Delta_{\underline{n},\gamma} \leq 0$, then $\frac{\partial}{\partial n}\Delta_{m,\gamma} < 0$ for all $m < \underline{n}$. This implies that $\Delta_{m,\gamma} > 0$ for all $m \in [0,\underline{n})$. Similarly, if there exists some $\overline{n} \in [0,T]$ such that $\Delta_{\overline{n},\gamma} = 0$ and $\frac{\partial}{\partial n}\Delta_{\overline{n},\gamma} \geq 0$, then $\frac{\partial}{\partial n}\Delta_{m,\gamma} > 0$ for all $m > \overline{n}$. This implies that $\Delta_{m,\gamma} > 0$ for all $m \in (\overline{n},T]$.

Hence, we may conclude that the function $\Delta_{n,\gamma}$ has at most two zeros in $[0,T]$.

CLAIM. *There exists a unique $\overline{n} \in (\frac{T}{2}, T)$ such that $\Delta_{\overline{n},\gamma} = 0$.*

PROOF OF CLAIM. Note first that $\Delta_{T,1} = \alpha_g^T - \alpha_b^T > 0$ since $\alpha_g > \alpha_b$. In addition, note that $\Delta_{T,\frac{1}{2}} = \frac{1}{2}\left[\alpha^T + (1-\alpha)^T\right]_{\alpha_b}^{\alpha_g}$. However,

$$\frac{\partial}{\partial\alpha}\left[\alpha^T + (1-\alpha)^T\right] = T\left[\alpha^{T-1} - (1-\alpha)^{T-1}\right] > 0 \text{ for all } \alpha > \frac{1}{2}.$$

Therefore, $\Delta_{T,\frac{1}{2}} > 0$. But since $\Delta_{T,\gamma}$ is linear in γ, this implies that $\Delta_{T,\gamma} > 0$ for all $\gamma \in [\frac{1}{2}, 1]$.

Next, consider

$$\Delta_{\frac{T}{2},\gamma} = \left[\gamma\alpha^{\frac{T}{2}}(1-\alpha)^{T-\frac{T}{2}} + (1-\gamma)\alpha^{T-\frac{T}{2}}(1-\alpha)^{\frac{T}{2}}\right]_{\alpha_b}^{\alpha_g} = \left[(\alpha(1-\alpha))^{\frac{T}{2}}\right]_{\alpha_b}^{\alpha_g}.$$

Since $\alpha(1-\alpha)$ is strictly decreasing on $(\frac{1}{2}, 1)$, we have $\Delta_{\frac{T}{2},\gamma} < 0$ for all $\gamma \in [\frac{1}{2}, 1]$.

Finally, because $\Delta_{n,\gamma}$ is continuous in n, there must exist some $\overline{n} \in (\frac{T}{2}, T)$ such that $\Delta_{\overline{n},\gamma} = 0$. Moreover, the convexity argument above implies that this $\overline{n}$ is the unique zero in $(\frac{T}{2}, T)$. ♢

The existence of a second zero is not guaranteed; in particular, there exists some $\underline{n} \in [0, \frac{T}{2})$ with $\Delta_{\underline{n},\gamma} = 0$ if, and only if, $\Delta_{0,\gamma} \geq 0$. (Note that $\underline{n} = 0$ in the boundary case where $\Delta_{0,\gamma} = 0$.) Again, the convexity argument above implies that this is the unique zero below $\frac{T}{2}$.

CLAIM. *Suppose there exists some $\underline{n} < \frac{T}{2}$ with $\Delta_{\underline{n},\gamma} = 0$. Then $\underline{n} < T - \overline{n}$.*

PROOF OF CLAIM. We can write

$$\Delta_{\underline{n},\gamma} = \gamma\Delta_{\underline{n},1} + (1-\gamma)\Delta_{T-\underline{n},1}, \text{ where } \Delta_{\underline{n},1} = \left[\alpha^{\underline{n}}(1-\alpha)^{T-\underline{n}}\right]_{\alpha_b}^{\alpha_g}.$$

Note, however, that

$$\frac{\partial}{\partial\alpha}\left[\alpha^{\underline{n}}(1-\alpha)^{T-\underline{n}}\right] = \underline{n}\alpha^{\underline{n}-1}(1-\alpha)^{T-\underline{n}} - (T-\underline{n})\alpha^{\underline{n}}(1-\alpha)^{T-\underline{n}-1} = (\underline{n} - \alpha T)\alpha^{\underline{n}-1}(1-\alpha)^{T-\underline{n}-1}.$$

Since $\underline{n} < \frac{T}{2}$, this expression is strictly negative whenever $\alpha \in (\frac{1}{2}, 1)$; since $1 > \alpha_g > \alpha_b > \frac{1}{2}$, this implies that $\Delta_{\underline{n},1} < 0$.

Thus, $\Delta_{T-\underline{n},1} > 0$ since $\Delta_{\underline{n},\gamma} = \gamma\Delta_{\underline{n},1} + (1-\gamma)\Delta_{T-\underline{n},1} = 0$. But since $\gamma \in [\frac{1}{2}, 1]$, this implies that $\Delta_{T-\underline{n},\gamma} = \gamma\Delta_{T-\underline{n},1} + (1-\gamma)\Delta_{\underline{n},1} > 0$, which is only possible if $T - \underline{n} > \overline{n}$. ♢

Thus, there exist $\overline{n} \in (\frac{T}{2}, T)$ and $\underline{n} < T - \overline{n}$ (where $\underline{n} < 0$ if $\Delta_{0,\gamma} < 0$) such that, for all $n \in [0, T]$,

$$\Delta_{n,\gamma} \begin{cases} > 0 & \text{if } n > \overline{n} \text{ or } n < \underline{n}, \\ = 0 & \text{if } n = \overline{n} \text{ or } n = \underline{n}, \\ < 0 & \text{if } \overline{n} > n > \underline{n}. \end{cases}$$

The first-best policy x^{FB} described in the theorem follows immediately. ■

PROOF OF THEOREM 2. When the principal uses a period-t prediction mechanism, her payoff is simply the difference in prediction-matching probabilities between the type-g and type-b agents.

To that end, recall the following notation:

$$\beta_{m,n,\theta} := \gamma\alpha_\theta^m(1-\alpha_\theta)^{n-m} + (1-\gamma)\alpha_\theta^{n-m}(1-\alpha_\theta)^m \text{ and } \Delta_{m,n} := \beta_{m,n,g} - \beta_{m,n,b}.$$

Note that $\binom{n}{m}\beta_{m,n,\theta}$ is the probability that exactly m out of n signals with precision α_θ match the public precision-γ outcome. Thus, for any $k \geq 0$, we can write the principal's payoff from using a period-$2k$ or -$(2k+1)$ prediction mechanism as

$$\Pi(2k) := \frac{1}{2}\binom{2k}{k}\Delta_{k,2k} + \sum_{j=k+1}^{2k}\binom{2k}{j}\Delta_{j,2k} \text{ and } \Pi(2k+1) := \sum_{j=k+1}^{2k+1}\binom{2k+1}{j}\Delta_{j,2k+1},$$

respectively. Finally, define $\delta(n) := \Pi(n) - \Pi(n-1)$. Note that since $\Pi(0) = 0$ and $\Pi(1) > 0$, we know that $\delta(1) > 0$.

CLAIM. *For any $k \geq 1$, both the principal and agent (of either type) are indifferent between the $(2k-1)$-period and $2k$-period prediction mechanisms.*

PROOF OF CLAIM. In the $(2k-1)$-period prediction mechanism, a type-θ agent is hired if k or more signals match the outcome. Partitioning that event into the case where exactly k signals match and the case where at least $k+1$ signals match, we can write the probability of hiring a type-θ agent in the $(2k-1)$-period prediction mechanism as

$$\binom{2k-1}{k}\beta_{k,2k-1,\theta} + \sum_{j=k+1}^{2k-1}\binom{2k-1}{j}\beta_{j,2k-1,\theta}.$$

In the $2k$-period prediction mechanism, on the other hand, a type-θ agent is hired with probability $\frac{1}{2}$ if exactly k signals match the principal's, and with certainty if $k+1$ or more signals match. Focusing on the first $2k-1$ periods, this implies that three events may lead to the agent being hired:

- at least $k+1$ of the first $2k-1$ signals match the public outcome, in which case the agent is hired regardless of the realization of the $2k$th signal;
- exactly k of the first $2k-1$ signals match, in which case the agent is hired with probability 1 if the $2k$th signal matches, and with probability $\frac{1}{2}$ if it does not; and
- exactly $k-1$ of the first $2k-1$ signals match, in which case the agent is hired with probability $\frac{1}{2}$ if the $2k$th signal matches, and is not hired otherwise.

Therefore, the probability of hiring a type-θ agent in the $2k$-period prediction mechanism is

$$\begin{aligned}
&\sum_{j=k+1}^{2k-1}\binom{2k-1}{j}\beta_{j,2k-1,\theta} + \binom{2k-1}{k}\left(\frac{1}{2}\beta_{k,2k,\theta} + \beta_{k+1,2k,\theta}\right) + \binom{2k-1}{k-1}\left(\frac{1}{2}\beta_{k,2k,\theta}\right) \\
&= \sum_{j=k+1}^{2k-1}\binom{2k-1}{j}\beta_{j,2k-1,\theta} + \binom{2k-1}{k}\left(\beta_{k,2k,\theta} + \beta_{k+1,2k,\theta}\right) \\
&= \sum_{j=k+1}^{2k-1}\binom{2k-1}{j}\beta_{j,2k-1,\theta} + \binom{2k-1}{k}\beta_{k,2k-1,\theta},
\end{aligned}$$

where the first equality follows from the fact that $\binom{2k-1}{k} = \binom{2k-1}{k-1}$, and the second from the observation that $\beta_{k,2k,\theta} + \beta_{k+1,2k,\theta} = \beta_{k-1,2k,\theta}$.

Thus, a type-θ agent is hired with exactly the same probability in the $2k$- and $(2k-1)$-period prediction mechanisms, and so is indifferent between the two; this also implies that $\delta(2k) = 0$. $\diamondsuit$

CLAIM. *For any $k > 0$, $\delta(2k+1) = \binom{2k}{k} \left[\alpha^k(1-\alpha)^k(\gamma\alpha + (1-\gamma)(1-\alpha) - \frac{1}{2})\right]_{\alpha_b}^{\alpha_g}$.*

PROOF OF CLAIM. In the $(2k+1)$-period prediction mechanism, a type-θ agent is hired if $k+1$ or more signals match the public outcome. Focusing on the first $2k$ periods, this implies that two events may lead to the agent being hired:

- at least $k+1$ of the first $2k$ signals match the outcome, in which case the agent is hired regardless of the realization of the $(2k+1)$th signal; or
- exactly k of the first $2k$ signals match, in which case the agent is hired with probability 1 if the $(2k+1)$th signal matches, and is not hired otherwise.

Therefore, the probability of hiring a type-θ agent in the $(2k+1)$-period prediction mechanism is

$$\sum_{j=k+1}^{2k} \binom{2k}{j} \beta_{j,2k,\theta} + \binom{2k}{k} \beta_{k+1,2k+1,\theta}.$$

In the $2k$-period prediction mechanism, an agent is hired with probability $\frac{1}{2}$ if exactly k signals match the outcome, and with certainty if at least $k+1$ match, so the probability of hiring type θ is

$$\frac{1}{2}\binom{2k}{k}\beta_{k,2k,\theta} + \sum_{j=k+1}^{2k} \binom{2k}{j} \beta_{j,2k,\theta}.$$

Thus, the difference between these two probabilities is

$$\binom{2k}{k}\beta_{k+1,2k+1,\theta} - \frac{1}{2}\binom{2k}{k}\beta_{k,2k,\theta} = \binom{2k}{k}\alpha_\theta^k(1-\alpha_\theta)^k\left(\gamma\alpha_\theta + (1-\gamma)(1-\alpha_\theta) - \frac{1}{2}\right).$$

Since the principal's payoff is the difference between the type-g and type-b agents' payoffs, this yields the desired result. $\diamondsuit$

The result above therefore implies that $\delta(2k+1)$ is proportional to

$$z(k) := \left[\alpha^k(1-\alpha)^k(\gamma\alpha + (1-\gamma)(1-\alpha) - \frac{1}{2})\right]_{\alpha_b}^{\alpha_g} = \left[\frac{1}{2}\alpha^k(1-\alpha)^k(2\gamma-1)\,(2\alpha-1)\right]_{\alpha_b}^{\alpha_g}.$$

There is a unique k^* such that $z(k^*) = 0$; expanding the expression above and taking logs yields

$$k^* = \ln\left(\frac{2\alpha_b - 1}{2\alpha_g - 1}\right) \Big/ \ln\left(\frac{\alpha_g(1-\alpha_g)}{\alpha_b(1-\alpha_b)}\right).$$

Furthermore, note that

$$z'(k) = \left[\frac{1}{2}\alpha^k(1-\alpha)^k \ln(\alpha(1-\alpha))(2\gamma-1)\,(2\alpha-1)\right]_{\alpha_b}^{\alpha_g}.$$

Since $\alpha_g > \alpha_b > \frac{1}{2}$, we must have $\alpha_g(1-\alpha_g) < \alpha_b(1-\alpha_b)$, implying that $z'(k^*) < 0$. By continuity and the fact that $z(\cdot)$ has a unique root, we must have $z(k) > 0$ for all $k < k^*$ and $z(k) < 0$ for all $k > k^*$. Of course, this implies that $\delta(2k+1) > 0$ for all $k < k^*$ and $\delta(2k+1) < 0$ for all $k > k^*$. ■

PROOF OF LEMMA 1. Fix any incentive compatible direct mechanism $\{\chi_h(\theta, s^T), \chi_l(\theta, s^T)\}$ with payoff Π, and define the alternative mechanism $\{x_h(s^T), x_l(s^T)\}$ by

$$x_r(s^T) := \chi_r(g, s^T) \text{ for all } r \in \{h, l\} \text{ and all } s^T \in \{h, l\}^T.$$

Denote by $\mu(\tilde{s}^T | s^T, \sigma)$ the probability that an agent who observes signals s^T and follows strategy $\sigma \in \Sigma$ reports the sequence $\tilde{s}^T$, where Σ is the set of all dynamic reporting strategies adapted to the signal process (as defined in Section 1.2). The principal's payoff from $\{x_h(\cdot), x_l(\cdot)\}$ is then

$$\Pi' = \frac{1}{2} \sup_{\sigma^g \in \Sigma} \left\{ \sum_{(r, s^T)} \Pr(r, s^T | g) \sum_{\tilde{s}^T} \mu(\tilde{s}^T | s^T, \sigma^g) \chi_r(g, \tilde{s}^T) \right\}$$
$$- \frac{1}{2} \sup_{\sigma^b \in \Sigma} \left\{ \sum_{(r, s^T)} \Pr(r, s^T | b) \sum_{\tilde{s}^T} \mu(\tilde{s}^T | s^T, \sigma^b) \chi_r(g, \tilde{s}^T) \right\}.$$

Note, however, that incentive compatibility of the original mechanism implies that the type-g agent finds truthful reporting of signals to be optimal, implying that

$$\Pi' = \frac{1}{2} \sum_{(r, s^T)} \Pr(r, s^T | g) \chi_r(g, s^T) - \frac{1}{2} \sup_{\sigma^b \in \Sigma} \left\{ \sum_{(r, s^T)} \Pr(r, s^T | b) \sum_{\tilde{s}^T} \mu(\tilde{s}^T | s^T, \sigma^b) \chi_r(g, \tilde{s}^T) \right\}.$$

In addition, incentive compatibility of the original mechanism implies that forcing the type-b agent to misreport his initial type and then re-optimize reduces his expected utility; this implies that

$$\Pi' \geq \frac{1}{2} \sum_{(r, s^T)} \Pr(r, s^T | g) \chi_r(g, s^T) - \frac{1}{2} \sum_{(r, s^T)} \Pr(r, s^T | b) \chi_r(b, s^T) =: \Pi.$$

Thus, since the principal's objective is *decreasing* in the utility of the type-b agent, the new mechanism $\{x_h(\cdot), x_l(\cdot)\}$ improves the principal's payoff. As $\{\chi_h(\theta, \cdot), \chi_l(\theta, \cdot)\}$ was an arbitrary incentive compatible mechanism, it is without loss to restrict attention to mechanisms that solicit only the agent's signals and in which the type-g agent is incentivized to report truthfully. ■

PROOF OF LEMMA 2. Trivial contracts are trivially incentive compatible: if the hiring decision does not depend on the agent's reports, then there is no incentive for the agent (of either type) to misreport any of his signals.

So fix any nontrivial deterministic and incentive-compatible contract $x_h, x_l : \{h, l\}^T \to \{0, 1\}$. Incentive-compatibility and nontriviality of this contract imply that there is no sequence of signals $s^T \in \{h, l\}^T$ such that $x_h(s^T) = x_l(s^T) = 1$; if there were such a sequence, then the agent would always have an incentive to report it and guarantee his hiring (unless the contract were an "always hire" trivial contract). Similarly, there is no sequence $s^T \in \{h, l\}^T$ such that $x_h(s^T) = x_l(s^T) = 0$; if there were such a sequence, then agent would never be willing to report it truthfully (unless the contract were a "never hire" trivial contract).

Note that, by backward induction, there must be some latest period $T' \leq T$ and history of reports $\hat{s}^{T'-1} \in \{h, l\}^{T'-1}$ such that the agent's period-T' report is pivotal; that is,

$$(x_h(\hat{s}^{T'-1}, h, \cdot), x_l(\hat{s}^{T'-1}, h, \cdot)) \neq (x_h(\hat{s}^{T'-1}, l, \cdot), x_l(\hat{s}^{T'-1}, l, \cdot)).$$

To see this, start in the period T. If there is no such $\hat{s}^{T-1}$, then the final-period report *never* affects the principal's hiring decision, which must then depend only on the reports from the first $T-1$ periods. Proceeding in this manner yields T' and a $\hat{s}^{T'-1}$. (Note that $T' \geq 1$ since otherwise the contract does not depend on the agent's report, contradicting the assumption that it is nontrivial.) Since periods $T'+1$ through T do not affect the hiring decision, we can without loss overload notation and describe the contract as two functions $x_h, x_l : \{h, l\}^{T'} \to \{0, 1\}$.

As argued above, nontriviality and incentive compatibility imply that

$$(x_h(\hat{s}^{T'-1}, h), x_l(\hat{s}^{T'-1}, h)), (x_h(\hat{s}^{T'-1}, l), x_l(\hat{s}^{T'-1}, l)) \in \{(1,0), (0,1)\}.$$

Since, by construction, we know that $(x_h(\hat{s}^{T'-1}, h), x_l(\hat{s}^{T'-1}, h)) \neq (x_h(\hat{s}^{T'-1}, l), x_l(\hat{s}^{T'-1}, l))$, it must then be the case that

$$(x_h(\hat{s}^{T'-1}, h), x_l(\hat{s}^{T'-1}, h)) = (1,0) \text{ and } (x_h(\hat{s}^{T'-1}, l), x_l(\hat{s}^{T'-1}, l)) = (0,1).$$

This follows from incentive compatibility, and the fact that the agent's posterior beliefs are such that $\Pr(r = h | \hat{s}^{T'-1}, h) > \Pr(r = h | \hat{s}^{T'-1}, l)$. Further, these beliefs must be such that

$$\Pr(r = h | \hat{s}^{T'-1}, h) \geq \frac{1}{2} \text{ and } \Pr(r = h | \hat{s}^{T'-1}, h) \leq \frac{1}{2},$$

as otherwise the pivotality of the period-T' report following history $\hat{s}^{T'-1}$ would lead to a violation of incentive compatibility.

Now consider any other history $\tilde{s}^{T'} \in \{h, l\}^{T'}$. Non-triviality and incentive compatibility again imply that $(x_h(\tilde{s}^{T'}), x_l(\tilde{s}^{T'})) \in \{(1,0), (0,1)\}$. We claim that we must have $(x_h(\tilde{s}^{T'}), x_l(\tilde{s}^{T'})) = (1,0)$ whenever $\Pr(r = h | \tilde{s}^{T'}) > \frac{1}{2}$ and $(x_h(\tilde{s}^{T'}), x_l(\tilde{s}^{T'})) = (0,1)$ whenever $\Pr(r = h | \tilde{s}^{T'}) < \frac{1}{2}$. To see why this must be true, suppose the contrary and note that this must yield a violation of incentive compatibility. In particular, consider the alternative agent strategy of always reporting $\hat{s}^{T'-1}$ in the first $T'-1$ periods regardless of his true signals, and then choosing a period-t report that matches his posterior belief; that is, he reports h if $\Pr(r = h | \tilde{s}^{T'}) > \frac{1}{2}$, l if $\Pr(r = h | \tilde{s}^{T'}) < \frac{1}{2}$, and chooses arbitrarily if $\Pr(r = h | \tilde{s}^{T'}) = \frac{1}{2}$. Such a strategy increases the agent's payoff over truthful reporting as it guarantees that the agent is hired precisely at the outcome he thinks more likely (whereas truthful reporting may lead to being hired only in the less likely state).

Finally, note that $\Pr(r = h | \tilde{s}^{T'}) > \frac{1}{2}$ if, and only if, $\sum_{\tau \leq T'} \mathbb{1}_h(s_\tau) > \frac{T'}{2}$. Thus, the (arbitrarily-chosen) nontrivial deterministic and incentive-compatible contract x_h, x_l is equivalent to a period-T' prediction mechanism. Therefore, *any* deterministic nontrivial and incentive-compatible contract is a period-t prediction mechanism for some $1 \leq t \leq T$. ■

PROOF OF THEOREM 3. Recall that Lemma 1 establishes that it is without loss to consider only mechanisms that induce the type-g agent to report her signals truthfully (and allowing the type-b agent to optimally misreport). Therefore, Lemma 2 greatly simplifies the class of mechanisms over which the principal must optimize. In particular, the principal must either abandon screening entirely (that is, employ a trivial mechanism, which yields a payoff of zero) or employ a period-t prediction mechanism for some $1 \leq t \leq T$.

Of course, Theorem 2 showed that the principal's payoff, within this class of mechanisms, is increasing in t until reaching a maximum at some $\overline{T}$. Therefore, the optimal deterministic mechanism is a period-$\widetilde{T}$ prediction mechanism, where $\widetilde{T} := \min\{T, \overline{T}\}$. ■

PROOF OF THEOREM 4. The result follows immediately from the argument in the main text. ■

PROOF OF LEMMA 3. Recall from the proof of Theorem 1 that we can write the principal's problem when the agent's signals are observable as

$$\max_{x(\cdot)} \left\{ \frac{1}{2} \sum_{k=0}^{T} \binom{T}{k} \Delta_{k,T} x(k) \right\},$$

where $x(k)$ denote the principal's hiring decision when she observes k signals that match the public outcome, and where

$$\Delta_{k,T} := \Big[\gamma \alpha^k (1-\alpha)^{T-k} + (1-\gamma)\alpha^{T-k}(1-\alpha)^k \Big]_{\alpha_b}^{\alpha_g}.$$

The solution to this linear program depends entirely on the signs of $\Delta_{k,T}$. We now focus on signing these terms when $T = 3$:

- $\Delta_{0,3}|_{\gamma=1} = \big[(1-\alpha)^3\big]_{\alpha_b}^{\alpha_g} < 0$ and $\Delta_{0,3}|_{\gamma=\frac{1}{2}} = \big[\frac{1}{2}(\alpha^3 + (1-\alpha)^3)\big]_{\alpha_b}^{\alpha_g} > 0$;
- $\Delta_{1,3}|_{\gamma=1} = \big[\alpha(1-\alpha)^2\big]_{\alpha_b}^{\alpha_g} < 0$ and $\Delta_{1,3}|_{\gamma=\frac{1}{2}} = \big[\frac{1}{2}\alpha(1-\alpha)\big]_{\alpha_b}^{\alpha_g} < 0$;
- $\Delta_{2,3}|_{\gamma=1} = \big[\alpha^2(1-\alpha)\big]_{\alpha_b}^{\alpha_g}$ is ambiguously signed (it may be positive or negative), while $\Delta_{2,3}|_{\gamma=\frac{1}{2}} = \big[\frac{1}{2}\alpha(1-\alpha)\big]_{\alpha_b}^{\alpha_g} < 0$; and
- $\Delta_{3,3}|_{\gamma=1} = \big[\alpha^3\big]_{\alpha_b}^{\alpha_g} > 0$ and $\Delta_{3,3}|_{\gamma=\frac{1}{2}} = \big[\frac{1}{2}(\alpha^3 + (1-\alpha)^3)\big]_{\alpha_b}^{\alpha_g} > 0$.

Since $\Delta_{k,T}$ is linear in γ, we can unambiguously sign $\Delta_{1,3} < 0$ and $\Delta_{3,3} > 0$; therefore, we must have $x^{FB}(3) = 1$ and $x^{FB}(1) = 0$; the principal always hires the agent when all three of his signals match the realized outcome, and never hires the agent when only one of his signals match the realized outcome.

By the same logic, it is *not* possible to unambiguously sign $\Delta_{0,3}$ and $\Delta_{2,3}$; however, we can characterize the solution x^{FB} for the various feasible sign combinations:

- If $\Delta_{0,3} < 0$ and $\Delta_{2,3} < 0$, then the solution must be such that $x^{FB}(0) = x^{FB}(1) = x^{FB}(2) = 0$ and $x^{FB}(3) = 1$; that is, the principal hires the agent if, and only if, all three of her signals are accurate (so $\overline{n} = 3$ and $\underline{n} = -1$).
- If $\Delta_{0,3} \geq 0$ and $\Delta_{2,3} < 0$, then the solution must be such that $x^{FB}(0) = x^{FB}(3) = 1$ and $x^{FB}(1) = x^{FB}(2) = 0$; that is, the principal hires the agent if, and only if, all three of her signals are consistent (so $\overline{n} = 3$ and $\underline{n} = 0$).
- If $\Delta_{0,3} < 0$ and $\Delta_{2,3} \geq 0$, then the solution must be such that $x^{FB}(0) = x^{FB}(1) = 0$ and $x^{FB}(2) = x^{FB}(3) = 1$; that is, the principal hires the agent if, and only if, a majority (at least two out of three) of her signals are accurate (so $\overline{n} = 2$ and $\underline{n} = -1$).

Note that the fourth possible sign combination (both $\Delta_{0,3} \geq 0$ and $\Delta_{2,3} \geq 0$) is not feasible. To see why, suppose that α_g and α_b are such that $\Delta_{2,3}|_{\gamma=1} \geq 0$ (otherwise, $\Delta_{2,3} < 0$ for all γ and we are done). Thus, as γ goes from $\frac{1}{2}$ to 1, $\Delta_{0,3}$ crosses from positive to negative while $\Delta_{2,3}$ goes from negative to positive. Let γ^* be such that $\Delta_{0,3}|_{\gamma=\gamma^*} = 0$; that is,

$$\gamma^* = \frac{\left[\alpha^3\right]_{\alpha_b}^{\alpha_g}}{\left[(\alpha^3 - (1-\alpha)^3)\right]_{\alpha_b}^{\alpha_g}}.$$

Then

$$\begin{aligned}\Delta_{2,3}|_{\gamma=\gamma^*} &= \gamma^* \left[\alpha^2(1-\alpha)\right]_{\alpha_b}^{\alpha_g} + (1-\gamma^*)\left[\alpha(1-\alpha)^2\right]_{\alpha_b}^{\alpha_g} \\ &= -\frac{(\alpha_g - \alpha_b)(\alpha_g + \alpha_b - 1)[(2\alpha_g - 1)^2 + (2\alpha_b - 1)^2 + (2\alpha_g - 1)(2\alpha_b - 1)]}{(\alpha_g + \alpha_b - 1)^2 + (1-\alpha_g)^2 + (1-\alpha_b)^2 + \alpha_g + \alpha_b} < 0,\end{aligned}$$

where the inequality follows from the fact that $1 > \alpha_g > \alpha_b > \frac{1}{2}$. Therefore, whenever $\Delta_{0,3} \geq 0$ (that is, whenever $\gamma \leq \gamma^*$), we must have $\Delta_{2,3} < 0$.

Thus, the first-best mechanism when $T = 3$ takes on one of the three desired forms. ■

PROOF OF THEOREM 5. We begin by recalling that Lemma 1 shows that it is without loss of generality for the principal to offer a contract of the form $x_r : \{h, l\}^T \to [0, 1]$, $r \in \{h, l\}$, such that the type-g agent is incentivized to report her signals truthfully while the type-b agent is free to misreport optimally. Therefore, letting Σ denote the set of all dynamic reporting strategies that are adapted to the signal process and $\mu(\tilde{s}^T|s^T, \sigma)$ the probability that an agent who observes signals s^T and follows strategy $\sigma \in \Sigma$ reports the sequence $\tilde{s}^T$, we can write the principal's problem as

$$\begin{aligned}&\max_{x_h, x_l} \left\{ \frac{1}{2} \sum_{(r,s^T)} \Pr(r, s^T|\theta = g) x_r(s^T) - \frac{1}{2} \sup_{\sigma^b \in \Sigma} \left\{ \sum_{(r,s^T)} \Pr(r, s^T|\theta = b) \sum_{\tilde{s}^T} \mu(\tilde{s}^T|s^T, \sigma^b) x_r(\tilde{s}^T) \right\} \right\} \qquad (\mathcal{P}) \\ &\text{s.t.} \sum_{(r,s^T)} \Pr(r, s^T|\theta = g) x_r(s^T) \geq \sum_{(r,s^T)} \Pr(r, s^T|\theta = g) \sum_{\tilde{s}^T} \mu(\tilde{s}^T|s^T, \sigma') x_r(\tilde{s}^T) \text{ for all } \sigma' \in \Sigma.\end{aligned}$$

Note that the constraint is simply the type-g agent's incentive compatibility condition, whereas the type-b agent's optimal reporting strategy has been incorporated into the objective function.

We will proceed to the solution of problem ($\mathcal{P}$) as follows:

- We define a relaxed problem with a restricted set of strategies available to the type-b agent.
- We will then argue that the solution to this relaxed problem features truthful reporting at certain histories by the type-b agent.
- We then incorporate the corresponding incentive compatibility constraints into a further relaxation of the problem, which we then solve.
- Finally, we demonstrate that our proposed solution is indeed feasible in the original problem, in the sense that the strategy we impose on the type-b agent's behavior in the relaxed problem is optimal given the identified solution.

We begin by restricting the set of possible misreports of the type-b agent. Denote by $\widehat{\Sigma} \subset \Sigma$ the set of strategies where, for all $s^3 \in \{h, l\}^3$ and any $s_2', s_3' \in \{h, l\}$,

$$\mu(\tilde{s}^3|s^3, \sigma) > 0 \text{ if, and only if } \begin{cases} \tilde{s}^T = (s_1, s_2, s_3) \text{ and } s_1 = s_2 = s_3, \\ \tilde{s}^T = (s_1, s_2, s_3') \text{ and } s_1 = s_2 \neq s_3, \text{ or} \\ \tilde{s}^T = (s_1, s_2', s_3) \text{ and } s_1 \neq s_2. \end{cases}$$

Thus, any strategy $\sigma \in \widehat{\Sigma}$ reports truthfully at all histories except possibly those where the agent first observes a contradictory signal. With this in hand, define the relaxed problem

$$\max_{x_h,x_l}\left\{\frac{1}{2}\sum_{(r,s^T)}\Pr(r,s^T|\theta=g)x_r(s^T)-\frac{1}{2}\sup_{\sigma^b\in\widehat{\Sigma}}\left\{\sum_{(r,s^T)}\Pr(r,s^T|\theta=b)\sum_{\tilde{s}^T}\mu(\tilde{s}^T|s^T,\sigma^b)x_r(\tilde{s}^T)\right\}\right\}$$
$$\text{s.t.}\sum_{(r,s^T)}\Pr(r,s^T|\theta=g)x_r(s^T)\geq\sum_{(r,s^T)}\Pr(r,s^T|\theta=g)\sum_{\tilde{s}^T}\mu(\tilde{s}^T|s^T,\sigma')x_r(\tilde{s}^T)\text{ for all }\sigma'\in\Sigma. \tag{$\mathcal{R}$}$$

CLAIM. *The solution to the relaxed problem ($\mathcal{R}$) yields the principal a higher payoff than the original problem ($\mathcal{P}$).*

PROOF OF CLAIM. Consider any solution x_r^* to problem ($\mathcal{P}$). Since $\widehat{\Sigma}\subset\Sigma$, we must have

$$\sup_{\sigma^b\in\widehat{\Sigma}}\left\{\sum_{(r,s^T)}\Pr(r,s^T|\theta=b)\sum_{\tilde{s}^T}\mu(\tilde{s}^T|s^T,\sigma^b)x_r^*(\tilde{s}^T)\right\}$$
$$\leq\sup_{\sigma^b\in\Sigma}\left\{\sum_{(r,s^T)}\Pr(r,s^T|\theta=b)\sum_{\tilde{s}^T}\mu(\tilde{s}^T|s^T,\sigma^b)x_r^*(\tilde{s}^T)\right\},$$

which implies that the maximal payoff from ($\mathcal{P}$) is achievable in ($\mathcal{R}$). $\Diamond$

Now further relax the problem by dropping the incentive compatibility constraints for the type-g agent; that is, consider the problem

$$\max_{x_h,x_l}\left\{\frac{1}{2}\sum_{(r,s^T)}\Pr(r,s^T|\theta=g)x_r(s^T)-\frac{1}{2}\sup_{\sigma^b\in\widehat{\Sigma}}\left\{\sum_{(r,s^T)}\Pr(r,s^T|\theta=b)\sum_{\tilde{s}^T}\mu(\tilde{s}^T|s^T,\sigma^b)x_r(\tilde{s}^T)\right\}\right\} \tag{$\mathcal{R}'$}$$

and note that (since it is less constrained) the solution to ($\mathcal{R}'$) yields the principal a higher payoff than that of ($\mathcal{R}$).

CLAIM. *There is a solution to ($\mathcal{R}'$) such that the type-b agent reports his signals truthfully at all histories.*

PROOF OF CLAIM. Suppose, by way of contradiction, that there is a solution x_r^* to ($\mathcal{R}'$) in which the type-b agent who has observed $s^2=(i,j)$ strictly prefers to misreport $s_2=j$ as $\tilde{s}_2=i$ for some $i,j\in\{h,l\}$ with $i\neq j$.

Since the preference is strict, it must be the case that the expected probability of being hired after reporting one of the sequences (i,j,i) or (i,j,j) is strictly less than one (otherwise, the agent would optimally report the second signal j truthfully). This implies, however, that only the type-g agent (who always reports truthfully in ($\mathcal{R}'$)) ever reports sequences (i,j,i) and (i,j,j). Therefore, the alternative hiring rule x_r^{**} defined by

$$x_r^{**}(\hat{s}^T):=\min\left\{x_r^*(\hat{s}^T)+\varepsilon\mathbb{1}_{\{(i,j,i),(i,j,j)\}}(\hat{s}^T),1\right\}$$

for sufficiently small $\varepsilon>0$ strictly increases the probability that the principal hires the type-g agent without influencing the strategy of the type-b agent. This, of course, increases the principal's payoff, contradicting the assumption that x_r^* solves ($\mathcal{R}'$).

An identical argument applies when $s^3 = (i,i,j)$. (Note that this argument can be applied separately across these two types of sequences since compound misreports are ruled out in $\widehat{\Sigma}$.) $\Diamond$

This argument implies that, instead of incorporating the type-b agent's problem into the objective function as in ($\mathcal{R}'$), we can instead incorporate the *solution* (truthful reporting) to that problem while also imposing the requisite incentive compatibility constraints. Thus, ($\mathcal{R}'$) is equivalent to

$$\begin{aligned} &\max_{x_h,x_l} \left\{ \frac{1}{2} \sum_{(r,s^T)} \Pr(r,s^T|\theta=g) x_r(s^T) - \frac{1}{2} \sum_{(r,s^T)} \Pr(r,s^T|\theta=b) x_r(s^T) \right\} \\ &\text{s.t.} \sum_{(r,s^T)} \Pr(r,s^T|\theta=g) x_r(s^T) \geq \sum_{(r,s^T)} \Pr(r,s^T|\theta=b) \sum_{\tilde{s}^T} \mu(\tilde{s}^T|s^T,\sigma') x_r(\tilde{s}^T) \text{ for all } \sigma' \in \widehat{\Sigma}. \end{aligned} \tag{$\mathcal{R}''$}$$

Since this relaxed problem is separable in histories conditioned on the agent's first signal (as we have assumed truthful reporting of the first signal), we can solve the problem separately for each of the two cases $s_1 \in \{h,l\}$. Formally, when the first signal is $s_1 = h$, we can write ($\mathcal{R}''$) as

$$\begin{aligned} \max_{x_r} &\left\{ \begin{array}{l} \Delta_{3,3} x_h(h,h,h) + \Delta_{2,3}[x_h(h,h,l) + x_h(h,l,h) + x_l(h,l,l)] \\ + \Delta_{1,3}[x_h(h,l,l) + x_l(h,h,l) + x_l(h,l,h)] + \Delta_{0,3} x_l(h,h,h) \end{array} \right\} \\ \text{s.t. } &\beta_{2,3,b} x_h(h,l,h) + \beta_{1,3,b} x_l(h,l,h) + \beta_{1,3,b} x_h(h,l,l) + \beta_{2,3,b} x_l(h,l,l) \\ &\qquad \geq \beta_{2,3,b} x_h(h,h,h) + \beta_{1,3,b} x_l(h,h,h) + \beta_{1,3,b} x_h(h,h,l) + \beta_{2,3,b} x_l(h,h,l), \\ &\beta_{2,3,b} x_h(h,h,l) + \beta_{1,3,b} x_l(h,h,l) \geq \beta_{2,3,b} x_h(h,h,h) + \beta_{1,3,b} x_l(h,h,h). \end{aligned} \tag{$\mathcal{R}''_h$}$$

CLAIM. *Suppose $\Delta_{2,3} \geq 0$. Then the solution to ($\mathcal{R}''_h$) is given by*

$$\begin{aligned} &x_h(h,h,h) = x_h(h,h,l) = 1, x_l(h,h,h) = x_l(h,h,l) = 0, \\ &x_h(h,l,h) = x_l(h,l,l) = 1, x_l(h,l,h) = x_h(h,l,l) = 0. \end{aligned} \tag{A.1}$$

PROOF OF CLAIM. We will proceed by showing that there exist multipliers λ and μ corresponding to the two incentive compatibility constraints in ($\mathcal{R}''_h$) such that the conjectured solution (A.1) satisfies the Karush-Kuhn-Tucker conditions. These conditions may be written as:

$$x_h(h,h,h): \quad \Delta_{3,3} - \lambda\beta_{2,3,b} - \mu\beta_{2,3,b} \geq 0 \tag{A.2}$$

$$x_h(h,h,l): \quad \Delta_{2,3} - \lambda\beta_{1,3,b} + \mu\beta_{2,3,b} \geq 0 \tag{A.3}$$

$$x_l(h,h,h): \quad \Delta_{0,3} - \lambda\beta_{1,3,b} - \mu\beta_{1,3,b} \leq 0 \tag{A.4}$$

$$x_l(h,h,l): \quad \Delta_{1,3} - \lambda\beta_{2,3,b} + \mu\beta_{1,3,b} \leq 0 \tag{A.5}$$

$$x_h(h,l,h), x_l(h,l,l): \quad \Delta_{2,3} + \lambda\beta_{2,3,b} \geq 0 \tag{A.6}$$

$$x_l(h,l,h), x_h(h,l,l): \quad \Delta_{1,3} + \lambda\beta_{1,3,b} \leq 0 \tag{A.7}$$

The directions of the inequalities above are determined by the feasibility constraint that each variable $x_r(\cdot)$ lies between 0 and 1.

Note that, at the conjectured solution, the first constraint (corresponding to period-two incentive compatibility) reduces to

$$\beta_{2,3,b} \geq \beta_{1,3,b}.$$

Of course, this inequality holds strictly, and so the constraint is slack. Therefore, we must have

$$\lambda = 0.$$

In addition, recall (from the proof of Lemma 3), that $\Delta_{3,3} > 0 > \Delta_{1,3}$ and that $\Delta_{0,3} < 0$ whenever $\Delta_{2,3} \geq 0$ (as was assumed). Therefore, it is easy to see that choosing

$$\mu = 0$$

leads to the satisfaction of all the KKT conditions above—which are, of course, both necessary and sufficient for the linear program ($\mathcal{R}_h''$). ◊

CLAIM. *Suppose $\Delta_{2,3} < 0$. Then the solution to ($\mathcal{R}_h''$) is given by*

$$\begin{aligned} &x_h(h,h,h) = x_h(h,h,l) = 1, x_l(h,h,h) = x_l(h,h,l) = 0, \\ &x_h(h,l,h) = x_l(h,l,l) = \frac{\beta_{1,3,b} + \beta_{2,3,b}}{2\beta_{2,3,b}}, x_l(h,l,h) = x_h(h,l,l) = 0. \end{aligned} \tag{A.8}$$

PROOF OF CLAIM. We will proceed by showing that there exist multipliers λ and μ corresponding to the two incentive compatibility constraints in ($\mathcal{R}_h''$) such that the conjectured solution (A.8) satisfies the Karush-Kuhn-Tucker conditions. These conditions may be written as:

$$x_h(h,h,h): \qquad \Delta_{3,3} - \lambda\beta_{2,3,b} - \mu\beta_{2,3,b} \geq 0 \tag{A.9}$$

$$x_h(h,h,l): \qquad \Delta_{2,3} - \lambda\beta_{1,3,b} + \mu\beta_{2,3,b} \geq 0 \tag{A.10}$$

$$x_l(h,h,h): \qquad \Delta_{0,3} - \lambda\beta_{1,3,b} - \mu\beta_{1,3,b} \leq 0 \tag{A.11}$$

$$x_l(h,h,l): \qquad \Delta_{1,3} - \lambda\beta_{2,3,b} + \mu\beta_{1,3,b} \leq 0 \tag{A.12}$$

$$x_h(h,l,h), x_l(h,l,l): \qquad \Delta_{2,3} + \lambda\beta_{2,3,b} = 0 \tag{A.13}$$

$$x_l(h,l,h), x_h(h,l,l): \qquad \Delta_{1,3} + \lambda\beta_{1,3,b} \leq 0 \tag{A.14}$$

The directions of the inequalities above are determined by the feasibility constraint that each variable $x_r(\cdot)$ lies between 0 and 1.

Note first that (A.13) implies that (since $\Delta_{2,3} < 0$) we must have

$$\lambda = -\frac{\Delta_{2,3}}{\beta_{2,3,b}} > 0.$$

Substituting this value into (A.14) yields

$$\beta_{2,3,b}\Delta_{1,3} \leq \beta_{1,3,b}\Delta_{2,3},$$

which is easily verified to hold. In addition, we can rewrite (A.9) and (A.12) as

$$\mu \leq \frac{\Delta_{3,3} + \Delta_{2,3}}{\beta_{2,3,b}} \text{ and } \mu \leq -\frac{\Delta_{2,3} + \Delta_{1,3}}{\beta_{1,3,b}},$$

respectively. Clearly, choosing

$$\mu = \min\left\{\frac{\Delta_{3,3} + \Delta_{2,3}}{\beta_{2,3,b}}, -\frac{\Delta_{2,3} + \Delta_{1,3}}{\beta_{1,3,b}}\right\}$$

satisfies both of these conditions. It remains to be shown that this choice of μ satisfies (A.10) and (A.11).

So suppose first that $\mu = \frac{\Delta_{3,3}+\Delta_{2,3}}{\beta_{2,3,b}} \leq -\frac{\Delta_{2,3}+\Delta_{1,3}}{\beta_{1,3,b}}$. Then we can rewrite (A.10) as

$$0 \leq \Delta_{2,3} + \frac{\Delta_{2,3}}{\beta_{2,3,b}}\beta_{1,3,b} + \frac{\Delta_{3,3}+\Delta_{2,3}}{\beta_{2,3,b}}\beta_{2,3,b} = \frac{(\alpha_g - \alpha_b)(2\gamma-1)(1-\gamma+(2\gamma-1)\alpha_b + \alpha_g(1-\alpha_g))}{1-\gamma+(2\gamma-1)\alpha_b},$$

and likewise rewrite (A.11) as

$$0 \geq \Delta_{0,3} + \frac{\Delta_{2,3}}{\beta_{2,3,b}}\beta_{1,3,b} - \frac{\Delta_{3,3}+\Delta_{2,3}}{\beta_{2,3,b}}\beta_{1,3,b} = -\frac{(\alpha_g - \alpha_b)(2\gamma-1)((\alpha_g-\alpha_b)^2 + 3\alpha_b(1-\alpha_b))}{1-\gamma+(2\gamma-1)\alpha_b}.$$

It is straightforward to see that both of these inequalities hold $1 \geq \alpha_g > \alpha_b \geq \frac{1}{2}$ and $1 > \gamma > \frac{1}{2}$.

On the other hand, suppose that $\mu = -\frac{\Delta_{2,3}+\Delta_{1,3}}{\beta_{1,3,b}} \leq \frac{\Delta_{3,3}+\Delta_{2,3}}{\beta_{2,3,b}}$. Then we can rewrite (A.10) as

$$-\frac{\Delta_{2,3} + \frac{\beta_{1,3,b}}{\beta_{2,3,b}}\Delta_{2,3}}{\beta_{2,3,b}} \leq \frac{\Delta_{2,3}+\Delta_{1,3}}{\beta_{1,3,b}}.$$

Note, however, that (A.14) implies that $-\frac{\beta_{1,3,b}}{\beta_{2,3,b}}\Delta_{2,3} \leq -\Delta_{1,3}$. Therefore, since (as is simple to verify) $\beta_{1,3,b} < \beta_{2,3,b}$, this inequality is satisfied. Finally, we can write (A.11) as

$$0 \geq \Delta_{0,3} + \frac{\Delta_{2,3}}{\beta_{2,3,b}}\beta_{1,3,b} + \frac{\Delta_{2,3}+\Delta_{1,3}}{\beta_{1,3,b}}\beta_{1,3,b} = \frac{(\alpha_b - \alpha_g)(2\gamma-1)((1-\alpha_g)^2 + (\alpha_g-\alpha_b) + \gamma(2\alpha_b-1))}{1-\gamma+(2\gamma-1)\alpha_b}.$$

Again, the inequality is satisfied since $1 > \alpha_g > \alpha_b > \frac{1}{2}$ and $1 > \gamma > \frac{1}{2}$.

Thus, the conjectured solution, along with λ and μ as defined above, satisfy the KKT conditions. Of course, these conditions are both necessary and sufficient for the linear program ($\mathcal{R}''_h$). $\diamondsuit$

Finally, it remains to be shown that the conjectured solutions to ($\mathcal{R}''_h$) above solve the unrelaxed problem ($\mathcal{P}$). Note that the original problem ($\mathcal{P}$) imposes incentive compatibility constraints on the type-g agent while the relaxed problem assumed truthful reporting; likewise, the original problem allowed the type-b agent to optimally misreport while the relaxed problem imposed incentive compatibility constraints on two histories and assumed truthful reporting at the others. Therefore, it suffices to show that the conjectured behavior in the relaxed problem is indeed optimal in the unrelaxed one.

CLAIM. *Suppose the principal chooses the either of the mechanisms described in* (A.1) *or* (A.8). *Then it is optimal for the agent to always report her private signals truthfully.*

PROOF OF CLAIM. We begin by noting that the solution in (A.1) corresponds to a period-3 prediction mechanism, as it deterministically hires the agent if a majority of his reported signals match the eventual outcome. Lemma 2 then immediately implies that this mechanism induces truthful reporting for both the type-g and type-b agents.

We now turn to the solution in (A.8), which can be implemented by offering the agent the option in period two to either make a prediction immediately (and be hired, if correct, with probability 1) or to make a prediction in period three (and be hired, if correct, with probability $\rho := \frac{\beta_{1,3,b}+\beta_{2,3,b}}{2\beta_{2,3,b}} < 1$). Note that there is an onto mapping from the set of signal-reporting strategies to the set of prediction strategies in this option implementation. In particular, truthful reporting of signals in (A.8) corresponds to making a sincere prediction in period two if both signals match, and otherwise making a sincere prediction in period three. Hence, showing that this conjectured behavior is optimal for the agent is sufficient for showing the optimality of truthful signal signal reporting in (A.8).

To see why this behavior is optimal for the agent, note first that observing two matching signals in periods one and two yields the agent enough information to make a prediction in period three: regardless of whether the third signal matches or not, he will make the same prediction. Since $\rho < 1$, a period-2 prediction yields the agent a strictly higher payoff than postponing. On the other hand, suppose that the agent has observed a pair of mismatched signals in the first two periods, leaving him with a uniform posterior over states. This implies that an early prediction (of either h or l) yields the type-θ agent an expected payoff of

$$\frac{1}{2}\gamma + \frac{1}{2}(1-\gamma) = \frac{1}{2}.$$

Postponing the prediction to period three (and then making a sincere prediction that follows the third private signal) yields the type-θ agent an expected payoff of

$$(\gamma\alpha_\theta + (1-\gamma)(1-\alpha_\theta))\,\rho = \frac{1}{2}\left(\frac{\gamma\alpha_\theta + (1-\gamma)(1-\alpha_\theta)}{\gamma\alpha_b + (1-\gamma)(1-\alpha_b)}\right).$$

Clearly, a type-b agent with mixed signals in period two is indifferent about delay, whereas a type-g agent with mixed signals in period two strictly prefers to delay his prediction since $\alpha_g > \alpha_b$. This implies that the mechanism in (A.8) is incentive compatible for both types of the agent.[18] $\lozenge$

Thus, the assumed behavior for the agent in the relaxed problem ($\mathcal{R}''$) is in fact a best response to the principal's proposed mechanism. This implies that the conjectured solution indeed solves the original problem ($\mathcal{P}$). ■

PROOF OF THEOREM 6. Recall from the proof of Theorem 2 that both the principal and agent (of either type) are indifferent between the $(2k-1)$-period and $2k$-period prediction mechanisms; therefore, assume without loss that $\overline{T}$ is odd, and let $\bar{k}$ be such that $\overline{T} = 2\bar{k}+1$ (and therefore, since $T > \overline{T}+1$, we have $T \geq 2\bar{k}+3$).[19]

CLAIM. $\Delta_{\bar{k}+2,2\bar{k}+3} < 0$.

PROOF OF CLAIM. Recall from the proof of Theorem 2 that we defined $\delta(n)$ to be the difference between the principal's expected payoff from an n-period and an $(n-1)$-period prediction mechanism. Since $\overline{T}$ is the optimal length for a prediction mechanism, Theorem 2 implies that $0 > \delta(\overline{T}+2) = \delta(2\bar{k}+3) = \delta(2(\bar{k}+1)+1)$.

However, the second claim in that proof showed that

$$\begin{aligned}\delta(2(\bar{k}+1)+1) &= \binom{2(\bar{k}+1)}{\bar{k}+1}\left[\alpha^{\bar{k}+1}(1-\alpha)^{\bar{k}+1}\left(\gamma\alpha + (1-\gamma)(1-\alpha) - \frac{1}{2}\right)\right]_{\alpha_b}^{\alpha_g} \\ &= \binom{2(\bar{k}+1)}{\bar{k}+1}\left[\gamma\alpha^{\bar{k}+2}(1-\alpha)^{\bar{k}+1} + (1-\gamma)\alpha^{\bar{k}+1}(1-\alpha)^{\bar{k}+2}\right]_{\alpha_b}^{\alpha_g}\end{aligned}$$

[18] Since the private signals and the public outcome are (positively) correlated with the underlying state, insincere predictions (that is, those that contradict the agent's private signals) are clearly dominated.

[19] Note that the argument that follows applies immediately to $\overline{T}$ even, so the proposed bound $T > \overline{T}+1$ continues to be sufficient for the optimality of randomization in that case.

$$- \binom{2(\bar{k}+1)}{\bar{k}+1} \frac{1}{2} \left[\gamma\alpha^{\bar{k}+1}(1-\alpha)^{\bar{k}+1} + (1-\gamma)\alpha^{\bar{k}+1}(1-\alpha)^{\bar{k}+1} \right]_{\alpha_b}^{\alpha_g}$$
$$= \binom{2(\bar{k}+1)}{\bar{k}+1} \left(\Delta_{\bar{k}+2,2\bar{k}+3} - \frac{1}{2}\Delta_{\bar{k}+1,2\bar{k}+2} \right).$$

But $\Delta_{\bar{k}+1,2\bar{k}+2} = (\alpha_g(1-\alpha_g))^{\bar{k}+1} - (\alpha_b(1-\alpha_b))^{\bar{k}+1} < 0$ since $\alpha_g > \alpha_b > \frac{1}{2}$. Therefore, to avoid contradicting the fact that $\delta(\overline{T}+2) < 0$, we must have $\Delta_{\bar{k}+2,2\bar{k}+3} < 0$. ◊

Now consider the alternative mechanism defined by

$$\hat{x}_r(s^T) := \begin{cases} 1 & \text{if } \sum_{\tau=1}^{2\bar{k}+2} \mathbb{1}_r(s_\tau) \geq \bar{k}+2, \\ \rho & \text{if } \sum_{\tau=1}^{2\bar{k}+2} \mathbb{1}_r(s_\tau) = \bar{k}+1 \text{ and } s_{2\bar{k}+3} = r, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\rho := \frac{\beta_{1,3,b} + \beta_{2,3,b}}{2\beta_{2,3,b}} = \frac{1}{2(\gamma\alpha_b + (1-\gamma)(1-\alpha_b))}$$

is the same probability as in the 3-period optimal stochastic mechanism described in Theorem 5. Essentially, $\hat{x}_r(\cdot)$ does not solicit any information from the agent until period $2\bar{k}+2$. At that point, it offers the agent the option of either making an immediate prediction in period $2\bar{k}+2$ or waiting one period until $2\bar{k}+3$ to make a prediction. The agent is hired with probability 1 if his early prediction is correct, with probability ρ if the late prediction is correct, and with probability 0 if his prediction is incorrect.

Clearly, if the agent has at least $\bar{k}+2$ identical signals in the first $2\bar{k}+2$ periods, he will continue to have a strict majority of that signal in period $2\bar{k}+3$; therefore, his recommendation will be the same in both periods, but delaying lowers the probability of being hired if the recommendation is correct. Therefore, such an agent will choose to make an immediate prediction in period $2\bar{k}+2$.

On the other hand, an agent with exactly $\bar{k}+1$ of each signal in period $2\bar{k}+2$ would prefer to wait until the next period before making a prediction. Notice that ρ is chosen to leave the type-b agent indifferent between guessing immediately and waiting for one additional signal, while (since $\alpha_g > \alpha_b$) the type-g agent's more informative signal gives him a strict incentive to delay.

Thus, it remains to be shown that the stochastic mechanism $\hat{x}_r(\cdot)$ defined above yields the principal a higher payoff than the period-$\overline{T}$ prediction mechanism.

As shown in the proof of Theorem 2, the principal's payoff of the period-$\overline{T}$ prediction mechanism (for $\overline{T} = 2\bar{k}+1$) equals that of the period-$(2\bar{k}+2)$ prediction mechanism. In that latter mechanism, the agent is hired with probability 1 when he observes at least $\bar{k}+2$ signals that match the outcome, with probability $\frac{1}{2}$ when he observes exactly $\bar{k}+1$ signals that match the outcome, and with probability 0 otherwise.

Therefore, the difference in the principal's payoff between $\hat{x}_r$ and the period-$\overline{T}$ prediction mechanism arises precisely from the situation where the agent has observed exactly $\bar{k}+1$ of each signal by period $2\bar{k}+2$, and therefore chooses to postpone predicting under $\hat{x}_r$. This leads to a payoff differential of

$$\binom{2\bar{k}+2}{\bar{k}+1} \left(\rho\Delta_{\bar{k}+2,2\bar{k}+3} - \frac{1}{2}\Delta_{\bar{k}+1,2\bar{k}+2} \right),$$

since there are exactly $\binom{2\bar{k}+2}{\bar{k}+1}$ signal sequences that lead the agent to be exactly tied in $2\bar{k}+2$ periods. The deterministic prediction mechanism will hire the agent with probability $\frac{1}{2}$ if he is exactly tied (due to the agent mixing when indifferent), whereas $\hat{x}_r$ hires the agent with probability ρ if the final signal matches (yielding a net payoff of $\rho\Delta_{\bar{k}+2,2\bar{k}+3}$).

Note, however, that

$$\begin{aligned}
\rho\Delta_{\bar{k}+2,2\bar{k}+3} - \frac{1}{2}\Delta_{\bar{k}+1,2\bar{k}+2} &= \frac{\Delta_{\bar{k}+2,2\bar{k}+3}}{2(\gamma\alpha_b + (1-\gamma)(1-\alpha_b))} - \frac{1}{2}\Delta_{\bar{k}+1,2\bar{k}+2} \\
&= \frac{[\gamma\alpha^{\bar{k}+2}(1-\alpha)^{\bar{k}+1} + (1-\gamma)\alpha^{\bar{k}+1}(1-\alpha)^{\bar{k}+2}]_{\alpha_b}^{\alpha_g}}{2(\gamma\alpha_b + (1-\gamma)(1-\alpha_b))} - \frac{1}{2}\Delta_{\bar{k}+1,2\bar{k}+2} \\
&= \frac{\left[(\alpha(1-\alpha))^{\bar{k}+1}(\gamma\alpha + (1-\gamma)(1-\alpha))\right]_{\alpha_b}^{\alpha_g}}{2(\gamma\alpha_b + (1-\gamma)(1-\alpha_b))} - \frac{1}{2}\left[(\alpha(1-\alpha))^{\bar{k}+1}\right]_{\alpha_b}^{\alpha_g} \\
&= \frac{1}{2}(\alpha_g(1-\alpha_g))^{\bar{k}+1}\left(\frac{\gamma\alpha_g + (1-\gamma)(1-\alpha_g)}{\gamma\alpha_b + (1-\gamma)(1-\alpha_b)} - 1\right) > 0.
\end{aligned}$$

Therefore, the mechanism $\hat{x}_r$ defined above, which nontrivially randomizes in $2\bar{k}+3$ periods, achieves a strictly higher revenue than the optimal deterministic mechanism, a period-$(2\bar{k}+1)$ recommendation mechanism. ∎

PROOF OF THEOREM 7. Note first that the revelation principle (see Lemma B.1 in our supplementary Appendix B.1) implies that—when the principal has commitment power—it is without loss of generality to restrict attention to incentive compatible direct mechanisms $\chi : \Lambda \times S^T \to \{0,1\}^2$, where we write $\chi(\cdot) = (\chi_h(\cdot), \chi_l(\cdot)) \in \{0,1\}^2$ for the principal's hiring decision given outcomes h and l, respectively.

So fix any nontrivial and incentive compatible direct mechanism χ, and note that we must have $\chi(\lambda, s^T) \in \{(1,0),(0,1)\}$ for all $(\lambda, s^T) \in \Lambda \times S^T$. Note that if $\chi(\hat{\lambda}, \hat{s}^T) = (1,1)$ for some reports $\hat{\lambda}, \hat{s}^T$, then incentive compatibility requires that the agent is *always* hired, regardless reports (otherwise he would deviate by always reporting $\hat{\lambda}, \hat{s}^T$). Similarly, if $\chi(\hat{\lambda}, \hat{s}^T) = (0,0)$ for some $(\hat{\lambda}, \hat{s}^T)$, then incentive compatibility requires that the agent is *never* hired, regardless of his reports (otherwise he would deviate by never reporting $\hat{\lambda}, \hat{s}^T$).

Now fix any $\lambda \in \Lambda$, and let t_λ be the largest period such that $\chi(\lambda, \cdot)$ is measurable with respect to the first t_λ reports; that is, $\chi(\lambda, s^T) = \chi(\lambda, \hat{s}^T)$ for all $s^T = (s^{t_\lambda}, s_{t_\lambda+1}, \ldots, s_T)$ and $\hat{s}^T = (s^{t_\lambda}, \hat{s}_{t_\lambda+1}, \ldots, \hat{s}_T)$ that coincide in their first t_λ periods. Since periods $t_\lambda + 1$ through T do not affect the hiring decision given an initial period report of λ, we abuse notation somewhat and write $\chi(\lambda, s^{t_\lambda})$ to denote the principal's hiring rule.

The definition of t_λ as the final period in which the agent's reported signal potentially changes the hiring decision as a function of the ultimate outcomes implies the existence of $\hat{s}, \hat{s}' \in S_{t_\lambda}$ and $\hat{s}^{t_\lambda - 1} \in \prod_{\tau=1}^{t_\lambda} S_\tau$ such that

$$\chi(\lambda, \hat{s}^{t_\lambda-1}, \hat{s}) = (1,0) \neq (0,1) = \chi(\lambda, \hat{s}^{t_\lambda-1}, \hat{s}').$$

Moreover, we must have

$$\Pr(r = h|\lambda, \hat{s}^{t_\lambda-1}, \hat{s}) \geq \frac{1}{2} \geq \Pr(r = h|\lambda, \hat{s}^{t_\lambda-1}, \hat{s}');$$

if this did not hold, the pivotality of the period-t_λ report following history $(\lambda, \hat{s}^{t_\lambda - 1})$ would lead to a violation of incentive compatibility.

This implies that the agent who initially observes signal $\lambda \in \Lambda$ has a strategy which guarantees that he is always hired at the outcome he thinks more likely in period t_λ: simply report $\hat{s}^{t_\lambda - 1}$ regardless of signals seen in the first $t_\lambda - 1$ periods, and then report either $\hat{s}$ or $\hat{s}'$ in period t_λ based on his true signals and his posterior expectation of the most likely outcome. Therefore, incentive compatibility implies that the continuation mechanism $\chi(\lambda, \cdot)$ must be payoff equivalent (for an agent who initially observes signal $\lambda \in \Lambda$) to making a prediction at period t_λ.

Finally, note that the signal structure is such that the agent, regardless of his initial private signal, weakly prefers to make a prediction as late as possible. Therefore, by incentive compatibility of the initial signal report, it must be the case that the agent observing $\lambda \in \Lambda$ is always ex ante indifferent between being asked to make a prediction in t_λ or in $t^* := \max_{\lambda' \in \Lambda}\{t_{\lambda'}\}$. As a result, the principal is indifferent between offering the direct mechanism χ or a period-t^* prediction mechanism.

To see that this outcome (and hence payoffs) remains implementable in the game without commitment, note that if the principal ignores all reports of the agent except that in period t^* (hiring if, and only if, the period-t^* prediction matches the ultimate outcome), it is a best response by the agent to babble in all periods except t^*. Of course, this babbling justifies the principal ignoring the reports in those periods. Meanwhile, hiring the agent after a correct period-t^* prediction is also sequentially rational for the principal; if it were not, then the mechanism's payoff in the full commitment model would be negative, contradicting its optimality. Thus, as in Theorem 4 for the baseline model, the lack of commitment does not change the outcomes or payoffs. ■

Appendix B. Supplementary Proofs for Online Publication

B.1. *A Revelation Principle for Deterministic Dynamic Mechanisms*

A deterministic mechanism $\mathcal{M}$ in our environment is simply a sequence of message spaces $M_0, M_1, \ldots, M_T$ (where we will let $M^t := \times_{\tau=0}^{t} M_\tau$ denote the set of period-t sequences of messages) and a decision rule $x : M^T \times \{h, l\} \to \{0, 1\}$.

Given a mechanism $\mathcal{M} = (M^T, x)$, the agent's reporting strategy μ is a sequence of rules

$$\mu_t : \theta \times S^t \times M^{t-1} \to \Delta(M_t),$$

where we write

$$\mu_t(m_t | \theta, s_1, \ldots, s_t, m_0, m_1, \ldots, m_{t-1})$$

to denote the probability of sending message $m_t \in M_t$ when the agent's private information is (θ, s^t) and she has already sent messages m^{t-1}. (Note that, as in any sequential game, the agent's strategy must specify the messages that she sends in some period t even after sequences of messages m^{t-1} that are not in the support of her strategy.)

A mechanism is a *direct mechanism* if $M_0 = \Theta$ and $M_t = S$ for all $t = 1, \ldots, T$.

Lemma B.1. *Consider an equilibrium μ of a game induced by a deterministic mechanism $\mathcal{M} = (M^T, x)$. Then there exists a deterministic direct mechanism $\widehat{\mathcal{M}} = (\theta \times S^T, \chi)$ that induces an equilibrium $\hat{\mu}$ with truthful revelation. Moreover, the principal's expected payoff under $\hat{\mu}$ in $\widehat{\mathcal{M}}$ is (weakly) greater than her expected payoff under μ in $\mathcal{M}$.*[20]

Proof. Consider a deterministic mechanism $\mathcal{M} = (M^T, x)$ and equilibrium reporting strategy μ.

Fix an arbitrary period $t \in \{0, 1, \ldots, T\}$, and let $\lambda_t := (\theta, s^t)$ denote the agent's period-t (private) history of type and signals. For each $\lambda_t \in \Lambda_t := \Theta \times S^t$ and each $m^{t-1} \in M^{t-1}$, define

$$M_t^{\lambda_t, m^{t-1}} := \left\{ m \in M_T | \mu_t(m_t | \lambda_t, m^{t-1}) > 0 \right\}$$

to be the set of equilibrium period-t messages sent by the agent with positive probability when her private type is λ_t and she has already reported messages m^{t-1}. Note that, by definition of equilibrium, it must therefore be the case that

$$\sum_{\substack{r \in \{h,l\} \\ s_{t+1}^T \in S^{T-t} \\ m_{t+1}^T \in M_{t+1} \times \cdots \times M_T}} \Pr(r, s_{t+1}^T | \lambda_t) \begin{bmatrix} \mu_{t+1}(m_{t+1} | (\lambda_t, s_{t+1}), (m^{t-1}, m_t)) \\ \quad \times \cdots \times \\ \qquad \mu_T(m_T | (\lambda_t, s_{t+1}^T), (m^{t-1}, m_t, m_{t+1}^T)) \end{bmatrix} x(m^{t-1}, m_t, m_{t+1}^T, r)$$

$$\geq \sum_{\substack{r \in \{h,l\} \\ s_{t+1}^T \in S^{T-t} \\ m_{t+1}^T \in M_{t+1} \times \cdots \times M_T}} \Pr(r, s_{t+1}^T | \lambda_t) \begin{bmatrix} \mu_{t+1}(m_{t+1} | (\lambda_t, s_{t+1}), (m^{t-1}, m'_t)) \\ \quad \times \cdots \times \\ \qquad \mu_T(m_T | (\lambda_t, s_{t+1}^T), (m^{t-1}, m'_t, m_{t+1}^T)) \end{bmatrix} x(m^{t-1}, m'_t, m_{t+1}^T, r)$$

[20]This result extends Strausz's (2003) deterministic revelation principle (in terms of payoffs) for a single agent to our dynamic environment.

for all $m_t \in M_t^{\lambda_t, m^{t-1}}$ and $m_t' \in M_t$, and where the above holds with equality when $m_t' \in M_t^{\lambda_t, m^{t-1}}$.

So define $\widehat{M}_t^{\lambda_t, m^{t-1}}$ to be the set of all messages that yield the principal her highest payoff from type λ_t among the messages that are sent with positive probability in equilibrium; that is,

$$\widehat{M}_t^{\lambda_t, m^{t-1}} := \underset{m_t' \in M_t^{\lambda_t, m^{t-1}}}{\operatorname{argmax}} \left\{ \sum_{\substack{r \in \{h,l\} \\ s_{t+1}^T \in S^{T-t} \\ m_{t+1}^T \in M_{t+1} \times \cdots \times M_T}} \Pr(r, s_{t+1}^T | \lambda_t) \left[\begin{array}{l} \mu_{t+1}(m_{t+1} | (\lambda_t, s_{t+1}), (m^{t-1}, m_t')) \\ \quad \times \cdots \times \\ \quad\quad \mu_T(m_T | (\lambda_t, s_{t+1}^T), (m^{t-1}, m_t', m_{t+1}^T)) \end{array} \right] \right. \\ \left. x(m^{t-1}, m_t', m_{t+1}^T, r) \left[\mathbb{1}_g(\theta) - \mathbb{1}_b(\theta) \right] \right\}.$$

With this in hand, define the mechanism $\mathcal{M}' := (M_0, \ldots, M_{t-1}, \Lambda_t, M_{t+1}, \ldots, M_T, x')$, where for all $m^{t-1} \in M^{t-1}$ and all $\lambda_t \in \Lambda_t$, we let

$$x'(m^{t-1}, \lambda_t, m_{t+1}^T, r) := x(m^{t-1}, \hat{m}_t^{\lambda_t, m^{t-1}}, m_{t+1}^T, r) \text{ for an arbitrary } \hat{m}_t^{\lambda_t, m^{t-1}} \in \widehat{M}_t^{\lambda_t, m^{t-1}}.$$

Thus, the (also deterministic) mechanism $\mathcal{M}'$ is identical to $\mathcal{M}$ in all periods except period t, where the agent is asked to report her entire private history up to that point; the mechanism then "translates" the reported private history into its corresponding principal-optimal period-t message chosen by the equilibrium μ. Since μ is an equilibrium reporting strategy in mechanism $\mathcal{M}$, then the strategy μ' defined by

$$\mu_\tau'(m_\tau | \theta, s^\tau, m^{\tau-1}) := \mu_\tau(m_\tau | \theta, s^\tau, m^{\tau-1}) \text{ for all } \tau < t;$$

$$\mu_t'(\lambda_t | \theta, s^t, m^{t-1}) := \begin{cases} 1 & \text{if } \lambda_t = (\theta, s^t), \\ 0 & \text{otherwise;} \end{cases} \text{ and}$$

$$\mu_\tau'(m_\tau | \theta, s^\tau, (m^{t-1}, \lambda_t, m_{t+1}^\tau)) := \mu_\tau(m_\tau | \theta, s^\tau, (m^{t-1}, \hat{m}_t^{\lambda_t, m^{t-1}}, m_{t+1}^\tau)) \text{ for all } \tau > t,$$

is by construction an equilibrium reporting strategy in mechanism $\mathcal{M}'$. (Note that μ' is identical to μ for all period $\tau < t$; optimally reports the private history truthfully in period t, which corresponds to an optimal message from μ; and follows the equilibrium continuation play of μ after any period-t report, truthful or otherwise.) Moreover, the agent's expected payoff is unchanged, while the principal's payoff is (weakly) higher in the equilibrium μ' of the new mechanism $\mathcal{M}'$.

Note, however, that the period t that we chose above was entirely arbitrary. Therefore, we can define a new (and still determinstic) mechanism $\mathcal{M}'' := (\Lambda_0, \Lambda_1, \ldots, \Lambda_T, x'')$ by iteratively applying the procedure above $T+1$ times, starting in the final period T and working backwards until we reach period 0. Note, however, that the message spaces induced by this iterative procedure contain some redundancy, in that the agent is asked to re-report her *entire* private history each period. However, the procedure above also generates a truthful equilibrium μ'' in which the agent *truthfully* re-reports that history in each period; this implies that (in equilibrium) misreports occur with zero probability.

Thus, we define a dynamic *direct* mechanism $\widehat{\mathcal{M}} := (\Theta, S, \ldots, S, \hat{x})$ in which the agent is asked to report only her *new* private information in each period and the decision rule $\hat{x}$ is defined by

$$\hat{x}(\theta, s^T, r) := x''((\theta), (\theta, s^1), (\theta, s^2), \ldots, (\theta, s^T), r) \text{ for all } (\theta, s^T) \in \Theta \times S^T \text{ and } r \in \{h, l\}.$$

Note that since the iterative procedure above preserves the deterministic nature of the decision rule, $\widehat{\mathcal{M}}$ is also deterministic; in addition, since the set of reporting strategies under $\widehat{\mathcal{M}}$ is a subset of those in $\mathcal{M}''$ (but still contains the equilibrium strategy of truthful reporting after all possible histories), the new direct mechanism $\widehat{\mathcal{M}}$ is incentive compatible. This also implies that the agent's payoff is the same as in the original mechanism $\mathcal{M}$, while the principal's payoff under $\widehat{\mathcal{M}}$ is (weakly) greater. ■

B.2. *Optimal Static Mechanism*

PROOF OF THEOREM 8. Before proceeding, note that Lemma 1 applies immediately in this setting with a single period-T report.

CLAIM. *It is without loss of generality to consider contracts such that $x_r(s^T) = x_r(\hat{s}^T)$ for all $s^T, \hat{s}^T$ such that $\sum_t \mathbb{1}_h(s_t) = \sum_t \mathbb{1}_h(\hat{s}_t)$.*

PROOF OF CLAIM. Suppose there exists some $\tilde{s}^T, \hat{s}^T \in \{h, l\}^T$ with $\sum_t \mathbb{1}_h(\tilde{s}_t) = \sum_t \mathbb{1}_h(\hat{s}_t)$ but $x_r(\tilde{s}^T) \neq x_r(\hat{s}^T)$ for some $r \in \{h, l\}$. Since signals are conditionally i.i.d., the agent has identical posterior beliefs $q_\theta = \Pr(\omega = H | \tilde{s}^T, \theta) = \Pr(\omega = H | \hat{s}^T, \theta)$ about the underlying state of the world after observing $\tilde{s}^T$ or $\hat{s}^T$.

Since the contract must be incentive compatible for the type-g agent, he must prefer reporting $\tilde{s}^T$ truthfully to misreporting $\tilde{s}^T$ as $\hat{s}^T$, implying

$$\begin{aligned} q_g(\gamma x_h(\tilde{s}^T) + (1-\gamma) x_l(\tilde{s}^T)) \\ + (1-q_g)(\gamma x_l(\tilde{s}^T) + (1-\gamma) x_h(\tilde{s}^T)) \end{aligned} \geq \begin{aligned} q_g(\gamma x_h(\hat{s}^T) + (1-\gamma) x_l(\hat{s}^T)) \\ + (1-q_g)(\gamma x_l(\hat{s}^T) + (1-\gamma) x_h(\hat{s}^T)). \end{aligned}$$

The agent must also prefer reporting $\hat{s}^T$ truthfully to misreporting $\hat{s}^T$ as $\tilde{s}^T$, implying

$$\begin{aligned} q_g(\gamma x_h(\hat{s}^T) + (1-\gamma) x_l(\hat{s}^T)) \\ + (1-q_g)(\gamma x_l(\hat{s}^T) + (1-\gamma) x_h(\hat{s}^T)) \end{aligned} \geq \begin{aligned} q_g(\gamma x_h(\tilde{s}^T) + (1-\gamma) x_l(\tilde{s}^T)) \\ + (1-q_g)(\gamma x_l(\tilde{s}^T) + (1-\gamma) x_h(\tilde{s}^T)). \end{aligned}$$

Of course, these two inequalities jointly imply that the type-g agent with belief q_g is indifferent between reporting $\tilde{s}^T$ or $\hat{s}^T$.

So consider the alternative mechanism $\{\hat{x}_h(\cdot), \hat{x}_l(\cdot)\}$ defined by, for $r = h, l$,

$$\hat{x}_r(s^T) := \begin{cases} x_r(\tilde{s}^T) & \text{if } s^T = \hat{s}^T \\ x_r(s^T) & \text{otherwise.} \end{cases}$$

Thus, $\{\hat{x}_h(\cdot), \hat{x}_l(\cdot)\}$ simply "deletes" the option of reporting as $\hat{s}^T$ and replaces it by the report of $\tilde{s}^T$. Since the original mechanism $\{x_h(\cdot), x_l(\cdot)\}$ was incentive compatible for the type-g agent and the type-g agent who observed $\hat{s}^T$ was indifferent between the two reports, $\{\hat{x}_h(\cdot), \hat{x}_l(\cdot)\}$ is also incentive compatible for the type-g agent. Moreover, $\{\hat{x}_h(\cdot), \hat{x}_l(\cdot)\}$ leaves the ex ante expected payoff of the type-g agent unchanged.

Meanwhile, the type-b agent's ex ante expected payoff is (weakly) lower under $\{\hat{x}_h(\cdot), \hat{x}_l(\cdot)\}$ than under $\{x_h(\cdot), x_l(\cdot)\}$ since there is one fewer potential report available to him. Since the principal's payoff is increasing in U_g and decreasing in U_b, this implies that $\{\hat{x}_h(\cdot), \hat{x}_l(\cdot)\}$ (weakly) raises the principal's expected payoff. ◊

With this property in hand, we abuse notation somewhat and write $x_r(n)$ to denote $x_r(s^T)$, where $n = \sum_t \mathbb{1}_h(s_t)$. We also write $q_\theta(n)$ to denote the associated posterior belief $\Pr(\omega = H | s^T, \theta)$.

CLAIM. *It is without loss of generality to consider symmetric contracts in which $x_h(n) = x_l(T-n)$ for all $n = 0, 1, \ldots, T$.*

PROOF OF CLAIM. Fix any contract $\{x_h(\cdot), x_l(\cdot)\}$ that is incentive compatible for the type-g agent, and define the alternative contract $\{\hat{x}_h(\cdot), \hat{x}_l(\cdot)\}$ by

$$\hat{x}_h(n) := x_l(T-n) \text{ and } \hat{x}_l(n) := x_h(T-n) \text{ for all } n = 0, 1, \ldots, T.$$

Then the expected utility of a type-θ agent who observes s^T with $\sum_t \mathbb{1}_h(s_t) = n$ but reports n' is

$$\begin{aligned}
\widehat{U}_\theta(n'|n) &= q_\theta(n)(\gamma \hat{x}_h(n') + (1-\gamma)\hat{x}_l(n')) + (1 - q_\theta(n))(\gamma \hat{x}_l(n') + (1-\gamma)\hat{x}_h(n')) \\
&= q_\theta(n)(\gamma x_l(T-n') + (1-\gamma) x_h(T-n')) \\
&\quad + (1 - q_\theta(n))(\gamma x_h(T-n') + (1-\gamma) x_l(T-n')) \\
&= (1 - q_\theta(T-n))(\gamma x_l(T-n') + (1-\gamma) x_h(T-n')) \\
&\quad + q_\theta(T-n)(\gamma x_h(T-n') + (1-\gamma) x_l(T-n')) \\
&= U_\theta(T-n'|T-n).
\end{aligned}$$

Letting $\sigma^\theta(\cdot)$ denote type-θ's optimal strategy under the original mechanism $\{x_h(\cdot), x_l(\cdot)\}$, this implies that type-θ's optimal reporting strategy $\hat{\sigma}^\theta(\cdot)$ under the new contract $\{\hat{x}_h(\cdot), \hat{x}_l(\cdot)\}$ is

$$\hat{\sigma}^\theta(n) = T - \sigma^\theta(T-n).$$

In particular, the type-g incentive compatibility of the original mechanism (that is, $\sigma^g(n) = n$ for all n) implies that $\hat{\sigma}^g(n) = n$ for all n. Moreover, the symmetry of the signal distributions implies that the agent's expected utility (conditional on quality) is the same across both mechanisms (that is, $\widehat{U}_g = U_g$ and $\widehat{U}_b = U_b$), so the principal's expected payoff is

$$\widehat{\Pi} := \frac{1}{2}\widehat{U}_g - \frac{1}{2}\widehat{U}_b = \frac{1}{2}U_g - \frac{1}{2}U_b.$$

Now define the (symmetric) mechanism $\{\overline{x}_h(\cdot), \overline{x}_l(\cdot)\}$ by

$$\overline{x}_r(n) := \frac{x_r(n) + \hat{x}_r(n)}{2} \text{ for all } n.$$

Then the expected utility of a type-θ agent who observes s^T with $\sum_t \mathbb{1}_h(s_t) = n$ but reports n' is

$$\begin{aligned}
\overline{U}_\theta(n'|n) &= q_\theta(n)(\gamma \overline{x}_h(n') + (1-\gamma)\overline{x}_l(n')) + (1 - q_\theta(n))(\gamma \overline{x}_l(n') + (1-\gamma)\overline{x}_h(n')) \\
&= \frac{1}{2}U_\theta(n'|n) + \frac{1}{2}\widehat{U}_\theta(n'|n).
\end{aligned}$$

Since $U_g(n|n) \geq U_g(n'|n)$ and $\widehat{U}_g(n|n) \geq \widehat{U}_g(n'|n)$ for all $n, n' \in \{0, 1, \ldots, T\}$, it must also be the case that $\overline{U}_g(n|n) \geq \overline{U}_g(n'|n)$ for all n and n'; that is, this new symmetric mechanism is type-g incentive compatible. This also implies that the type-g expected utility is unchanged, so $\overline{U}_g = U_g$. On the other hand, note that

$$\overline{U}_b := \sum_n \Pr(n|\theta = b) \sup_{n'} \{\overline{U}_b(n'|n)\}$$

$$= \sum_n \Pr(n|\theta = b) \sup_{n'} \left\{ \frac{1}{2} U_b(n'|n) + \frac{1}{2} \widehat{U}_b(n'|n) \right\}$$
$$\leq \sum_n \Pr(n|\theta = b) \sup_{n'} \left\{ \frac{1}{2} U_b(n'|n) \right\} + \sum_n \Pr(n|\theta = b) \sup_{n'} \left\{ \frac{1}{2} \widehat{U}_b(n'|n) \right\}$$
$$= \frac{1}{2} U_b + \frac{1}{2} \widehat{U}_b = U_b.$$

Thus, the new symmetric mechanism leaves the type-g agent's expected utility unchanged while decreasing that of the type-b agent, thereby increasing the principal's payoff. $\Diamond$

We now move to an equivalent posterior-space setting where, instead of focusing on the signals received by an agent, we consider the posterior beliefs induced by those signals. (Note that this is equivalent due to the two lemmas above as well as the one-to-one mapping between the number of h signals and the agent's posterior belief.) We denote the agent's posterior beliefs that the state of the world is $\omega = H$ by $q \in [0,1]$, and let F_θ denote the distribution of type-θ's posterior beliefs.

CLAIM. *The distributions F_θ are symmetric about $\frac{1}{2}$; that is, $F_\theta(q) = 1 - F_\theta(1-q)$ for all $q \in [0,1]$ and $\theta \in \{g, b\}$. In addition, the type-g agent puts more mass on extremal posteriors than the type-b agent, so $F_g(q) \geq F_b(q)$ for all $q \in (0, \frac{1}{2})$.*

PROOF OF CLAIM. To see that the distributions are symmetric, note that the symmetry of the signal-generating process implies that, for all $n = 0, \ldots, T$, it is equally likely for the number of h signals observed by the agent to equal n or to equal $T - n$; moreover, it is straightforward to show that $q_\theta(n) = 1 - q_\theta(T-n)$.

To see that the second property holds, note that the probability an agent with signal precision α observes signals s^T with $n \leq \sum_t \mathbb{1}_h(s_t) \leq T - n$ is

$$\pi(n, T, \alpha) = \sum_{k=n}^{T-n} \binom{T}{k} \left[\frac{1}{2} \alpha^k (1-\alpha)^{T-k} + \frac{1}{2} \alpha^{T-k} (1-\alpha)^k \right]$$
$$= \frac{1}{2} \sum_{k=n}^{T-n} \binom{T}{k} \alpha^k (1-\alpha)^{T-k} + \frac{1}{2} \sum_{k=n}^{T-n} \binom{T}{T-k} \alpha^{T-k} (1-\alpha)^k = \sum_{k=n}^{T-n} \binom{T}{k} \alpha^k (1-\alpha)^{T-k}.$$

Note that

$$\frac{\partial \pi(n, T, \alpha)}{\alpha} = \sum_{k=n}^{T-n} \binom{T}{k} \alpha^{k-1} (1-\alpha)^{T-k-1} (k - T\alpha)$$
$$= \sum_{k=n}^{T-n} \binom{T}{k} k \alpha^{k-1} (1-\alpha)^{T-k-1} - \sum_{k=n}^{T-n} \binom{T}{k} T \alpha^k (1-\alpha)^{T-k-1}$$
$$= \sum_{k=n}^{T-n} \binom{T-1}{k-1} T \alpha^{k-1} (1-\alpha)^{T-k-1} - \sum_{k=n}^{T-n} \binom{T}{k} T \alpha^k (1-\alpha)^{T-k-1}$$
$$= \frac{T}{1-\alpha} \left(\sum_{k=n}^{T-n} \binom{T-1}{k-1} \alpha^{k-1} (1-\alpha)^{T-k} - \sum_{k=n}^{T-n} \binom{T}{k} \alpha^k (1-\alpha)^{T-k} \right)$$
$$= \frac{T}{1-\alpha} \left(\sum_{k=n-1}^{T-n-1} \binom{T-1}{k} \alpha^k (1-\alpha)^{T-k-1} - \pi(n, T, \alpha) \right).$$

Now recall that $\pi(n, T, \alpha)$ is the probability of observing between n and $T-n$ signals equal to h. There are three possible ways in which this event can occur:

- $\sum_{t=1}^{T-1} \mathbb{1}_h(s_t) = n-1$ and $s_T = h$, occurring with probability $\alpha\binom{T-1}{n-1}\alpha^{n-1}(1-\alpha)^{T-n}$;
- $n \leq \sum_{t=1}^{T-1} \mathbb{1}_h(s_t) \leq T-n-1$, occurring with probability $\sum_{k=n}^{T-n-1}\binom{T-1}{k}\alpha^k(1-\alpha)^{T-k-1}$; or
- $\sum_{t=1}^{T-1} \mathbb{1}_h(s_t) = T-n$ and $s_T = l$, occurring with probability $(1-\alpha)\binom{T-1}{T-n}\alpha^{T-n}(1-\alpha)^{n-1}$.

Since $\pi(n, T, \alpha)$ is the sum of these three probabilities, we can rewrite the expression above as

$$\begin{aligned}\frac{\partial \pi(n,T,\alpha)}{\alpha} &= \frac{T}{1-\alpha}\Bigg(\sum_{k=n-1}^{T-n-1}\binom{T-1}{k}\alpha^k(1-\alpha)^{T-k-1} - \alpha\binom{T-1}{n-1}\alpha^{n-1}(1-\alpha)^{T-n} \\ &\qquad\qquad - \sum_{k=n}^{T-n-1}\binom{T-1}{k}\alpha^k(1-\alpha)^{T-k-1} - (1-\alpha)\binom{T-1}{T-n}\alpha^{T-n}(1-\alpha)^{n-1}\Bigg) \\ &= \frac{T}{1-\alpha}\left((1-\alpha)\binom{T-1}{n-1}\alpha^{n-1}(1-\alpha)^{T-n} - (1-\alpha)\binom{T-1}{T-n}\alpha^{T-n}(1-\alpha)^{n-1}\right) \\ &= T\binom{T-1}{n-1}\left(\alpha^{n-1}(1-\alpha)^{T-n} - \alpha^{T-n}(1-\alpha)^{n-1}\right).\end{aligned}$$

It is easy to see that this expression is negative whenever $\alpha \geq \frac{1}{2}$ and $n \leq \frac{T}{2}$, thereby implying that the type-g agent is less likely to observe an "intermediate" number of h signals than the type-b agent; that is, since $\alpha_g > \alpha_b$, the type-g agent is more likely to observe extremal numbers of h signals than the type-b agent.

Finally, note that $q_g(n) \leq q_b(n)$ for $n \leq \frac{T}{2}$ and $q_g(n) \geq q_b(n)$ for $n \geq \frac{T}{2}$; therefore, the posteriors induced by these more extremal signals are themselves more extreme. This implies $F_g(q) \geq F_b(q)$ for all $q \in (0, \frac{1}{2})$ and $F_g(q) \leq F_b(q)$ for all $q \in (\frac{1}{2}, 1)$, as desired. ◊

So now consider the principal's problem in this setting. Applying our results above and treating the agent's posterior as his type, the principal offers a mechanism $\{x_h(q), x_l(q)\}$ that must be incentive compatible for the type-g agent.

With this in mind, let $U_\theta(q'|q)$ denote the expected payoff of an agent who is of type θ, has posterior q, and reports q':

$$U_\theta(q'|q) := (\gamma q + (1-\gamma)(1-q))x_h(q') + ((1-\gamma)q + \gamma(1-q))x_h(1-q').$$

Note that U_θ is, in fact, independent of the agent's type θ; this implies that whenever the mechanism is incentive compatible for the type-g agent, it will also be incentive compatible for the type-b agent. Combining this observation with the symmetry property derived above (which implies $x_h(q) = x_l(1-q)$ for all q), we write the agent's (type-independent) indirect utility as

$$\begin{aligned}U(q'|q) &= (\gamma q + (1-\gamma)(1-q))x_h(q') + ((1-\gamma)q + \gamma(1-q))x_h(1-q') \\ &= ((2\gamma-1)q + (1-\gamma))x_h(q') + (\gamma - (2\gamma-1)q)x_h(1-q') \\ &= ((2\gamma-1)q - \gamma)(x_h(q') - x_h(1-q')) + x_h(q').\end{aligned}$$

The principal's problem is then to

$$\max_{x_h}\left\{\int_0^1 U(q|q)d[F_g(q) - F_b(q)]\right\} \text{ s.t. } U(q|q) \geq U(q'|q) \text{ for all } q, q' \in [0,1].$$

The incentive compatibility constraint implies that we must have both $U(q|q) \geq U(q'|q)$ and $U(q'|q') \geq U(q|q')$ for all $q, q' \in [0,1]$. Summing these incentive constraints yields

$$(2\gamma - 1)(q - q')\left[(x_h(q) - x_h(1-q)) - (x_h(q') - x_h(1-q'))\right] \geq 0.$$

This implies that $x_h(q) - x_h(1-q)$ must be nondecreasing in q, which in addition implies that $x_h(q) - x_h(1-q) \geq 0$ for all $q \geq \frac{1}{2}$.

The standard "sandwich" arguments can be used to further characterize incentive compatible mechanisms. Letting $U^*(q) := U(q|q)$ for all q, we have

$$U^*(q) \geq U^*(q') + (2\gamma - 1)(q - q')(x_h(q') - x_h(1-q')).$$

Reversing the roles of q and q' above and summing the resulting inequalities yields

$$(2\gamma - 1)(q - q')(x_h(q') - x_h(1-q')) \leq U^*(q) - U^*(q') \leq (2\gamma - 1)(q - q')(x_h(q) - x_h(1-q)).$$

Since $-1 \leq x_h(q) - x_h(1-q) \leq 1$, $U^*(q)$ is Lipschitz continuous. In addition, $x_h(q) - x_h(1-q)$ is monotone and therefore continuous almost everywhere, and so $U^*(q)$ is differentiable almost everywhere. Applying the Envelope Theorem,

$$\frac{dU^*(q)}{dq} = (2\gamma - 1)(x_h(q) - x_h(1-q))$$

at every point of continuity of $x_h(q) - x_h(1-q)$ (which is almost everywhere).

We now integrate the principal's objective function by parts. (This is proper since U^* is absolutely continuous and the distribution functions F_θ are monotone.) Note that

$$\begin{aligned}
\int_0^1 U^*(q)d[F_g(q) - F_b(q)] &= \left[U^*(q)(F_g(q) - F_b(q))\right]_0^1 - \int_0^1 \frac{dU^*(q)}{dq}(F_g(q) - F_b(q))dq \\
&= -\int_0^1 (x_h(q) - x_h(1-q))(F_g(q) - F_b(q))dq \\
&= -2\int_{\frac{1}{2}}^1 (x_h(q) - x_h(1-q))(F_g(q) - F_b(q))dq,
\end{aligned}$$

where the final step follows from the symmetry of the distributions about $\frac{1}{2}$.

Recall that $F_g(q) - F_b(q) \leq 0$ for all $q \geq \frac{1}{2}$; therefore, since $x_h(q) - x_h(1-q)$ is constrained by feasibility to lie within $[-1, 1]$, the objective function is easily maximized pointwise by setting $x_h(q) - x_h(1-q) = 1$ for all $q > \frac{1}{2}$, yielding the solution

$$x_h^*(q) = x_l^*(1-q) = \begin{cases} 0 & \text{if } q < \frac{1}{2}, \\ \frac{1}{2} & \text{if } q = \frac{1}{2}, \\ 1 & \text{if } q > \frac{1}{2}. \end{cases}$$

It is easy to see (by observation) that this mechanism does indeed satisfy the full set of incentive compatibility constraints, implying that it is indeed optimal. Of course, this is precisely equivalent to a period-T prediction mechanism: after observing all T signals, the agent reports to the principal whether they view state H or state L as more likely, and the agent is hired if (and only if) their prediction matches the principal's signal $r \in \{h, l\}$. ∎

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