(Ir)rational Exuberance: Optimism, Ambiguity and Risk

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Abstract

We propose a rational model of (ir)rational exuberance in asset markets. That is, the behavior of bulls and bears is rational in the standard economic sense of agents maximizing utility subject to a budget constraint, defined by market prices and the agent's income. As observed by Keynes (1930): "The market price will be fixed at the point at which the sales of the bears and the purchases of the bulls are balanced." This equilibration of optimistic and pessimistic beliefs of investors is a consequence of investors maximizing Keynesian utilities subject to budget constraints defined by market prices and the investor's income.

Keynesian Utilities

Keynesian utilities represent the investor's preferences for optimism. Bulls are optimistic and believe that market prices will go up, but bears are pessimistic and believe that market prices will go down. Hence bulls buy long and bears sell short. Keynesian utilities are defined as the composition of the investor's preferences for risk and her preferences for ambiguity, where we assume preferences for risk and preferences for ambiguity are independent.

If $U(x)$ denotes preferences for risk, then U maps state-contingent claims x to state-utility vectors $y = U(x)$. If $J(y)$ denotes preferences for ambiguity, then J maps state-utility vectors $U(x)$ to subjective values $J \circ U(x)$

$$x \longrightarrow J \circ U(x) \text{ is the composition of } U \text{ and } J\text{, denoted } J \circ U(x).$$

Types of Keynesian Utilities

In the following 2×2 contingency table on the types of Keynesian utilities, the rows are ambiguity-averse and ambiguity-seeking preferences and the columns are risk-averse and risk-seeking preferences. The cells are the investor's preferences for optimism and pessimism. The diagonal cells of the table are the symmetric Keynesian utilities and the off-diagonal cells of the table are the asymmetric Keynesian utilities. Bears are pessimistic and have concave Keynesian utilities. Bulls are optimistic and have convex Keynesian utilities

Table 1

Keynesian Preferences	Risk-Averse	Risk-Seeking
Ambiguity-Averse	Bears	Asymmetric
Ambiguity-Seeking	Asymmetric	Bulls

Legendre–Fenchel Conjugates of Keynesian Utilities

.For pessimistic utility functions, we invoke the Legendre–Fenchel biconjugate for concave functions, where

$$J \circ U(x) \equiv \min_{\pi \in R^N_{++}} \left[\sum \pi \cdot x + J^*(\pi)\right]$$

and $J^*(\pi)$ is a smooth concave function on R^N_{++}, the Legendre–Fenchel conjugate of $J \circ U(x)$, where

$$J^*(\pi) \equiv \min_{x \in R^N_{+}} \left[\sum \pi \cdot x + J \circ U(x)\right]$$

For optimistic utility functions, we invoke the Legendre–Fenchel biconjugate for convex functions, where

$$J \circ U(x) \equiv \max_{\pi \in R^N_{++}} \left[\sum \pi \cdot x + J^*(\pi)\right]$$

and $J^*(\pi)$ is a smooth convex function on R^N_{++}, the Legendre–Fenchel conjugate of $J \circ U(x)$, where

$$J^*(\pi) \equiv \max_{x R^N_{+}} \left[\sum \pi \cdot x + J \circ U(x)\right].$$

Monotone Maps and Convex (Concave) Utilities

If $F(y)$ is a vector-valued map from R^N into R^N,then F is strictly, monotone increasing (decreasing) if for all x and $y \in R^N$:

$$[x - y] \cdot [F(x) - F(y)] > 0 \ (< 0)$$

$J \circ U(x)$ is strictly convex (concave) in x iff $\nabla_x J \circ U(x)$ is a strictly, monotone increasing (decreasing) map of x.

Betting Odds for Bears and Bulls

It follows from the envelope theorem, that for bears

$$\begin{aligned}\nabla_x J \circ U(x) &= \arg \max_{\pi \in R^N_{++}} \left[\sum \pi \cdot x + J^*(\pi)\right] = \widehat{\pi}, \text{ where} \\ J \circ U(x) &= \max_{\pi \in R^N_{++}} \left[\sum \pi \cdot x + J^*(\pi)\right] = \sum \widehat{\pi} \cdot x + J^*(\widehat{\pi})\right]\end{aligned}$$

and for bulls

$$\begin{aligned}\nabla_x J \circ U(x) &= \arg \min_{\pi \in R^N_{++}} \left[\sum \pi \cdot x + J^*(\pi)\right] = \widehat{\pi}, \text{ where} \\ J \circ U(x) &= \min_{\pi \in R^N_{++}} \left[\sum \pi \cdot x + J^*(\pi)\right] = \sum \widehat{\pi} \cdot x + J^*(\widehat{\pi})].\end{aligned}$$

The expectations of investors today regarding the payoffs of the state-contingent claim x tomorrow is the normalized marginal subjective value of x :

$$\frac{\nabla_x J \circ U(x)}{\|\nabla_x J \circ U(x)\|_1} = \frac{\widehat{\pi}}{\|\widehat{\pi}\|_1} \in \Delta^0,$$

A Two Parameter Family of Keynesian Utilities

Let $u(x_s) \equiv x_s^{\beta}$ If $\beta \leq 1$, then $u(x_s)$ is concave in x_s. If $\alpha \leq 1$, then $j(u(x_s)) \equiv (u(x_s))^{\alpha}$ is concave in $u(x_s)$. Hence $j \circ u(x_s) \equiv (x_s)^{\beta\alpha}$ is concave in x_s, i.e., $j \circ u(x_s)$ is pessimistic If $\beta \geq 1$, then $u(x_s)$ is convex in x_s.If $\alpha \geq 1$, then $j(u(x_s)) \equiv (u(x_s))^{\alpha}$ is convex in $u(x_s)$. Hence $j \circ u(x_s) \equiv (x_s)^{\beta\alpha}$ is convex in x_s i.e., $j \circ u(x_s)$ is optimistic. Consider the following additively separable utility functions on the space of state-contingent claims $x \equiv (x_1, x_2, ..., x_N)$,where $U(x) \equiv (u(x_1), u(x_2), ..., u(x_N))$:

$$J \circ U(x) \equiv \sum_{s=1}^{s=N} j \circ u(x_s) \text{ where } j \circ u(x_s) \equiv (x_s)^{\beta\alpha}$$

A Quadratic Family of Keynesian Utilities

We propose quadratic specifications of preferences for risk and preferences for ambiguity, defined by scalar proxies for risk and ambiguity: β and α. Concave quadratic utility functions were introduced by Shannon and Zame (2002) in their analysis of indeterminacy in infinite dimension general equilibrium models. $f(x)$ is a concave quadratic function if for all y and z:

$$f(y) < f(z) + \nabla f(z) \cdot (y - z) - \frac{1}{2}K \|y - z\|^2, \text{ where } K > 0.$$

Concave Quadratic Utilities for Risk and Ambiguity

$J \circ U(x)$ is the composition of a smooth, concave quadratic map $U(x)$, where $U(x)$ is a negative definite diagonal $N \times N$ matrix for each $x \in R^N_{++}$ and a smooth, concave quadratic function $J(y)$, where $J : R^N \rightarrow R$. If $u : R_+ \rightarrow R_+$, then

$$U(x) \equiv (u(x_1), u(x_2), ..., u(x_N))$$

is the state-utility vector for the state-contingent claim

$$x = (x_1, x_2, ..., x_N).$$

Gradients of Composite Functions as Hadamard Products

If $z = [z_1, z_2, ..., z_N]$ and $w = [w_1, w_2, ..., w_N]$, then

$$z \cdot w \equiv [z_1 w_1, z_2 w_2, ..., z_N w_N]$$

is the Hadamard or pointwise product of z and w. If we define the gradient of state-utility vector $U(x)$ as the vector

$$\nabla_x U(x) \equiv [\partial u(x_1), \partial u(x_2), ..., \partial u(x_N)]$$

then by the chain rule

$$\nabla_x J \circ U(x) = [\nabla_x U(x)] \cdot [\nabla_{U(x)} J(U(x))].$$

If

$$G(x) = z(x) \cdot w(x),$$

where $z(x)$ and $w(x) \in R^N_{++}$, then Bentler and Lee (1978) state and Magnus and Neudecker (1985) prove that

$$\nabla_x G(x) = \nabla_x z(x) \mathrm{diag}(w(x)) + \nabla_x w(x) \mathrm{diag}(z(x)).$$

The Hessian for Keynesian Utilities

$$\nabla_x^2 J \circ U(x) = \nabla_x([\nabla_x U(x)] \cdot [\nabla_{U(x)} J(U(x)]))$$

$$= [\nabla_{U(x)}^2 J(U(x))](\mathrm{diag}[\nabla_x U(x)])^2 + [\nabla_x^2 U(x)]\mathrm{diag}[\nabla_{U(x)} J(U(x))].$$

If $U(x)$ is a concave quadratic map and $J(y)$ is a convex quadratic function, then

$$\nabla_x^2 U(x) = -\mathrm{diag}(\beta) < 0$$

$$\nabla_y^2 J(y) = \mathrm{diag}(\alpha) > 0.$$

If A and B are diagonal $N \times N$ matrices then $A - B$ is negative semidefinite iff $E \leq F$.

Hence $\nabla_x^2 J \circ U(x)$ is negative semidefinite iff:

$$\mathrm{diag}(\alpha)\mathrm{diag}[\nabla_x U(x)]^2 - \mathrm{diag}(\beta)\mathrm{diag}[\nabla_{U(x)} J(U(x))] \leq 0.$$

Keynesian Utilities for Bulls

Theorem

If $J \circ U(x)$, *is the composition of* $U(x)$ *and* $J(y)$,*where (a)* $(y_1, y_2, ..., y_N) \equiv y = U(x) \equiv (u(x_1), u(x_2), ..., u(x_N))$ *is a monotone, smooth, convex, diagonal quadratic map from* R^N_{++} *onto* R^N_{++} *,with the proxy for risk,* $\beta > 0$*, (b)* $J(y)$ *is a monotone, smooth, convex quadratic function from* R^N_{++} *into* R*,with the proxy for ambiguity,* $\alpha > 0$*, (c)*

$$\nabla^2_x J \circ U(x) = \mathrm{diag}(\alpha)(\mathrm{diag}[\nabla_x U(x)])^2 + \mathrm{diag}(\beta)\mathrm{diag}[\nabla_{U(x)} J(U(x))]$$

then $J \circ U(x)$ *is convex on* R^N_{++}.

The Optimal Investment Problem for Bulls

If the investor's income today is I and she is endowed with convex Keynesian utilities, $W_{Bulls}(x)$, then her optimal investment problem is (P):

$$\max\{W_{Bulls}(x) \mid -x_1 \leq 0, \ -x_2 \leq 0, \ p \cdot x - I \leq 0\}$$

where the Fritz John Lagrangian for constrained maximization

$$L(x_1, x_2, \lambda_0, \lambda_1, \lambda_2, \lambda_3) \equiv \lambda_0 W_{Bulls}(x) - \lambda_1[-x_1] - \lambda_2[-x_2] - \lambda_3[p \cdot x - I].$$

Theorem

[Fritz John]: If x^* is a local maximizer of (P) then there exists multipliers $\lambda^* \equiv (\lambda_0^*, \lambda_1^*, \lambda_2^*, \lambda_3^*) \gneq 0$ such that:

$$\lambda_0^*(\partial_{x_1} W_{Bulls}(x^*), \partial_{x_2} W_{Bulls}(x^*)) = (-\lambda_1^* + \lambda_3^* p_1, -\lambda_2^* + \lambda_3^* p_2),$$

where $\lambda_0^* = 1$, by Theorems 19.12 in Simon and Blume.

Betting Odds of Bulls and Market Odds

If $x^* = (0, x_2^*)$, then

$$\lambda_0^*(\partial_{x_1} W_{Bulls}((0, x_2^*)), \partial_{x_2} W_{Bulls}((0, x_2^*)) = (-\lambda_1^* + \lambda_3^* p_1,\ \lambda_3^* p_2)$$

It follows that some bulls are more optimistic than the market that tomorrow's state of the world is state 2. That is,

$$\frac{\partial_{x_2} W_{Bulls}((0,x_2^*))}{\partial_{x1} W_{Bulls}((0,x_2^*))} = \frac{\lambda_3^* p_2}{-\lambda_1^* + \lambda_3^* p_1} > \frac{p_2}{p_1}.$$

Betting Odds of Bulls and Market Odds [Continued]

If $x^* = (x_1^*, 0)$, then

$$\lambda_0^*(\partial_{x_1} W_{Bulls}((x_1^*, 0)), \partial_{x_2} W_{Bulls}((x_1^*, 0)) = (\lambda_3^* p_1, -\lambda_2^* + \lambda_3^* p_2)$$

It follows that the other bulls are more optimistic than the market that tomorrow's state of the world is state 1:

$$\frac{\partial_{x_1} W_{Bulls}((x_1^*,0))}{\partial_{x_2} W_{Bulls}((x_1^*,0))} = \frac{\lambda_3^* p_1}{-\lambda_2^* + \lambda_3^* p_2} > \frac{p_1}{p_2}.$$

Keynesian Utilities for Bears

Theorem

If $J \circ U(x)$, is the composition of $U(x)$ and $J(y)$,where (a) $(y_1, y_2, ..., y_N) \equiv y = U(x) \equiv (u(x_1), u(x_2), ..., u(x_N))$ is a monotone, smooth, concave, diagonal quadratic map from R^N_{++} onto R^N_{++}, with the proxy for risk, $-\beta < 0$, (b) $J(y)$ is a monotone, smooth,concave quadratic function from R^N_{++} into R, with the proxy for ambiguity, $-\alpha < 0$, (c)

$$\nabla^2_x J \circ U(x) = -\mathrm{diag}(\alpha)(\mathrm{diag}[\nabla_x U(x)])^2 - \mathrm{diag}(\beta)\mathrm{diag}[\nabla_{U(x)} J(U(x))]$$

then $J \circ U(x)$ is concave on R^N_{++}.

The Optimal Investment Problem for Bears

If the investor's income today is I and she is endowed with concave Keynesian utilities $W_{Bears}(x)$, then her optimal investment problem is (P):

$$\max\{W_{Bears}(x) \mid x_1 \geq 0,\ x_2 \geq 0,\ I - p \cdot x \geq 0\}$$

where the KKT Lagrangian for constrained maximization

$$L(x_1, x_2, \lambda) \equiv W_{Bears}(x) + \lambda_3[I - p \cdot x] + \lambda_1 x_1 + \lambda_2 x_2.$$

Theorem

[Karush-Kuhn-Tucker] *If Slater's constraint qualification is satisfied then* x^* *is a maximizer of* (P), *where* $x^* \in R^N_+$, *iff there exists a multipliers* $\lambda^* \equiv (\lambda_3^*, \lambda_1^*, \lambda_2^*) \gneqq 0$ *such that:*

$$(\partial_{x_1} W_{Bears}(x^*), \partial_{x_2} W_{Bears}(x^*)) = (\lambda_3^* p_1 - \lambda_1^*, \lambda_3^* p_2 - \lambda_2^*).$$

Betting Odds of Bears and Market Odds

If $x^* = (0, x_2^*)$, then

$$(\partial_{x_1} W_{Bears}((0, x_2^*)), \partial_{x_2} W_{Bears}((0, x_2^*)) = (\lambda_3^* p_1 - \lambda_1^*, \lambda_3^* p_2) \text{ and}$$

It follows that some bears are more pessimistic than the market that tomorrow's state of the world is state 1. That is,

$$\frac{\partial_{x_1} W_{Bears}((0, x_2^*))}{\partial_{x_2} W_{Bears}((0, x_2^*))} = \frac{\lambda_3^* p_1 - \lambda_1^*}{\lambda_3^* p_2} < \frac{p_1}{p_2}.$$

Betting Odds of Bears and Market Odds [Continued]

If $x^* = (x_1^*, 0)$, then

$$(\partial_{x_1} W_{Bears}((x_1^*, 0)), \partial_{x_2} W_{Bears}((x_1^*, 0)) = (\lambda_3^* p_1, \lambda_3^* p_2 - \lambda_2^*)$$

It follows that the other bears are more pessimistic than the market that tomorrow's state of the world is state 2. That is,

$$\frac{\partial_{x_2} W_{Bears}((x_1^*,0))}{\partial_{x_1} W_{Bears}((x_1^*,0))} = \frac{\lambda_3^* p_2 - \lambda_2^*}{\lambda_3^* p_1} < \frac{p_2}{p_1}.$$

Trades Between Bulls and Bears

Theorem

At the market prices (p_1, p_2), some bulls trade Arrow–Debreu state-contingent claims for state 2 with some bears for Arrow–Debreu state-contingent claims for state 1. That is,

$$\frac{\partial_{x_2} W_{Bulls}((0, x_2^*))}{\partial_{x_1} W_{Bulls}((0, x_2^*))} > \frac{p_2}{p_1} \geq \frac{\partial_{x_2} W_{Bears}((x_1^*, 0))}{\partial_{x_1} W_{Bears}((x_1^*, 0))}.$$

At the market prices (p_1, p_2), other bulls trade Arrow–Debreu state-contingent claims for state 1 with other bears for Arrow–Debreu state-contingent claims for state 2. That is,

$$\frac{\partial_{x_1} W_{Bulls}((x_1^*, 0))}{\partial_{x_2} W_{Bulls}((x_1^*, 0))} > \frac{p_1}{p_2} \geq \frac{\partial_{x_1} W_{Bears}((0, x_2^*))}{\partial_{x_2} W_{Bears}((0, x_2^*))}.$$

Existence of Equilibrium in an Edgeworth Box

Here is an example of a competitive equilibrium in an exchange economy with two states of the world. There is a continuum of bulls indexed on $[0, 1]$ and a continuum of bears indexed on $[0, 1]$. The sum of the average endowments of the bulls, Θ_{Bulls}, and the average endowments of the bears, Θ_{Bears}, define the average social endowment $\Theta \equiv \Theta_{Bulls} + \Theta_{Bears}$. We construct the associated Edgeworth box, where the X-axis is the payoff of the average social endowment in state 1 and the Y-axis is the payoff of the average social endowment in state 2. Zero is the origin of the positive orthant for bulls, i.e., $x \geq 0$ and Θ, the average social endowment, is the origin of the positive orthant for bears, i.e., $y \leq \Theta$.

Existence of Equilibrium in an Edgeworth Box [Continued]

If $p = (p_1, p_2)$ is a vector of positive state prices where $p \cdot \Theta_{Bulls} \equiv I$, $p \cdot \Theta_{Bears} \equiv J$, and a fraction $\rho \in (0,1)$ of bulls who demand the asset with payoffs $(I/p_1, 0)$ and a fraction $(1-\rho) \in (0,1)$ of bulls who demand the asset with payoffs $(0, I/p_2)$, then aggregate demand of the bulls at state prices p is

$$z \equiv \left(\rho \frac{I}{p_1}, (1-\rho)\frac{I}{p_2}\right).$$

In the Edgeworth box, z is a point on the interior of the budget line $p \cdot x = I$, where $x = (x_1, x_2)$ is a state-contingent claim in the positive orthant for bulls. In this example, if every bear maximizes utility subject to the budget constraint $p \cdot y = J$, where $y = \Theta - x$ and z is a state-contingent claim in the positive orthant for the bulls , then $[p; z, \Theta - z]$ is a competitive equilibrium in the exchange economy, where all bears are endowed with the same concave utility function $U(y)$ and

$$\Theta - z = \arg \max_{p \cdot y = J} U(y).$$

Optimality and Existence of Equilibrium in Asset Markets

Optimality follows from Aumann's celebrated (1964) core equivalence theorem for exchange economies with a continuum of traders, where traders may be endowed with nonconcave utility functions, e.g., bulls, but consumption sets are assumed to be bounded below,i.e.,short sales are not allowed. Existence follows from Aumann's (1966) existence theorem for exchange economies with a continuum of traders, where traders may be endowed with nonconcave utility functions,e.g.,bulls, but consumption sets are assumed to be bounded below,i.e.,short sales are not allowed.

Asymmetric Keynesian Utilities

For asymmetric Keynesian utilities there exists a state-contingent claim $\widehat{x}$, "the reference point," where for quadratic utilities of ambiguity and quadratic utilities of risk, $J \circ U(x)$ is concave or pessimistic on

$$[\widehat{x}, +\infty] \equiv \{x \in R^N_+ : x \geq \widehat{x}\}$$

and $J \circ U(x)$ is convex or optimistic on

$$(0, \widehat{x}] \equiv \{x \in R^N_+ : x \leq \widehat{x}\}.$$

Risk-Seeking and Ambiguity-Averse Keynesian Utilities

Theorem

If $J \circ U(x)$, is the composition of $U(x)$ and $J(y)$,where (a) $(y_1, y_2, ..., y_N) \equiv y = U(x) \equiv (u(x_1), u(x_2), ..., u(x_N))$ is a monotone, smooth, concave, diagonal quadratic map from R^N_{++} onto R^N_{++} ,with the proxy for risk, $-\beta < 0$ (b) $J(y)$ is a monotone, smooth, convex quadratic function from R^N_+ into R,with the proxy for ambiguity, $\alpha > 0$,(c)

$$\nabla^2_x J \circ U(x) = \mathrm{diag}(\alpha)(\mathrm{diag}[\nabla_x U(x)])^2 - \mathrm{diag}(\beta)\mathrm{diag}[\nabla_{U(x)} J(U(x))]$$

then there exists a reference point $\hat{x}$ such that the financial market data D is rationalized by the composite function $J \circ U(x)$ with two domains of convexity: $(\hat{x}, +\infty]$ and $(0, \hat{x}]$, where $J \circ U(x)$ is concave on $(\hat{x}, +\infty]$ and $J \circ U(x)$ is convex on $(0, \hat{x}]$.

Risk-Averse and Ambiguity-Seeking Keynesian Utilities

Theorem

If $J \circ U(x)$, is the composition of $U(x)$ and $J(y)$,where (a) $(y_1, y_2, ..., y_N) \equiv y = U(x) \equiv (u(x_1), u(x_2), ..., u(x_N))$ is a monotone, smooth, convex, diagonal quadratic map from R^N_{++} onto R^N_{++} with the proxy for risk, $\beta > 0$, (b) $J(y)$ is a monotone, smooth, concave quadratic function from R^N_+ into R with the proxy for risk,$-\alpha < 0$, (c)

$$\nabla_x^2 J \circ U(x) = -\mathrm{diag}(\alpha)(\mathrm{diag}[\nabla_x U(x)])^2 + \mathrm{diag}(\beta)\mathrm{diag}[\nabla_{U(x)} J(U(x))]$$

then there exists a reference point $\widehat{x}$ such that the financial market data D is rationalized by the composite function $J \circ U(x)$ with two domains of convexity: $(\widehat{x}, +\infty]$ and $(0, \widehat{x}]$, where $J \circ U(x)$ is concave on $(\hat{x}, +\infty]$ and $J \circ U(x)$ is convex on $(0, \hat{x}]$.

Asymmetric Keynesian Utilities and Prospect Theory

Here is the 2×2 contingency table for an investor endowed with asymmetric Keynesian utilities. We divide R_+^N into the standard 4 quadrants with the reference point, $\widehat{x}$, as the origin:

Table 2	
$\nabla_x^2 J \circ U(x)$ is indefinite	$J \circ U(x)$ is concave
on Quadrant II	on Quadrant I
$J \circ U(x)$ is convex	$\nabla_x^2 J \circ U(x)$ is indefinite
on Quadrant III	on Quadrant IV

That is, the investor is a bull for "losses," quadrant III, but a bear for "gains," quadrant I. In prospect theory, preferences for risk have a similar "shape," see figure 10 in Kahneman (2011).