Dynamic Tax Reforms: Note on the Stochastic Model

Mikhail Golosov, Aleh Tsyvinski and Nicolas Werquin*

October 2, 2013

In this note we derive the formulas for the welfare effects of tax reforms in the stochastic model. The notations are the same as in the deterministic model (see Golosov, Tsyvinski and Werquin 2013). For the clarity of the exposition, we consider the case where the horizon is $T = 2$ periods. The individual characteristics θ_1, θ_2 are interpreted as productivity shocks, and individuals are also heterogeneous in initial capital stock k_0. For simplicity, we assume that the interest is deterministic, i.e. known with certainty by each individual in period one. Details and the proofs of the results of this section are gathered in the Appendix.

1 Environment

1.1 The Model

In period one, an individual knows his first-period type, or productivity, $\theta_1 \in [0, \infty)$, and his initial capital stock $k_0 \in \mathbb{R}$. He then chooses his first-period consumption $c_1 \geq 0$, labor income $z_1 \geq 0$, and savings $k_1 \in \mathbb{R}$ to carry over to period two, subject to a budget constraint. In period two, he draws his second-period type $\theta_2 \in [0, \infty)$. For all $\theta_1 \in \mathbb{R}_+$, the second-period type θ_2 is drawn from a distribution $F_{2|1}(\theta_2 | \theta_1)$ whose density $f_{2|1}(\theta_2 | \theta_1)$ is strictly positive on the interval $[\underline{\theta}_2, \bar{\theta}_2]$. For simplicity we assume that $[\underline{\theta}_2, \bar{\theta}_2] = \mathbb{R}_+$. The individual then chooses his second-period consumption $c_2 \geq 0$ and labor income $z_2 \geq 0$, subject to a budget constraint. In each period t, the individual receives flow utility from consumption and labor given by $u(c_t, l_t) = u(c_t, z_t/\theta_t)$. Given his initial draw (k_0, θ_1), he chooses $c_1(k_0, \theta_1)$, $k_1(k_0, \theta_1)$, and $\{c_2(k_0, \theta_1, \theta_2) : \theta_2 \in \mathbb{R}_+\}$ in order to maximize the expected discounted value of his utility. The components of the choice vector X of an individual with initial capital and productivity (k_0, θ_1) are the choices of first-period labor income $z_1(k_0, \theta_1)$, savings $k_1(k_0, \theta_1)$, and all the values $\{z_2(k_0, \theta_1, \theta_2) : \theta_2 \in \mathbb{R}_+\}$ corresponding to the possible draws of θ_2 in period two.

In each period $t = 1, 2$, the government levies a tax T_t. The first-period tax function T_1 is a function of the individual's first-period labor income z_1 and capital income $r_2 k_1$ only.[1] The

*Golosov: Princeton University; Tsyvinski and Werquin: Yale University

[1] In the stochastic model, the government cannot tax second-period labor income z_2 in period one, as z_2 depends on the value of θ_2 that the individual will draw in period two, and hence is not known in period one. On the other hand, since the interest rate r is deterministic, the individual's capital income rk_1 is known with

second-period tax function T_2 is a function of the individual's entire history of labor incomes $\{z_1, z_2\}$ and capital income $r_2 k_2$. The assumptions about the tax functions are identical to those we made in the deterministic model.

1.1.1 The Individual's Problem.

An individual with type θ_1 in period 1 solves the following problem,

$$
\begin{aligned}
\mathscr{U}\left(k_0, \theta_1\right) \equiv & \max u\left(c_1, z_1/\theta_1\right) + \beta \mathbb{E}_{\theta_2}\left[u\left(c_2, z_2/\theta_2\right) | \theta_1\right] \\
& \text{s.t.} \quad c_1 + k_1 = z_1 + k_0 - T_1\left(z_1, r k_1\right) \\
& \text{and} \quad c_2\left(\theta_2\right) = z_2\left(\theta_2\right) + (1+r) k_1 - T_2\left(z_1, z_2\left(\theta_2\right), r k_1\right), \quad \forall \theta_2 \in \mathbb{R}_+,
\end{aligned} \tag{1}
$$

so that $\mathscr{U}\left(k_0, \theta_1\right)$ denotes the individual's indirect utility. We can rewrite this problem as follows

$$
\begin{aligned}
\mathscr{U}\left(k_0, \theta_1\right) \equiv & \max u\left(c_1, z_1/\theta_1\right) + \beta \mathbb{E}_{\theta_2}\left[u\left(c_2, z_2/\theta_2\right) | \theta_1\right] \\
& \text{s.t.} \quad c_1 = \left(1 - \tau_{1,z_1}\right) z_1 - \left(1 + \tau_{1,rk_1} r\right) k_1 + R_1 \\
& \text{and} \quad c_2 = -\tau_{2,z_1} z_1 + \left(1 - \tau_{2,z_2}\right) z_2 + \left(1 + \left(1 - \tau_{2,rk_1}\right) r\right) k_1 + R_2, \quad \forall \theta_2.
\end{aligned} \tag{2}
$$

where the marginal rates are defined as

$$
\begin{aligned}
\tau_{1,x_j} & \equiv \frac{\partial T_1}{\partial x_j}\left(z_1, r k_1\right), \quad \forall x_j \in \left\{z_1, r k_1\right\}, \\
\tau_{2,x_j}\left(z_1, \mathbf{x}_2^2, r k_1\right) & \equiv \frac{\partial T_2}{\partial x_j}\left(z_1, \mathbf{x}_2^2, r k_1\right), \quad \forall x_j \in \left\{z_1, z_2, r k_1\right\},
\end{aligned} \tag{3}
$$

and the virtual incomes are defined as

$$
\begin{aligned}
R_1 & \equiv \tau_{1,z_1} z_1 + \tau_{1,rk_1} r k_1 + k_0 - T_1\left(z_1, r k_1\right), \\
R_2\left(z_1, \mathbf{x}_2^2, r k_1\right) & \equiv \tau_{2,z_1} z_1 + \tau_{2,z_2} \mathbf{x}_2^2 + \tau_{2,rk_1} r k_1 - T_2\left(z_1, \mathbf{x}_2^2, r k_1\right).
\end{aligned} \tag{4}
$$

The first-order conditions of the individual's problem (1), which we derive in the Appendix, define implicitely a Marshallian (uncompensated) earnings supply functions and a capital income function, where z_1, $z_2\left(\theta_2\right)$ for all $\theta_2 \in \mathbb{R}_+$, and rk_1, depend on the marginal and net-of-tax rates $1 - \tau_{1,z_1}$, $\left\{\tau_{2,z_1}\left(z_1, \mathbf{x}_2^2, rk_1\right)\right\}_{\mathbf{x}_2^2 \in \mathbb{R}_+}$, $\left\{1 - \tau_{2,z_2}\left(z_1, \mathbf{x}_2^2, rk_1\right)\right\}_{\mathbf{x}_2^2 \in \mathbb{R}_+}$, τ_{1,rk_1}, $\left\{1 - \tau_{2,rk_1}\left(z_1, \mathbf{x}_2^2, rk_1\right)\right\}_{\mathbf{x}_2^2 \in \mathbb{R}_+}$, and on the virtual incomes R_1, $\left\{R_2\left(z_1, \mathbf{x}_2^2, rk_1\right)\right\}_{\mathbf{x}_2^2 \in \mathbb{R}_+}$, defined respectively in (3) and (4). Since θ_2, and hence z_2 and $T_2(\cdot,\cdot,\cdot)$, are unknown when z_1 and rk_1 are chosen, the two decision variables (z_1, rk_1) depend on the set of all possible marginal tax rates and virtual incomes that the individual may end up facing in period two, depending on his type θ_2. Thus, z_1 and rk_1 depend on the whole set $\left\{\left(\tau_2\left(z_1, \mathbf{x}_2^2, rk_1\right), R_2\left(z_1, \mathbf{x}_2^2, rk_1\right)\right) : \mathbf{x}_2^2 \in \mathbb{R}_+\right\}$, parametrized by the possible values $\mathbf{x}_2^2$ of second-period incomes that the individual may end up choosing in period two.

certainty in period one, even though it is realized in period two.

Moreover, even though z_2 is chosen after a value of θ_2 has been drawn (say θ_2^*), $z_2(\theta_2^*)$ does not depend only on the marginal tax rate and virtual income that he ends up actually facing (i.e., $\tau_2(z_1, z_2(\theta_2^*), rk_1)$), unless the utility function has no income effects. This is because z_1 and rk_1, which have been chosen before the draw (taking into account the probabilities of all possible draws of θ_2), are not in general the optimal values given this particular draw θ_2^*, and this in turn affects the choice of $z_2(\theta_2^*)$. We thus obtain that for all $\theta_2^* \in \mathbb{R}_+$, $z_2(\theta_2^*)$ depends on the entire set of marginal tax rates and virtual incomes $\left\{\left(\tau_2\left(z_1, \mathbf{x}_2^2, rk_1\right), R_2\left(z_1, \mathbf{x}_2^2, rk_1\right)\right) : \mathbf{x}_2^2 \in \mathbb{R}_+\right\}$. In particular, when we perturb the tax function in the second period, $T_2(\cdot,\cdot,\cdot)$, at a given point $\mathbf{x}^2 = \left(z_1, \mathbf{x}_2^2, rk_1\right)$, *all* the choice variables, $(z_1, \{z_2(\theta_2) : \theta_2 \in \mathbb{R}_+\}, rk_1)$, ajdust, even if the individual turns out not to be affected at all by the perturbation (i.e., even if $z_2(\theta_2^*) \neq \mathbf{x}_2^2$). This is the main conceptual difficulty that needs to be addressed in the stochastic model. Note finally that if the utility function has no income effects, then for all θ_2^*, $z_2(\theta_2^*)$ depends only on $\tau_2(z_1, z_2(\theta_2^*), rk_1)$.

We define the vector of income choices X as[2]

$$X(k_0, \theta_1) = \begin{pmatrix} z_1(k_0, \theta_1) \\ z_2(k_0, \theta_1, \underline{\theta}_2) \\ \vdots \\ z_2(k_0, \theta_1, \theta_2) \\ \vdots \\ z_2\left(k_0, \theta_1, \bar{\theta}_2\right) \\ rk_1(k_0, \theta_1) \end{pmatrix}. \tag{5}$$

It is the vector of choice variables of an individual who has type θ_1 in period one, and a continuum of possible types θ_2 in period two, drawn from the distribution $F(\theta_2 | \theta_1)$ on $\left[\underline{\theta}_2, \bar{\theta}_2\right]$. Thus, in period one, this individual chooses his first-period labor income $z_1(k_0, \theta_1)$, his capital income $rk_1(k_0, \theta_1)$, and all the values of second-period labor incomes $z_2(k_0, \theta_1, \theta_2)$ corresponding to the possible draws of θ_2.

1.2 Elasticities and Income Effect Parameters.

It is important to note that there are many more marginal tax rates and virtual incomes that are relevant for the individual than there were in the deterministic model. As we explained above, z_1, $\{z_2(\theta_2) : \theta_2 \in \mathbb{R}_+\}$ and rk_1 all depend on the whole set of marginal tax rates and virtual incomes $\left\{\left(\tau_2\left(z_1, \mathbf{x}_2^2, k_1\right), R_2\left(z_1, \mathbf{x}_2^2, k_1\right)\right) : \mathbf{x}_2^2 \in \mathbb{R}_+\right\}$, parametrized by the possible values $\mathbf{x}_2^2$ of second-period incomes that the individual may end up choosing in period two. We thus need to define the following elasticities and income effect parameters. First, we define

[2] Note that this "vector" has a continuum of rows. It will also be the case for the "matrices" that will be defined later. However, we show in the Appendix that all the usual operations on vectors and matrices naturally generalize to this case.

the elasticities of labor incomes z_1, $\{z_2(\theta_2) : \theta_2 \in \mathbb{R}_+\}$ and capital income rk_1 with respect to the marginal tax rates on z_1 and rk_1 that the individual faces in period one: $\tau_{1,z_1}, \tau_{1,rk_1}$. We then define the elasticities of z_1, $\{z_2(\theta_2) : \theta_2 \in \mathbb{R}_+\}$ and rk_1 with respect to all the marginal tax rates $\left\{\tau_{2,z_1}\left(\mathbf{x}^2\right), \tau_{2,z_2}\left(\mathbf{x}^2\right), \tau_{2,rk_1}\left(\mathbf{x}^2\right) : \mathbf{x}^2 = \left(z_1, \mathbf{x}_2^2, rk_1\right) \in \mathbb{R}_+^2 \times \mathbb{R}\right\}$ that the individual can possibly face in period two, depending on the possible values $\mathbf{x}_2^2$ of second-period incomes that the individual may end up choosing. Similarly we first define the income effect parameters of z_1, $\{z_2(\theta_2) : \theta_2 \in \mathbb{R}_+\}$ and rk_1 with respect to the individual's virtual income in period one, R_1. We then define the income effect parameters of z_1, $\{z_2(\theta_2) : \theta_2 \in \mathbb{R}_+\}$ and rk_1 with respect to all the virtual incomes that the individual can possibly face in period two, $\left\{R_2\left(\mathbf{x}^2\right) : \mathbf{x}^2 = \left(z_1, \mathbf{x}_2^2, rk_1\right) \in \mathbb{R}_+^2 \times \mathbb{R}\right\}$.

These elasticities are new to the literature on taxation, and have not been estimated empirically. Whether they are quantitatively significant is an empirical question, but we show that they matter theoretically. We derive explicit, closed-form expressions for all these elasticities, as we did in the deterministic setting.

1.2.1 Uncompensated Elasticities.

Consider an individual with type (k_0, θ_1) in the first period, which leads him to choose a vector X of incomes and savings. Let $X_1 = (z_1, rk_1)$ be his choice of first-period labor income and capital income. To keep notations concise, we let, for all $y \in \{z_1, rk_1\}$, $\tau_{1,y} = 1 - \tau_{1,z_1}$ if $y = z_1$ and $\tau_{1,y} = 1 + \tau_{1,rk_1} r$ if $y = rk_1$. Similarly, we let, for all $y \in \{z_1, z_2, rk_1\}$, $\tau_{2,y}\left(\mathbf{x}^2\right) = \tau_{2,z_1}\left(\mathbf{x}^2\right)$ if $y = z_1$, $\tau_{2,z_2}\left(\mathbf{x}^2\right) = 1 - \tau_{2,z_2}\left(\mathbf{x}^2\right)$ if $y = z_2$, and $\tau_{2,y}\left(\mathbf{x}^2\right) = 1 + \left(1 - \tau_{2,rk_1}\left(\mathbf{x}^2\right)\right) r$ if $y = rk_1$.

We define this individual's uncompensated elasticities of first-period labor income z_1, capital income rk_1, and second-period labor incomes $z_2(\theta_1, \theta_2)$, with respect to the first-period marginal tax rates at point $\mathbf{x}^1 = \left(\mathbf{x}_1^1, \mathbf{x}_2^1\right) = (z_1, rk_1)$, as[3]

$$\begin{aligned} \zeta_{x,\tau_{1,y}}^{u,(X_1)} &= \frac{\tau_{1,y}}{x} \frac{\partial x}{\partial \tau_{1,y}}, \quad \forall x \in \{z_1, rk_1\}, \\ \zeta_{z_2,\tau_{1,y}}^{u,(X)} &= \frac{\tau_{1,y}}{z_2} \frac{\partial z_2}{\partial \tau_{1,y}}, \quad \forall z_2(k_0, \theta_1, \theta_2) \in \mathbb{R}_+. \end{aligned} \tag{6}$$

We define this individual's uncompensated elasticities of first-period labor income z_1, capital rk_1, and second-period labor incomes $z_2(k_0, \theta_1, \theta_2)$, with respect to the second-period marginal

[3]These elasticities depend in general directly on the individual's types $(k_0, \theta_1, \theta_2)$. Indeed, it is possible that two individuals with different types choose the same vector of incomes and savings. In this case, their elasticities of z_1, rk_1 and $z_2(k_0, \theta_1, \theta_2)$, will be different. For simplicity, and as we did in the deterministic case, we assume throughout the paper that for all $X_1 = (z_1, rk_1) \in \mathbb{R}_+ \times \mathbb{R}$, there is at most one vector (k_0, θ_1) such that $X_1(k_0, \theta_1) = (z_1, rk_1)$, and that for all $X = (z_1, z_2, rk_1) \in \mathbb{R}_+^2 \times \mathbb{R}$, there is at most one vector $(k_0, \theta_1, \theta_2)$ such that $X(k_0, \theta_1, \theta_2) = (z_1, z_2, rk_1)$. Thus $\zeta_{x,\tau_{1,y}}^{u}(k_0, \theta_1)$ can simply be written $\zeta_{x,\tau_{1,y}}^{u}(X_1(k_0, \theta_1))$, or $\zeta_{x,\tau_{1,y}}^{u,(X_1)}$. Similarly, $\zeta_{z_2,\tau_{1,y}}^{u}(k_0, \theta_1, \theta_2)$ can simply be written $\zeta_{z_2,\tau_{1,y}}^{u}(X(k_0, \theta_1, \theta_2))$, or $\zeta_{x,\tau_{1,y}}^{u,(X)}$.

tax rates *at point* $\mathbf{x}^2 = \left(\mathbf{x}_1^2, \mathbf{x}_2^2, \mathbf{x}_3^2\right) = \left(z_1, \mathbf{x}_2^2, rk_1\right)$, as[4]

$$\begin{aligned} \zeta^{u,(X_1)}_{x,\tau_{2,y}(\mathbf{x}^2)} &= \frac{\tau_{2,y}\left(\mathbf{x}^2\right)}{x} \frac{\partial x}{\partial \tau_{2,y}\left(\mathbf{x}^2\right)}, \quad \forall x \in \left\{z_1, rk_1\right\}, \\ \zeta^{u,(X)}_{z_2,\tau_{2,y}(\mathbf{x}^2)} &= \frac{\tau_{2,y}\left(\mathbf{x}^2\right)}{z_2} \frac{\partial z_2}{\partial \tau_{2,y}\left(\mathbf{x}^2\right)}, \quad \forall z_2\left(k_0, \theta_1, \theta_2\right) \in \mathbb{R}_+. \end{aligned} \tag{7}$$

1.2.2 Income Effect Parameters.

Consider an individual with type (k_0, θ_1) in the first period, which leads him to choose a vector X of incomes and savings. Let $X_1 = (z_1, rk_1)$ be his choice of first-period labor income and capital income. We define this individual's marginal propensities to earn and save out of first-period non-wage income in period as

$$\begin{aligned} \eta^{(X_1)}_{x,R_1} &= \tau_{1,x} \frac{\partial x}{\partial R_1}, \quad \forall x \in \left\{z_1, rk_1\right\}, \\ \eta^{(X)}_{z_2,R_1} &= \frac{\partial z_2}{\partial R_1}, \quad \forall z_2\left(k_0, \theta_1, \theta_2\right) \in \mathbb{R}_+, \end{aligned} \tag{8}$$

where $\tau_{1,x} = 1 - \tau_{1,z_1}$ if $x = z_1$, and $\tau_{1,x} = 1 + \tau_{1,rk_1} r$ if $x = rk_1$.

We define this individual's marginal propensities to earn and save out of second-period non-wage income *at point* $\mathbf{x}^2 = \left(\mathbf{x}_1^2, \mathbf{x}_2^2, \mathbf{x}_3^2\right) = \left(z_1, \mathbf{x}_2^2, rk_1\right)$, as

$$\begin{aligned} \eta^{(X_1)}_{x,R_2(\mathbf{x}^2)} &= \tau_{2,x}\left(\mathbf{x}^2\right) \frac{\partial x}{\partial R_2\left(\mathbf{x}^2\right)}, \quad \forall x \in \left\{z_1, rk_1\right\}, \\ \eta^{(X)}_{z_2,R_2(\mathbf{x}^2)} &= \left(1 - \tau_{2,z_2}\left(\mathbf{x}^2\right)\right) \frac{\partial z_2}{\partial R_2\left(\mathbf{x}^2\right)}, \quad \forall z_2\left(k_1, \theta_1, \theta_2\right) \in \mathbb{R}_+, \end{aligned} \tag{9}$$

where $\tau_{2,x}\left(\mathbf{x}^2\right) = \tau_{2,z_1}\left(\mathbf{x}^2\right)$ if $x = z_1$, and $\tau_{2,x}\left(\mathbf{x}^2\right) = 1 + \left(1 - \tau_{2,rk_1(\mathbf{x}^2)}\right) r$ if $x = rk_1$.

[4] To define these elasticities formally, we need to approximate the tax function $T_2\left(\mathbf{x}_1^2, \mathbf{x}_2^2, \mathbf{x}_3^2\right)$ on the cube $\left[\mathbf{x}_1^2, \mathbf{x}_1^2 + d\bar{x}\right] \times \left[\mathbf{x}_2^2, \mathbf{x}_2^2 + d\bar{x}\right] \times \left[\mathbf{x}_3^2, \mathbf{x}_3^2 + d\bar{x}\right]$ by a hyperplane, $T_2\left(\mathbf{x}^2\right) = \tau_{2,z_1}\mathbf{x}_1^2 + \tau_{2,z_2}\mathbf{x}_2^2 + \tau_{2,rk_1} r\mathbf{x}_3^2 + R_2$. Then, we increase the first component of its gradient, $\partial T_2/\partial z_1 = \tau_{2,z_1}$, by $d\tau_{2,z_1}$ in the cube. Taking the limit as $d\bar{x} \to 0$ of the change in $X = \left(z_1, \left\{z_2\left(\theta_2\right) : \theta_2 \in \mathbb{R}_+\right\}, rk_1\right)$, where $(z_1, rk_1) = \left(\mathbf{x}_1^2, \mathbf{x}_3^2\right)$, following this perturbation, divided by $(d\bar{x})^3$, gives the elasticities $\zeta^u_{z_1,\tau_{2,z_1}(\mathbf{x}^2)}$, $\left\{\zeta^u_{z_2,\tau_{2,z_1}(\mathbf{x}^2)} : z_2 \in \mathbb{R}_+\right\}$, $\zeta^u_{rk_1,\tau_{2,z_1}(\mathbf{x}^2)}$.

We finally define the following vectors of income effect parameters,

$$\boldsymbol{\eta}_{X,R_1}^{(X)} = \begin{pmatrix} \frac{\eta_{z_1,R_1}}{1-\tau_{1,z_1}} \\ \eta_{z_2,R_1}^{(\underline{\theta}_2)} \\ \vdots \\ \eta_{z_2,R_1}^{(\theta_2)} \\ \vdots \\ \eta_{z_2,R_1}^{(\bar{\theta}_2)} \\ \frac{\eta_{rk_1,R_1}}{1+\tau_{1,rk_1}r} \end{pmatrix}, \quad \boldsymbol{\eta}_{X,R_2(\mathbf{x}^2)}^{(X)} = \begin{pmatrix} \frac{\eta_{z_1,R_2(\mathbf{x}^2)}}{\tau_{2,z_1}(\mathbf{x}^2)} \\ \frac{\eta_{z_2,R_2(\mathbf{x}^2)}^{(\underline{\theta}_2)}}{1-\tau_{2,z_2}(\mathbf{x}^2)} \\ \vdots \\ \frac{\eta_{z_2,R_2(\mathbf{x}^2)}^{(\theta_2)}}{1-\tau_{2,z_2}(\mathbf{x}^2)} \\ \vdots \\ \frac{\eta_{z_2,R_2(\mathbf{x}^2)}^{(\bar{\theta}_2)}}{1-\tau_{2,z_2}(\mathbf{x}^2)} \\ \frac{\eta_{rk_1,R_2(\mathbf{x}^2)}}{1+\left(1-\tau_{2,rk_1(\mathbf{x}^2)}\right)r} \end{pmatrix}, \tag{10}$$

where $\eta_{z_2,R_1}^{(\theta_2)}$ denotes $\eta_{z_2,R_1}^{(X(k_0,\theta_1,\theta_2))} = \eta_{z_2,R_1}^{(z_1,z_2(k_0,\theta_1,\theta_2),rk_1)}$.

1.2.3 Compensated Elasticities.

Consider an individual with type (k_0, θ_1) in the first period, which leads him to choose a vector X of incomes and savings. Let $X_1 = (z_1, rk_1)$ be his choice of first-period labor income and capital income. We define this individual's matrix of compensated elasticities $\boldsymbol{\zeta}_{X,\tau_1}^{c,(X)}$ with respect to the first-period marginal tax rates $\tau_{1,z_1}\left(\mathbf{x}^1\right), \tau_{1,rk_1}\left(\mathbf{x}^1\right)$, at point $\mathbf{x}^1 = \left(\mathbf{x}_1^1, \mathbf{x}_2^1\right) = (z_1, rk_1)$, as

$$\boldsymbol{\zeta}_{X,\tau_1}^{c,(X)} = \begin{pmatrix} \frac{-z_1}{1-\tau_{1,z_1}}\zeta_{z_1,1-\tau_{1,z_1}}^{c} & 0 & \cdots & 0 & \cdots & 0 & \frac{z_1}{1+\tau_{1,rk_1}}\zeta_{z_1,1+\tau_{1,rk_1}}^{c} \\ \frac{-z_2(\underline{\theta}_2)}{1-\tau_{1,z_1}}\zeta_{z_2,1-\tau_{1,z_1}}^{c,(\underline{\theta}_2)} & 0 & \cdots & 0 & \cdots & 0 & \frac{z_2(\underline{\theta}_2)}{1+\tau_{1,rk_1}}\zeta_{z_2,1+\tau_{1,rk_1}}^{c,(\underline{\theta}_2)} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ \frac{-z_2(\theta_2)}{1-\tau_{1,z_1}}\zeta_{z_2,1-\tau_{1,z_1}}^{c,(\theta_2)} & 0 & \cdots & 0 & \cdots & 0 & \frac{z_2(\theta_2)}{1+\tau_{1,rk_1}}\zeta_{z_2,1+\tau_{1,rk_1}}^{c,(\theta_2)} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ \frac{-z_2(\bar{\theta}_2)}{1-\tau_{1,z_1}}\zeta_{z_2,1-\tau_{1,z_1}}^{c,(\bar{\theta}_2)} & 0 & \cdots & 0 & \cdots & 0 & \frac{z_2(\bar{\theta}_2)}{1+\tau_{1,rk_1}}\zeta_{z_2,1+\tau_{1,rk_1}}^{c,(\bar{\theta}_2)} \\ \frac{-rk_1}{1-\tau_{1,z_1}}\zeta_{rk_1,1-\tau_{1,z_1}}^{c} & 0 & \cdots & 0 & \cdots & 0 & \frac{rk_1}{1+\tau_{1,rk_1}}\zeta_{rk_1,1+\tau_{1,rk_1}}^{c} \end{pmatrix}, \tag{11}$$

where there are a continuum of columns of zeros (corresponding to $\theta_2 \in \left[\underline{\theta}_2, \bar{\theta}_2\right]$), and where $\zeta_{z_2,\tau_{1,y}}^{c,(\theta_2)}$ denotes $\zeta_{z_2,\tau_{1,y}}^{c,(X(k_1,\theta_1,\theta_2))} = \zeta_{z_2,\tau_{1,y}}^{c,(z_1,z_2(k_1,\theta_1,\theta_2),rk_1)}$.

We define this individual's matrix of compensated elasticities $\boldsymbol{\zeta}_{X,\tau_2(\mathbf{x}^2)}^{c,(X)}$ with respect to the second-period marginal tax rates $\tau_{2,z_1}\left(\mathbf{x}^2\right), \tau_{2,z_2}\left(\mathbf{x}^2\right), \tau_{2,rk_1}\left(\mathbf{x}^2\right)$, at point $\mathbf{x}^2 = \left(\mathbf{x}_1^2, \mathbf{x}_2^2, \mathbf{x}_3^2\right) =$

$\left(z_1, \mathbf{x}_2^2, rk_1\right)$, as

$$\boldsymbol{\zeta}^{c,(X)}_{X,\tau_2(\mathbf{x}^2)} =$$

$$\begin{pmatrix}
\frac{z_1}{\tau_{2,z_1}(\mathbf{x}^2)}\zeta^c_{z_1,\tau_{2,z_1}(\mathbf{x}^2)} & 0 & \cdots & 0 & \frac{-z_1}{1-\tau_{2,z_2}(\mathbf{x}^2)}\zeta^c_{z_1,1-\tau_{2,z_2}(\mathbf{x}^2)} & 0 & \cdots & 0 & \frac{-z_1}{1-\tau_{2,rk_1}(\mathbf{x}^2)}\zeta^c_{z_1,1-\tau_{2,rk_1}(\mathbf{x}^2)} \\
\frac{z_2(\underline{\theta}_2)}{\tau_{2,z_1}(\mathbf{x}^2)}\zeta^{c,(\underline{\theta}_2)}_{z_2,\tau_{2,z_1}(\mathbf{x}^2)} & 0 & \cdots & 0 & \frac{-z_2(\underline{\theta}_2)}{1-\tau_{2,z_2}(\mathbf{x}^2)}\zeta^{c,(\underline{\theta}_2)}_{z_2,1-\tau_{2,z_2}(\mathbf{x}^2)} & 0 & \cdots & 0 & \frac{-z_2(\underline{\theta}_2)}{1-\tau_{2,rk_1}(\mathbf{x}^2)}\zeta^{c,(\underline{\theta}_2)}_{z_2,1-\tau_{2,rk_1}(\mathbf{x}^2)} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{z_2(\theta_2)}{\tau_{2,z_1}(\mathbf{x}^2)}\zeta^{c,(\theta_2)}_{z_2,\tau_{2,z_1}(\mathbf{x}^2)} & 0 & \cdots & 0 & \frac{-z_2(\theta_2)}{1-\tau_{2,z_2}(\mathbf{x}^2)}\zeta^{c,(\theta_2)}_{z_2,1-\tau_{2,z_2}(\mathbf{x}^2)} & 0 & \cdots & 0 & \frac{-z_2(\theta_2)}{1-\tau_{2,rk_1}(\mathbf{x}^2)}\zeta^{c,(\theta_2)}_{z_2,1-\tau_{2,rk_1}(\mathbf{x}^2)} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{z_2(\bar{\theta}_2)}{\tau_{2,z_1}(\mathbf{x}^2)}\zeta^{c,(\bar{\theta}_2)}_{z_2,\tau_{2,z_1}(\mathbf{x}^2)} & 0 & \cdots & 0 & \frac{-z_2(\bar{\theta}_2)}{1-\tau_{2,z_2}(\mathbf{x}^2)}\zeta^{c,(\bar{\theta}_2)}_{z_2,1-\tau_{2,z_2}(\mathbf{x}^2)} & 0 & \cdots & 0 & \frac{-z_2(\bar{\theta}_2)}{1-\tau_{2,rk_1}(\mathbf{x}^2)}\zeta^{c,(\bar{\theta}_2)}_{z_2,1-\tau_{2,rk_1}(\mathbf{x}^2)} \\
\frac{rk_1}{\tau_{2,z_1}(\mathbf{x}^2)}\zeta^c_{rk_1,\tau_{2,z_1}(\mathbf{x}^2)} & 0 & \cdots & 0 & \frac{-rk_1}{1-\tau_{2,z_2}(\mathbf{x}^2)}\zeta^c_{rk_1,1-\tau_{2,z_2}(\mathbf{x}^2)} & 0 & \cdots & 0 & \frac{-rk_1}{1-\tau_{2,rk_1}(\mathbf{x}^2)}\zeta^c_{rk_1,1-\tau_{2,rk_1}(\mathbf{x}^2)}
\end{pmatrix}, \tag{12}$$

where the only interior column of this matrix that has non-zero components is the column indexed by θ_2^*, where θ_2^* is such that $z_2(k_0, \theta_1, \theta_2^*) = \mathbf{x}_2^2$. That is, it is the column indexed by the value θ_2^* which is such that, if the individual draws this type θ_2^* in period two, his choice of second-period income will be the point where the second-period marginal tax function is perturbed, i.e. $(z_1, z_2(k_0, \theta_1, \theta_2^*), rk_1) = \mathbf{x}^2$.

First, note that these matrices have a continuum of rows and columns. However, we show in the Appendix that the usual matrix operations extend naturally to this case. Second, we haven't rigorously defined the compensated demands. It is not obvious as in the deterministic case, because markets are incomplete in the stochastic case. However, we can define the compensated elasticities from the uncompensated elasticities and the income effect parameters from the Slutsky equations which we present in the next paragraph (see Appendix).

1.2.4 Slutsky Equations.

The Slutsky equations, which define formally the compensated elasticities from the uncompensated elasticities and the income effect parameters, are all written explicitly in the Appendix.

1.2.5 Explicit Expressions for the elasticities and income effect parameters.

We derive explicit expressions for all these elasticities and income effect parameters in the Appendix.

1.3 Social Welfare

Let $F_1(k_0, \theta_1)$ be the joint c.d.f. of (k_0, θ_1) on $\mathbb{R} \times \mathbb{R}_+$. The government chooses the tax functions $T_1(\cdot, \cdot)$, $T_2(\cdot, \cdot, \cdot)$. We define social welfare function as

$$\begin{aligned}\mathscr{W}|_{\{T_1,T_2\}} =& \frac{1}{p} \int_{\mathbb{R}\times\mathbb{R}_+} G\left(\mathscr{U}(k_0, \theta_1)|_{\{T_1,T_2\}}\right) dF_1(k_0, \theta_1) \\ &+ \int_{\mathbb{R}\times\mathbb{R}_+} \left[T_1(X_1(k_0, \theta_1)) + \delta \int_{\mathbb{R}_+} T_2(X(k_0, \theta_1, \theta_2)) \, dF(\theta_2 | \theta_1)\right] dF_1(k_0, \theta_1),\end{aligned} \tag{13}$$

where p denote the shadow value of public funds.

We define the marginal social welfare weights in both periods as follows (details and intuitions are given in the Appendix):

$$\begin{aligned} g_1\left(\bar{X}_1(k_0, \theta_1)\right) &\equiv \frac{1}{p} G'(\mathscr{U}(k_0, \theta_1)) \, u_{c1}(k_0, \theta_1), \\ g_2\left(\bar{X}_2(k_0, \theta_1, \theta_2)\right) &\equiv \frac{\beta\delta^{-1}}{p} G'(\mathscr{U}(k_0, \theta_1)) \, u_{c2}(k_0, \theta_1, \theta_2), \end{aligned} \tag{14}$$

where X_1 denotes the individual's choice vector in period one, and X_2 denotes his choice vector in period two, after the second-period type θ_2 has been drawn. To define these weights we use our assumption that there is at most one type vector that leads an individual to choose a particular income vector in each period.

Intuitively, the marginal welfare weight $g_1\left(\bar{X}_1\right)$ represents the value, in terms of public funds, of giving one additional dollar in the first period, uniformly to all individuals who have first-period income and savings $\bar{X}_1 = \left(\bar{z}_1, r\bar{k}_1\right)$. Similarly, the marginal welfare weight $g_2\left(\bar{X}_2\right)$ represents the value, in terms of public funds, of giving one additional dollar in the second period, uniformly to all individuals who have first-period income, (realized) second-period income, and savings $\bar{X}_1 = \left(\bar{z}_1, \bar{z}_2, r\bar{k}_1\right)$.

2 Behavioral Responses to Perturbations

We now derive the behavioral responses to the multilinear perturbations we have defined in Section 3. The formulas we obtain are more complex than in the deterministic setting, due to the additional degree of conceptual complexity, described above, inherent to the stochastic setting. Remarkably, however, we show that we can define the elasticity matrices (as well as the gradients and Hessians of the tax functions) in a way that allows to write the formula in a similar compact and empirically estimable form as (25) in the deterministic model. Moreover, we provide a heuristic derivation of this formula that, although conceptually more difficult than in the deterministic case, follows the same steps and the intuition of the static case.

Consider an individual with type (k_0, θ_1) in the first period, which leads him to choose a vector X of incomes and savings. Let $X_1 = (z_1, rk_1)$ be his choice of first-period income and savings. We define the following matrices

$$DT_1(z_1, rk_1) = \begin{pmatrix} \frac{\partial T_1}{\partial z_1}(z_1, rk_1) \\ 0 \\ \vdots \\ 0 \\ \frac{\partial T_1}{\partial (rk_1)}(z_1, rk_1) \end{pmatrix}, \quad DT_2\left(z_1, \mathbf{x}_2^2, rk_1\right) = \begin{pmatrix} \frac{\partial T_2}{\partial z_1}\left(z_1, \mathbf{x}_2^2, rk_1\right) \\ 0 \\ \vdots \\ 0 \\ \frac{\partial T_2}{\partial z_2}\left(z_1, \mathbf{x}_2^2, rk_1\right) \\ 0 \\ \vdots \\ 0 \\ \frac{\partial T_2}{\partial (rk_1)}\left(z_1, \mathbf{x}_2^2, rk_1\right) \end{pmatrix}, \tag{15}$$

where the only non-zero element in the (continuum of) interior rows of $DT_2\left(z_1, \mathbf{x}_2^2, rk_1\right)$ is in the row indexed by θ_2^*, where θ_2^* is such that $z_2(k_0, \theta_1, \theta_2^*) = \mathbf{x}_2^2$.

We also define

$$D^2T_1(z_1, rk_1) = \begin{pmatrix} \frac{\partial^2 T_1}{\partial z_1^2}(z_1, rk_1) & 0 & \cdots & 0 & \frac{\partial^2 T_1}{\partial z_1 \partial (rk_1)}(z_1, rk_1) \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ \frac{\partial^2 T_1}{\partial z_1 \partial (rk_1)}(z_1, rk_1) & 0 & \cdots & 0 & \frac{\partial^2 T_1}{\partial (rk_1)^2}(z_1, rk_1) \end{pmatrix}, \tag{16}$$

and

$$D^2T_2\left(\mathbf{x}^2\right) = \begin{pmatrix} \frac{\partial^2 T_2}{\partial z_1^2}\left(\mathbf{x}^2\right) & 0 & \cdots & 0 & \frac{\partial^2 T_2}{\partial z_1 \partial z_2}\left(\mathbf{x}^2\right) & 0 & \cdots & 0 & \frac{\partial^2 T_2}{\partial z_1 \partial (rk_1)}\left(\mathbf{x}^2\right) \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{\partial^2 T_2}{\partial z_1 \partial z_2}\left(\mathbf{x}^2\right) & 0 & \cdots & 0 & \frac{\partial^2 T_2}{\partial z_2^2}\left(\mathbf{x}^2\right) & 0 & \cdots & 0 & \frac{\partial^2 T_2}{\partial z_2 \partial (rk_1)}\left(\mathbf{x}^2\right) \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \frac{\partial^2 T_2}{\partial z_1 \partial (rk_1)}\left(\mathbf{x}^2\right) & 0 & \cdots & 0 & \frac{\partial^2 T_2}{\partial z_2 \partial (rk_1)}\left(\mathbf{x}^2\right) & 0 & \cdots & 0 & \frac{\partial^2 T_2}{\partial (rk_1)^2}\left(\mathbf{x}^2\right) \end{pmatrix}, \tag{17}$$

where the only non-zero elements in the (continuum of) interior rows (resp., columns) of $D^2T_2\left(\mathbf{x}^2\right)$ are in the row (resp., column) indexed by θ_2^*, where θ_2^* is such that $z_2(k_0, \theta_1, \theta_2^*) = \mathbf{x}_2^2$.

We now derive the behavioral responses to general perturbations of the tax system. The proofs of these results are all collected in the Appendix. Here we just describe the heuristic derivation.

Elasticity Effect. Consider an individual with type (k_0, θ_1) in the first period, which leads him to choose a vector X of incomes and savings, as in (5). Let $X_1 = (z_1, rk_1)$ be his choice of first-period income and savings. Assume that X belongs to a region of the space $\mathbb{R}_+ \times \mathbb{R}_+^{[0,\infty)} \times \mathbb{R}$ where the gradient of the tax function $T_1(\cdot,\cdot)$ is perturbed by the amount $d\tau_1 = (d\tau_{1,z_1}, d\tau_{1,rk_1})'$, and the gradient of the tax function $T_2(\cdot,\cdot,\cdot)$ is perturbed by the amount $d\tau_2 = (d\tau_{2,z_1}, d\tau_{2,z_2}, d\tau_{2,rk_1})'$. More precisely, assume that the gradient of $T_2(\cdot,\cdot,\cdot)$ is perturbed by $d\tau_2$ at the point $\mathbf{x}^2 = (z_1, \mathbf{x}_2^2, rk_1)$. (More rigorously, we perturb on the cube $[z_1, z_1 + d\bar{x}] \times [\mathbf{x}_2^2, \mathbf{x}_2^2 + d\bar{x}] \times [rk_1, rk_1 + d\bar{x}]$.) Let θ_2^* be the second period type such that $z_2(k_0, \theta_1, \theta_2^*) = \mathbf{x}_2^2$. Define the vectors $d\tau_1, d\tau_2(\mathbf{x}^2)$ as the change in the matrices $DT_1(z_1, rk_1)$ and $DT_2(\mathbf{x}^2)$ (defined in (15)) due to the perturbation, i.e.,

$$\begin{aligned} d\tau_1 &= \begin{pmatrix} d\tau_{1,z_1} & 0 & \dots & 0 & d\tau_{1,rk_1} \end{pmatrix}', \\ d\tau_2(\mathbf{x}^2) &= \begin{pmatrix} d\tau_{2,z_1} & 0 & \dots & 0 & d\tau_{2,z_2} & 0 & \dots & 0 & d\tau_{2,rk_1} \end{pmatrix}', \end{aligned} \tag{18}$$

where the only interior row of $d\tau_2(\mathbf{x}^2)$ that has a non-zero component is the row indexed by θ_2^*.

This perturbation has an elasticity effect which produces a small change dX for the individual $X(k_0, \theta_1)$. This change is the consequence of two effects. First, there is a direct compensated effect due to the exogenous increase in the marginal tax rates. This effect is equal to $\boldsymbol{\zeta}_{X,\tau_1}^{c,(X)} d\tau_1$ for the perturbation of the first-period tax function, and to $\boldsymbol{\zeta}_{X,\tau_2(\mathbf{x}^2)}^{c,(X)} d\tau_2(\mathbf{x}^2)$ for the perturbation of the second-period tax function, where $\boldsymbol{\zeta}_{X,\tau_1}^{c,(X)}, \boldsymbol{\zeta}_{X,\tau_2(\mathbf{x}^2)}^{c,(X)}$ are the compensated elasticity matrices defined in (11),(12). Note that $\boldsymbol{\zeta}_{X,\tau_2(\mathbf{x}^2)}^{c,(X)} d\tau_2(\mathbf{x}^2)$ is equal to

$$\frac{\partial X}{\partial \tau_{2,z_1}(\mathbf{x}^2)} d\tau_{2,z_1} - \frac{\partial X}{\partial (1 - \tau_{2,z_2}(\mathbf{x}^2))} d\tau_{2,z_2} - \frac{\partial X}{\partial (1 - \tau_{2,rk_1}(\mathbf{x}^2))} d\tau_{2,rk_1}.$$

All the components of X, including those with $z_2(\theta_2') \neq \mathbf{x}_2^2$ (i.e., $\theta_2' \neq \theta_2^*$), are directly affected by the perturbation at the point $\mathbf{x}^2$, as was explained above.

Second, there is an indirect effect due to the shift of the taxpayer along the tax function by dX, which induces an endogenous additional change in marginal rates. This change is equal to $d(DT_1(z_1, rk_1)) = (D^2T_1(z_1, rk_1))\, dX$ for the first-period tax function, where $D^2T_1(z_1, rk_1)$ is the matrix defined in (16). Thus the first-period indirect change in the choice vector is equal to $\boldsymbol{\zeta}_{X,\tau_1}^{c,(X)} (D^2T_1(X_1))\, dX$. For the second-period tax function, this indirect change is more complex. Consider a second-period income $z_2' \equiv z_2(k_0, \theta_1, \theta_2')$, with θ_2' not necessarily equal to θ_2^* (i.e., z_2' not necessarily equal to $\mathbf{x}_2^2$). Since z_2' changes due to the direct effect described above, it induces a change in the second-period marginal tax function at the point (z_1, z_2', rk_1), equal to $d(DT_2(z_1, z_2', rk_1)) = (D^2T_2(z_1, z_2', rk_1))\, dX$, where $D^2T_2(z_1, z_2', rk_1)$ is the matrix defined in (17). This in turn induces an indirect change in the vector X. This indirect change is equal to

$\boldsymbol{\zeta}^{c,(X)}_{X,\tau_2(\mathbf{x}^{2\prime})}\left(D^2T_2\left(z_1,z_2',rk_1\right)\right)dX$, where $\boldsymbol{\zeta}^{c,(X)}_{X,\tau_2(\mathbf{x}^{2\prime})}$ denotes the matrix $\boldsymbol{\zeta}^{c,(X)}_{X,\tau_2\left(z_1,z_2',k_2\right)}$. This effect must be summed over all possible values of θ_2' (i.e., of $z_2'=\mathbf{x}_2^{2\prime}$).

Therefore, we obtain that the individual changes his vector X in response to the perturbation by an amount

$$dX=\left[I-\boldsymbol{\zeta}^{c,(X)}_{X,\tau_1}\left(D^2T_1\left(X_1\right)\right)-\int_0^\infty \boldsymbol{\zeta}^{c,(X)}_{X,\tau_2(\mathbf{x}^{2\prime})}\left(D^2T_2\left(\mathbf{x}^{2\prime}\right)\right)d\mathbf{x}_2^{2\prime}\right]^{-1}$$
$$\times\left[\boldsymbol{\zeta}^{c,(X)}_{X,\tau_1}d\tau_1+\boldsymbol{\zeta}^{c,(X)}_{X,\tau_2(\mathbf{x}^2)}d\tau_2\left(\mathbf{x}^2\right)\right],$$

where I is the identity matrix (with a continuum of interior rows and columns).

Income Effect. Consider an individual with type (k_0,θ_1) in the first period, which leads him to choose a vector X of incomes and savings. Let $X_1=(z_1,rk_1)$ be his choice of first-period income and savings. Assume that X belongs to a region of the space $\mathbb{R}_+\times\mathbb{R}_+^{[0,\infty)}\times\mathbb{R}$ where the period-one virtual income R_1 is perturbed by the amount $dR_1\in\mathbb{R}$, and the period-two virtual income is perturbed by the amount dR_2 at the point $\mathbf{x}^2=\left(z_1,\mathbf{x}_2^2,rk_1\right)$. (More rigorously, we perturb on the cube $[z_1,z_1+d\bar{x}]\times\left[\mathbf{x}_2^2,\mathbf{x}_2^2+d\bar{x}\right]\times[rk_1,rk_1+d\bar{x}]$.) Let θ_2^* be the second period type such that $z_2\left(k_0,\theta_1,\theta_2^*\right)=\mathbf{x}_2^2$.

This perturbation induces a small change dX in the choice vector of individual X. This change is the consequence of two effects. First, there is a direct income effect due to the exogenous increase in virtual incomes. It is equal to $\boldsymbol{\eta}^{(X)}_{X,R_1}dR_1$ for the perturbation of the first-period virtual income, and to $\boldsymbol{\eta}^{(X)}_{X,R_2(\mathbf{x}^2)}dR_2\left(\mathbf{x}^2\right)$ for the perturbation of the second-period virtual income at point $\mathbf{x}^2$, where the vectors $\boldsymbol{\eta}^{(X)}_{X,R_1},\boldsymbol{\eta}^{(X)}_{X,R_2(\mathbf{x}^2)}$ are defined in (10). Second, there is an indirect elastic effect due to the shift of the taxpayer on the tax function by dX, which induces an endogenous change in marginal rates similar to those described in the previous paragraph. Therefore, the individual changes his vector X in response to the perturbation by an amount

$$dX=\left[I-\boldsymbol{\zeta}^{c,(X)}_{X,\tau_1}\left(D^2T_1\left(X_1\right)\right)-\int_0^\infty \boldsymbol{\zeta}^{c,(X)}_{X,\tau_2(\mathbf{x}^{2\prime})}\left(D^2T_2\left(\mathbf{x}^{2\prime}\right)\right)d\mathbf{x}_2^{2\prime}\right]^{-1}$$
$$\times\left[\boldsymbol{\eta}^{(X)}_{X,R_1}dR_1+\boldsymbol{\eta}^{(X)}_{X,R_2(\mathbf{x}^2)}dR_2\left(\mathbf{x}^2\right)\right].$$

Individual Responses to Perturbations. We summarize these results in the following Proposition, which is proved in the Appendix:

Proposition 1. *Consider an individual with type (k_0,θ_1) in the first period, which leads him to choose a vector X of incomes and savings. Let $X_1=(z_1,rk_1)$ be his choice of first-period income and savings. Assume that X belongs to a region of the space where the gradient of the tax function $T_1(\cdot,\cdot)$ is perturbed by the amount $d\tau_1=(d\tau_{1,z_1},d\tau_{1,rk_1})'$, the gradient of the tax function $T_2(\cdot,\cdot,\cdot)$ is perturbed by the amount $d\tau_2=(d\tau_{2,z_1},d\tau_{2,z_2},d\tau_{2,rk_1})'$ at the point $\mathbf{x}^2=\left(z_1,\mathbf{x}_2^2,rk_1\right)$, the period-one virtual income R_1 is perturbed by the amount $dR_1\in\mathbb{R}$, and the period-two virtual* *income is perturbed by the amount dR_2 at the point $\mathbf{x}^2 = \left(z_1, \mathbf{x}_2^2, rk_1\right)$. Let θ_2^* be the second period type such that $z_2\left(k_0, \theta_1, \theta_2^*\right) = \mathbf{x}_2^2$. Define the vectors $d\tau_1, d\tau_2\left(\mathbf{x}^2\right)$ as in (18). Then the individual changes his vector X in response to the perturbation by an amount*

$$dX = \left[I - \boldsymbol{\zeta}_{X,\tau_1}^{c,(X)}\left(D^2 T_1\left(X_1\right)\right) - \int_0^\infty \boldsymbol{\zeta}_{X,\tau_2(\mathbf{x}^{2\prime})}^{c,(X)}\left(D^2 T_2\left(\mathbf{x}^{2\prime}\right)\right) d\mathbf{x}_2^{2\prime}\right]^{-1} \times \left[\boldsymbol{\zeta}_{X,\tau_1}^{c,(X)} d\tau_1 + \boldsymbol{\zeta}_{X,\tau_2(\mathbf{x}^2)}^{c,(X)} d\tau_2\left(\mathbf{x}^2\right) + \boldsymbol{\eta}_{X,R_1}^{(X)} dR_1 + \boldsymbol{\eta}_{X,R_2(\mathbf{x}^2)}^{(X)} dR_2\left(\mathbf{x}^2\right)\right]. \tag{19}$$

Proof. See the Appendix. □

3 Welfare Gains of Tax Reforms

As in the deterministic case, the welfare gains of tax reforms, i.e. the change in social welfare (13) due to the perturbations (20) defined in Section 3 of the deterministic model, are given by the sum of (*i*) a mechanical gain in tax revenue (net of the welfare loss) due to the increase in the tax liability faced by individuals above the point of the perturbation (e.g., individuals who earn $z_1 \geq \bar{z}_1$ and $z_2 \geq \bar{z}_2$ if the second-period tax function is perturbed at point $(\bar{z}_1, \bar{z}_2)$, similar to what shown in Figure 3); (*ii*) the elasticity effects, due to the increases in the marginal tax rates in the corresponding regions (similar to the dark shaded bands in Figure 3), and the behavioral responses that these changes induce on the choice vectors X of individuals; and (*iii*) an income effect due to the lump-sum increase in the tax liability in the corresponding region (similar to the light shaded band in Figure 3), and the behavioral responses that this change induces on the choice vectors X of individuals. Importantly, in the stochastic model, if the tax function is perturbed only for some values of second-period income z_2 (e.g., for $z_2 \geq \bar{z}_2$), then *all* the individuals adjust their income vector X, not just those who end up earning $z_2 \geq \bar{z}_2$ in period two. This is because, in period one, every individual has a positive probability of drawing a second-period type that would induce him to choose $z_2 \geq \bar{z}_2$ in period two. Other than that, the intuitions and the proofs parallel those of the deterministic setting.

For simplicity, we only show how the formulas of Section 6 of the deterministic model, i.e. (40), (45), (55) and (57), write in the stochastic model. We thus make the same assumptions as in Section 6, namely, that the utility function has no income effects, $u(c, l) = u(c - v(l))$, the planner maximizes revenue, and the initial tax system is separable between labor and capital incomes, capital is taxed linearly only in period two, and labor income z_t is taxed only in period t, with an age-independent labor income tax schedule.

Under these assumptions, we show in the Appendix, using the explicit expressions for the elasticities and income effect parameters, the following relationships between the income effect parameters of capital income w.r.t. an increase in the first- and second-period virtual income in the stochastic and the deterministic models:

$$0 < \frac{\partial (rk_1)}{\partial R_1} < \left.\frac{\partial (rk_1)}{\partial R_1}\right|^{Det}, \text{ and } \int_{\mathbf{x}_2^2=0}^{\infty} \frac{\partial (rk_1)}{\partial R_2(\mathbf{x}^2)} d\mathbf{x}_2^2 < \left.\frac{\partial (rk_1)}{\partial R_2}\right|^{Det} < 0, \tag{20}$$

and between the compensated elasticities of capital income w.r.t. an increase in the net-of capital income tax rate in the stochastic and the deterministic models:

$$0 < \int_{\mathbf{x}_2^2=0}^{\infty} \frac{\partial (rk_1^c)}{\partial (1-\tau_{2,rk_1}(\mathbf{x}^2))} d\mathbf{x}_2^2 < \left.\frac{\partial (rk_1^c)}{\partial (1-\tau_{2,rk_1})}\right|^{Det}. \tag{21}$$

That is, the effect on capital income of a lump-sum increase in income in the first-period income is smaller in the stochastic model than in the deterministic model. Similarly, the effect on capital income of an *certain* increase in the second-period virtual income (i.e., a change that occurs for all the values of the individual's second-period labor income) in the stochastic model, is smaller than the second-period income effect parameter in the deterministic model. Finally, the compensated capital income elasticity is also smaller in the stochastic model than its counterpart in the deterministic model. The explicit expressions for the elasticities in the Appendix show precisely how these expressions differ.

We show in the Appendix that the revenue gains of separable perturbations of labor incomes in periods one and two have formal expressions that are identical to their counterparts (40) in the deterministic model. However, the inequalities (20) about the income effect parameters imply that the revenue gains of the perturbations are closer to the static gains (37) in the stochastic model than in the deterministic model (smaller gains from decreasing the tax rates in period one, and smaller gains from increasing the tax rate in period two, starting from the static optimum). This implies that the amount of age-dependence shown in Figure 5 is an upper bound of the optimal amount of age-dependence in the stochastic model. Similarly, the formula giving the revenue gains of perturbing the capital income tax rate is identical to the deterministic formula (45), but the inequalities (21) imply that the gains from increasing the tax rates are strictly larger in the stochastic model than in the deterministic model. Thus, the optimal capital income tax rates computed in the deterministic model are a lower bound of the optimal rates in the stochastic model. We finally derive the revenue gains from introducing history-dependence and joint taxation of labor and capital incomes in the stochastic model.

A Proofs for the Stochastic Model

A.0.1 First-order conditions of the individual's problem

It is easy to check that the first order conditions of the original problem (1) and the linearized problem (2) are the same. They write

$$\begin{aligned}
&[c_1]: && u_c\left(c_1, z_1/\theta_1\right) = \lambda_1, \\
&[z_1]: && \frac{1}{\theta_1} u_l\left(c_1, z_1/\theta_1\right) + \lambda_1\left(1-\tau_{1,z_1}\right) - \int_0^\infty \lambda_2\left(\theta_2\right)\tau_{2,z_1} d\theta_2 = 0, \\
&[c_2(\theta_2)]: && \beta u_c\left(c_2, z_2/\theta_2\right) f_{2|1}\left(\theta_2 \left|\theta_1\right.\right) = \lambda_2\left(\theta_2\right), \\
&[z_2(\theta_2)]: && \beta\frac{1}{\theta_2} u_l\left(c_2, z_2/\theta_2\right) f_{2|1}\left(\theta_2 \left|\theta_1\right.\right) + \lambda_2\left(\theta_2\right)\left(1-\tau_{2,z_2}\right) = 0, \\
&[k_1]: && \lambda_1\left(1+\tau_{1,rk_1} r\right) = \int_0^\infty \lambda_2\left(\theta_2\right)\left(1+\left(1-\tau_{2,rk_1}\right) r\right) d\theta_2,
\end{aligned} \tag{22}$$

and the budget constraints write

$$\begin{aligned}
&c_1 = \left(1-\tau_{1,z_1}\right) z_1 - \left(1+\tau_{1,rk_1} r\right) k_1 + R_1, \\
&c_2 = -\tau_{2,z_1} z_1 + \left(1-\tau_{2,z_2}\right) z_2 + \left(1+\left(1-\tau_{2,rk_1}\right) r\right) k_1 + R_2, \quad \forall \theta_2 \in \mathbb{R}_+.
\end{aligned} \tag{23}$$

We thus obtain, after rearranging the first order conditions (22):

$$\begin{aligned}
-\frac{1}{\theta_1} u_l\left(c_1, z_1/\theta_1\right) &= \left(1-\tau_{1,z_1}\right) u_c\left(c_1, z_1/\theta_1\right) - \beta \mathbb{E}_{\theta_2}\left[\tau_{2,z_1} u_c\left(c_2, z_2/\theta_2\right) \left|\theta_1\right.\right], \\
-\frac{1}{\theta_2} u_l\left(c_2, z_2/\theta_2\right) &= \left(1-\tau_{2,z_2}\right) u_c\left(c_2, z_2/\theta_2\right), \quad \forall \theta_2 \in \mathbb{R}_+, \\
0 &= \left(1+\tau_{1,rk_1} r\right) u_c\left(c_1, z_1/\theta_1\right) - \beta \mathbb{E}_{\theta_2}\left[\left(1+\left(1-\tau_{2,rk_1}\right) r\right) u_c\left(c_2, z_2/\theta_2\right) \left|\theta_1\right.\right].
\end{aligned} \tag{24}$$

We can express these first order conditons in terms of $z_1, \{z_2 : \theta_2 \in \mathbb{R}_+\}, rk_1$ only, using the budget constraints, as follows. The intratemporal condition in period one writes

$$\begin{aligned}
&-\frac{1}{\theta_1} u_l\left\{\left(1-\tau_{1,z_1}\right) z_1 - \left(1+\tau_{1,rk_1} r\right) k_1 + R_1, z_1/\theta_1\right\} \\
&= \left(1-\tau_{1,z_1}\right) u_c\left\{\left(1-\tau_{1,z_1}\right) z_1 - \left(1+\tau_{1,rk_1} r\right) k_1 + R_1, z_1/\theta_1\right\} \\
&\quad - \beta \mathbb{E}_{\theta_2}\left[\tau_{2,z_1} u_c\left\{-\tau_{2,z_1} z_1 + \left(1-\tau_{2,z_2}\right) z_2 + \left(1+\left(1-\tau_{2,rk_1}\right) r\right) k_1 + R_2, z_2/\theta_2\right\} \left|\theta_1\right.\right].
\end{aligned} \tag{25}$$

The intratemporal condition in period two and state $\theta_2 \in \mathbb{R}_+$ writes

$$\begin{aligned}
&-\frac{1}{\theta_2} u_l\left\{-\tau_{2,z_1} z_1 + \left(1-\tau_{2,z_2}\right) z_2 + \left(1+\left(1-\tau_{2,rk_1}\right) r\right) k_1 + R_2, z_2/\theta_2\right\} \\
&= \left(1-\tau_{2,z_2}\right) u_c\left\{-\tau_{2,z_1} z_1 + \left(1-\tau_{2,z_2}\right) z_2 + \left(1+\left(1-\tau_{2,rk_1}\right) r\right) k_1 + R_2, z_2/\theta_2\right\}.
\end{aligned} \tag{26}$$

The intertemporal condition writes

$$\begin{aligned}&\left(1+\tau_{1,rk_1} r\right) u_c\left\{\left(1-\tau_{1,z_1}\right) z_1-\left(1+\tau_{1,rk_1} r\right) k_1+R_1, z_1/\theta_1\right\}\\=&\beta\mathbb{E}_{\theta_2}\left[\left(1+\left(1-\tau_{2,rk_1}\right) r\right) u_c\left\{-\tau_{2,z_1} z_1+\left(1-\tau_{2,z_2}\right) z_2+\left(1+\left(1-\tau_{2,rk_1}\right) r\right) k_1+R_2, z_2/\theta_2\right\}|\theta_1\right].\end{aligned} \tag{27}$$

For the proof of Proposition 11, it will be useful to rewrite them in terms of the tax functions and their partial derivatives, rather than the marginal tax rates and virtual incomes. The intratemporal condition in period one writes

$$\begin{aligned}&-\frac{1}{\theta_1} u_l\left\{z_1+k_0-k_1-T_1\left(z_1, rk_1\right), z_1/\theta_1\right\}\\=&\left(1-\frac{\partial T_1\left(z_1, rk_1\right)}{\partial z_1}\right) u_c\left\{z_1+k_0-k_1-T_1\left(z_1, rk_1\right), z_1/\theta_1\right\}\\&-\beta\mathbb{E}_{\theta_2}\left[\frac{\partial T_2\left(z_1, z_2, rk_1\right)}{\partial z_1} u_c\left\{z_2+(1+r) k_1-T_2\left(z_1, z_2, rk_1\right), z_2/\theta_2\right\}|\theta_1\right].\end{aligned} \tag{28}$$

The intratemporal condition in period two and state $\theta_2 \in \mathbb{R}_+$ writes

$$\begin{aligned}&-\frac{1}{\theta_2} u_l\left\{z_2+(1+r) k_1-T_2\left(z_1, z_2, rk_1\right), z_2/\theta_2\right\}\\=&\left(1-\frac{\partial T_2\left(z_1, z_2, rk_1\right)}{\partial z_2}\right) u_c\left\{z_2+(1+r) k_1-T_2\left(z_1, z_2, rk_1\right), z_2/\theta_2\right\}.\end{aligned} \tag{29}$$

The intertemporal condition writes

$$\begin{aligned}&\left(1+\frac{\partial T_1\left(z_1, rk_1\right)}{\partial\left(rk_1\right)} r\right) u_c\left\{z_1+k_0-k_1-T_1\left(z_1, rk_1\right), z_1/\theta_1\right\}\\=&\beta\mathbb{E}_{\theta_2}\left[\left(1+\left(1-\frac{\partial T_2\left(z_1, z_2, rk_1\right)}{\partial\left(rk_1\right)}\right) r\right) u_c\left\{z_2+(1+r) k_1-T_2\left(z_1, z_2, rk_1\right), z_2/\theta_2\right\}|\theta_1\right].\end{aligned} \tag{30}$$

A.0.2 Slutsky Equations

Consider an individual with first period labor income and savings z_1, rk_1, and let $\mathbf{x}^2 = \left(z_1, \mathbf{x}_2^2, rk_1\right)$. We define, for $x \in \{z_1, z_2(\theta_2), rk_1\}$, the compensated demands as

$$
\begin{aligned}
\frac{\partial x^c}{\partial (1-\tau_{1,z_1})} &= \frac{\partial x}{\partial (1-\tau_{1,z_1})} - \frac{\partial x}{\partial R_1} z_1, \\
\frac{\partial x^c}{\partial \tau_{1,rk_1}} &= \frac{\partial x}{\partial \tau_{1,rk_1}} + \frac{\partial x}{\partial R_1}(rk_1), \\
\frac{\partial x^c}{\partial \tau_{2,z_1}(\mathbf{x}^2)} &= \frac{\partial x}{\partial \tau_{2,z_1}(\mathbf{x}^2)} + \frac{\partial x}{\partial R_2(\mathbf{x}^2)} z_1, \\
\frac{\partial x^c}{\partial (1-\tau_{2,z_2}(\mathbf{x}^2))} &= \frac{\partial x}{\partial (1-\tau_{2,z_2}(\mathbf{x}^2))} - \frac{\partial x}{\partial R_2(\mathbf{x}^2)} \mathbf{x}_2^2, \\
\frac{\partial x^c}{\partial (1-\tau_{2,rk_1}(\mathbf{x}^2))} &= \frac{\partial x}{\partial (1-\tau_{2,rk_1}(\mathbf{x}^2))} - \frac{\partial x}{\partial R_2(\mathbf{x}^2)}(rk_1).
\end{aligned}
$$

Therefore, we obtain the compensated demand elasticities, with $\tilde{\tau}_{1,x} = 1 - \tau_{1,z_1}$ if $x = z_1$, $\tilde{\tau}_{1,x} r = 1 + \tau_{1,rk_1} r$ if $x = rk_1$, and $\tilde{\tau}_{1,x} = 1$ if $x = z_2(\theta_2)$, and with $\tilde{\tau}_{2,x}\left(\mathbf{x}^2\right) = 1 - \tau_{2,z_2}\left(\mathbf{x}^2\right)$ if $x = z_2(\theta_2)$, $\tilde{\tau}_{2,x}\left(\mathbf{x}^2\right) = \tau_{2,z_1}\left(\mathbf{x}^2\right)$ if $x = z_1$, and $\tilde{\tau}_{2,x}\left(\mathbf{x}^2\right) r = 1 + \left(1 - \tau_{2,rk_1}\left(\mathbf{x}^2\right)\right) r$ if $x = rk_1$,

$$
\begin{aligned}
\zeta^c_{x,1-\tau_{1,z_1}} &= \zeta^u_{x,1-\tau_{1,z_1}} - \frac{(1-\tau_{1,z_1}) z_1}{\tilde{\tau}_{1,x} x} \eta_{x,R_1}, \\
\zeta^c_{x,\tilde{\tau}_{1,rk_1}} &= \zeta^u_{x,\tilde{\tau}_{1,rk_1}} + \frac{\tilde{\tau}_{1,rk_1} rk_1}{\tilde{\tau}_{1,x} x} \eta_{x,R_1}, \\
\zeta^c_{x,\tau_{2,z_1}(\mathbf{x}^2)} &= \zeta^u_{x,\tau_{2,z_1}(\mathbf{x}^2)} + \frac{\tau_{2,z_1}\left(\mathbf{x}^2\right) z_1}{\tilde{\tau}_{2,x}\left(\mathbf{x}^2\right) x} \eta_{x,R_2(\mathbf{x}^2)}, \\
\zeta^c_{x,1-\tau_{2,z_2}(\mathbf{x}^2)} &= \zeta^u_{x,1-\tau_{2,z_2}(\mathbf{x}^2)} - \frac{\left(1-\tau_{2,z_2}\left(\mathbf{x}^2\right)\right) \mathbf{x}_2^2}{\tilde{\tau}_{2,x}\left(\mathbf{x}^2\right) x} \eta_{x,R_2(\mathbf{x}^2)}, \\
\zeta^c_{x,\tilde{\tau}_{2,rk_1}} &= \zeta^u_{x,\tilde{\tau}_{2,rk_1}} - \frac{\tilde{\tau}_{2,rk_1}\left(\mathbf{x}^2\right) rk_1}{\tilde{\tau}_{2,x}\left(\mathbf{x}^2\right) x} \eta_{x,R_2(\mathbf{x}^2)}.
\end{aligned}
\tag{31}
$$

A.0.3 Elasticities and income effect parameters: Explicit expressions

To obtain explicit expressions for the elasticities, we differentiate the first-order conditions (25), (26) and (27) with respect to $\tau_{p,y}$, where $\tau_{p,y} \in \left\{\tau_{1,z_1}, \tau_{1,rk_1}, R_1, \tau_{2,z_1}\left(\mathbf{x}^2\right), \tau_{2,z_2}\left(\mathbf{x}^2\right), \tau_{2,rk_1}\left(\mathbf{x}^2\right), R_2\left(\mathbf{x}^2\right)\right\}$. Everywhere we denote by θ_2^* the second-period type such that $z_2(\theta_1, \theta_2^*) = \mathbf{x}_2^2$ (i.e., such that

$$
(z_1(k_0, \theta_1), z_2(k_0, \theta_1, \theta_2^*), rk_1(k_0, \theta_1)) = \mathbf{x}^2,
$$

where $\mathbf{x}^2$ is the point at which the marginal tax functions $\tau_{2,y}$ are perturbed).

As a preliminary step before differentiating the first order conditions, we let $u_{t,a}$ denote the partial derivative of the period-t flow utility u with respect to its variable a (consumption or leisure), and we can write the partial derivatives of $u_{t,a}$ with respect to the marginal tax rates as:

$$\begin{aligned}\frac{\partial u_{1,a}}{\partial \tau_{p,y}} =& \frac{\partial}{\partial \tau_{p,y}} u_a \left\{ (1-\tau_{1,z_1}) z_1 - (1+\tau_{1,rk_1} r) k_1 + R_1, z_1/\theta_1 \right\} \\ =& \left\{ (1-\tau_{1,z_1}) u_{1,ac} + \frac{1}{\theta_1} u_{1,al} \right\} \frac{\partial z_1}{\partial \tau_{p,y}} + \left\{ -\frac{1}{r} (1+\tau_{1,rk_1} r) u_{1,ac} \right\} \frac{\partial (rk_1)}{\partial \tau_{p,y}} \\ & + \{-y u_{p,ac}\} \mathbf{1}_{\{\tau_{p,y}=\tau_{1,y}\}} + \{u_{p,ac}\} \mathbf{1}_{\{\tau_{p,y}=R_1\}},\end{aligned} \tag{32}$$

and, for all $\theta_2 \in \mathbb{R}_+$,

$$\begin{aligned}\frac{\partial u_{2,a}}{\partial \tau_{p,y}} =& \frac{\partial}{\partial \tau_{p,y}} u_a \left\{ -\tau_{2,z_1} z_1 + (1-\tau_{2,z_2}) z_2 + (1+(1-\tau_{2,rk_1}) r) k_1 + R_2, z_2/\theta_2 \right\} \\ =& \{-\tau_{2,z_1} u_{2,ac}\} \frac{\partial z_1}{\partial \tau_{p,y}} + \left\{ (1-\tau_{2,z_2}) u_{2,ac} + \frac{1}{\theta_2} u_{2,al} \right\} \frac{\partial z_2}{\partial \tau_{p,y}} + \left\{ \frac{1}{r} (1+(1-\tau_{2,rk_1}) r) u_{2,ac} \right\} \frac{\partial (rk_1)}{\partial \tau_{p,y}} \\ & + \left\{ -y u_{p,ac} \left(\mathbf{x}^2 (\theta_2^*) \right) \right\} \mathbf{1}_{\left\{\tau_{p,y}=\tau_{2,y}\left(\mathbf{x}^2(\theta_2^*)\right)\right\}} + \left\{ u_{p,ac} \left(\mathbf{x}^2 (\theta_2^*) \right) \right\} \mathbf{1}_{\left\{\tau_{p,y}=R_2\left(\mathbf{x}^2(\theta_2^*)\right)\right\}}.\end{aligned} \tag{33}$$

Differentiating the first-order conditions w.r.t. $\tau_{p,y}$ using (32) and (33) then yields

$$\begin{aligned} 0 =& \frac{1}{\theta_1} \frac{\partial u_{1,l}}{\partial \tau_{p,y}} + (1-\tau_{1,z_1}) \frac{\partial u_{1,c}}{\partial \tau_{p,y}} - \beta \mathbb{E}_{\theta_2} \left[\tau_{2,z_1} \frac{\partial u_{2,c}}{\partial \tau_{p,y}} \,|\theta_1 \right] \\ & - u_{1,c} \mathbf{1}_{\left\{\tau_{p,y}=\tau_{1,z_1}\right\}} - \beta u_{2,c} \left(\mathbf{x}^2 (\theta_2^*) \right) f_{2|1} \left(\theta_2^* \,|\theta_1 \right) \mathbf{1}_{\left\{\tau_{p,y}=\tau_{2,z_1}\left(\mathbf{x}^2(\theta_2^*)\right)\right\}}, \\ 0 =& \frac{1}{\theta_2} \frac{\partial u_{2,l}}{\partial \tau_{p,y}} + (1-\tau_{2,z_2}) \frac{\partial u_{2,c}}{\partial \tau_{p,y}} - u_{2,c} \left(\mathbf{x}^2 (\theta_2^*) \right) \mathbf{1}_{\left\{\tau_{p,y}=\tau_{2,z_2}\left(\mathbf{x}^2(\theta_2^*)\right)\right\}}, \quad \forall \theta_2 \in \mathbb{R}_+, \\ 0 =& (1+\tau_{1,rk_1} r) \frac{\partial u_{1,c}}{\partial \tau_{p,y}} - \beta \mathbb{E}_{\theta_2} \left[(1+(1-\tau_{2,rk_1}) r) \frac{\partial u_{2,c}}{\partial \tau_{p,y}} \,|\theta_1 \right] \\ & + r u_{1,c} \mathbf{1}_{\left\{\tau_{p,y}=\tau_{1,rk_1}\right\}} + \beta r u_{2,c} \left(\mathbf{x}^2 (\theta_2^*) \right) f_{2|1} \left(\theta_2^* \,|\theta_1 \right) \mathbf{1}_{\left\{\tau_{p,y}=\tau_{2,rk_1}\left(\mathbf{x}^2(\theta_2^*)\right)\right\}}.\end{aligned} \tag{34}$$

Thus, the differentiated intratemporal first order condition in period one writes:

$$\begin{aligned} & \left\{ (1-\tau_{1,z_1}) \left((1-\tau_{1,z_1}) u_{1,cc} + \frac{1}{\theta_1} u_{1,cl} \right) + \frac{1}{\theta_1} \left((1-\tau_{1,z_1}) u_{1,cl} + \frac{1}{\theta_1} u_{1,ll} \right) + \beta \mathbb{E}_{\theta_2} \left[\tau_{2,z_1}^2 u_{2,cc} \,|\theta_1 \right] \right\} \frac{\partial z_1}{\partial \tau_{p,y}} \\ & + \int_0^\infty \left\{ -\beta \tau_{2,z_1} \left((1-\tau_{2,z_2}) u_{2,cc} + \frac{1}{\theta_2} u_{2,cl} \right) f_{2|1} \left(\theta_2 \,|\theta_1 \right) \right\} \frac{\partial z_2 (\theta_2)}{\partial \tau_{p,y}} d\theta_2 \\ & + \left\{ -\left(\frac{1}{r} + \tau_{1,rk_1} \right) \left((1-\tau_{1,z_1}) u_{1,cc} + \frac{1}{\theta_1} u_{1,cl} \right) - \beta \mathbb{E}_{\theta_2} \left[\tau_{2,z_1} \left(\frac{1}{r} + (1-\tau_{2,rk_1}) \right) u_{2,cc} \,|\theta_1 \right] \right\} \frac{\partial (rk_1)}{\partial \tau_{p,y}} \\ =& \left\{ y \left((1-\tau_{1,z_1}) u_{1,cc} + \frac{1}{\theta_1} u_{1,cl} \right) + u_{1,c} \mathbf{1}_{\{y=z_1\}} \right\} \mathbf{1}_{\{\tau_{p,y}=\tau_{1,y}\}} + \left\{ -(1-\tau_{1,z_1}) u_{1,cc} - \frac{1}{\theta_1} u_{1,cl} \right\} \mathbf{1}_{\{\tau_{p,y}=R_1\}} \\ & + \left\{ -\beta \left[y \tau_{2,z_1} \left(\mathbf{x}^2 (\theta_2^*) \right) u_{2,cc} \left(\mathbf{x}^2 (\theta_2^*) \right) - u_{2,c} \left(\mathbf{x}^2 (\theta_2^*) \right) \mathbf{1}_{\{y=z_1\}} \right] f_{2|1} \left(\theta_2^* \,|\theta_1 \right) \right\} \mathbf{1}_{\left\{\tau_{p,y}=\tau_{2,y}\left(\mathbf{x}^2(\theta_2^*)\right)\right\}} \\ & + \left\{ \tau_{2,z_1} \left(\mathbf{x}^2 (\theta_2^*) \right) \beta u_{2,cc} \left(\mathbf{x}^2 (\theta_2^*) \right) f_{2|1} \left(\theta_2^* \,|\theta_1 \right) \right\} \mathbf{1}_{\left\{\tau_{p,y}=R_2\left(\mathbf{x}^2(\theta_2^*)\right)\right\}}.\end{aligned} \tag{35}$$

Similarly, the differentiated intratemporal first order condition in period two and state $\theta_2 \in \mathbb{R}_+$

writes:

$$
\begin{aligned}
&\left\{\tau_{2,z_1}\left(\left(1-\tau_{2,z_2}\right)u_{2,cc}+\frac{1}{\theta_2}u_{2,cl}\right)\right\}\frac{\partial z_1}{\partial \tau_{p,y}}\\
&+\left\{-\left(1-\tau_{2,z_2}\right)\left(\left(1-\tau_{2,z_2}\right)u_{2,cc}+\frac{1}{\theta_2}u_{2,cl}\right)-\frac{1}{\theta_2}\left(\left(1-\tau_{2,z_2}\right)u_{2,cl}+\frac{1}{\theta_2}u_{2,ll}\right)\right\}\frac{\partial z_2\left(\theta_2\right)}{\partial \tau_{p,y}}\\
&+\left\{-\left(\frac{1}{r}+\left(1-\tau_{2,rk_1}\right)\right)\left(\left(1-\tau_{2,z_2}\right)u_{2,cc}+\frac{1}{\theta_2}u_{2,lc}\right)\right\}\frac{\partial \left(rk_1\right)}{\partial \tau_{p,y}}\\
=&\left\{-y\left(\left(1-\tau_{2,z_2}\left(\mathbf{x}^2\left(\theta_2^*\right)\right)\right)u_{2,cc}\left(\mathbf{x}^2\left(\theta_2^*\right)\right)+\frac{1}{\theta_2^*}u_{2,lc}\left(\mathbf{x}^2\left(\theta_2^*\right)\right)\right)-u_{2,c}\left(\mathbf{x}^2\left(\theta_2^*\right)\right)\mathbf{1}_{\{y=z_2\}}\right\}\mathbf{1}_{\left\{\tau_{p,y}=\tau_{2,y}\left(\mathbf{x}^2\left(\theta_2^*\right)\right)\right\}}\\
&+\left\{\left(1-\tau_{2,z_2}\left(\mathbf{x}^2\left(\theta_2^*\right)\right)\right)u_{2,cc}\left(\mathbf{x}^2\left(\theta_2^*\right)\right)+\frac{1}{\theta_2^*}u_{2,lc}\left(\mathbf{x}^2\left(\theta_2^*\right)\right)\right\}\mathbf{1}_{\left\{\tau_{p,y}=R_2\left(\mathbf{x}^2\left(\theta_2^*\right)\right)\right\}}.
\end{aligned}
\tag{36}
$$

Finally, the differentiated intertemporal first order condition writes:

$$
\begin{aligned}
&\left\{\left(\frac{1}{r}+\tau_{1,rk_1}\right)\left(\left(1-\tau_{1,z_1}\right)u_{1,cc}+\frac{1}{\theta_1}u_{1,cl}\right)+\beta\mathbb{E}_{\theta_2}\left[\tau_{2,z_1}\left(\frac{1}{r}+\left(1-\tau_{2,rk_1}\right)\right)u_{2,cc}\left|\theta_1\right.\right]\right\}\frac{\partial z_1}{\partial \tau_{p,y}}\\
&+\int_0^\infty\left\{-\beta\left(\frac{1}{r}+\left(1-\tau_{2,rk_1}\right)\right)\left(\left(1-\tau_{2,z_2}\right)u_{2,cc}+\frac{1}{\theta_2}u_{2,cl}\right)f_{2|1}\left(\theta_2\left|\theta_1\right.\right)\right\}\frac{\partial z_2}{\partial \tau_{p,y}}d\theta_2\\
&+\left\{-\left(\frac{1}{r}+\tau_{1,rk_1}\right)^2u_{1,cc}-\beta\mathbb{E}_{\theta_2}\left[\left(\frac{1}{r}+\left(1-\tau_{2,rk_1}\right)\right)^2u_{2,cc}\left|\theta_1\right.\right]\right\}\frac{\partial\left(rk_1\right)}{\partial \tau_{p,y}}\\
=&\left\{y\left(\frac{1}{r}+\tau_{1,rk_1}\right)u_{1,cc}-u_{1,c}\mathbf{1}_{\{y=rk_1\}}\right\}\mathbf{1}_{\{\tau_{p,y}=\tau_{1,y}\}}+\left\{-\left(\frac{1}{r}+\tau_{1,rk_1}\right)u_{1,cc}\right\}\mathbf{1}_{\{\tau_{p,y}=R_1\}}\\
&+\left\{-\beta\left[y\left(\frac{1}{r}+\left(1-\tau_{2,rk_1}\left(\mathbf{x}^2\left(\theta_2^*\right)\right)\right)\right)u_{2,cc}\left(\mathbf{x}^2\left(\theta_2^*\right)\right)+u_{2,c}\left(\mathbf{x}^2\left(\theta_2^*\right)\right)\mathbf{1}_{\{y=rk_1\}}\right]f_{2|1}\left(\theta_2^*\left|\theta_1\right.\right)\right\}\mathbf{1}_{\left\{\tau_{p,y}=\tau_{2,y}\left(\mathbf{x}^2\left(\theta_2^*\right)\right)\right\}}\\
&+\left\{\left(\frac{1}{r}+\left(1-\tau_{2,rk_1}\left(\mathbf{x}^2\left(\theta_2^*\right)\right)\right)\right)\beta u_{2,cc}\left(\mathbf{x}^2\left(\theta_2^*\right)\right)f_{2|1}\left(\theta_2^*\left|\theta_1\right.\right)\right\}\mathbf{1}_{\left\{\tau_{p,y}=R_2\left(\mathbf{x}^2\left(\theta_2^*\right)\right)\right\}}.
\end{aligned}
\tag{37}
$$

We define the matrix A as

$$
A=\begin{pmatrix}
a_{z_1,z_1} & a_{z_1,\underline{\theta}_2} & \cdots & a_{z_1,\theta_2} & \cdots & a_{z_1,\bar{\theta}_2} & a_{z_1,k_1}\\
a_{\underline{\theta}_2,z_1} & a_{\underline{\theta}_2,\underline{\theta}_2} & 0 & \cdots & \cdots & 0 & a_{\underline{\theta}_2,k_1}\\
\vdots & 0 & \ddots & \ddots & 0 & \vdots & \vdots\\
a_{\theta_2,z_1} & \vdots & \ddots & a_{\theta_2,\theta_2} & \ddots & \vdots & a_{\theta_2,k_1}\\
\vdots & \vdots & 0 & \ddots & \ddots & 0 & \vdots\\
a_{\bar{\theta}_2,z_1} & 0 & \cdots & \cdots & 0 & a_{\bar{\theta}_2,\bar{\theta}_2} & a_{\bar{\theta}_2,k_1}\\
a_{k_1,z_1} & a_{k_1,\underline{\theta}_2} & \cdots & a_{k_1,\theta_2} & \cdots & a_{k_1,\bar{\theta}_2} & a_{k_1,k_1}
\end{pmatrix},
\tag{38}
$$

where the coefficients on the first line are given by

$$\begin{aligned}
a_{z_1,z_1} &= (1-\tau_{1,z_1})\left((1-\tau_{1,z_1})\, u_{1,cc} + \frac{1}{\theta_1}u_{1,cl}\right) + \frac{1}{\theta_1}\left((1-\tau_{1,z_1})\, u_{1,cl} + \frac{1}{\theta_1}u_{1,ll}\right) + \beta\mathbb{E}_{\theta_2}\left[\tau_{2,z_1}^2 u_{2,cc}\,|\theta_1\right], \\
a_{z_1,\theta_2} &= -\beta\tau_{2,z_1}\left((1-\tau_{2,z_2})\, u_{2,cc} + \frac{1}{\theta_2}u_{2,cl}\right) f_{2|1}\left(\theta_2\,|\theta_1\right), \quad \forall\theta_2\in\mathbb{R}_+, \\
a_{z_1,k_1} &= -\left(\frac{1}{r}+\tau_{1,rk_1}\right)\left((1-\tau_{1,z_1})\, u_{1,cc} + \frac{1}{\theta_1}u_{1,cl}\right) - \beta\mathbb{E}_{\theta_2}\left[\tau_{2,z_1}\left(\frac{1}{r}+(1-\tau_{2,rk_1})\right) u_{2,cc}\,|\theta_1\right],
\end{aligned}$$

the coefficients on the last line are given by

$$\begin{aligned}
a_{k_1,z_1} &= \left(\frac{1}{r}+\tau_{1,rk_1}\right)\left((1-\tau_{1,z_1})\, u_{1,cc} + \frac{1}{\theta_1}u_{1,cl}\right) + \beta\mathbb{E}_{\theta_2}\left[\tau_{2,z_1}\left(\frac{1}{r}+(1-\tau_{2,rk_1})\right) u_{2,cc}\,|\theta_1\right], \\
a_{k_1,\theta_2} &= -\beta\left(\frac{1}{r}+(1-\tau_{2,rk_1})\right)\left((1-\tau_{2,z_2})\, u_{2,cc} + \frac{1}{\theta_2}u_{2,cl}\right) f_{2|1}\left(\theta_2\,|\theta_1\right), \quad \forall\theta_2\in\mathbb{R}_+, \\
a_{k_1,k_1} &= -\left(\frac{1}{r}+\tau_{1,rk_1}\right)^2 u_{1,cc} - \beta\mathbb{E}_{\theta_2}\left[\left(\frac{1}{r}+(1-\tau_{2,rk_1})\right)^2 u_{2,cc}\,|\theta_1\right],
\end{aligned}$$

and the coefficients on the interior lines are all equal to zero except on the edges and the diagonal, with for all $\theta_2 \in \mathbb{R}_+$,

$$\begin{aligned}
a_{\theta_2,1} &= \tau_{2,z_1}\left((1-\tau_{2,z_2})\, u_{2,cc} + \frac{1}{\theta_2}u_{2,cl}\right), \\
a_{\theta_2,\theta_2} &= -(1-\tau_{2,z_2})\left((1-\tau_{2,z_2})\, u_{2,cc} + \frac{1}{\theta_2}u_{2,cl}\right) - \frac{1}{\theta_2}\left((1-\tau_{2,z_2})\, u_{2,cl} + \frac{1}{\theta_2}u_{2,ll}\right), \\
a_{\theta_2,k_1} &= -\left(\frac{1}{r}+(1-\tau_{2,rk_1})\right)\left((1-\tau_{2,z_2})\, u_{2,cc} + \frac{1}{\theta_2}u_{2,cl}\right).
\end{aligned}$$

We define the vectors $B_{\tau_{1,y}}$, B_{R_1}, and $B^c_{\tau_{1,y}}$, for $y \in \{z_1, rk_1\}$, as

$$B_{\tau_{1,y}} = \begin{pmatrix} b_1^{\tau_{1,y}} \\ 0 \\ \vdots \\ 0 \\ b_{k_1}^{\tau_{1,y}} \end{pmatrix}, B_{R_1} = \begin{pmatrix} b_1^{R_1} \\ 0 \\ \vdots \\ 0 \\ b_{k_1}^{R_1} \end{pmatrix}, B^c_{\tau_{1,y}} = \begin{pmatrix} b_1^{c,\tau_{1,y}} \\ 0 \\ \vdots \\ 0 \\ b_{k_1}^{c,\tau_{1,y}} \end{pmatrix}, \tag{39}$$

where the coefficients of $B_{\tau_{1,y}}$ are given by

$$\begin{aligned}
b_1^{\tau_{1,y}} &= u_{1,c}\mathbf{1}_{\{y=z_1\}} + \left((1-\tau_{1,z_1})\, u_{1,cc} + \frac{1}{\theta_1}u_{1,cl}\right) y, \\
b_{k_1}^{\tau_{1,y}} &= -u_{1,c}\mathbf{1}_{\{y=rk_1\}} + \left(\left(\frac{1}{r}+\tau_{1,rk_1}\right) u_{1,cc}\right) y,
\end{aligned}$$

the coefficients of B_{R_1} are given by

$$\begin{aligned}
b_1^{R_1} &= -\left(\left(1-\tau_{1,z_1}\right)u_{1,cc} + \frac{1}{\theta_1}u_{1,cl}\right), \\
b_{k_1}^{R_1} &= -\left(\frac{1}{r} + \tau_{1,rk_1}\right)u_{1,cc},
\end{aligned}$$

and the coefficients of $B^c_{\tau_{1,y}}$ are given by

$$\begin{aligned}
b_1^{c,\tau_{1,y}} &= u_{1,c}\mathbf{1}_{\{y=z_1\}}, \\
b_{k_1}^{c,\tau_{1,y}} &= -u_{1,c}\mathbf{1}_{\{y=rk_1\}}.
\end{aligned}$$

Moreover we define the matrix $B^c_{\tau_1}$ whose first column is the vector $B^c_{\tau_{1,z_1}}$, whose interior columns are all zero, and whose last column is the vector $B^c_{\tau_{1,rk_1}}$. Thus $B^c_{\tau_1}$ has only two non-zero components, $b_1^{c,\tau_{1,z_1}}$ in the first row and first column, and $b_{k_1}^{c,\tau_{1,rk_1}}$ in the last row and last column.

We also define the vectors $B_{\tau_{2,y}(\mathbf{x}^2)}$, $B_{R_2(\mathbf{x}^2)}$, and $B^c_{\tau_{2,y}(\mathbf{x}^2)}$, for $y \in \{z_1, z_2, rk_1\}$ and $\mathbf{x}^2 = \left\{z_1, \mathbf{x}_2^2\left(\theta_2^*\right), rk_1\right\}$ (meaning that the individuals who choose (z_1, rk_1) in the first period, will actually face the tax change in period 2 if their second-period type is θ_2^*, so that their second period income is $z_2 = \mathbf{x}_2^2\left(\theta_2^*\right)$, at the point where the tax function is perturbed), as

$$B_{\tau_{2,y}(\mathbf{x}^2)} = \begin{pmatrix} b_1^{\tau_{2,y}(\mathbf{x}^2)} \\ 0 \\ \vdots \\ 0 \\ b_{\theta_2^*}^{\tau_{2,y}(\mathbf{x}^2)} \\ 0 \\ \vdots \\ 0 \\ b_{k_1}^{\tau_{2,y}(\mathbf{x}^2)} \end{pmatrix}, B_{R_2(\mathbf{x}^2)} = \begin{pmatrix} b_1^{R_2(\mathbf{x}^2)} \\ 0 \\ \vdots \\ 0 \\ b_{\theta_2^*}^{R_2(\mathbf{x}^2)} \\ 0 \\ \vdots \\ 0 \\ b_{k_1}^{R_2(\mathbf{x}^2)} \end{pmatrix}, B^c_{\tau_{2,y}(\mathbf{x}^2)} = \begin{pmatrix} b_1^{c,\tau_{2,y}(\mathbf{x}^2)} \\ 0 \\ \vdots \\ 0 \\ b_{\theta_2^*}^{c,\tau_{2,y}(\mathbf{x}^2)} \\ 0 \\ \vdots \\ 0 \\ b_{k_1}^{c,\tau_{2,y}(\mathbf{x}^2)} \end{pmatrix}, \tag{40}$$

where the coefficients of $B_{\tau_{2,y}(\mathbf{x}^2)}$ are given by

$$\begin{aligned}
b_1^{\tau_{2,y}(\mathbf{x}^2)} &= \beta\left(u_{2,c}\mathbf{1}_{\{y=z_1\}} - \tau_{2,z_1}u_{2,cc}y\right)\Big|_{\mathbf{x}^2} f_{2|1}\left(\theta_2^* \left|\theta_1\right.\right), \\
b_{\theta_2^*}^{\tau_{2,y}(\mathbf{x}^2)} &= \left(-u_{2,c}\mathbf{1}_{\{y=z_2\}} - \left(\left(1-\tau_{2,z_2}\right)u_{2,cc} + \frac{1}{\theta_2^*}u_{2,cl}\right)y\right)\Bigg|_{\mathbf{x}^2}, \\
b_{k_1}^{\tau_{2,y}(\mathbf{x}^2)} &= \beta\left(-u_{2,c}\mathbf{1}_{\{y=rk_1\}} - \left(\frac{1}{r} + \left(1-\tau_{2,rk_1}\right)\right)u_{2,cc}y\right)\Bigg|_{\mathbf{x}^2} f_{2|1}\left(\theta_2^* \left|\theta_1\right.\right),
\end{aligned}$$

the coefficients of $B_{R_2(\mathbf{x}^2)}$ are given by

$$\begin{aligned}
b_1^{R_2(\mathbf{x}^2)} &= \beta \left.\left(\tau_{2,z_1} u_{2,cc}\right)\right|_{\mathbf{x}^2} f_{2|1}\left(\theta_2^* \left|\theta_1\right.\right), \\
b_{\theta_2^*}^{R_2(\mathbf{x}^2)} &= \left.\left(\left(1-\tau_{2,z_2}\right) u_{2,cc} + \frac{1}{\theta_2^*} u_{2,cl}\right)\right|_{\mathbf{x}^2}, \\
b_{k_1}^{R_2(\mathbf{x}^2)} &= \beta \left.\left(\left(\frac{1}{r} + \left(1-\tau_{2,rk_1}\right)\right) u_{2,cc}\right)\right|_{\mathbf{x}^2} f_{2|1}\left(\theta_2^* \left|\theta_1\right.\right),
\end{aligned}$$

and the coefficients of $B^c_{\tau_{2,y}(\mathbf{x}^2)}$ are given by

$$\begin{aligned}
b_1^{c,\tau_{2,y}(\mathbf{x}^2)} &= \beta \left. u_{2,c}\right|_{\mathbf{x}^2} f_{2|1}\left(\theta_2^* \left|\theta_1\right.\right) \mathbf{1}_{\{y=z_1\}}, \\
b_{\theta_2^*}^{c,\tau_{2,y}(\mathbf{x}^2)} &= - \left. u_{2,c}\right|_{\mathbf{x}^2} \mathbf{1}_{\{y=z_2\}}, \\
b_{k_1}^{c,\tau_{2,y}(\mathbf{x}^2)} &= -\beta \left. u_{2,c}\right|_{\mathbf{x}^2} f_{2|1}\left(\theta_2^* \left|\theta_1\right.\right) \mathbf{1}_{\{y=rk_1\}}.
\end{aligned}$$

Note that the only interior row of each of the vectors $B_{\tau_{2,y}(\mathbf{x}^2)}, B_{R_2(\mathbf{x}^2)}, B^c_{\tau_{2,y}(\mathbf{x}^2)}$ that has a non-zero component is the row indexed by θ_2^* (such that $z_2\left(\theta_2^*\right) = \mathbf{x}_2^2$ for the individual who chooses $(z_1, rk_1) = \left(\mathbf{x}_1^2, \mathbf{x}_3^2\right)$, where $\mathbf{x}^2 = \left(\mathbf{x}_1^2, \mathbf{x}_2^2, \mathbf{x}_3^2\right)$ is the point at which the tax function is perturbed) and where all the components of these vectors are evaluated at $\theta_2 = \theta_2^*$ and $X_2 = (z_1, z_2, rk_1) = \left(\mathbf{x}_1^2, \mathbf{x}_2^2, \mathbf{x}_3^2\right)$.

We finally define the matrix $B^c_{\tau_2(\mathbf{x}^2)}$ whose first column is the vector $B^c_{\tau_{2,z_1}(\mathbf{x}^2)}$, whose only non-zero interior column, indexed by θ_2^*, is the vector $B^c_{\tau_{2,z_2}(\mathbf{x}^2)}$, and whose last column is the vector $B^c_{\tau_{2,z_2}(\mathbf{x}^2)}$. Thus $B^c_{\tau_2(\mathbf{x}^2)}$ has only three non-zero components, $b_1^{c,\tau_{2,z_1}(\mathbf{x}^2)}$ in the first row and first column, $b_{\theta_2^*}^{c,\tau_{2,z_2}(\mathbf{x}^2)}$ in the $\left(\theta_2^*\right)^{th}$ row and $\left(\theta_2^*\right)^{th}$ column, and $b_{k_1}^{c,\tau_{2,rk_1}(\mathbf{x}^2)}$ in the last row and last column.

The system of differentiated first order conditions (35), (36), and (37) then writes, in matrix form, as

$$A \times \left(\partial X / \partial \tau_{p,y}\right) = B_{\tau_{p,y}},$$

for any $\tau_{p,y} \in \left\{\tau_{1,z_1}, \tau_{1,rk_1}, R_1, \tau_{2,z_1}\left(\mathbf{x}^2\right), \tau_{2,z_2}\left(\mathbf{x}^2\right), \tau_{2,rk_1}\left(\mathbf{x}^2\right), R_2\left(\mathbf{x}^2\right)\right\}$. We show in the next subsection that the matrices with a continuum of rows and columns can be inverted in a natural way. We thus obtain the partial derivatives of the (uncompensated) demands:

$$\frac{\partial X}{\partial \tau_{p,y}} = \begin{pmatrix} \partial z_1 / \partial \tau_{p,y} \\ \partial z_2\left(\underline{\theta}_2\right) / \partial \tau_{p,y} \\ \vdots \\ \partial z_2\left(\bar{\theta}_2\right) / \partial \tau_{p,y} \\ \partial\left(rk_1\right) / \partial \tau_{p,y} \end{pmatrix} = A^{-1} \times B_{\tau_{p,y}}. \tag{41}$$

Moreover, from the Slutsky equations (31), for any $\tau_{p,y} \in \left\{\tau_{1,z_1}, \tau_{1,rk_1}, \tau_{2,z_1}\left(\mathbf{x}^2\right), \tau_{2,z_2}\left(\mathbf{x}^2\right), \tau_{2,rk_1}\left(\mathbf{x}^2\right)\right\}$, we obtain the partial derivatives of the compensated demands:

$$\frac{\partial X^c}{\partial \tau_{p,y}} = \frac{\partial X}{\partial \tau_{p,y}} + \frac{\partial X}{\partial R_p} y = \begin{pmatrix} \partial z_1^c / \partial \tau_{p,y} \\ \partial z_2^c \left(\underline{\theta}_2\right) / \partial \tau_{p,y} \\ \vdots \\ \partial z_2^c \left(\bar{\theta}_2\right) / \partial \tau_{p,y} \\ \partial \left(r k_1^c\right) / \partial \tau_{p,y} \end{pmatrix} = A^{-1} \times B^c_{\tau_{p,y}}. \tag{42}$$

Finally, using the definitions of the matrices $B^c_{\tau_1}$ and $B^c_{\tau_2(\mathbf{x}^2)}$, we have

$$\begin{aligned} &\boldsymbol{\zeta}^{c,(X)}_{X,\tau_1} = A^{-1} B^c_{\tau_1}, \\ &\boldsymbol{\zeta}^{c,(X)}_{X,\tau_2(z_1,z_2(\theta_2),rk_1)} = A^{-1} B^c_{\tau_2(z_1,z_2(\theta_2),rk_1)}, \\ &\boldsymbol{\eta}^{(X)}_{X,R_1} = A^{-1} B_{R_1}, \\ &\boldsymbol{\eta}^{(X)}_{X,R_2(\mathbf{x}^2)} = A^{-1} B_{R_2(\mathbf{x}^2)}, \end{aligned} \tag{43}$$

which gives explicit expressions for the matrices of elasticities and vectors of income effect parameters.

A.0.4 Matrix Operations

We define the product of two matrices M and N with a continuum of interior rows and columns (indexed by the values $\theta_2 \in \left[\underline{\theta}_2, \bar{\theta}_2\right) = \mathbb{R}_+$), one row at the top and one column on the left (indexed by z_1), and one row at the bottom and one column on the right (indexed by k_1), as

$$[M \times N]_{i,j} = m_{i,z_1} n_{z_1,j} + \int_{\underline{\theta}_2}^{\bar{\theta}_2} m_{i,\theta_2'} n_{\theta_2',j} d\theta_2' + m_{i,k_1} n_{k_1,j}, \quad \forall i,j \in \{z_1, \theta_2, k_1\}.$$

Consider for instance the matrix A defined in (38), whose coefficients are $a_{\theta_2,\theta_2'} = 0$ for all $\theta_2 \in \mathbb{R}_+$ with $\theta_2 \neq \theta_2'$. The equation that we write formally $A \times X = B$ (X being a vector or a matrix) means that X is the solution to the corresponding system of (a continuum of) equations. It easy to solve this system for $B = I$, where I is the identity matrix. In this case, we denote the solution to the system as $X \equiv A^{-1}$, the inverse of the matrix A. Straightforward algebra shows that, letting $\lambda_{i,j} = a_{i,j} - \int_0^\infty \frac{a_{i,\theta_2} a_{\theta_2,j}}{a_{\theta_2,\theta_2}} d\theta_2$ for $i,j \in \{z_1, k_1\}$ and $D = \lambda_{z_1,z_1}\lambda_{k_1,k_1} - \lambda_{z_1,k_1}\lambda_{k_1,z_1}$, the coefficients of the matrix A^{-1} are given by

$$\left[A^{-1}\right]_{z_1,z_1} = \frac{1}{D}\lambda_{k_1,k_1}, \left[A^{-1}\right]_{z_1,k_1} = \frac{-1}{D}\lambda_{z_1,k_1}, \left[A^{-1}\right]_{k_1,z_1} = \frac{1}{D}\lambda_{k_1,z_1}, \left[A^{-1}\right]_{k_1,k_1} = \frac{-1}{D}\lambda_{z_1,z_1},$$
$$\left[A^{-1}\right]_{\theta_2,z_1} = \frac{1}{D}\left(\lambda_{k_1,z_1}\frac{a_{\theta_2,k_1}}{a_{\theta_2,\theta_2}} - \lambda_{k_1,k_1}\frac{a_{\theta_2,z_1}}{a_{\theta_2,\theta_2}}\right), \left[A^{-1}\right]_{z_1,\theta_2} = \frac{1}{D}\left(\lambda_{z_1,k_1}\frac{a_{k_1,\theta_2}}{a_{\theta_2,\theta_2}} - \lambda_{k_1,k_1}\frac{a_{z_1,\theta_2}}{a_{\theta_2,\theta_2}}\right),$$
$$\left[A^{-1}\right]_{\theta_2,k_1} = \frac{-1}{D}\left(\lambda_{z_1,z_1}\frac{a_{\theta_2,k_1}}{a_{\theta_2,\theta_2}} - \lambda_{z_1,k_1}\frac{a_{\theta_2,z_1}}{a_{\theta_2,\theta_2}}\right), \left[A^{-1}\right]_{k_1,\theta_2} = \frac{-1}{D}\left(\lambda_{z_1,z_1}\frac{a_{k_1,\theta_2}}{a_{\theta_2,\theta_2}} - \lambda_{k_1,z_1}\frac{a_{z_1,\theta_2}}{a_{\theta_2,\theta_2}}\right),$$
$$\left[A^{-1}\right]_{\theta_2,\theta_2'} = \frac{\mathbf{1}_{\{\theta_2=\theta_2'\}}}{a_{\theta_2,\theta_2}} + \frac{1}{D}\left[\frac{a_{\theta_2',k_1}}{a_{\theta_2',\theta_2'}}\left(\lambda_{z_1,z_1}\frac{a_{k_1,\theta_2}}{a_{\theta_2,\theta_2}} - \lambda_{k_1,z_1}\frac{a_{z_1,\theta_2}}{a_{\theta_2,\theta_2}}\right) + \frac{a_{\theta_2',z_1}}{a_{\theta_2',\theta_2'}}\left(\lambda_{k_1,k_1}\frac{a_{z_1,\theta_2}}{a_{\theta_2,\theta_2}} - \lambda_{z_1,k_1}\frac{a_{k_1,\theta_2}}{a_{\theta_2,\theta_2}}\right)\right].$$

Moreover, we can then easily show that using this definition for A^{-1}, X is the solution to the system of equations $X = A^{-1} \times B$, where the multiplication is defined as above. Therefore, the multiplication and inverse of A, and thus expressions as, e.g., (43), are well defined.

A.0.5 Elasticities and Income Effect Parameters: Explicit Expressions with the Simplifying Assumptions of Section 6

Suppose, as in Section 6 of the deterministic model, that the flow utility has no income effects, $u(c,l) = u(c - v(l))$, that the initial tax system is between z_1, z_2, and rk_1, and separable consists of a linear capital income tax $\tau_{2,rk_1} \times (rk_1)$ levied in period two,[5] and a labor income tax schedule $T_{t,z_t}(z_t)$ in each period t.

Under these assumptions, the matrix A is diagonal,[6] so that A^{-1} is diagonal with coefficients

$$a_{z_1,z_1}^{-1} = -\theta_1^2\left(v_1''u_1'\right)^{-1}, \quad a_{\theta_2,\theta_2}^{-1} = \theta_2^2\left(v_2''u_2'\right)^{-1} \text{ for all } \theta_2 \in \mathbb{R}_+,$$
$$a_{k_1,k_1}^{-1} = -r^2\left\{u_1'' + \beta\mathbb{E}_{\theta_2}\left[\left(1 + \left(1 - \tau_{2,rk_1}\right)r\right)^2 u_2'' \,|\theta_1\right]\right\}^{-1}.$$

Moreover, each of the vectors $B^c_{\tau_{1,y}}$, $B^c_{R_1}$, $B^c_{\tau_{2,y}(\mathbf{x}^2)}$, $B^c_{R_2(\mathbf{x}^2)}$ has only one non-zero component, with

$$b_{z_1}^{c,\tau_{1,z_1}} = u_1',\ \ b_{k_1}^{c,\tau_{1,rk_1}} = -u_1',\ \ b_{k_1}^{R_1} = -\frac{1}{r}u_1'',\ \ b_{k_1}^{R_2(\mathbf{x}^2)} = \frac{\beta}{r}\left(1 + \left(1 - \tau_{2,rk_1}\right)r\right) u_2''\big|_{\mathbf{x}^2} f_{2|1}\left(\theta_2^*\,|\theta_1\right),$$
$$b_{z_1}^{c,\tau_{2,z_1}(\mathbf{x}^2)} = \beta\, u_2'\big|_{\mathbf{x}^2} f_{2|1}\left(\theta_2^*\,|\theta_1\right),\ \ b_{\theta_2^*}^{c,\tau_{2,z_2}(\mathbf{x}^2)} = -\, u_2'\big|_{\mathbf{x}^2},\ \ b_{k_1}^{c,\tau_{2,rk_1}(\mathbf{x}^2)} = -\beta\, u_2'\big|_{\mathbf{x}^2} f_{2|1}\left(\theta_2^*\,|\theta_1\right).$$

Therefore, we obtain that the elasticities and income effect parameters of labor incomes are given

[5] Note that, since we assume for simplicity that r is deterministic, their capital income tax liability is known to individuals with certainty in period one, when they choose how much to save.

[6] As in the deterministic model, in a model with more than two periods, and hence more than one choice of capital income, only the upper-left block of the matrix A (i.e., the rows and columns corresponding to labor incomes) would be diagonal.

by, for all $\theta_2 \in \mathbb{R}_+$,

$$\begin{aligned} &\frac{\partial z_1^c}{\partial \tau_{1,y}} = \frac{-\theta_1^2}{v_1''}\mathbf{1}_{\{y=z_1\}}, \quad \frac{\partial z_2^c(\theta_2)}{\partial \tau_{1,y}} = 0, \quad \frac{\partial z_1}{\partial R_1} = \frac{\partial z_2(\theta_2)}{\partial R_1} = \frac{\partial z_1}{\partial R_2(\mathbf{x}^2)} = \frac{\partial z_2(\theta_2)}{\partial R_2(\mathbf{x}^2)} = 0, \\ &\frac{\partial z_1^c}{\partial \tau_{2,y}(\mathbf{x}^2)} = -\frac{\theta_1^2}{v_1''}\frac{\beta u_2'(\mathbf{x}^2) f_{2|1}(\theta_2^* | \theta_1)}{u_1'}\mathbf{1}_{\{y=z_1\}}, \quad \frac{\partial z_2^c(\theta_2)}{\partial \tau_{2,y}(\mathbf{x}^2)} = \frac{-\theta_2^{*2}}{v_2''(\mathbf{x}_2^2)}\mathbf{1}_{\{y=z_2,\theta_2=\theta_2^*\}}, \end{aligned} \tag{44}$$

and the elasticities and income effect parameters of capital incomes are given by, for all $\theta_2 \in \mathbb{R}_+$,

$$\begin{aligned} &\frac{\partial(rk_1^c)}{\partial \tau_{1,y}} = \frac{1}{D}r^2 u_1' \mathbf{1}_{\{y=rk_1\}}, \quad \frac{\partial(rk_1^c)}{\partial \tau_{2,y}(\mathbf{x}^2)} = \frac{1}{D}\beta r^2 u_2'(\mathbf{x}^2) f_{2|1}(\theta_2^* | \theta_1)\mathbf{1}_{\{y=rk_1\}}, \\ &\frac{\partial(rk_1)}{\partial R_1} = \frac{1}{D}r u_1'', \quad \frac{\partial(rk_1)}{\partial R_2(\mathbf{x}^2)} = \frac{-1}{D}\beta(1+(1-\tau_{2,rk_1})r) u_2''(\mathbf{x}^2) f_{2|1}(\theta_2^* | \theta_1), \end{aligned} \tag{45}$$

with

$$D \equiv u_1'' + \beta(1+(1-\tau_{2,rk_1})r)^2 \mathbb{E}_{\theta_2}\left[u_2'' | \theta_1\right].$$

These elasticities are similar to those found in the deterministic (in the case $T = 2$), except that the terms u_2'' in the expressions of the capital income elasticities are replaced with their expectation conditional on θ_1. In particular, when the utility function has no income effects and the initial tax system is separable, labor incomes z_1 and $\{z_2(\theta_2) : \theta_2 \in \mathbb{R}_+\}$ depend only on the marginal tax rate on labor income that they are actually facing. Thus, there will be no difference between the stochastic and the deterministic model, as will be seen below, as far as the responses of labor incomes to perturbations are concerned. Capital income, however, reacts to a perturbation of $\tau_{2,z_2}(\mathbf{x}^2)$ even if the individual ends up not being affected by the perturbation, because of the positive probability, from the point of view of period one, that he may end up drawing the type θ_2^*, and hence choosing the income $z_2 = \mathbf{x}_2^2$ at which the tax function τ_{2,z_2} is perturbed. Hence a perturbation of $\tau_{2,z_2}(\mathbf{x}^2)$ induces a change in savings of all individuals. In the deterministic model, on the other hand, this perturbation induced a response only of those who were going to face the new tax rate, i.e. with second-period income $\mathbf{x}_2^2$. In the stochastic model, these capital income elasticities are weighted by the probability that the individual ends up being affected by the perturbation, i.e. the density $f_{2|1}(\theta_2^* | \theta_1)$. Thus, a perturbation of $\tau_{2,z_2}(\mathbf{x}^2)$ acts as an income effect for all the individuals, weighted by their probability, conditional on their first-period type θ_1, that $\theta_2 = \theta_2^*$.

In particular, if the utility function is CRRA, we have

$$\begin{aligned} u_1'' &= -\sigma\left\{u_1'\right\}^{1+1/\sigma} = -\sigma\left\{\beta(1+(1-\tau_{2,rk_1})r)\mathbb{E}_{\theta_2}\left[u_2' | \theta_1\right]\right\}^{1+1/\sigma} \\ &\geq \beta^{1+1/\sigma}(1+(1-\tau_{2,rk_1})r)^{1+1/\sigma}\mathbb{E}_{\theta_2}\left[-\sigma\left\{u_2'\right\}^{1+1/\sigma} | \theta_1\right] \\ &= \beta^{1+1/\sigma}(1+(1-\tau_{2,rk_1})r)^{1+1/\sigma}\mathbb{E}_{\theta_2}\left[u_2'' | \theta_1\right], \end{aligned}$$

where the second line follows from Jensen's inequality. Therefore, we obtain the following relationship between the income effect parameter w.r.t. an increase in the first-period virtual income in the deterministic and the stochastic model:

$$\frac{\partial (rk_1)}{\partial R_1} = \frac{ru_1''}{u_1'' + \beta \left(1 + \left(1 - \tau_{2,rk_1}\right) r\right)^2 \mathbb{E}_{\theta_2}\left[u_2'' \left| \theta_1\right.\right]}$$
$$\leq \frac{r}{1 + \beta^{-1/\sigma} \left(1 + \left(1 - \tau_{2,rk_1}\right) r\right)^{1-1/\sigma}} = \left.\frac{\partial (rk_1)}{\partial R_1}\right|^{Det}.$$

Similarly, comparing the effect on capital income of an *certain* increase in the second-period virtual income (i.e., that occurs for any value of the individual's second-period labor income) to the second-period income effect parameter in the deterministic model, yields:

$$\int_{\mathbf{x}_2^2=0}^{\infty} \frac{\partial (rk_1)}{\partial R_2 (\mathbf{x}^2)} d\mathbf{x}_2^2 = \frac{-\beta \left(1 + \left(1 - \tau_{2,rk_1}\right) r\right) \mathbb{E}_{\theta_2}\left[u_2'' \left| \theta_1\right.\right]}{u_1'' + \beta \left(1 + \left(1 - \tau_{2,rk_1}\right) r\right)^2 \mathbb{E}_{\theta_2}\left[u_2'' \left| \theta_1\right.\right]}$$
$$\leq \frac{-\beta^{-1/\sigma} \left(1 + \left(1 - \tau_{2,rk_1}\right) r\right)^{-1/\sigma}}{1 + \beta^{-1/\sigma} \left(1 + \left(1 - \tau_{2,rk_1}\right) r\right)^{1-1/\sigma}} = \left.\frac{\partial (rk_1)}{\partial R_2}\right|^{Det}.$$

Finally, we have the following relationships between the compensated elasticity of capital income in the stochastic and the deterministic models:

$$\int_{\mathbf{x}_2^2=0}^{\infty} \frac{\partial (rk_1^c)}{\partial \left(1 - \tau_{2,rk_1}(\mathbf{x}^2)\right)} d\mathbf{x}_2^2 = \frac{-\beta r^2 \mathbb{E}_{\theta_2}\left[u_2' \left| \theta_1\right.\right]}{u_1'' + \beta \left(1 + \left(1 - \tau_{2,rk_1}\right) r\right)^2 \mathbb{E}_{\theta_2}\left[u_2'' \left| \theta_1\right.\right]}$$
$$\leq \left(\frac{-u_1'}{u_1''}\right) \frac{r^2 \left(1 + \left(1 - \tau_{2,rk_1}\right) r\right)^{-1}}{1 + \beta^{-1/\sigma} \left(1 + \left(1 - \tau_{2,rk_1}\right) r\right)^{1-1/\sigma}} = \left.\frac{\partial (rk_1^c)}{\partial \left(1 - \tau_{2,rk_1}\right)}\right|^{Det}.$$

Thus, all the elasticities and income effect parameters are smaller in the stochastic model than their counterparts in the deterministic model.

Finally, these expressions and the characteristics of the initial tax system assumed in Section 6 imply that the matrix

$$\hat{A} \equiv \left[I - \boldsymbol{\zeta}_{X,\tau_1}^{c,(X)} \left(D^2 T_1 (X_1)\right) - \int_0^{\infty} \boldsymbol{\zeta}_{X,\tau_2(\mathbf{x}^{2\prime})}^{c,(X)} \left(D^2 T_2 \left(\mathbf{x}^{2\prime}\right)\right) d\mathbf{x}_2^{2\prime}\right]^{-1}$$

is diagonal, with components given by

$$\hat{a}_{z_1,z_1} = \left\{1 + \frac{z_1}{1 - \tau_{1,z_1}} \zeta_{z_1,1-\tau_{1,z_1}}^{c,(X_1)} T_1'' (z_1)\right\}^{-1},$$
$$\hat{a}_{\theta_2,\theta_2} = \left\{1 + \frac{z_2 (\theta_2)}{1 - \tau_{2,z_2}\left(\mathbf{x}_2^2 (\theta_2)\right)} \zeta_{z_2,1-\tau_{2,z_2}\left(\mathbf{x}_2^2(\theta_2)\right)}^{c,(\theta_2)} T_2'' \left(\mathbf{x}_2^2 (\theta_2)\right)\right\}^{-1}$$
$$\hat{a}_{k_1,k_1} = 1.$$

Therefore, the behavioral response of the individual $X (k_0, \theta_1)$ to any perturbation, given by formula (19), is given by

$$\begin{aligned}
dz_1 &= \frac{-z_1 \zeta^{c,(X_1)}_{z_1,1-\tau_{1,z_1}} d\tau_{1,z_1} + z_1 \left(1 - T_1'\left(z_1\right)\right) \frac{1}{\tau_{2,z_1}(\mathbf{x}^2)} \zeta^{c,(X_1)}_{z_1,\tau_{2,z_1}(\mathbf{x}^2)} d\tau_{2,z_1}\left(\mathbf{x}^2\right)}{1 - T_1'\left(z_1\right) + z_1 \zeta^{c}_{z_1,1-\tau_{1,z_1}} T_1''\left(z_1\right)}, \\
dz_2\left(\theta_2\right) &= \frac{-z_2\left(\theta_2^*\right) \zeta^{c,(\mathbf{x}^2)}_{z_2,1-\tau_{2,z_2}(\mathbf{x}^2)} d\tau_{2,z_2}\left(\mathbf{x}^2\right)}{1 - T_2'\left(z_2\left(\theta_2^*\right)\right) + z_2\left(\theta_2^*\right) \zeta^{c,(\mathbf{x}^2)}_{z_2,1-\tau_{2,z_2}(\mathbf{x}^2)} T_2''\left(z_2\left(\theta_2^*\right)\right)}, \quad \forall \theta_2 \in \mathbb{R}_+, \\
d\left(rk_1\right) &= rk_1 \zeta^{c,(X_1)}_{rk_1,\tau_{1,rk_1}} d\tau_{1,rk_1} + \eta^{(X_1)}_{rk_1,R_1} dR_1 \\
&\quad + \frac{-rk_1}{1-\tau_{2,rk_1}} \zeta^{c,(X_1)}_{rk_1,1-\tau_{2,rk_1}(\mathbf{x}^2)} d\tau_{2,rk_1}\left(\mathbf{x}^2\right) + \frac{\eta^{(X_1)}_{rk_1,R_2(\mathbf{x}^2)}}{1 + \left(1-\tau_{2,rk_1}\right) r} dR_2\left(\mathbf{x}^2\right),
\end{aligned} \tag{46}$$

where as usual, θ_2^* is the second period type such that $\left(z_1\left(k_0,\theta_1\right), z_2\left(k_0,\theta_1,\theta_2^*\right), rk_1\left(k_0,\theta_1\right)\right) = \mathbf{x}^2$.

A.0.6 Social Marginal Welfare Weights

We now provide the intuition for the definition of the marginal social welfare weights defined in (14), which summarize the government's redistributive tastes in the model.

Consider individuals with first period type $\left(\bar{k}_0,\bar{\theta}_1\right)$ and first-period choice vector $\bar{X}_1$ under the initial tax system, such that $X_1\left(\bar{k}_0,\bar{\theta}_1\right) = \bar{X}_1$. Similarly, consider one of those individuals, with (realized) second period type $\left(\bar{k}_0,\bar{\theta}_1,\bar{\theta}_2\right)$ and (realized) choice vector $\bar{X}_2$ under the initial tax system, such that $X_2\left(\bar{k}_0,\bar{\theta}_1,\bar{\theta}_2\right) = \bar{X}_2$. Suppose that the planner gives lump-sum an additional income $e_1\left(\bar{k}_0,\bar{\theta}_1\right)$ in period 1 to all the individuals $\left(\bar{k}_0,\bar{\theta}_1\right)$, and an additional income $e_2\left(\bar{k}_0,\bar{\theta}_1,\bar{\theta}_2\right)$ in period 2 to the individual $\left(\bar{k}_0,\bar{\theta}_1,\bar{\theta}_2\right)$. Individuals with first period type $\left(\bar{k}_0,\bar{\theta}_1\right)$ have an indirect utility $\mathscr{U}\left(\bar{k}_0,\bar{\theta}_1,e_1,e_2\left(\bar{\theta}_2\right)\right)$ given by

$$\begin{aligned}
\mathscr{U}\left(\bar{k}_0,\bar{\theta}_1,e_1,e_2\left(\bar{\theta}_2\right)\right) \equiv & \max u\left(c_1, z_1/\theta_1\right) + \beta \mathbb{E}_{\theta_2}\left[u\left(c_2, z_2/\theta_2\right)|\theta_1\right] \\
& \text{s.t.} \quad c_1 + k_1 = z_1 + k_0 - T_1\left(z_1, rk_1\right) + e_1 \\
& \text{and} \quad c_2\left(\theta_2\right) = z_2\left(\theta_2\right) + (1+r)k_1 - T_2\left(z_1, z_2\left(\theta_2\right), rk_1\right) + e_2 \mathbf{1}_{\left\{\theta_2 = \bar{\theta}_2\right\}}.
\end{aligned}$$

By the envelope theorem, the change in the indirect utility of any individual $\left(k_0,\theta_1\right)$ following the reception of the additional income by individuals $\left(\bar{k}_0,\bar{\theta}_1\right)$ in period 1, and $\left(\bar{k}_0,\bar{\theta}_1,\bar{\theta}_2\right)$ in period 2, is given by

$$\begin{aligned}
\frac{d\mathscr{U}\left(k_0,\theta_1\right)}{de_1\left(\bar{k}_0,\bar{\theta}_1\right)} &= \lambda_1\left(\bar{k}_0,\bar{\theta}_1\right) \mathbf{1}_{\left\{(k_0,\theta_1) = \left(\bar{k}_0,\bar{\theta}_1\right)\right\}} = u_{c_1}\left(\bar{k}_0,\bar{\theta}_1\right) \mathbf{1}_{\left\{(k_0,\theta_1) = \left(\bar{k}_0,\bar{\theta}_1\right)\right\}}, \\
\frac{d\mathscr{U}\left(k_0,\theta_1\right)}{de_2\left(\bar{k}_0,\bar{\theta}_1,\bar{\theta}_2\right)} &= \lambda_2\left(\bar{k}_0,\bar{\theta}_1,\bar{\theta}_2\right) \mathbf{1}_{\left\{(k_0,\theta_1) = \left(\bar{k}_0,\bar{\theta}_1\right)\right\}} = \beta u_{c_2}\left(\bar{k}_0,\bar{\theta}_1,\bar{\theta}_2\right) f_{2|1}\left(\bar{\theta}_2\left|\bar{\theta}_1\right.\right) \mathbf{1}_{\left\{(k_0,\theta_1) = \left(\bar{k}_0,\bar{\theta}_1\right)\right\}}.
\end{aligned}$$

(The ex-ante welfare of all the individuals who have potentially the type $\bar{\theta}_2$ in period two, i.e., individuals with type $\left(\bar{k}_0, \bar{\theta}_1\right)$ in period one, increases, weighted by the probability $f_{2|1}\left(\bar{\theta}_2 \left|\bar{\theta}_1\right.\right)$ that they end up actually drawing this type.)

We define the period-1 marginal social welfare weight associated with type $\left(\bar{k}_0, \bar{\theta}_1\right)$ as the change in the social welfare $\mathscr{W}$ when this individual is given an additional unit of income in period 1. That is,

$$
\begin{aligned}
g_1\left(\bar{k}_0, \bar{\theta}_1\right) &\equiv \frac{d}{de_1\left(\bar{k}_0, \bar{\theta}_1\right)} \left\{ \frac{1}{p} \int_{\mathbb{R}\times\mathbb{R}_+} G\left(\mathscr{U}\left(k_0, \theta_1\right)|_{\mathscr{T}}\right) dF_1\left(k_0, \theta_1\right) \right\} \\
&= \frac{1}{p} \int_{\mathbb{R}\times\mathbb{R}_+} G'\left(\mathscr{U}\left(k_0, \theta_1\right)\right) \frac{d\mathscr{U}\left(k_0, \theta_1\right)}{de_1\left(\bar{k}_0, \bar{\theta}_1\right)} dF_1\left(k_0, \theta_1\right) \\
&= \frac{1}{p} \int_{\mathbb{R}\times\mathbb{R}_+} G'\left(\mathscr{U}\left(\bar{k}_0, \bar{\theta}_1\right)\right) u_{c_1}\left(\bar{k}_0, \bar{\theta}_1\right) \mathbf{1}_{\left\{\left(k_0, \theta_1\right) = \left(\bar{k}_0, \bar{\theta}_1\right)\right\}} dF_1\left(k_0, \theta_1\right) \\
&= \frac{1}{p} G'\left(\mathscr{U}\left(\bar{k}_0, \bar{\theta}_1\right)\right) u_{c_1}\left(\bar{k}_0, \bar{\theta}_1\right) f_1\left(\bar{k}_0, \bar{\theta}_1\right),
\end{aligned}
$$

We then define the period-1 marginal social welfare weight associated with income $\bar{X}_1$ as

$$
\begin{aligned}
g_1\left(\bar{X}_1\right) &\equiv \frac{1}{h_{X_1}\left(\bar{X}_1\right)} \int_{\mathbb{R}\times\mathbb{R}_+} g_1\left(k_0, \theta_1\right) \mathbf{1}_{\left\{X_1\left(k_0, \theta_1\right) = \bar{X}_1\right\}} dk_0 d\theta_1 \\
&= \frac{1}{h_{X_1}\left(\bar{X}_1\right)} \int_{\Theta} \frac{1}{p} G'\left(\mathscr{U}\left(k_0, \theta_1\right)\right) u_{c_1}\left(k_0, \theta_1\right) \mathbf{1}_{\left\{X_1\left(k_0, \theta_1\right) = \bar{X}_1\right\}} dF_1\left(k_0, \theta_1\right).
\end{aligned}
$$

Using the assumption made above that there is a unique draw $\left(k_0, \theta_1\right)$, namely $\left(\bar{k}_0, \bar{\theta}_1\right)$, such that an individual with this particular draw chooses the vector $\bar{X}_1$, i.e. $X_1\left(\bar{k}_0, \bar{\theta}_1\right) = \bar{X}_1$, we can write $g_1\left(\bar{X}_1\right)$ simply as

$$
g_1\left(\bar{X}_1\right) = \frac{1}{p} G'\left(\mathscr{U}\left(\bar{X}_1\right)\right) u_{c_1}\left(\bar{X}_1\right),
$$

which is definition (14).

Similarly, we define the period-2 marginal social welfare weight associated with type $\left(\bar{k}_0, \bar{\theta}_1, \bar{\theta}_2\right)$ as the change in the social welfare $\mathscr{W}$ when this individual is given an additional unit of income in period 2. That is,

$$
\begin{aligned}
g_2\left(\bar{k}_0, \bar{\theta}_1, \bar{\theta}_2\right) &\equiv \frac{\delta^{-1}}{p} \frac{d}{de_2\left(\bar{k}_0, \bar{\theta}_1, \bar{\theta}_2\right)} \left\{ \int_{\mathbb{R}\times\mathbb{R}_+} G\left(\mathscr{U}\left(k_0, \theta_1\right)|_{\mathscr{T}}\right) dF_1\left(k_0, \theta_1\right) \right\} \\
&= \frac{\delta^{-1}}{p} \int_{\mathbb{R}\times\mathbb{R}_+} G'\left(\mathscr{U}\left(k_0, \theta_1\right)\right) \frac{d\mathscr{U}\left(k_0, \theta_1\right)}{de_2\left(\bar{k}_0, \bar{\theta}_1, \bar{\theta}_2\right)} dF_1\left(k_0, \theta_1\right) \\
&= \frac{\delta^{-1}}{p} \int_{\mathbb{R}\times\mathbb{R}_+} G'\left(\mathscr{U}\left(\bar{k}_0, \bar{\theta}_1\right)\right) \left\{\beta u_{c_2}\left(\bar{k}_0, \bar{\theta}_1, \bar{\theta}_2\right) f\left(\bar{\theta}_2 \left|\bar{\theta}_1\right.\right)\right\} \mathbf{1}_{\left\{\left(k_0, \theta_1\right) = \left(\bar{k}_0, \bar{\theta}_1\right)\right\}} dF_1\left(k_0, \theta_1\right) \\
&= \frac{\beta\delta^{-1}}{p} G'\left(\mathscr{U}\left(\bar{k}_0, \bar{\theta}_1\right)\right) u_{c_2}\left(\bar{k}_0, \bar{\theta}_1, \bar{\theta}_2\right) f_1\left(\bar{k}_0, \bar{\theta}_1\right) f_{2|1}\left(\bar{\theta}_2 \left|\bar{\theta}_1\right.\right).
\end{aligned}
$$

We then define the period-2 marginal social welfare weight associated with income $\bar{X}_2$ as

$$\begin{aligned} g_2\left(\bar{X}_2\right) &\equiv \frac{1}{h_{X_2}\left(\bar{X}_2\right)} \int_{\mathbb{R}\times\mathbb{R}_+^2} g_1\left(k_0,\theta_1,\theta_2\right) \mathbf{1}_{\left\{X_2(k_0,\theta_1,\theta_2)=\bar{X}_2\right\}} dk_0 d\theta_1 d\theta_2 \\ &= \frac{\beta\delta^{-1}}{p} \frac{1}{h_{X_2}\left(\bar{X}_2\right)} \int_{\Theta} G'\left(\mathscr{U}\left(k_0,\theta_1\right)\right) u_{c_2}\left(k_0,\theta_1,\theta_2\right) \mathbf{1}_{\left\{X_2(k_0,\theta_1,\theta_2)=\bar{X}_2\right\}} f_{2|1}\left(\theta_2\left|\theta_1\right.\right) f_1\left(k_0,\theta_1\right) dk_0 d\theta_1 d\theta_2. \end{aligned}$$

Using the assumption made above that there is a unique draw (k_0,θ_1,θ_2), namely $\left(\bar{k}_0,\bar{\theta}_1,\bar{\theta}_2\right)$, such that an individual with this particular draw chooses the vector $\bar{X}_2$, i.e. $X_2\left(\bar{k}_0,\bar{\theta}_1,\bar{\theta}_2\right)=\bar{X}_2$, we can write $g_2\left(\bar{X}_2\right)$ simply as

$$g_2\left(\bar{X}_2\right) = \frac{\beta\delta^{-1}}{p} G'\left(\mathscr{U}\left(\bar{X}_1\right)\right) u_{c_2}\left(\bar{X}_2\right),$$

which is definition (14).

A.0.7 Proof of Proposition 11

We now prove the formula (19) of Proposition 11.

Consider an individual who chooses the income vector $X = \left(z_1, \left\{z_2\left(\theta_2\right) : \theta_2 \in \mathbb{R}_+\right\}, rk_1\right)$ before the perturbation. This choice vector satisfies the first order conditions (28), (29) and (30). Suppose that X belongs to a region of the space where the first-period marginal tax rates and virtual income τ_{1,z_1}, τ_{1,rk_1}, R_1 are perturbed by the respective amounts $d\tau_{1,z_1}$, $d\tau_{1,rk_1}$, dR_1, and the second-period marginal tax rates and virtual income $\tau_{2,z_1}\left(\mathbf{x}^2\right)$, $\tau_{2,z_2}\left(\mathbf{x}^2\right)$, $\tau_{2,rk_1}\left(\mathbf{x}^2\right)$, $R_2\left(\mathbf{x}^2\right)$ are perturbed at point $\mathbf{x}^2 = \left(z_1, \mathbf{x}_2^2, rk_1\right)$ by the respective amounts $d\tau_{2,z_1}$, $d\tau_{2,z_2}$, $d\tau_{2,rk_1}$, dR_2. Thus, the tax functions $T_1\left(\cdot,\cdot\right)$ and $T_2\left(\cdot,\cdot,\cdot\right)$ are replaced by the tax functions $\tilde{T}_1\left(\cdot,\cdot\right)$ and $\tilde{T}_2\left(\cdot,\cdot,\cdot\right)$, with

$$\begin{aligned} \tilde{T}_1\left(z_1,rk_1\right) &= T_1\left(z_1,rk_1\right) + dR_1, \\ \frac{\partial \tilde{T}_1}{\partial x_j}\left(z_1,rk_1\right) &= \frac{\partial T_1}{\partial x_j}\left(z_1,rk_1\right) + d\tau_{1,x_j}, \end{aligned}$$

for all $x_j \in \left\{z_1, rk_1\right\}$, and

$$\begin{aligned} \tilde{T}_2\left(z_1,z_2,rk_1\right) &= T_2\left(z_1,z_2,rk_1\right) + dR_2 \mathbf{1}_{\left\{z_2=\mathbf{x}_2^2\right\}}, \\ \frac{\partial \tilde{T}_2}{\partial x_j}\left(z_1,z_2,rk_1\right) &= \frac{\partial T_2}{\partial x_j}\left(z_1,z_2,rk_1\right) + d\tau_{2,x_j} \mathbf{1}_{\left\{z_2=\mathbf{x}_2^2\right\}}, \end{aligned}$$

for all $x_j \in \left\{z_1, z_2, rk_1\right\}$. As usual, we define θ_2^* such that the individual with first-period incomes $\left(z_1, rk_1\right)$ chooses $z_2\left(\theta_1,\theta_2^*\right) = \mathbf{x}_2^2$ if he draws the type θ_2^* in period two. We assume that the change dX is first order in dR_t and $d\tau_{t,x_j}$, i.e. individuals do not jump discretely in response to the perturbation.

Following the perturbation, the individual chooses a new vector $X + dX$ which satisfies the following new first order conditions. Denoting $\tilde{z}_1 \equiv z_1 + dz_1$, $\tilde{k}_1 \equiv k_1 + dk_1$ and $\tilde{z}_2(\theta_2) \equiv z_2(\theta_2) + dz_2(\theta_2)$ for all $\theta_2 \in \mathbb{R}_+$, the perturbed first-period intratemporal condition writes:

$$\begin{aligned}
&-\frac{1}{\theta_1} u_l \left\{ \tilde{z}_1 + k_0 - \tilde{k}_1 - T_1\left(\tilde{z}_1, r\tilde{k}_1\right) - dR_1, \tilde{z}_1/\theta_1 \right\} \\
&= \left(1 - \frac{\partial T_1\left(\tilde{z}_1, r\tilde{k}_1\right)}{\partial z_1} - d\tau_{1,z_1}\right) u_c \left\{ \tilde{z}_1 + k_0 - \tilde{k}_1 - T_1\left(\tilde{z}_1, r\tilde{k}_1\right) - dR_1, \tilde{z}_1/\theta_1 \right\} \\
&\quad - \beta \mathbb{E}_{\theta_2}\left[\left(\frac{\partial T_2\left(\tilde{z}_1, \tilde{z}_2(\theta_2), r\tilde{k}_1\right)}{\partial z_1} + d\tau_{2,z_1} \mathbf{1}_{\left\{z_2(\theta_2) = \mathbf{x}_2^2\right\}} \right) \right. \\
&\qquad \left. \times u_c \left\{ \tilde{z}_2(\theta_2) + (1+r)\tilde{k}_1 - T_2\left(\tilde{z}_1, \tilde{z}_2(\theta_2), r\tilde{k}_1\right) - dR_2 \mathbf{1}_{\left\{z_2(\theta_2) = \mathbf{x}_2^2\right\}}, \tilde{z}_2(\theta_2)/\theta_2 \right\} | \theta_1 \right]
\end{aligned} \tag{47}$$

The perturbed second-period intratemporal condition writes, for all $\theta_2 \in \mathbb{R}_+$:

$$\begin{aligned}
&-\frac{1}{\theta_2} u_l \left\{ \tilde{z}_2(\theta_2) + (1+r)\tilde{k}_1 - T_2\left(\tilde{z}_1, \tilde{z}_2(\theta_2), r\tilde{k}_1\right) - dR_2 \mathbf{1}_{\left\{z_2(\theta_2) = \mathbf{x}_2^2\right\}}, \tilde{z}_2(\theta_2)/\theta_2 \right\} \\
&= \left(1 - \frac{\partial T_2\left(\tilde{z}_1, \tilde{z}_2(\theta_2), r\tilde{k}_1\right)}{\partial z_2} - d\tau_{2,z_2} \mathbf{1}_{\left\{z_2(\theta_2) = \mathbf{x}_2^2\right\}}\right) \\
&\qquad \times u_c \left\{ \tilde{z}_2(\theta_2) + (1+r)\tilde{k}_1 - T_2\left(\tilde{z}_1, \tilde{z}_2(\theta_2), r\tilde{k}_1\right) - dR_2 \mathbf{1}_{\left\{z_2(\theta_2) = \mathbf{x}_2^2\right\}}, \tilde{z}_2(\theta_2)/\theta_2 \right\}.
\end{aligned} \tag{48}$$

The perturbed intertemporal condition writes:

$$\begin{aligned}
0 &= \left(1 + \frac{\partial T_1\left(\tilde{z}_1, r\tilde{k}_1\right)}{\partial (rk_1)} r + d\tau_{1,rk_1} r\right) u_c \left\{ \tilde{z}_1 + k_0 - \tilde{k}_1 - T_1\left(\tilde{z}_1, r\tilde{k}_1\right) - dR_1, \tilde{z}_1/\theta_1 \right\} \\
&\quad - \beta \mathbb{E}_{\theta_2}\left[\left(1 + \left(1 - \frac{\partial T_2\left(\tilde{z}_1, \tilde{z}_2(\theta_2), r\tilde{k}_1\right)}{\partial (rk_1)} - d\tau_{2,rk_1} \mathbf{1}_{\left\{z_2(\theta_2) = \mathbf{x}_2^2\right\}} \right) r \right) \right. \\
&\qquad \left. \times u_c \left\{ \tilde{z}_2(\theta_2) + (1+r)\tilde{k}_1 - T_2\left(\tilde{z}_1, \tilde{z}_2(\theta_2), r\tilde{k}_1\right) - dR_2 \mathbf{1}_{\left\{z_2(\theta_2) = \mathbf{x}_2^2\right\}}, \tilde{z}_2(\theta_2)/\theta_2 \right\} | \theta_1 \right].
\end{aligned} \tag{49}$$

We now linearize this system of equations around the initial point X with first-order Taylor approximations in $d\tau_{1,z_1}$, $d\tau_{1,rk_1}$, dR_1,$d\tau_{2,z_1}$, $d\tau_{2,z_2}$, $d\tau_{2,rk_1}$, dR_2, using the fact that the system of first order conditions holds both before and after the perturbation (the individual re-optimizes by computing the solution to his new problem following the perturbation). The proof uses similar arguments as in the proof for the deterministic case, hence we only highlight its main steps.

The difference between the perturbed and the initial first-period intratemporal conditions writes, to a first order around the initial vector X,

$$\begin{aligned}
&[a_{z_1,z_1}]\,dz_1 + \int_0^{\infty} [a_{z_1,\theta_2}]\,dz_2(\theta_2)\,d\theta_2 + [a_{z_1,k_1}]\,d(rk_1) \\
&- \left\{ \left[B^c_{\tau_{1,z_1}}\right]_{z_1} \left(\left[D^2T_1(X_1)\right]_{z_1,z_1} dz_1 + \left[D^2T_1(X_1)\right]_{z_1,k_1} d(rk_1) \right) \right\} \\
&- \int_0^{\infty} \left\{ \left[B^c_{\tau_{2,z_1}(z_1,z_2(\theta_2),rk_1)}\right]_{z_1} \left(\left[D^2T_2(z_1,z_2(\theta_2),rk_1)\right]_{z_1,z_1} dz_1 \right.\right. \\
&\qquad \left.\left. + \left[D^2T_2(z_1,z_2(\theta_2),rk_1)\right]_{z_1,\theta_2} dz_2(\theta_2) + \left[D^2T_2(z_1,z_2(\theta_2),rk_1)\right]_{z_1,k_1} d(rk_1) \right) \right\} d\theta_2 \\
&= \left[B^c_{\tau_{1,z_1}}\right]_{z_1} d\tau_{1,z_1} \left[B^c_{\tau_{2,z_1}(\mathbf{x}^2)}\right]_{z_1} d\tau_{2,z_1} - [B_{R_1}]_{z_1} dR_1 - \left[B_{R_2(\mathbf{x}^2)}\right]_{z_1} dR_2.
\end{aligned} \tag{50}$$

The difference between the perturbed and the initial second-period intratemporal conditions writes, to a first order around the initial vector X, for all $\theta_2 \in \mathbb{R}_+$,

$$\begin{aligned}
&[a_{\theta_2,1}]\,dz_1 + [a_{\theta_2,\theta_2}]\,dz_2(\theta_2) + [a_{\theta_2,k_1}]\,d(rk_1) \\
&- \left\{ \left[B^c_{\tau_{2,z_2}(z_1,z_2(\theta_2),rk_1)}\right]_{\theta_2} \left(\left[D^2T_2(z_1,z_2(\theta_2),rk_1)\right]_{\theta_2,z_1} dz_1 \right.\right. \\
&\qquad \left.\left. + \left[D^2T_2(z_1,z_2(\theta_2),rk_1)\right]_{\theta_2,\theta_2} dz_2(\theta_2) + \left[D^2T_2(z_1,z_2(\theta_2),rk_1)\right]_{\theta_2,k_1} d(rk_1) \right) \right\} \\
&= \left[B^c_{\tau_{2,z_2}(\mathbf{x}^2)}\right]_{\theta_2^*} \mathbf{1}_{\{\theta_2=\theta_2^*\}} d\tau_{2,z_2} - \left[B_{R_2(\mathbf{x}^2)}\right]_{\theta_2^*} \mathbf{1}_{\{\theta_2=\theta_2^*\}} dR_2.
\end{aligned} \tag{51}$$

Finally, the difference between the perturbed and the initial intertemporal conditions writes, to a first order around the initial vector X,

$$\begin{aligned}
&[a_{k_1,z_1}]\,dz_1 + \int_0^{\infty} [a_{k_1,\theta_2}]\,dz_2(\theta_2)\,d\theta_2 + [a_{k_1,k_1}]\,d(rk_1) \\
&- \left\{ \left[B^c_{\tau_{1,rk_1}}\right]_{k_1} \left(\left[D^2T_1(X_1)\right]_{k_1,z_1} dz_1 + \left[D^2T_1(X_1)\right]_{k_1,k_1} d(rk_1) \right) \right\} \\
&- \int_0^{\infty} \left\{ \left[B^c_{\tau_{2,rk_1}(z_1,z_2(\theta_2),rk_1)}\right]_{k_1} \left(\left[D^2T_2(z_1,z_2(\theta_2),rk_1)\right]_{k_1,z_1} dz_1 \right.\right. \\
&\qquad \left.\left. + \left[D^2T_2(z_1,z_2(\theta_2),rk_1)\right]_{k_1,\theta_2} dz_2(\theta_2) + \left[D^2T_2(z_1,z_2(\theta_2),rk_1)\right]_{k_1,k_1} d(rk_1) \right) \right\} d\theta_2 \\
&= \left[B^c_{\tau_{2,rk_1}(\mathbf{x}^2)}\right]_{k_1} d\tau_{2,rk_1} + \left[B^c_{\tau_{1,rk_1}}\right]_{k_1} d\tau_{1,rk_1} - [B_{R_1}]_{k_1} dR_1 - \left[B_{R_2(\mathbf{x}^2)}\right]_{k_1} dR_2.
\end{aligned} \tag{52}$$

Introducing in equations (50), (51) and (52) the coefficients that are equal to zero from the matrices $D^2T_1(X_1)$, $D^2T_2(\mathbf{x}^2)$, $B^c_{\tau_1}$, $B^c_{\tau_2(z_1,z_2(\theta_2),rk_1)}$, $B^c_{\tau_2(\mathbf{x}^2)}$, and the vectors $d\tau_1$, $d\tau_2(\mathbf{x}^2)$, B_{R_1}, $B_{R_2(\mathbf{x}^2)}$, we can rewrite this system as follows. The first-period intratemporal condition becomes

$$
\begin{aligned}
&[a_{z_1,z_1}]\,dz_1+\int_0^\infty [a_{z_1,\theta_2}]\,dz_2(\theta_2)\,d\theta_2+[a_{z_1,k_1}]\,d(rk_1)\\
&-\left\{\sum_{i=z_1,\theta_2''\in\mathbb{R}_+,k_1}\left(\left[B_{\tau_1}^c\right]_{z_1,i}\left[D^2T_1(X_1)\right]_{i,z_1}dz_1\right.\right.\\
&\qquad\left.\left.+\int_0^\infty\left[B_{\tau_1}^c\right]_{z_1,i}\left[D^2T_1(X_1)\right]_{i,\theta_2'}dz_2(\theta_2')\,d\theta_2'+\left[B_{\tau_1}^c\right]_{z_1,i}\left[D^2T_1(X_1)\right]_{i,k_1}d(rk_1)\right)\right\}\\
&-\int_0^\infty\left\{\sum_{i=z_1,\theta_2''\in\mathbb{R}_+,k_1}\left(\left[B_{\tau_2(z_1,z_2(\theta_2),rk_1)}^c\right]_{z_1,i}\left[D^2T_2(z_1,z_2(\theta_2),rk_1)\right]_{z_1,z_1}dz_1\right.\right.\\
&\qquad+\int_0^\infty\left[B_{\tau_2(z_1,z_2(\theta_2),rk_1)}^c\right]_{z_1,i}\left[D^2T_2(z_1,z_2(\theta_2),rk_1)\right]_{z_1,\theta_2'}dz_2(\theta_2')\,d\theta_2'\\
&\qquad\left.\left.+\left[B_{\tau_2(z_1,z_2(\theta_2),rk_1)}^c\right]_{z_1,i}\left[D^2T_2(z_1,z_2(\theta_2),rk_1)\right]_{z_1,k_1}d(rk_1)\right)\right\}d\theta_2\\
=&\sum_{i=z_1,\theta_2''\in\mathbb{R}_+,k_1}\left[B_{\tau_1}^c\right]_{z_1,i}[d\tau_1]_i+\sum_{i=z_1,\theta_2''\in\mathbb{R}_+,k_1}\left[B_{\tau_2(\mathbf{x}^2)}^c\right]_{z_1,i}\left[d\tau_2\left(\mathbf{x}^2\right)\right]_i\\
&\qquad-[B_{R_1}]_{z_1}dR_1-\left[B_{R_2(\mathbf{x}^2)}\right]_{z_1}dR_2\left(\mathbf{x}^2\right).
\end{aligned}
\tag{53}
$$

The second-period intratemporal condition becomes

$$
\begin{aligned}
&[a_{\theta_2,1}]\,dz_1+[a_{\theta_2,\theta_2}]\,dz_2(\theta_2)+[a_{\theta_2,k_1}]\,d(rk_1)\\
&-\left\{\sum_{i=z_1,\theta_2''\in\mathbb{R}_+,k_1}\left(\left[B_{\tau_1}^c\right]_{\theta_2,i}\left[D^2T_1(X_1)\right]_{i,z_1}dz_1\right.\right.\\
&\qquad\left.\left.+\int_0^\infty\left[B_{\tau_1}^c\right]_{\theta_2,i}\left[D^2T_1(X_1)\right]_{i,\theta_2}dz_2(\theta_2)\,d\theta_2+\left[B_{\tau_1}^c\right]_{\theta_2,i}\left[D^2T_1(X_1)\right]_{i,k_1}d(rk_1)\right)\right\}\\
&-\left\{\sum_{i=z_1,\theta_2''\in\mathbb{R}_+,k_1}\left(\left[B_{\tau_2(z_1,z_2(\theta_2),rk_1)}^c\right]_{\theta_2,i}\left[D^2T_2(z_1,z_2(\theta_2),rk_1)\right]_{i,z_1}dz_1\right.\right.\\
&\qquad+\left[B_{\tau_2(z_1,z_2(\theta_2),rk_1)}^c\right]_{\theta_2,i}\left[D^2T_2(z_1,z_2(\theta_2),rk_1)\right]_{i,\theta_2}dz_2(\theta_2)\\
&\qquad\left.\left.+\left[B_{\tau_2(z_1,z_2(\theta_2),rk_1)}^c\right]_{\theta_2,i}\left[D^2T_2(z_1,z_2(\theta_2),rk_1)\right]_{i,k_1}d(rk_1)\right)\right\}\\
=&\sum_{i=z_1,\theta_2''\in\mathbb{R}_+,k_1}\left[B_{\tau_1}^c\right]_{\theta_2,i}[d\tau_1]_i+\sum_{i=z_1,\theta_2''\in\mathbb{R}_+,k_1}\left[B_{\tau_2(\mathbf{x}^2)}^c\right]_{\theta_2,i}\left[d\tau_2\left(\mathbf{x}^2\right)\right]_i\\
&\qquad-[B_{R_1}]_{\theta_2}dR_1-\left[B_{R_2(\mathbf{x}^2)}\right]_{\theta_2}dR_2\left(\mathbf{x}^2\right).
\end{aligned}
\tag{54}
$$

The intertemporal condition becomes

$$
\begin{aligned}
&[a_{k_1,z_1}]\, dz_1 + \int_0^\infty [a_{k_1,\theta_2}]\, dz_2(\theta_2)\, d\theta_2 + [a_{k_1,k_1}]\, d(rk_1) \\
&- \left\{ \sum_{i=z_1,\theta_2''\in\mathbb{R}_+,k_1} \left(\left[B^c_{\tau_1}\right]_{k_1,i} \left[D^2T_1(X_1)\right]_{i,z_1} dz_1 \right.\right. \\
&\qquad \left.\left. + \int_0^\infty \left[B^c_{\tau_1}\right]_{k_1,i} \left[D^2T_1(X_1)\right]_{i,\theta_2} dz_2(\theta_2)\, d\theta_2 + \left[B^c_{\tau_1}\right]_{k_1,i} \left[D^2T_1(X_1)\right]_{i,k_1} d(rk_1) \right) \right\} \\
&- \int_0^\infty \left\{ \sum_{i=z_1,\theta_2''\in\mathbb{R}_+,k_1} \left(\left[B^c_{\tau_2(z_1,z_2(\theta_2),rk_1)}\right]_{k_1,i} \left[D^2T_2(z_1,z_2(\theta_2),rk_1)\right]_{i,z_1} dz_1 \right.\right. \\
&\qquad + \int_0^\infty \left[B^c_{\tau_2(z_1,z_2(\theta_2),rk_1)}\right]_{k_1,i} \left[D^2T_2(z_1,z_2(\theta_2),rk_1)\right]_{i,\theta_2'} dz_2\left(\theta_2'\right) d\theta_2' \\
&\qquad \left.\left. + \left[B^c_{\tau_2(z_1,z_2(\theta_2),rk_1)}\right]_{k_1,i} \left[D^2T_2(z_1,z_2(\theta_2),rk_1)\right]_{i,k_1} d(rk_1) \right) \right\} d\theta_2 \\
= &\sum_{i=z_1,\theta_2''\in\mathbb{R}_+,k_1} \left[B^c_{\tau_1}\right]_{k_1,i} [d\tau_1]_i + \sum_{i=z_1,\theta_2''\in\mathbb{R}_+,k_1} \left[B^c_{\tau_2(\mathbf{x}^2)}\right]_{k_1,i} \left[d\tau_2\left(\mathbf{x}^2\right)\right]_i \\
&\qquad - [B_{R_1}]_{k_1} dR_1 - \left[B_{R_2(\mathbf{x}^2)}\right]_{k_1} dR_2\left(\mathbf{x}^2\right).
\end{aligned}
\tag{55}
$$

Therefore, the system (53), (54) and (55) writes, in matrix form,

$$
\begin{aligned}
&A \times dX - B^c_{\tau_1}\left(D^2T_1(X_1)\right) dX - \int_0^\infty B^c_{\tau_2(z_1,z_2(\theta_2),rk_1)} \left(D^2T_2(z_1,z_2(\theta_2),rk_1)\right) dX d\theta_2 \\
=&B^c_{\tau_1} d\tau_1 + B^c_{\tau_2(\mathbf{x}^2)} d\tau_2\left(\mathbf{x}^2\right) - B_{R_1} dR_1 - B_{R_2(\mathbf{x}^2)} dR_2.
\end{aligned}
\tag{56}
$$

We showed that that we can invert the system (56) by multiplying it by the matrix A^{-1}, to obtain:

$$
\begin{aligned}
&\left\{ I - \boldsymbol{\zeta}^{c,(X)}_{X,\tau_1}\left(D^2T_1(X_1)\right) - \int_0^\infty \boldsymbol{\zeta}^{c,(X)}_{X,\tau_2(z_1,z_2(\theta_2),rk_1)} \left(D^2T_2(z_1,z_2(\theta_2),rk_1)\right) d\theta_2 \right\} dX \\
=&\boldsymbol{\zeta}^{c,(X)}_{X,\tau_1} d\tau_1 + \boldsymbol{\zeta}^{c,(X)}_{X,\tau_2(\mathbf{x}^2)} d\tau_2\left(\mathbf{x}^2\right) - \boldsymbol{\eta}^{(X)}_{X,R_1} dR_1 - \boldsymbol{\eta}^{(X)}_{X,R_2(\mathbf{x}^2)} dR_2\left(\mathbf{x}^2\right).
\end{aligned}
\tag{57}
$$

We can finally invert the system (57) to obtain dX, as in formula (19).

A.0.8 Welfare Effects of Tax Reforms

We derived (equation (46)) the expression for vector dX, defined by (19), under the assumptions of this subsection. We use this vector to derive the revenue gains of tax reforms in the sequel.

A.0.9 Separable Perturbations of the First-Period Labor Income Tax Schedule

Consider a separable perturbation of the first-period labor income tax schedule at point $\bar{z}_1$. That is, we increase the marginal tax rate $T'_{1,z_1}(z_1)$ by $d\tau_{1,z_1}$ on $[\bar{z}_1, \bar{z}_1 + d\bar{x}]$, and increase the total tax liability $T_{1,z_1}(z_1)$ lump sum by dR_1 on $[\bar{z}_1 + d\bar{x}, \infty)$.

Consider first individuals with first-period type (k_0, θ_1) such that $X_1(k_0, \theta_1) = (z_1, rk_1)$, with $z_1 \in [\bar{z}_1, \bar{z}_1 + d\bar{x}]$ when the initial tax system is in place. In response to the perturbation, they change their first-period labor income z_1 by the amount

$$dz_1|_{(\bar{z}_1, rk_1)} = \frac{-\bar{z}_1 \zeta^{c,(\bar{z}_1, rk_1)}_{z_1, 1-\tau_{1,z_1}}}{1 - T'_{1,z_1}(\bar{z}_1) + \bar{z}_1 \zeta^{c,(\bar{z}_1, rk_1)}_{z_1, 1-\tau_{1,z_1}} T''_{1,z_1}(\bar{z}_1)} d\tau_{1,z_1}. \tag{58}$$

Their capital income rk_1 does not change, i.e. $d(rk_1) = 0$, and for any second-period type θ_2 that they may draw, their choice of second period labor income $z_2(\theta_2)$ will be the same as the one they would have chosen in the absence of the perturbation, i.e. $dz_2(\theta_2) = 0$ for all $\theta_2 \in \mathbb{R}_+$.

Now consider individuals with first-period type (k_0, θ_1) such that $X_1(k_0, \theta_1) = (z_1, rk_1)$, with $z_1 \geq \bar{z}_1 + d\bar{x}$ when the initial tax system is in place. In response to the perturbation, they change their capital income rk_1 by the amount

$$d(rk_1)|_{(z_1, rk_1)} = \eta^{(z_1, rk_1)}_{rk_1, R_1}(-dR_1), \tag{59}$$

and their first- and second-period incomes do not change.

Using (58) and (59), we obtain that the total effect of the perturbation on government revenue is

$$\begin{aligned}\frac{\Gamma_{1,z_1}(\bar{z}_1)}{dR_1} =& [1 - H_{z_1}(\bar{z}_1)] \\ &+ \int_{\mathbb{R}} T'_{1,z_1}(\bar{z}_1) \left\{ \frac{dz_1|_{(\bar{z}_1, rk_1)}}{d\tau_{1,z_1}} \right\} h_{rk_1|z_1}(rk_1 | \bar{z}_1) h_{z_1}(\bar{z}_1) d(rk_1) \\ &- \int_{\bar{z}_1}^{\infty} \int_{\mathbb{R}} \delta\tau_k \left\{ \frac{d(rk_1)|_{(z_1, rk_1)}}{-dR_1} \right\} h_{X_1}(z_1, rk_1) dz_1 d(rk_1).\end{aligned} \tag{60}$$

Using the fact that the compensated elasticity $\zeta^{c,(\bar{z}_1, rk_1)}_{z_1, 1-\tau_{1,z_1}} = (\theta_1 v'_1) / (\bar{z}_1 v''_1)$ does not depend on rk_1, the integral in the second line of (60) is equal to

$$\begin{aligned}&\int_{\mathbb{R}} T'_{1,z_1}(\bar{z}_1) \left\{ \frac{dz_1|_{(\bar{z}_1, rk_1)}}{d\tau_{1,z_1}} \right\} h_{rk_1|z_1}(rk_1 | \bar{z}_1) h_{z_1}(\bar{z}_1) d(rk_1) \\ =& T'_{1,z_1}(\bar{z}_1) \frac{\bar{z}_1 \zeta^{c,(\bar{z}_1)}_{z_1, 1-\tau_{1,z_1}}}{1 - T'_{1,z_1}(\bar{z}_1) + \bar{z}_1 \zeta^{c,(\bar{z}_1)}_{z_1, 1-\tau_{1,z_1}} T''_{1,z_1}(\bar{z}_1)} h_{z_1}(\bar{z}_1).\end{aligned} \tag{61}$$

Moreover, letting $\mathbb{E}\left[\eta^{(X_1)}_{rk_1, R_1} | z_1 \geq \bar{z}_1\right]$ denote the average income effect parameter of capital income w.r.t. period-one virtual income, among individuals with first-period labor income $z_1 \geq \bar{z}_1$, the integral in the third line of (60) is equal to

$$
\begin{aligned}
-\delta\tau_k \int_{\bar{z}_1}^{\infty} \int_{\mathbb{R}} \eta_{rk_1,R_1}^{(z_1,rk_1)} \frac{h_{X_1}(z_1, rk_1)}{1 - H_{z_1}(\bar{z}_1)} dz_1 d(rk_1) &= -\delta\tau_k \int_{\bar{z}_1}^{\infty} \bar{\eta}_{rk_1,R_1}^{(z_1)} \frac{h_{z_1}(z_1)}{1 - H_{z_1}(\bar{z}_1)} dz_1 \\
&= -\delta\tau_k \mathbb{E}\left[\eta_{rk_1,R_1}^{(X_1)} | z_1 \geq \bar{z}_1\right].
\end{aligned}
\tag{62}
$$

If the utility function is CARA, the income effect parameters $\eta_{rk_1,R_1}^{(z_1,rk_1)}$ are constant, independent of z_1 and rk_1, and hence this term is simply equal to $-\delta\tau_k\eta_{rk_1,R_1}$. If the utility function is CRRA, however, the income effect parameters $\eta_{rk_1,R_1}^{(z_1,rk_1)}$ are not constant (as they were in the deterministic model).

Therefore, substituting (61) and (62) in (60) normalizing the revenue gains by $1 - H_{z_1}(\bar{z}_1)$, we obtain

$$
\begin{aligned}
\frac{\gamma_{1,z_1}(\bar{z}_1)}{dR_1} =& 1 - \frac{T'_{1,z_1}(\bar{z}_1)}{1 - T'_{1,z_1}(\bar{z}_1) + \bar{z}_1 \zeta_{z_1,1-\tau_{1,z_1}}^{c,(\bar{z}_1)} T''_{1,z_1}(\bar{z}_1)} \zeta_{z_1,1-\tau_{1,z_1}}^{c,(\bar{z}_1)} \frac{\bar{z}_1 h_{z_1}(\bar{z}_1)}{1 - H_{z_1}(\bar{z}_1)} \\
& - \delta\tau_k \int_{\bar{z}_1}^{\infty} \bar{\eta}_{rk_1,R_1}^{(z_1)} \frac{h_{z_1}(z_1)}{1 - H_{z_1}(\bar{z}_1)} dz_1.
\end{aligned}
\tag{63}
$$

Formula (63) shows that the revenue effect of perturbing the first-period labor income tax rate in the stochastic model is formally identical to the effect in the deterministic model, given by (40) with $T = 2$. However, the inequalities (20) and (21) imply that for all z_1, we have $0 < \bar{\eta}_{rk_1,R_1}^{(z_1)} < \bar{\eta}_{rk_1,R_1}^{(z_1),Det}$. Thus, starting from the static optimum, the revenue gains from decreasing the labor income tax rates in period one are smaller in the stochastic model, than in the deterministic model. This implies that the computations of Figure 5 give an upper bound of the optimal amount of age-dependence in a stochastic economy: the tax schedules should be closer to the static optimum in the early periods than predicted by the deterministic model.

A.0.10 Separable Perturbations of the Second-Period Labor Income Tax Schedule

Consider a separable perturbation of the second-period labor income tax schedule at point $\bar{z}_2$. That is, we increase the marginal tax rate $T'_{2,z_2}(z_2)$ by $d\tau_{2,z_2}$ on $[\bar{z}_2, \bar{z}_2 + d\bar{x}]$, and increase the total tax liability $T_{2,z_2}(z_2)$ lump sum by dR_2 on $[\bar{z}_2 + d\bar{x}, \infty)$. That is, in the notations of this section, the perturbation of the virtual income is $dR_2(\mathbf{x}^2) = (-dR_2)\mathbf{1}_{\{\mathbf{x}_2^2 \geq \bar{z}_2\}}$.

Consider an individual with first-period type (k_0, θ_1), such that $X_1(k_0, \theta_1) = (z_1, rk_1)$. This individual has a positive probability of drawing the second-period type $\theta_2^*(k_0, \theta_1)$, which induces him to choose second-period labor income $z_2(\theta_2^*(k_0, \theta_1)) \in [\bar{z}_2, \bar{z}_2 + d\bar{x}]$ when the initial tax system is in place. That is, he will be directly affected by the perturbation with positive probability. He changes his second-period income $z_2(\theta_2)$ relative to the income he would have chosen in the absence of a perturbation only if his second-period type is drawn in the set of θ_2's such that $z_2(\theta_2(k_0,\theta_1)) \in [\bar{z}_2, \bar{z}_2 + d\bar{x}]$ (see explicit expressions (44), and (45) for the elasticities). Note moreover that the elasticity $\zeta^{c,(\mathbf{x}^2)}_{z_2,1-\tau_{2,z_2}(\mathbf{x}^2)} = -\theta_2^* v_2'(\mathbf{x}_2^2) / (\mathbf{x}_2^2 v_2''(\mathbf{x}_2^2))$, and thus the change $dz_2(\theta_2^*)$, depend only on the individual's second-period income $\mathbf{x}_2^2$.[7] Thus, any individual with second-period income $z_2 = \bar{z}_2$ before the perturbation changes this income, in response to the perturbation, by the amount

$$dz_2|_{\bar{z}_2} = \frac{-\bar{z}_2 \zeta^{c,(\bar{z}_2)}_{z_2,1-\tau_{2,z_2}} d\tau_{2,z_2}}{1 - T_2'(\bar{z}_2) + \bar{z}_2 \zeta^{c,(\bar{z}_2)}_{z_2,1-\tau_{2,z_2}} T_2''(\bar{z}_2)}. \tag{64}$$

Any individual with first-period type (k_0, θ_1), such that $X_1(k_0,\theta_1) = (z_1, rk_1)$, has a positive probability of drawing the second-period type $\theta_2(k_0,\theta_1)$, which induces him to choose second-period labor income $z_2(\theta_2(k_0,\theta_1)) \in [\bar{z}_2 + d\bar{x}, \infty)$ when the initial tax system is in place. He changes his capital income rk_1 (chosen in period one) relative to the capital income he would have chosen in the absence of a perturbation, because of the positive probability that he ends up being poorer in the second period, by the amount

$$\begin{aligned} d(rk_1)|_{X_1} &= \int_{\mathbb{R}_+^2 \times \mathbb{R}} \frac{\eta^{(X_1)}_{rk_1,R_2(\mathbf{x}^2)}}{1 + (1-\tau_{2,rk_1}) r} dR_2(\mathbf{x}^2)\, d\mathbf{x}^2 \\ &= \frac{-1}{1 + (1-\tau_k) r} \left\{ \int_{\bar{z}_2}^{\infty} \eta^{(X_1)}_{rk_1,R_2(X_1,\mathbf{x}_2^2)} d\mathbf{x}_2^2 \right\} dR_2. \end{aligned} \tag{65}$$

Note that the explicit expressions for the income effect parameters $\eta^{(X_1)}_{rk_1,R_2(\mathbf{x}^2)}$ show that the change in capital income of an individual in response to an additional dollar given to him if his income in period two is $z_2 \geq \bar{z}_2$, i.e. the integral in (65), is equal to

$$\begin{aligned} \int_{\bar{z}_2}^{\infty} \eta^{(X_1)}_{rk_1,R_2(X_1,\mathbf{x}_2^2)} d\mathbf{x}_2^2 &= \frac{-\beta(1+(1-\tau_k)r)^2 \mathbb{E}\left[u_2'' \mathbf{1}_{\{z_2 \geq \bar{z}_2\}} | X_1\right]}{u_1'' + \beta(1+(1-\tau_k)r)^2 \mathbb{E}\left[u_2'' | X_1\right]} \\ &= (1 - H_{z_2}(\bar{z}_2)) \frac{-\beta(1+(1-\tau_k)r)^2 \mathbb{E}\left[u_2'' | X_1, z_2 \geq \bar{z}_2\right]}{u_1'' + \beta(1+(1-\tau_k)r)^2 \mathbb{E}\left[u_2'' | X_1\right]}. \end{aligned}$$

It is equal to the change in capital income of the individual with first-period income vector X_1, if he receives an additional dollar in the states of the world where he chooses a second-period income larger than $\bar{z}_2$. We then denote by $\bar{\eta}_{rk_1,R_2(z_2 \geq \bar{z}_2)}$ the change in aggregate capital income in the economy, when an additional dollar is distributed lump-sum in period two, uniformly among all the individuals whose labor income in period two is above $\bar{z}_2$, irrespective of their first-period

[7] It also depends on the second-period type θ_2^*, but there is a one-to-one map between θ_2^* and z_2, independent of z_1, rk_1, given by the second-period intratemporal first order condition, $v'(z_1/\theta_1) = \theta_1(1 - T_1'(z_1))$.

income choices. That is,

$$\bar{\eta}_{rk_1,R_2(z_2\geq\bar{z}_2)} \equiv \frac{1}{1-H_{z_2}(\bar{z}_2)} \int_{\mathbb{R}_+}\int_{\mathbb{R}} \left\{ \int_{\bar{z}_2}^{\infty} \eta_{rk_1,R_2\left(X_1,\mathbf{x}_2^2\right)}^{(X_1)} d\mathbf{x}_2^2 \right\} h_{X_1}(z_1, rk_1)\, dz_1 d(rk_1). \quad (66)$$

Using (64) and (65), the change in government revenue due to the perturbation is thus given by

$$\begin{aligned} \frac{\Gamma_{2,z_2}(\bar{z}_2)}{dR_2} =& \delta\left[1-H_{z_2}(\bar{z}_2)\right] + \delta T'_{2,z_2}(\bar{z}_2) \left\{ \frac{dz_2|_{\bar{z}_2}}{d\tau_{2,z_2}} \right\} h_{z_2}(\bar{z}_2) \\ & - \int_{\mathbb{R}_+}\int_{\mathbb{R}} \delta\tau_k \left\{ \frac{d(rk_1)|_{(z_1,rk_1)}}{-dR_2} \right\} h_{X_1}(z_1, rk_1)\, dz_1 d(rk_1). \end{aligned} \quad (67)$$

Normalizing by $1-H_{z_2}(\bar{z}_2)$, and using the definition (66) of the average income effect parameter, we can rewrite the revenue gains (67) as

$$\begin{aligned} & \gamma_{2,z_2}(\bar{z}_2)/(\delta dR_2) \\ =& 1 - \frac{T'_{2,z_2}(\bar{z}_2)}{1-T'_2(\bar{z}_2) + \bar{z}_2\zeta_{z_2,1-\tau_{2,z_2}}^{c,(\bar{z}_2)} T''_2(\bar{z}_2)} \zeta_{z_2,1-\tau_{2,z_2}}^{c,(\bar{z}_2)} \frac{\bar{z}_2 h_{z_2}(\bar{z}_2)}{1-H_{z_2}(\bar{z}_2)} - \frac{\tau_k}{1+(1-\tau_k)\,r} \bar{\eta}_{rk_1,R_2(z_2\geq\bar{z}_2)}, \end{aligned} \quad (68)$$

Formula (68) shows that the revenue effect of perturbing the second-period labor income tax rate in the stochastic model is formally similar to the effect in the deterministic model, given by (40) with $T=2$. However, using the explicit expressions for the income effect parameters in the deterministic and the stochastic models to obtain inequalities similar to (20) and (21), we obtain that the savings effect in the stochastic setting, $\bar{\eta}_{rk_1,R_2(z_2\geq\bar{z}_2)}$, is strictly larger than in the deterministic setting, $\int_{\bar{z}_2}^{\infty} \bar{\eta}_{rk_1,R_2}^{(z_2)} h_{z_2}(z_2)/(1-H_{z_2}(\bar{z}_2))\, dz_2$. Thus, starting from the static optimum, the revenue gains from increasing the labor income tax rates in period two are smaller in the stochastic model, than in the deterministic model. This implies that the computations of Figure 5 give an upper bound of the optimal amount of age-dependence in a stochastic economy: the tax schedules should be closer to the static optimum in the late periods than predicted by the deterministic model.

A.0.11 Separable Perturbations of the Capital Income Tax Schedule

Consider a separable perturbation of the capital income tax schedule at point $r\bar{k}_1$. That is, we increase the marginal tax rate $T'_{2,rk_1}(rk_1) = \tau_k$ by $d\tau_{2,rk_1}$ on $\left[r\bar{k}_1, r\bar{k}_1 + d\bar{x}\right]$, and increase the total tax liability $T_{2,rk_1}(rk_1)$ lump-sum by dR_k on $\left[r\bar{k}_1 + d\bar{x}, \infty\right)$.

Consider an individual with first-period type (k_0, θ_1) such that $X_1(k_0,\theta_1) = (z_1, rk_1)$, with $rk_1 \in \left[r\bar{k}_1, r\bar{k}_1 + d\bar{x}\right]$ when the initial tax system is in place. In response to the perturbation, he changes his capital income rk_1 by the amount

$$\begin{aligned} d\left(rk_1\right)\big|_{\left(z_1,r\bar{k}_1\right)} &= \frac{-r\bar{k}_1}{1+\left(1-\tau_k\right)r}\left\{\int_{\mathbb{R}}\zeta^{c,\left(z_1,r\bar{k}_1\right)}_{rk_1,1-\tau_{2,rk_1}\left(z_1,r\bar{k}_1,\mathbf{x}_2^2\right)}d\mathbf{x}_2^2\right\}d\tau_{2,rk_1} \\ &\equiv \frac{-r\bar{k}_1}{1+\left(1-\tau_k\right)r}\zeta^{c,\left(z_1,r\bar{k}_1\right)}_{rk_1,1-\tau_{2,rk_1}}d\tau_{2,rk_1}, \end{aligned} \tag{69}$$

where $\zeta^{c,\left(z_1,r\bar{k}_1\right)}_{rk_1,1-\tau_{2,rk_1}}$ denotes the compensated capital income elasticity of individual $X_1=\left(z_1,r\bar{k}_1\right)$, w.r.t. to an increase in the capital income tax rate at level $r\bar{k}_1$ (irrespective of the individual's labor incomes, i.e. separable perturbation). The explicit expression for this elasticity is given by

$$\frac{\partial\left(rk_1^c\right)}{\partial\left(1-\tau_{2,rk_1}\right)} = \frac{-r^2\left(1+\left(1-\tau_k\right)r\right)^{-1}u_1'}{u_1''+\beta\left(1+\left(1-\tau_k\right)r\right)^2\mathbb{E}\left[u_2''\left|X_1\right.\right]}. \tag{70}$$

We finally denote by $\bar{\zeta}^{c,\left(r\bar{k}_1\right)}_{rk_1,1-\tau_{2,rk_1}}$ the average compensated elasticity of savings among individuals with capital income $r\bar{k}_1$, that is,

$$\bar{\zeta}^{c,\left(r\bar{k}_1\right)}_{rk_1,1-\tau_{2,rk_1}} \equiv \int_{\mathbb{R}_+}\zeta^{c,\left(z_1,r\bar{k}_1\right)}_{rk_1,1-\tau_{2,rk_1}}h_{z_1|rk_1}\left(z_1\left|r\bar{k}_1\right.\right)dz_1. \tag{71}$$

Consider an individual with first-period type (k_0,θ_1) such that $X_1(k_0,\theta_1)=(z_1,rk_1)$, with $rk_1\geq r\bar{k}_1+d\bar{x}$ when the initial tax system is in place. In response to the perturbation, he changes his capital income rk_1 by the amount

$$d\left(rk_1\right)\big|_{X_1} = -\left\{\int_0^\infty\frac{\eta^{(X_1)}_{rk_1,R_2\left(X_1,\mathbf{x}_2^2\right)}}{1+\left(1-\tau_{2,rk_1}\right)r}d\mathbf{x}_2^2\right\}dR_k \equiv \frac{-1}{1+\left(1-\tau_k\right)r}\eta^{(z_1,rk_1)}_{rk_1,R_2}dR_k, \tag{72}$$

where $\eta^{(z_1,rk_1)}_{rk_1,R_2}$ denotes the income effect parameter of individual $X_1=(z_1,rk_1)$ w.r.t. a *certain* increase in his period-two virtual income (irrespective of his second-period labor income). The explicit expression for this income effect parameter is

$$\eta^{(X_1)}_{rk_1,R_2} = \frac{-\beta\left(1+\left(1-\tau_{2,rk_1}\right)r\right)^2\mathbb{E}_{\theta_2}\left[u_2''\left|\theta_1\right.\right]}{u_1''+\beta\left(1+\left(1-\tau_{2,rk_1}\right)r\right)^2\mathbb{E}_{\theta_2}\left[u_2''\left|\theta_1\right.\right]}, \tag{73}$$

We finally denote by $\mathbb{E}\left[\eta^{(X_1)}_{rk_1,R_2}\left|rk_1\geq r\bar{k}_1\right.\right]$ the average income effect parameter of capital income w.r.t. a certain increase period-two virtual income, among individuals with capital income $rk_1\geq r\bar{k}_1$. That is,

$$\begin{aligned} \mathbb{E}\left[\eta^{(X_1)}_{rk_1,R_2}\left|rk_1\geq r\bar{k}_1\right.\right] &\equiv \int_{\mathbb{R}_+}\int_{r\bar{k}_1}^\infty\eta^{(z_1,rk_1)}_{rk_1,R_2}h_{z_1|rk_1}\left(z_1\left|rk_1\right.\right)\frac{h_{rk_1}\left(rk_1\right)}{1-H_{rk_1}\left(r\bar{k}_1\right)}dz_1d\left(rk_1\right). \\ &= \int_{r\bar{k}_1}^\infty\bar{\eta}^{(rk_1)}_{rk_1,R_2}\frac{h_{rk_1}\left(rk_1\right)}{1-H_{rk_1}\left(r\bar{k}_1\right)}d\left(rk_1\right). \end{aligned} \tag{74}$$

Using (69) and (72), we obtain that the total effect of the perturbation on government revenue is thus given by

$$\frac{\Gamma_{1,rk_1}\left(r\bar{k}_1\right)}{dR_k} = \delta\left[1-H_{rk_1}\left(r\bar{k}_1\right)\right]+\delta\tau_k\int_{\mathbb{R}_+}\left\{\frac{d\left(rk_1\right)|_{\left(z_1,r\bar{k}_1\right)}}{d\tau_{2,rk_1}}\right\}h_{z_1|rk_1}\left(z_1\left|r\bar{k}_1\right.\right)h_{rk_1}\left(r\bar{k}_1\right)dz_1$$
$$-\delta\tau_k\int_{r\bar{k}_1}^{\infty}\int_{\mathbb{R}_+}\left\{\frac{d\left(rk_1\right)|_{\left(z_1,rk_1\right)}}{-dR_k}\right\}h_{X_1}\left(z_1,rk_1\right)dz_1 d\left(rk_1\right). \tag{75}$$

Normalizing by $1-H_{rk_1}\left(r\bar{k}_1\right)$ and using the definitions of the average compensated elasticities and income effect parameters (71) and (74), we can rewrite these revenue gains (75) as

$$\gamma_{1,rk_1}\left(r\bar{k}_1\right)/\left(\delta dR_k\right)$$
$$=1-\frac{\tau_k}{1+\left(1-\tau_k\right)r}\bar{\zeta}^{c,\left(r\bar{k}_1\right)}_{rk_1,1-\tau_{2,rk_1}}\frac{r\bar{k}_1 h_{rk_1}\left(r\bar{k}_1\right)}{1-H_{rk_1}\left(r\bar{k}_1\right)}-\frac{\tau_k}{1+\left(1-\tau_k\right)r}\int_{r\bar{k}_1}^{\infty}\bar{\eta}^{\left(rk_1\right)}_{rk_1,R_2}\frac{h_{rk_1}\left(rk_1\right)}{1-H_{rk_1}\left(r\bar{k}_1\right)}d\left(rk_1\right). \tag{76}$$

Formula (76) shows that the revenue effect of perturbing the capital income tax rate in the stochastic model is formally identical to the effect in the deterministic model, given by (45) with $T=2$. However, the inequalities (20) and (21) show that the average compensated capital income elasticity in the stochastic model, $\bar{\zeta}^{c,\left(r\bar{k}_1\right)}_{rk_1,1-\tau_{2,rk_1}}$, is (positive and) smaller than its counterpart in the deterministic model, $\bar{\zeta}^{c,\left(r\bar{k}_1\right),Det}_{rk_1,1-\tau_{2,rk_1}}$. Similarly, the average income effect parameters in the stochastic model, $\bar{\eta}^{\left(rk_1\right)}_{rk_1,R_2}$, are (negative and) smaller than their counterparts in the deterministic model, $\bar{\eta}^{\left(rk_1\right),Det}_{rk_1,R_2}$. Thus, the revenue gains from increasing the capital income tax rates in period two in the stochastic model are larger than in the deterministic model. On the one hand, the increase in the tax rate induces a smaller decrease in capital income (in the stochastic model) for individuals with $rk_1=r\bar{k}_1$. On the other hand, the increase in the lump-sum tax liability induces a larger increase in capital income (in the stochastic model) for individuals with $rk_1\geq r\bar{k}_1$. Both effect imply that the optimal capital income tax rate in the deterministic model is a lower bound of the optimal tax rate in the stochastic model.

A.0.12 Joint Perturbations: History Dependence

Consider a joint perturbation of the first- and second-period labor income tax schedules in period two. That is, we increase the marginal tax rate $T'_{2,z_1}\left(z_1\right)$ by $d\tau_{2,z_1}$ on $[\bar{z}_1,\bar{z}_1+d\bar{x}]\times[\bar{z}_2,\infty)$, increase the marginal tax rate $T'_{2,z_2}\left(z_2\right)$ by $d\tau_{2,z_2}$ on $[\bar{z}_1,\infty)\times[\bar{z}_2,\bar{z}_2+d\bar{x}]$, and increase the total tax liability $T_{2,z_2}\left(z_2\right)$ lump-sum by dR_2 on $[\bar{z}_1+d\bar{x},\infty)\times[\bar{z}_2+d\bar{x},\infty)$.

Consider an individual with first-period income $z_1\in[\bar{z}_1,\bar{z}_1+d\bar{x}]$ and capital income $rk_1\in\mathbb{R}$. He has positive probability of drawing a second-period type that leads him to choose $z_2\in[\bar{z}_2,\infty)$

in period two. Thus he changes his first period labor income z_1 by the amount

$$dz_1|_{(\bar{z}_1,rk_1)} = \frac{\bar{z}_1\left(1-T_1'\left(\bar{z}_1\right)\right)}{1-T_1'\left(\bar{z}_1\right)+\bar{z}_1\zeta^{c,(\bar{z}_1)}_{z_1,1-\tau_{1,z_1}}T_1''\left(\bar{z}_1\right)}\left\{\int_{\bar{z}_2}^{\infty}\frac{\zeta^{c,(\bar{z}_1,rk_1)}_{z_1,\tau_{2,z_1}\left(\bar{z}_1,\mathbf{x}_2^2,rk_1\right)}}{\tau_{2,z_1}\left(\bar{z}_1,\mathbf{x}_2^2,rk_1\right)}d\mathbf{x}_2^2\right\}d\tau_{2,z_1}. \tag{77}$$

Note that the explicit expression of the elasticity $\zeta^{c,(X_1)}_{z_1,\tau_{2,z_1}\left(X_1,\mathbf{x}_2^2\right)}$ imply that the integral term in (77) is equal to

$$\int_{\bar{z}_2}^{\infty}\frac{\zeta^{c,(\bar{z}_1,rk_1)}_{z_1,\tau_{2,z_1}\left(\bar{z}_1,\mathbf{x}_2^2,rk_1\right)}}{\tau_{2,z_1}\left(\bar{z}_1,\mathbf{x}_2^2,rk_1\right)}d\mathbf{x}_2^2 = -\frac{\theta_1^2}{v_1''\left(\bar{z}_1\right)}\frac{\beta\mathbb{E}\left[u_2'\mathbf{1}_{\{z_2\geq\bar{z}_2\}}\left|X_1\right.\right]}{u_1'}. \tag{78}$$

Note that in the stochastic model, the elasticity $\zeta^{c,(X_1)}_{z_1,\tau_{2,z_1}\left(X_1,\mathbf{x}_2^2\right)}$, i.e. the response of first-period income z_1 to an increase in period two of the first-period labor income tax rate τ_{2,z_1}, does not depend only on first-period income z_1, as was the case in the deterministic model, unless this tax change is certain, i.e. it occurs for all choices of z_2 (that is, if $\bar{z}_2 = 0$ in the integral of (78)). If this is the case, then the elasticity is the same as in the deterministic model. We denote by $\bar{\zeta}^{c,(\bar{z}_1)}_{z_1,\tau_{2,z_1}(z_2\geq\bar{z}_2)}$ the aggregate change in capital income, among individuals with labor income $\bar{z}_1$, when the tax rate τ_{2,z_1} that they face in period two increases if their second-period labor income z_2 is larger than $\bar{z}_2$. That is,

$$\bar{\zeta}^{c,(\bar{z}_1)}_{z_1,\tau_{2,z_1}(z_2\geq\bar{z}_2)} \equiv \int_{\bar{z}_2}^{\infty}\int_{\mathbb{R}}\frac{\zeta^{c,(\bar{z}_1,rk_1)}_{z_1,\tau_{2,z_1}\left(\bar{z}_1,\mathbf{x}_2^2,rk_1\right)}}{\tau_{2,z_1}\left(\bar{z}_1,\mathbf{x}_2^2,rk_1\right)}h_{rk_1|z_1}\left(rk_1\left|\bar{z}_1\right.\right)d\mathbf{x}_2^2 d\left(rk_1\right). \tag{79}$$

Consider an individual with first-period income $z_1 \geq \bar{z}_1 + d\bar{x}$ and capital income $rk_1 \in \mathbb{R}$. He has positive probability of drawing a second-period type that leads him to choose $z_2 \in [\bar{z}_2, \bar{z}_2 + d\bar{x}]$ in period two. If he does (and only if he does, once the second-period type has been drawn), then in period two he changes his second-period labor income $\bar{z}_2$ by the amount

$$dz_2|_{(z_1,\bar{z}_2,rk_1)} = \frac{-\bar{z}_2\zeta^{c,(z_1,\bar{z}_2,rk_1)}_{z_2,1-\tau_{2,z_2}}}{1-T_{2,z_2}'\left(\bar{z}_2\right)+\bar{z}_2\zeta^{c,(z_1,\bar{z}_2,rk_1)}_{z_2,1-\tau_{2,z_2}}T_{2,z_2}''\left(\bar{z}_2\right)}d\tau_{2,z_2} \tag{80}$$

But the compensated elasticity $\zeta^{c,(z_1,\bar{z}_2,rk_1)}_{z_2,1-\tau_{2,z_2}} = -\bar{\theta}_2 v_2'\left(\bar{z}_2\right)/v_2''\left(\bar{z}_2\right)$ depends only on the individual's second period income $\bar{z}_2$, and can thus be written simply as $\zeta^{c,(\bar{z}_2)}_{z_2,1-\tau_{2,z_2}}$.

Finally, an individual with first period income $z_1 \geq \bar{z}_1 + d\bar{x}$ and capital income $rk_1 \in \mathbb{R}$ has a positive probability of drawing a second-period type that leads him to choose $z_2 \geq \bar{z}_2 + d\bar{x}$ in period two. Thus, this individual will change his capital income, by the amount

$$d\left(rk_1\right)|_{(z_1,rk_1)} = \frac{1}{1+\left(1-\tau_k\right)r}\left\{\int_{\bar{z}_2}^{\infty}\eta^{(z_1,rk_1)}_{rk_1,R_2\left(z_1,\mathbf{x}_2^2,rk_1\right)}d\mathbf{x}_2^2\right\}\left(-dR_2\right). \tag{81}$$

The explicit expressions for the income effect parameters imply that the integral in (81) is equal to

$$\int_{\bar{z}_2}^{\infty} \eta_{rk_1,R_2\left(z_1,\mathbf{x}_2^2,rk_1\right)}^{(z_1,rk_1)} d\mathbf{x}_2^2 = \frac{\beta r^2 \left(1+(1-\tau_k)\, r\right) \mathbb{E}\left[u_2' \mathbf{1}_{\{z_2\geq \bar{z}_2\}} \left| X_1\right.\right]}{u_1'' + \beta \left(1+(1-\tau_k)\, r\right)^2 \mathbb{E}\left[u_2'' \left| X_1 \right.\right]}. \tag{82}$$

This term is equal to the change in capital income of individual X_1 if he receives an additional dollar lump-sum if he chooses second-period labor income $z_2 \geq \bar{z}_2$. Finally, we denote by $\bar{\eta}_{rk_1,R_2(z_1\geq\bar{z}_1,z_2\geq\bar{z}_2)}$ the change in aggregate capital income in the economy, when an additional dollar is distributed lump-sum in period two, uniformly among all the individuals whose labor income in period one is above $\bar{z}_1$ and labor income in period two is above $\bar{z}_2$, i.e.

$$\bar{\eta}_{rk_1,R_2(z_1\geq\bar{z}_1,z_2\geq\bar{z}_2)} \equiv \frac{1}{\bar{H}_{z_1,z_2}\left(\bar{z}_1,\bar{z}_2\right)} \int_{\bar{z}_1}^{\infty} \int_{\bar{z}_2}^{\infty} \int_{\mathbb{R}} \eta_{rk_1,R_2\left(X_1,\mathbf{x}_2^2\right)}^{(X_1)} h_{X_1}\left(z_1,rk_1\right) dz_1 d\mathbf{x}_2^2 d\left(rk_1\right). \tag{83}$$

Using (77), (80) and (81), the change in government revenue due to the perturbation thus writes:

$$\begin{aligned}
\frac{\Gamma_{2,(z_1,z_2)}\left(\bar{z}_1,\bar{z}_2\right)}{dR_2} =& \delta\left[1-H_{z_1,z_2}\left(\bar{z}_1,\bar{z}_2\right)\right] \\
&+ \int_{\mathbb{R}} T_{1,z_1}'\left(\bar{z}_1\right) \left\{ \frac{dz_1|_{(\bar{z}_1,rk_1)}}{d\tau_{2,z_1}} \right\} h_{z_1}\left(\bar{z}_1\right) h_{rk_1|z_1}\left(rk_1 \left|\bar{z}_1\right.\right) d\left(rk_1\right) \\
&+ \int_{\bar{z}_1}^{\infty} \int_{\mathbb{R}} \delta T_{2,z_2}'\left(\bar{z}_2\right) \left\{ \frac{dz_2|_{(z_1,\bar{z}_2,rk_1)}}{d\tau_{2,z_2}} \right\} h_{z_2}\left(\bar{z}_2 \left| z_1, rk_1\right.\right) h_{X_1}\left(z_1,rk_1\right) dz_1 d\left(rk_1\right) \\
&- \int_{\bar{z}_1}^{\infty} \int_{\mathbb{R}} \delta\tau_k \left\{ \frac{d\left(rk_1\right)|_{(z_1,rk_1)}}{-dR_2} \right\} h_{X_1}\left(z_1,rk_1\right) dz_1 d\left(rk_1\right).
\end{aligned} \tag{84}$$

Normalizing by $\bar{H}_{z_1,z_2}\left(\bar{z}_1,\bar{z}_2\right)$ and using the definitions of the average compensated elasticities and income effect parameters (79) and (83), we can rewrite the revenue gains (84) as

$$\begin{aligned}
\frac{\gamma_{2,(z_1,z_2)}\left(\bar{z}_1,\bar{z}_2\right)}{\delta dR_2} =& 1 - \frac{\tau_k}{1+(1-\tau_k)\, r} \bar{\eta}_{rk_1,R_2(z_1\geq\bar{z}_1,z_2\geq\bar{z}_2)} \\
&+ \delta^{-1} \frac{\left(1-T_1'\left(\bar{z}_1\right)\right) T_{1,z_1}'\left(\bar{z}_1\right)}{1-T_1'\left(\bar{z}_1\right) + \bar{z}_1 \zeta_{z_1,1-\tau_{1,z_1}}^{c,(\bar{z}_1)} T_1''\left(\bar{z}_1\right)} \bar{\zeta}_{z_1,\tau_{2,z_1}(z_2\geq\bar{z}_2)}^{c,(\bar{z}_1)} \frac{\bar{z}_1 h_{z_1}\left(\bar{z}_1\right)}{\bar{H}_{z_1,z_2}\left(\bar{z}_1,\bar{z}_2\right)} \\
&- \frac{T_{2,z_2}'\left(\bar{z}_2\right)}{1-T_{2,z_2}'\left(\bar{z}_2\right) + \bar{z}_2 \zeta_{z_2,1-\tau_{2,z_2}}^{c,(\bar{z}_2)} T_{2,z_2}''\left(\bar{z}_2\right)} \bar{\zeta}_{z_2,1-\tau_{2,z_2}}^{c,(\bar{z}_2)} \frac{\bar{z}_2 \int_{\bar{z}_1}^{\infty} h_{z_1,z_2}\left(z_1,\bar{z}_2\right) dz_1}{\bar{H}_{z_1,z_2}\left(\bar{z}_1,\bar{z}_2\right)},
\end{aligned} \tag{85}$$

But the formulas giving the revenue gains of the corresponding separable perturbations imply, letting $\bar{\zeta}_{z_1,\tau_{2,z_1}}^{c,(\bar{z}_1)} \equiv \bar{\zeta}_{z_1,\tau_{2,z_1}(z_2\geq 0)}^{c,(\bar{z}_1)}$ (i.e. the perturbation occurs with certainty to individuals with

$z_1 = \bar{z}_1$),

$$\frac{\delta^{-1}\left(1-T_1'\left(\bar{z}_1\right)\right)T_{1,z_1}'\left(\bar{z}_1\right)}{1-T_1'\left(\bar{z}_1\right)+\bar{z}_1\zeta_{z_1,1-\tau_{1,z_1}}^{c,(\bar{z}_1)}T_1''\left(\bar{z}_1\right)} = -\frac{1}{\bar{\zeta}_{z_1,\tau_{2,z_1}}^{c,(\bar{z}_1)}}\left(\frac{\bar{z}_1 h_{z_1}\left(\bar{z}_1\right)}{1-H_{z_1}\left(\bar{z}_1\right)}\right)^{-1}\left(1-\frac{\tau_k}{1+\left(1-\tau_k\right)r}\bar{\eta}_{rk_1,R_2(z_1\geq\bar{z}_1)}-\frac{\gamma_{2,z_1}\left(\bar{z}_1\right)}{\delta dR_2}\right), \tag{86}$$

and

$$\frac{T_{2,z_2}'\left(\bar{z}_2\right)}{1-T_2'\left(\bar{z}_2\right)+\bar{z}_2\zeta_{z_2,1-\tau_{2,z_2}}^{c,(\bar{z}_2)}T_2''\left(\bar{z}_2\right)} = \frac{1}{\zeta_{z_2,1-\tau_{2,z_2}}^{c,(\bar{z}_2)}}\left(\frac{\bar{z}_2 h_{z_2}\left(\bar{z}_2\right)}{1-H_{z_2}\left(\bar{z}_2\right)}\right)^{-1}\left(1-\frac{\tau_k}{1+\left(1-\tau_k\right)r}\bar{\eta}_{rk_1,R_2(z_2\geq\bar{z}_2)}-\frac{\gamma_{2,z_2}\left(\bar{z}_2\right)}{\delta dR_2}\right). \tag{87}$$

Thus, substituting (86) and (87) in (85), the revenue gains of the joint perturbations are equal to

$$\frac{\gamma_{2,(z_1,z_2)}\left(\bar{z}_1,\bar{z}_2\right)}{\delta dR_2} = W_1\frac{\gamma_{2,z_1}\left(\bar{z}_1\right)}{\delta dR_2}+W_2\frac{\gamma_{2,z_2}\left(\bar{z}_2\right)}{\delta dR_2}+\left\{1-W_1\left(\frac{1+S_1}{1+S_{1,2}}\right)-W_2\left(\frac{1+S_2}{1+S_{1,2}}\right)\right\}\left(1+S_{1,2}\right). \tag{88}$$

where the weights W_1 and W_2 are defined by

$$\begin{aligned} W_1 &= \frac{\bar{\zeta}_{z_1,\tau_{2,z_1}(z_2\geq\bar{z}_2)}^{c,(\bar{z}_1)}}{\bar{\zeta}_{z_1,\tau_{2,z_1}}^{c,(\bar{z}_1)}}\left(\frac{h_{z_1}\left(\bar{z}_1\right)}{1-H_{z_1}\left(\bar{z}_1\right)}\right)^{-1}\left(\frac{\int_{\bar{z}_2}^{\infty}h_{z_1,z_2}\left(\bar{z}_1,z_2\right)dz_2}{\bar{H}_{z_1,z_2}\left(\bar{z}_1,\bar{z}_2\right)}\right)\left(\frac{h_{z_1}\left(\bar{z}_1\right)}{\int_{\bar{z}_2}^{\infty}h_{z_1,z_2}\left(\bar{z}_1,z_2\right)dz_2}\right), \\ &= \frac{1}{\mathbb{P}\left(z_2\geq\bar{z}_2\,|\bar{z}_1\right)}\frac{\bar{\zeta}_{z_1,\tau_{2,z_1}(z_2\geq\bar{z}_2)}^{c,(\bar{z}_1)}}{\bar{\zeta}_{z_1,\tau_{2,z_1}}^{c,(\bar{z}_1)}}\frac{\left[\lambda_{z_1,z_2}\left(\bar{z}_1,\bar{z}_2\right)\right]_1}{\lambda_{z_1}\left(\bar{z}_1\right)}, \\ W_2 &= \left(\frac{h_{z_2}\left(\bar{z}_2\right)}{1-H_{z_2}\left(\bar{z}_2\right)}\right)^{-1}\left(\frac{\int_{\bar{z}_1}^{\infty}h_{z_1,z_2}\left(z_1,\bar{z}_2\right)dz_1}{\bar{H}_{z_1,z_2}\left(\bar{z}_1,\bar{z}_2\right)}\right) = \frac{\left[\lambda_{z_1,z_2}\left(\bar{z}_1,\bar{z}_2\right)\right]_2}{\lambda_{z_2}\left(\bar{z}_2\right)}, \end{aligned} \tag{89}$$

and the "savings effects" $S_1, S_2, S_{1,2}$ are defined by

$$\begin{aligned} S_1 &= -\frac{\tau_k}{1+\left(1-\tau_k\right)r}\bar{\eta}_{rk_1,R_2(z_1\geq\bar{z}_1)},\ S_2 = -\frac{\tau_k}{1+\left(1-\tau_k\right)r}\bar{\eta}_{rk_1,R_2(z_2\geq\bar{z}_2)}, \\ S_{1,2} &= -\frac{\tau_k}{1+\left(1-\tau_k\right)r}\bar{\eta}_{rk_1,R_2(z_1\geq\bar{z}_1,z_2\geq\bar{z}_2)}. \end{aligned} \tag{90}$$

Formula (88) shows that the revenue gains of introducing history-dependence in the tax system, in the stochastic model, are given by a formula that resembles its counterpart (55) in the deterministic model with, however, several differences. As in the deterministic case, the weight W_2, given by (89), is equal to the ratio of the second component of the multivariate hazard rate of the joint distribution of labor incomes, to the hazard rate of the univariate distribution of second-period labor income. However, the weight W_1, given by (89), is equal to the corresponding ratio (first component of the multivariate hazard, to the hazard of the first-period labor income distribution), multiplied by a ratio of conditional to unconditional elasticity, where the unconditional elasticity is weighted by the probability that the individual $\bar{z}_1$ will indeed be affected by the tax change, i.e. $z_2 \geq \bar{z}_2$. This is because in the stochastic model, *all* the individuals with first-period income $z_1 = \bar{z}_1$, are potentially affected by the tax change, and hence react to the perturbation by changing their first period income z_1. Finally, these weights are multiplied by the ratios of "savings effects" (90), which comes from the fact that, even with a CRRA utility function, individuals react differently to perturbations of their lump-sum tax liability, depending on the states of the world in this perturbation takes place.

A.0.13 Joint Perturbations: Joint Taxation of Labor and Capital Incomes

Consider a joint perturbation of the capital income and second-period labor income tax schedules in period 2. That is, increase the marginal tax rate $T'_{2,z_2}(z_2)$ by $d\tau_{2,z_2}$ on $[\bar{z}_2, \bar{z}_2 + d\bar{x}] \times \left[r\bar{k}_1, \infty\right)$, increase the marginal tax rate $T'_{2,rk_1}(rk_1)$ by $d\tau_{2,rk_1}$ on $[\bar{z}_2, \infty) \times \left[r\bar{k}_1, r\bar{k}_1 + d\bar{x}\right]$, and increase the total tax liability $T_{2,z_2}(z_2)$ lump sum by dR_2 on $[\bar{z}_2 + d\bar{x}, \infty) \times \left[r\bar{k}_1 + d\bar{x}, \infty\right)$.

Consider first an individual with capital income $rk_1 \in \left[r\bar{k}_1, r\bar{k}_1 + d\bar{x}\right]$, and with first-period labor income $z_1 \in \mathbb{R}_+$. He has positive probability of drawing a second-period type that leads him to choose $z_2 \in [\bar{z}_2, \infty)$ in period two. Thus he changes his capital income rk_1 by the amount

$$d\left(rk_1\right)\big|_{\left(z_1, r\bar{k}_1\right)} = \frac{-r\bar{k}_1}{1 + (1 - \tau_k)\, r} \left\{ \int_{\bar{z}_2}^{\infty} \zeta_{rk_1, 1-\tau_{2,rk_1}\left(\mathbf{x}_2^2\right)}^{c,\left(z_1, r\bar{k}_1\right)} d\mathbf{x}_2^2 \right\} d\tau_{2,rk_1}, \tag{91}$$

where the explicit expressions for the compensated elasticities imply that the integral in (91) is equal to

$$\int_{\bar{z}_2}^{\infty} \zeta_{rk_1, 1-\tau_{2,rk_1}\left(\mathbf{x}_2^2\right)}^{c,\left(z_1, r\bar{k}_1\right)} d\mathbf{x}_2^2 = \frac{1}{r\bar{k}_1} \frac{\beta r^2 \left(1 + (1 - \tau_k)\, r\right) \mathbb{E}\left[u_2' \mathbf{1}_{\{z_2 \geq \bar{z}_2\}} \left| X_1 \right.\right]}{u_1'' + \beta \left(1 + (1 - \tau_k)\, r\right)^2 \mathbb{E}\left[u_2'' \left| X_1 \right.\right]}. \tag{92}$$

We denote by $\zeta_{rk_1, 1-\tau_{2,rk_1}(z_2 \geq \bar{z}_2)}^{c,\left(r\bar{k}_1\right)}$ the aggregate change in capital income, among individuals with capital income $r\bar{k}_1$, when the net-of capital income tax rate $1 - \tau_{2,rk_1}$ that they face in period two increases if their second-period labor income z_2 is larger than $\bar{z}_2$. That is,

$$\zeta_{rk_1, 1-\tau_{2,rk_1}(z_2 \geq \bar{z}_2)}^{c,\left(r\bar{k}_1\right)} \equiv \int_{\mathbb{R}} \int_{\bar{z}_2}^{\infty} \zeta_{rk_1, 1-\tau_{2,rk_1}\left(\mathbf{x}_2^2\right)}^{c,\left(z_1, r\bar{k}_1\right)} h_{z_1 | rk_1} \left(z_1 \left| r\bar{k}_1\right.\right) dz_1 d\mathbf{x}_2^2. \tag{93}$$

Consider now an individual with capital income $k_1 \geq \bar{k}_1 + d\bar{x}$ and first-period labor income $z_1 \in \mathbb{R}_+$. He has positive probability of drawing a second-period type that leads him to choose $z_2 \in [\bar{z}_2, \bar{z}_2 + d\bar{x}]$ in period two. If he does (and only if he does, once second-period type is revealed), then in period two he changes his second-period labor income $\bar{z}_2$ by the amount

$$dz_2|_{(z_1,\bar{z}_2,rk_1)} = \frac{-\bar{z}_2 \zeta^{c,(z_1,\bar{z}_2,rk_1)}_{z_2,1-\tau_{2,z_2}}}{1 - T'_{2,z_2}(\bar{z}_2) + \bar{z}_2 \zeta^{c,(z_1,\bar{z}_2,rk_1)}_{z_2,1-\tau_{2,z_2}} T''_{2,z_2}(\bar{z}_2)} d\tau_{2,z_2}. \tag{94}$$

Note that the compensated elasticity $\zeta^{c,(z_1,\bar{z}_2,rk_1)}_{z_2,1-\tau_{2,z_2}}$ depends only on the individual's second period income, and can thus simply be written as $\zeta^{c,(\bar{z}_2)}_{z_2,1-\tau_{2,z_2}}$.

Finally, consider again an individual with capital income $k_1 \geq r\bar{k}_1 + d\bar{x}$ and first-period labor income $z_1 \in \mathbb{R}_+$. He has positive probability of drawing a second-period type that leads him to choose $z_2 \geq \bar{z}_2 + d\bar{x}$ in period two. Thus, this individual will also change his capital income, by the amount

$$d(rk_1)|_{(z_1,rk_1)} = \frac{1}{1 + (1-\tau_{2,rk_1})r} \left\{ \int_{\bar{z}_2}^{\infty} \eta^{(z_1,rk_1)}_{rk_1,R_2(z_1,\mathbf{x}_2^2,rk_1)} d\mathbf{x}_2^2 \right\} (-dR_2), \tag{95}$$

where the explicit expressions for the income effect parameters imply that the integral in (95)

$$\int_{\bar{z}_2}^{\infty} \eta^{(z_1,rk_1)}_{rk_1,R_2(z_1,\mathbf{x}_2^2,rk_1)} d\mathbf{x}_2^2 = \frac{\beta(1+(1-\tau_k)r)^2 \mathbb{E}\left[u''_2 \mathbf{1}_{\{z_2 \geq \bar{z}_2\}} | X_1\right]}{u''_1 + \beta(1+(1-\tau_k)r)^2 \mathbb{E}_{\theta_2}\left[u''_2(z_1, z_2, rk_1)|\theta_1\right]}. \tag{96}$$

It is equal to the change in capital income of the individual $X_1 = (z_1, rk_1)$ when the second-period virtual income is increased on the interval $[\bar{z}_2, \infty)$. Finally, we denote by $\bar{\eta}_{rk_1,R_2(z_2\geq\bar{z}_2,rk_1\geq r\bar{k}_1)}$ the aggregate change in capital in the economy if one dollar is distributed lump-sum and uniformly in period two among all the individuals whose second-period labor income and capital income are respectively above $\bar{z}_2$ and above $r\bar{k}_1$, that is,

$$\bar{\eta}_{rk_1,R_2(z_2\geq\bar{z}_2,rk_1\geq r\bar{k}_1)} \equiv \frac{1}{\bar{H}_{z_2,rk_1}(\bar{z}_2, r\bar{k}_1)} \int_{\mathbb{R}} \int_{\bar{z}_2}^{\infty} \int_{r\bar{k}_1}^{\infty} \eta^{(z_1,rk_1)}_{rk_1,R_2(z_1,\mathbf{x}_2^2,rk_1)} h_{X_1}(z_1, rk_1)\, dz_1 d\mathbf{x}_2^2 d(rk_1). \tag{97}$$

Using (91), (94) and (95), we obtain that the change in government revenue due to the perturbation thus writes:

$$\begin{aligned}
\frac{\Gamma_{2,(,z_2,rk_1)}(\bar{z}_2, r\bar{k}_1)}{dR_2} =& \delta\left[1 - H_{z_2,rk_1}(\bar{z}_2, r\bar{k}_1)\right] \\
&+ \delta \int_{\mathbb{R}} \tau_k \left\{ \frac{d(rk_1)|_{(z_1,r\bar{k}_1)}}{d\tau_{2,rk_1}} \right\} h_{rk_1}(r\bar{k}_1)\, h_{z_1|rk_1}(z_1|r\bar{k}_1)\, dz_1 \\
&+ \int_{r\bar{k}_1}^{\infty} \int_{\mathbb{R}} \delta T'_{2,z_2}(\bar{z}_2) \left\{ \frac{dz_2|_{(z_1,\bar{z}_2,rk_1)}}{d\tau_{2,z_2}} \right\} h_{z_2}(\bar{z}_2|z_1, rk_1)\, h_{X_1}(z_1, rk_1)\, dz_1 d(rk_1) \\
&- \int_{r\bar{k}_1}^{\infty} \int_{\mathbb{R}} \delta\tau_k \left\{ \frac{d(rk_1)|_{(z_1,rk_1)}}{-dR_2} \right\} h_{X_1}(z_1, rk_1)\, dz_1 d(rk_1).
\end{aligned} \tag{98}$$

Normalizing by $\bar{H}_{z_2,rk_1}\left(\bar{z}_2, r\bar{k}_1\right)$ and using the definitions of the average compensated elasticities and income effect parameters (93) and (97), we can rewrite these revenue gains as

$$\begin{aligned}\frac{\gamma_{2,(z_2,rk_1)}\left(\bar{z}_2, r\bar{k}_1\right)}{\delta dR_2} =&1 - \frac{\tau_k}{1+(1-\tau_k)\,r}\bar{\eta}_{rk_1,R_2\left(z_2\geq\bar{z}_2,rk_1\geq r\bar{k}_1\right)} \\ &- \frac{\tau_k}{1+(1-\tau_k)\,r}\zeta^{c,\left(r\bar{k}_1\right)}_{rk_1,1-\tau_{2,rk_1}(z_2\geq\bar{z}_2)}\frac{r\bar{k}_1 h_{rk_1}\left(r\bar{k}_1\right)}{\bar{H}_{z_2,rk_1}\left(\bar{z}_2, r\bar{k}_1\right)} \\ &- \frac{T'_{2,z_2}\left(\bar{z}_2\right)}{1-T'_{2,z_2}\left(\bar{z}_2\right)+\bar{z}_2\zeta^{c,(\bar{z}_2)}_{z_2,1-\tau_{2,z_2}}T''_{2,z_2}\left(\bar{z}_2\right)}\zeta^{c,(\bar{z}_2)}_{z_2,1-\tau_{2,z_2}}\frac{\bar{z}_2\int_{r\bar{k}_1}^{\infty}h_{z_2,rk_1}\left(\bar{z}_2,rk_1\right)d\left(rk_1\right)}{\bar{H}_{z_2,rk_1}\left(\bar{z}_2, r\bar{k}_1\right)},\end{aligned} \tag{99}$$

But the formulas giving the revenue gains of the corresponding separable perturbations imply, letting $\bar{\zeta}^{c,(\bar{z}_1)}_{z_1,\tau_{2,z_1}} \equiv \bar{\zeta}^{c,(\bar{z}_1)}_{z_1,\tau_{2,z_1}(z_2\geq 0)}$ (i.e. the perturbation occurs with certainty to individuals with $z_1 = \bar{z}_1$),

$$\begin{aligned}&\frac{T'_{2,z_2}\left(\bar{z}_2\right)}{1-T'_2\left(\bar{z}_2\right)+\bar{z}_2\zeta^{c,(\bar{z}_2)}_{z_2,1-\tau_{2,z_2}}T''_2\left(\bar{z}_2\right)} \\ =&\frac{1}{\zeta^{c,(\bar{z}_2)}_{z_2,1-\tau_{2,z_2}}}\left(\frac{\bar{z}_2 h_{z_2}\left(\bar{z}_2\right)}{1-H_{z_2}\left(\bar{z}_2\right)}\right)^{-1}\left(1-\frac{\tau_k}{1+(1-\tau_k)\,r}\bar{\eta}_{rk_1,R_2(z_2\geq\bar{z}_2)}-\frac{\gamma_{2,z_2}\left(\bar{z}_2\right)}{\delta dR_2}\right),\end{aligned} \tag{100}$$

and

$$\begin{aligned}&\frac{\tau_k}{1+(1-\tau_k)\,r} \\ =&\frac{1}{\bar{\zeta}^{c,\left(r\bar{k}_1\right)}_{rk_1,1-\tau_{2,rk_1}}}\left(\frac{r\bar{k}_1 h_{rk_1}\left(r\bar{k}_1\right)}{1-H_{rk_1}\left(r\bar{k}_1\right)}\right)^{-1}\left(1-\frac{\tau_k}{1+(1-\tau_k)\,r}\mathbb{E}\left[\eta^{(X_1)}_{rk_1,R_2}\,\middle|\,rk_1\geq r\bar{k}_1\right]-\frac{\gamma_{1,rk_1}\left(r\bar{k}_1\right)}{\delta dR_k}\right).\end{aligned} \tag{101}$$

Substituting (100) and (101) in (99), we obtain that the revenue gains of the joint perturbations are equal to

$$\begin{aligned}&\frac{\gamma_{2,(z_2,rk_1)}\left(\bar{z}_2, r\bar{k}_1\right)}{\delta dR_2} \\ =&W_k\frac{\gamma_{1,rk_1}\left(r\bar{k}_1\right)}{\delta dR_k}+W_2\frac{\gamma_{2,z_2}\left(\bar{z}_2\right)}{\delta dR_2}+\left\{1-W_k\left(\frac{1+S_k}{1+S_{2,k}}\right)-W_2\left(\frac{1+S_2}{1+S_{2,k}}\right)\right\}\left(1+S_{2,k}\right),\end{aligned} \tag{102}$$

where the weights W_k and W_2 are defined by

$$W_k = \frac{\zeta^{c,\left(r\bar{k}_1\right)}_{rk_1,1-\tau_{2,rk_1}(z_2\geq\bar{z}_2)}}{\bar{\zeta}^{c,\left(r\bar{k}_1\right)}_{rk_1,1-\tau_{2,rk_1}}} \left(\frac{h_{rk_1}\left(r\bar{k}_1\right)}{1-H_{rk_1}\left(r\bar{k}_1\right)}\right)^{-1} \left(\frac{\int_{\bar{z}_2}^{\infty} h_{z_2,rk_1}\left(z_2,r\bar{k}_1\right)dz_2}{\bar{H}_{z_2,rk_1}\left(\bar{z}_2,r\bar{k}_1\right)}\right) \left(\frac{h_{rk_1}\left(r\bar{k}_1\right)}{\int_{\bar{z}_2}^{\infty} h_{z_2,rk_1}\left(z_2,r\bar{k}_1\right)dz_2}\right),$$

$$= \frac{1}{\mathbb{P}\left(z_2\geq\bar{z}_2\left|r\bar{k}_1\right.\right)} \frac{\zeta^{c,\left(r\bar{k}_1\right)}_{rk_1,1-\tau_{2,rk_1}(z_2\geq\bar{z}_2)}}{\bar{\zeta}^{c,\left(r\bar{k}_1\right)}_{rk_1,1-\tau_{2,rk_1}}} \frac{\left[\lambda_{z_2,rk_1}\left(\bar{z}_2,r\bar{k}_1\right)\right]_2}{\lambda_{rk_1}\left(r\bar{k}_1\right)},$$

$$W_2 = \left(\frac{h_{z_2}\left(\bar{z}_2\right)}{1-H_{z_2}\left(\bar{z}_2\right)}\right)^{-1} \left(\frac{\int_{r\bar{k}_1}^{\infty} h_{z_2,rk_1}\left(\bar{z}_2,rk_1\right)d\left(rk_1\right)}{\bar{H}_{z_2,rk_1}\left(\bar{z}_2,r\bar{k}_1\right)}\right) = \frac{\left[\lambda_{z_2,rk_1}\left(\bar{z}_2,r\bar{k}_1\right)\right]_1}{\lambda_{z_2}\left(\bar{z}_2\right)}, \tag{103}$$

and the "savings effects" $S_k, S_2, S_{k,2}$ are defined by:

$$S_k = -\frac{\tau_k}{1+\left(1-\tau_k\right)r}\mathbb{E}\left[\eta^{(X_1)}_{rk_1,R_2}\left|rk_1\geq r\bar{k}_1\right.\right],\ S_2 = -\frac{\tau_k}{1+\left(1-\tau_k\right)r}\bar{\eta}_{rk_1,R_2(z_2\geq\bar{z}_2)},$$
$$S_{2,k} = -\frac{\tau_k}{1+\left(1-\tau_k\right)r}\bar{\eta}_{rk_1,R_2\left(z_2\geq\bar{z}_2,rk_1\geq r\bar{k}_1\right)}. \tag{104}$$

Formula (102) shows that the revenue gains of jointly taxing labor and capital incomes in period two in the stochastic model are given by the usual ratios: hazard rates (multivariate vs. univariate), compensated elasticities (conditional vs. unconditional), savings effects, and the probability that the individual ends up being affected by the perturbation, i.e. $z_2 \geq \bar{z}_2$.