Amplification of Uncertainty in Illiquid Markets *

(JOB MARKET PAPER)

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Abstract

This paper argues that the capacity of financial markets to aggregate information is diminished in times of distress, resulting in countercyclical economic uncertainty. I build upon a rational expectations equilibrium model that delivers this result from the combination of (i) countercyclical funding constraints faced by informed financial intermediaries, and (ii) the dispersed nature of information in the economy. During downturns, informed traders become increasingly exposed to non-fundamental price fluctuations (noise trading risk), which reduces information-based trading and the informativeness of asset prices. Uncertainty can spike quite dramatically as conditions deteriorate due to amplification mechanisms that arise from the dispersed nature of information, and the presence of information externalities in a dynamic environment. I show that heightened uncertainty leads to increased risk premia, Sharpe ratios, and stock price volatility even when attitude towards risk and the unconditional volatility of fundamentals remain constant. I also trace the implications for real investment decisions when firms learn about productivity from the observation of stock prices: uncertainty affects welfare by reducing the accuracy of investment and also reduces its level as a precautionary response of firms. The mechanism outlined suggests that the success of public liquidity provision in stabilizing markets depends crucially on the distribution of liquidity across agents.

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1 Introduction

One of the most salient features of recessions is the pervasiveness of uncertainty. Evidence including the dispersion and mean errors of analyst’s estimations and the behavior of volatility option indexes, suggests that predicting economic outcomes is more difficult during contractions.\footnote{See Veronesi (1999), Rich and Tracy (2003) and Bollerslev and Zhou (2007).} Since uncertainty deters consumption and investment of households and firms, a thorough understanding of its determinants is key for guiding policy that can contain the severity and persistence of slumps.

This paper makes three main contributions. First, I offer a tractable model that explains countercyclical uncertainty from the diminished capacity of distressed financial markets to aggregate information about economic fundamentals. \footnote{The concept of uncertainty tackled in this paper is closer to the idea of risk—the variance of economic fundamentals given available information—than to Knightian uncertainty—the concept of unmeasurable risk.} Second, I argue that cycle-dependent information aggregation can explain the time-variation of expected returns (the risk premia) and the price of risk (Sharpe ratios), as well as countercyclical stock price volatility. Third, I trace the implications for real allocations in a production economy with investment where firms learn information about productivity from asset prices.

The model I propose combines two literatures that are central in finance and macroeconomics, but seldom analyzed jointly: the \textit{rational expectations equilibrium} (REE) analysis of information aggregation (Grossman and Stiglitz (1980); Hellwig (1980))\footnote{See also Diamond and Verrecchia (1981).} and the \textit{limits of arbitrage} (De Long, Shleifer, Summers and Waldmann (1990); Shleifer and Vishny (1997)). As in a noisy REE model, I consider agents that react to private and heterogeneous information by trading in financial markets with agents that have other trading motives (noise trading), resulting in the partial aggregation of information about fundamentals into asset prices. The limits of arbitrage qualify the extent of such aggregation by underscoring the limitations to trade that arise from funding constraints in actual markets. I argue that because funding constraints are more binding in contractions than during expansions, information aggregation through asset prices is procyclical.

Most work on the limits of arbitrage link asset pricing phenomena to funding fragility during episodes of financial turmoil\footnote{See Kyle and Xiong (2001), Krishnamurthy and He (2008), and Brunnermeier and Pedersen (2009)} \footnote{See also Diamond and Verrecchia (1981).}. Since it is difficult to embed constraints observed in financial markets within models of information aggregation, most of the literature sidesteps the informational role of prices by assuming exogenous information asymmetries across agents. I build a model where financial intermediaries face early redemptions from clients, which heightens trading risks as liquidation values can diverge from fundamentals due to noise trading. Focusing on dynamic risk considerations triggered by withdrawals is both realistic and a useful analytical simplification that allows me to provide closed-form solutions for the informativeness of asset prices as a function
of underlying economic conditions. Moreover, spikes in economic uncertainty can be quite pronounced when conditions deteriorate due to amplification mechanisms that arise from the dispersed nature of information, and the presence of information externalities in a dynamic environment.

The second contribution of this paper is to trace the asset pricing implications of countercyclical uncertainty. I argue that the model can shed light on the observed time-variation of expected returns and Sharpe ratios. As economic conditions and the informativeness of prices deteriorate, traders demand larger compensations for holding risk. Moreover, higher expected returns translate into heightened price variability in the presence of noise trading, consistent with countercyclical stock market volatility. Importantly, the model delivers these results even when the attitude towards risk and the unconditional volatility of fundamentals remain constant.

As a third contribution, I study the implications for real allocations in an production economy with investment. I show that when firms learn information about productivity conditions from stock prices, heightened noise results in clear welfare losses by reducing the accuracy of investment decisions. Moreover, I provide a closed-form solution for the investment problem in which the level of investment is negatively affected by the variance of fundamentals, conditional on asset prices. This result is well in line with the literature on partial investment irreversibilities that predicts a fall in investment during periods of high economic uncertainty (Bernanke (1993); Dixit and Pindyck (1994); Bertola and Caballero (1994)), and recent supporting evidence (Bloom, Bond and Van Reenen (2007); Bloom (2009)).

This paper relates to three other broad strands of literature that cover countercyclical uncertainty, asset pricing models of asymmetric information and financial constraints, and time-varying risk premia.

First, several papers have studied countercyclical uncertainty and business cycle learning dynamics. While Van Niewerburgh and Veldkamp (2006) and Angeletos and La'o (2008) focus on neoclassical production economies with no role for financial markets, Veronesi (1999) discusses learning in financial markets in a model of regime shifts, but in a representative agent framework without financial constraints.

Second, a growing literature featuring models of asymmetric information and financial constraints is gaining a central stage in asset pricing. Xiong (2001), and Kyle and Xiong (2001) build equilibrium models where arbitrageurs’ wealth losses destabilize asset prices. While their effect comes from increased risk-aversion, Krishnamurthy and He (2008) stress the impact on the contracting problem between households and intermediaries. Brunnermeier and Pedersen (2009) focus on the perverse feedback between funding liquidity and non-fundamental volatility (market

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6Harvey (1989); Ferson and Harvey (1991).
7Schwert (1989); Bollerslev, Chou and Kroner (1992); Campbell, Lettau, Malkiel and Xu (2001)
8Dow and Gorton (1997); Dow and Rahi (2003); Goldstein and Guembel (2008)
liquidity). These papers accurately predict several moments of pricing data, but at a cost of tractability that renders learning from prices infeasible. Barlevy and Veronesi (2003) and Yuan (2005) simultaneously deal with funding constraints and learning to explain stock market crashes. Two important additions of my approach are the inclusion of dispersed information and dynamic risk considerations. While the former is key to understand the implications of limited information aggregation on real allocations, both elements add interesting amplification forces.

Third, a well established literature has studied time-varying risk premia. Campbell and Cochrane (1999) build a representative agent model with habit formation (non time-separable preferences), and argue that the effective risk aversion of households spikes as consumption falls towards habit levels in recessions. The alternative explanation stresses exogenous time-variations in the volatility (conditional heteroskedasticity) of the dividend generating process of risky assets (Barsky and DeLong (1993); Bansal and Yaron (2004)). While the attitude towards risk and the exogenous amount of it are likely to be higher in recessions, I argue that uncertainty arising from the endogenous fluctuation of asset price informativeness can play an important role by itself. To my knowledge, no related paper has formally studied the connection between endogenous information aggregation and time-variation in the price of risk.

The remainder of the paper is structured as follows. The next section describes a simple REE model in which funding constraints exogenously limit market participation to a subset of informed traders, with the aim of elucidating the broad link between real market constraints and the cyclical variation of price informativeness. Section 3 introduces dynamic risk in a three stage model, where informed traders are financial intermediaries that invest on behalf of households. Informed traders choose asset positions in a first stage, but are forced to unwind a fraction of these in the second stage to meet redemptions at a price affected by noise trading shocks. At this stage, noise trading and liquidations are absorbed uninformed traders, modeled as rational investors who learn from prices. An interesting element of the equilibrium is that noise trading risk—the price impact of a given supply shock—is endogenous to informed trading, since price informativeness affects the willingness of uninformed traders to absorb supply, for a given price change. Endogeneity of noise trading risk amplifies the effect of illiquidity on uncertainty, and can result in multiple trading equilibria. In section 4, I discuss the asset pricing implications of the model, while section 5 discusses real allocation consequences when stock prices provide information for real investment decisions. Section 6 discusses further testable predictions of the model and offers brief remarks for policy. Section 7 concludes.

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10For a recent example see Gärdenau and Pedersen (2009).

11I am indebted to Andy Atkeson for pointing this out.
2 A noisy REE model with financial constraints

2.1 Setup

This section illustrates how financial constraints can hinder information aggregation using a simple REE model of an endowment economy with asymmetric information, which will be extended in subsequent sections. There are two stages: 1, and 2. The consumption good is produced in the random amount \( \theta \) at stage 2. I refer to \( \theta \) as the dividend, or economic fundamentals interchangeably, which follows a normal distribution with zero mean and variance \( \lambda^{-1} \).

The economy is populated by traders that exchange claims on the risky endowment through shares in a financial market that opens in stage 1. In stage 2, \( \theta \) is revealed and traders are paid according to their net positions, and consume.

Traders can be either informed traders or noise traders. There are a continuum of informed traders indexed by \( i \in [0, 1] \), and each is endowed with one share. At stage 1, they observe a private signal \( s_i \) about the value of the dividend:

\[
s_i = \theta + \epsilon_i; \quad \epsilon_i \sim N\left(0, \lambda^{-1}_\epsilon\right),
\]

The signal consists of the true realization of \( \theta \) plus idiosyncratic noise \( \epsilon_i \), which is identically and independently distributed across traders conditional on \( \theta \). Informed traders have CARA preferences with risk aversion \( \gamma \) over the consumption of terminal wealth at stage 2: \( U(W_{i,2}) = -\exp(-\gamma \cdot W_{i,2}) \).

Noise traders have other (unmodeled) trading motives and supply the random amount of \( n \) shares, with \( n \sim N\left(0, \lambda^{-1}_n\right) \).

To illustrate the impact of funding constraints on information aggregation, suppose informed traders have a funding status \( f_i \in \{0, 1\} \), so that only traders with \( f_i = 1 \) can exchange claims on the financial market. Funding liquidity \( F \equiv \int f_idi \) measures the fraction \( F \in (0, 1] \) of informed traders that are allowed to trade. Limits to trading can arise from losses on prior positions. If the wealth (capital) of traders is low enough, they are likely to face constraints in raising funds for further trading. Alternatively, agents that use informed traders as intermediaries might face liquidity needs that force them to make withdrawals. I explore the latter channel in section 3, but for now I take \( F \) as exogenous, focusing on the informational properties of the share price that result from its variation. I assume that liquid traders (\( f_i = 1 \)) can borrow at a riskless rate normalized at zero.
2.2 Equilibrium

A competitive equilibrium is defined by 1) a share price function $P_1(\theta, n; F)$; 2) demands schedules by informed traders $t_i = t(s_i, P_1; f_i, F)$; and 3) a set of prior beliefs $H(\theta)$, and posterior beliefs $H(\theta | s_i, P_1)$ such that $\forall i \in [0, 1]$: (i) If $f_i = 0$, $t_i = 0$ and if $f_i = 1$ asset demands are optimal given posterior beliefs and aggregate funding liquidity $F$; (ii) The share price clears the market; and (iii) Posterior beliefs are updated through Bayes law.

The price function $P_1(\cdot)$ depends on the realization of the fundamental ($\theta$), noise trading ($n$), and aggregate funding liquidity $F$. Condition (i) states that informed traders who can trade ($f_i = 1$) maximize expected utility given posterior beliefs and aggregate funding conditions, since $F$ will affect the informativeness of the share price. Condition (ii) imposes market clearing for any realization of the noisy asset supply, while (iii) restricts beliefs to follow Bayes rule: the conditional distribution of $D$ is updated from the observation of signals $s_i$ and the share price $12 P_1$.

The solution method follows three steps, as is standard in noisy REE settings (Grossman (1976)). First, I conjecture that the price function is linear in the shocks:

$$ P_1(\theta, n, \cdot) = A_1 + A_2 \cdot \theta + A_3 \cdot n $$

so that informed traders back out a noisy signal about $\theta$ from the observation of price labeled $\hat{p}_1$: the informational content of the price,

$$ \hat{p}_1 \equiv \frac{P_1 - A_1}{A_2} = \theta - \Delta \cdot n $$

where $\Delta = -A_3/A_2$. From expression (3), $\hat{p}_1$ given $\theta$ is distributed normally with mean $\theta$ and variance $\lambda_1^{-1} = \lambda_n^{-1} \cdot \Delta^2$. The variance of the price signal is the product of two terms: the variance of noise trading shocks ($\lambda_n^{-1}$) and the noise amplifier $\Delta$. The latter captures the response of the price to innovations in fundamentals relative to noise trading (the ratio $-A_3/A_2$). When high, noise trading has a large impact on the price, which becomes a poor aggregator of dispersed information about the fundamental $\theta$.

Informed traders’ posterior beliefs of $\theta$ depend on private signals and the market-clearing price. Applying the projection theorem (Appendix A), the first two moments of informed traders’ posterior beliefs are given by

$$ \mathbb{E}[\theta | s_i, \hat{p}_1] = a_0 \cdot s_i + a_1 \cdot \hat{p}_1 $$

$$ \forall[\theta | s_i, \hat{p}_1] = [\lambda_\theta + \lambda_\epsilon + \lambda_n/\Delta^2]^{-1} $$

12The dependence of posterior beliefs on equilibrium prices does not literally require traders to simultaneously observe the equilibrium price while they submit demands, since price-contingent demand schedules are allowed.
where $a_0$, and $a_1$ are the Bayesian weights assigned to the private and public signals, which depend on their relative precision.

The second step is to compute the optimal demands that follow from the posterior beliefs characterized by (4). Informed traders’ terminal wealth is given by $W_{i,2} = t_i \cdot (\theta - P_1) + \theta$, which conditional on demand $t_i$ and the information set $\{s_i, \hat{p}_1\}$ is normally distributed. Maximizing the expectation of exponential utility is then equivalent to maximizing a mean-variance utility augmented by risk-aversion ($\gamma$), leading to demands

$$t_i = \frac{\mathbb{E}[\theta | s_i, \hat{p}_1] - P_1}{\Sigma} - 1$$

where $\Sigma = \gamma \cdot \mathbb{V}[\theta | s_i, \hat{p}_1]$ is the risk-aversion adjusted variance, common across traders. Demand schedules in (5) are proportional to expected returns, tempered by risk. The $-1$ term reflects a hedging motive from the initial endowment.

To solve the linear equilibrium the third step imposes market-clearing:

$$F \cdot \int t_i \, di = n$$

where the left hand side is the aggregate demands of informed traders, which equals the noisy asset supply. Solving for $P_1$ yields the coefficients in (2) through the method of undetermined coefficients. These are functions of the primitive parameters and aggregate funding liquidity; $F$. The following proposition summarize the main results.

**Proposition 1 (Equilibrium):** There exists a unique linear equilibrium\(^\text{13}\) price function $P_1$:

$$P_1 = \tilde{A}_1 + \tilde{A}_2 \cdot \theta + \tilde{A}_3 \cdot n$$

with coefficients

$$\tilde{A}_1 = -\Sigma; \quad \tilde{A}_2 = a_0 + a_1; \quad \tilde{A}_3 = -\Sigma \frac{a_0 + a_1}{a_0}$$

where $\Sigma$, $a_0$ and $a_1$ are given by (4) as a function of the noise amplifier $\Delta$:

$$\Delta (F) = \frac{\gamma}{\lambda} \cdot \frac{1}{F}$$

**Proof.** In appendix A. ■

Expression (7) gives the noise amplifier $\Delta (\cdot)$ as a function of funding liquidity $F$. Figure 1 plots the impact of funding on $\Delta (\cdot)$, and the conditional variance of fundamentals $\mathbb{V}[\theta | s, \hat{p}_1]$. As

\(^{13}\)I ignore whether there exist other, non-linear equilibria.
$F \to 1$, all informed traders impound their private information into the price. The noise amplifier reaches its lower bound $\Delta (1) = \gamma \cdot \lambda^{-1}$, an increasing function of the variance of idiosyncratic noise in private signals $\lambda^{-1}$ and risk aversion ($\gamma$).\(^{14}\)

Conversely, as $F \to 0$ the asset price loses informational value completely ($\Delta \to \infty$). Since information is dispersed, a higher value of $\Delta$ increases the conditional variance of fundamentals for all traders. Economic uncertainty, defined as the variance of the dividend conditional on information, is decreasing in funding liquidity: $\partial V[\theta | \cdot] / \partial F < 0$ (expression (4)).

### 3 A noisy REE model with dynamic risk

This section shows that the effects of funding constraints introduced in the simple model of section 2 are amplified when informed traders face funding constraints going forward: even if current financing is available, traders might fear tightening of constraints in the near future. The following quote captures the essence of the argument:

"Arbitrageurs can become most constrained .. when the mispricing they have bet against gets even worse... the fear of this scenario would make them more cautious when they put on their

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\(^{14}\)This case corresponds to the economy analyzed by Grossman and Stiglitz (1980) when the unit measure of traders choose to become informed. In the present model, the relevant margin is not information acquisition but constraints on the participation of informed traders.
initial trades, and hence less effective in bringing about market efficiency” (Shleifer and Vishny (1997), pg 37).

Mounting evidence suggests that this “fear of illiquidity” mechanism is important empirically, particularly when asset prices fall at the outset of contractions. Several studies document predictability of mutual fund net inflows from lagged fund performance\textsuperscript{15}. In the time series, Warther (1995) documents a positive correlation between mutual fund performance and aggregate inflows into the industry. Reliance on debt raises similar concerns. Brunnermeier and Pedersen (2009) argue that margin requirements tighten when counterparty-risk increases during crises. Moreover, rolling-over short-term debt becomes increasingly difficult as liquidity dries up\textsuperscript{16} and can escalate to virtual market freezes\textsuperscript{17} as illustrated by the subprime episode. In short, all sources of financing become increasingly fragile as crises unfold.

In practice, funding constraints arise both from the behavior of financial intermediaries’ creditors (debt) and its clients (capital). I focus on the latter. Modeling risky debt is burdensome because it requires some type of limited liability constraint (Bernanke and Gertler (1989); Holmstrom and Tirole (1997)), a property at odds with CARA preferences, which allow negative consumption. However, CARA preferences permit tractable information aggregation in a risk-averse setting. Focusing exclusively on capital constraints enables me to use a more tractable and parsimonious model and analyze the implications of price informativeness on risk premia.\textsuperscript{18}

The model presented below studies dynamic risk\textsuperscript{19} in a setting with two trading stages. At stage 1, a risk-averse informed trader raises funding from a continuum of clients, trading on their behalf for a fraction of the profits. Clients provide funding for buying or shorting shares, but can withdraw funds at stage 2 depending on idiosyncratic liquidity shocks. This forces informed traders to liquidate the corresponding fraction of positions at a price that can differ from fundamentals due to noise trading. The unwinding of informed traders’ positions and noise trading at stage 2 is absorbed by uninformed traders, modeled as rational investors who learn from prices. A key observation is that although the variance of noise trading is an exogenous parameter, its impact on stage 2 prices is endogenous to informed traders’ decisions: the less they react to private information, the less revealing equilibrium prices are. Higher uncertainty of uninformed traders heightens the price impact of asset liquidations and random supply shocks, making informed traders even less willing to put on their initial trades. Endogeneity of noise trading risk adds an interesting

\textsuperscript{15}Chevalier and Ellison (1997); Wermers (1999); Coval and Stafford (2007).
\textsuperscript{16}See Acharya, Gale and Yorulamzer (2009).
\textsuperscript{17}See Gorton and Metrick (2009).
\textsuperscript{18}Xiong (2001) and Kyle and Xiong (2001) derive limited arbitrage from decreased risk-tolerance of CRRA traders following wealth losses. Power utility is much less tractable for modeling information aggregation, however. See Mertens (2009) for a discussion on non-linear methods to overcome this problem.
\textsuperscript{19}He and Wang (1995) analyze volume patterns and dynamic trading strategies in a noisy REE setting in which traders face the risk of fewer trading opportunities as the finite terminal date approaches.
amplification force and can result in multiple trading equilibria.

3.1 Setup

There are three stages: 1, 2 and 3. The single risky asset in the economy pays a liquidation value of $D$ in stage 3, which follows

$$D = D_{-1} + \theta + \mu$$

where $D_{-1} = \bar{D} + \theta_{-1}$ is common knowledge at stage 1, given by a mean $\bar{D}$ plus the realization of the lagged dividend innovation $\theta_{-1}$. Both $\theta_{-1}$ and $\theta$ are drawn independently from a normal distribution with zero mean and variance $\lambda_{\theta}^{-1}$. Henceforth, I will refer to $\theta_{-1}$ as lagged economic conditions. The innovation $\theta + \mu$ becomes common knowledge at stage 3, but part of the uncertainty about $\theta$ will reduced by trading in prior stages. The term $\mu$ is a white noise, normally distributed variable\textsuperscript{20} with variance $\lambda_{\mu}^{-1}$.

Agents

There are two kinds of agents: traders and clients. Traders can be of three types; informed, uninformed, or noise traders. There are a continuum of informed traders indexed by $i \in [0, 1]$ born at stage 1 with CARA preferences ($\gamma$) over the consumption of terminal wealth $W_{i,3}$. At stage 1, each observes a private signal about $\theta$ given by (1). Uninformed traders are born in stage 2 in unit mass, and have CARA($\gamma_u$) preferences over the consumption of terminal wealth $W_{u,3}$. They have no private information about $\theta$ but make rational inferences from the asset price. For simplicity, I assume they have enough wealth to finance trading positions at stage 2. Noise traders are born in stages 1 and 2 in masses $n_1$ and $n_2$, drawn independently from a zero-mean normal distribution with variance $\lambda_n^{-1}$.

Financial intermediation arises from the need to finance purchases or sales of the risky asset. I treat long and short positions symmetrically by assuming both require funding the entire price\textsuperscript{21}. Informed traders have no endowments and cannot borrow at the riskless rate, so they must raise funding from clients to take asset positions at stage 1. I assume simple linear contracts: to raise an amount $X$, an informed trader receives 1 unit of the consumption good from a continuum (mass $X$) of clients, who can choose whether they want to settle their investment at stage 2 through early withdrawals, or maintain it until stage 3. I assume clients are in relatively large mass so that informed traders can always raise funding from a continuum of them. The informed trader retains a fraction $1 - c$ of trading profits.

\textsuperscript{20}This additional payoff component will play a role in the equilibrium selection of the model, as described below.

\textsuperscript{21}As discussed by Brunnermeier and Pedersen (2009), short positions also require capital, since borrowing shares require withholding the proceeds of the sale plus the margin requirement in deposit accounts. This provides a safeguard for the lender against counterparty risk.
Clients have risk-neutral preferences\textsuperscript{22} and can only participate in the asset market through informed traders. At stage 2, clients receive an endowment which is an increasing function of lagged economic conditions $g(\theta_{-1})$ plus a client-specific liquidity shock $\ell_j \sim \mathcal{N}(L, \lambda^{-1} \ell)$. Liquidity shocks are unknown to all agents in stage 1. In the spirit of Diamond and Dybvig (1983) and the ensuing the bank-runs literature, clients with low values of the liquidity shock will withdraw funding from informed traders. In particular, client $j$ redeems at stage 2 whenever $g(\theta_{-1}) + \ell_j < 0$. Considering the simple function $g(\theta_{-1}) = \theta_{-1}$, the fraction of positions that informed traders will keep until stage 3, which I label funding liquidity $F$, is given by

$$F(\theta_{-1}) = 1 - \Pr (\theta_{-1} + \ell_j < 0) = \Phi \left( \sqrt{\lambda \ell} \cdot (L + \theta_{-1}) \right) \quad (8)$$

while the complement $1 - F$ must be liquidated at stage 2 to meet redemptions. For tractability, I assume withdrawal decisions are independent of the price at stage 2, and that informed traders do not voluntarily change net positions at this stage. Without these assumptions, normality of conditional wealth breaks down, and the CARA framework is no longer tractable.\textsuperscript{23}

**Asset market** I now describe the timing of trading in the asset market, which is summarized in Figure 2. Claims on $D$ are exchanged in the financial market that opens at stages 1 and 2. At stage 1, informed traders submit price-contingent asset demand schedules. Although individual clients’ liquidity shocks are unknown at stage 1, with the law of large numbers informed traders can perfectly forecast the fraction of early withdrawals $(1 - F)$ from the knowledge of $\theta_{-1}$. Demand schedules depend on private signals $s_i$ and anticipated funding liquidity $F$: $t_i = t(s_i, P_1; F)$.

I assume the average supply of shares in stage 1 is unity, so the economy has aggregate risk. In addition, noise traders supply the random amount of $n_1$ shares. A market auctioneer then selects a price $P_1$ for the shares, at which all price-contingent demands can be executed.

At stage 2 informed traders are forced to unwind a fraction $1 - F$ of their positions to meet withdrawals. Uninformed traders also bid for the shares at stage 2 by submitting demand schedules to the market auctioneer. Uninformed traders condition both on the price at stage 2; $P_2$, as well as on $P_1$ and funding liquidity: $t_u = t(P_1, P_2; F)$. The second draw of noise traders supply a total of $n_2$ shares. The market auctioneer then selects the price $P_2$ at which all uninformed demands can be executed given the realization of the noisy supply and informed traders’ early liquidations. At stage 3, $\theta + \mu$ is revealed and agents consume net positions.

\textsuperscript{22}Risk neutrality ensures clients will always be willing to engage in financial contracts, since informed traders make positive profits in expectations.

\textsuperscript{23}Voluntary liquidations at stage 2 would skew stage 2 prices, conditional on information at stage 1, breaking the normality of traders wealth. These assumptions do not compromise the main qualitative results however, since what matters is that funding constraints place some limits in the positions that traders can take in the future, affecting their incentives to trade in the present.
3.2 Equilibrium

A competitive sequential equilibrium is defined by 1) a sequence of share price functions \( P_1 (\theta, n_1; F) \), \( P_2 (\theta, n_1, n_2; F) \); 2) demands by informed \( t_i = t (s_i, P_1; F) \) and uninformed traders \( t_u = t (P_1, P_2; F) \); and 3) a set of prior beliefs \( H (D | D_{-1}) \) for all agents, posterior beliefs \( H (D | D_{-1}; s_i, P_1, F) \)
and \( H (P_2 | D_{-1}; s_i, P_1, F) \) for informed traders, and \( H (D | D_{-1}; P_1, P_2, F) \) for uninformed traders such that, \( \forall \ i \in [0, 1] \), and \( u \): (i) Asset demands are optimal given funding liquidity \( F \), and posterior beliefs \( H (D | \cdot) \) and \( H (P_2 | \cdot) \) (ii) The asset price clears the market at each stage; and (iii) Posterior beliefs are updated using Bayes law.

The main object of the equilibrium are the price functions \( \{ P_1 (\cdot), P_2 (\cdot) \} \), which depend on the realization of the shocks up to each stage, and funding liquidity. Under condition (i), informed traders maximize expected utility given expected withdrawals and posterior beliefs about the dividend and price \( P_2 \), since a fraction \( 1 - F \) of positions will pay according to the latter. Condition (ii) imposes market clearing at each trading round, for all realization of the noisy asset supplies. Condition (iii) imposes Bayesian updating of the conditional distributions of \( D \) and \( P_2 \) on all available information, including prices. I now briefly sketch the main steps to solve the equilibrium, with details relegated to Appendix B.

**Step 1: Price conjectures and beliefs**

**Conjecture 1** (*\( P_1 \) is affine*): There exists an equilibrium at stage 1, where \( P_1 \) is a linear combination of the dividend’s prior expectation \( (D_{-1}) \), the dividend innovation \( (\theta) \), and the noise trading shock \( (n_1) \);

\[
P_1 = A_0 + A_1 \cdot D_{-1} + A_2 \cdot \theta + A_3 \cdot n_1
\]  

(9)

Under the linear equilibrium of Conjecture 1, \( P_1 \) is informationally equivalent to a noisy public signal about \( \theta \); \( \tilde{\theta}_1 \equiv \theta - \Delta \cdot n_1 \), where \( \Delta = -A_3/A_2 \). The noise in the signal depends on the
variance of noise trading and the noise amplifier $\Delta$. Since informed traders must liquidate some positions at stage 2, they also form beliefs about $P_2$. I raise a similar conjecture about the price function $P_2(\cdot)$:

**Conjecture 2 (P_2 is affine):** There exists an equilibrium at stage 2, where $P_2$ is a linear combination of the dividend’s prior expectation ($D_{-1}$), the shocks $\{\theta, n_1, n_2\}$, the share price $P_1$, and its informational content $\tilde{p}_1$:

$$P_2 = B_0 + B_1 \cdot D_{-1} + B_2 \cdot \theta + B_3 \cdot n_1 + B_4 \cdot n_2 + B_5 \cdot \tilde{p}_1 + B_6 \cdot P_1$$  \hspace{1cm} (10)

Informed traders’ posterior beliefs about the dividend $D$ and the price $P_2$ depend on private information and the public signal $\tilde{p}_1$. The projection theorem gives the first two moments of informed traders’ posterior beliefs regarding $D$. While the conditional moments of $\theta$ are given by expression (4), the mean and variance of $\mu$ are zero and $\lambda_\mu^{-1}$, respectively. Beliefs about price $P_2$ can be similarly computed from (10) using expression (4).

Uninformed traders also use share prices to form beliefs. As I discuss in appendix B, the equilibrium considered below implies an informational role for $P_2$ from which uninformed traders can make further updates about the realization of $\theta$. For simplicity, I assume in the text that uninformed traders only make inferences from $P_1$ through the endogenous signal $\tilde{p}_1$; i.e., uninformed traders process information in prices with a lag.\(^{24}\) Given information $\Omega_u : \{D_{-1}; P_1\} = \{D_{-1}; \tilde{p}_1\}$, the first two moments of uninformed traders’ beliefs are

$$\begin{align*}
\mathbb{E}[D \mid \Omega_u] &= D_{-1} + b_1 \cdot \tilde{p}_1 \\
\mathbb{V}[D \mid \Omega_u] &= \mathbb{V}[\theta \mid \Omega_u] + \mathbb{V}[\mu \mid \Omega_u] = \left[\lambda_\theta + \lambda_n / \Delta^2\right]^{-1} + \lambda_\mu^{-1};
\end{align*} \hspace{1cm} (11)$$

**Step 2: Optimal demands** Since informed traders’ face withdrawals at stage 2, terminal wealth is given by

$$W_{i,3} = (1 - c) t_i [F \cdot (D - P_1) + (1 - F) \cdot (P_2 - P_1)]$$

From (10), wealth follows a normal distribution, conditional on posterior information at stage 1. Maximizing expected utility leads to demands

$$t_i = \frac{F \cdot \mathbb{E}[D \mid \Omega_i] + (1 - F) \cdot \mathbb{E}[P_2 \mid \Omega_i] - P_1}{\Sigma} \hspace{1cm} (12)$$

where $\Sigma = (1 - c) \gamma [F^2 \mathbb{V}[D \mid \Omega_i] + (1 - F)^2 \mathbb{V}[P_2 \mid \Omega_i] + 2F(1 - F) \text{Cov}[D, P_2 \mid \Omega_i]]$ is the risk aversion-adjusted variance. Asset demands of informed traders depend on expectations about the

\(^{24}\)An additional public signal in $P_2$ does not change the qualitative results of the model, but adds significant complexity to its solution.
dividend and price $P_2$. Importantly, response to private signals is tempered by $P_2$ volatility.

Uninformed traders participate at stage 2, choosing demands to maximize the expected utility over the terminal wealth $W_u = t_u \cdot (D - P_2)$:

$$t_u = \frac{\mathbb{E}[D|\Omega_u] - P_2}{\Sigma_u}$$

with risk-aversion adjusted variance $\Sigma_u = \gamma_u \cdot \mathbb{V}[D|\Omega_u]$. Uninformed traders’ demands are proportional to the expected profit per share $(\mathbb{E}[D|\Omega_u] - P_2)$, tempered by the risk-aversion adjusted variance; $\Sigma_u$.

**Step 3: Market clearing** I now solve the coefficients in the price conjectures. I impose the market-clearing condition at stage 2:

$$t_u - (1 - F) \cdot \int t_i \, di = n_2$$

where the left hand side is given by uninformed demands plus the unwinding of informed traders’ positions from stage 1, which must equal the second draw of noise trading $n_2$. Solving for $P_2$ gives the coefficients in Conjecture 2.

The market clearing condition at stage 1 corresponds to

$$\int t_i \, di = 1 + n_1$$

where the left hand side contains aggregate demands by informed traders, which must equal the unit supply of shares plus noise trading. Solving for $P_1$ yields the coefficients in Conjecture 1, as a function of the coefficients in Conjecture 2. I summarize the main properties of the equilibrium in the following proposition, referring proofs to Appendix B.

**Proposition 2 (Existence and uniqueness):** (i) All linear price equilibria satisfy the system of equations:

\begin{align}
\Delta &= \frac{\Sigma}{a_0 (F + (1 - F) B_2)^2}, \\
\Sigma &= (1 - c) \gamma \left\{ \mathbb{V} \left[ \theta | s, \tilde{p}_1 \right] (F + (1 - F) B_2)^2 + (1 - F)^2 \Sigma_u \lambda^{-1} + F^2 \lambda^{-1} \right\}, \\
\Sigma_u &= \gamma_u \left\{ \mathbb{V} \left[ \theta | \tilde{p}_1 \right] + \lambda^{-1} \right\}, \\
B_2 &= \frac{- \Sigma_u (1 - F) F \cdot a_0}{\Sigma + \Sigma_u (1 - F)^2 \cdot a_0}
\end{align}

The profit splitting rule acts as a risk-aversion moderator: the lower is the fraction of profits that corresponds to informed traders $(1 - c)$, the more aggressive they trade.

I ignore whether there exist other, non-linear equilibria.

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25 The profit splitting rule acts as a risk-aversion moderator: the lower is the fraction of profits that corresponds to informed traders $(1 - c)$, the more aggressive they trade.

26 I ignore whether there exist other, non-linear equilibria.
where $F = F(\theta_{-1})$ is given by equation (8), $\mathbb{V}[\theta | s, \tilde{p}_1]$ and $\mathbb{V}[\theta | \tilde{p}_1]$ are functions of $\Delta$ given by (4) and (11), and $a_0 = \lambda_v \mathbb{V}[\theta | s, \tilde{p}_1]$. (ii) A linear equilibrium always exists; and (iii) There exists $\delta_0 > 0$, s.t. whenever the sufficient condition $\lambda_{\mu} < \delta_0$ holds, the equilibrium is unique. If $\lambda_{\mu} > \delta_0$, additional equilibria might exist.

Proposition 2 validates the conjectures made in (9) and (10): beliefs based on the proposed price functions lead to asset demands that sustain such beliefs in equilibrium. Equilibrium multiplicity is an interesting property of the two-stage trading environment that arises from higher-order beliefs of informed traders: in order to predict price $P_2$, they form expectations about uninformed traders’ demands, and thus uninformed beliefs.

In one of these equilibria, informed traders respond aggressively to private signals, and the price $P_1$ is nearly revealing. Facing low risk, uninformed traders absorb the supply of the asset with little effect on price $P_2$. Low $P_2$ volatility, in turn, sustains the optimality of informed traders’ aggressive response to information.

Conversely, if informed traders react mildly to private information, a noisy $P_1$ reduces uninformed traders’ willingness to absorb supply, increasing $P_2$ volatility. High payoff uncertainty of early liquidations sustains the mild response of informed traders to information at stage 1. This leads to an equilibrium with higher noise in prices –the noisy equilibrium.

The inclusion of $\mu$ in the dividend equation adds uncertainty which cannot be mitigated through information aggregation in the financial market. Consequently, the near-revealing equilibrium vanishes for a high enough variance of $\mu$: no matter how precise the price signal is, uninformed traders conditional variance of $D$ is above $\lambda_{\mu}^{-1}$. This bounds the volatility of $P_2$ from below so that aggressive trading of informed traders in the near-revealing equilibrium is not optimal. In what follows, I restrict attention to the noisy equilibrium, which exists for all values of $\lambda_{\mu}$.

I now state the relation between funding liquidity and information aggregation in the asset price (proven in Appendix B):

**Proposition 3 (Liquidity and noise):** There exists $\delta_1 > 0$, s.t. if the sufficient condition $\lambda_{\mu} < \delta_1$ holds, there exists a threshold $\overline{\theta} < \infty$ s. t. the noise amplifier $\Delta(\cdot)$ is strictly decreasing in $\theta_{-1}$, for all values of lagged economic conditions $\theta_{-1} < \overline{\theta}$. Moreover, $\partial \overline{\theta} / \partial \gamma_n > 0$ and $\partial \overline{\theta} / \partial \lambda_{-1}^{-1} > 0$.

Figure 3 plots the impact of the lagged dividend on funding liquidity $F$ and the noise amplifier $\Delta(\theta_{-1})$ (Table 1 specifies the benchmark parameters used in all the figures below). As economic

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27 Since multiple linear equilibria might exist for a subset of parameter values, it is necessary to impose the additional requirement that traders share common beliefs about which equilibrium is selected. Equilibrium coordination is essential for asset prices to provide unique signals about $\theta$ to different traders. If agents coordinate on different equilibria, nonlinear information aggregation would result, rendering the CARA-gaussian setup intractable.

28 It is worthwhile noting that the impact of funding liquidity $F$ on the noise amplifier $\Delta$ is qualitatively similar in both equilibria.
Table 1

<table>
<thead>
<tr>
<th>Baseline parameters</th>
<th></th>
<th>Baseline parameters</th>
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</thead>
<tbody>
<tr>
<td>Unconditional variance of $\theta$</td>
<td>$\lambda_{\theta}^{-1} = 2$</td>
<td>Average dividend</td>
<td>$\bar{D} = 5$</td>
</tr>
<tr>
<td>Unconditional variance of $\mu$</td>
<td>$\lambda_{\mu}^{-1} = 0.4$</td>
<td>Liquidity shock</td>
<td>$L = 6; \lambda_{l}^{-1} = 50$</td>
</tr>
<tr>
<td>Variance of private signals</td>
<td>$\lambda_{e}^{-1} = 2$</td>
<td>Risk-aversion</td>
<td>$\gamma = 1; \gamma u = 2$</td>
</tr>
<tr>
<td>Noise trading shock variance</td>
<td>$\lambda_{n}^{-1} = 2.5$</td>
<td>Profit sharing rule</td>
<td>$c = 0.9$</td>
</tr>
</tbody>
</table>

Figure 3: Liquidity and Noise

Left panel - Aggregate funding liquidity ($F$): fraction of positions held by informed traders until stage 3. Right panel - Noise amplifier ($\Delta$): the ratio of $P_1$ reaction to noise and fundamentals ($A3/A2$ in Conjecture 1).
conditions deteriorate and future withdrawals become more of a concern, informed traders hold back on their initial trades limiting the aggregation of private information in the asset price.

3.3 Illiquidity and the amplification of uncertainty

To understand the mechanisms involved in the amplification of uncertainty, consider an informed trader who expects a high dividend at stage 3 and must choose her initial trade in stage 1. If she expects a relatively large fraction of withdrawals $1 - F$ in stage 2, she will respond moderately to private information because liquidations of positions is risky in the presence of noise trading, but also because she will be dumping her shares together with all other informed traders. This reversal of positions causes an opposite price pressure in $P_2$ which reduces expected profits. I refer to the first consideration as the noise risk effect, and to the second as the trade reversal effect.

Figure 4 shows the path of prices after a one-standard deviation positive innovation in $\theta$. The left panel corresponds to a case where the lagged dividend was relatively low (-1 st. dev.), while the right panel considers a relatively high value of $\theta_{-1}$ (+1 st. dev.). The solid line in both panels is the simulated trajectory of prices when both realizations of the noise trading shock are zero, as an illustration of the trade reversal effect\(^{29}\). The dashed lines plot the additional price impact of a positive innovation in noise trading $n_2$. The figure makes clear that both effects become more pervasive in illiquid markets.

Equation (16a) can be decomposed to illustrate the impact of each of these effects in the noise amplifier $\Delta$. The noise risk effect is captured by the ratio $\Sigma/a_0$ in (16a) (left panel of Figure 5). The term $a_0 = \lambda \cdot \mathbb{V}[\theta | s, \tilde{p}_1]$ is the response\(^{30}\) of informed traders’ expectations about $\theta$ to innovations in private signals, so that the ratio $a_0/\Sigma$ reflects how strongly demands of informed traders react to private signals once the tempering effect of risk ($\Sigma$) is accounted for. In the simple setting of section 2, this ratio is a constant given by $\lambda/\gamma$, since an increase in informed traders’ risk (say, from a noisier asset price) was exactly canceled out by a higher reliance on private signals. That is, the amount of private information impounded into the price per trader is constant in the single trading stage model.

When informed traders face additional noise risk in price $P_2$ this cancelation of effects breaks down. Under the restrictions of Proposition 3, $\Sigma/a_0$ is decreasing in funding liquidity. When funding tightens, higher exposure to noise at stage 2 increases $\Sigma$ by a larger extent than the conditional variance $\mathbb{V}[\theta | s, \tilde{p}_1]$, raising $\Sigma/a_0$ and the noise amplifier $\Delta$.

The trade reversal effect is captured by the second term in the denominator of (16a): $F + (1 - F) B_2$ (Right panel of Figure 5). When funding is tight, informed traders expect a low

\(^{29}\)The trade reversal effect relates to the mean of $P_2$ conditional on the expectations of $\theta$. Conditioning only on prior information $D_{-1}$, the expected trade reversal effect is zero.

\(^{30}\)See Appendix A.
Figure 4: Illiquidity and Dynamic Risk

Right panel – Price trajectory for low lagged fundamentals \((\theta_{-1} = -1 \text{ st. dev.})\). Solid line plots the mean path of prices \((n_1 = n_2 = 0)\). The dashed line plots the reaction in \(P_2\) from a 1 st. dev. innovation in \(n_2\). Left panel – Price trajectory for high lagged fundamentals \((\theta_{-1} = 1 \text{ st. dev.})\). Solid line plots the mean path of prices \((n_1 = n_2 = 0)\). The dashed line plots the reaction in \(P_2\) from a 1 st. dev. innovation in \(n_2\). Both panels assume a positive 1 st. dev. innovation in \(\theta\).

Figure 5: Amplification Effects–Noise Risk and Trade Reversals

Left panel – Noise risk effect: the ratio between the risk aversion-adjusted variance of informed traders \((R)\) and the weight of private information in the expectations of \(\theta\). Right panel – Trade reversals effect: underreaction to private information due to anticipated discount of \(P_2\) w.r.t. fundamentals \((F + (1-F)B_2 < F + (1-F))\).
fraction of positions to pay according to the actual dividend \( D \), and therefore impound less private information into the price (lower term \( F \)). However, prices at stage 2 also depend on \( D \) since uninformed traders’ demands will be high when prices at stage 1 suggest a strong dividend. This is captured by the \((1 - F) B_2\) term, where \( B_2 \) is the effect of the fundamental \( \theta \) on the price \( P_2 \) (equation (10)) once other partial effects are factored out. But from expression (16d), note that \( B_2 \) is actually negative: \(-1 < B_2 < 0\), implying that a decrease in \( F \) also lowers the term \((1 - F) B_2\).

Intuitively, for every share informed traders decide to buy at stage 1, they anticipate selling a fraction \( 1 - F \) at stage 2. Uninformed traders will require a price drop to absorb trade reversals, with a discount that is increasing in the total amount of shares downloaded into the market. This provides an additional incentive for informed traders to underreact to private information, which is what the negative value of \( B_2 \) represents. This result resembles a fire sales argument: to the extent traders anticipate possible asset liquidations in the future, they will respond less aggressively to current trading opportunities.

Both these effects can also be interpreted as information externalities that arise among traders. When informed traders fear a large liquidation at stage 2, they respond mildly to private information, increasing the conditional variance of fundamentals of all traders. Interestingly, the social inefficiency of private decisions hits back on informed traders. In particular, heightened uncertainty of uninformed traders make them reluctant to absorb supply of the asset at stage 2, increasing the volatility of \( P_2 \) in reaction to noise trading (the noise risk effect), but also the premium for holding informed traders’ liquidations, thus lowering its conditional mean (the trade reversal effect). Since both effects multiply each other in the denominator of \( \Delta (\theta - 1) \) in expression (16a), the spike in the noise amplifier can be quite marked, as is apparent from Figure 3.

Figure 6 plots the effect of lagged economic conditions on the dividend’s conditional variance, for both informed and uninformed traders. The residual variance of fundamentals given available information is decreasing on \( \theta - 1 \) through its effect on funding liquidity, an effect particularly pronounced for uninformed traders that lack private information.

### 4 Asset pricing implications

Two salient features of asset prices during economic slumps are the increase in expected excess returns of risky assets—the risk premia—and spikes in the volatility of stock markets. I argue in this section that countercyclical price informativeness is consistent with both observations. Moreover, I stress the importance of modeling endogenous informativeness of prices in a context where information is heterogeneous across agents that possess private knowledge about fundamentals, since under certain conditions the asset pricing effects can be substantially larger than predicted by common information benchmarks.
4.1 Risk premia

Excess returns of risky assets are predictable at low frequencies. Fama and French (1989) document that variables related to business cycle conditions, such as default and term spreads, track excess returns of corporate bonds and stocks in a similar way as the dividend yield. This suggest that return predictability reflects cyclical variation in the price of risk, an interpretation supported by the literature on volatility tests that cannot otherwise explain the “excess volatility” of price/dividend ratios.31

Countercyclical price informativeness suggests one possible explanation for time variation in risk premia. I view this mechanism as complementary to the existing theories that focus on cyclical variation in the attitude towards risk (Campbell and Cochrane (1999)), or an exogenous conditional heteroskedastic dividend process (Barsky and DeLong (1993); Bansal and Yaron (2004)). The informational channel suggests that financial markets are poor aggregators of information about the value of future dividends when traders are funding constrained. As the increase in risk goes up, the corresponding reward for bearing it moves in the same direction. Importantly, this is true in the model even when the risk aversion of agents as well as the unconditional variance of dividends are held constant.

I compute the risk premium as the expected return on holding a position on the risky asset

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between stages 1 and 3, conditioning on prior information $D_{-1}$. Conditional on the realization of the shocks $\{\theta, n_1\}$, the holding period return is given by

$$
\text{Ret}_{1,3} \equiv \frac{D - P_1}{\bar{D}} = \frac{\theta - A_2 \cdot \bar{p}_1 - A_0}{\bar{D}}
$$

The risk premium is just the average of $\text{Ret}_{1,3}$ over the joint normal distribution of $\{\theta, n_1\}$:

$$
RP \equiv \mathbb{E} [\text{Ret}_{1,3} \mid D_{-1}] = \frac{-A_0}{\bar{D}} = \frac{[\Sigma + \Sigma_u (1 - F)^2]}{\bar{D}}
$$

where $A_0$ is the intercept of the price function in Conjecture 1.

Note that the risk aversion-adjusted variance of uninformed traders; $\Sigma_u$, also appears in expression (17), although I define the risk premium as the discount required by informed traders at stage 1. The intuition is that informed traders will require a high premium when they expect price $P_2$ to be more volatile, which results from the unwillingness of uninformed traders to absorb noise when $\Sigma_u$ is high. The next proposition states the conditions under which the risk premium in this economy is strictly decreasing in lagged economic conditions $\theta_{-1}$. I provide all proofs of this section in Appendix C.

**Proposition 4 (Risk-premium):** Under the parameter restrictions of propositions 2 and 3, the risk premium is strictly decreasing in lagged economic conditions for all $\theta_{-1} < \bar{\theta}$.

The left panel of Figure 7 plots the negative relation between the risk premia and $\theta_{-1}$. As $\theta_{-1}$ falls and funding tightens, the rise in the noise amplifier $\Delta$ increases the conditional variance of fundamentals. Correspondingly, traders will require a higher return for holding the asset. Note that tighter funding not only increases the risk premium by directly raising the share of asset liquidated early on but it also increases the conditional variance of the remaining positions that pay according to $D$. This is a direct implication of a dispersed information environment where all traders learn from prices.

The right panel of Figure 6 plots the Sharpe ratio: the quotient between the risk-premia and the conditional standard deviation of the dividend return,

$$
SR \equiv \frac{RP}{\sqrt{\mathbb{V} [D - P_1 \mid \Omega_i] / D^2}} = \frac{-A_0}{\sqrt{\mathbb{V} [D \mid \Omega_i]}}
$$

which is decreasing in $\theta_{-1}$ as well, as made precise by the next proposition:

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\[32\] Since prices can be negative with positive probability, I compute expected returns using the unconditional mean $\bar{D}$ on the denominator, which actually understates the cyclical variation in risk premia that would result from using $P_1$ instead.
Proposition 5 (Sharpe ratio): Under the parameter restrictions of propositions 2 and 3, the Sharpe ratio is strictly decreasing in lagged economic conditions for all $\theta_{-1} < \bar{\theta}'$.

Propositions 4 and 5 follows directly from CARA preferences, in the presence of aggregate risk (average asset supply =1). Market clearing implies that in expected terms (when noise trading $n_1$ is zero), the expected profit per share is proportional to the risk aversion-adjusted variance of informed traders $\Sigma$. Dividing by the standard deviation of $D$ conditional on information at stage 1, gives a Sharpe ratio that is proportional to the conditional standard deviation of the dividend, and therefore decreasing in $\theta_{-1}$.

The benchmark parameters in Table 1 imply a mean expected excess return on equity of 5.2%. Of course, this figure depends on an arbitrary choice of parameters, mainly the value of the average dividend $\bar{D}$. What is less arbitrary from the figure is the considerable variation of the equity premium, which oscillates by a factor of 8 within the 3 standard deviations range of lagged economic conditions. Importantly, the Sharpe ratio follows a similar countercyclical behavior, so that the slope of the mean-variance frontier is higher in contractions as documented in the data.

Of course, these results should only be interpreted qualitatively, since a single-period CARA framework is hardly suited for a quantitative asset pricing discussion. Within these limitations, it is still worth noting that the model has the potential to generate an interesting time-variation in the forecastability of returns and the price of risk.

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4.2 Price volatility

I now discuss the impact of funding liquidity on the volatility of price $P_1$ conditional on prior information; $\mathbb{V}[P_1 | D_{-1}]$. I focus on $P_1$ because $P_2$ behaves much like $P_1$ except for scale effects. The left panel in Figure 8 shows the relation between funding liquidity, and price volatility. As funding liquidity tightens, volatility spikes considerably. The key driver of the increase in the conditional variance of both informed and uninformed traders. Since in equilibrium the supply of shares unloaded by noise traders must always be absorbed, higher risk premia translates into a larger price response to noise $n_1$.

The right panel of figure 8 decomposes price variability into its two sources. In the absence of noise, only dividend innovations affect the price volatility, which would then match the unconditional variance of $\theta$; $\lambda_{\theta}^{-1}$. The interaction between noise and risk-aversion, however, prevents full revelation from the observation of the prices, introducing non-fundamental volatility coming from supply innovations. For relatively low values of $\theta_{-1}$, this second source of volatility becomes dominant, and price volatility drops as economic conditions improve. As the price becomes more informative, traders also weight less the prior belief $\mathbb{E}[\theta] = 0$, which makes the price sensitive to innovations in fundamentals. The latter effect can dominate for large enough $\theta_{-1}$, depending on parameter choices. 34

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34 This depends mainly on the variance of noise trading $\lambda_{n}^{-1}$ and risk aversion ($\gamma$ and $\gamma_U$). An increase in either will expand the region in which volatility is strictly decreasing.
As a digression, statements about price volatility must be weighted by the limitations inherent in a static environment. In a fully dynamic context, the mapping between endogenous uncertainty and volatility is likely to be more involved. Since prices must eventually reflect dividend innovations at some frequency in the data, comparative statics about price informativeness is likely to affect the timing of price movements in anticipation of dividends, but not price volatility defined over low enough frequencies.  

The analysis above is probably best suited to make predictions about relatively high frequency stock market fluctuations, closely tied to the presence of noise. Interestingly, this suggests that spikes in volatility during recessions should coincide with increased forecastability at high frequencies due to return reversals, a testable prediction I expand on below.

Wang (1993) studies an intertemporal equilibrium model of asymmetric information where price fluctuations are driven by dividend innovations and noise. He finds that increasing information asymmetry (or the uncertainty about fundamentals, averaged across agents) can indeed generate spikes in price volatility when noise trading is important. His conclusions are confirmed in the static analysis provided here, as volatility is indeed decreasing in price informativeness.

4.3 Amplification effects of dispersed information

Although asymmetric information models with financial constraints have gained central importance in the asset pricing literature over the last decade, few studies have analyzed the informational properties of equilibrium. Two exceptions are Barlevy and Veronesi (2003) and Yuan (2005), who explain stock market crashes as resulting from the interactions between uninformed traders who learn from prices, and funding constraints of informed traders. In these papers however, information among those who posses it is common, so that prices only convey information about fundamentals from more to less informed traders in a strictly hierarchical information structure. Importantly, prices do not aggregate dispersed information about fundamentals. In the model presented above, information aggregation implies the asset price provides a new signal of dividend value for all traders.

I highlight the magnification of the asset pricing effects expanded above by considering an economy identical to that in section 3, except informed traders observe the same private signal s about the dividend innovation θ:

\[ s_i = s_j = s = \theta + \epsilon, \quad \forall i \in [0, 1]; \quad \text{with} \quad \epsilon \sim N\left(0, \lambda^{-1}\right) \]

I solve for the equilibrium of this economy in Appendix C, limiting the discussion here to the main

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35 I thank Jeremy Stein pointing this out.
36 Campbell and Kyle (1993) also provide an intertemporal model of asymmetric information. Although uncertainty about asset returns and price volatility are countercyclical, they consider an exogenous information structure.
contrasts with the dispersed information benchmark.

With common private signals, informed traders learn nothing from the asset price. As economic conditions worsen and funding tightens, informed traders are subject to higher risk from early liquidations, but not from the portion of their portfolios that pay according to the dividend. As the left panel in figure 9 shows, this translates into a milder reaction of the risk premium with respect to economic conditions. Since price volatility is intimately tied to required returns, the results also extend to the variance of $P_1$.

Naturally, this contrast depends on the assumed parameters. If the dispersion of private signals ($\lambda^{-1}_e$) is very low, or the variance of noise trading shocks ($\lambda^{-1}_n$) is very large, both cases will exhibit roughly similar asset pricing sensitivity to changes in economic conditions. Intuitively, when private information is very precise, informed traders have little to learn from prices. Moreover, when noise trading shocks are too volatile, the market mechanism is unable to provide valuable information about fundamentals to begin with.

Taking a stand on which starting point is empirically more relevant is out of the scope of this paper, since the most important parameters do not seem to have clear empirical counterparts to feed a calibration exercise. I therefore merely point out that, if the dispersed nature of information is of central importance in financial markets, the effects of funding illiquidity will tend to be magnified with respect to the case where knowledge of economic conditions is constant.
5 Real investment implications

Countercyclical uncertainty not only matters for understanding the behavior of asset prices, but it is likely to play an important role in real allocations decisions through its impact on aggregate investment. Several authors have pointed out that higher risk premia directly reduces investment by raising firms’ cost of capital. \(^{37}\) In this section I explore a different channel by considering the investment problem of a firm that learns information about its productivity –more broadly interpreted as future demand conditions for its products, or the likely success of an investment project– from the price of its shares in the stock market. This learning channel is usually referred to as the feedback\(^ {38}\) literature: firms learn information about the likely success of projects by observing their stock prices (or an industrial average, or even aggregate indexes) to the extent that prices aggregate information which can be of value to the firm, but is originally dispersed across agents in the economy. Although recent empirical work documents a link between asset prices and investment working through information flows (Chen, Goldstein and Jiang (2007)), the relevance of this channel is far from being settled (Morck, Shleifer and Vishny (1990)).

Another view which fits more naturally with the single asset framework in this paper is to interpret “fundamentals” as aggregate demand in the economy, and aggregate investment as the representative firm’s investment decision. The asset market then plays a coordination role between households and firms, conveying information about consumption plans of the former into production decisions of the latter.

I capture the impact of uncertainty on investment decisions following the insights from Bernanke (1993), Dixit and Pindyck (1994), Bertola and Caballero (1994) and the recent evidence in Bloom et al. (2007) and Bloom (2009). This literature highlights how partial irreversibilities generate a “real option value” on investment decisions, creating a wedge between the marginal product of capital which justifies investment and disinvestment. This wedge generates a region of inaction, where investment is fixed for some range of the underlying state variables. When uncertainty increases, the option value of delaying decisions is raised, expanding the inaction region. In simple terms, firms prefer to wait until the dust settles before undertaking investment decisions that will later be regretted.

Below I present a tractable model which exhibits the feature that a spike in uncertainty coming from a noisier asset price increases the expected losses from investment, which leads to a reduction in the scale of investment with respect to the level attained in a deterministic environment. This reduction is increasing in the variance of fundamentals, conditional on information.

\(^{37}\) See Lettau and Ludvigson (2002), and more recently Hassan and Mertens (2008).
\(^{38}\) See Dow and Gorton (1997); Subrahmanyam and Titman (1999); Dow and Rahi (2003); Goldstein and Guembel (2008).
5.1 Setup

Technology I consider a single period. A firm produces a consumption good using capital \( k \geq 0 \). Investment translates into units of the consumption good given the realization of a random productivity parameter \( \theta \) through the profit function \( \Pi(\theta, k) \):

\[
\Pi(\theta; k) = \theta \cdot k - \frac{1}{2} k^2 - L(\theta; k), \quad \text{with}
\]

\[
L(\theta; k) = \left[ \frac{a}{2} (\theta - k)^2 \right] k
\]

where \( \theta \) is normally distributed with mean of zero and variance \( \lambda^{-1}_\theta \).

Profits are adapted from a standard quadratic costs of investment function modified to include the loss function \( L(\theta; k) \) which is quadratic in the difference \( \theta - k \). This term captures the cost of investment being "out of line" with respect to the ex-post optimal investment which, as I will show in a moment, is given by \( \theta \). The term is multiplied by \( k \), making losses proportional to investment—a given deviation from the ex-post optimum should be more costly for a larger investment scale.

The asset market and investment Modeling feedback effects between asset prices and investment is a complex task. A rigorous approach needs to deal with the fact that prices simultaneously reflect dispersed information about fundamentals and the reaction of investment to prices, since investment affects expected dividends. Albagli, Hellwig and Tsyvinski (2009) explicitly consider such interaction in an environment that allows for differential information observed by the firm and traders, but where the firm still learns valuable information from its stock price. In this paper I follow the simpler route of assuming a separation between the profits of the firm in (19), and the dividend paid out to shareholders (see for instance Subrahmanyam and Titman (1999)). In particular, shares pay the terminal dividend \( \theta \) so that the asset price will aggregate information about \( \theta \) and affect the investment decision, but the latter will not be incorporated in traders beliefs about dividend value.

I consider the simple trading environment of section 2 where only a fraction \( F \) of traders participate in the asset market. I focus on characterizing the firm’s investment decision given the information about \( \theta \) contained in the share price of expression (2), with the noise amplifier given by expression (7).

5.2 The investment problem

The firm maximizes expected profits in (19), given its posterior beliefs \( H(\theta \mid \Omega_f) \). To keep the analysis simple, I assume the firm only learns information about \( \theta \) from its share price, so that \( \Omega_f : \{\tilde{p}_1\} \), where \( \tilde{p}_1 \) is given by equation (3). It is straightforward to include private information
in the firm’s beliefs, but this will not change the qualitative results discussed here. The firm’s problem then reduces to

\[
\max_k \mathbb{E} [\Pi \mid \Omega_f] = k \cdot \mathbb{E} [\theta \mid \bar{p}_1] - \frac{1}{2} k^2 - \frac{a}{2} k \cdot \mathbb{E} [(\theta - k)^2 \mid \bar{p}_1]
\]

The f.o.c. of this problem is derived in Appendix D, and leads to an optimal investment of

\[
k^* = \frac{2}{3} \mathbb{E} [\theta \mid \bar{p}_1] - \frac{1}{3a} + \sqrt{\left( \frac{1}{3a} + \frac{\mathbb{E} [\theta \mid \bar{p}_1]}{3} \right)^2 - \frac{1}{3} \mathbb{V} [\theta \mid \bar{p}_1]}
\]

The original problem is a polynomial of order three in \(k\), with a negative coefficient next to the cubic term. The global solution of this problem is therefore always at \(k = -\infty\), which is ruled out by the requirement \(k \geq 0\). The positive root in the f.o.c., when real, is the local maximum of the problem. But note that for some values of \(\mathbb{E} [\theta \mid \bar{p}_1]\), the squared term inside the square root can fall below \(1/3 \cdot \mathbb{V} [\theta \mid \bar{p}_1]\). The complex root then indicates the inexistence of a local maximum for \(k \geq 0\), leading to the corner solution \(k = 0\).

5.3 Market illiquidity and real investment

To understand the impact of varying price informativeness on investment decisions, consider for a moment a perfect information benchmark in which \(\mathbb{V} [\theta \mid \bar{p}_1] \to 0\). From expression (20), this yields the simple investment decision \(k^* = \theta\). An analogous result obtains in the standard quadratic problem with uncertainty, but no loss term: \(k^* = \mathbb{E} [\theta \mid \bar{p}_1]\). In contrast, the loss function \(L (\theta; k)\) introduces an additional cost of investment “mistakes” – my proxy for partial irreversibilities in a static model– which holds back investment in a more uncertain environment (higher \(\mathbb{V} [\theta \mid \bar{p}_1]\)). Figure 10 plots the implications of uncertainty in the underinvestment of the firm relative to its expectations about fundamentals. For each value of the expected productivity \(\theta\) (dotted line), the firm chooses a lower investment level (solid line), a gap that widens as funding in the asset market tightens through its impact on price informativeness.

Modeling the reaction of investment to changes in both the first and second moments of productivity is important for explaining the growth rate or steepness asymmetry – the empirical regularity that investment tends to contract sharply at the outset of a slump but builds up only gradually when growth resumes. In the model above, investment asymmetry results from endogenous information aggregation in asset prices: when a boom turns into a bust, investment will be lower not only because the conditional first moment \(\mathbb{E} [\theta \mid \bar{p}_1]\) is likely to be lower but also because uncertainty about it is larger when traders in the asset market face tighter constraints. Investment thus falls sharply, contributing to the steep fall on GDP that characterize economic contractions.
Two theoretical papers bear a similar prediction about investment dynamics. Van Nieuwerburgh and Veldkamp (2006) use a DSGE real business cycle model where capital and labor amplify the effect of unobserved productivity in total output, which is also affected by an unobservable disturbance. This creates asymmetric learning dynamics as larger hiring of inputs during expansions raises the signal-to-noise ratio of observable output, improving inferences about productivity. Chamley and Gale (1994) build a model where investment generates positive information externalities between players, introducing a motive for strategic delay that is consistent with investment data. However, neither paper discusses the role of asset markets in generating endogenous uncertainty about fundamentals.

The above analysis suggests endogenous uncertainty about fundamentals can have welfare effects that go beyond redistribution of wealth between asset market participants. To the extent that real resources are guided by asset prices, funding liquidity affects economic efficiency by reducing the precision of investment. Moreover, as the model shows in convenient closed form, the first moment of investment falls below the expected value of fundamentals as a precautionary measure.

The next proposition states these welfare results formally. The appropriate concept of firm value is the ex-ante expectation of firm’s profits, since the interest here is how production outcomes will be affected by investment decisions over all possible realizations of $\tilde{p}_1$. 

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Proposition 6 (firm value): The unconditional expected profit of the firm is strictly increasing in funding liquidity $F$;

$$\frac{\partial E[\Pi(\theta, k)]}{\partial F} > 0$$

Proof. In Appendix D. ■

The intuition of the result follows directly from decomposing the impact of $F$ into its effect on average profits per unit of capital (the intensive margin), and its effect on total investment (the extensive margin). The proof shows that the reduction in uncertainty brought along by a higher level of $F$ increases expected average profits as the firm’s decision becomes more accurate -the intensive margin of profits increases. Moreover, as the level of investment increases with $F$ the extensive margin also contributes in raising the value of the firm, since average expected profits are positive, conditional on $k^* > 0$.

6 Testable predictions and policy implications

6.1 Testable predictions

The model offered advances some new asset pricing predictions. First, it predicts that the autocorrelation of stock returns in high-frequency data associated with high trading volume should decline (or become more negative) when funding conditions are tight. As Campbell, Grossman and Wang (1993) and Wang (1994) argue, returns explained by noise trading tend to reverse at high frequencies, which is supported empirically using high trading volume days as a proxy for noise trading. The analysis above implies that such reversals should be stronger during contractions, since noisy trading has a larger impact on asset prices when uncertainty is higher.

Pastor and Stambaugh (2003) find an analogous pattern for the cross-section of stocks returns. Stocks with higher “liquidity betas” exhibit larger return reversal associated with trading volume. An interesting extension would be to test for differential impacts across the business cycle, or conditional on institutional investors’ funding restrictions.

The analysis also suggests models of information aggregation may have some bite in explaining certain asset pricing anomalies, such as the post-earnings announcements drift (PEAD). A learning approach to this anomaly (Hong, Lim and Stein (2000)) suggests drifts could reflect the slow diffusion of information. It follows that PEAD should be more pronounced when funding constraints limit information aggregation. A finding in this direction is provided by Chordia and Shivakumar (2006), who document that the profitability of PEAD strategies are significant negative predictors of future economic activity. While this result challenges the interpretation of PEAD excess returns as compensation for risk–since they actually provide hedges according to this evidence–it is con-
sistent with this alternative interpretation in which high PEAD returns reflect slow incorporation of information into prices during periods of financial turmoil.

Finally, the model has direct implications on the precision and relative dispersion of professional forecasts. If forecasters follow Bayesian updating rules when making predictions (about either macroeconomic aggregates, or individual firms’ profits), they will tend to base those predictions less in information inferred from asset markets during contractions, when prices become less reliable. One should expect a shift towards private sources of information, and therefore an increase in the cross-sectional dispersion of forecasts. Since overall uncertainty is higher, one should also expect higher mean prediction errors. Both predictions seem to hold in the data (Veronesi (1999); Van Niewerburgh and Veldkamp (2006)).

6.2 Policy implications

The funding problems of banks and financial institutions in the midst of the sub-prime crisis underscores the importance of prompt public liquidity provision. The model suggest that the success of interventions, however, will depends on how liquidity is distributed across players.

The argument developed above explains uncertainty and non-fundamental volatility as the joint product of funding constraints and dispersed information. In models that considers only the former, the size of the liquidity pool is a sufficient statistic for the success of the policy, since knowledge about the environment is not the element driving the gap between prices and fundamentals.

Things change dramatically if information is dispersed. Indeed, risk-averse traders who happen to find themselves sitting on a pile of cash may have little use for it when uncertainty is an endogenous state-variable. If the problem is a group problem –the failure of the market in aggregating disseminated pieces of information though trading– it is intuitively very appealing that the solution should be a group solution.

A second issue relates to the optimal taxation of dividends vs. capital gains from trading. In the model, prices become less informative because funding constraints shorten the effective trading horizon of informed traders, exposing them to additional risk. A marginal decrease in tax rates applied to dividends—with the corresponding increase in capital gains if one wishes to maintain fiscal neutrality—would tend to increase the incentives to react to private information about fundamentals, and could potentially reduce equilibrium uncertainty. This insight applies more generally to models that stress other reasons for the short trading horizons, such as Froot, Scharfstein and Stein (1992).
7 Conclusion

I develop a tractable model in which the diminished capacity of distressed financial markets to aggregate information explains countercyclical uncertainty. Building on a standard noisy REE model with dispersed information, I incorporate funding constraints on informed traders that are more likely to bind in periods of financial distress. Countercyclical risk premia, Sharpe ratios, and stock price volatility follow directly from this mechanism even when the attitude towards risk and the unconditional volatility of fundamentals remain constant.

I argue that adding dispersed information and dynamic risk considerations into the analysis delivers strong internal amplification mechanisms. Moreover, dispersed information and the endogenous aggregation capacity of financial markets is the appropriate conceptual benchmark for understanding the impact of uncertainty in real investment decisions and for guiding policy actions.

Future work may proceed in several directions. First, a quantitative assessment about the contribution of the informational mechanism for asset pricing and real investment phenomena seems in order. This is likely to be a complicated task, since the appropriate benchmark of CRRA utility (or even Epstein-Zin preferences) calls for the use of nonlinear methods to tract information aggregation—but one that should be tackled nonetheless to gauge the relevance of endogenous price informativeness. Second, extending the framework to multiple assets and allowing traders to learn from a richer set of signals can provide answers regarding the comovement of individual stocks and aggregate market indexes. As Morck, Yeung and Yu (2000) argue, comovement seems to increase during volatile markets in the time-series, and it is also higher in countries with less developed financial systems. Finally, the model can easily fit additional information about economic conditions—such as public news—whose impact on stock prices might endogenously depend on the business cycle.
8 Appendix

8.1 Appendix A

Signal extraction problem: projection theorem  The inference problem analyzed in sections 2 and 3, generally speaking, amounts to forecasting a \( N \times 1 \) vector \( X \) (with unconditional mean \( E[X] \)) from the observation of a \( M \times 1 \) vector of correlated signals \( \Omega \) (with unconditional mean \( E[\Omega] \)). If \( X \) and \( \Omega \) are jointly normally distributed, and \( \Sigma_{XX}, \Sigma_{\Omega\Omega} \) and \( \Sigma_{X\Omega} \) are the variance-covariance matrices of \( X, \Omega \) and between \( X \) and \( \Omega \) respectively, then the projection theorem gives the following results for the conditional moments of \( X \mid \Omega \):

\[
\begin{align*}
E[X \mid \Omega] &= E[X] + \Sigma_{\Omega\Omega}^{-1} (\Omega - E[\Omega]), \\
\text{Var}[X \mid \Omega] &= \Sigma_{X\Omega} \Sigma_{\Omega\Omega}^{-1} (\Sigma_{X\Omega})'.
\end{align*}
\]

Applied to the specific problem analyzed in the text, \( X \) corresponds to \( \theta \) (with \( E[\theta] = 0 \)), and the vector of signals becomes \( \Omega_i = \{s_i, \tilde{p}_1\} = \{\theta + s_i, \theta - \Delta \cdot n_1\} \) for informed traders, and \( \Omega_u = \{\tilde{p}_1\} = \{\theta + s_i\} \) for the uninformed, where \( \tilde{p}_1 \) is an object of equivalent informational content as \( P_1 \). The Bayesian weights of informed traders \( \{a_0, a_1\} \) and uninformed traders \( \{b_1\} \) are given by

\[
\begin{align*}
a_0 &= \lambda_e \cdot \text{Var}[\theta \mid \Omega_i] ;
\quad a_1 = \lambda_n \Delta^{-2} \cdot \text{Var}[\theta \mid \Omega_i], \\
b_1 &= \lambda_n \Delta^{-2} \cdot \text{Var}[\theta \mid \Omega_u]
\end{align*}
\]

where

\[
\text{Var}[\theta \mid \Omega_i] = \left[ \lambda_\theta + \lambda_e + \lambda_n / \Delta^2 \right]^{-1},
\quad \text{Var}[\theta \mid \Omega_u] = \left[ \lambda_\theta + \lambda_n / \Delta^2 \right]^{-1}.
\]

Note that the conditional second moments of \( \theta \) do not depend on the trader-specific private signal \( s_i \), and are therefore common across traders of the same type.

For proving Proposition 2, it will be of use to state the conditional expectation of the noise trading shock, \( n_1 \). Applying the theorem once again, we find

\[
E[n_1 \mid s_i, \tilde{p}_1] = (a_0 / \Delta) \cdot s_i - ((1 - a_1) / \Delta) \cdot \tilde{p}_1
\]

(21)

Proof of Proposition 1: To solve the proposed linear equilibrium of section 2, I replace informed traders beliefs computed in (4) in demands from (5). This allows to write the market-
clearing condition (6) as

\[
n = F \cdot \left( \int \frac{a_0 \cdot s_i + a_1 \cdot \tilde{p}_1 - P_1}{\Sigma} di - 1 \right), \text{ or} \]

\[
F + n = F \cdot \frac{a_0 \cdot \theta + a_1 \cdot \tilde{p}_1 - P_1}{\Sigma}
\]

where the second line follows from the zero mean of idiosyncratic noise \( \epsilon_i \). This gives price \( P_1 \) as

\[
P_1 = -\Sigma + a_0 \cdot \theta + a_1 \cdot \tilde{p}_1 - \Sigma / F \cdot n
\]

Comparing with the price conjecture in 2 (method of undetermined coefficients) yields the results in the proposition. QED.

8.2 Appendix B

Proof of Proposition 2: Part (i)

To solve the proposed linear equilibrium of section 3, I replace informed traders beliefs about \( \theta \) and \( P_2 \) using (4) and (10), and uninformed beliefs about \( D \) using (11), in the market-clearing condition at stage 2:

\[
1 + n_2 = \frac{D_{-1} + b_1 \cdot \tilde{p}_1 - P_2}{\Sigma_u} - \frac{(1 - F)}{\Sigma} \cdot \int \left\{ F \left( D_{-1} + \mathbb{E}[\theta \mid s, \tilde{p}_1]\right) + (1 - F) \left( B_0 + B_1 D_{-1} + B_2 \mathbb{E}[\theta \mid s, \tilde{p}_1] + B_3 \mathbb{E}[n_1 \mid s, \tilde{p}_1] + B_5 \tilde{p}_1 + B_6 P_1 \right) \right\} di
\]

where \( \mathbb{E}[\theta \mid s, \tilde{p}_1] = a_0 \cdot s_i + a_1 \cdot \tilde{p}_1 \), and \( \mathbb{E}[n_1 \mid s, \tilde{p}_1] \) is given by 21. This given price \( P_2 \) as a function of the variables \( \{\theta, n_1, n_2, \tilde{p}_1, P_1\} \), which can be compared to the price conjecture in (10) to obtain (through the method of undetermined coefficients)

\[
\begin{align*}
B_0 &= B_3 = 0; \quad B_1 + B_8 = 1, \\
B_2 &= -\frac{\Sigma_u \cdot a_0 (1 - F) \cdot F}{\Sigma + \Sigma_u a_0 (1 - F)^2}; \quad B_4 = -\Sigma_u; \\
B_6 &= -\frac{\Sigma_u a_1 (1 - F) (F + (1 - F)B_2)}{\Sigma + \Sigma_u (1 - F)^2}
\end{align*}
\]

Similarly, the market-clearing conditions at stage 1 implies a price \( P_1 \) as a function of \( \{D_{-1}, \theta, n_1, \tilde{p}_1\} \).
Regrouping the observed terms in $P_{1} (D_{-1}, \theta, n_{1}, \tilde{p}_{1})$ into the left hand side allows to solve for $\tilde{p}_{1}$:

\[
P_{1} + \left[ \Sigma + \Sigma_{u} (1 - F)^2 \right] - D_{-1} - \tilde{p}_{1} \left[ \frac{\Sigma + \Sigma_{u} (1 - F)^2}{\Sigma} \right] (Fa_{1} + (1 - F)B_{6}) = \theta - n_{1} \frac{\Sigma}{a_{0} \cdot (F + (1 - F)B_{2})} \equiv \tilde{p}_{1}
\]

which implies that the noise amplifier $\Delta$ is given by equation (16a), as stated in the proposition.

To compute the risk-aversion adjusted variance of informed traders; $\Sigma$, note that the conditional covariance between $P_{2}$ and $D$ is $B_{2} \mathcal{V}[\theta | s, \tilde{p}_{1}]$. From the definition of $\Sigma$ in the text, this gives equation (16b). The expression for $\Sigma_{u}$ comes from (11). This completes the four equations in the proposition that any linear equilibrium must satisfy, in the unknowns $\{\Delta, \Sigma, \Sigma_{u}, B_{2}\}$.

**Part (ii)**

The system of equations in Proposition 2 can be reduced to two equations in the unknowns $\{\Delta, \Sigma\}$:

\[
\Sigma (\Sigma, \Delta) = (1 - c) \gamma (1 - F)^2 \gamma_{u}^{2} \left[ \lambda_{\theta} + \lambda_{n}/\Delta^{2} \right]^{-1} + \lambda_{\mu}^{-1} \lambda_{n}^{-1}
\]

\[
\Delta (\Sigma, \Delta) = \frac{\Sigma_{u} \left( \left[ \lambda_{\theta} + \lambda_{n}/\Delta^{2} \right]^{-1} + \lambda_{\mu}^{-1} \right) (1 - F)^2 \lambda_{\epsilon} \left[ \lambda_{\theta} + \lambda_{\epsilon} + \lambda_{n}/\Delta^{2} \right]^{-1} \lambda_{\epsilon}^{-1}}{\left[ \lambda_{\theta} + \lambda_{\epsilon} + \lambda_{n}/\Delta^{2} \right]^{-1}}
\]

Proving existence of equilibria amounts to showing that the loci of combinations $(\Sigma, \Delta)$ that satisfy each equation intersects at least once, for all parameter values. I provide figure A1 as a graphical intuition of the proof below. I begin with the loci $\Sigma = \hat{\Sigma}(\Sigma, \Delta)$ that satisfies equation (23), for a given value of $\Delta$. First, note that the derivative of the RHS of (23) w.r.t. $\Sigma$ is less than unity, and that $\hat{\Sigma}(0, \Delta) > 0$. Since the derivative of the left hand side (23) is one, there is a unique value $\Sigma^{*}(\Delta)$ that satisfies (23), for each $\Delta$. To characterize the loci $\Sigma = \hat{\Sigma}(\Sigma, \Delta)$ in the $(\Delta, \Sigma)$ space, note that its intercept is given by

\[
\hat{\Sigma}(\Sigma, 0) \equiv \Sigma = (1 - c) \gamma \lambda_{\mu}^{-1} \left[ (1 - F)^2 \gamma_{u}^{2} \lambda_{\mu}^{-1} \lambda_{n}^{-1} + F^{2} \right]
\]

and the limit $\hat{\Sigma}(\Sigma, \Delta \rightarrow \infty) \equiv \Sigma$ solves

\[
\Sigma = (1 - c) \gamma \cdot \left\{ (1 - F)^2 \gamma_{u}^{2} \left[ \lambda_{\theta}^{-1} + \lambda_{\mu}^{-1} \right]^{2} \lambda_{n}^{-1} + F^{2} \left[ \lambda_{\mu}^{-1} + \frac{\Sigma^{2}}{\left[ \Sigma + \gamma_{u} \left( \lambda_{\theta}^{-1} + \lambda_{\mu}^{-1} \right) (1 - F)^2 \lambda_{\epsilon} \left[ \lambda_{\theta} + \lambda_{\epsilon} \right]^{-1} \lambda_{\epsilon} \right]^{2}} \right] \right\}
\]

\[34\]
with $\Sigma > \Sigma$. This upper limit $\Sigma$ ensures that the implicit function $\Sigma^*(\Delta)$ becomes concave in $\Delta$, for $\Delta > \Delta_1$. Without imposing any parameter restrictions, these results imply the function $\Sigma^*(\Delta)$ always has a solution $\forall \Delta \in \mathbb{R}^+$, and that $\Sigma^*(\Delta) < \Sigma$.

Similarly, the loci $\Delta = \hat{\Delta}(\Sigma, \Delta)$ that characterizes the implicit solution $\Delta^*(\Sigma)$ in equation (24) is continuous in $\Sigma \in \mathbb{R}^+$, and always has a solution $\Delta^*(\Sigma) > 0$. It follows that represented in the $(\Sigma, \Delta)$ space, the loci $\Sigma^*(\Delta)$ from equation one will intersect the loci $\Delta^*(\Sigma)$ at least once, since the former is a locally convex (for $\Delta > \Delta_1$), continuous correspondence with image $\Delta \in [0, \infty)$ on the domain $(0, \Sigma)$, and the latter is a continuous, positive-valued function $\forall \Sigma \in \mathbb{R}^+$. This implies that, over the interval $\Sigma \in [\hat{\Sigma}, \Sigma)$, equations (23) and (24) will always be satisfied simultaneously for at least one pair $(\Sigma^*, \Delta^*)$. QED.

**Part (iii)**

Finding conditions for equilibrium uniqueness amounts to finding the parameter subspace for which the loci $\Sigma^*(\Delta)$ and $\Delta^*(\Sigma)$ from equations (23) and (24) intersect only once. For this I must first restrict $\Delta^*(\Sigma)$ to be single-valued (since $\Sigma^*(\Delta)$ is single-valued without restrictions, from the analysis in part (ii)). $\Delta^*(\Sigma)$ is single-valued whenever the derivative of the right hand side of (24) w.r.t. $\Delta$ is less than one. It is simple to show that this derivative is actually negative whenever $\gamma_u \lambda_{-1}^{-1} (2 + \lambda_{-1}^{-1}) > 1$, or $\lambda_{-1} < \kappa_0 (\gamma_u, \lambda_{-1})$.

To see under what conditions these single-valued functions (assuming $\lambda_{-1} < \kappa_0 (\cdot)$) cross only once, it is necessary to characterize the intercepts and slopes of the implicit functions $\Sigma^*(\Delta)$ and $\Delta^*(\Sigma)$, which I do in the $(\Sigma, \Delta)$ space. Since both functions are continuous, I can use the implicit
function theorem for the latter task. Starting with $\Delta^* (\Sigma)$, its intercept $\Delta$ solves

$$\Delta = \gamma_u (1 - F)^2 \left[ \lambda^{-1}_\mu + (\lambda_\theta + \lambda_n / \Delta^2)^{-1} \right]$$

which is increasing $\lambda^{-1}_\mu$. To find the slope, I totally differentiate the function

$$f : \Delta - \hat{\Delta} (\Sigma, \Delta) = 0, \quad \frac{d\Delta}{d\Sigma} = \frac{\partial f}{\partial \Sigma} / \frac{\partial f}{\partial \Delta} = \frac{\partial \hat{\Delta} (\cdot) / \partial \Sigma}{1 - \partial \hat{\Delta} (\cdot) / \partial \Delta}$$

to find $d\Delta / d\Sigma > 0$ under the condition already stated ($\lambda _\mu < \kappa_0 (\cdot)$). Moreover, the function is concave, with a slope that approaches $(\lambda_\epsilon + \lambda_\theta) / \lambda_\epsilon$ as $\Sigma \to \infty$.

Regarding $\Sigma^* (\Delta)$, I already established that $\Sigma^* (0) = R$, so that $\Sigma^* (\Delta)$ does not intercept the $\Delta$ axis (since $\Sigma \geq 0$). With respect to its slope, I totally differentiate the function

$$f : \Sigma - \hat{\Sigma} (\Sigma, \Delta) = 0, \quad \frac{d\Sigma}{d\Delta} = -\frac{\partial f}{\partial \Delta} / \frac{\partial f}{\partial \Sigma} = \frac{\partial \Sigma (\cdot) / \partial \Delta}{1 - \partial \Sigma (\cdot) / \partial \Sigma}$$

to find with some algebra that a sufficient condition for a positive slope is

$$2 \gamma (1 - c) [F^2 \lambda^{-1}_\mu + (1 - F)^2 \gamma_u^2 \lambda^{-1}_\mu \lambda^{-1}_n] > F^2 \lambda_n,$$

or $\lambda _\mu < \kappa_1 (\cdot)$. Recall as well that this function is concave for $\Delta > \Delta_1$, which implies a convex loci in the $(\Sigma, \Delta)$ space, with $\Sigma < \Sigma$. These results together imply one can always find a finite value $\lambda _\mu = \kappa_2$, s.t. $\forall \lambda _\mu < \kappa_2$, the concave function $\Delta^* (\Sigma)$ intersects only once with the (locally) convex function $\Sigma^* (\Delta)$. This case corresponds to the solid line $\Delta^* (\Sigma)$ in figure A1. In contrast, if $\lambda _\mu \to 0$, then it cannot be granted that the two loci will not intersect more than once (dotted line). Therefore, whenever $\lambda _\mu \leq \delta_0$ with $\delta_0 \equiv \min (\kappa_0 (\cdot), \kappa_1 (\cdot), \kappa_2)$, the equilibrium in Proposition 2 is unique. QED.

**Proof of Proposition 3:** I will follow a similar reasoning as in the previous proof, finding conditions under which an increase in $F$ moves the loci $\Sigma^* (\Delta)$ and $\Delta^* (\Sigma)$ in a direction that implies an unambiguous decrease in $\Delta (F)$. First, note that the intercept of the implicit function $\Delta^* (\Sigma)$ given by (27) is decreasing in $F$, which can be shown by totally differentiating

$$f : \Delta - \hat{\Delta} (F, \Delta) = 0, \quad \frac{d\Delta}{dF} = -\frac{\partial f}{\partial F} / \frac{\partial f}{\partial \Delta} = \frac{\partial \hat{\Delta} (\cdot) / \partial F}{1 - \partial \hat{\Delta} (\cdot) / \partial \Delta}$$
which is negative whenever $\partial \hat{\Delta} (\cdot) / \partial \Delta < 1$. With some algebra, this can be translated into the sufficient condition $2\gamma_u (1 - F)^2 \left( \lambda_0 + \lambda_n/\Delta^2 \right)^{-1} < \Delta$, which is always satisfied by some $\lambda_\mu \leq \kappa_3$, with $\kappa_3 > 0$, since $\Delta > \lambda_\mu^{-1}$. Regarding the slope of $\Delta^* (\Sigma)$, I totally differentiate the function

$$f : \Delta - \hat{\Delta} (F, \Delta) = 0,$$

$$\frac{d\Delta}{dF} = -\frac{\partial f / \partial F}{\partial f / \partial \Delta} = \frac{\partial \hat{\Delta} (\cdot) / \partial F}{1 - \partial \hat{\Delta} (\cdot) / \partial \Delta}$$

to find $d\Delta/dF < 0$ whenever $1 - \partial \hat{\Delta} (\cdot) / \partial \Delta > 0$, or when condition $\lambda_\mu \leq \kappa_0 (\cdot)$ in the previous proof holds.

It remains to find the conditions under which the intercept and the slope of the implicit function $\Sigma^* (\Delta)$ have a negative w.r.t. $F$. The derivative of the intercept $R$ in equation (25) is proportional to $F - \gamma_u (1 - F) \lambda_\mu^{-1} \lambda_n^{-1}$, which is always negative for values of $F$ below $\bar{F}_1 \equiv \gamma_u^2 \lambda_\mu^{-1} \lambda_n^{-1}/ (1 + \gamma_u^2 \lambda_\mu^{-1} \lambda_n^{-1})$. With respect to the slope of $\Sigma^* (\Delta)$, I totally differentiate the function

$$f : \Sigma - \hat{\Sigma} (\Sigma, F) = 0,$$

$$\frac{d\Sigma}{dF} = -\frac{\partial f / \partial F}{\partial f / \partial \Sigma} = \frac{\partial \Sigma (\cdot) / \partial F}{1 - \partial \Sigma (\cdot) / \partial \Sigma}$$

which amounts to finding restrictions under which $\partial \Sigma (\cdot) / \partial F < 0$. But note that there always exists a value $F > 0$, s.t. the derivative of the right hand side of equation (23) is negative w.r.t. $F$, which can be easily shown by evaluating $\partial \Sigma (\cdot) / \partial F$ at $F = 0$. Moreover, we can find a lower bound $\bar{F}_2$ s.t. the derivative is negative at $F \leq \bar{F}_2$. Taking the partial of equation (23) w.r.t $F$ gives

$$\frac{\partial \Sigma (\cdot)}{\partial F} \propto -\lambda_n^{-1} + \frac{F}{1 - F} \frac{\left[ \mathcal{V} [\theta | \Omega_u] \cdot \Gamma (F) + \lambda_\mu^{-1} \right]}{\gamma_u \left[ \mathcal{V} [\theta | \Omega_u] + \lambda_\mu^{-1} \right]} + 2 \frac{F^2}{1 - F} \Gamma (F) a_0 \mathcal{V} [\theta | \Omega_i]$$

$$< -\lambda_n^{-1} + \frac{F}{1 - F} \frac{1}{\gamma_u} + 2 \frac{F^2}{1 - F} \left[ \lambda_\mu + \lambda_\theta \right]^{-1},$$

where $\Gamma (F) \equiv \Sigma^2 / \left[ \Sigma + \gamma_u \left[ \mathcal{V} [\theta | \Omega_u] + \lambda_\mu^{-1} \right] (1 - F)^2 a_0 \right]^2 < 1$, so that $\bar{F}_2 > \bar{F}$, where $\bar{F}$ makes the last term in the inequality equal to zero. It follows that both the derivative of the intercept, and the slope of $\Sigma^* (\Delta, \cdot)$ is negative for all $F \leq \bar{F} \equiv \min (\bar{F}_1, \bar{F}_2)$. Note that both thresholds $\bar{F}_1$ and $\bar{F}_2$ are increasing functions of the variance of noise trading shocks $\lambda_n^{-1}$, and the risk aversion of uninformed traders increase.

Finally, since there is a one-to-one, monotonically increasing relation between $\theta_{-1}$ and $F$ (equation (8)), it follows that the threshold $\bar{F}$ can be mapped into a threshold $\bar{\theta}$, s.t., $\Delta' (\theta_{-1}) < 0$, $\forall \theta_{-1} \leq \bar{\theta} \equiv \lambda_\mu^{-1/2} \cdot \Phi^{-1} (\bar{F})$, with $\partial \theta / \partial \lambda_n^{-1} > 0$, and $\partial \theta / \partial \gamma_u$. In summary, whenever the sufficient
conditions $\lambda_\mu < \delta_1 \equiv \min (\kappa_0 (\cdot), \kappa_3)$, and $\theta_{-1} < \bar{\theta} (\lambda_n^{-1}, \gamma_u)$ hold, $\Delta' (\theta_{-1}) < 0$. QED.

**Equilibrium when uninformed traders learn from $P_2$:** From Conjecture 2, uninformed traders can back out an additional noisy signal about $\theta$:

$$
\tilde{p}_2 \equiv \frac{P_2 - B_0 - B_1 \cdot D_{-1} - B_5 \cdot \bar{p}_1 - B_6 \cdot P_1}{B_2} = \theta + \frac{B_3}{B_2} n_1 + \frac{B_4}{B_2} n_2
$$

The equilibrium solution proceeds in the same steps as before, with the difference that uninformed traders’ beliefs now correspond to

$$
\mathbb{E} [\theta | \tilde{p}_1, \tilde{p}_2] = b_1 \tilde{p}_1 + b_2 \tilde{p}_2,
$$

$$
\mathbb{V} [\theta | \tilde{p}_1, \tilde{p}_2] = \left[ \lambda_0 + \lambda_n \left( \Delta^{-2} + \left( \frac{B_3}{B_2} \right)^2 + \left( \frac{B_4}{B_2} \right)^2 \right)^{-1} \right]
$$

which will imply a different solution for the linear coefficients of Conjecture 2. These are now given by

$$
\begin{align*}
B_0 &= B_3 = 0; & B_1 + B_5 &= 1, \\
B_2 &= \frac{b_2 - \Sigma_u \cdot a_0 (1 - F) \cdot F}{\Sigma + \Sigma_u a_0 (1 - F)^2}; & B_4 &= \Sigma_u \left[ \frac{b_2 \Sigma - \Sigma_u a_0 (1 - F) \cdot F}{\Sigma_u a_0 (1 - F) (F + (1 - F)b_2)} \right]; \\
B_6 &= \frac{b_1 \Sigma - \Sigma_u a_1 (1 - F) (F + (1 - F)B_2)}{\Sigma + \Sigma_u (1 - F)^2}
\end{align*}
$$

The equilibrium is now characterized by a system of seven equations: $\Delta$ from 16a; $B_2$ and $B_4$ from (30); $b_2$ given by

$$
b_2 = \frac{\lambda_n (B_2/B_4)^2}{\lambda_0 + \lambda_n (\Delta^{-2} + (B_2/B_4)^2)}
$$

from the projection theorem; $\Sigma$ and $\Sigma_u$ from (16b) and (29), and $a_0 = \lambda_n \mathbb{V} [\theta | s, \tilde{p}_1]$ as before.

I have not been able to provide an analytically find specific parameter conditions that can ensure uniqueness. Figure A2 plots one equilibrium simulated for starting values of the iteration in the neighborhood of the equilibrium in which uninformed traders do not learn from $P_2$. The results are fairly intuitive: first, since $P_2$ provides additional information, the conditional variance of uninformed traders is lower in this case (panel B), which increases their willingness to absorb informed traders’ liquidations and noise. This alleviates to some extent the noise risk and trade reversal effects discussed in section 3, increasing the incentives for informed traders to react to information at stage 1. Consequently, the noise amplifier $\Delta$ at stage 1 is lower as well. The new equilibrium behaves qualitatively very similar to the previous one, and I suspect that similar
restrictions on the variance of $\mu$ can lead to equilibrium uniqueness, but the precise characterization is out of the scope of the present paper.

### 8.3 Appendix C

**Proof of Proposition 4:** This proof follows directly from the proof of Proposition 3. Taking the partial derivative of (17) w.r.t. $\theta_{-1}$ gives

$$\frac{\partial RP}{\partial \theta_{-1}} = \left[ \frac{\partial \Sigma}{\partial F} + \frac{\partial \Sigma_u}{\partial F} \left( 1 - F \right)^2 - 2 \Sigma_u (1 - F) \right] \frac{\partial F}{\partial \theta_{-1}}$$

Recall that under the condition $F \leq \bar{F}$, both the intercept and slope of the implicit function $\Sigma^*(\Delta, \cdot)$ are decreasing in $F$. I also showed that the derivative and intercept of the implicit function $\Delta^*(\Sigma, \cdot)$ are decreasing in $F$ under the restriction $\lambda_\mu < \delta_1$. It follows that the solution $\Sigma^*(\Delta, \cdot)$ has a negative derivative w.r.t. $F$ under these restrictions, which takes care of the term $\partial \Sigma / \partial F$. Moreover, since $\Sigma_u$ is increasing in the conditional variance $\mathbb{V}[\theta | \Omega_u] = [\lambda_\theta + \lambda_n / \Delta^2]^{-1}$, it follows that under the same restrictions $\partial \Sigma_u / \partial F < 0$. Finally, $\partial F / \partial \theta_{-1} > 0$ from equation (8).

QED.
Proof of Proposition 5: The derivative of the Sharpe ratio w.r.t. \(\theta_{-1}\) is proportional to

\[
-2 (1 - F) \gamma_u^2 \left( \frac{\mathbb{V}[\mathcal{\theta} | \Omega_n]}{\mathbb{V}[\mathcal{\theta} | \Omega_i]} + \lambda_{\mu}^{-1} \right)^{1/2} L_n^{-1} - (1 - F)^2 \gamma_u^2 \lambda^{-1} \frac{\left( \mathbb{V}[\mathcal{\theta} | \Omega_n] + \lambda_{\mu}^{-1} \right)^{1/2}}{\Delta (\mathbb{V}[\mathcal{\theta} | \Omega_i] + \lambda_{\mu}^{-1})^{1/2}} \left[ 4 \mathbb{V}[\mathcal{\theta} | \Omega_n] - \mathbb{V}[\mathcal{\theta} | \Omega_i] \right] \frac{\mathbb{V}[\mathcal{\theta} | \Omega_n] + \lambda_{\mu}^{-1}}{\mathbb{V}[\mathcal{\theta} | \Omega_i] + \lambda_{\mu}^{-1}} \\
+ 2 F \frac{\Gamma (F) \mathbb{V}[\mathcal{\theta} | \Omega_i] + \lambda_{\mu}^{-1}}{\mathbb{V}[\mathcal{\theta} | \Omega_i] + \lambda_{\mu}^{-1}} + \frac{F^2}{\mathbb{V}[\mathcal{\theta} | \Omega_i] + \lambda_{\mu}^{-1}} \cdot \left[ \Gamma (F) \frac{\partial\mathbb{V}[\mathcal{\theta} | \Omega_i]}{\partial \Delta} \cdot \Delta' (\cdot) + \Gamma' (F) \frac{\partial\mathbb{V}[\mathcal{\theta} | \Omega_i]}{\partial \Delta} \right] \\
- \frac{1}{2} \left( \mathbb{V}[\mathcal{\theta} | \Omega_i] + \lambda_{\mu}^{-1} \right)^{-1/2} \left( \Gamma (F) \mathbb{V}[\mathcal{\theta} | \Omega_i] + \lambda_{\mu}^{-1} \right) \frac{\partial\mathbb{V}[\mathcal{\theta} | \Omega_i]}{\partial \Delta} \cdot \Delta' (\cdot) \right]
\]

where the first term is negative, and the second can be shown to be negative under the restriction \(\lambda_{\mu} < \delta\). The third term is positive, while the fourth is of ambiguous sign, depending on the value of \(F\). It follows that it is possible to find a value \(\bar{F}_3\) s.t. \(\forall F < \bar{F}_3\), the whole expression is negative. This can be translated into a corresponding threshold for \(\theta_{-1}\), such that \(\partial S \Sigma / \partial \theta_{-1} < 0 \forall \theta_{-1} \leq \bar{\theta}' \equiv \lambda_{\ell}^{-1/2} \cdot \Phi^{-1} (\bar{F}_3)\), as stated in the proposition. QED.

Equilibrium characterization in the common information economy I proceed in similar steps as in section 3. First, I conjecture the linear price functions

\[
P_1 = A_o + A_1 \cdot D_{-1} + A_2 \cdot (\mathcal{\theta} + \epsilon) + A_3 \cdot n_1 \\
P_2 = B_0 + B_1 \cdot D_{-1} + B_2 \cdot (\mathcal{\theta} + \epsilon) + \\
+ B_4 \cdot n_2 + B_5 \cdot P_1 + B_6 \cdot \tilde{p}_1
\]

where I have already used the fact that the partial effect of \(n_1\) on \(P_2\) is zero (\(B_3 = 0\), as in previous section). The key difference in this setting is that aggregating across informed traders will not wash out the common noise \(\epsilon\), so that prices will inherit \(s = \mathcal{\theta} + \epsilon\). Another distinction is that informed traders can perfectly infer the noise trading shock \(n_1\), since they observe the price as well as the common signal \(\epsilon\). I use these facts to replace the conditional moments of \(\mathcal{\theta}\) and \(P_2\) in informed traders’ demands.

Using the market-clearing condition at stage 2 allows to solve for the \(B\)'s coefficients of the conjecture about \(P_2\), which I then replace in the market-clearing condition at stage 1. From the resulting \(P_1\), I can recover the public signal

\[
\tilde{p}_1 = (\mathcal{\theta} + \epsilon) - \Delta \cdot n_1
\]
where $\Delta$ is now given by

$$\Delta = \frac{\Sigma}{F \cdot a_0 + (1 - F) \cdot B_2}$$

Note that with common information however, uninformed traders are the only agents making inferences from the asset price $P_1$, which now captures both noise trading shocks and the common noise $\epsilon$.

The solution for the coefficients in price $P_2$ are given by

$$B_0 = 0; B_1 = \frac{\Sigma - \Sigma_u \cdot (1 - F) F}{\Sigma + \Sigma_u (1 - F)^2}; B_2 = \frac{-\Sigma_u (1 - F) F \cdot a_0}{\Sigma + \Sigma_u (1 - F)^2}$$

$$B_4 = -\Sigma_u; B_5 = \frac{\Sigma_u \cdot (1 - F)}{\Sigma + \Sigma_u (1 - F)^2}; B_6 = \frac{b_1 \Sigma}{\Sigma + \Sigma_u (1 - F)^2};$$

with $b_1 = \lambda_\epsilon \lambda_n / [\lambda_\theta (\lambda_n + \lambda_\epsilon \Delta^2) + \lambda_\epsilon \lambda_n]$

What remains is finding the expressions for the total risk (weighted by risk-aversion) faced by informed traders; $\Sigma$. Note that, conditional on the signal $\epsilon$, the conditional variance of $P_2$ is just $(\Sigma_u)^2 \cdot \lambda_n^{-1}$, which gives

$$\Sigma = \gamma \left[ F^2 \left( \frac{1}{\lambda_\theta + \lambda_\epsilon} + \frac{1}{\lambda_\mu} \right) + (1 - F)^2 (\Sigma_u)^2 \cdot \lambda_n^{-1} \right]$$

where $\Sigma_u = \gamma_u \left[ (\lambda_\epsilon^{-1} + \lambda_n^{-1} \Delta^2)^{-1} + \lambda_\mu^{-1} \right]$. This last expression, together with (31), allows to solve for $\Delta$.

### 8.4 Appendix D

**First order condition of the investment problem:** I can rewrite the investment problem in the maximand as

$$\mathbb{E} [\theta \mid \tilde{p}_1] - \frac{1}{2} k^2 - \frac{a}{2} k \cdot \mathbb{E} [\theta^2 - 2\theta k + k^2 \mid \tilde{p}_1]$$

$$= \mathbb{E} [\theta \mid \tilde{p}_1] - \frac{1}{2} k^2 - \frac{a}{2} k \cdot [\mathbb{E} [\theta \mid \tilde{p}_1] + (\mathbb{E} [\theta \mid \tilde{p}_1])^2 - 2\mathbb{E} [\theta \mid \tilde{p}_1] k + k^2]$$

taking the derivative w.r.t. $k$ leads to a quadratic equation in $k$, which can be solved to yield the result in expression (20).

**Proof of Proposition 6:** I first show that conditional on $\tilde{p}_1$, the firm does weakly better on average when the precision of information is larger - i.e., the expected profit is increasing in $F$. I
can write the firm’s profit expectation as
\[
E[\Pi(\theta, k) | \tilde{p}_1] = k^* \cdot \left\{ \frac{E[\Pi(\theta, k) | \tilde{p}_1]}{k^*} \right\}
\]
and its derivative with respect to \( F \) as
\[
\frac{\partial E[\Pi(\theta, k) | \tilde{p}_1]}{\partial F} = \frac{\partial k^*}{\partial F} \cdot \frac{E[\Pi(\theta, k) | \tilde{p}_1]}{k^*} + k^* \cdot \frac{\partial \left\{ \frac{E[\Pi(\theta, k) | \tilde{p}_1]}{k^*} \right\}}{\partial k^*} / \partial k^*
\]

I distinguish two possible cases. First, if either expectations on fundamentals are too low, and/or the conditional variance of fundamentals is too large, the firm does not invest, and expected profits do not change with marginal increases in \( F \). In the second situation, investment \( k^* \) is positive, which is the only case when the firm expects the average profit per unit of \( k^* \) to be positive as well. This makes the first term of the last expression strictly positive, for \( k^* > 0 \).

The second term requires more analysis. Its partial derivative w.r.t. \( F \) can be written as
\[
= -\frac{k^*}{2} \frac{\partial V[\theta | \tilde{p}_1]}{\partial F} \left\{ \frac{\partial k^*}{\partial V[\theta | \tilde{p}_1]} + a \left( 1 + 2 \left( E[\theta | \tilde{p}_1] - k^* \right) \frac{\partial \left( E[\theta | \tilde{p}_1] - k^* \right)}{\partial V[\theta | \tilde{p}_1]} \right) \right\}
\]
Since \( \partial V[\theta | p_1] / \partial F < 0 \), it suffices to show that the term in brackets is strictly positive, for \( k^* > 0 \). We can write
\[
\frac{\partial k^*}{\partial V[\theta | \tilde{p}_1]} = -\frac{1}{6} (SQR)^{-1/2}, \quad \text{and} \quad \frac{\partial \left( E[\theta | \tilde{p}_1] - k^* \right)}{\partial V[\theta | \tilde{p}_1]} = \frac{1}{6} (SQR)^{-1/2}
\]
where
\[
SQR = \left( \frac{1}{3a} + \frac{E[\theta | \tilde{p}_1]}{3} \right)^2 - \frac{1}{3} V[\theta | \tilde{p}_1]
\]
which is strictly positive under the requirement \( k^* > 0 \). This allows to rewrite the bracket term as
\[
\frac{1}{6} (SQR)^{-1/2} \left[ \frac{2E[\theta | \tilde{p}_1]}{3} + \frac{2}{3a} - 2 (SQR)^{1/2} \right] + a
\]
which can be further manipulated by replacing the solution \( k^* \) from (20), to yield
\[
k^* + 3 \left( \frac{1}{3a} - (SQR)^{1/2} \right) + a = k^* + 3 \left( \frac{2E[\theta | \tilde{p}_1]}{3} - k^* \right) + a
\]
\[
= 2 \left( E[\theta | \tilde{p}_1] - k^* \right) + a
\]
which is always strictly positive, for any arbitrarily small variance \( V[\theta | \tilde{p}_1] \), since \( E[\theta | \tilde{p}_1] - k^* > 0 \).
It remains to show that $F$ raises the unconditional profit expectation. This can be restated as the integral of conditional profits, over the distribution of $\tilde{P}_1$. Labeling the cdf of $\tilde{P}_1$ by $G(\tilde{P}_1)$, one can write

$$\frac{\partial \mathbb{E}\left[\Pi(\theta, k) \mid \tilde{P}_1\right]}{\partial F} = \int \left(\frac{\partial \mathbb{E}\left[\Pi(\theta, k) \mid \tilde{P}_1\right]}{\partial F}\right) dG(\tilde{P}_1) > 0$$

which follows from the fact that the conditional result $\frac{\partial \mathbb{E}\left[\Pi(\theta, k) \mid \tilde{P}_1\right]}{\partial F} \geq 0$ holds for any realization of the price $P_1$. Since at least for some realizations the firm will choose a positive level of investment, it follows that the impact of $F$ on firm value is strictly positive. QED.
References


