Lecture 12

Optimal Control Theory

Imagine that you control an all-terrain vehicle that must go from point \( x_0 \) to point \( x_1 \). There are varied types of ground between \( x_0 \) and \( x_1 \), hilly, flat, sandy, marshy, grassy, and so on. The objective is to choose the route and speed so as to minimize total fuel consumption. You control direction and throttle, which are equivalent to a vector \( u \) in the set

\[
\{ u \in \mathbb{R}^2 \mid ||u|| \leq r \} = U,
\]

where the positive number \( r \) corresponds to the maximum throttle setting. Direction is the same as the direction of the vector \( u \). Your control setting at time \( t \) is \( u(t) \in U \) and your location is \( x(t) \in \mathbb{R}^2 \). Your rate of progress is determined by the differential equations

\[
\frac{dx_1(t)}{dt} = g_1(x_1(t), x_2(t), u_1(t), u_2(t)) \quad \text{and} \quad \frac{dx_2(t)}{dt} = g_2(x_1(t), x_2(t), u_1(t), u_2(t)).
\]

In vector form, these equations become \( dx(t)/dt = g(x(t), u(t)) \). Your velocity at time \( t \), \( dx(t)/dt \), depends on \( x(t) \), because the kind of terrain you are in depends on your position, \( x(t) \). The rate of fuel consumption at time \( t \) is \( f(x(t), u(t)) \). You start at time 0. The problem is to choose a time \( T \) and a control path \( u:[0, T] \to U \) so that if the function \( x:[0, T] \to \mathbb{R}^2 \) satisfies the differential equation \( dx(t)/dt = g(x(t), u(t)) \) and the initial condition \( x(0) = x_0 \), then

\[
x(T) = x_1 \quad \text{and you minimize fuel consumption} \int_{0}^{T} f(x(t), u(t)) \, dt \quad \text{or maximize} \int_{0}^{T} - f(x(t), u(t)) \, dt.
\]

This problem can be generalized and made more rigorous as follows. Let the possible controls be a closed subset \( U \) of \( R^k \). An admissible control is a piecewise continuous function \( u:[0, T] \to U \), where \( T > 0 \).

Definition: A function \( u:[0, T] \to U \) is piecewise continuous if

1) \( u \) is continuous except at a finite number of values of \( t \) and

2) if \( t \) is a point of discontinuity of \( u \), then both the left and right-hand limits,
\[ \lim_{t \to 1^-} u(t) \text{ and } \lim_{t \to 1^+} u(t), \text{ exist (and are therefore finite).} \]

**Example:** A piecewise continuous function.

![Graph of a piecewise continuous function]

**Example:** A non-piecewise continuous function.

\[
u(t) = \begin{cases} 
\frac{1}{t}, & \text{if } 0 \leq t < 1 \\
1 - t, & \text{if } 1 \leq t \leq 2 \\
1, & \text{if } t \geq 2.
\end{cases}
\]
The function \( u \) in the previous example is not piecewise continuous, because \( \lim_{\Delta t \to 0, \Delta t > 0} u(t) \) does not exist.

**Example:** Another non-piecewise continuous function.

\[
    u(t) = \begin{cases} 
        \sin(t^{-1}), & \text{if } 0 < t \leq 1 \\
        0, & \text{if } t = 0. 
    \end{cases}
\]

This function is not piecewise continuous because \( \lim_{t \to 0, t > 0} u(t) \) does not exist.

**Notation:** Let \( \mathcal{A} \) denote the set of admissible controls. That is,

\[
    \mathcal{A} = \{ u : [0, T] \to U \mid T > 0 \text{ and } u \text{ is piecewise continuous} \}.
\]

Note that the set \( U \) is fixed, but the end time \( T \) may be variable.

The optimization problem to be analyzed is described using functions \( f : \mathbb{R}^n \times U \to \mathbb{R} \) and \( g_n : \mathbb{R}^n \times U \to \mathbb{R} \), for \( n = 1, \ldots, N \), which are assumed to be continuously differentiable with respect to \( x_1, \ldots, x_N \) and continuous with respect to \( x_1, \ldots, x_{N-1}, u_1, \ldots, u_K \). Let \( g(x, u) = (g_1(x, u), \ldots, g_N(x, u)) \), where \( x = (x_1, \ldots, x_N) \) and \( u = (u_1, \ldots, u_K) \). The problem under consideration is

\[
    \max_{\mathcal{A}} \int_0^T f(x(t), u(t)) \, dt \\
    \text{s.t. } \frac{dx(t)}{dt} = g(x(t), u(t)), \text{ for all } t, \\
    x(0) = x_0 \text{, and } x(T) = x_1.
\]

(12.1)

We seek necessary conditions for optimality. I find these conditions in a non-rigorous but suggestive way using the Kuhn-Tucker theorem. Assume that \( f \) and the functions \( g_n \) are concave, as they often are in economic applications. Divide the time interval \([0, T]\) into \( M \) intervals of length \( \Delta t = T/M \), where \( M \) is a large positive integer. Assume that \( N = K = 1 \), so that \( x(t) \) and \( u(t) \) are numbers. Let us look at the behavior of \( x \) and \( u \) at times \( m\Delta t \), for \( m = 0, 1, \ldots, M \). Problem 12.1 can be then be approximated by the following finite dimensional problem.
\[
\max \sum_{m=0}^{M-1} \Delta t \ f(x(m\Delta t), u(m\Delta t)) \\
\text{s.t. } x(m\Delta t) - x((m-1)\Delta t) \\
\quad \leq \Delta t \ g(x((m-1)\Delta t), u((m-1)\Delta t)), \\
\quad \text{for } m = 1, \ldots, M, \\
x(0) \leq x_0 \text{ and } -x(M\Delta t) \leq -x_1.
\] (12.2)

The objective function \( \sum_{m=0}^{M-1} \Delta t f( x( m \Delta t ) , u( m \Delta t ) ) \) is a Riemann sum approximating the integral \( \int_0^T f(x(t), u(t)) \, dt \). The constraints are inequalities rather than equations, so that we may apply the Kuhn-Tucker theorem. Since \( f \) is concave and the functions

\[
x(m\Delta t) - x((m-1)\Delta t) - \Delta t g((m-1)\Delta t), u((m-1)\Delta t))
\]

are convex, we may use this theorem, provided the constraint qualification applies. Since the approach taken is intended to be suggestive rather than rigorous, let's just assume that this qualification applies. Then the Kuhn-Tucker theorem implies that there exist non-negative numbers \( \lambda(0), \ldots, \lambda(M) \) and a non-negative number \( \beta \) such that

\[
\lambda(0) = 0, \text{ if } x(0) < x_0,
\]
\[
\lambda(m) = 0, \text{ if } x(m\Delta t) - x((m-1)\Delta t) < \Delta t g(x((m-1)\Delta t), u((m-1)\Delta t)), \\
\quad \text{for } m = 1, \ldots, M, \text{ and}
\]
\[
\beta = 0, \text{ if } -x(M\Delta t) < -x_1,
\]

and the vector \( u(0), u(\Delta t), \ldots, u((M-1)\Delta t), x(0), x(\Delta t), \ldots, x(M\Delta t) \) maximizes the Lagrangian

\[
\mathcal{L} = \sum_{m=0}^{M-1} \Delta t \ f(x(m\Delta t), u(m\Delta t)) \\
\quad - \sum_{m=1}^{M} \lambda(m) \left[ x(m\Delta t) - x((m-1)\Delta t) - \Delta t g(x((m-1)\Delta t), u((m-1)\Delta t)) \right] \\
\quad + \beta x((M\Delta t) - \lambda(0) x(0) \\
\quad = \sum_{m=0}^{M-1} \left[ \Delta t f(x(m\Delta t), u(m\Delta t)) + \lambda(m+1) \Delta t g(x(m\Delta t), u(m\Delta t)) \right]
\] (12.3)
\[- \sum_{m=1}^{M} \lambda(m) \left[ x(m \Delta t) - x((m-1) \Delta t) \right] + \beta x(M \Delta t) - \lambda(0) x(0) .\]

I now simplify the expression for the Lagrangian by using an analogue of integration by parts. Recall corollary 5.16 to the fundamental theorem of calculus that says that if the function \( F: [0, T] \rightarrow \mathbb{R} \) is continuously differentiable, then \( \int_{0}^{t} \frac{dF(s)}{ds} ds = F(t) - F(0) \), for all \( t \) such that \( 0 \leq t \leq T \). Integration by parts is based on the equations

\[
F(T) G(T) - F(0) G(0) = \int_{0}^{T} \frac{d}{dt} \left[ F(t) G(t) \right] dt
\]

\[
= \int_{0}^{T} \left( \frac{dF(t)}{dt} G(t) + F(t) \frac{dG(t)}{dt} \right) dt
\]

\[
= \int_{0}^{T} \frac{dF(t)}{dt} G(t) dt + \int_{0}^{T} F(t) \frac{dG(t)}{dt} dt.
\]

The first equation follows from the fundamental theorem of calculus, and the second follows from Leibniz's rule for the differentiation of the product of two functions. If we rearrange the above equations, we obtain the following equation, which is called integration by parts.

\[
\int_{0}^{T} \frac{dF(t)}{dt} G(t) dt = F(T) G(T) - F(0) G(0) - \int_{0}^{T} F(t) \frac{dG(t)}{dt} dt.
\]

Since the Lagrangian 12.1 has differences rather than derivatives and sums in place of integrals, we must somehow make the same replacements in the equation for integration by parts. If \( y_1, y_2, \ldots \) is a sequence of numbers, its first difference is the sequence \( \Delta y_1, \Delta y_2, \ldots \), where \( \Delta y_m = y_{m+1} - y_m \), for all \( m \). Leibniz's rule for first differences is

\[
\Delta(x y)_m = x \Delta y_m + y \Delta x_m + (x y_{m+1} - x y_m)
\]

\[
= y_{m+1} \Delta x_m + x \Delta y_m .
\]

The analogue of the fundamental theorem of calculus equation

\[
\int_{0}^{T} \frac{d}{dt} \left[ F(t) G(t) \right] dt = F(T) G(T) - F(0) G(0)
\]

is
\[ \sum_{m=1}^{M} \Delta(x_{m}, y_{m}) = (x_{2} - x_{1}) + (x_{3} - x_{2}) + \ldots + (x_{M+1} - x_{M}) = x_{M+1} y_{M} - x_{1} y_{1}. \]

Putting these equations together, we see that

\[ x_{M+1} y_{M+1} - x_{1} y_{1} = \sum_{m=1}^{M} \Delta(x_{m}, y_{m}) = \sum_{m=1}^{M} \Delta x_{m} + \sum_{m=1}^{M} \Delta y_{m}. \]

This equation in turn implies that

\[ -\sum_{m=1}^{M} \Delta x_{m} = \sum_{m=1}^{M} \Delta y_{m} - x_{1} y_{1} + x_{M+1} y_{M+1}, \]

which is a discrete time analogue of integration by parts. Let \( x_{m} = x((m-1)\Delta t) \) and \( y_{m} = \lambda(m-1) \). Then the equation just derived implies that

\[ -\sum_{m=1}^{M} \lambda(m) [x(m\Delta t) - x((m-1)\Delta t)] = -\sum_{m=1}^{M} \lambda(m) \Delta x((m-1)\Delta t) \]

\[ = \sum_{m=1}^{M} x((m-1)\Delta t) \Delta \lambda(m-1) - \lambda(M) x(M\Delta t) + \lambda(0) x(0). \]

If we substitute this equation into the expression 12.3 for the Lagrangian, we obtain

\[ \mathcal{L} = \sum_{m=0}^{M-1} \Delta t [f(x(m\Delta t), u(m\Delta t)) + \lambda(m+1)g(x(m\Delta t), u(m\Delta t))] \]

\[ + \sum_{m=1}^{M} x((m-1)\Delta t) \Delta \lambda(m-1) - \lambda(M) x(M\Delta t) + \lambda(0) x(0) + \beta x(M\Delta t) - \lambda(0) x(0) \]

\[ \mathcal{L} = \sum_{m=0}^{M-1} \Delta t [f(x(m\Delta t), u(m\Delta t)) + \lambda(m+1)g(x(m\Delta t), u(m\Delta t))] \]

\[ + \sum_{m=1}^{M} x((m-1)\Delta t) \Delta \lambda(m-1) + [\beta - \lambda(M)] x(T) \]

\[ \mathcal{L} = \sum_{m=0}^{M-1} \Delta t [f(x(m\Delta t), u(m\Delta t)) + \lambda(m+1)g(x(m\Delta t), u(m\Delta t))] \]

\[ + \sum_{m=1}^{M} x(m\Delta t) \Delta \lambda(m) + [\beta - \lambda(M)] x(T), \]

where I have used the fact that \( M\Delta t = T \).
According to the Kuhn-Tucker theorem, the Lagrangian $\mathcal{L}$ is maximized with respect to $u(m\Delta t)$ and $x(m\Delta t)$, for all $m$. Therefore $u(m\Delta t)$ solves the problem

$$\max_{u \in \mathcal{U}} \left[ f(x(m\Delta t), u) + \lambda(m+1) g(x(m\Delta t), u) \right], \quad (12.4)$$

for $m = 0, \ldots, M-1$. If we maximize the Lagrangian with respect to $x(m\Delta t)$, we obtain the first order conditions

$$\Delta \lambda(m) + \Delta t \frac{\partial}{\partial x} \left[ f(x(m\Delta t), u(m\Delta t)) + \lambda(m+1) g(x(m\Delta t), u(m\Delta t)) \right] = 0,$$

for $m = 0, \ldots, M-1$. That is,

$$\frac{\lambda(m+1) - \lambda(m)}{\Delta t} = -\frac{\partial}{\partial x} \left[ f(x(m\Delta t), u(m\Delta t)) + \lambda(m+1) g(x(m\Delta t), u(m\Delta t)) \right].$$

Think of $\lambda$ as a function of the continuous time variable $t$, so that $\lambda(t)$ is its value at time $t$. Replace $\lambda(m)$ by $\lambda(m\Delta t)$ and let $\Delta t$ go to zero while increasing $m$ so that $m\Delta t$ approaches a value $t$ as $m$ goes to infinity. The previous equation becomes

$$\frac{\lambda((m+1)\Delta t) - \lambda(m\Delta t)}{\Delta t} = -\frac{\partial}{\partial x} \left[ f(x(m\Delta t), u(m\Delta t)) + \lambda((m+1)\Delta t) g(x(m\Delta t), u(m\Delta t)) \right].$$

Taking the limit as $\Delta t$ goes to zero and assuming that $\lambda$ is differentiable, we obtain the equation

$$\frac{d\lambda(t)}{dt} = -\frac{\partial}{\partial x} \left[ f(x(t), u(t)) + \lambda(t) g(x(t), u(t)) \right],$$

for all $t$. Similarly because $u(m\Delta t)$ solves problem 12.4, it follows that $u(t)$ solves the problem

$$\max_{u \in \mathcal{U}} \left[ f(x(t), u) + \lambda(t) g(x(t), u) \right],$$

for all $t$.

If we assume that the constraints in problem 12.2 all hold with equality at the optimum, we see that

$$\frac{dx(t)}{dt} = g(x(t), u(t)), $$

for all $t$. 

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for all $t$.

These conclusions may be summarized using the Hamiltonian function, which is a continuous time instantaneous analogue of the Lagrangian of problems with finitely many variables. (In continuous time, there are infinitely many variables, namely $x(t)$ and $u(t)$, for all values of $t$.) The Hamiltonian function is defined to be

$$H(x, u, \lambda) = f(x, u) + \lambda g(x, u),$$

We may summarize our intuitively derived findings as follows. Let $u(t)$ be the optimal control at time $t$. For all $t$,

$$u(t) \text{ solves the problem } \max_{u \in U} H(x(t), u, \lambda(t)),$$

$$\frac{dx(t)}{dt} = \frac{\partial}{\partial \lambda} H(x(t), u(t), \lambda(t)), \text{ and }$$

$$\frac{d\lambda(t)}{dt} = -\frac{\partial}{\partial x} H(x(t), u(t), \lambda(t)).$$

The equations

$$\frac{dx}{dt} = \frac{\partial H}{\partial \lambda} \text{ and } \frac{d\lambda}{dt} = -\frac{\partial H}{\partial x}$$

are called the Hamiltonian system. The fact that $u(t)$ maximizes $H(x(t), u, \lambda(t))$ with respect to $u$ is called the maximum principle.

The maximum principle implies that

$$\frac{\partial}{\partial u} H(x(t), u(t), \lambda(t)) = 0,$$

provided $H$ is differentiable with respect to $u$. Moreover if in addition the function $u(t)$ is differentiable, then

$$\frac{d}{dt} H(x(t), u(t), \lambda(t)) = \frac{\partial H}{\partial x} \frac{dx}{dt} + \frac{\partial H}{\partial u} \frac{du}{dt} + \frac{\partial H}{\partial \lambda} \frac{d\lambda}{dt}$$

$$= \frac{\partial H}{\partial x} \frac{\partial H}{\partial \lambda} \left(0\right) \frac{du}{dt} + \frac{\partial H}{\partial \lambda} \left(-\frac{\partial H}{\partial x}\right) = 0,$$

so that $H$ is constant along the optimal path. This assertion holds even if the function $u(t)$ is not differentiable, but it does not apply if $f$ or $g$ depends directly on time $t$. 
If \( x(t) \) and \( u(t) \) are vectors, then the above statements remain true, when suitably stated. More precisely, a necessary condition for optimality is that there exist piecewise differentiable functions \( \lambda_1(t), \ldots, \lambda_N(t) \) such that \((\lambda_1(t), \ldots, \lambda_N(t)) \neq 0\), for all \( t \), and such that

\[
H(x, u, \lambda) = H(x_1, \ldots, x_N, u_1, \ldots, u_k, \lambda_1, \ldots, \lambda_N)
\]

\[
= f(x_1, \ldots, x_N, u_1, \ldots, u_k) + \sum_{n=1}^{N} \lambda_n g(x_1, \ldots, x_N, u_1, \ldots, u_k),
\]

then, for all \( t \),

\[
\frac{\partial x_n(t)}{\partial t} = \frac{\partial}{\partial \lambda_n} H(x(t), u(t), \lambda(t)) \quad \text{and}
\]

\[
\frac{\partial \lambda_n(t)}{\partial t} = -\frac{\partial}{\partial x_n} H(x(t), u(t), \lambda(t)),
\]

for \( n = 1, \ldots, N \), and the optimal control \( u(t) \) solves the problem

\[
\max_{u \in \mathcal{U}} H(x(t), u, \lambda(t)),
\]

for all \( t \). Furthermore

\[
\frac{\partial}{\partial t} H(x(t), u(t), \lambda(t)) = 0.
\]

These statements are true even if \( f \) and the functions \( g_k \) are not concave. The variables

\( x_1, \ldots, x_N \)

are called state variables, and the variables \( \lambda_1, \ldots, \lambda_N \) are called costate, dual, conjugate, or auxiliary variables. These dual variables need not be non-negative, as they are in Kuhn-Tucker theory.

Suppose that \( N = K = 1 \) and that \( u = dx/dt \), so that the rate of change of \( x \) is the control. Then \( g(x, u) = u \), so that \( H(x, u) = f(x, u) + \lambda u \). Assume that \( f \) is differentiable. The maximum principle implies that

\[
\lambda = -\frac{\partial f(x, u)}{\partial u}.
\]

The Hamiltonian system implies that
\[ \frac{dx}{dt} = -\frac{\partial H(x, u)}{\partial x} = -\frac{\partial f(x, u)}{\partial x}. \]

These two equations imply that
\[ \frac{d}{dt} \frac{\partial f(x, u)}{\partial u} = \frac{\partial f(x, u)}{\partial x}. \]

If we substitute \( dx/dt \) for \( u \), this equation becomes
\[ \frac{d}{dt} \frac{\partial f(x, dx/dt)}{\partial x} = \frac{\partial f(x, dx/dt)}{\partial x}. \]

This is known as Euler's equation. There follows an application of this equation to a simple example.

**Example 12.1:** Imagine that our object is to bring a point on a line to 0 by controlling its velocity. The loss at any moment in time is the square of the distance from 0 plus the square of the speed of the point. The loss associated with speed may be thought of as the cost of fuel consumed. If \( x(t) \) is the distance from 0 at any moment, then the loss is \( x^2(t) + (dx(t)/dt)^2 \). Suppose we have a finite amount of time \( T \) over which to minimize the loss. Then the control problem is

\[
\begin{align*}
\min_{\dot{x}(t)/dt} & \int_0^T \left[ x^2(t) + (dx(t)/dt)^2 \right] dt \\
\text{s.t. } x(0) &= x_0 \text{ and} \\
x(T) &= 0, 
\end{align*}
\]

where \( x_0 \) is given and \( dx(t)/dt \) is understood to be a piecewise continuous function of \( t \). This problem is, of course, the same as

\[
\begin{align*}
\max_{\dot{x}(t)/dt} & \int_0^T \left[ -x^2(t) + (dx(t)/dt)^2 \right] dt \\
\text{s.t. } x(0) &= x_0 \text{ and} \\
x(T) &= 0. 
\end{align*}
\]

In order to apply Euler's equation, let \( f(x, dx/dt) = -x^2 - (dx/dt)^2 \). Since
\[ \frac{\partial f(x, dx/dt)}{\partial (dx/dt)} = -2dx/dt \text{ and } \frac{\partial f(x, dx/dt)}{\partial x} = -2x, \]
Euler's equation implies that

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\[-2 \frac{d}{dt} \left( \frac{dx}{dt} \right) = -2x, \]

so that \( \frac{d^2x(t)}{dt^2} = x(t) \), for all \( t \). Any solution of this differential equation has the form

\[ x(t) = ae^t + be^{-t}, \]

for some numbers \( a \) and \( b \).

The maximum principle implies that

\[ \lambda(t) = -\frac{\partial f(x(t), dx(t)/dt)}{\partial (dx/dt)} = 2 \frac{x(t)}{dt} = 2ae^t - 2be^{-t}. \]

Recall that the Hamiltonian system yields two ordinary differential equations,

\[
\begin{align*}
\frac{dx(t)}{dt} &= g(x(t), u(t)) \quad \text{and} \\
\frac{d\lambda(t)}{dt} &= -\frac{\partial}{\partial x} \left[ f(x(t), u(t)) + \lambda(t) g(x(t), u(t)) \right],
\end{align*}
\]

where \( u(t) = dx(t)/dt \). The solution of these two equations requires two initial conditions, one for \( x(0) \) and one for \( \lambda(0) \). We are given the value \( x(0) = x_0 \). We have to choose \( \lambda(0) \). In this example, these initial conditions are

\[ x_0 = x(0) = a + b \quad \text{and} \]

\[ \lambda(0) = 2a - 2b, \]

conditions which determine the parameters \( a \) and \( b \). Since \( b = x_0 - a \),

\[ \lambda(0) = 2a - 2x_0 + 2a = 4a - 2x_0, \]

so that the choice of \( \lambda(0) \) fixes \( a \) and hence \( b \). In order to pin down the choice of \( a \) or \( \lambda(0) \), we use the condition that \( x(T) = 0 \), i.e.,

\[ x(T) = ae^T + (x_0 - a)e^{-T} = 0, \]

so that \( a(e^{2T} - 1) + x_0 = 0 \), which implies that
\[ a = \frac{X_0}{e^{2T} - 1}. \]

Therefore

\[ b = x_0 - a = x_0 + \frac{X_0}{e^{2T} - 1} = \frac{e^{2T}x_0}{e^{2T} - 1}, \]

and so the solution is

\[ x(t) = -\frac{X_0}{e^{2T} - 1} e^t + \frac{e^{2T}x_0}{e^{2T} - 1} e^{-t}. \]

Suppose now that we want to minimize the objective function

\[ \int_0^T \left[ x^2(t) + \left( \frac{dx(t)}{dt} \right)^2 \right] dt \]

with respect to \( T \). To this end, we calculate the value of the problem with horizon \( T \), namely,

\[ V(T) = \min_{dx(t)/dt \atop 0} \int_0^T \left[ x^2(t) + \left( \frac{dx(t)}{dt} \right)^2 \right] dt \]

s.t. \( x(0) = x_0 \) and

\[ x(T) = 0. \]

Notice that

\[ x^2(t) = (ae^t + be^{-t})^2 = a^2e^{2t} + 2ab + b^2e^{-2t} \]

and

\[ \left( \frac{dx(t)}{dt} \right)^2 = (ae^t - be^{-t})^2 = a^2e^{2t} - 2ab + b^2e^{-2t}. \]

Hence

\[ x^2(t) + \left( \frac{dx(t)}{dt} \right)^2 = 2a^2e^{2t} + 2b^2e^{-2t}, \]

and so
\[ V(T) = \int_0^T \left[ x^2(t) + \left( \frac{dx(t)}{dt} \right)^2 \right] dt = \int_0^T \left[ 2a^2e^{2t} + 2b^2e^{-2t} \right] dt = a^2e^{2T} - b^2e^{-2T} \]

\[ = a^2(e^{2T} - 1) - b^2(e^{-2T} - 1) = a^2(e^{2T} - 1) + b^2(1 - e^{-2T}) \]

Substituting \( \frac{x_0}{e^{2T} - 1} \) for a and \( \frac{e^{2T}x_0}{e^{2T} - 1} \) for b, we find that

\[ V(T) = \frac{x_0^2}{(e^{2T} - 1)^2} (e^{2T} - 1) + \frac{e^{2T}x_0^2}{(e^{2T} - 1)^2} (1 - e^{-2T}) \]

\[ = \frac{x_0^2}{e^{2T} - 1} + \frac{x_0^2}{1 - e^{-2T}} \]

This function is clearly decreasing with respect to T, so that the optimal choice of T is \( T = \infty \), which implies that \( a = 0 \) and \( b = x_0 \), so that the solution is

\[ x(t) = x_0 e^{-t} \]

**Example:** This is an example of an optimal control problem that has no solution. Modify the previous example by assuming that there is no cost to changing the value of \( x \). The problem then becomes

\[ \max_{dx(1)/dt \leq 0} \int_0^T -x^2(t) \, dt \]

\[
\text{s.t. } x(0) \text{ is given.}
\]

The obvious thing to do in this problem is to get to 0 as fast as possible. No matter how big is \( dx(1)/dt \) over the interval \([0, T]\) and no matter how small \( T \) is, you can do better by reducing \( T \) and increasing the size of \( dx(t)/dt \) during the period \([0, T]\). Since \( dx(t)/dt \) cannot be infinite, there is no optimal solution unless \( x(0) = 0 \). This ends the example.

The maximum principle and the Hamiltonian system describe the evolution of \((x(t), u(t), \lambda(t))\). The Hamiltonian system is an ordinary differential equation involving 2N variables \((x_1, \ldots, x_N, \lambda_1, \ldots, \lambda_N)\) and so needs 2N initial conditions to determine a solution. These 2N conditions are the equations
\[ x(0) = x_0, \ldots, x(0) = x_N, x(T) = x_1, \ldots, x(T) = x_{1N} \]

Given \( x(t), x(T), \lambda(t), \lambda(T) \) at each time \( t \), \( u(t) \) is determined by the maximum principle. If \( T \) is fixed, then \( \lambda(0) = (\lambda_0, \ldots, \lambda_N) \) is determined by the 2N initial conditions and the maximum principle. If \( T \) is variable, it must be chosen artfully so as to maximize the objective function \( \int_0^T f(x(t), u(t)) \, dt \) of problem 12.1.

The Value Function in Optimal Control Theory

Kuhn-Tucker theory tells us that the Kuhn-Tucker coefficients are subgradients of the value function. The value function at time 0 is

\[
V(x, 0) = \max_{u \in A} \int_0^T f(x(t), u(t)) \, dt
\]

subject to

\[
\frac{dx(t)}{dt} = g(x(t), u(t)), \text{ for all } t \in [0, T]
\]

and for \( n = 1, \ldots, N \), and

\[
x(0) = x_0 \text{ and } x(T) = x_T,
\]

where we think of \( x(t) \) as an \( N \)-vector and \( u(t) \) as a \( K \)-vector. Then the \( N \)-vector \( \lambda(0) \) is the derivative, \( \frac{\partial}{\partial x_0} V(x, 0) \), of \( V(x, 0) \) with respect to \( x_0 \) if \( V \) is differentiable with respect to \( x_0 \).

Suppose that \((x, u)\) solves the above maximization problem and let \( \lambda(t) \) be the corresponding conjugate function. The value function at time \( t \) is

\[
V(x, t) = \max_{u \in A} \int_t^T f(x(s), u(s)) \, ds
\]

subject to

\[
\frac{dx(s)}{ds} = g(x(s), u(s)), \text{ for all } s \in [t, T]
\]

and for \( n = 1, \ldots, N \), and

\[
x(t) = x_0 \text{ and } x(T) = x_T.
\]

Then \( \lambda(t) = \frac{\partial}{\partial x} V(x(t), t) \), provided \( V(x, t) \) is differentiable with respect to \( x \).

We can gain further insight from using the value function \( V(x, t) \), if we assume it is differentiable with respect to both \( x \) and \( t \). For simplicity of exposition, I assume that \( N = 1 \), so
that $x$ is a number rather than an $N$-vector. Let $(x(t), u(t))$ be an optimal program with conjugate function $\lambda(t)$. By the fundamental theorem of calculus and the chain rule of differentiation,

$$-f(x(t), u(t)) = \frac{dV(x(t), t)}{dt} = \frac{\partial V(x(t), t)}{\partial x} \frac{dx(t)}{dt} + \frac{\partial V(x(t), t)}{\partial t},$$

$$= \lambda(t) g(x(t), u(t)) + \frac{\partial V(x(t), t)}{\partial t},$$

so that

$$\frac{\partial V(x(t), t)}{\partial t} = -f(x(t), u(t)) - \lambda(t) g(x(t), u(t)) = -H(x(t), u(t), \lambda(t))$$

$$= -\max_{u \in U} H(x(t), u, \lambda(t)),$$

where the last equation follows from the maximum principle. This equation may be rewritten as

$$\frac{\partial V(x(t), t)}{\partial t} = -\max_{u \in U} \left[ f(x(t), u) + \frac{\partial V(x(t), t)}{\partial x} g(x(t), u) \right],$$

which is known as the Hamilton-Jacobi-Bellman equation. It is what corresponds in continuous time models to the Bellman equation in discrete time models. In some problems, this partial differential equation can be solved to obtain $V$. Its solution over the whole state space is a necessary and sufficient condition for the existence of an optimum, provided the value function is differentiable.