APPROXIMATE VICKREY AUCTIONS

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1. Abstract

In a socially efficient or "competitive" equilibrium in a market for homogenous goods , firms produce until their marginal cost of production is equal to the prevailing market price; identically, firms on the demand side consume until their marginal utility, or profit, is equal to that price. In imperfectly competitive markets for homogenous goods, firms may attempt to sell (buy) less than their socially efficient quantity in order to increase (decrease) the price of both marginal and infra-marginal units and increase their total profits. This strategic behavior, or distortion, leads to deadweight loss. The extent to which firms are able to distort the market depends both on the elasticity of demand and on the elasticity of supply from other firms. In a series of pioneering papers, William Vickrey sought to overcome this inefficiency by severing the connection between each firm's bids and the price it faces for infra-marginal units. These Vickrey mechanisms have theoretically attractive properties but have been widely criticized as impractical and are, in fact, rare or nonexistent in real-world auctions, although many ascending auctions resemble the simplest of these mechanisms. This paper shows that the usual framework of uniform price, multi-unit homogenous goods auctions is a first-order approximation of the two-sided auction described in Vickrey (1961). The paper examines some properties of a second-order approximation involving quadratic payments.

2. Background

2.1. **Uniform-Price Markets and Market Power.** In the typical symmetric Cournot oligopoly model for homogenous goods, n identical firms perfectly observe market demand and then simultaneously choose production quantities, facing first order conditions

$$0 = \pi_i'(q) = P(q; q_{-i}) + P'(q; q_{-i}) \cdot q - MC_i(q)$$

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Notably, P and P' take the vector of other players quantities q_{-i} as exogenously given and consequently P' is a function only of demand curvature.

Firms set $P(q) = MC_i(q) - P'(q) \cdot q$, and since an increase in the quantity produced by a firm decreases the market clearing price this just means that each firm set prices above its marginal cost. Prices are distorted more by firms which produce greater quantities, and also if demand is more elastic.

In the Bertrand "price-setting" model, each firm chooses a price at which to offer arbitrarily many units of a good. With perfect information this model leads to the competitive outcome if there are at least two firms with the lowest production costs among firms in the market.

In many real-life uniform price auctions for homogenous goods, firm bidding and behavior does not quite match that found in this model. For example, in electricity markets[1] firms submit a series of bids tracing out a supply schedule over various possible prices. If marginal costs are strictly increasing and demand is decreasing then there is a unique market clearing price where demand intersects marginal supply. Firms still exercise market power by misstating their own cost curve which leads to an underestimate of aggregate supply. This "supply function bidding" is an intermediate between the Cournot model, in which residual supply is completely inelastic, and the Bertrand model, in which residual supply is infinitely elastic. Inasmuch as market power in these auctions is decreasing in the elasticity of the residual supply, it is natural that the deadweight loss from the "supply function equilibria" which arise due to this form of bidding should be intermediate between the Cournot and Bertrand models.

Supply function bids are a natural response to uncertainty by each firm about the level of demand or the level of their competitors' supply curves. Klemperer and Meyer (1989), in a widely cited paper, describe a unique supply function equilibrium in uniform price auctions

[1] See Hortacsu (2008), Cramton (2002)

when (supply-side) firms have uncertainty about the level of demand. The authors confirm that the quantity distortion from this equilibrium is less than in the Cournot model but positive. Without uncertainty, the first-order condition for a supply function equilibrium is

$$0 = \pi_i'(q) = P_{-i}(q) + P_{-i}'(q) \cdot q - mc_i(q)$$

In particular the P_{-i}, P_{-i}' no longer take q_{-i} as vectors which are chosen independently of q_i. Under uncertainty, each firm maximizes the expected value of $\pi_i'(q)$ over a known distribution of the level of demand. In a real understatement, the authors note that the resulting firm behavior is not "convenient analytically."[2] Each firm needs to take into account the shape of the sum of its opponents' bids as well as demand. If there are several symmetric firms and we assume a symmetric equilibrium then the problem is somewhat more tractable because all of the firms are responding to the same residual demand curve. If firms are asymmetric then the equilibrium conditions are especially unappealing, because each firm chooses a best response to a different residual demand curve determined by its opponents bids. As a result, it is difficult or impossible to write down exact expressions for optimal firm behavior.

Despite the computational difficulty, these supply-curve-oligopoly markets exist in practice as well as in the theoretical literature. Some papers have sought to understand some of their features. One approach is to try to directly estimate the slope of the residual market supply, P_{-i}', which is a measure of markups and therefore of both market power and inefficiency. Prete and Hobbs (2015) interpolate piecewise linear supply and demand curves from finitely many submitted bids and then computes P_{-i}' in terms of the empirically estimated residual demand.[3]

2.2. **Vickrey Auctions.** In uniform price auctions, "market power" is reflected in misrepresented supply curves and inefficiently low quantities.[4] The goal of this misrepresentation

[2]Klemperer and Meyer (1989); Farrell and Shapiro (2010) are even more pessimistic.

[3]There are many interesting empirical questions which extend naturally from Klemperer and Meyer's treatment of the topic. Do firms really act as if they have uncertainty about level, but not slope, of demand? Do firms face uncertainty about their competitors, and how do they respond to this uncertainty? What kinds of supply curves seem to be bid in practice? These are questions I hope to pursue in the future.

[4]If there is imperfect competition on both sides of a market and firms face increasing (decreasing) marginal costs (utilities) then the clearing price might be on either side of the competitive (or, social planner's) price.

is to influence the prices of infra-marginal units of the good and to shift consumer surplus to producers. In a series of beautiful papers, Vickrey describes mechanisms in which bidders have no control over the prices of their units and therefore (in theory) no incentive to bid anything other than their true marginal costs.

The simplest and best known of these auctions is the "sealed-bid, second-price" auction for an indivisible good (often called simply a Vickrey auction). In this auction, bidders submit a single bid for the good being auctioned. The highest bidder wins – but pays the amount of the second highest bid to the auctioneer. An attractive feature is that it is an equilibrium for each bidder to bid honestly, since the level of any given bidder's bid cannot affect the price of the good conditional on that bidder turning out to be the winner. If bidders are risk-neutral with independent private values, this auction is revenue equivalent to other efficient auctions (i.e. ones in which the bidder with the highest valuation wins) and is easy to participate in, since bidders do not gain anything from investigating the distribution of competitors' values or their bidding strategies. Bidders do their best and the one with the highest private value gains the good at a price that made this bidder perfectly willing to bid honestly. On the other hand, it is well known that this scheme is not in general revenue maximizing; the seller can do much better with other schemes. In particular, if bidders are not symmetric, a revenue-maximizing auctioneer will want to take steps to encourage bids from "weak" bidders who might otherwise stay away or buy in a secondary market.

Vickrey (1961) describes an auction for multi-unit homogenous goods which is efficient and in which truthful reporting of cost schedules is an equilibrium. This is an improvement over the equilibrium bidding strategy developed in Klemperer and Meyer on two fronts. First, there is no deadweight loss. Second, as in the second price auction, each bidder gains nothing from investigating the demand curve or her opponents' supply curves. Any expense incurred by firms gathering information about their competitors is purely deadweight loss, at least in the independent private values (or costs) framework. Making participation easier - that is, making bidders less sensitive to information about market features other than their

Still, the oligopoly quantity is guaranteed to be lower than the competitive quantity, resulting in deadweight loss.

own cost curves - is a critical goal in mechanism design.[5]

The Vickrey mechanism for a two-sided homogenous goods auction is as follows. An auctioneer solicits supply and demand schedules from participants. The auctioneer then finds the intersection of aggregate supply and aggregate demand, which determines the equilibrium quantity and the distribution of that quantity on both sides of the market. For each marginal unit of supply (demand), firms are awarded (charged) the residual market clearing price when their supply (demand) curve is replaced by a completely inelastic demand supply (demand) curve for the quantity supplied (demanded) up to that marginal unit.
In particular, suppose firms submit supply curves $s_i(p)$, and take aggregate demand to be $D(p)$. If supply curves are upward sloping and demand is downward sloping, so that $s_i'(p) > 0$ and $D'(p) < 0$, then there is a unique equilibrium price p^* so that

$$D(p^*) = \sum_i s_i(p^*)$$

We define q_i^*, the equilibrium quantity produced or sold by firm i which is active on the supply side, as

$$q_i^* := s_i(p^*)$$

In the Klemperer and Meyer model, firm i is awarded $q_i^* \cdot p^*$. In the Vickrey auction, firm i is awarded for the nth unit of production the price which would be the market clearing price if that firm restricted its production to $n-1$ units, taking all other firms' supply functions as fixed. That is, for the nth unit the firm receives

$$P_{-i}(n) := p \text{ s.t. } D(p) - (n-1) = \sum_{j \neq i} s_j(p)$$

This discrete unit case is clumsy; in the continuous case, the payment for the marginal unit x is

$$P_{-i}(x) := p \text{ s.t. } D(p) - x = \sum_{j \neq i} s_j(p)$$

[5]Ausubel and Milgrom (2004) describes this as one of the most important virtues of Vickrey-type mechanisms. Cramton (2002) suggests that one advantage that uniform price auctions have over discriminatory auctions is that the former are easier for small firms to participate in. Lower participation costs lead to greater participation, especially by small bidders. In general this leads to better efficiency and revenues.

and the total payment is

$$V_i(q_i^*) = \int_0^{q_i^*} P_{-i}(x)\ dx$$

In particular, $P_{-i}(x)$ is decreasing in x, so that each firm is paid more for its first unit of production than for its last; $P_{-i}(x) > p^* \forall 0 \leq x < q_i^*, P_{-i}(q_i^*) = p^*$, so that units are not sold at anything like a uniform price; and $P_{-i}(x) \neq P_{-j}(x)$ in general, so firms do not necessarily receive the same payment for equal production.

This auction has very attractive properties. As in the second-price auction, truthful bidding is (at least weakly) an equilibrium strategy, because each firm's own bid enters in the calculation made to determine quantity q_i^* but not in the calculation of the payment $V_i(q_i^*)$. The outcome is the efficient one, since the equilibrium quantity and price are determined from truthfully reported aggregate supply and demand curves. Moreover, as noted earlier, the mechanism eliminates the incentive for participants to invest in acquiring information about the rest of the market.

Despite these promising qualities, Vickrey auctions are very rare in practice, except in the trivial case of conventional auctions in which bidders need not bid more than a small increment higher than the next highest bidder.[6] Rothkopf (2007) gives a laundry list of arguments for the notion that the Vickrey-Clarke-Groves process (a generalization of the auction described in Vickrey (1961) is impractical. Two stand out as the most compelling in the homogenous goods case: "Revenue Deficiency" and "Weak Equilibria."

One of the biggest problems with the Vickrey auction is apparent from the statement above that $V_i(q_i^*) > q_i^* \cdot p^*$, so that even if demand is competitive and the total receipts from the sale of q_i^* units are $q_i^* \cdot p^*$, the auction is revenue deficient. In some two-sided markets firms on the demand side also have market power, so that their 'Vickrey payments," $V^j(q_i)$, are strictly less than $q^{i*} \cdot p^*$, and thus increase the deficiency. As Ausubel and Milgrom

[6] eBay Inc. offers something very close to a truthful-revelation implementation of the second price auction in some ascending internet auctions.

(2004) note, perhaps unnecessarily, revenue is one of the primary goals of an auction. Market power is undesirable because it leads to deadweight loss. If the auctioneer is assumed to be operated by, or at the behest of, the government, so that revenue deficiencies which have to be made up by the auctioneer necessarily bring on taxation, then a mechanism which requires subsidies of potentially unbounded magnitude (i.e. could total the entire size of the market) cannot be celebrated for eliminating deadweight loss; the former can amount to the entire size of the market, while the latter is necessarily a fraction of that market. For example, if there are 5 firms each of which is capacity constrained at 10 units of production - i.e. can produce up to 10 units at a low cost but can only produce the 10th to 13th units at arbitrarily high costs - and demand is perfectly inelastic at 60, then each firm receives arbitrarily high payments, inasmuch as $P_{-i}(5)$ is arbitrarily large. If demand is more elastic then this problem is smaller, but so too is the deadweight loss associated with imperfect competition. One way to understand the revenue deficit is that a monopolist seller facing competitive demand captures the entire surplus from trade; meanwhile, each buyer pays just the market clearing price, so that the deficit is the entire consumer surplus.

The second problem is related to this revenue deficiency. It is that the equilibrium strategies are only very weakly incentivized "away from the market clearing price." If firms have a reasonably good picture of the market, i.e. have reasonably tight bounds on both their own equilibrium quantities and the market clearing price, they can freely manipulate their supply curves away from the relevant regions. The second price auction is revenue equivalent to other efficient mechanisms, and technically the Vickrey auction for homogenous goods shares this property: truthful bidding is an equilibrium, and that equilibrium is revenue maximal among efficient mechanisms.[7] Levin and Skyrzypacz (2016) criticize combinatorial clock auctions, one common dynamic implementation of Vickrey auctions, as allowing other inefficient equilibria and smaller-than-expected payments. Their observation is that when bids which set prices do not affect allocations, as is the case in the last round of the combinatorial clock auction, there is room for a broad set of equilibria other than the truthful

[7]Ausubel and Milgrom (2004).

one.[8] In the combinatorial clock auction the final round of bidding explicitly cannot change the allocation resulting from the auction,[9] while in Vickrey's original formulation there is a single supply function bid which determines both the allocations and payments. But if firms are reasonably certain about market conditions, so that they know with some confidence what the clearing price and their own quantity will be, some regions of the supply function bids they submit will with near certainty not change the allocation while still affecting the prices faced by their opponents. In the preceding example, five capacity constrained firms extract large revenues because their capacity constraint was coincidentally near their equilibrium quantity, the residual supply curve was very inelastic, and consequently the residual price would be as large as possible if any firm restricted production.[10] Levin and Skyrzypacz suggest that in many situations, five such firms which are not genuinely capacity constrained could (nearly) risklessly collude and bid as if they were constrained, raising total revenues to these producers. In particular, if firms "know" that they will not be assigned more than $\overline{q_i}$ units in any equilibrium they can fabricate a capacity constraint at $\overline{q_i}$.

In this paper, I do not formally discuss how firms come to know things like $\overline{q_i}$, but it is clear that the prize for collusion in the Vickrey auction is potentially very large. Absent other regulatory mechanisms there is a low risk to this kind of misrepresentation. One aggressive regulatory tactic is to encourage entry by a new firm (possibly one operated by the regulating body itself) into a given market. If firms are colluding in a Cournot game, entry by a new firm drives down the market price and likely leaves the would-be colluders producing less

[8]In the more general setting considered by those authors, those equilibria include some which are inefficient. Ausubel (2004) proposes an ascending auction which is always efficient in the homogenous goods case, so that the inefficient-equilibrium criticism is not fully applicable to that case, but in footnote 27 of their paper Levin and Skyrzypacz suggest that these auctions are still vulnerable to revenue reduction resulting from dishonesty.

[9]This last round of bidding is linked to previous rounds through activity rules, but these implementation details are tangential to this paper

[10]Recall that the Vickrey payment is $\int_0^{q_i^*} P_{-i}(x)\,dx$. $P_{-i}(x)$ is bounded above by $D^{-1}(\sum_{j\neq i} q_j^* + x)$, but is typically general smaller if competing firms increase their quantity in response to the reduced competition from firm i. If firms are capacity constrained around q_j^*, i.e. if $(s^j)^{-1}(q_j^* + \delta)$ is very large relative to δ, then competing firms do not increase their quantity in response to a reduction in q_i and the upper bound is tight.

than their optimal quantities. In this Vickrey setting, however, the new entry amounts to a surprise downward shock in residual demand, shifting the colluding firms towards lower quantities. But those firms misrepresented their ability to supply *greater* quantities, so that the resulting equilibrium would still be the post-entry efficient. In other words, collusion is potentially profitable and less prone to punishment by a regulator.[11]

Vickrey auctions can be expensive (revenue deficient) for two reasons. The payments above the market clearing price are perfect compensation for market power, and if firms in a market have significant market power, as in the case when demand is inelastic and firms are capacity constrained, then these payments will naturally be quite high. This case may be hopeless from an efficiency standpoint; if true market power is large then these payments might be unreasonably large relative to the deadweight loss due to uniform price auctions. But even if true market power is not very large, firms may be able to cheaply misrepresent their supply curves and inflate payments to other firms on the same side of the market, collectively earning profits as if true market power were large while taking on little risk. Vickrey auctions are also more efficient than uniform price auctions, and the natural question is whether a more gentle tradeoff between efficiency and revenue deficiency exists.

One purpose of the present paper is to suggest an auction which is "in between" the uniform price auction and the Vickrey auction; given some reasonable assumptions, this compromise is more efficient than the uniform price procedure, has limited revenue deficiency, and avoids reliance on the "non-local" properties of firms' bids in determining the size of payments to market participants. A starting point is to recast the uniform price auction with supply function bids[12] as a first order approximation of the Vickrey auction given firms with limited market power. Next, I will consider the properties of the corresponding second order approximation. Where possible I will give a characterization of its properties; the discussion

[11]A regulator which arbitrarily eliminated an existing firm rather than introducing a new one would make the collusion more costly to the remaining firms in that the resulting allocation would be inefficient. This kind of action from a regulator seems unlikely since the elimination of an existing firm makes the market less competitive and therefore increases the revenue deficiency.

[12]The one considered in Klemperer and Meyer (1989).

is necessarily conjectural, though promising. Finally, I will discuss the implementation of this second order approximation.

3. Local Approximations to the Vickrey Auction

3.1. **Taylor Series.** From our notation above, each firm i receives a Vickrey payment

$$V_i(q) = \int_0^q P_{-i}(x)\ dx$$

We can take a Taylor expansion of V_i around $q = 0$:

$$V_i(q) = V(0) + q \cdot V'(0) + \frac{q^2}{2!} \cdot V''(0) + \frac{q^3}{3!} \cdot V'''(0) + \ldots$$

We should certainly have $V(0) = 0$, and applying the fundamental theorem of calculus this is

$$V(q) = q \cdot P_{-i}(0) + \frac{q^2}{2!} \cdot P_i'(0) + \frac{q^3}{3!} \cdot P_i''(0) + \ldots$$

It would be nice to say that the first term in this series is $V(q) = q \cdot P_{-i}(0) + \ldots$ and stop there. But $P_{-i}(0)$, the prevailing market price when firm i produces 0, depends critically on the very "non-local" properties of other firms' supply curves that we hope to avoid. We would like to replace $P_{-i}^{(m)}(0)$ by $P_{-i}^{(m)}(q)$ everywhere.

To do that we take the Taylor expansion for $P_{-i}^{(m)}(x)$ around $x = q$, e.g.

$$P_{-i}(x) = P_{-i}(q) + (x - q) \cdot P_{-i}'(q) + \frac{(x-q)^2}{2!} \cdot P_{-i}''(q) + \ldots$$

and then evaluate these expansions at $x = 0$. So we have:

$$P_{-i}(0) = P_{-i}(q) - q \cdot P_{-i}'(q) + \frac{q^2}{2!} \cdot P_{-i}''(q) + O(q^3)$$

$$P_{-i}'(0) = P_{-i}'(q) - q \cdot P_{-i}''(q) + O(q^2)$$

Etcetera.

Consequently we can rewrite

$$V_i(q) = q \cdot (P_{-i}(q) - q \cdot P_{-i}'(q) + \frac{q^2}{2!} P_{-i}''(q) + O(q^3))$$

$$+ \frac{q^2}{2!} \cdot (P_{-i}'(q) - q \cdot P_{-i}''(q) + O(q^2)) + \frac{q^3}{3!} \cdot (P_{-i}''(q) + O(q)) + \ldots$$

$$\implies V_i(q) = q \cdot P_{-i}(q) - \frac{q^2}{2!} \cdot P'_{-i}(q) + \frac{q^3}{3!} \cdot P''_{-i}(q) + O(q^4)$$

3.2. **First Order Approximation: Uniform Price Auction.** Here we have our first result: the first order approximation of $V_i(q)$ is

$$V_i^1(q) = q \cdot P_{-i}(q)$$

which is recognizable as the standard uniform-price-auction payment. As we know, the first order condition is

$$(\pi_i^1)' = (V_i^1)'(q) - C_i'(q) = 0 \implies q \cdot P'_{-i}(q) + P_{-i}(q) - C_i'(q) = 0$$

$$\implies P_{-i}(q) - C_i'(q) = -q \cdot P'_{-i}(q)$$

$C_i'(q)$ is marginal cost, so a more suggestive phrasing of this equation is

$$P - MC = -qP'_{-i}(q)$$

The deadweight loss in this auction is clearly increasing in the incentive to misreport marginal costs, which here is $-qP'_{-i}(q)$, but the nature of the relationship between that term and the deadweight loss is not immediately clear; discussion follows in section 3.4.

3.3. **Second Order Approximation: Quadratic Payments.** The second order approximation of $V_i(q)$ is

$$V_i^2(q) = q \cdot P_{-i}(q) - \frac{q^2}{2!} \cdot P'_{-i}(q)$$

Recall that $P'_{-i}(q) < 0$ and is increasing in magnitude in the elasticity of other firms' supply and decreasing in magnitude in the elasticity of demand. Before, we took $q_i \cdot P'_{-i}(q_i)$ to be a rough measure of firm i's market power. Interpreted in this light, we can write

$$V_i^2(q) = q[P_{-i}(q) - \frac{q}{2} \cdot P'_{-i}(q)]$$

So the per-unit payment a firm receives is the "market-clearing price" plus a bonus proportional to the firm's market power. This market-clearing per-unit price is the one faced by a firm with no market power, such as one with arbitrarily small quantity.

With this pricing scheme, each firm faces FOC

$$(\pi_i^2)'(q) = P_{-i}(q) + q \cdot P'_{-i}(q) - q \cdot P'_{-i}(q) - \frac{q^2}{2} \cdot P''_{-i}(q) - C'_i(q) = 0$$

$$\implies P_{-i}(q) - C'_i(q) = \frac{q^2}{2} \cdot P''_{-i}(q)$$

It follows that the comparable incentive to misreport marginal costs is $\frac{q^2}{2} \cdot P''_{-i}(q)$.[13] For the common functional form of linear demand and quadratic costs it can be easily shown that $P''_{-i}(q) = 0$, so that truthful reporting is an equilibrium. It is plausible that the quadratic payments lead to more efficient equilibria in more general cases, although we will see that this statement amounts to some numerical restrictions on the supply and demand curves.

3.4. **Deadweight Loss Considerations.** How can we compare deadweight loss under uniform and quadratic payments when we cannot pin down precise equilibrium prices and quantities? Informally, the Klemperer and Meyer result is that in a uniform price auction with uncertainty about the level of demand, firms submit supply curves that are shifted to the right by the expected value of their "incentive gap" $P - C'_i = -q \cdot P'_{-i}(q)$ over some distribution of the unknown component of demand. Inasmuch as the expectation and precise equilibrium bidding strategies cannot be computed, it is useful instead to focus on the relative size of $P - C'_i$ under the uniform and quadratic schemes.

Suppose that we did know the equilibrium bidding strategy of each firm i under each payment policy k[14] producing some $P - MC$, so that firm i submits a curve $(p, s_{i,k}(p))$ with

$$s_{i,k}(p) - (C'_i)^{-1}(p) = g_{i,k}(p)$$

the rightward shift of firm i's supply curve at point p. Then at a point p aggregate supply is shifted to the right by $\sum_i g_{i,k}(p)$. If aggregate supply and demand are linear,[15] deadweight

[13]It seems plausible that this is a smaller incentive to dissemble than $q \cdot P'_{-i}(q)$, especially if q is small or P_{-i} is well behaved. The Klemperer and Meyer approach shows that classifying demand and supply forms for which this statement is true requires some work.

[14]$k = 1$ is the uniform price auction, $k = 2$ is the quadratic payment scheme.

[15]This is the case when costs are quadratic and demand is linear.

loss is a triangle with

$$DWL = \frac{1}{2} \cdot [\text{Horizontal Shift in Aggregate Supply}]^2 \cdot [S'(P^*)^{-1} - D'(P^*)^{-1}]^{-1}$$

$$= \frac{1}{2}[\sum g_{i,k}(p)]^2 \cdot [S'(P^*)^{-1} - D'(P^*)^{-1}]^{-1}$$

Here we can write that P^* is the efficient price, although the assumption of linearity means that S' and D' are constant functions. If supply and demand are not linear then we can still write the approximation

$$DWL_k \approx \frac{1}{2} \cdot [\sum g_{i,k}(p)]^2 \cdot [S'(P^*)^{-1} - D'(P^*)^{-1}]^{-1}$$

This approximation is worse if S' and D' are very different at the supply function equilibrium price and quantity, but S' and D' do not depend in any way on the payment policy or bidding strategies of the firms. All things equal, then, the ratio of deadweight loss under two payment policies 2 and 1 is

$$DWL_2/DWL_1 \approx \frac{[\sum_i g_{i,2}(p)]^2}{[\sum_i g_{i,1}(p)]^2}$$

The Klemperer and Meyer result is that each $g_{i,k}$ is something like the average incentive gap, so a rough measure of the reduction in deadweight loss is the square of the ratio of the incentive gap faced by each participant

$$(\frac{(P-C')_2}{(P-C')_1})^2$$

In particular, in section 4.2 we will write that policy 2 is more efficient than policy 1 if

$$|\frac{(P-C')_2}{(P-C')_1}| < 1$$

4. Properties of the Second-Order Approximation (Quadratic Payments)

Quadratic payments of this sort certainly solve the problem of the Vickrey auction's reliance on weak-equilibrium bids. The auctioneer need only estimate $P'_{-i}(q)$ for each participant, and it is the result of a straightforward calculation in section 4.1. that the auctioneer only needs estimates of the level and slope of each bidder's supply function (inverse marginal cost curve) at their assigned quantity. Unlike the first order approximation, which was the revenue neutral uniform price auction, the quadratic payments are revenue deficient. Section 4.3. will introduce some reasonably simple bounds on the total payments made in this system.

4.1. **Two Derivatives of P_{-i}.** In notation resembling that in Klemperer and Meyer, let s^i be the supply curve submitted by firm i, where a point on this curve $p, s^i(p)$ represents a willingness to sell $s^i(p)$ units when the market clearing price is p.[16] The market clearing condition is

$$D(p) = \sum_i s^i(p)$$

The residual market price function is defined by

$$D(P_{-i}(q)) - q = \sum_{j \neq i} s^j(P_{-i}(q))$$

Taking a derivative implicitly, we have

$$P'_{-i}(q) \cdot D(P_{-i}(q)) - 1 = P'_{-i}(q)[\sum_{j \neq i} (s^j)'(P_{-i}(q))$$

$$\implies P'_{-i}(q) = [D'(P) - \sum_{j \neq i} (s^j)'(P)]^{-1}$$

Recall that $P'_{-i}(q) < 0$, which corresponds to both $D'(P) < 0$ and $(s^j)'(P) > 0$ because demand is assumed to be decreasing and marginal costs to be increasing. As stated informally before, P'_{-i} is large when demand and supply are inelastic ($D'(P)$, $(s^j)'(P)$ close to 0, respectively).

We can take a second derivative to get

$$P''_{-i}(q) \cdot D(P_{-i}(q)) + P'_{-i}(q)^2 \cdot D''(P_{-i}(q)) = P''_{-i}(q)[\sum_{j \neq i} (s^j)'(P_{-i}(q))] + P'_{-i}(q)^2[\sum_{j \neq i} (s^j)''(P_{-i}(q))]$$

$$\implies P''_{-i}(q)[D(P) - \sum_{j \neq i} (s^j)'(P)] = P'_{-i}(q)^2[\sum_{j \neq i} (s^j)''(P) - D''(P)]$$

$$\implies P''_{-i}(q) = P'_{-i}(q)^3[\sum_{j \neq i} (s^j)''(P) - D''(P)]$$

[16]In the uniform price setting, the interpretation of the bid as a willingness to sell $s^i(p)$ units for the unit price of p is more straightforward, but in the quadratic payment setting this is not correct.

4.2. **Comments on the Efficiency of Quadratic Payment Equilibria.** To reiterate, in a departure from the analytic tractability of Bertrand and Cournot competition, supply function bidding prevents us from writing down most equilibria under quadratic payments explicitly. Nonetheless, quadratic payments lead to an efficient equilibrium in one special case and there are several observations which suggest that quadratic payments often lead to more efficient equilibria than the uniform price auction.

Following the discussion in section 3.4, one expression which is almost equivalent to quadratic payments leading to a more efficient equilibrium than uniform pricing is[17]

$$\left|\frac{P - C_i' \text{ under quadratic payments}}{P - C_i' \text{ under uniform pricing}}\right| < 1$$

Note that the sign of this ratio is in general uncertain. The quadratic reward is quite large and is proportional in part to the steepness of the residual supply curve. If firm i can shift its opponents dramatically toward some steeper part of their supply curve by increasing q then firm i might overproduce (i.e. set price below marginal cost) to increase the quadratic portion of its payment.

Formally (denoting $\sum_{j\neq i} s_j = S_{-i}$) the ratio is

$$\frac{\frac{q^2}{2}\cdot P''_{-i}(q)}{-q\cdot P'_{-i}(q)} = -\frac{q}{2}(P'_{-i}(q))^2\cdot[S''_{-i}(p) - D''(p)]$$

$$= \frac{-qP'_{-i}(q)}{2}\cdot\frac{S''_{-i}(p) - D''(p)}{D'(p) - S'_{-i}(p)}$$

Here I can make several suggestive comments about the magnitude of this expression. First, $-qP'_{-i}(q)$ is the "amount of market power" in the uniform price auction, so for reasonably competitive markets (i.e. those in which firms have only a little market power)

[17]As before, certainty requires that we take an expectation over each firm's uncertainty of the level of demand. My discussion has almost entirely elided the nature of this uncertainty. Still, in the expressions that follow it is natural to think of this expected value as being over "reasonable values of the market price" given a firms' production. If each firm is not too powerful, and/or demand is reasonably elastic, it seems plausible that this expectation is not that different from a point estimate. Put differently, the range of reasonable market prices is not that wide.

$-qP'_{-i}(q)2$ will be fairly small. Second, as noted before, the sign of $\frac{S''_{-i}(p)-D''(p)}{D'(p)-S'_{-i}(p)}$ is indeterminate. We know that $-qP'_{-i}(q) > 0$ and $D'(p) - S'_{-i}(p) < 0$ but $S''_{-i}(p) - D''(p)$ is an unfamiliar expression; we can certainly come up with reasons for supply and demand to be convex or concave. One optimistic (but exaggerated) interpretation of this ambiguity is that it contributes to the possibility that the quadratic payments are perfectly efficient; if firms themselves are uncertain about whether they should submit supply curves above or below their true marginal costs, then they might submit the truth. A more substantial version is the following result:

If oligopolistic firms face linear, downward sloping demand and quadratic costs (so, linear marginal costs) then it is an equilibrium in the quadratic payments auction for the firms to submit their true cost curves.

Proof: If all of the firms $j \neq i$ submit linear marginal costs then $s_j(p)$ is linear $\implies S''_{-i}(p) = 0$ uniformly. Similarly linear demand implies $D''(p) = 0$. So firm i's first order condition is

$$P_{-i}(q) = C_i'(q) - \frac{q^2}{2}P''_{-}(q) = C_i'(q)$$

and consequently reports its true inverse marginal cost curve.[18]

Klemperer and Meyer show that there is a unique equilibrium in the uniform price auction in this case and that it is (obviously) not the efficient one. We might hope that this generalizes in some natural way to firms with piecewise linear supply curves and linear demand; it strongly suggests that supply and demand which are "almost linear," i.e. which have very small second derivatives, result in nearly efficient quantities.[19]

4.3. **Magnitude of Revenue Deficiency.** In some cases quadratic payments have an equilibrium with socially efficient quantities, and there is suggestive evidence that there are equilibria in more general cases which are reasonably efficient (i.e. more efficient than the uniform

[18]A natural question is whether we can show that this is the *unique* equilibrium under quadratic payments.

[19]There are many extensions to consider here. Suppose firms are restricted to a small, finite number of line segments to describe their supply curves, as might be realistic in e.g. electricity auctions to reduce computational complexity. Is there a clear result in this case? Again, these are questions for future work.

price equilibrium). Efficiency is the "good" property of Vickrey auctions. The "bad" property is that the cost of that efficiency is potentially very high, even in the absence of collusive behavior. In return for the residual inefficiency we might hope that we can place bounds on the revenue deficit under quadratic payments. I give two approaches to this problem in the symmetric case.

In the previous section we saw that

$$P'_{-i}(q) = [D'(P) - \sum_{j \neq i} (s^j)'(P)]^{-1}$$

Suppose there are n symmetric firms. Then

$$P'_{-i}(q) = [D'(P) - \frac{n-1}{n} \cdot (n(s^j)'(P))]^{-1}$$

One approach is to argue that

$$\frac{\partial P_{-i}}{\partial q} \cdot \frac{Q}{P} = \frac{1}{\epsilon_{RD}}$$

is itself a meaningful inverse elasticity - the "inverse elasticity of residual demand." Prete and Hobbs (2015) attempts to compute the absolute value of this inverse elasticity[20] for electricity markets in California between 1998 and 2000 in two ways, and finds that the monthly moving average is often close to 0 and never greater than 3.5, with yearly averages ranging from 0.135 to 1.345. A caveat for using these estimates is that the bidders submitted supply schedules in a uniform price auction, so they are somewhat distorted from what we would see under quadratic payments; still it is not unreasonable to think of these numbers as being of the right magnitude.

A second approach is to bound $P'_{-i}(q)$ in terms of the inverse elasticities of supply and demand. Note that we can define $P'_0(q) = [D'(P) - \sum (s^i)'(P)]^{-1}$ with $P'_0(q) \leq \frac{n}{n-1} \cdot P'_{-i}(q)$ (recall that all of these are negative) so that

$$-q_i^2 \cdot P'_{-i}(q) \leq -q_i^2 \cdot P'_0(q) \cdot \frac{n}{n-1}$$

[20]The discussion here elides over the difference between $\frac{Q}{P}$ and $\frac{Q-q_i}{P}$, but there is little reason to think that this distinction meaningfully affects the conclusion.

By the AM-HM inequality

$$-[D'(P) - S'(P)]^{-1} \leq -[\frac{\frac{1}{D'(P)} - \frac{1}{S'(P)}}{4}]$$

$$\implies -P_0'(q) \cdot \frac{Q}{P} \leq -[\frac{\epsilon_D^{-1} - \epsilon_S^{-1}}{4}]$$

There is nothing special about this ratio $\frac{n}{n-1}$ and the assumption of symmetry; if firms are broadly similar we can make looser estimates. This approach is of course limited if exactly one of the firms is too powerful, e.g. if there is a single firm with (locally) almost perfectly elastic supply and other inelastic firms. If this is unlikely, as seems reasonable in many functional forms (and is certainly implied by symmetry) then we can bound the subsidy paid to each firm i uniformly by

$$-q_i^2 \cdot P_0'(q) \cdot \kappa$$

for some "dissimilarity parameter" κ (again, in the case of symmetry, $\kappa = \frac{n}{n-1}$).

Then we have two upper bounds for the total subsidy paid, using either

$$-P_0'(q) \cdot \frac{Q}{P} = -\frac{1}{\epsilon_{RD}} \text{ or } -P_0'(q) \cdot \frac{Q}{P} \leq -[\frac{\epsilon_D^{-1} - \epsilon_S^{-1}}{4}]$$

The subsidy is bounded above by

$$-\frac{1}{2}\sum_i q_i^2 \cdot P_0'(q) \cdot \kappa = -\frac{\kappa}{2}\sum_i q_i^2 \cdot \frac{P}{Q^2} \cdot P_0'(q) \cdot \frac{Q}{P}$$

Taking $H = \sum q_i^2/Q^2$ to be the Herfindahl-Hirschman Index,

$$\text{Total subsidies} \leq \frac{\kappa}{2} \cdot H \cdot (P \cdot Q) \cdot [-P_0'(q) \cdot \frac{Q}{P}]$$

Now $P \cdot Q$ is the size of the entire market; κ is a "dissimilarity parameter" which is reasonably close to 1 if firms are similar and there are at least 3 of them; H is the HHI; and $P_0'(q) \cdot \frac{Q}{P}$ can be bounded by inverse supply and demand elasticities or estimated outright as above.

Our motivation was a concern that these subsidies might be large relative to the market. For reasonable values of the HHI and inverse elasticities, however, this is clearly not the case. For example,

$$|\frac{\kappa}{2} \cdot H \cdot [P_0'(q) \cdot \frac{Q}{P}]| = 0.12 \cdot [P_0'(q) \cdot \frac{Q}{P}]$$

in the case of a 5-firm symmetric oligopoly (roughly similar to the electricity market considered by Prete and Hobbs (2015)). Using the Prete and Hobbs estimates for the inverse elasticity $|\frac{1}{\epsilon_{RD}}|$,[21] we get typical yearly subsidies consisting of less than 10% of the total market size.

On the other hand, using the inverse elasticity bound and assuming $|\epsilon_D|, |\epsilon_S| \geq \frac{1}{2}$ then the subsidy is no larger than $0.5 \cdot \kappa \cdot H$. In symmetric oligopolies with $n \in [5, 10]$ we get an upper bound on the size of the revenue deficiency between 5% and 12% of the total market.

5. Additional Considerations for Quadratic Payments

5.1. **Implementation Details.** In theory, subtly different auction formats can lead to different equilibria. An area of active research is whether this theory is reflected in real auction outcomes, and consequently how large of an impact auction design choices make in real marketplaces. Fabra et al. (2006) is pessimistic about the relevance of the supply function equilibrium considered by Klemperer and Meyer (1989) to electricity markets in practice, because in all real-world markets bidders are restricted to a finite number of bids. Essentially all practical implementations of auction mechanisms restrict bids to finitely many price-quantity pairs, either by insisting on discrete prices (minimum tick sizes in treasury auctions), by limiting the number of bids per bidder (as in electricity auctions, according to Fabra et al. (2006)), or some combination of the two. A potential criticism, then, is that a supply function bidding version of the quadratic equilibrium is unrealistic and the inefficiency which is combated by the quadratic payment does not arise in other settings.

If bidders are able to submit fully flexible supply function curves as bids then there is no problem. The format of firms' bids is identical as under the uniform price auction of the same type and the auctioneer observes s'_i, D' directly from those bids and can use them to determine payments.

[21]See Table 2 in the Prete and Hobbs paper; this is really a back of the envelope calculation which is at best suggestive.

Suppose instead the auctioneer restricts - or permits - the bidders to submit much less information than a full supply curve.[22] Unlike in a uniform price auction, the auctioneer must estimate $s_i'(p)$ at the market clearing price, so in general might solicit different kinds of bids for the quadratic payment mechanism than she would in a uniform price auction.

One approach is as follows. The auctioneer commits to a starting price p_0 and a price increment Δ. Then the auctioneer solicits bids consisting of pairs $s_j(p_m), s_j'(p_m)$ at each $p_m = p_0 + m \cdot \Delta$. The auction would conclude when supply and demand cross, with the equilibrium quantities determined by both components of the bids. WLOG, assume $\Delta > 0$, so that this is an ascending auction. Then the auction terminates at p_n for which $\sum s_i(p_n) > D(p_n)$ but $\sum s_i(p_{n-1}) < D(p_{n-1})$. In this setting, the auctioneer has a linear approximation of the supply curve: it is the line passing through $(p_n, \sum s_i(p_n))$ with positive slope $\sum s_i'(p_n)$; moreover she knows the true shape of downward-sloping demand (or has an equivalent linear approximation). These two lines determine a unique clearing price and quantity. This approach is flexible enough to accommodate ascending and descending auctions and allows firms to submit directly their elasticities. One disadvantage is that the equilibrium price calculated from the linear approximation(s) of supply (and demand) may be below p_{n-1}; the auctioneer could implement constraints on the magnitude of s_i' to mitigate this problem.

A closely related approach is to take bids consisting of just the level of supply at each p_m, and then to estimate $s_i'(p_m)$ by its left approximation, i.e. $\frac{s_i(p_{m-1})-s_i(p_m)}{p_{m-1}-p_m}$. This avoids any need for activity rules.

The pressing question about these ascending auctions is the one raised by Fabra et al., namely, whether they preserve the equilibrium properties of the supply function bidding

[22] The motive behind such a restriction is usually computational. In some electricity markets, for example, the clearing process involves an algorithm whose running time is rapidly increasing in the number of bids. An alternative solution would be for firms to face a cost which is increasing in the complexity of their bid which could represent bid preparation costs or a cost imposed by the auctioneer who want to limit the complexity of her role in the market.

implementation. There are two reasons to believe the affirmative answer. First, recent laboratory experiments described in Brandts et al. (2014) suggest that market power appears in a way that more closely resembles the supply function equilibrium from Klemperer and Meyer than the equilibria predicted by Fabra et al. (2006) and later work, in spite of the theoretical problems posed by non-continuous bidding functions. Second, the thrust of the criticism is that with finitely many bids, competition takes on a Bertrand flavor; except at finitely many jumps, quantities do not vary with the clearing price. In these implementations of the quadratic payment auction, however, both quantities and prices are affected at the margin by firms' bids.

5.2. **Bidder Participation Costs.** Recall that a most attractive property of the Vickrey auction is that honest reporting is an equilibrium strategy and therefore it is easy to be a bidder: firms have no incentive to investigate market conditions if they know their own supply or demand curve and therefore are willing to bid even if they cannot invest in developing a strategy. It is unclear whether that is true of the quadratic payments auction. This paper has offered suggestive (and in some cases conclusive) evidence that bidding honestly is a reasonable strategy for participants in such an auction, but in general truthful reporting is not an equilibrium; moreover the payment scheme is more convoluted than a uniform price auction, so it may not even be the case that these auctions are more attractive to bidders than ones offering uniform prices. If a significant portion of the efficiency gains from a Vickrey auction come from attracting more bidders, then, this second-order approximation may not be viable.

6. Conclusion

In uniform price auctions, firms with market power buy or sell less than their efficient quantity in order to increase their expected profitability. The first-order correction of this market power is a per-unit payment proportional to a firm's market power; the total payment received or extracted from each firm is therefore a quadratic function of the equilibrium quantity allocated to that firm. Uniform price payments and quadratic payments are first- and second-order approximations, respectively, of an auction described in Vickrey (1961) which perfectly mitigates market power, albeit at potentially high cost in terms of revenue. At least in some common cases, quadratic payments lead to auctions which are more efficient than ones with uniform prices. The revenue deficiency resulting from those payments is not unboundedly large, although the appendix shows that it may be larger in magnitude than the increase in efficiency. Vickrey imagines a government or other auctioneer "counterspeculating" to combat firms with market power. Even a first-order "counterspeculation" can be expensive, so that the inefficiency of taxation to fund any such project is likely a critical consideration. That may be one reason that this kind of revenue-deficient mechanism is rare or even nonexistent in practice.

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7. Appendix: Linear Demand and Quadratic Cost

With the strong assumption of linear demand and quadratic costs, it is feasible to specify equilibrium bids, quantities and payments. Quadratic costs imply linear supply function bids, and it was shown in section 4.2 that this supply and demand structure leads to an equilibrium in which firms bid truthfully.

Suppose there are n identical firms with costs $C_i(q) = c \cdot \frac{q^2}{2}$ facing demand $D(p) = \tilde{A} - B \cdot p$, where $\tilde{A}$ is drawn from some distribution.[23]

Each firm has, and therefore bids, inverse marginal cost curve $s_i(p) = \frac{p}{c}$; each firm faces $S_{-i}(p) = \frac{(n-1)p}{c}$. The equilibrium price solves $\tilde{A} - B \cdot p = \frac{np}{c}$

$$\implies p^* = \frac{c\tilde{A}}{n + cB},\; q_i^* = \frac{p^*}{c} = \frac{\tilde{A}}{n + cB}$$

Inverse demand is

$$P(q) = \frac{\tilde{A} - q}{B}$$

The total surplus from trade is

$$[\int_0^{nq_i^*} \frac{\tilde{A} - q}{B}\, dq] - n \cdot C_i(q_i^*) = nq_i^* \cdot \frac{\tilde{A}}{B} - \frac{(nq_i^*)^2}{2B} - \frac{c \cdot n}{2} \cdot (q_i^*)^2$$

$$= \frac{1}{2B}[2 \cdot n \cdot q_i^* \cdot \tilde{A} - n(n + c \cdot B)(q_i^*)^2] = \frac{1}{2B}[2 \cdot n \cdot q_i^* \cdot \tilde{A} - n \cdot q_i^* \cdot \tilde{A}]$$

$$= \frac{1}{2B}[n \cdot q_i^* \cdot \tilde{A}]$$

The quadratic payments are proportional to $-P'_{-i}(q_i) = [D'(p^*) - \sum_{j \neq i} (s^j)'(p^*)]^{-1}$

$$= -[-B - \frac{n-1}{c}]^{-1} = \frac{1}{B + \frac{n-1}{c}} = \frac{c}{(n-1) + cB} = \frac{n + cB}{(n-1) + cB} \cdot \frac{p^*}{\tilde{A}}$$

Total quadratic payments are therefore

$$(q_i^*)^2 \cdot n \cdot \frac{n + cB}{(n-1) + cB} \cdot \frac{p^*}{\tilde{A}}$$

For comparison, symmetric firms under Cournot solve

$$P(q) + P'(q) \cdot q = MC_i(q)$$

[23] As in Klemperer and Meyer, if firms know A, the level of demand, then there is no reason for them to submit elastic curves; but as in that paper we assume that firms know B, the slope of demand, perfectly

$$\implies (\frac{\tilde{A} - nq}{B}) - \frac{1}{B} \cdot q = cq \implies q_i^C = \frac{\tilde{A}}{n + cB + 1}$$

Klemperer and Meyer show that the quantity produced under the supply function equilibrium is bounded below by the Cournot quantity, so we can obtain an upper bound on the efficiency gain from moving to the efficient quantity under the quadratc payment auction as

$$\int_{nq_i^C}^{nq_i^*} \frac{\tilde{A} - q}{B} \, dq - n[C_i(q_i^*) - C_i(q_i^C)]$$

Demand elasticity is

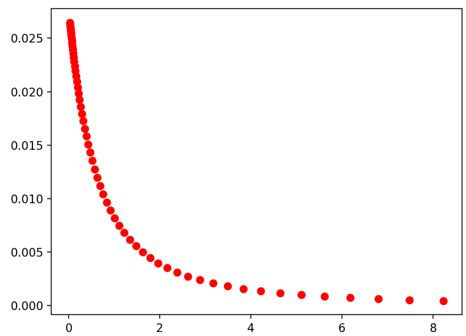
$$\frac{\partial D(p)}{\partial p} \cdot \frac{p^*}{Q^*} = -B \cdot \frac{p^*}{nq^*} = -\frac{cB}{n}$$

so it is sufficient to fix A, c, n and vary B to see how large the payment and efficiency bound is over differently elastic demand.

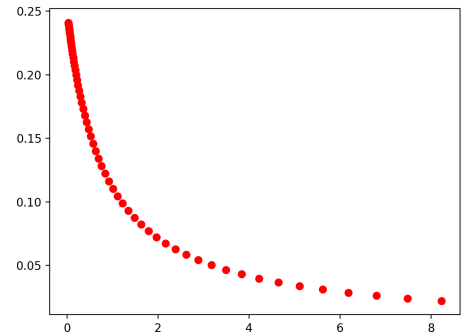
Figure 1 is a plot of the upper bound of the efficiency improvement under the quadratic payment policy divided by the total surplus attained under an efficient allocation with $A = 30, n = 5, c = 1$ and B ranging over $[1.1^{-20}, 1.1^{40}]$. The $x-$axis is the absolute value of the demand elasticity $|-\frac{cB}{n}| = -\frac{B}{5}$. Figure 2 is a plot of the size of the total quadratic payment as a fraction of the total market size $p^* \cdot n \cdot q^*$. Figure 3 is a plot of the quadratic payments as a fraction of the total surplus from trade. Figure 4 is a plot of the quadratic deficit divided by the upper bound of the efficiency improvement.

When demand is very inelastic, the Cournot equilibrium represents a more dramatic departure from the efficient equilibrium in efficiency terms, although given five symmetric firms, the total deadweight loss is quite small (with the most inelastic demand, slightly more than 2.5% of the total gains from trade are lost). Similarly, the revenue deficit from quadratic payments is decreasing in the elasticity of demand, which is reasonable in light of the bound on the size of these payments in terms of inverse elasticities derived in section 4.3.

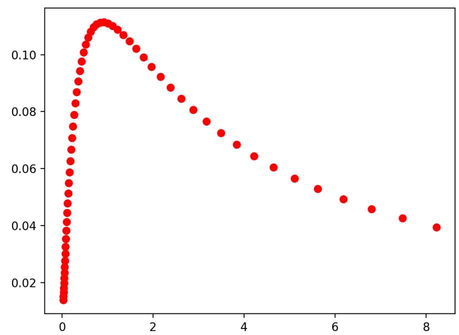
Figures 3 and 4 paint a more interesting picture. If demand is very inelastic, the deficit from the quadratic subsidy may be smaller than the efficiency gain, suggesting that a quadratic payment auction, as described in previous sections, could improve social welfare. In more elastic regions of demand, however, the deficit created is much larger than an upper



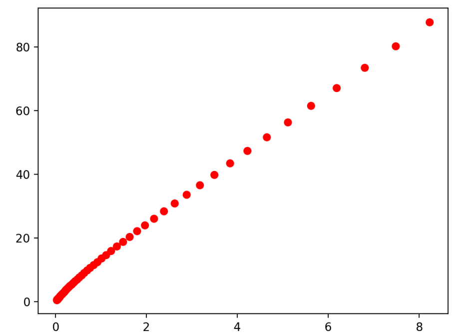
(A) Figure 1: Efficiency gain as a fraction of the total surplus



(B) Figure 2: Deficit from quadratic payments as a fraction of the total market



(C) Figure 3: Deficit as a fraction of the total surplus



(D) Figure 4: Deficit from quadratic payment divided by efficiency gain

bound on the efficiency gained from switching to a quadratic payment auction. Notably, in the extremely inelastic demand cases, quadratic payments lead to a significant transfer from producers to consumers because the market shrinks by much more than the total quadratic payment.