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Conflict Leads to Cooperation in Nash Bargaining

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Abstract

We consider a Nash demand game where N players come to the bargaining table with requests for coalition partners and a potentially generated resource. We show that group learning leads to complete cooperation and an interior core allocation with probability one. Our arguments highlight group dynamics and demonstrate how destructive group behaviors - exclusion, divide and conquer tactics, and scapegoating - can propel groups toward beneficial and self-enforcing cooperation.

Keywords: Nash bargaining, learning, core, group conflict
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1 Introduction

This paper develops a noncooperative model of multilateral bargaining in which group learning leads to convergence of allocations to the subset of the core which is *strictly* self-enforcing. The literature on noncooperative bargaining foundations for the core (which consists of *weakly* self-enforcing allocations) has a rich history, including Chatterjee, Dutta, Ray & Sengupta (1993), Perry & Reny (1994), and Konishi & Ray (2003). In contrast to such papers, which typically examine the equilibria of dynamic bargaining games with forward-looking players, we are interested in the learning process that results from repeated play of a static bargaining game that extends the canonical two-player demand model of Nash (1950).

In our model, N players to come to the bargaining table with demands for both a potentially generated resource and coalition partners. The groups that form must be mutually compatible in terms of resource and partner requests, and their ability to produce the resource is governed by a convex and strictly superadditive characteristic function. By permitting players to include or exclude other players from their coalition, our simple model can capture interesting and realistic group dynamics.

In particular, group settings often display inefficient and destructive behaviors. Individuals can be excluded from groups or steal away other players' partners. Groups can scapegoat a member to make them absorb the impact of a group failure, or take advantage of individuals who are desperate and alone. Individual greed may lead to internal strife, and one group can instigate conflict within another in order to divide and conquer it. The novelty of this paper is the construction of a learning process by which these destructive behaviors act in concert to propel groups towards strictly self-enforcing cooperation. Consequently, this paper may be contrasted with Agastya (1997), which finds convergence to weakly self-enforcing (core) allocations through learning in a demand bargaining model where players can only submit resource requests; coalitions are not modeled and essentially arise out of a "black box" which this paper attempts to open.[1]

[1]A form of coalition selection is permitted by Arnold & Schwalbe (2002), but they restrict the role of group dynamics by assuming that players in blocking coalitions can play randomly (so noncore allocations are immediately unstable) and that groups cannot split (exclusion is not permitted).

2 Nash bargaining with N players

Consider a group $I = \{1, 2, \ldots, N\}$ of $N \geq 3$ players. Denoting by $\mathcal{I}$ the set of all possible coalitions, we describe the resources a particular group may obtain by a convex and strictly superadditive characteristic function $v : \mathcal{I} \to \mathbb{R}$.[2]

In our demand game, players come to the bargaining table with two requests. First, as in the standard two-player Nash demand game, player i requests some amount $d_i \in [v(i), v(I)]$ of the resource for herself. Second, player i also specifies a list of players $P_i \in \mathcal{I}$ with whom she is willing to form a coalition.[3] The list of all resource and partner requests submitted is given by (d, P), where $d = (d_1, d_2, \ldots, d_N)$ and $P = (P_1, P_2, \ldots, P_N)$.

Not every combination of resource and partner requests is feasible. Letting $\Pi(I)$ denote the set of all coalition structures (i.e., partitions of I), a particular coalition structure $\pi \in \Pi(I)$ will be *feasible* if all of its coalitions are *mutually compatible* and *demand-feasible*. Mutual compatibility requires that for each group $S \in \pi$, no member $j \in S$ is excluded from the partner list of some other player in that group (i.e., there is no $i \in S$ such that $j \notin P_i$). Demand-feasibility is the simple condition that for each coalition $S \in \pi$ containing at least two players, the total amount of resource requested, $\sum_{i \in S} d_i$, does not exceed the total amount of resource available, $v(S)$.

We assume that a player i who remains unpartnered in π will receive $v(i)$ in spite of her resource request. Consequently, there is always a feasible coalition structure: namely, the trivial coalition structure where every player is alone. When more than one coalition structure is feasible, we assume that mutually compatible and demand-feasible groups form when possible. Formally, defining the norm $\rho : \Pi(I) \to \{1, 2, \ldots, N\}$ of a coalition structure to be the number of coalitions formed, we assume that the coalition structure that forms is chosen according to a fixed probability distribution $F \in \Delta\Pi(I)$, conditional on the set of feasible coalition structures with minimal ρ-norm.[4]

Each player has a strictly increasing utility $u_i : \mathbb{R}_+ \to \mathbb{R}$ over the resource. The utility of player i under the requests (d, P) and the coalition structure π is $u_i(d_i)$, if π specifies a

[2]Convexity means that for all $S, T \subseteq I$, $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$. Strict superadditivity means that if $S \cap T = \emptyset$, $v(S \cup T) > v(S) + v(T)$.

[3]We assume for notational simplicity that player i's list always includes herself.

[4]E.g., when $N = 4$, then $\rho(\{(1, 2, 3), (4)\}) = 2$. We assume the distribution F has full support.

nontrivial coalition for i, and $u_i(v(i))$ otherwise. Prior to knowing which π forms, a player considers her F-expected utility over all feasible coalition structures of minimal ρ-norm.

3 Enforceability through exclusion

The *core* of a cooperative game with characteristic function v is the set of self-enforcing allocations $\text{Core}(v) = \left\{ d \mid \sum_{i \in I} d_i = v(I) \text{ and } \sum_{i \in S} d_i \geq v(S) \ \forall \ S \subset I \right\}$. We will be interested in the set of all strictly self-enforcing allocations (i.e., the *interior* of the core, obtained by using strict inequalities above), which we denote $\text{Core}^*(v)$.[5] In fact, each interior core allocation corresponds to a strict Nash equilibrium outcome of our demand game.

Theorem 1. *(d, P) is a strict Nash equilibrium outcome of the demand game if and only if $d \in \text{Core}^*(v)$ and $P_i = I$ for all i.*

That an interior core allocation and $P_i = I$ must be a strict Nash equilibrium outcome is clear: any deviation surely yields a player strictly less of the resource. To understand why being at an interior core allocation is necessary, it is helpful to make the following observations.

Observation 1. *In any strict Nash equilibrium, it must be that $P_i = I$ for all i, that the grand coalition is not strictly demand-feasible, and that $d_i > v(i)$ for all i.*

Indeed, switching from P_i to any P'_i with $P_i \subset P'_i$ does at least as well: either the resulting coalition structure has the same norm (in which case there is a weak increase in the probability that i will be in a nontrivial coalition), or the norm decreases (in which case i must have a partner, otherwise that coalition structure would have been feasible before). Moreover, even though the grand coalition must be mutually compatible, it cannot be strictly feasible because players would want to increase their resource requests.

Instead of concentrating on demand requests, our proof concentrates on when the players have incentives to *exclude* others, building upon what we call the **exclusion principle**: *you should never exclude a player who can steal away members of your coalition and leave you alone.* Excluding such a player increases your probability of remaining unpartnered and receiving only $v(i)$, thereby lowering your expected utility.

[5] Convexity implies that the core is nonempty, and superadditivity implies that it has a nonempty interior.

Proof of Theorem 1. We now show that repeated use of the exclusion principle implies that I cannot contain any demand-feasible subgroup. This would complete the proof, since being at a strict Nash equilibrium would then require that $\sum_{i \in \mathcal{I}} d_i = v(I)$. Suppose by contradiction that a demand-feasible subgroup does exist. If there is exactly one such subgroup, then any player inside it may safely exclude any player outside it. Therefore, there must be more than one feasible subgroup under d. Let $\hat{I}$ be the collection of players who have some feasible subgroup. We aim to show that $\hat{I}$ is strictly demand-feasible: for if $\hat{I} = I$, then the grand coalition is strictly feasible, and if $\hat{I} \subset I$, then the minimal norm rule ensures that any player within $\hat{I}$ may safely exclude any player outside $\hat{I}$.

Suppose that $\hat{I}$ is not feasible and that S_1, the largest feasible subgroup of $\hat{I}$, has size s. To prevent any player $i \in S_1$ from excluding any player $j \notin S_1$, it must be the case (by the exclusion principle) that j must have a feasible subgroup S_2 containing some of i's partners in S_1. For it to be possible that i could remain alone in a a feasible coalition structure of minimal norm if she excludes j, it must be the case that j's potential coalition S_2 also has size s. To see this, note that no subgroup strictly outside of S_1 can be feasible, else the union of the two would be feasible; and that for i to remain alone, the norm cannot increase when j steals i's partners. The same exclusion principle holds for players in this next feasible subgroup S_2, and so on and so forth. Let $\{S_n\}_{1 \leq n \leq \hat{N}}$ denote the collection of all the feasible subgroups of size s. This collection must satisfy two properties:

(1) No player can be in every largest feasible subgroup (i.e., $\bigcap_{n \in \{1,\dots,\hat{N}\}} S_n = \emptyset$).

(2) If $S_n \neq S_{n'}$, then $S_n \cap S_{n'}$ is a feasible subgroup.

Property (1) follows from the exclusion principle. Property (2) is a result of the following simple observation[6] and the fact that s is the size of the largest subgroup.

Observation 2. *Suppose that the resource request vector d has only strictly individually rational requests. If the two subgroups S_n and $S_{n'}$ are demand-feasible and $S_n \cap S_{n'}$ is demand-infeasible or empty, then $S_n \cup S_{n'}$ is strictly demand-feasible.*

[6]The proof of this result follows directly from the fact that convexity of v is equivalent to the condition that for every $S, T \subseteq \mathcal{I}$, $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$.

By property (2), $S_1 \cap S_2$ must be a feasible subgroup, else the subgroup $S_1 \cup S_2$ would be feasible and of size larger than s. Inductively, for every $k \leq \hat{N}$, $\cap_{j=1}^{k} S_j$ must be a feasible subgroup, else $S_k \cup \left(\cap_{j=1}^{k-1} S_j \right)$ would be feasible and of size larger than s. But then $\cap_{j=1}^{\hat{N}} S_j$ must be nonempty, contradicting property (1) and completing the proof. □

4 Divide, conquer and cooperate

We now consider the N-player demand bargaining game played over time $t = 1, 2, \ldots$.[7] The players are boundedly rational and care only about the list of resource and partner requests (d, P) submitted in the previous period. Typically (with probability $1 - \nu$ very close to one), a player choose a myopic best response to the previous period's demands.[8] With a small probability ν, however, she does not bother to update her strategy and makes the same requests as in the last period. Therefore, at each point in time, the previous period's requests (d, P) serve as the state of the game. For technical convenience, we restrict each player i's resource requests to the set $[v(i), v(I)]_K$ of K-place decimal fractions in $[v(i), v(I)]$.[9] The evolution of the game then defines a finite-state Markov chain over the state space of players' partner and resource requests. We are interested in how the group learns to play over time.

Theorem 2. *For sufficiently large K, the bargaining game converges with probability one to a state where $d \in \text{Core}^*(v)$ and $P_i = I$ for all i.*

Clearly, any strict Nash equilibrium corresponds to an absorbing state of the dynamic process. Therefore, to prove this theorem we need only show that from any other state, the process can reach an interior core allocation with positive probability.

The intuition for this result is as follows. If players in a group cannot agree on an interior core allocation, then they may split into factions. Consequently, players may reach a situation where they are partitioned into mutually exclusive blocs, each of which agrees on an interior core allocation of their group. If any of these blocs consists of a lone player, then that player

[7] We can either imagine that these players are successive generations, that their identities are fixed over time, or some combination of the two. The arguments also extend immediately to the case of multiple parallel populations that each population samples or more general matching technologies.

[8] When there are multiple best responses, the player breaks ties with a distribution having full support. Whether or not a player best responds in a period happens independently of other players and of past play.

[9] We assume there is K^* such that the values of v are K^*-place decimal fractions and that $K \geq K^*$. It will be the case that best responses are always in $[v(i), v(I)]_K$.

is desperate to join an existing group and offers to accept strictly less than her marginal contribution to some group. If that group takes advantage of her offer, an interior core allocation of the enlarged group is created. With only nontrivial blocs remaining, one group S may instigate conflict over resources within another group S' by inviting it to join and then rescinding the invitation - after the invitees have all responded greedily. With the abandoned group S' unable to agree on a feasible allocation, one member is scapegoated and bears the burden of lowering her request. If this happens repeatedly, the scapegoat eventually leaves the group and joins S, creating an interior core allocation. The divide and conquer process continues until S becomes the grand coalition.

Let us develop this more formally. As a preliminary step in the proof, consider groups which are alienated from other players. Formally, suppose that the game is at a state (d, P) where d is not an interior core allocation and there exists a group of players $S \subseteq I$ such that every member of S is excluded by every player outside S ($P_i \cap S = \emptyset$ for all $i \notin S$), and vice-versa ($P_i \subseteq S$ for all $i \in S$). We argue that there is a positive probability that either the players in S come to agree on an allocation in the interior core of their group, or disintegrate into factions. To state this, we introduce the notation $d|_S$ and $v|_S$ for the restrictions of the allocation and characteristic function to the group S.

Lemma 1. *There is positive probability that the game moves to a state (d', P') where either the players in S all agree to an allocation in the interior core of S (i.e., $d'|_S \in \text{Core}^*(v|_S)$) or a faction $T \subset S$ has broken away from S (i.e., $P'_i = T$ for all $i \in T$).*

The proof of this lemma, which is in the appendix, builds on the exclusion technique developed earlier. Notice that if groups which cannot agree on an interior core allocation split into factions, then repeated application of Lemma 1 leads to the observation that from any nonabsorbing state the game can transition to a state (d^*, P^*) where the coalition structure is composed of mutually exclusive blocs, each in equilibrium with itself.

Observation 3. *It is possible to reach a coalition structure π^* where every group is alienated from players outside it and agrees on an allocation in the interior core of their group (i.e., for all $S \in \pi^*$, $P^*_i = S$ for $i \in S$, and if S is nonsingleton, then $d^*|_S \in \text{Core}^*(v|_S)$)*

If this coalition structure π^* is the trivial one $\{(1), (2), \ldots, (N)\}$, then an interior core allocation is only a step away, for the players are indifferent among all requests. If π^* is a nontrivial coalition structure then the situation is a bit trickier. However, because of the following result we can assume that every bloc consists of at least two players. Indeed, if j is unpartnered and desperate, she can join an existing group S (and create an interior core allocation for $S \cup \{j\}$) by offering to accept strictly less than her marginal contribution.

Lemma 2. *For large enough K, the game can reach a state $(\tilde{d}, \tilde{P})$ where S and j cooperate on an interior core allocation (i.e., $\tilde{P}_i = S \cup \{j\}$ for all $i \in S \cup \{j\}$ and $\tilde{d}|_{S\cup\{j\}} \in \mathrm{Core}^*(v|_{S\cup\{j\}})$) and $(\tilde{d}, \tilde{P})$ is the same as (d^*, P^*) for all other individuals.*

The states in Lemmas 1 and 2 correspond to weak Nash equilibria. We now exhibit a path of play to a strict Nash equilibrium outcome (an interior core allocation) using destructive group behaviors. Suppose that there exist two distinct blocs S and S' (otherwise we are done). We argue that S may use a **divide and conquer** procedure to extract players from S' who will then join S *à la* Lemma 2. Essentially, *one group causes another to break down by instigating internal strife over resources.*

To see how this may happen, suppose that the members of S and S' mutually invite each other; that is, simultaneously, every $i \in S \cup S'$ requests $(\tilde{d}_i, S \cup S')$. If the players in S do not update their strategy in the next period, while every player $j \in S'$ greedily best responds with the request $(\tilde{d}_j + v(S \cup S') - v(S) - v(S'), S \cup S')$, then S' is of no use to S.

Lemma 3. *No member of S' may now feasibly join a coalition with any member of S.*

Suppose that the members of S now choose to abandon the members of S'. That is, suppose that each $i \in S$ responds with (d_i^*, S) and that the members of S' do not move. Since S' had been at an interior core allocation, their members are unable to form any feasible coalitions with each other. In fact, if there is any player $k \in S'$ who is unable to obtain a payoff bigger than $v(k)$ by lowering her request, she may as well exit the coalition by setting $P_k = \{k\}$ and eventually join S *à la* Lemma 2.

Otherwise, at least one of the members of S' will need to lower her request. Let us consider what happens when this burden falls on one individual. Fix a **scapegoat** $j \in S'$ and assume that *all players in $S' \setminus \{j\}$ do not change their requests, but that the scapegoat j lowers her request.* Suppose that j can obtain her best-response payoff by creating a coalition with just a subgroup of S'; then she may as well modify her resource request accordingly and set $P_j = S''$, where $S'' \subset S'$ is the smallest subgroup of S' with which j may obtain her best payoff. Note that the resulting allocation would be in $\text{Core}^*(v|_{S''})$. This group S'' could safely break away in the next period, and $S' \setminus S''$ could then itself split or reach an interior core allocation as Lemma 1 prescribes.

If the scapegoat j can only obtain a payoff larger than $v(j)$ by creating a coalition with the entire group S', then the resulting allocation will be in $\text{Core}^*(v|_{S'})$. But now suppose the process repeats itself with the same scapegoat: S and S' mutually invite each other, S' responds greedily, S abandons S', and j bears the burden of lowering her request. This need only be repeated a finite number of times before the scapegoat j's best response is to break away - at which point she may join S *à la* Lemma 2. Furthermore, S can repeat this divide and conquer process against other groups until it grows to become the grand coalition and an interior core allocation is reached.

5 Discussion

This paper demonstrates how inefficient group behaviors can propel groups toward strictly self-enforcing cooperative outcomes. In essence, we have shown that the destructive behaviors used here to achieve cooperation are too destructive to sustain endless cycles of their use. To refine our prediction of interior core convergence further, we can introduce a small possibility of mistakes as in Kandori, Mailath & Rob (1993) to show that the outcomes persisting in the long run correspond to those allocations within the interior of the core that minimize the maximum individual wealth. [10] This corresponds to a long run lexicographic social preference for strict enforceability (primarily) and wealth equity (secondarily).

[10] See the supplement posted on the author's webpage for the proof. Agastya (1999) has a similar result for the core, though the sampling procedures used differ. These approaches build on Young (1993).

Appendix

Proof of Lemma 1. Throughout, assume without loss that $d_i > v(i)$, $P_i = S$ for all $i \in S$ unless stated otherwise, $\sum_{i\in S} d_i \geq v(S)$, and S and $I \setminus S$ mutually exclude each other.

Imagine first that S contains no feasible subgroups. If $\sum_{j\in S} d_j = v(S)$, then condition (1) of the lemma is satisfied and the proof is complete. If instead $\sum_{j\in S} d_j > v(S)$ and no subgroups are feasible, then whenever only one individual k best responds and the rest remain inert, one of three things may happen: (a) the resulting allocation could be in the interior core of the restricted game (again satisfying condition (1)), (b) the resulting allocation (d, P) would no longer be strictly individually rational - then $P_k = \{k\}$ is a best response for k and $P_i = S \setminus \{k\}$ becomes a best response for $i \in S \setminus \{k\}$ in the subsequent period (satisfying condition (2) of the lemma), or (c) the resulting allocation is strictly individually rational for players in S, $\sum_{j\in S} d_j \geq v(S)$, and some subgroup of S is feasible. Consider the only nontrivial case, (c). Define the largest group size $s_d = \max_{\{T\subset S: \sum_{j\in T} d_j \leq v(T)\}} |T|$ and the collection $\mathcal{T}_d = \{\ T \subset S \mid \sum_{i\in T} d_i \leq v(T) \text{ and } |T| = s_d\ \}$. There are two subcases.

Case (i). There is $T \in \mathcal{T}_d$ such that for all $i \in T$, d_i is a best response to (d, P). If a state satisfying condition (2) cannot be reached, no player $j \in T$ may be indifferent between $P_j = T$ and $P_j = S$: if j best-responds with (d_j, T) and all others play the same best response, then (d_k, T) would be a best response for every $k \in T$ in the following period and condition (2) would be satisfied. So (d_j, S) must be strictly preferred to (d_j, T) for every $k \in T$; this implies that for each $j \in T$, there is a feasible group of size s_d containing another member of T but not j. No player in T can be in every feasible group of size s_d that contains a member of T; and the intersection of these groups of size s_d must be feasible, else a bigger group is feasible. A contradiction can be found as in the proof of Theorem 1.

Case (ii). For all $T \in \mathcal{T}_d$, there is $i \in T$ such that d_i is not a best response to (d, P). For each (d, P) we may partition the members of S into the following three groups:

$$T_{(d,P)} = \{i \in S \mid d_i \text{ is a best response to } (d, P)\ \},$$

$$T^+_{(d,P)} = \{i \in S \setminus T_{(d,P)} \mid \text{ there is a best response } d^*_i \text{ to } (d, P) \text{ with } d^*_i > d_i\}, \text{ and}$$

$$T^-_{(d,P)} = \{i \in S \setminus (T_{(d,P)} \cup T^+_{(d,P)}) \mid \text{ there is a best response } d^*_i \text{ to } (d, P) \text{ with } d^*_i < d_i\}.$$

Beginning at state (d, P), let all players in $T_{(d,P)} \cup T^-_{(d,P)}$ be inert and let all players in $T^+_{(d,P)}$ raise their requests. Call the resulting state (d', P'). If $T^+_{(d',P')} = \emptyset$, stop; otherwise this can be repeated a finite number of times until $T^+_{(\tilde{d},\tilde{P})} = \emptyset$ in the resulting state $(\tilde{d}, \tilde{P})$.

Suppose that a state satisfying condition (2) cannot be reached. The outcome of every player in S's best response to $(\tilde{d}, \tilde{P})$ must be strictly individually rational, else some $k \in S$ could best respond by setting $P_k = \{k\}$ and a state satisfying condition (2) might result. Also, $T \in \mathcal{T}_{\tilde{d}} \Rightarrow T \not\subseteq T_{(\tilde{d},\tilde{P})}$, otherwise one returns to Case (1). Therefore, $T^-_{(\tilde{d},\tilde{P})} \neq \emptyset$.

The first task is to show that under $(\tilde{d}, \tilde{P})$ and the assumption that condition (2) cannot be satisfied, S cannot have any feasible subgroups. Suppose that there is at least one feasible subgroup of S, and again denote by $s_{\tilde{d}}$ the size of the largest such subgroup. $T^+_{(\tilde{d},\tilde{P})} = \emptyset$ and $T \in \mathcal{T}_{\tilde{d}} \Rightarrow T \not\subseteq T_{(\tilde{d},\tilde{P})}$, so some $i \in T^-_{(\tilde{d},\tilde{P})}$ must be both included and excluded from feasible subgroups of S of size $s_{\tilde{d}}$. If she were never excluded, lowering her request would not be a best response. Note once more that no player can be in every feasible subgroup of size $s_{\tilde{d}}$ (because condition (2) cannot be satisfied), and that the intersection of any two such subgroups must be a feasible subgroup (because no subgroup of size larger than $s_{\tilde{d}}$ is feasible and S is not strictly feasible). The same argument as in Case (1) leads to the desired contradiction.

Hence, S lacks feasible subgroups under $(\tilde{d}, \tilde{P})$. Choose some $k \in T^-_{(\tilde{d},\tilde{P})}$ to best respond and let all others be inert. The best response of k has $d^*_k = \max_{T \subseteq S, k \in T} v(T) - \sum_{j \in T \setminus \{k\}} \tilde{d}_j$. If $T^* \in \arg\max_{T \subseteq S, k \in T} v(T) - \sum_{j \in T \setminus \{k\}} \tilde{d}_j$ for some $T^* \neq S$, then (d^*_k, T^*) is optimal for k. Next period, $(\tilde{d}_j, T^*)$ will be a best response for each $j \in T^* \setminus \{k\}$, a contradiction to the assumption that condition (2) cannot be satisfied. Therefore, S forms and no subgroups of S will be feasible, i.e. a state satisfying (1) will be reached. $\square$

Proof of Lemma 2. Any request is a best response for j; and those in S are indifferent about inviting players who exclude them. Fix $m \in \mathbf{Z}_+$. Suppose that in the same period, player j requests $(v(S \cup \{j\}) - v(S) - m \cdot 10^{-K}, S \cup \{j\})$ and each $i \in S$ requests $(d_i, S \cup \{j\})$. Next period, some $k \in S$ requests $(d_k + m \cdot 10^{-K}, S \cup \{j\})$ and players in $(S \cup \{j\}) \setminus \{k\}$ don't move. It remains to verify that the resulting allocation d' has $d'|_{S \cup \{j\}} \in \text{Core}^*(v|_{S \cup \{j\}})$.

Clearly $\sum_{i \in S} d_i = v(S)$. Define $\varepsilon^* = \min_{S \cap T = \emptyset} v(S \cup T) - v(S) - v(T)$, which is positive by strict superadditivity, and assume K is large enough that $m \cdot 10^{-K} < \varepsilon^*$. The assumption on

K guarantees that $d'_j > v(j)$ is satisfied. For any $S' \subset S$, we must show $S' \cup \{j\}$ is infeasible. If $k \in S'$, this is trivial by convexity and the fact that $d|_S \in \text{Core}^*(v|_S)$:

$$\sum_{i \in S' \cup \{j\}} d'_i = \sum_{i \in S'} d_i + v(S \cup \{j\}) - v(S) > v(S') + v(S \cup \{j\}) - v(S) \geq v(S') + v(S' \cup \{j\}) - v(S')$$

If $k \notin S'$, then the infeasibility requirement is satisfied when $\sum_{i \in S'} d_i > v(S') + m \cdot 10^{-K}$; for then convexity and $d'_i = d_i$ for $i \in S'$ ensure that $v(S' \cup \{j\}) < d'_j + \sum_{i \in S'} d'_i$ because

$$m \cdot 10^{-K} < v(S' \cup \{j\}) - v(S') + \sum_{i \in S'} d_i - v(S' \cup \{j\}) \leq v(S \cup \{j\}) - v(S) + \sum_{i \in S'} d_i - v(S' \cup \{j\})$$

A technical issue arises only when $\hat{S} = \left\{S' \subset S, S' \neq \emptyset \mid \sum_{i \in S'} d_i \leq v(S') + m \cdot 10^{-K}\right\}$ is nonempty and such that $\cap_{S' \in \hat{S}} S' = \emptyset$. If $\cap_{S' \in \hat{S}} S' \neq \emptyset$, simply let the best-responding player k be in $\cap_{S' \in \hat{S}} S'$. We will now show that $\exists\ K^{**} \in \mathbf{Z}_+$ such that $\cap_{S' \in \hat{S}} S' = \emptyset$ is impossible whenever $K \geq \max\left\{K^*, K^{**}\right\}$. Let $K^{**} = [\log \frac{|\hat{S}|(m+1)+m}{\varepsilon^*}] + 1$ and suppose that $\cap_{S' \in \hat{S}} S' = \emptyset$. Convexity necessitates that $\sum_{i \in S' \cap S''} d_i \leq v(S' \cap S'') + (2m+1) \cdot 10^{-K}$, otherwise $S' \cup S''$ is strictly feasible, a contradiction to $d|_S \in \text{Core}^*(v|_S)$. Consider some $S' \in \hat{S}$ and take $S'' \subset S$ such that $S' \not\subseteq S''$ and $S'' \not\subseteq S'$. If $\sum_{i \in S''} d_i = v(S'') + r \cdot 10^{-K}$ for some $r \leq |\hat{S}|(m+1)+m$, then by convexity it must be that $\sum_{i \in S' \cap S''} d_i \leq v(S' \cap S'') + (r+m+1) \cdot 10^{-K}$ to avoid the contradiction that $S' \cup S''$ is strictly feasible. Consider two distinct $S_1, S_2 \in \hat{S}$ and let $T_1 = S_1 \cap S_2$. $T_1 \neq \emptyset$, else $S_1 \cup S_2$ is strictly feasible. If $T_1 \neq S_1, S_2$ then $\sum_{i \in T_1} d_i \leq v(T_1) + (2m+1) \cdot 10^{-K}$; otherwise, if $T_1 = S_1$ then $\sum_{i \in T_1} d_i = \sum_{i \in S_1} d_i$, and similarly for the case $T_1 = S_2$. In either case, $\sum_{i \in T_1} d_i \leq v(T_1) + (2m+1) \cdot 10^{-K}$ is the upper bound of interest. Inductively define $T_n = T_{n-1} \cap S_{n+1}$ for $2 \leq n \leq R = |\hat{S}| - 1$. If $T_n = \emptyset$, one obtains a contradiction. We are concerned with the case $T_n \neq T_{n-1}, S_{n+1}$ to get the upper bound on $\sum_{i \in T_n} d_i$, which by convexity is $\sum_{i \in T_n} d_i \leq v(T_n) + [(n+2)m + n + 1] \cdot 10^{-K}$. The final intersection $T_R = T_{R-1} \cap S_{R+1}$ must be empty and a contradiction arises. □

Proof of Lemma 3. Let $d' = \tilde{d} + [v(S \cup S') - v(S) - v(S')] \cdot 1_{S'}$, where 1_X is the usual indicator function. First, we prove the following intermediate result using convexity: *take nonempty* $A, A', B \subset I$ *with* $A \cap B = \emptyset$ *and* $A' \subset A$; *and let* d *be such that* $d|_A \in \text{Core}^*(v|_A)$. *If* $A' \cup B$

is a feasible coalition under d, then $A \cup B$ is strictly feasible under d. To see this, note that by convexity, $v(A \cup B) - v(A) \geq v(A' \cup B) - v(A')$. By assumption, both $\sum_{i \in A'} d_i > v(A')$ and $\sum_{i \in A' \cup B} d_i \leq v(A' \cup B)$. Hence $v(A \cup B) - v(A) > \sum_{i \in B} d_i$. Noting that $\sum_{i \in A} d_i = v(A)$ completes the proof of the claim.

We claim that for any $\emptyset \neq S'' \subseteq S'$, $\sum_{i \in S''} d'_i > v(S \cup S'') - v(S)$. To see this, note that by convexity, strict superadditivity, and $\tilde{d}|_{S'} \in \text{Core}^*(v|_{S'})$,

$$\begin{aligned}
&\sum_{i \in S''} \tilde{d}_i + |S''|[v(S \cup S') - v(S) - v(S')] - v(S \cup S'') + v(S) \\
&\geq \sum_{i \in S''} \tilde{d}_i + |S''|[v(S \cup S'') - v(S) - v(S'')] - v(S \cup S'') + v(S) \\
&\geq (|S''| - 1)[v(S \cup S'') - v(S) - v(S'')]
\end{aligned}$$

The lemma now follows from the contrapositive of the intermediate claim. □

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