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Foundations of Intrinsic Habit Formation

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Abstract

We provide theoretical foundations for several common (nested) representations of intrinsic linear habit formation. Our axiomatization introduces an intertemporal theory of *weaning* a decision-maker from her habits using the device of compensation. We clarify differences across specifications of the model, provide measures of habit-forming tendencies, and suggest methods for axiomatizing time-nonseparable preferences.

Keywords: time-nonseparable preferences, linear habit formation, weaning, compensated separability, gains monotonicity
JEL classification: C60, D11, D90

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"Soon I'll be fed up with the (theory of) relativity...Even such a thing fades away when one is too involved with it." – *Albert Einstein*

1 Introduction

Does one's valuation for a good depend on its frequency of consumption? Will someone accustomed to certain levels of comfort and quality come to demand the same? And is an increase in consumption always beneficial, even if it is only temporary? Because questions such as these cannot be properly addressed in the standard intertemporally separable model of choice, the literature in varied fields of economics has seen a surge in models incorporating intertemporal nonseparability through habit formation. By presuming a correlation between an individual's prior consumption levels (her *intrinsic habit*) and her enjoyment of present and future consumption, such models have had success in accounting for notable phenomena that more traditional theory has been unable to explain.[1]

The literature on habit formation has, however, been unable to come to a consensus on a single formulation of intertemporal dependence; and in some cases, the predictions of the most commonly utilized models disagree.[2] Related to this difficulty is the scarcity of theoretical work examining the underpinnings of such preferences. While there is a large axiomatic literature on static reference dependence, there is little understanding of dynamic settings where the reference point changes endogenously, as is the case in habit formation.[3] By clarifying the implications for choice behavior, such work would help illuminate why one utility representation of habit formation might be more reasonable than another; or why the commonly used incarnations are reasonable at all. We contribute to the literature in that theoretical vein.

[1]Variations of the model of *intrinsic linear habit formation* we axiomatize have shed light on data indicating individuals are far more averse to risk than expected (e.g., Constantinides (1990) on the equity premium); suggested why consumption growth is connected strongly to income, but only weakly to interest rates (see Boldrin, Christiano & Fisher (2001) for a real business cycles model with habit formation and intersectoral inflexibilities); and explained the consumption contractions seen before exchange rate stabilization programs collapse (Uribe (2002)).

[2]While intrinsic linear habits is the most common model, other models posit habits that are extrinsic (Abel (1990)'s "catching up with the Joneses" effect), nonlinear (Campbell & Cochrane (1999)), or affect the discount factor (Shi & Epstein (1993)). A common nonlinear model specifies a linear habit aggregator that divides consumption (Carroll, Overland & Weil (2000)); this model is criticized by Wendner (2003) for having counterintuitive implications for consumption growth.

[3]We contribute to this axiomatic literature, particularly Neilson (2006), which specifies the first component of a bundle as the reference point. By contrast, we do not assume a particular reference point but derive an infinite sequence of endogenously changing reference points.

We formulate a theory of history-dependent intertemporal choice, describing a decision-maker (DM) by a family of continuous preference relations over future consumption, each corresponding to a possible consumption history. Our representative DM is dynamically consistent given her consumption history, can be *weaned* from her habits using special streams of *compensation*, and satisfies a separability axiom appropriate for time-nonseparable preferences. Though she is fully rational, her history dependent behavior violates the axioms of Koopmans (1960), upon which the standard theory of discounted utility rests. Instead, our theory lays the foundation for the model of *linear habit formation*, in which a DM evaluates consumption at each point in time with respect to a reference point that is generated linearly from her consumption history. Suppose the DM's time-0 habit is $h = (\ldots, h_3, h_2, h_1)$, where h_k denotes consumption k periods ago. If she consumes the stream $c = (c_0, c_1, c_2, \ldots)$, her time-$t$ habit will be $h^{(t)} = (h, c_0, c_1, \ldots, c_{t-1})$, where $h_k^{(t)}$ denotes consumption k periods prior to time t. The DM then evaluates the stream c using the utility function

$$U_h(c) = \sum_{t=0}^{\infty} \delta^t u\Big(c_t - \sum_{k=1}^{\infty} \lambda_k h_k^{(t)}\Big).$$

In this model, the time-t habit $(h, c_0, c_1, \ldots, c_{t-1})$ that results from consuming c under initial habit h is *aggregated* into the DM's period-t reference point by taking a weighted average using the *habit formation coefficients* $\{\lambda_k\}_{k\geq 1}$. These coefficients satisfy a geometric decay property ensuring that the influence of past consumption fades over time. A number of variations of this model are prevalent in the applied literature. We provide foundations for this general formulation and some common specializations, clarifying the behavioral differences across the nested specifications and providing various measures of habit-forming tendencies.

Although our DM has discounted utility over *habit-adjusted* consumption streams of the form $(c_0 - \sum_{k=1}^{\infty} \lambda_k h_k, c_1 - \lambda_1 c_0 - \sum_{k=1}^{\infty} \lambda_{k+1} h_k, \ldots)$, the problem at hand has a quite different nature than that of Koopmans', who imposes the axioms of discounted utility on the *real* consumption space. By contrast, the space of habit-adjusted consumption streams is hypothetical and depends on the DM's habit-formation coefficients. Axioms imposed on the real consumption space must both elicit the manner of habit-adjustment and embed it into the utility representation as the history-dependent "inner utility" $c_t - \sum_{k=1}^{\infty} \lambda_k h^{(t)}$ that is evaluated before the "outer utility" $\sum_{t=0}^{\infty} \delta^t u(\cdot)$ is applied.

To resolve this problem, we develop a *compensation-based* theory of intertemporal choice that succeeds in disentangling the effects of habit formation and time-preference. Just as classical Hicksian income compensation separates income and substitution effects, we propose intertemporal consumption compensation in our main axiom, *Habit Compensation*, to identify both habit and time-preference. An increase in the DM's habit has similar effects as a change in intertemporal prices, and by compensating the DM for this change with a decreasing stream of the habit-forming good (*weaning*) we can elicit subjective reference points from choice behavior. Our approach suggests a means to derive axiomatic foundations for discounted utility representations on spaces defined by subjective reference points.

This paper is related to a growing literature on forward-looking habit formation, beginning with the seminal work of Becker & Murphy (1988) on rational addiction. Although Koopmans (1960) uncovered foundations for intertemporally separable preferences, this literature has not found axiomatic foundations for a structured model of habit-forming preferences over consumption streams, such as those used in applied work. Rustichini & Siconolfi (2005) propose axiomatic foundations for a model of dynamically consistent habit formation which, unlike this paper, does not offer a particular structure for the utility or form of habit aggregation. Gul & Pesendorfer (2007) study self-control problems by considering preferences over menus of consumption streams of addictive goods, rather than over the streams themselves. Shalev (1997) provides a foundation for a special case of loss aversion, which, like the classical representation, is time-inconsistent (Tversky & Kahneman (1991)). Our representation can accommodate a dynamically consistent model of loss aversion where the period-utility takes the well-known "S"-shaped form. Such a model would resolve various anomalies of intertemporal choice; as Camerer & Loewenstein (2004) note, many effects "are consistent with stable, uniform, time discounting once one measures discount rates with a more realistic utility function."

This paper is organized as follows. We present the framework in Section 2 and the main axioms in Section 3. We discuss the main representation theorem and its proof in Section 4. In Sections 5-7 we examine the behavioral implications of some common restrictions on the model.

2 The framework

We consider a DM facing an infinite-horizon decision problem in which a single habit-forming good is consumed in every period $t \in \mathbb{N} = \{0, 1, 2, \ldots\}$ from the same set $Q = \mathbb{R}_+$. A consumption level $q \in Q$ may be interpreted as a choice of either quantity or quality of the good.

The DM chooses an infinite stream of consumption $c = (c_0, c_1, c_2, \ldots)$ from the set of bounded consumption streams $C = \{c \in \times_{t=0}^{\infty} Q \mid \sup_t c_t < \infty\}$, where c_t is the consumption level prescribed for t periods into the future. Date 0 is always interpreted to be the current date. We consider C as a metric subspace of $\times_{t=0}^{\infty} Q$ endowed with the product metric $\rho^C(c, c') = \sum_{t=0}^{\infty} \frac{1}{2^t} \frac{|c_t - c'_t|}{1+|c_t - c'_t|}$.[4]

The DM's preferences over the space of consumption streams C depend on her consumption history, her *habit*. The set of possible habits is time-invariant and given by the space of bounded streams $H = \{h \in \times_{k=\infty}^{1} Q \mid \sup_k h_k < \infty\}$. Each habit $h \in H$ is an infinite stream denoting prior consumption and is written as $h = (\ldots, h_3, h_2, h_1)$, where h_k denotes the consumption level of the DM k periods ago. We endow the space H with the sup metric $\rho^H(h, h') = \sup_k |h_k - h'_k|$.

The DM realizes that her future tastes will be influenced by her consumption history. Starting from any given initial habit $h \in H$, consuming the stream $c \in C$ will result in the date-t habit $(h, c_0, c_1, \ldots, c_{t-1})$. Consequently, the DM's habit, and therefore her preferences, may undergo an infinite succession of changes endogenously induced from her choice of consumption stream. The DM's preferences given a habit $h \in H$ are denoted by $\succeq_h$ and are defined on the consumption space C. Each such preference is a member of the family $\succeq = \{\succeq_h\}_{h \in H}$. We assume that the DM's preference depends on consumption history but not on calendar time.

Our setup explicitly presumes histories are infinite because this assumption is commonly invoked in the literature. Alternatively, one may assume that the DM's preferences are affected only by her last $K \geq 3$ consumption levels.[5] The notation in our analysis would remain the same so long as current and future habits are truncated after K components; that is, (h, c_0) would denote the habit $(h_{K-1}, \ldots, h_2, h_1, c_0)$. Finally, while our framework is one of riskless choice, the analysis can be extended immediately to lotteries over consumption streams by imposing the von Neumann-Morganstern axioms on lotteries and our axioms on the degenerate lotteries.

We collect here some useful notation. We reserve the variable $k \in \{1, 2, 3, \ldots\}$ to signify a period of previous consumption and the variable $t \in \{0, 1, 2, \ldots\}$ to

[4] Since $\times_{t=0}^{\infty} Q$ endowed with ρ^C is a topologically separable metric space, so is C when viewed as a metric subspace. Ensuring that C is separable in this manner allows us to concentrate on the structural elements of habit formation. Alternatively we could impose separability directly as in Rustichini & Siconolfi (2005). Bleichrodt, Rohde & Wakker (2007) is representative of a literature that concentrates on relaxing assumptions about the consumption space, including separability.

[5] $K \geq 3$ is required only for the proof of time-additivity.

signify a period of impending consumption. The notation $c+c'$ (or $h+h'$) refers to usual vector addition. As is customary, ${}^tc$ denotes $(c_t, c_{t+1}, c_{t+2}, \ldots)$ and c^t denotes $(c_0, c_1, \ldots, c_t)$. If $c' \in C$ we write $(c^t, {}^{t+1}c+c')$ to denote $(c_0, c_1, \ldots, c_t, c_{t+1}+c'_0, c_{t+2}+c'_1, \ldots)$. For $\alpha \in \mathbb{R}$ we use the similar notation α^t to signify the t-period repetition $(\alpha, \alpha, \ldots, \alpha)$ and $(c^t, {}^{t+1}c+\alpha)$ to compactly denote $(c_0, c_1, \ldots, c_t, c_{t+1}+\alpha, c_{t+2}+\alpha, \ldots)$ whenever the resulting stream is in C. At times it will be convenient to let hq denote the habit (h, q) that forms after consuming q under habit h (similarly for hc^t). The zero habit $(\ldots, 0, 0)$ is denoted by $\bar{0}$. Finally, $h \geq h'$ (or $c \geq c'$) means $h_k \geq h'_k$ for all k (or $c_t \geq c'_t$ for all t), with at least one strict inequality.

3 The main axioms

This section presents axioms of choice behavior that are necessary and sufficient for a linear habit formation representation. The roles that these axioms play in the proof of the representation theorem are discussed in Section 4.

The following axioms are imposed for all $h \in H$. The first three axioms are familiar in the theory of rational choice over consumption streams, and the fourth is a simple technical condition to ensure that the DM's preferences are non-degenerate. As usual, $\succ_h$ denotes the asymmetric part of $\succeq_h$.

Axiom PR (Preference Relation) $\succeq_h$ is a complete and transitive binary relation.

Axiom C (Continuity) For all $c \in C$, $\{c' : c' \succ_h c\}$ and $\{c' : c \succ_h c'\}$ are open.

Axiom DC (Dynamic Consistency) For any $q \in Q$ and $c, c' \in C$, $(q, c) \succeq_h (q, c')$ if and only if $c \succeq_{hq} c'$.

Axiom S (Sensitivity) There exist $c \in C$ and $\alpha > 0$ such that $c + \alpha \not\sim_h c$.

Axioms PR and C together require that the DM's choices are derived from a continuous preference relation, thereby ensuring a continuous utility representation on our separable space. Axiom DC further assumes that the DM's preferences are dynamically consistent in a history-dependent manner, in the sense that given the relevant histories, she will not change her mind tomorrow about the consumption stream she chooses today. Axiom DC is weak enough to accommodate a number of observed time-discounting anomalies, but strong enough to ensure that dynamic programming techniques can be used to solve the DM's choice problem and that the DM's welfare can be analyzed unambiguously.[6] Axiom S is a non-degeneracy

[6]Without DC, it becomes more difficult to interpret the DM's choices for the future and discuss the welfare implications of her choices; the DM's choice may need to be modeled through

condition requiring that there is some consumption stream that can be uniformly increased in a manner that does not leave it indifferent to the original. It is a much weaker condition than monotonicity, which we address in Section 5, and allows for the possibility that due to habit formation, the DM is worse off under a uniform increase in consumption.

Our main structural axiom of habit formation provides a revealed-preference theory of *weaning* a DM from her habits. To state the axiom, we define the set of ordered pairs of consumption histories $\mathcal{H} = \{(h', h) \in H \times H \mid h' \leq h\}$. We say that habits $(h', h) \in \mathcal{H}$ *agree on* k if $h_k = h'_k$. Similarly, we say that the habits $(h', h) \in \mathcal{H}$ agree on a subset of indices $\mathbb{K} \subset \{1, 2, \ldots\}$ if they agree on each $k \in \mathbb{K}$. The axiom has three parts, two of which play central roles. The first, weaning, says that for any ordered pair of habits, there is a decreasing "compensating stream" that compensates the DM for having the higher habit. The second, compensated separability, says that if a compensating stream that is received in the future compensates the DM for variations in prior consumption, preferences over current consumption are independent of the future consumption stream.

Axiom HC (Habit Compensation) There is a collection $\{d^{h',h}\}_{(h',h)\in\mathcal{H}}$ of strictly positive consumption streams such that

(i) (Weaning). Each $d^{h',h}$ is a weakly decreasing stream and uniquely satisfies

$$c \succeq_{h'} c' \text{ iff } c + d^{h',h} \succeq_h c' + d^{h',h} \quad \forall\, c, c' \in C.$$

(ii) (Compensated Separability). For any $c, \hat{c} \in C$, $t \geq 0$ and $h' \leq hc^t, h\hat{c}^t$,

$$(c^t, d^{h',hc^t}) \succeq_h (\hat{c}^t, d^{h',h\hat{c}^t}) \text{ iff } (c^t, \bar{c} + d^{h',hc^t}) \succeq_h (\hat{c}^t, \bar{c} + d^{h',h\hat{c}^t}) \quad \forall\, \bar{c} \in C.$$

(iii) (Independence of Irrelevant Habits). For any $\hat{k}$, $(h', h) \in \mathcal{H}$ that agree on $\hat{k}$, and $q \in Q$, if $\hat{h}'_k = \begin{cases} h'_k & \text{if } k \neq \hat{k} \\ q & \text{if } k = \hat{k} \end{cases}$ and $\hat{h}_k = \begin{cases} h_k & \text{if } k \neq \hat{k} \\ q & \text{if } k = \hat{k} \end{cases}$ then $d^{h',h} = d^{\hat{h}',\hat{h}}$.

Formally, Axiom HC(i) says that for any $h \geq h'$, there exists a unique *compensating stream* $d^{h',h}$ such that when we endow the DM with $d^{h',h}$ *at the larger*

an equilibrium concept rather than as a decision problem. An equilibrium notion for dynamic reference dependence is studied in Köszegi & Rabin (2008), where the utility over sequences of consumption and beliefs is technically consistent but beliefs are forced to be determined rationally in a *personal equilibrium* (see Köszegi & Rabin (2006)).

habit h, her choice behavior at h is identical to her choice behavior at the smaller habit h', *without this endowment*.[7] As illustrated in Figure 1, HC(i) establishes that the indifference curves for habit h' are translated up by the strictly positive stream $d^{h',h}$ into indifference curves for habit h.[8] Because $d^{h',h}$ is a consumption stream of the habit-forming good, the amount with which the DM is compensated in any period must account not only for her original habit, but also for habits generated by compensation received in previous periods. In theory, this could lead to an increasing need for compensation over time. Since $d^{h',h}$ serves as the baseline consumption level which induces the DM with habit h to behave as if she has habit h', the requirement that $d^{h',h}$ is weakly decreasing formalizes the sense in which the DM can be "weaned" from her habit: the DM receives the highest levels of compensation today, because the effect of her habit today is sufficiently stronger than it will be tomorrow.

Axiom HC(ii) considers the effect of compensation received midstream. Suppose a DM with habit h compares consumption streams having one of two possible consumption paths for periods 0 through t: c^t or $\hat{c}^t$. Which path the DM chooses affects her habit, and therefore her preferences, at time $t+1$. But if, starting in period $t+1$, the DM is compensated to behave as if she has some lower habit h' (using the appropriate choice of either d^{h',hc^t} or $d^{h',h\hat{c}^t}$), then the DM evaluates any common continuation path $\bar{c}$ starting from time $t+1$ from the perspective of habit h', regardless of what she has already consumed. Axiom HC(ii) says that the DM's choice between the two infinite streams is determined by the values of the consumption stream up to time t, as long as these streams agree on their continuation path. That is, receiving the appropriate compensation starting from period t blocks the channel through which consumption prior to t affects future preferences; the future becomes "separable" from the past. Consequently, Axiom HC(ii) may be viewed as a generalization of separability for time-nonseparable preferences, and would be satisfied by the standard model of discounted utility if all the compensating streams were identically zero.

Axiom HC(iii) ensures that if $(h', h) \in \mathcal{H}$ agree on some k, then the compen-

[7] Given the existence of compensating streams, uniqueness corresponds to a regularity or non-degeneracy condition on preferences for any fixed habit: if compensation is not unique for some pair $(\hat{h}', \hat{h})$, then for every $h \geq \hat{h}$, there are nonzero $\bar{c} \neq \bar{\bar{c}} \in C$ such that for any $c, c' \in C$, we have $c + \bar{c} \succeq_h c' + \bar{c}$ if and only if $c + \bar{\bar{c}} \succeq_h c' + \bar{\bar{c}}$. As the representation theorem shows, this rules out period-utilities that are essentially periodic functions (see Figure 2).

[8] Moreover, while it is not evident from the picture, the two pictured indifference curves correspond to the same utility levels under their respective habits; hence the analogy to Hicksian income compensation.

sation needed to wean the DM from h to h' is independent of the period-k habit level. Thus, an element of a habit that is unchanged does not affect weaning.

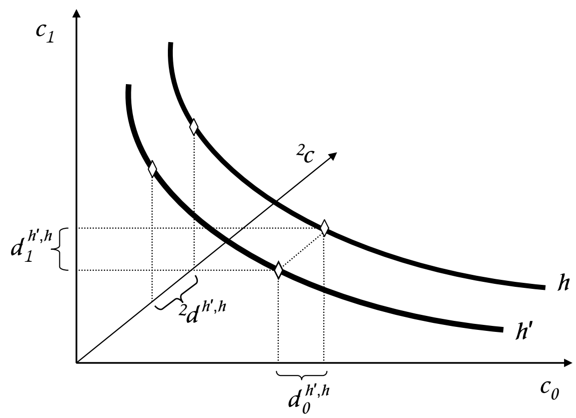

Figure 1: HC(i) applied to an h'-indifference curve on (c_0, c_1), for given ^2c

Finally, we require two additional technical conditions on the DM's initial level of compensation. These conditions concern the strength of the DM's memory and rule out degenerate representations of the preferences we seek. First, we require that the initial compensation needed for a habit goes to zero as that habit becomes more distant in memory: i.e., for any habit $h \in H$ we have $\lim_{t\to\infty} d_0^{\bar{0},h0^t} = 0$. In counterpoint, the second condition states that for any fixed prior date of consumption, we can find two habits that differ widely enough on that date to generate any initial level of compensation: i.e., for any $q > 0$ and k, there exist $(h', h) \in \mathcal{H}$ that agree on $\mathbb{N} \setminus \{k\}$ and satisfy $d_0^{h',h} = q$.[9] We say the DM's memory is *non-degenerate* if these two conditions hold.

Axiom NDM (Non-Degenerate Memory) The DM's memory is non-degenerate.

4 The main representation theorem

We now present our main theorem, which offers a precise characterization of the preferences that satisfy our axioms of habit formation. The utility representation obtained is a dynamically consistent and additive model of intrinsic linear habit formation that has featured prominently in the applied literature. The representation theorem requires a weak acyclicity condition on period utilities, but otherwise

[9]The first condition is required *only for histories of infinite length*: it rules out an undesirable term inside the utility that depends only on tail elements of the habit. The second condition rules out degenerate solutions of a critical functional equation.

permits any choice of continuous period utility. We say that $u : \mathbb{R} \to \mathbb{R}$ is *quasi-cyclic* if there exist $\alpha \in \mathbb{R}$ and $\beta, \gamma > 0$ such that $u(x+\gamma) = \beta u(x) + \alpha$ for all $x \in \mathbb{R}$, and *cyclic* if it is quasi-cyclic with $\beta = 1$. See Figure 2 in the appendix for an illustration of quasi-cyclic functions.

Theorem 1 (Main representation)**.** *The family of preference relations $\succeq$ satisfies Axioms PR, C, DC, S, HC, and NDM if and only if there exist a discount factor $\delta \in (0,1)$, habit formation coefficients $\{\lambda_k\}_{k\geq 1} \in \mathbb{R}$, and a period-utility $u : \mathbb{R} \to \mathbb{R}$ such that for every $h \in \mathcal{H}$, $\succeq_h$ can be represented by*

$$U_h(c) = \sum_{t=0}^{\infty} \delta^t u\Big(c_t - \sum_{k=1}^{\infty} \lambda_k h_k^{(t)}\Big), \ \textit{with } h^{(t)} = (h, c_0, c_1, \ldots, c_{t-1}), \tag{1}$$

where the habit formation coefficients $\{\lambda_k\}_{k\geq 1}$ are unique and satisfy

$$\lambda_k \in (0,1) \ \textit{and} \ \frac{\lambda_{k+1}}{\lambda_k} \leq 1 - \lambda_1 \ \textit{for all } k \geq 1; \tag{2}$$

and the period-utility $u(\cdot)$ is continuous, unique up to positive affine transformation, and is not cyclic (and is not quasi-cyclic if $\sum_{k=1}^{\infty} \lambda_k < 1$).

In Section 4.1 we examine why this utility representation satisfies Axiom HC, which provides some insight into our constructive proof of the theorem in Appendix B.1. In Section 4.2 we give an overview of some of the key steps in the construction.

The representation in Theorem 1 may be seen as a model of dynamic reference dependence: the linear *habit aggregator* $\varphi : H \to \mathbb{R}$ defined by

$$\varphi(h^{(t)}) = \sum_{k=1}^{\infty} \lambda_k h_k^{(t)} \tag{3}$$

determines the reference point against which date-t consumption is evaluated. The representation has two main features. First, the DM transforms each consumption stream c into a habit-adjusted stream $(c_0 - \varphi(h), c_1 - \varphi(h, c_0), c_2 - \varphi(h, c_0, c_1), \ldots)$; we denote this transformation by $g(h, c)$ and call it the DM's "inner utility." The DM then applies a discounted "outer utility" U^*, given by $\sum_{t=0}^{\infty} \delta^t u(\cdot)$, to evaluate the habit-adjusted stream. The DM's utility U_h over consumption streams is then given by $U^*(g(h, \cdot))$. Because the habit formation coefficients in Theorem 1 are positive, the representation implies that utility is history dependent. If the DM's history is assumed to be finite and of length K, only the first K habit formation coefficients would be positive.

A standard discounted utility maximizer, for whom all the habit formation coefficients would equal zero, would satisfy all our axioms if the compensating streams were identically zero. We may include the standard model by relaxing Axiom HC to include the possibility that all the compensating streams are identically zero, but avoid doing so to simplify exposition. The other restriction in this representation is the acyclicity requirement on the period utility; some functions violating this requirement are illustrated in Figure 2 in the appendix. Observe that if the DM's period-utility were linear (hence cyclic) in the representation above, then her choice behavior would be observationally equivalent to that in a model without habit formation. More generally, if the DM's period-utility violates the acyclicity requirement, then we cannot pin down the transformation of her preferences from one habit to another; that is, acyclicity ensures that compensating streams are unique. In light of Figure 2, a quasi-cyclic function, unless it is linear, would not fall into the class of period-utilities regularly considered in economic models.[10] Consequently, the compensating streams are unique for essentially all applications.

Theorem 1 may also be viewed as obtaining foundations for a *log-linear* representation $U_h(c) = \sum_{t=0}^{\infty} \delta^t u\Big(\frac{c_t}{\varphi(h^{(t)})}\Big)$, where $\varphi(\hat{h}) = \prod_{k=1}^{\infty} \hat{h}_k^{\lambda_k}$ and $\frac{\lambda_{k+1}}{\lambda_k} \leq 1 - \lambda_1$, if we reinterpret the framework so that the DM cares about, and forms habits over, *consumption growth rates* instead of consumption levels.[11] Assuming consumption is bounded below by $\varepsilon > 0$, in such a model the DM forms habits over the *logarithms* of her past consumption levels $(\ldots, \log h_2, \log h_1)$ and her preferences are defined over streams of logarithms of consumption $(\log c_0, \log c_1, \ldots)$. The axioms would need to be reinterpreted in this new setting; for example, in Axiom HC(i), the DM would need to be compensated in terms of rates of consumption growth rather than using consumption levels.

4.1 Why the representation satisfies Axiom HC

Consider a DM who can be represented as in Theorem 1. Why does this DM satisfy Axiom HC, and how would the compensating streams look?

Consider any ordered pair of habits $(h', h) \in \mathcal{H}$. At time t, the DM's period-utility is $u\big(c_t - \varphi(h', c^{t-1})\big)$ if she has habit h', while it is $u\big(c_t - \varphi(h, c^{t-1})\big)$ if she has habit h. However, there is a simple relationship between these two period-utilities

[10] A quasi-cyclic function has a period and repeats itself (up to scaling). Unless it is affine, it cannot be both smooth and concave; nor can it have a finite and nonzero number of kinks.

[11] Such a model is proposed by Kozicki & Tinsley (2002) and is particularly appealing in light of Wendner (2003), which shows the counterintuitive implications of a common model in which the argument of the period-utility is current consumption divided by a *linear* habit stock.

obtained by adding and subtracting $\varphi(h, c^{t-1})$:

$$u\big(c_t - \varphi(h', c^{t-1})\big) = u\big(c_t + [\varphi(h, c^{t-1}) - \varphi(h', c^{t-1})] - \varphi(h, c^{t-1})\big). \tag{4}$$

Since the habit aggregator $\varphi(\cdot)$ is strictly increasing and linear, the bracketed term $\varphi(h, c^{t-1}) - \varphi(h', c^{t-1})$ is strictly positive and equal to $\varphi\big(h - h', 0^t\big)$.

Axiom HC(i) says that whenever the DM is endowed with $d^{h',h}$ at habit h, her utility from any stream c is the same as her utility from c under the lower habit h', without compensation. We use (4) to construct $d^{h',h}$ as follows. At time 0, we provide the DM with the amount $d_0^{h',h} = \varphi(h - h')$. As seen from (4), the DM's period-utility from consuming $c_0 + d_0^{h',h}$ under habit h at time 0 is the same as her period-utility from consuming c_0 under habit h'. To construct $d_1^{h',h}$, we must take into account that the DM was compensated with the habit-forming good: the actual time-0 consumption level under h in (4) is $c_0 + d_0^{h',h}$. The bracketed term in (4) is then $d_1^{h',h} = \varphi\big(h - h', \varphi(h - h')\big)$.

Continuing in this manner, at time t the compensating stream $d^{h',h}$ compensates for the original difference in habits as well as for compensation provided prior to t. Formally, $d^{h',h}$ has the recursive structure

$$d^{h',h} = \Big(\varphi(h-h'), \varphi\big(h-h', \varphi(h-h')\big), \varphi\big(h-h', \varphi(h-h'), \varphi(h-h', \varphi(h-h'))\big), \dots\Big), \tag{5}$$

where φ is linear. In the Appendix we prove this fundamental characterization of compensation directly from the axioms. In the special case that the habits involved differ only by the most recent element, (5) takes a particularly simple form:

$$\begin{aligned} d_0^{hq',hq} &= \lambda_1(q - q') \\ d_1^{hq',hq} &= \lambda_2(q - q') + \lambda_1 d_0^{hq',hq} \\ d_2^{hq',hq} &= \lambda_3(q - q') + \lambda_2 d_0^{hq',hq} + \lambda_1 d_1^{hq',hq} \\ &\vdots \end{aligned}$$

Then it is easy to see that $d^{hq',hq}$ is a weakly decreasing stream if $\frac{\lambda_{k+1}}{\lambda_k} \leq 1 - \lambda_1$; and if one knows $d^{hq',hq}$ then this triangular linear system recovers all the $\{\lambda_k\}_{k=1}^{\infty}$.

Because the argument of the period utility is linear, the construction of $d^{h',h}$ above delivers a compensating stream that is independent of the actual consumption stream c being evaluated. That is, linearity of the "inner utility" is critically related to the order of the quantifiers in Axiom HC(i). Indeed, HC(i) would be nearly unrestrictive if the compensation were allowed to depend on the choices involved without specifying any further properties. Note that Axiom HC(i) by itself does not require the manner of habit dependence to be homogenous across habits. Our construction of compensation still works if the habit formation coefficients depend on tail elements of the habit (e.g, $\lambda_{k,h} = \lambda_k \frac{\alpha + \limsup_{k'} h_{k'}}{\beta + \limsup_{k'} h_{k'}}$, where $\beta > \alpha > 0$). Tail dependence would only violate Axiom HC(iii), which requires homogeneity. Furthermore, the form of the "outer" utility is irrelevant: our construction remains valid so long as the DM evaluates a consumption stream c through $U^*(c_0 - \sum_{k=1}^{\infty} \lambda_k h_k, c_1 - \lambda_1 c_0 - \sum_{k=1}^{\infty} \lambda_{k+1} h_k, \ldots)$, where $U^* : \mathbb{R}^\infty \to \mathbb{R}$.

The special feature of our time-additive utility is that it satisfies Axiom HC(ii), which is a generalized separability axiom that restricts the "outer utility" U^* above to be additively separable (that is, $U^*(x_0, x_1, x_2, \ldots) = \sum_{s=0}^{\infty} u_s^*(x_s)$). To see why HC(ii) is implied by time-additivity, notice that if the DM receives compensation d^{h',hc^t} after consuming c^t, and $d^{h',h\hat{c}^t}$ after consuming $\hat{c}^t$, then comparing the streams $(c^t, \bar{c})$ and $(\hat{c}^t, \bar{c})$ reduces to comparing $\sum_{s=0}^{t} u_s^*(c_s - \varphi(h, c^{s-1}))$ and $\sum_{s=0}^{t} u_s^*(\hat{c}_s - \varphi(h, \hat{c}^{s-1}))$. This argument does not depend on stationarity or dynamic consistency (i.e., $u_s^*(\cdot) = \delta^s u(\cdot)$); if the DM naively used $\beta - \delta$ quasi-hyperbolic discounting, HC(ii) would still be satisfied. Moreover, HC(ii) does not require linearity of the "inner utility": the axiom would still be satisfied using a generalized notion of compensation that permits dependence on the consumption streams being evaluated, so long as the "outer utility" is time-additive.

4.2 Constructing the representation from the axioms

Here we offer an overview of our constructive proof in Appendix B.1, discussing some of the key steps in the argument. In Section 4.2.1 we show that the habit aggregator $\varphi(\cdot)$ is linear and that compensation has the recursive form in (5). In Section 4.2.2 we generate the DM's "inner utility." That is, we find the DM's manner of habit-adjustment, given by $c_t - \varphi(hc^{t-1})$ at each time t, and construct a preference relation $\succeq^*$ over habit-adjusted consumption streams that is equivalent to the DM's preferences over actual consumption streams. Finally, in Section 4.2.3 we discuss how to find a discounted utility representation for $\succeq^*$, which serves as the "outer utility" in the representation of each $\succeq_h$.

In the remainder of this section we will provide intuition for some of the arguments by imposing the strong restriction that habits are only one-period long. This allows us to convey the flavor of the arguments while sidestepping complications that arise from more intricate history dependence. We defer complete arguments, including topological considerations, to the appendix.

4.2.1 Determining the form of habit aggregation

In order to construct the utility representation, we must first determine how the DM's habits are aggregated into a single reference point. In view of (5), it is evident that our constructive proof should define the habit aggregator $\varphi(h)$ by $d_0^{\bar{0},h}$. Therefore, the first task at hand is to prove that our axioms imply that there exists a sequence of habit formation coefficients $\{\lambda_k\}_{k=1}^{\infty}$ such that $d_0^{\bar{0},h} = \sum_{k=1}^{\infty} \lambda_k h_k$. Second, we would like to prove the recursive structure in (5), for then $\{\lambda_k\}_{k=1}^{\infty}$ would fully characterize each $d^{h',h}$. To accomplish these tasks we must develop further properties of compensation from the axioms.

The underlying idea is best elucidated using one-period histories $q \in Q$. One-period histories allow us to avoid several complications that we must defer to the appendix; these include accounting for extended effects of compensation on future preferences, aggregating different periods in history, and showing that the habit-formation coefficients are homogeneous across all histories and are applied to updated histories in a stationary manner.[12] In this simplified setting, the desired results will follow from three claims:

(i) (Triangle Equality) For any $q'' < q' < q$, we have $d^{q'',q} = d^{q'',q'} + d^{q',q}$.

(ii) (Weak Invariance) For any q, q', we have $d^{q',q'+d_0^q} = d^{0,d_0^q}$.

(iii) (Recursion) For any q, we have $d^{0,d_0^q} = {}^1d^{0,q}$.

Then, by claim (i), $d^{q',q} = d^{0,q} - d^{0,q}$ for any $q' < q$. Defining $\varphi : Q \to \mathbb{R}_+$ by $\varphi(q) = d_0^{0,q}$ for $q > 0$ and $\varphi(0) = 0$, we have $d_0^{q',q'+d_0^q} = \varphi(q' + \varphi(q)) - \varphi(q')$. By claim (ii), we know that $d_0^{q',q'+d_0^q} = d_0^{0,d_0^q} = \varphi(\varphi(q))$. Therefore,

$$\varphi(\varphi(q)) = \varphi(q' + \varphi(q)) - \varphi(q') \quad \forall\, q, q' \in Q. \tag{6}$$

Since Axiom NDM implies that the range of $\varphi(q')$ is all of Q, the functional equation above is equivalent to a simple Cauchy equation, $\varphi(q + q') = \varphi(q) + \varphi(q')$ for all $q', q \in Q$. Because (i) implies that $\varphi(\cdot)$ is increasing, the solution to this functional equation is $\varphi(q) = \lambda q$ for some $\lambda > 0$. Iterated use of (iii) implies the recursive structure (5) in this setting.

[12] For example, one must rule out that even though $d_0^{\bar{0},h} = \sum_{k=1}^{\infty} \lambda_k h_k$, the k-th element of the initial history, h_k, always receives weight λ_k in the future.

We now prove claims (i)-(iii). For claim (i), observe that we wish to show

$$c + d^{q'',q'} + d^{q',q} \succeq_q c' + d^{q'',q'} + d^{q',q} \text{ if and only if } c \succeq_{q''} c' \text{ for all } c, c' \in C,$$

for then uniqueness of compensation would imply that $d^{q'',q'} + d^{q',q}$ is $d^{q'',q}$. By Axiom HC(i), $d^{q'',q'}$ satisfies $c \succeq_{q''} c'$ if and only if $c + d^{q'',q'} \succeq_{q'} c' + d^{q'',q'}$ for all $c, c' \in C$. But using Axiom HC(i) again on the RHS above, we also know that

$$c+d^{q'',q'} \succeq_{q'} c'+d^{q'',q'} \text{ if and only if } c+d^{q'',q'}+d^{q',q} \succeq_q c'+d^{q'',q'}+d^{q',q} \text{ for all } c, c' \in C,$$

completing the argument.

Now consider claims (ii) and (iii). Consider any $q, q' \in Q$ and any two $c, c' \in C$ such that $c_0 = c'_0 = q'$. By Axiom HC(i),

$$c \succeq_0 c' \text{ if and only if } c + d^{0,q} \succeq_q c' + d^{0,q}. \tag{7}$$

Applying Axiom DC to the RHS of (7),

$$c + d^{0,q} \succeq_q c' + d^{0,q} \text{ if and only if } {}^1c + {}^1d^{0,q} \succeq_{q'+d_0^{0,q}} {}^1c' + {}^1d^{0,q}. \tag{8}$$

But again by Axiom DC, $c \succeq_0 c'$ if and only if ${}^1c \succeq_{q'} {}^1c'$. Combining (7) and (8),

$${}^1c \succeq_{q'} {}^1c' \text{ if and only if } {}^1c + {}^1d^{0,q} \succeq_{q'+d_0^{0,q}} {}^1c' + {}^1d^{0,q}.$$

Since ${}^1c$ and ${}^1c'$ were arbitrary, it must be that ${}^1d^{0,q} = d^{q',q'+d_0^{0,q}}$. But ${}^1d^{0,q}$ is independent of q'. Setting $q' = 0$, this proves claim (iii). Moreover, $d^{q',q'+d_0^{0,q}}$ must be independent of q', proving claim (ii).

4.2.2 The habit-adjusted consumption space C^* and preference $\succeq^*$

Once we have constructed $\varphi(\cdot)$, we may construct the space of habit-adjusted consumption streams. To do this, we define the mapping $g : H \times C \to \mathbb{R}^\infty$ by

$$g(h, c) = (c_0 - \varphi(h), c_1 - \varphi(h, c_0), c_2 - \varphi(h, c_0, c_1), \ldots)$$

$C^* = g(H \times C)$ is the space of all possible habit-adjusted consumption streams, while $C^*_h = g(\{h\}, C)$ is the space of all h-adjusted consumption streams.[13] Intu-

[13]We endow $\mathbb{R}^\infty$ with the product topology; metrize $H \times C$ by $\rho^{H\times C}((h, c), (h', c')) = \rho^H(h, h') + \rho^C(c, c')$; and consider C^* as a metric subspace of $\mathbb{R}^\infty$.

itively, for any possible consumption stream c and habit h of the DM, the resulting habit-adjusted consumption stream $g(h, c)$ is "worse" the higher is the DM's habit h. Formally, it can be shown that $C^*_{h'} \subseteq C^*_h$ if $h \geq h'$ (i.e., the C^*_h's are nested).

We would like to construct a relation $\succeq^*$ on habit-adjusted consumption streams that is equivalent to the DM's preferences on real consumption streams, by defining

$$g(h, c) \succeq^* g(h, \hat{c}) \text{ if and only if } c \succeq_h \hat{c}. \tag{9}$$

By obtaining a utility representation U^* for $\succeq^*$ on the space C^*, we would have a representation U_h for each $\succeq_h$. We would simply transform each stream c by the habit-adjustment $g(h, \cdot)$ (the "inner utility") and then apply the "outer utility" U^*; more formally, $U_h(\cdot) = U^*(g(h, \cdot))$. However, before we can find a representation for $\succeq^*$, we must show that it is a continuous preference relation; and given that there are multiple pairs of streams and habits that map to the same habit-adjusted stream c^* we must also show that $\succeq^*$ is well-defined.

We illustrate that $\succeq^*$ is well-defined using one-period histories. If one fixes a particular habit q, we can uniquely reconstruct from any $c^* \in C^*_q$ the consumption stream c such that $g(q, c) = c^*$. Indeed, since $c^*_0 = c_0 - \lambda q$, we know $c_0 = c^*_0 + \lambda q$. Similarly, since $c^*_1 = c_1 - \lambda c_0$, we know $c_1 = c^*_1 + \lambda c^*_0 + \lambda^2 q$, and so on and so forth. Using the linear habit-aggregator $\varphi(\cdot)$, the stream c such that $g(q, c) = c^*$ is given by $(c^*_0 + \varphi(q), c^*_1 + \varphi(c^*_0 + \varphi(q)), \ldots)$.

To see that $\succeq^*$ is well-defined, notice that we may equivalently define $\succeq^*$ by

$$c^* \succeq^* \hat{c}^* \text{ iff } (c^*_0+\varphi(q), c^*_1+\varphi(c^*_0+\varphi(q)), \ldots) \succeq_q (\hat{c}^*_0+\varphi(q), \hat{c}^*_1+\varphi(\hat{c}^*_0+\varphi(q)), \ldots) \tag{10}$$

for some $q \in Q$ such that $c^*, \hat{c}^* \in C^*_q$. Suppose that $\succeq^*$ is not well-defined. That is, while the RHS of (10) holds for some q, there is a q' such that $c^*, \hat{c}^* \in C^*_{q'}$ and

$$(\hat{c}^*_0 + \varphi(q'), \hat{c}^*_1 + \varphi(\hat{c}^*_0 + \varphi(q')), \ldots) \succ_{q'} (c^*_0 + \varphi(q'), c^*_1 + \varphi(c^*_0 + \varphi(q')), \ldots).$$

Assume without loss that $q > q'$. Axiom HC(i) then implies that

$$(\hat{c}^*_0+\varphi(q'), \hat{c}^*_1+\varphi(\hat{c}^*_0+\varphi(q')), \ldots)+d^{q',q} \succ_q (c^*_0+\varphi(q'), c^*_1+\varphi(c^*_0+\varphi(q')), \ldots)+d^{q',q}.$$

But since $d^{q',q} = (\varphi(q - q'), \varphi(\varphi(q - q')), \ldots)$, the relation above is precisely

$$(\hat{c}^*_0 + \varphi(q), \hat{c}^*_1 + \varphi(\hat{c}^*_0 + \varphi(q)), \ldots) \succ_q (c^*_0 + \varphi(q), c^*_1 + \varphi(c^*_0 + \varphi(q)), \ldots),$$

which contradicts (10). Hence $\succeq^*$ is well-defined.

Given that $\succeq^*$ is well-defined, we can now show it is a preference relation. Because the C^*_q's are nested, for any three habit-adjusted consumption streams, there is $\bar{q}$ large enough that all three belong to $C^*_{\bar{q}}$. Therefore, $\succeq^*$ inherits completeness and transitivity from $\succeq_{\bar{q}}$; a more delicate argument proves that $\succeq^*$ also inherits continuity.

4.2.3 Obtaining a discounted "outer utility" representation

While the DM's preferences are neither additively separable nor dynamically consistent in a manner independent of history, we can prove that $\succeq^*$ does satisfy these properties, and therefore that $\succeq^*$ has a discounted utility representation U^*.

We leave a detailed discussion of the argument for additive separability, which is complex, to the Appendix. Given our other axioms, we show in the Appendix that Axiom HC(ii), which has the flavor of a separability axiom, implies that $\succeq^*$ satisfies the separability conditions of Gorman (1968) on C^*.[14] To prove that HC(ii) generates this complete set of separability conditions for $\succeq^*$ on C^* using our axioms on C requires that consumption histories be at least three periods long.

However, we can show here that $\succeq^*$ satisfies history-independent dynamic consistency, which gives the representation of $\succeq^*$ a recursive structure. Again, let us consider the special case of one-period histories. We would like to show that for any $c^*, \hat{c}^* \in C^*$ with $c^*_0 = \hat{c}^*_0$, $(c^*_0, {}^1c^*) \succeq^* (c^*_0, {}^1\hat{c}^*)$ if and only if ${}^1c^* \succeq^* {}^1\hat{c}^*$. To see this, note that $(c^*_0, {}^1c^*) \succeq^* (c^*_0, {}^1\hat{c}^*)$ if and only if

$$(c^*_0 + \varphi(q), c^*_1 + \varphi(c^*_0 + \varphi(q)), \ldots) \succeq_q (c^*_0 + \varphi(q), \hat{c}^*_1 + \varphi(c^*_0 + \varphi(q)), \ldots) \tag{11}$$

for some $q \in Q$ such that $c^*, \hat{c}^* \in C^*_q$. Because $\succeq_q$ satisfies Axiom DC, (11) holds if and only if

$$\begin{aligned}(c^*_1 + \varphi(c^*_0 + \varphi(q)), & c^*_2 + \varphi(c^*_1 + \varphi(c^*_0 + \varphi(q))), \ldots) \\ & \succeq_{c^*_0+\varphi(q)} (\hat{c}^*_1 + \varphi(c^*_0 + \varphi(q)), \hat{c}^*_2 + \varphi(\hat{c}^*_1 + \varphi(c^*_0 + \varphi(q))), \ldots).\end{aligned}$$

This means, by definition, that ${}^1c^* \succeq^* {}^1\hat{c}^*$. Hence the claim is proved.

[14] The only other paper of which we are aware that applies Gorman-type conditions to infinite streams in order to obtain a discounted utility representation is Bleichrodt, Rohde & Wakker (2007), which is unrelated to habit formation.

5 Desirable habit-forming goods

For cases in which the consumption good is a desirable one, we can strengthen the previous representation to one in which the period-utility is monotonic, as is typically assumed in the applied literature on habit formation.

Standard monotonicity says the DM is better off whenever consumption in any period is increased. This seemingly innocuous assumption may not be satisfied in a time-nonseparable model: a consumption increase also raises the DM's habit. We suggest a weakening that accommodates the possibility that a short-term consumption gain might not suffice to overcome the long-term utility loss. Our axiom considers an unambiguous "gain" to be an indefinite increase in consumption.[15]

Axiom GM (Gains Monotonicity) If $\alpha > 0$, $(c^t, {}^{t+1}c + \alpha) \succ c$ for all c, t.

Replacing Axiom S with GM ensures that the period-utility in Theorem 1 is increasing. The proof requires additional results found in the supplement.

Theorem 2 (Main representation with monotonic period-utility). *The family of preference relations $\succeq$ satisfies Axioms PR, C, DC, GM, HC, and NDM if and only if each $\succeq_h$ can be represented as in Theorem 1 using an increasing period-utility $u(\cdot)$ which is (i) strictly increasing on $(0, \infty)$ if $\sum_{k=1}^{\infty} \lambda_k < 1$ and (ii) strictly increasing on either $(-a, \infty)$ or $(-\infty, a)$ for some $a > 0$ if $\sum_{k=1}^{\infty} \lambda_k = 1$.*

Unlike monotonicity, Axiom GM does not contradict experimental evidence indicating that individuals may prefer receiving an increasing stream of consumption over one that is larger but fluctuates more (see Camerer & Loewenstein (2004) for a comprehensive survey). Instead, it suggests a guideline for when a larger stream should be preferred. Consider two consumption streams, c and c', with $c \geq c'$. We say that $c >_{GD} c'$, or c *gains-dominates* c', if c has larger period-to-period gains and smaller period-to-period losses: that is, $c_t - c_{t-1} \geq c'_t - c'_{t-1}$ $\forall\ t \geq 1$. The following result characterizes GM in terms of a preference for gains-dominating streams.

Proposition 1 (Respect of gains-domination). *A preference relation continuous in the product topology satisfies GM if and only if it respects gains-domination; that is, if and only if for any $c, c' \in C$, $c >_{GD} c'$ implies that $c \succ c'$.*

The proof is immediate after noting that a stream will gains-dominate another if and only if the difference between the two streams is positive and increasing; the

[15] By constrast, Shalev (1997)'s constant-tail monotonicity says (restricted to deterministic streams) that if a stream gives q from time t onwards, then raising q to some $q' > q$ from t onwards improves the stream. This is equivalent to saying that a weakly increasing (decreasing) consumption stream is at least as good (bad) as getting its worst (best) element constantly.

result follows from repeatedly applying Axiom GM to build the gains-dominating stream forward and using continuity in the product topology.

6 The autoregressive model and habit decay

A frequently used specification of the linear habit formation model posits an *autoregressive* specification of the habit aggregator that reduces the number of habit parameters to two. According to this model, there exist $\alpha, \beta > 0$ with $\alpha + \beta \leq 1$ such that the habit aggregator satisfies the autoregressive law of motion $\varphi(hq) = \alpha\varphi(h) + \beta q$ for all $h \in H$ and $q \in Q$.[16] In this section we examine the implications of this simplification for choice behavior. Specifically, we show that the autoregressive structure of the habit aggregator corresponds to an additional axiom that can calibrate the habit decay parameter α in that model.

Suppose a DM faces two possible consumption scenarios for period 0, *High* and *Low*. In the former, the DM consumes very much at $t = 0$; in the latter, she consumes very little. We may wonder whether the date-0 consumption level determined in these scenarios has an irreversible effect on the DM's future preferences. If the DM were to consume very little for some time after scenario *High*, and very much for some time after scenario *Low*, could the opposing effects cancel so that her preferences following each scenario eventually coincide? The next axiom describes a choice behavior for which such equilibration is possible.

Axiom IE (Immediate Equilibration) For all $c_0, \hat{c}_0 \in Q$, there exist $c_1, \hat{c}_1 \in Q$ such that for all $\bar{c}, \bar{\bar{c}} \in C$, $(c_0, c_1, \bar{c}) \succeq_h (c_0, c_1, \bar{\bar{c}})$ if and only if $(\hat{c}_0, \hat{c}_1, \bar{c}) \succeq_h (\hat{c}_0, \hat{c}_1, \bar{\bar{c}})$.

This says we can undo by tomorrow the effect of a difference in consumption today. Together, Axioms DC and IE imply that $\succeq_{hc_0c_1}$ and $\succeq_{h\hat{c}_0\hat{c}_1}$ are identical.

Axiom IE offers a comparative measure of habit decay. To see this, fix any period-0 consumption levels $\hat{c}_0 > c_0$ and consider the corresponding period-1 consumption levels $\hat{c}_1, c_1$ that are given by Axiom IE. If the DM's habits decay slowly then the effects of prior consumption linger strongly, so c_1 will have to be quite large and $\hat{c}_1$ will have to be quite small in order to offset the initial difference. More formally, for fixed $\hat{c}_0 > c_0$ one would expect that the difference $c_1 - \hat{c}_1$ in the period-1 consumption levels required by Axiom IE should be larger for those DM's whose habits decay more slowly.

This intuition is confirmed by the following representation theorem, which re-

[16] Such a model appears in Boldrin, Christiano & Fisher (1997) in our discrete time form and in Constantinides (1990), Schroder & Skiadas (2002) and Sundaresan (1989) in continuous time.

veals that Axiom IE corresponds to the autoregressive specification of habits, and that habits decay at the constant rate $\frac{c_1-\hat{c}_1}{\hat{c}_0-c_0}$.[17]

Theorem 3 (Autoregressive habit formation)**.** *The family of preference relations $\succeq$ satisfies Axioms PR, C, DC, S, HC, NDM and IE if and only if each $\succeq_h$ can be represented by $U_h(c) = \sum_{t=0}^{\infty} \delta^t u\big(c_t - \varphi(h, c_0, c_1, \ldots, c_{t-1})\big)$ as in Theorem 1 and there exist $\alpha, \beta > 0$ with $\alpha+\beta \leq 1$ such that the linear habit aggregator $\varphi(\cdot)$ satisfies the autoregressive law of motion*

$$\varphi(hq) = \alpha\varphi(h) + \beta q \quad \forall\, h \in H,\ q \in Q. \tag{12}$$

Moreover, for arbitrary choice of $c_0, \hat{c}_0$ in Axiom IE, the values of $c_1, \hat{c}_1$ given by Axiom IE calibrate the habit decay parameter: $\alpha = \frac{c_1-\hat{c}_1}{\hat{c}_0-c_0}$.[18]

The proof of Theorem 3, which appears in the Appendix, suggests a more general result. It can similarly be shown that a generalization of the autoregressive model that has n habit parameters corresponds to a generalized $n-1$ period version of equilibration in which it takes $n-1$ periods to equilibrate preferences after a single difference in consumption.

Clearly, for the simplest autoregressive model, the *geometric coefficients* model where the aggregator satisfies the law of motion $\varphi(hq) = (1-\lambda)\varphi(h) + \lambda q$, the choice experiment in Axiom IE immediately recovers the single parameter λ. Since this model corresponds to the special case $\alpha + \beta = 1$, the parameter λ is given by $1 - \frac{c_1-\hat{c}_1}{\hat{c}_0-c_0}$. Although the autoregressive model and its geometric specialization appear quite similar, we show in the next section that choice behavior can depend critically on whether $\alpha + \beta$ is equal to or smaller than one.

7 Persistent versus responsive habits

In this section we distinguish between two types of preferences that satisfy our axioms, those whose habits are *responsive* to weaning and those whose habits are *persistent*. Recall that Axiom HC(i) implies that the indifference curves for the preference $\succeq_{h'}$ are translated up by $d^{h',h}$ into indifference curves for $\succeq_h$, as

[17] Consider an alternative to IE: $\forall\, h$, $\exists\, q \in Q$ s.t. for all $\bar{c}, \bar{\bar{c}} \in C$, $\bar{c} \succeq_h \bar{\bar{c}}$ iff $(q, \bar{c}) \succeq_h (q, \bar{\bar{c}})$. This axiom would get the representation in Theorem 3 but would not calibrate the parameter α.

[18] For finite histories of length $K \geq 3$, the habit aggregator cannot be written in the form (12) but the result of Theorem 3 is unchanged: the ratio of successive habit formation coefficients $\frac{\lambda_{k+1}}{\lambda_k}$ is constant and given by $\frac{c_1-\hat{c}_1}{\hat{c}_0-c_0}$.

illustrated in Figure 1. Stated differently, $d^{h',h}$ measures the distance between the indifference curves of $\succeq_{h'}$ and $\succeq_h$. Whether the DM can be weaned using a quickly fading stream of compensation, or must be weaned using possibly high levels of consumption that fade slowly - or never at all - will determine the extent to which consumption affects her preferences. To capture this, we suggest the following simple characterization of the DM's habit-forming tendencies.

Definition 1. *The DM is responsive to weaning if she can always be weaned using a finite amount of compensation; that is, for every $(h', h) \in \mathcal{H}$, the total amount $\sum_{t=0}^{\infty} d_t^{h',h}$ is finite. The DM has persistent habits if she can never be weaned using a finite amount of compensation; that is, for every $(h', h) \in \mathcal{H}$, $\sum_{t=0}^{\infty} d_t^{h',h} = \infty$.*

We show that the value $\sum_{k=1}^{\infty} \lambda_k$ characterizes a DM's habits as responsive or persistent and can have a profound effect on the manner in which indifference curves are translated from one habit to another.[19]

Proposition 2 (Dichotomy)**.** *Suppose the DM satisfies our axioms. Then,*

(i) The DM's habits are persistent if $\sum_{k=1}^{\infty} \lambda_k = 1$. Moreover, for every $(h', h) \in \mathcal{H}$, the compensating stream $d^{h',h}$ is constant.

(ii) The DM's habits are responsive to weaning if $\sum_{k=1}^{\infty} \lambda_k < 1$. Moreover, if k^* is such that $\frac{\lambda_{k^*+1}}{\lambda_{k^*}} < 1 - \lambda_1$, then for every $(h', h) \in \mathcal{H}$, the compensating stream $d^{h',h}$ decays at least at the geometric rate $1 - \lambda_{k^*}(1 - \lambda_1 - \frac{\lambda_{k^*+1}}{\lambda_{k^*}})$ for all $t \geq k^*$; and if $h > h'$, $d^{h',h}$ is strictly decreasing for all t.

Observe that $\sum_{k=1}^{\infty} \lambda_k = 1$ if and only if $\frac{\lambda_{k+1}}{\lambda_k} = 1 - \lambda_1$ for every k. To illustrate the meaning of this result suppose that $\sum_{k=1}^{\infty} \lambda_k = 1 - \gamma$, where $\gamma > 0$ may be attributed to $\frac{\lambda_{k^*+1}}{\lambda_{k^*}}$ falling below $1 - \lambda_1$ for some small k^*. Even if γ is small, the effect of habits on choice behavior is quite different from that under persistent habits, *ceteris paribus*. Compensation rapidly decreases early on and the translation of the indifference map between two habits $(h', h) \in \mathcal{H}$ is much milder than it would be if habits were persistent (in which case the translation would be constant).

This difference is particularly pronounced within the class of autoregressive models discussed in the previous section. The autoregressive model corresponds to the restriction that $\frac{\lambda_{k+1}}{\lambda_k}$ is a constant given by α (and $\beta = \lambda_1$). If $\alpha + \beta < 1$, applying the result above with $k^* = 1$ indicates that compensation decreases immediately. While attention is not always paid to the value of $\alpha + \beta$ in the autoregressive

[19] The proof follows from Lemma 8.

model, this result suggests that this modeling decision should be taken with care. In particular, the following result shows that the choice of a period-utility should be made in conjunction with the choice of persistent or responsive habits.[20]

Proposition 3 (Persistent habits)**.** *Suppose the DM's habits are persistent. Then for any $\varepsilon > 0$, there are no $c \in C$ and habit $h \in H$ such that the argument of the DM's period-utility, $c_t - \varphi(hc^{t-1})$, is at least as large as ε for every t.*

To facilitate dynamic programming, the applied literature typically uses a period-utility satisfying an Inada condition $\lim_{x \to 0} u'(x) = \infty$. For such a period-utility, this result means that a persistent DM will have infinite marginal utility infinitely often from any bounded consumption stream. Moreover, a persistent DM cannot perfectly smooth her habit-adjusted consumption if her consumption is bounded.

8 Conclusion

In this paper we have introduced the device of compensating a DM for giving up her habits to provide axiomatic foundations for intrinsic linear habit formation. This approach has allowed us to clarify the behavioral differences across some prevalent specifications of this model in the applied literature.

Our axiomatization can be modified to accommodate other models of history dependence. For example, it is easy to extend our axioms to generate a multidimensional version of intrinsic linear habit formation (e.g., with one standard good and two habit-forming ones). By specifying compensation to be independent across goods, one may obtain the representation $\sum_{t=0}^{\infty} \delta^t u\big(c_t^1, c_t^2 - \varphi^2(h^2, c_0^2, \ldots, c_{t-1}^2), c_t^3 - \varphi^3(h^3, c_0^3, \ldots, c_{t-1}^3)\big)$, where the habit aggregator $\varphi^i(\cdot)$ for good $i = 2, 3$ is given by $\varphi^i(\hat{h}^i) = \sum_{k=1}^{\infty} \lambda_k^i \hat{h}_k^i$. Although consumption histories for each good are evaluated separately, the curvature of $u(\cdot)$ may imply that a DM's desire for a habit-forming good she has not tried before is intensified when another good for which she has a high habit is unavailable. In addition, if the definition of weaning is generalized so that compensation may depend on the DM's choice set, then the critical assumption generating linearity is relaxed. One may potentially place the appropriate axioms on compensation to axiomatize models of non-linear habit formation.

[20] This result follows from Lemma 31 in the supplement.

Appendix

A Illustrations of quasi-cyclicity

Figure 2: Violations of acyclicity. (a) $\beta = 1$; (b) $\beta > 1$; (c) $\beta = 1$ and affine.

B Proof of Theorem 1

Combined with the results in the supplement, this also proves Theorem 2.

B.1 Sufficiency

Axioms PR, C, DC, S, HC, and NDM are implicit in all hypotheses.

Results about the sequences $d^{h',h}$

Lemma 1 (Zero)**.** *For each h' there is no nonzero $\bar{c} \in C$ such that $c + \bar{c} \succeq_{h'} c' + \bar{c}$ iff $c \succeq_{h'} c'$ for all $c, c' \in C$. Consequently we may define $d^{h,h} = (0, 0, \ldots)$.*

Proof. If there were, then for any $h \geq h'$ both $\bar{c} + d^{h',h}$ and $d^{h',h}$ would compensate from h' to h, violating uniqueness. □

Lemma 2 (Triangle Equality)**.** *Let $h'' \geq h' \geq h$. Then $d^{h,h''} = d^{h,h'} + d^{h',h''}$.*

Proof. This is analogous to the proof on page 14 in the main text. □

By the triangle equality, $d^{h',h} = d^{\bar{0},h} - d^{\bar{0},h'}$. We abuse notation by writing d^h whenever $d^{\bar{0},h}$ is intended. For any $h \in H$, $q \in Q$ and $k \in \mathbb{N}$, the habit $h^{k,q} \in H$ is defined by $h^{k,q}_k = q$ and $h^{k,q}_t = h_t$ for every $t \neq k$. In particular, $\bar{0}^{k,q}$ is the habit which has q as the k-th element and 0 everywhere else.

Lemma 3 (Additive Separability)**.** $d^{h',h} = \sum_{k=1}^{\infty} \left(d^{\bar{0}^{k,h_k}} - d^{\bar{0}^{k,h'_k}} \right)$.

Proof. Let $h^0 = h'$ and for every k inductively define h^k by $h^k_k = h_k$ and $h^k_i = h^{k-1}_i$ for all $i \neq k$. We prove the lemma in three steps: (i) for any $(h', h) \in \mathcal{H}$, we may write $d^{h',h} = \sum_{k=1}^{\infty} d^{h^{k-1},h^k} + \lim_{K\to\infty} d^{h^K,h}$; (ii) each $d^{h^{k-1},h^k} = d^{\bar{0}^{k},h_k} - d^{\bar{0}^{k},h'_k}$; and (iii) $\lim_{K\to\infty} d^{h^K,h} = (0, 0, \ldots)$.

(i) Using iterated application of Lemma 2, observe that for habits $(h', h) \in \mathcal{H}$ that eventually agree (WLOG, suppose they agree on $\{t, t+1, \ldots\}$) we have $d^{h',h} = \sum_{k=1}^{t} d^{h^{k-1},h^k}$. Now consider arbitrary $(h', h) \in \mathcal{H}$. For any $K \in \mathbb{N}$ and any $c, c' \in C$, $c \succeq_{h'} c'$ iff $c + \sum_{k=1}^{K} d^{h^{k-1},h^k} \succeq_{h^K} c' + \sum_{k=1}^{K} d^{h^{k-1},h^k}$. But again by Weaning in Axiom HC, $c + \sum_{k=1}^{K} d^{h^{k-1},h^k} \succeq_{h^K} c' + \sum_{k=1}^{K} d^{h^{k-1},h^k}$ iff $c+\sum_{k=1}^{K} d^{h^{k-1},h^k} + d^{h^K,h} \succeq_h c' + \sum_{k=1}^{K} d^{h^{k-1},h^k} + d^{h^K,h}$. Therefore, for arbitrary K, $d^{h',h} = \sum_{k=1}^{K} d^{h^{k-1},h^k} + d^{h^K,h}$.

(ii) We now show that each d^{h^{k-1},h^k} is independent of the values of h' and h on $\mathbb{N} \setminus \{k\}$. In fact, we will show that for arbitrary $q' \leq q$ and $(\underline{h}, \bar{h}) \in \mathcal{H}$,

$$d^{\underline{h}^{k,q'},\underline{h}^{k,q}} = d^{\bar{h}^{k,q'},\bar{h}^{k,q}} \text{ if and only if } d^{\underline{h}^{k,q},\bar{h}^{k,q}} = d^{\underline{h}^{k,q'},\bar{h}^{k,q'}}. \tag{13}$$

To see this, use Lemma 2 to write $d^{\underline{h}^{k,q'},\bar{h}^{k,q}} = d^{\underline{h}^{k,q'},\bar{h}^{k,q'}} + d^{\bar{h}^{k,q'},\bar{h}^{k,q}}$ as well as $d^{\underline{h}^{k,q'},\bar{h}^{k,q}} = d^{\underline{h}^{k,q'},\underline{h}^{k,q}} + d^{\underline{h}^{k,q},\bar{h}^{k,q}}$. Combining these two expressions,

$$d^{\underline{h}^{k,q'},\bar{h}^{k,q'}} - d^{\underline{h}^{k,q},\bar{h}^{k,q}} = d^{\underline{h}^{k,q'},\underline{h}^{k,q}} - d^{\bar{h}^{k,q'},\bar{h}^{k,q}}.$$

This proves (13). By Axiom HC(iii), $d^{\underline{h}^{k,q'},\bar{h}^{k,q'}} = d^{\underline{h}^{k,q},\bar{h}^{k,q}}$. Since h^k and h^{k+1} agree on $\mathbb{N} \setminus \{k\}$, (13) implies that $d^{h^k,h^{k+1}} = d^{\bar{0}^{k,h'_k},\bar{0}^{k,h_k}}$. Now use the triangle equality.

(iii) Now we show that $\lim_{K\to\infty} d^{h^K,h} = (0, 0, \ldots)$. Since the habits h^K and h agree on $\{1, 2, \ldots, K\}$, iterated application of Axiom HC(iii) implies that for each K, $d^{h^K,h} = d^{h'0^K,h0^K}$. But by the triangle equality, $d^{h',h}$ is decreasing in h'. Hence $d^{h'0^K,h0^K} \leq d^{\bar{0},h0^K}$. Therefore,

$$(0, 0, \ldots) \leq \lim_{K\to\infty} d^{h^K,h} = \lim_{K\to\infty} d^{h'0^K,h0^K} \leq \lim_{K\to\infty} d^{\bar{0},h0^K} = (0, 0, \ldots),$$

where the last equality is due to Axiom NDM and $d^{h',h}$ decreasing in h'. $\square$

Lemma 4 (Weak Invariance)**.** *For any $q, \hat{q} \in Q$ and k, $d_0^{\bar{0}^{k,q},\bar{0}^{k,q+d_0^{\bar{0}^{k,\hat{q}}}}} = d_0^{\bar{0},\bar{0}^{k,d_0^{\bar{0}^{k,\hat{q}}}}}$.*

Proof. Consider any $c, c' \in C$ such that $(c_0, c_1, \ldots, c_{k-1})$ and $(c'_0, c'_1, \ldots, c'_{k-1})$ are both equal to $(q, 0, 0, \ldots, 0)$. According to Weaning,

$$c \succeq_{\bar{0}} c' \text{ iff } c + d^{\bar{0}^{k,\hat{q}}} \succeq_{\bar{0}^{k,\hat{q}}} c' + d^{\bar{0}^{k,\hat{q}}}. \tag{14}$$

Applying DC to the RHS of (14),

$$c + d^{\bar{0}^{k,\hat{q}}} \succeq_{\bar{0}^{k,\hat{q}}} c' + d^{\bar{0}^{k,\hat{q}}} \text{ iff } {}^k c + {}^k d^{\bar{0}^{k,\hat{q}}} \succeq_{(\bar{0}^{k,\hat{q}}, q + d_0^{\bar{0}^{k,\hat{q}}}, d_1^{\bar{0}^{k,\hat{q}}}, \ldots, d_{k-1}^{\bar{0}^{k,\hat{q}}})} {}^k c' + {}^k d^{\bar{0}^{k,\hat{q}}}. \tag{15}$$

But again by DC, this time applied to the LHS of (14),

$$c \succeq_{\bar{0}} c' \text{ iff } {}^k c \succeq_{\bar{0}^{k,q}} {}^k c'. \tag{16}$$

Combining expressions (15) and (16) using (14),

$${}^k c \succeq_{\bar{0}^{k,q}} {}^k c' \text{ iff } {}^k c + {}^k d^{\bar{0}^{k,\hat{q}}} \succeq_{(\bar{0}^{k,\hat{q}}, q + d_0^{\bar{0}^{k,\hat{q}}}, d_1^{\bar{0}^{k,\hat{q}}}, \ldots, d_{k-1}^{\bar{0}^{k,\hat{q}}})} {}^k c' + {}^k d^{\bar{0}^{k,\hat{q}}}. \tag{17}$$

Since both have a q in the k-th place, $\bar{0}^{k,q} \leq (\bar{0}^{k,\hat{q}}, q + d_0^{\bar{0}^{k,\hat{q}}}, d_1^{\bar{0}^{k,\hat{q}}}, \ldots, d_{k-1}^{\bar{0}^{k,\hat{q}}})$. As ${}^k c$ and ${}^k c'$ are arbitrary, ${}^k d^{\bar{0}^{k,\hat{q}}} = d^{\bar{0}^{k,q}, (\bar{0}^{k,\hat{q}}, q + d_0^{\bar{0}^{k,\hat{q}}}, d_1^{\bar{0}^{k,\hat{q}}}, \ldots, d_{k-1}^{\bar{0}^{k,\hat{q}}})}$. In particular, the choice of c, c' (which depended on q) does not affect $d^{\bar{0}^{k,\hat{q}}}$. This means ${}^k d^{\bar{0}^{k,\hat{q}}} = d^{(\bar{0}^{k,\hat{q}}, d_0^{\bar{0}^{k,\hat{q}}}, d_1^{\bar{0}^{k,\hat{q}}}, \ldots, d_{k-1}^{\bar{0}^{k,\hat{q}}})}$ as well. Canceling parts using Lemma 3 gives the desired conclusion. □

Construction of the habit aggregator

For each k define $\varphi_k : Q \to \mathbb{R}$ by $\varphi_k(q) = d_0^{\bar{0}^{k,q}}$ if $q > 0$ and $\varphi_k(0) = 0$. We naturally define $\varphi : H \to \mathbb{R}$ by $\varphi(h) = d_0^h = \sum_{k=1}^{\infty} \varphi_k(h_k)$.

Lemma 5 (Linearity)**.** *$\varphi_k(q) = \lambda_k q$ for some $\lambda_k > 0$ and for all $q \in Q$; and $d^{h',h} = d^{\bar{0},h-h'}$ for every $(h', h) \in \mathcal{H}$. This implies that $\varphi(h - h') = d_0^{h-h'} = d_0^{h',h}$*

Proof. By Lemma 2, we know that $\varphi_k(q + \varphi_k(\hat{q})) = \varphi_k(q) + d_0^{\bar{0}^{k,q}, \bar{0}^{k, q + d_0^{\bar{0}^{k,\hat{q}}}}}$ because

$$d_0^{\bar{0}^{k, q + d_0^{\bar{0}^{k,\hat{q}}}}} = d_0^{\bar{0}^{k,q}} + d_0^{\bar{0}^{k,q}, \bar{0}^{k, q + d_0^{\bar{0}^{k,\hat{q}}}}}.$$

But the last term above is $\varphi_k(\varphi_k(\hat{q}))$ because of Lemma 4, weak invariance. Then, by construction, $\varphi_k(\cdot)$ is additive on its image, i.e., for every k,

$$\varphi_k(\varphi_k(\hat{q}) + q) = \varphi_k(\varphi_k(\hat{q})) + \varphi_k(q) \quad \forall\, q, \hat{q} \in Q. \tag{18}$$

By Axiom NDM, $\varphi_k(\cdot)$ is onto Q.[21] Hence (18) is identical to a non-negativity restricted Cauchy equation (i.e., $f(x+y) = f(x) + f(y)$ for all $x, y \geq 0$) under the reparametrization $q' = \varphi_k(\hat{q})$. We know $\varphi_k(\cdot)$ is strictly monotone, so by Aczel & Dhombres (1989, Corollary 9), $\varphi_k(x) = \lambda x$ for some $\lambda > 0$. $\square$

Lemma 6 (Recursive Structure)**.** *For any $t \geq 0$ and $h \in H$, ${}^t d^h = d^{hd_0^h d_1^h \cdots d_{t-1}^h}$; hence $d_1^h = \varphi(h\varphi(h))$, $d_2^h = \varphi(h\varphi(h)\varphi(h\varphi(h)))$, etc.*

Proof. By strong induction. The lemma is true for $t = 0$: $d^h = d^h$. Assume that ${}^t d^h = d^{hd_0^h d_1^h \cdots d_{t-1}^h}$ for all t smaller than some $\hat{t}$. This implies that ${}^{\hat{t}+1}d^h = {}^1 d^{hd_0^h d_1^h \cdots d_{\hat{t}-1}^h}$. Using the inductive hypothesis with $hd_0^h d_1^h \cdots d_{\hat{t}-1}^h$ as the habit,

$$ {}^1 d^{hd_0^h d_1^h \cdots d_{\hat{t}-1}^h} = d^{hd_0^h d_1^h \cdots d_{\hat{t}-1}^h d_0^{hd_0^h d_1^h \cdots d_{\hat{t}-1}^h}}. $$

Once more by the inductive hypothesis, $d_{\hat{t}}^h = d_0^{hd_0^h d_1^h \cdots d_{\hat{t}-1}^h}$. Therefore, ${}^{\hat{t}+1}d^h$ is equal to $d^{hd_0^h d_1^h \cdots d_{\hat{t}}^h}$ as desired. $\square$

Lemma 7 (Geometric Decay)**.** *For all $h \in \mathcal{H}$, d^h is decreasing iff $\lambda_1 \in (0,1)$ and*

$$ \frac{\lambda_{k+1}}{\lambda_k} \leq 1 - \lambda_1 \quad \forall\, k \geq 1. \tag{19} $$

We remark that (19) clearly implies $\sum_{k=1}^{\infty} \lambda_k \leq 1$.

Proof. Lemmas 3, 5 and 6 together prove that each $d_t^{\bar{0}^{k,q}}$ may be written

$$ d_t^{\bar{0}^{k,q}} = q\lambda_{t+k} + \sum_{i=0}^{t-1} d_i^{\bar{0}^{k,q}} \lambda_{t-i}. \tag{20} $$

Therefore, for $t \geq 1$,

$$ d_{t-1}^{\bar{0}^{k,q}} - d_t^{\bar{0}^{k,q}} = \sum_{i=0}^{t-2} d_i^{\bar{0}^{k,q}} \lambda_{t-i-1} + q\lambda_{t-1+k} - \sum_{i=0}^{t-1} \bar{0}^{k,q} \lambda_{t-i} - \lambda_{t+k} q. \tag{21} $$

When $t = 1$, only the term $q(\lambda_k - \lambda_k \lambda_1 - \lambda_{k+1})$ remains in (21) for each k. Hence, the condition (19) holds if and only if $d_0^{\bar{0}^{k,q}} \geq d_1^{\bar{0}^{k,q}}$ for every k. Note that this also has the effect of implying $\lambda_1 < 1$, since $\lambda_k > 0$ for every k by Lemma 5. Now, we show that (19) guarantees that $d_{t-1}^{\bar{0}^{k,q}} \geq d_t^{\bar{0}^{k,q}}$ for every t. Indeed, rearranging (21)

[21] The solution of functional equation (18) is not fully characterized. Jarczyk (1991, pp. 52-61) proves continuous solutions must be affine. We know φ is a.e. continuous (without NDM).

and plugging in from (20), we obtain

$$\begin{aligned} d_{t-1}^{\bar{0}^{k},q} - d_{t}^{\bar{0}^{k},q} &= \sum_{i=0}^{t-2} d_i^{\bar{0}^{k},q}[\lambda_{t-i-1} - \lambda_{t-i}] + q[\lambda_{t-1+k} - \lambda_{t+k}] - \lambda_1 d_{t-1}^{\bar{0}^{k},q} \\ &= \sum_{i=0}^{t-2} d_i^{\bar{0}^{k},q}[\lambda_{t-i-1}(1-\lambda_1) - \lambda_{t-i}] + q[\lambda_{t-1+k}(1-\lambda_1) - \lambda_{t+k}]. \end{aligned}$$

Hence $d_{t-1}^{\bar{0}^{k},q} \geq d_{t}^{\bar{0}^{k},q}$ follows from condition (19). □

Lemma 8 (Persistent or Responsive)**.** *For any $h \in \mathcal{H}$,*

(i) If $\sum_{k=1}^{\infty} \lambda_k < 1$, d^h is infinitely summable. In particular, if for some $\varepsilon > 0$ there is k^* such that $\frac{\lambda_{k^*+1}}{\lambda_{k^*}} = 1 - \lambda_1 - \varepsilon$ then $\frac{d_t^h}{d_{t-1}^h} \leq 1 - \varepsilon\lambda_{k^*}$ for all $t \geq k^*$.

(ii) If $\sum_{k=1}^{\infty} \lambda_k = 1$ then d^h is a constant sequence.

Proof. For (i), let $\varepsilon = 1 - \lambda_1 - \frac{\lambda_{k^*+1}}{\lambda_{k^*}}$ and $x_{t,k^*} = \begin{cases} d_{t-1-k^*}^h & \text{if } t > k^* \\ h_{k^*+1-t} & \text{if } t \leq k^* \end{cases}$ Using the recursive construction of Lemma 6 and the fact that $\varphi(h0^t) = \sum_{k=t+1}^{\infty} \frac{\lambda_k}{\lambda_{k-t}} \lambda_{k-t} h_{k-t}$,

$$\frac{d_t^h}{d_{t-1}^h} = \frac{\varphi(h d_0^h \cdots d_{t-2}^h 0) + \lambda_1 d_{t-1}^h}{d_{t-1}^h} \leq \frac{(1-\lambda_1) d_{t-1}^h - \varepsilon x_{t,k^*} \lambda_{k^*} + \lambda_1 d_{t-1}^h}{d_{t-1}^h},$$

with equality if k^* uniquely satisfies $\frac{\lambda_{k+1}}{\lambda_k} < 1 - \lambda_1$. Since $d_{t-1-k^*}^h \geq d_{t-1}^h \ \forall\ t > k^*$,

$$\frac{d_t^h}{d_{t-1}^h} \leq \frac{(1-\lambda_1) d_{t-1}^h - \varepsilon d_{t-1-k^*}^h \lambda_{k^*} + \lambda_1 d_{t-1}^h}{d_{t-1}^h} = (1-\lambda_1) - \varepsilon \frac{d_{t-1-k^*}^h}{d_{t-1}^h} \lambda_{k^*} + \lambda_1 \leq 1 - \varepsilon\lambda_{k^*}.$$

For (ii), note that for all $q \in Q$, $\varphi(hq) = (1-\lambda_1)\varphi(h) + \lambda_1 q$. Therefore $\varphi(h\varphi(h)) = \varphi(h)$. The claim easily follows from induction and Lemma 6. □

Construction of the continuous preference relation $\succeq^*$

We use Axiom HC to construct a continuous map g from $H \times C$ into an auxiliary space C^*, as well as a continuous preference relation on C^* preserving $\succeq$. We endow the space $\times_{i=0}^{\infty}\mathbb{R}$ with the product topology and define the transformation $g : H \times C \to \times_{i=0}^{\infty}\mathbb{R}$ by $g(h, c) = (c_0 - \varphi(h), c_1 - \varphi(hc_0), c_2 - \varphi(hc_0c_1), \ldots)$. Let $C^* = g(H \times C)$ and $C_h^* = g(\{h\} \times C)$, for any $h \in H$, be the image and projected image under g, respectively. We shall consider C^* to be a metric subspace of $\times_{t=0}^{\infty}\mathbb{R}$, implying that C^* is a metric space in its own right. As a reminder, the spaces H and C are metrized by the sup metric $\rho^H(h,h') = \sup_k |h_k - h'_k|$ and the product metric $\rho^C(c,c') = \sum_{t=0}^{\infty} \frac{1}{2^t} \frac{|c_t - c'_t|}{1+|c_t - c'_t|}$ respectively. We metrize $H \times C$ by $\rho^{H\times C}((h,c),(h',c')) = \rho^H(h,h') + \rho^C(c,c')$.

Lemma 9 (Continuous Mapping)**.** *$g(\cdot,\cdot)$ is a continuous mapping; moreover, for any given $h \in H$, $g(h,\cdot)$ is a homeomorphism into C^*_h.*

Proof. The map is continuous in the topology if every component is. Linearity of φ implies that the t-th component can be written as $c_t - \varphi(h0^t) - \sum_{k=1}^{t} \lambda_k c_{t-k}$; as only there is only a finite sum of elements of c in each component, the map is continuous with respect to C. Using the sup metric it is clear that $\varphi(\cdot)$ is continuous with respect to H. The desired joint continuity is evident under the respective metrics. Finally, for any $h \in H$ we can directly exhibit the clearly continuous inverse $g^{-1}(h,\cdot) : C^*_h \to C$ defined by $g^{-1}(h,c^*)$ equal to

$$(c_0^* + \varphi(h), c_1^* + \varphi(h, c_0^* + \varphi(h)), c_2^* + \varphi\big(h, c_0^* + \varphi(h), c_1^* + \varphi(h, c_0^* + \varphi(h))\big), \ldots). \quad \square$$

Lemma 10 (Nestedness)**.** *$C^*_{h'} \subseteq C^*_h$ for any $(h',h) \in \mathcal{H}$.*

Proof. Take $(c_0 - \varphi(h'), c_1 - \varphi(h'c_0), c_2 - \varphi(h'c_0c_1), \ldots) \in C^*_{h'}$, so that $(c_0, c_1, c_2, \ldots) \in C$. For any $(h',h) \in \mathcal{H}$, $c + d^{h',h} \in C$. By Lemma 6 we know that $d^{h',h} = d^{h-h'} = (\varphi(h-h'), \varphi(h-h', \varphi(h'-h)), \ldots)$. Moreover, since φ is affine by Lemma 5,

$$\begin{aligned}(c_0 + &\varphi(h-h') - \varphi(h), c_1 + \varphi(h-h', \varphi(h-h')) - \varphi(h, c_0 + \varphi(h-h')), \ldots) \\ &= (c_0 + \varphi(h-h'-h), c_1 + \varphi(h-h'-h, \varphi(h-h') - c_0 - \varphi(h-h')), \ldots) \\ &= (c_0 - \varphi(h'), c_1 - \varphi(h'c_0), c_2 - \varphi(h'c_0c_1), \ldots) \in C^*_h. \qquad \square\end{aligned}$$

Lemma 11 (Topological Properties)**.** *C^* is separable, connected and convex.*

Proof. Connectedness follows from being the continuous image of a connected space. Convexity follows from convexity of C and H and linearity of $g(\cdot,\cdot)$. To see separability, construct the sequence $\{h^n\}_{n\in\mathbb{Z}}$ by $h^n = (\ldots, n, n, n)$. By Lemma 10, $C^* = \bigcup_{n\in\mathbb{Z}} C^*_{h^n}$. Since each $g(h^n,\cdot)$ is continuous, each $C^*_{h^n}$ is separable, being the continuous image of a separable space. Let $\tilde{C}^*_{h^n}$ denote the countable dense subset of each $C^*_{h^n}$. Then $\bigcup_{n\in\mathbb{Z}} \tilde{C}^*_{h^n}$ is a countable dense subset of C^*.[22] $\square$

We define a binary relation $\succeq^*$ on $C^* \times C^*$ by

$$g(h,c) \succeq^* g(h,\dot{c}) \text{ iff } c \succeq_h \dot{c}. \tag{22}$$

[22] Note H is not separable under the sup metric; if we were to make H separable by endowing it with the product topology instead, then g would not be continuous with respect to h.

Note that the definition of $\succeq^*$ can be rewritten as $c^* \succeq^* \dot{c}^*$ if and only if $c^*, \dot{c}^* \in C^*_h$ and $g^{-1}(h, c^*) \succeq_h g^{-1}(h, \dot{c}^*)$ for some $h \in H$.

Lemma 12 (Well-Definedness)**.** *The relation $\succeq^*$ is well-defined.*

Proof. Suppose there are h, h' and $c^*, \dot{c}^* \in C^*_h, C^*_{h'}$ with $g^{-1}(h, c^*) \succeq_h g^{-1}(h, \dot{c}^*)$ and $g^{-1}(h', \dot{c}^*) \succ_{h'} g^{-1}(h', c^*)$. We apply HC(i) to both relationships: $g^{-1}(h, c^*) + d^{h,\bar{h}} \succeq_{\bar{h}} g^{-1}(h, \dot{c}^*) + d^{h,\bar{h}}$ and $g^{-1}(h', \dot{c}^*) + d^{h',\bar{h}} \succ_{\bar{h}} g^{-1}(h', c^*) + d^{h',\bar{h}}$. But both $g^{-1}(h, c^*) + d^{h,\bar{h}}$ and $g^{-1}(h', c^*) + d^{h',\bar{h}}$ are equal to $g^{-1}(\bar{h}, c^*)$ (similarly for $\dot{c}^*$). Hence the statements above are contradictory. □

Lemma 13 (Continuous Preference)**.** *$\succeq^*$ is a continuous preference relation.*

Proof. The $C^*_{h'}$ are nested by Lemma 10. Thus for any $c^*, \dot{c}^*, \hat{c}^* \in C^*$, there is $h \in H$ large enough so that $c^*, \dot{c}^*, \hat{c}^* \in C^*_h$. Hence $\succeq^*$ inherits completeness and transitivity over $\{c^*, \dot{c}^*, \hat{c}^*\}$ from $\succeq_h$, which suffices since $c^*, \dot{c}^*, \hat{c}^*$ were arbitrary.

To prove that $\succeq^*$ is continuous in the product topology, we will show that the weak upper contour sets are closed; the argument for the weak lower contour sets is identical. Consider any sequence of streams $\{c^{*n}\}_{n\in\mathbb{Z}} \in C^*$ converging to some $c^* \in C^*$ and suppose there is $\hat{c}^* \in C^*$ such that $c^{*n} \succeq^* \hat{c}^*$ for all n. Take any h and c such that $g(h, c) = c^*$. By Lemma 9, g is continuous: for any ε-ball $Y \subset C^*$ around c^* there is a δ-ball $X \subset H \times C$ around (h, c) such that $g(X) \subset Y$. Because the sequence $\{c^{*n}\}$ converges to c^* there is a subsequence $\{c^{*m}\} \in Y$ also converging to c^*. By our use of the sup metric on H we know that any $(h', c') \in X$ must satisfy $h' \leq h + (\delta, \delta, \ldots)$. Then Lemma 10 ensures that for every m, $c^{*m} \in C^*_{h+(\ldots,\delta,\delta)}$. Take $\bar{h} \geq h + (\ldots, \delta, \delta)$ and large enough that $\hat{c}^* \in C^*_{\bar{h}}$. We may compare the corresponding streams in C under $\succeq_{\bar{h}}$. Using $g^{-1}(\bar{h}, \cdot)$ as defined in the proof of Lemma 9, take $\bar{c}^m = g^{-1}(\bar{h}, c^{*m})$ for each m, $\bar{c} = g^{-1}(\bar{h}, c^*)$, and $\hat{\bar{c}} = g^{-1}(\bar{h}, \hat{c}^*)$. Using the hypothesis and the definition of $\succeq^*$ we know that $\bar{c}^m \succeq_{\bar{h}} \hat{\bar{c}}$ for every m. Lemma 9 asserts that $g^{-1}(\bar{h}, \cdot)$ is continuous, so $\bar{c}^m$ converges to $\bar{c}$. Since $\succeq_{\bar{h}}$ is continuous by Axiom C, we know $\bar{c} \succeq_{\bar{h}} \hat{\bar{c}}$, proving that $c^* \succeq^* \hat{c}^*$. □

Standard results then imply $\succeq^*$ has a continuous representation $U^* : C^* \to \mathbb{R}$.

Lemma 14 (Koopmans Sensitivity)**.** *There exist $q^*, \hat{q}^* \in \mathbb{R}$, $c^* \in C^*$, and $t \in \mathbb{N}$ such that $(c^{*t-1}, q^*, {}^{t+1}c^*) \succ^* (c^{*t-1}, \hat{q}^*, {}^{t+1}c^*)$.*

Proof. Let $\alpha > 0$, $h \in H$, and $c \in C$ be such that $c + \alpha \not\sim_h c$. Since the compensating streams are (weakly) decreasing and $d_0^{\bar{0}\alpha} < \alpha$ for all $\alpha > 0$, we can write any positive constant stream as a staggered sum of streams of the form $(\alpha, d^{\bar{0}\alpha})$.

Formally, for any $\alpha > 0$ we can find a sequence of times $t^1 < t^2 < \cdots$ and positive numbers $\alpha > \alpha^1 > \alpha^2 > \cdots$ such that the stream $(\alpha, \alpha, \ldots)$ can be written as the consumption stream given by $(\alpha, d^{\bar{0}\alpha})$ starting at time 0, plus $(\alpha^1, d^{\bar{0}\alpha^1})$ starting at time t^1, plus $(\alpha^2, d^{\bar{0}\alpha^2})$ starting at time t^2, etc. Suppose by contradiction that $\forall$ $q^*, \hat{q}^* \in \mathbb{R}$, $c^* \in C^*$, and $t \in \mathbb{N}$, $(c^{*t-1}, q^*, {}^{t+1}c^*) \sim^* (c^{*t-1}, \hat{q}^*, {}^{t+1}c^*)$. Let $g(h, c) = c^*$ where h, c are given as initially stated. Then $(c^{*t-1}, c^*_t + \alpha, {}^{t+1}c^*) \sim^* c^*$ by hypothesis. By definition, this means that $g^{-1}(h, (c^{*t-1}, c^*_t + \alpha, {}^{t+1}c^*)) \sim_h g^{-1}(h, (c^*))$, or $(c^{t-1}, c_t + \alpha, {}^{t+1}c + d^{\bar{0}\alpha}) \sim_h c$. Iterative application of the indifference for $\alpha^1, \alpha^2, \ldots$ and product continuity imply that $c + (\alpha, \alpha, \ldots) \sim_h c$, violating Axiom S. □

Separability conditions for $\succeq^*$

We now prove that Compensated Separability suffices for the required additive separability conditions to hold by showing that the following mapping from C into C^* is surjective, so the needed conditions apply for all elements of C^*. For each t, define the "compensated consumption" map $\xi_t : H \times C \to C^*$ by

$$\xi_t(h, c) = g\big(h, (c^{t-1}, {}^{t}c + d^{h0^t, hc^{t-1}})\big). \tag{23}$$

To show ξ_t is surjective, we first prove the following auxiliary result.

Lemma 15 (Containment)**.** *For any $h \in H$, $t \geq 0$ and $c^t \in Q^{t+1}$, there exists $\hat{h} \in H$ large enough that $C^*_{hc^t} \subseteq C^*_{\hat{h}0^{t+1}}$.*

Proof. Since φ is linear and strictly increasing, we may choose $\hat{h} > h$ such that

$$\varphi(\hat{h}0^{t+1}) - \varphi(hc^t) \geq \sum_{s=0}^{t} (1 - \lambda_1)^{s+1} c_s. \tag{24}$$

Choose any $c^* \in C^*_{hc^t}$. Then, there is a $\dot{c} \in C$ such that $g(hc^t, \dot{c}) = c^*$. For it to also be true that $c^* \in C^*_{\hat{h}0^{t+1}}$ it must be that for some $\hat{c} \in C$,

$$\hat{c}_s - \varphi(\hat{h}0^{t+1}\hat{c}^{s-1}) = c^*_s = \dot{c}_s - \varphi(hc^t\dot{c}^{s-1}) \quad \forall s \geq 0, \tag{25}$$

where $c^{-1}, \dot{c}^{-1}$ are ignored for the case $s = 0$. We claim that we can construct a $\hat{c} \in C$ (nonnegative, bounded) by using (25) to recursively define $\hat{c}_s = \varphi(\hat{h}0^{t+1}\hat{c}^{s-1}) + \dot{c}_s - \varphi(hc^t\dot{c}^{s-1})$ for every $s \geq 0$.

Step (i): $\hat{c}$ is nonnegative. It suffices to show $\hat{c} \geq \dot{c}$. For $s = 0$ it is clearly true that $\hat{c}_0 \geq \dot{c}_0$, since we have chosen $\varphi(\hat{h}0^{t+1}) - \varphi(hc^t) \geq 0$ in (24). We proceed by strong induction, assuming $\hat{c}_{\hat{s}-1} \geq \dot{c}_{\hat{s}-1}$ for every $\hat{s} \leq s$. From (25), to prove $\hat{c}_s \geq \dot{c}_s$ we must show $\varphi(\hat{h}0^{t+1}\hat{c}^{s-1}) - \varphi(hc^t\dot{c}^{s-1}) \geq 0$. By the inductive hypothesis,

$$
\begin{aligned}
\varphi(\hat{h}0^{t+1}\hat{c}^{s-1}) - \varphi(hc^t\dot{c}^{s-1}) &= \varphi\big((\hat{h}-h)0^{t+s+1}\big) + \varphi\big(\bar{0}(\hat{c}_1-\dot{c}_1)\cdots(\hat{c}_{s-1}-\dot{c}_{s-1})\big) + \\
&\qquad \varphi\big(\bar{0}(\hat{c}_0-\dot{c}_0)0^{s-1}\big) - \varphi(\bar{0}c^t0^s) \\
&\geq \varphi\big(\bar{0}(\hat{c}_0-\dot{c}_0)0^{s-1}\big) - \varphi(\bar{0}c^t0^s) \\
&= \varphi\Big(\bar{0}\big(\varphi(\hat{h}0^{t+1}) - \varphi(hc^t)\big)0^{s-1}\Big) - \varphi(\bar{0}c^t0^s) \\
&\geq \lambda_s \sum_{i=0}^{t} (1-\lambda_1)^{i+1} c_i - \sum_{i=0}^{t} \lambda_{s+1+i} c_i
\end{aligned}
\tag{26}
$$

where the first inequality comes from the nonnegativity of φ; the equality comes from plugging in for $\hat{c}_0 - \dot{c}_0$ from (25); and the second inequality comes from (24) and Lemma 5. By Lemma 7, $\frac{\lambda_{s+1+i}}{\lambda_s} \leq (1-\lambda_1)^{i+1}$, hence 26 is positive.

Step (ii): $\hat{c}$ remains bounded. Since $\dot{c} \in C$ it is bounded, so it will suffice to show that the difference between $\hat{c}$ and $\dot{c}$ is bounded. Let us denote by y the quantity $\varphi\big((\hat{h}-h)0^{t+2}\big) + \varphi\big(\bar{0}(\hat{c}_0-\dot{c}_0)0\big)$. By construction, for every $s \geq 1$, $\hat{c}_s - \dot{c}_s$ is equal to the sum on the RHS of the first equality in (26). By the fading nature of compensation, all terms but $\varphi\big(\bar{0}(\hat{c}_1-\dot{c}_1)\cdots(\hat{c}_s-\dot{c}_s)\big)$ converge to 0 as s tends to infinity. In fact, for any h and t, $\varphi(h0^t) \leq (1-\lambda_1)^t\varphi(h)$. Consequently, the sum $\varphi\big((\hat{h}-h)0^{t+s+1}\big) + \varphi\big(\bar{0}(\hat{c}_0-\dot{c}_0)0^{s-1}\big)$ is no bigger than $(1-\lambda_1)^{s-1}y$ for any s. Let us drop the negative term $-\varphi(\bar{0}c^t0^s)$ in (26) to obtain an upper bound. By the definition of y, we see that $\hat{c}_1 - \dot{c}_1 \leq y$. We claim that for all $s \geq 1$, $\hat{c}_s - \dot{c}_s \leq y$. The proof proceeds by strong induction. Using the inductive hypothesis, $\hat{c}_s - \dot{c}_s \leq y(1-\lambda_1)^{s-1} + y\sum_{k=1}^{s-1}\lambda s$. But $\sum_{k=1}^{s-1}\lambda s \leq \lambda_1 \sum_{k=0}^{s-2}(1-\lambda_1)^k = 1-(1-\lambda_1)^{s-1}$, so $\hat{c}_s - \dot{c}_s \leq y$ as claimed. □

Lemma 16 (Surjectivity)**.** *Each ξ_t as defined in* (23) *is surjective.*

Proof. Fix any $c^* \in C^*$ and $t \geq 1$. By definition, there is $h \in H$ and $c \in C$ such that $g(h,c) = c^*$. That is, for every s, $c_s - \varphi(hc_0c_1\dots c_{s-1}) = c^*_s$. Fix this h and c.

We wish to show that there exist $\hat{h} \in H$ and $\hat{c} \in C$ such that $\xi_t(\hat{h},\hat{c}) = c^*$, i.e.

$$
(\ \hat{c}_0 - \varphi(\hat{h}), \hat{c}_1 - \varphi(\hat{h}\hat{c}_0), \dots, \hat{c}_{t-1} - \varphi(\hat{h}\hat{c}_0\dots\hat{c}_{t-2}), \hat{c}_t - \varphi(\hat{h}0^t), \hat{c}_{t+1} - \varphi(\hat{h}0^t\hat{c}_t), \dots\ ) = c^*. \tag{27}
$$

Because $c^* \in C^*_h$, ${}^tc^* \in C^*_{hc^{t-1}}$. Equation (27) suggests that we must show that ${}^tc^* \in C^*_{\hat{h}0^t}$ for some $\hat{h} \in H$. Lemma 15 provides a $\bar{c}$ and $\hat{h} > h$ s.t. $g(\hat{h}0^t, \bar{c}) = {}^tc^*$.

Moreover, since $\hat{h} > h$, $c^* \in C^*_{\hat{h}}$. Therefore, there exists $\bar{\bar{c}} \in C$ such that $g(\hat{h}, \bar{\bar{c}}) = c^*$ and in particular, $g(\hat{h}, \bar{\bar{c}})^{t-1} = c^{*t-1}$. Setting $\hat{c} = (\bar{\bar{c}}^{t-1}, {}^t\bar{c})$, we have $\xi_t(\hat{h}, \hat{c}) = c^*$. □

Lemma 17 (Separability)**.** $\succeq^*$ *satisfies the following separability conditions:*
(i) Take any $c^*, \hat{c}^* \in C^*$ with $c^*_0 = \hat{c}^*_0$. Then, for any $\bar{c}^*_0$ s.t. $(\bar{c}^*_0, {}^1c^*), (\bar{c}^*_0, {}^1\hat{c}^*) \in C^*$,

$$(c^*_0, {}^1c^*) \succeq^* (c^*_0, {}^1\hat{c}^*) \text{ iff } (\bar{c}^*_0, {}^1c^*) \succeq^* (\bar{c}^*_0, {}^1\hat{c}^*). \tag{28}$$

(ii) For any $t \geq 0$, $c^*, \hat{c}^*, \bar{c}^*, \bar{\bar{c}}^* \in C^*$ s.t. $(c^{*t}, \bar{c}^*)$, $(\hat{c}^{*t}, \bar{c}^*)$, $(c^{*t}, \bar{\bar{c}}^*)$, $(\hat{c}^{*t}, \bar{\bar{c}}^*) \in C^*$,

$$(c^{*t}, \bar{c}^*) \succeq^* (\hat{c}^{*t}, \bar{c}^*) \text{ iff } (c^{*t}, \bar{\bar{c}}^*) \succeq^* (\hat{c}^{*t}, \bar{\bar{c}}^*). \tag{29}$$

Proof. The proof of Condition (i) is analogous to the proof on page 16 for one-period histories. We now prove Condition (ii).

Find h large enough so that $(c^{*t}, \bar{c}^*)$, $(\hat{c}^{*t}, \bar{c}^*)$, $(c^{*t}, \bar{\bar{c}}^*)$, $(\hat{c}^{*t}, \bar{\bar{c}}^*) \in C^*_h$. Hence, there exist $\tilde{c}, \tilde{\tilde{c}}, \dot{c}, \ddot{c}$ such that $g(h, \tilde{c}) = (c^{*t}, \bar{c}^*)$, $g(h, \tilde{\tilde{c}}) = (\hat{c}^{*t}, \bar{c}^*)$, $g(h, \dot{c}) = (c^{*t}, \bar{\bar{c}}^*)$, and $g(h, \ddot{c}) = (\hat{c}^{*t}, \bar{\bar{c}}^*)$. Moreover, we must have $\tilde{c}^t = \dot{c}^t$ and $\tilde{\tilde{c}}^t = \ddot{c}^t$.

By Lemma 16, ξ_t is surjective. We claim there are $\hat{h}$ and $c, \hat{c}, \bar{c}, \bar{\bar{c}} \in C$ so that

$$\begin{aligned} &\xi_t(\hat{h}, (c^t, \bar{c})) = (c^{*t}, \bar{c}^*),\ \xi_t(\hat{h}, (\hat{c}^t, \bar{c})) = (\hat{c}^{*t}, \bar{c}^*), \\ &\xi_t(\hat{h}, (c^t, \bar{\bar{c}})) = (c^{*t}, \bar{\bar{c}}^*),\ \xi_t(\hat{h}, (\hat{c}^t, \bar{\bar{c}})) = (\hat{c}^{*t}, \bar{\bar{c}}^*). \end{aligned} \tag{30}$$

Recalling the construction in Lemma 15, choose $\hat{h} > h$ large enough so that

$$\varphi(\hat{h}0^{t+1}) \geq \max\Big\{ \sum_{s=0}^{t} (1-\lambda_1)^{s+1} \tilde{c}_s + \varphi(h\tilde{c}^t), \sum_{s=0}^{t} (1-\lambda_1)^{s+1} \tilde{\tilde{c}}_s + \varphi(h\tilde{\tilde{c}}^t) \Big\}.$$

Now that we have an $\hat{h}$ that will work uniformly for these four streams in C^*, note again from the construction in Lemma 15 that the required continuation streams depend only on $\tilde{c}^t = \dot{c}^t$ and $\tilde{\tilde{c}}^t = \ddot{c}^t$. Therefore, $\bar{c}$ and $\bar{\bar{c}}$ may be constructed as desired in (30). From the construction at the end of Lemma 16 and the fact that $\hat{h}$ has been chosen to work uniformly, c and $\hat{c}$ may be chosen to satisfy (30).

Consequently, using (30), the desired result (29) holds if and only if

$$\xi_t(\hat{h}, (c^t, \bar{c})) \succeq^* \xi_t(\hat{h}, (\hat{c}^t, \bar{c})) \text{ iff } \xi_t(\hat{h}, (c^t, \bar{\bar{c}})) \succeq^* \xi_t(\hat{h}, (\hat{c}^t, \bar{\bar{c}})),$$

which, using the definitions of ξ_t in (23) and $\succeq^*$, holds if and only if

$$(c^{t-1}, \bar{c} + d^{h0^t, hc^{t-1}}) \succeq_{\hat{h}} (\hat{c}^{t-1}, \bar{c} + d^{h0^t, h\hat{c}^{t-1}}) \text{ if and only if}$$
$$(c^{t-1}, \bar{\bar{c}} + d^{h0^t, hc^{t-1}}) \succeq_{\hat{h}} (\hat{c}^{t-1}, \bar{\bar{c}} + d^{h0^t, h\hat{c}^{t-1}}).$$

But this is immediately true by Compensated Separability. □

For each subset of indices $\mathbb{K} \subset \mathbb{N}$, we will define the projection correspondences $\iota_{\mathbb{K}} : C^* \rightsquigarrow \times_{i\in\mathbb{K}}\mathbb{R}$ by $\iota_{\mathbb{K}}(\hat{C}^*) = \{x \times_{i\in\mathbb{K}} \mathbb{R} \mid \exists\ c^* \in \hat{C}^* \text{ s.t. } c^*|_{\mathbb{K}} = x\ \}$, where $c^*|_{\mathbb{K}}$ denotes the restriction of the stream c^* to the indices in $\mathbb{K}$ (e.g., $c^*|_{\{3,4\}} = (c_3^*, c_4^*)$). For any $t \geq 0$ we will use C_t^* and ${}^tC^*$ to denote the projected spaces $\iota_{\{t\}}(C^*)$ and $\iota_{\{t,t+1,\ldots\}}(C^*)$, respectively. Since $g(\cdot,\cdot)$ is continuous the projected image C_t^* is connected for every t. Moreover each C_t^* is separable. It is evident by the arbitrariness of histories used to construct these spaces that for any t, ${}^tC^* = C^*$.

Lemma 18 (Product of Projections)**.** *Choose some t and $\hat{c}^* \in {}^tC^*$, and take $c_s^* \in C_s^*$ for every $0 \leq s \leq t$. Then $(c_0^*, c_1^*, \ldots, c_t^*, \hat{c}^*) \in C^*$.*

Proof. Pick $\hat{h} \in H$ and $\hat{c} \in C$ such that $\hat{c}^* \in C^*_{\hat{h}\hat{c}^t}$, and let $\tilde{c}_t^* = g(\hat{h}, \hat{c})|_{\{0,1,\ldots,t\}}$. Choose any $\varepsilon \geq \max\{0, \max_{0\leq i\leq t} \frac{\tilde{c}_i^* - c_i^*}{\sum_{k=i+1}^{\infty} \lambda_k}\}$ and set $h = \hat{h} + (\ldots, \varepsilon, \varepsilon)$. Recall the inverse function $g^{-1}(h, \cdot)$, which takes an element of C^* and returns an element of C. We do not know that $(c_0^*, c_1^*, \ldots, c_t^*, \hat{c}^*) \in C^*$, but we demonstrate that applying the transformation used in $g^{-1}(h,\cdot)$ to $(c_0^*, c_1^*, \ldots, c_t^*, \hat{c}^*)$ returns a nonnegative stream. Let us take $c^t = g^{-1}(h, (c_0^*, c_1^*, \ldots, c_t^*, \hat{c}^*))|_{\{0,1,\ldots,t\}}$. Since the $C^*_{h'}$ are nested and $h \geq \hat{h}$, it will suffice to prove that $c^t \geq \hat{c}^t$, for then $\hat{c}^* \in C^*_{hc^t}$ and there is a $\bar{c} \in C$ such that $g(h, (c^t, \bar{c})) = (c_0^*, c_1^*, \ldots, c_t^*, \hat{c}^*)$. Using the transformation, $c^t \geq \hat{c}^t$ if and only if $c_0^* + \varphi(h) \geq \tilde{c}_0^* + \varphi(\hat{h})$, $c_1^* + \varphi(hc_0) \geq \tilde{c}_1^* + \varphi(\hat{h}\hat{c}_0)$, up through $c_t^* + \varphi(hc_0 \ldots c_{t-1}) \geq \tilde{c}_t^* + \varphi(\hat{h}\hat{c}_0 \ldots \hat{c}_{t-1})$. But this can be seen using induction, the choices of ε and h, and the fact that φ is linear and strictly increasing. □

We have proved that $C^* = C_0^* \times C_1^* \times C_2^* \times C^*$ and that $\succeq^*$ is continuous and sensitive (stationarity implies essentiality of all periods). Hence C^* is a product of separable and connected spaces. We now use the result of Gorman (1968, Theorem 1), which requires that each of C_0^*, C_1^*, C_2^* and C^* be arc-connected and separable. We have shown separability; and arc-connectedness follows from being a path-connected Hausdorff space (a convex space is path-connected, and a metric space is Hausdorff). Gorman's Theorem 1 asserts that the set of separable indices is closed under unions, intersections, and differences. Condition (29) implies separability of $\{(0),(1)\}$ and stationarity implies separability of $\{(1,2,3,\ldots)\}$ and

$\{(2,3,4,\ldots)\}$, etc..[23] Repeated application of Gorman's theorem implies Debreu's additive separability conditions for $n=4$ and we may conclude (Fishburn (1970, Theorem 5.5)) that there exist $u_0, u_1, u_2 : \mathbb{R} \to \mathbb{R}$ and $U_3 : C^* \to \mathbb{R}$ (all continuous and unique up to a similar positive linear transformation) such that $c^* \succeq^* \hat{c}^*$ iff $u_0(c_0^*) + u_1(c_1^*) + u_2(c_2^*) + U_3({}^3c^*) \geq u_0(\hat{c}_0^*) + u_1(\hat{c}_1^*) + u_2(\hat{c}_2^*) + U_3({}^3\hat{c}^*)$.

$\succeq_h$ can be represented as in (1)

Lemma 19 (Representation)**.** *For some continuous* $u(\cdot)$ *and* $\delta \in (0,1)$*,* $U_h(c) = \sum_{t=0}^{\infty} \delta^t u(c_t - \sum_{k=1}^{\infty} \lambda_k h_k^{(t)})$*, where* $h^{(t)} = (h, c_0, c_1, \ldots, c_{t-1})$*.*

Proof. $\succeq^*$ is a continuous, stationary, and sensitive preference relation; and can be represented in the form $u_0(\cdot) + u_1(\cdot) + u_2(\cdot) + U_3(\cdot)$ on the space $C^* = C_0^* \times C_1^* \times C_2^* \times C^*$, with the additive components continuous and unique up to a similar positive affine transformation. There is also additive representability on $C^* = C_0^* \times C_1^* \times C^*$, with the additive components again unique up to a similar positive linear transformation. By stationarity, $u_0(\cdot)+u_1(\cdot)+[u_2(\cdot)+U_3(\cdot)]$ and $u_1(\cdot)+u_2(\cdot)+U_3(\cdot)$ are both additive representations on $C^* = C_0^* \times C_1^* \times C^*$. Thus, $\exists\ \delta > 0$ and $\beta_1, \beta_2, \beta_3 \in \mathbb{R}$ s.t. $u_1(\cdot) = \delta u_0(\cdot) + \beta_1$, $u_2(\cdot) = \delta u_1(\cdot) + \beta_2 = \delta^2 u_0(\cdot) + \delta\beta_1 + \beta_2$, and for any $c^* \in C^*$, $U_3(c^*) = \delta[u_2(c_0^*) + U_3({}^1c^*)] + \beta_3 = \delta^3 u_0(c_0^*) + \delta U_3({}^1c^*) + \beta_3 + \delta\beta_2 + \delta^2\beta_1$. Each $c \in C$ and $h \in H$ is bounded and $\sum_{k=1}^{\infty} \lambda_k \leq 1$, so for each $c^* \in C^*$ $\exists\ \bar{x}, \underline{x} \in \mathbb{R}$ such that $\underline{x} \leq c_t^* \leq \bar{x}\ \forall\ t$. By Tychonoff's theorem $[\underline{x}, \bar{x}]^\infty$ is compact in $\times_{i=0}^{\infty}\mathbb{R}$ and therefore $[\underline{x}, \bar{x}]^\infty \cap C^*$ is compact in C^*. Given $\underline{x}$ and $\bar{x}$, continuity of $u_0(\cdot)$ and $U_3(\cdot)$ ensures they remain uniformly bounded on $[\underline{x}, \bar{x}]$ and $[\underline{x}, \bar{x}]^\infty \cap C^*$, respectively. Using iterative substitution $U^*(c^*) = \sum_{t=0}^{\infty} \delta^t u(c_t^*)$, where $u(\cdot) = u_0(\cdot)$ is continuous and $\delta \in (0,1)$ by product continuity. To represent $\succeq_h$ as in (1) we then transform each $c \in C$ by $g(h, \cdot)$ into an argument of U^*. □

The felicity u is not (quasi-)cyclic

We first prove the following auxiliary result.[24]

Lemma 20 (Rewriting)**.** *Consider any sequence* $\{\gamma_t\}_{t\in\mathbb{N}}$ *and* $h \in H$*. If* $\bar{c} \in \times_{t=0}^{\infty}\mathbb{R}$

[23] Because (29) hold for all t it is an even stronger hypothesis than necessary; also, for any t, $\{(t, t+1, t+2, \ldots)\}$ is strictly sensitive by dynamic consistency.

[24] For technical convenience, the statement of this lemma allows an extension of the definition of compensation to negative "histories;" hence if $\gamma < 0$ then $d^{(\bar{0},\gamma)} = -d^{(\bar{0},-\gamma)}$.

satisfies $\bar{c}_t = \varphi(h\bar{c}^{t-1}) + \gamma_t$ *for every* t *then each* $\bar{c}_t$ *may be alternatively written as*

$$\bar{c}_t = \gamma_t + d^h_t + \sum_{s=0}^{t-1} d_s^{\bar{0}\gamma_{t-s-1}}. \tag{31}$$

Proof. It is clearly true for $t = 0$. Suppose (31) holds for every $t \leq T - 1$. Then

$$\begin{aligned}\bar{c}_T &= \gamma_T + \varphi(h\bar{c}^{T-1}) \\ &= \gamma_T + \varphi(h, \gamma_0 + d^h_0, \gamma_1 + d^h_1 + d_0^{\bar{0}\gamma_0}, \dots, \gamma_{T-1} + d^h_{T-1} + \sum_{s=0}^{T-2} d_s^{\bar{0}\gamma_{T-s-2}}) \\ &= \gamma_T + \varphi(hd^h_0 \cdots d^h_{T-1}) + \sum_{s=0}^{T-1} \varphi(\bar{0}\gamma_s d_0^{\bar{0}\gamma_s} \cdots d_{T-2-s}^{\bar{0}\gamma_s}) \\ &= \gamma_T + d^h_T + \sum_{s=0}^{T-1} d_s^{\bar{0}\gamma_{T-s-1}},\end{aligned}$$

where the second-to-last equality follows from using the recursive characterization given in Lemma 6 and reversing the order of summation. □

Lemma 21 (Acyclicity)**.** $u(\cdot)$ *is not cyclic, and is not quasi-cyclic if* $\sum_{k=1}^{\infty} \lambda_k < 1$.

Proof. The two cases are examined separately.
Case (i): $\sum_{k=1}^{\infty} \lambda_k < 1$. Suppose that u is quasi-cyclic, so there exists $\gamma, \beta > 0$ and $\alpha \in \mathbb{R}$ such that $u(x + \gamma) = \beta u(x) + \alpha$ for every $x \in \mathbb{R}$. Apply Lemma 20 with $\gamma_t = \gamma$ for every t and recall the summability of per-period compensation from Lemma 8. These results imply that $\bar{c}$ as defined in Lemma 20 remains bounded, i.e. $\bar{c} \in C$. Moreover $\bar{c}_0 = \gamma$, so c is nonzero. We claim this $\bar{c}$ is exactly ruled out in Lemma 1, a contradiction. By the representation $c + \bar{c} \succeq_h c' + \bar{c}$ iff

$$\sum_{t=0}^{\infty} \delta^t u\big(c_t + \bar{c}_t - \varphi(hc^{t-1}) - \varphi(\bar{0}\bar{c}^{t-1})\big) \geq \sum_{t=0}^{\infty} \delta^t u\big(c'_t + \bar{c}_t - \varphi(hc'^{t-1}) - \varphi(\bar{0}\bar{c}^{t-1})\big).$$

Consider the t-th term $u\big(c_t + \bar{c}_t - \varphi(hc^{t-1}) - \varphi(\bar{0}\bar{c}^{t-1})\big)$. By construction of $\bar{c}$, this term is equal to $u\big(c_t - \varphi(hc^{t-1}) + \gamma\big) = \beta u(c_t - \varphi(hc^{t-1})) + \alpha$. Since $\beta > 0$, it becomes evident that $c + \bar{c} \succeq_h c' + \bar{c}$ iff $c \succeq_h c'$ for any $c, c' \in C$.
Case (ii): $\sum_{k=1}^{\infty} \lambda_k = 1$. Suppose that u is cyclic. Then there exists $\gamma > 0$ and $\alpha \in \mathbb{R}$ such that $u(x + \gamma) = u(x) + \alpha$ for every $x \in \mathbb{R}$. In this case, simply choose $\bar{c}_0 = \gamma$ and $\bar{c}_t = \varphi(\bar{0}\bar{c}^{t-1})$ for every $t \geq 1$. Clearly $\bar{c} \in C$. It is easy to check that $c + \bar{c} \succeq_h c' + \bar{c}$ iff $c \succeq_h c'$ for any $c, c' \in C$, violating Lemma 1. □

B.2 Necessity

The constructive proof of sufficiency has proved all but uniqueness of compensation.

Lemma 22 (Unique Compensation). *Given the representation, for every* $(h', h) \in \mathcal{H}$ *there is a unique* $d \in C$ *satisfying* $c + d \succeq_h c' + d$ *iff* $c \succeq_{h'} c'$ *for every* $c', c \in C$.

Proof. Suppose both $d^{h',h}$ as constructed earlier and some $d \in C$, $d \neq d^{h',h}$ satisfyy the condition. By the representation for $\succeq_{h'}$, both the utility functions $\sum_{t=0}^{\infty} \delta^t u\big(c_t - \varphi(h'c^{t-1}) + d_t - \varphi((h-h')d^{t-1})\big)$ and $\sum_{t=0}^{\infty} \delta^t u\big(c_t - \varphi(h'c^{t-1})\big)$ represent $\succeq_{h'}$. Using the uniqueness of the additive representation, there exist $\beta > 0$ and a sequence $\{\alpha_t\}_{t\geq 0}$ such that for any $c \in C$,

$$u\big(c_t - \varphi(h'c^{t-1}) + d_t - \varphi((h-h')d^{t-1})\big) = \beta u\big(c_t - \varphi(h'c^{t-1})\big) + \alpha_t.$$

Let $\gamma_t = d_t - \varphi((h-h')d^{t-1})$ for every t; we must show $\gamma_t = 0$ for all t. For any $x \in \mathbb{R}$ and any t, there is $c \in C$ such that $c_t - \varphi(h'c^{t-1}) = x$. Indeed, if $x \geq 0$ choose $c_s = 0$ for $s < t$ and $c_t = \varphi(h'0^t) + x$; if $x < 0$, choose $c_s = 0$ for $s < t-1$, $c_{t-1} = \frac{x}{\lambda_1}$, and $c_t = \varphi(h'0^t)$. Hence $u(x + \gamma_t) = \beta u(x) + \alpha_t$ for all x, t.

Suppose that $\sum_{k=1}^{\infty} \lambda_k < 1$. Consider the first nonzero γ_t. If it is positive then u is quasi-cyclic, a contradiction. If $\gamma_t < 0$, then rearranging and changing variables gives $u(x + |\gamma_t|) = \frac{1}{\beta} u(x) - \frac{\alpha_t}{\beta}$. Hence u is quasi-cyclic, a contradiction.

Now suppose $\sum_{k=1}^{\infty} \lambda_k = 1$. If some $\gamma_t = 0$ then $u(x)(1-\beta) = \alpha_t$ for all x, implying that $\beta = 1$ and u is cyclic, a contradiction. Hence $\gamma_t \neq 0$ for every t. We aim to show there exist $t, \hat{t}$ such that $\gamma_t \neq \gamma_{\hat{t}}$. If instead $\gamma_t = \gamma$ for every t, then we know that $\gamma > 0$ from Lemma 26 in the supplemental Appendix. That lemma says that for any $\gamma < 0$, there does not exist a stream $c \in C$ and history $\hat{h} \in H$ such that $g(\hat{h}, c) \leq (\gamma, \gamma, \ldots)$ (apply the lemma with $\hat{h} = h - h'$ and $c = d$). But if $\gamma > 0$, then $d_t = \varphi((h-h')d^{t-1}) + \gamma$ cannot be in C, a contradiction. To see this, observe by Lemma 8 that $d_{t-1}^{\bar{0}\gamma} = \lambda_1 \gamma > 0$ when $\sum_{k=1}^{\infty} \lambda_k = 1$; then apply Lemma 20 to see d grows unboundedly.

Hence there exist nonzero $\gamma_t \neq \gamma_{\hat{t}}$ such that $u(x+\gamma_t) = \beta u(x) + \alpha_t$ and $u(x+\gamma_{\hat{t}}) = \beta u(x) + \alpha_{\hat{t}}$ for all x. Plugging $x + \gamma_{\hat{t}}$ into the first equation and $x + \gamma_t$ into the second implies $\beta u(x+\gamma_t) + \alpha_{\hat{t}} = u(x + \gamma_t + \gamma_{\hat{t}}) = \beta u(x + \gamma_{\hat{t}}) + \alpha_t$ for all x. Suppose WLOG that $\gamma_t > \gamma_{\hat{t}}$. By changing variables we see that for all x $u(x + \tilde{\gamma}) = u(x) + \tilde{\alpha}$, where $\tilde{\gamma} = \gamma_t - \gamma_{\hat{t}}$ and $\tilde{\alpha} = \frac{\alpha_t - \alpha_{\hat{t}}}{\beta}$. But then u is cyclic, a contradiction. □

C Proof of Theorem 3

If $\sum_{k=1}^{\infty}\lambda_k = 1$, then $\frac{\lambda_{k+1}}{\lambda_k} = 1-\lambda_1$ for every k and clearly $\varphi(hq) = (1-\lambda_1)\varphi(h)+\lambda_1 q$.

For the particular h and $c_0, \hat{c}_0 \in Q$ from Axiom IE find the corresponding $c_1, \hat{c}_1$. Axioms IE and DC together imply that $\succeq_{hc_0c_1}$ and $\succeq_{h\hat{c}_0\hat{c}_1}$ are equivalent preferences, both representable as in (1) according to Theorem 1. By the uniqueness of additive representations up to positive affine transformation, there exist a $\rho > 0$ and a σ_i for every $i \geq 0$ such that for each $\bar{c} \in C$,

$$u\big(\bar{c}-\varphi(h00\bar{c}^{i-1})-\lambda_{i+1}c_1-\lambda_{i+2}c_0\big) = \rho u\big(\bar{c}-\varphi(h00\bar{c}^{i-1})-\lambda_{i+1}\hat{c}_1-\lambda_{i+2}\hat{c}_0\big)+\sigma_i. \quad (32)$$

For each i, let $\gamma_i = \lambda_{i+1}c_1 + \lambda_{i+2}c_0 - \lambda_{i+1}\hat{c}_1 - \lambda_{i+2}\hat{c}_0$.

If $\sum_{k=1}^{\infty}\lambda_k < 1$, then $\gamma_i = 0$ for every i since u cannot be quasi-cyclic. For the case $\sum_{k=1}^{\infty}\lambda_k = 1$, we note that $\rho = 1$ must hold. Since $\frac{\lambda_{i+1}}{\lambda_i} \leq 1-\lambda_1 \in (0,1)$, both $|\lambda_{i+1}\hat{c}_1+\lambda_{i+2}c_0|$ and $|\lambda_{i+1}\hat{c}_1+\lambda_{i+2}\hat{c}_0|$ tend to zero as i goes to infinity. As previously noted, for any i and $x \in \mathbb{R}$ we may find a $\bar{c} \in C$ such that $x = \bar{c} - \varphi(h00\bar{c}^{i-1})$. Then, by (32) and continuity of $u(\cdot)$, $\lim_{i\to\infty}\sigma_i = (1-\rho)u\big(x\big)$ for any $x \in \mathbb{R}$. Since the RHS depends on x while the LHS does not, we must have $\rho = 1$. Since u cannot be cyclic when $\sum_{k=1}^{\infty}\lambda_k = 1$, we have $\gamma_i = 0$ for every i in that case too.

Since $\gamma_i = 0$ for every i, we have $\frac{\lambda_{i+1}}{\lambda_i} = \frac{c_1-\hat{c}_1}{\hat{c}_0-c_0}$ for all $i \geq 1$. Then

$$\varphi(hq) = \sum_{k=2}^{\infty}\lambda_k h_{k-1} + \lambda_1 q = \sum_{k=2}^{\infty}\frac{\lambda_k}{\lambda_{k-1}}\lambda_{k-1}h_{k-1} + \lambda_1 q = \frac{c_1-\hat{c}_1}{\hat{c}_0-c_0}\varphi(h) + \lambda_1 q.$$

Now define $\alpha = \frac{c_1-\hat{c}_1}{\hat{c}_0-c_0}$ and $\beta = \lambda_1$. Clearly $\alpha+\beta \leq 1$ since $\frac{\lambda_{i+1}}{\lambda_i} \leq 1-\lambda_1$. □

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