On the finite sample properties of kernel-based integrated variance estimators*

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Abstract

The presence of market microstructure noise in high-frequency asset price data renders the classical realized variance estimator inconsistent for the object of interest, namely, the integrated variance of the underlying efficient price process (Bandi and Russell, 2004a, and Zhang et al., 2004). In recent research, HAC-type estimators have been proposed that are either theoretically consistent (Barndorff-Nielsen et al., 2005, and Zhang et al., 2004) or "near-consistent" (Hansen and Lunde, 2004a), even when realistic market microstructure noise plays a role.

This paper studies the finite sample properties of these estimators. We find that the existing asymptotic approximations to the estimators' finite sample properties are unsatisfactory. However, in the spirit of Bandi and Russell's optimal sampling methods for realized variance (Bandi and Russell, 2004a, 2004b), we show how to optimize the finite sample performance of the estimators on the basis of a conditional (on the volatility path of the underlying price process) mean-squared error criterion.

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1 Introduction

The asymptotic consistency of HAC-type variance estimators relies on a limiting condition requiring the number of autocovariances to diverge to infinity as the ratio (ϕ, say) between the number of autocovariances and the number of observations goes to zero. As noticed as early as Neave (1970), while this condition is "mathematically convenient," it might lead to inaccurate asymptotic approximations to the estimators' final sample properties. In effect, the ratio ϕ is fixed in any given sample. For a given ϕ, the magnitude of the finite sample mean-squared error (MSE) of HAC-type variance estimators can be substantial and very different from asymptotic approximations relying on a vanishing ϕ.

Some important recent contributions on integrated variance estimation by virtue of noisy high-frequency asset price data rely on a similar asymptotic condition for "near-consistency" or consistency (Bandorff-Nielsen et al., 2005, Hansen and Lunde, 2004a, and Zhang et al., 2004). These estimates are subject to the same observation; applied researchers are necessarily forced to select a value of ϕ. This paper shows that for a given ϕ the finite sample properties of HAC-type variance estimators do not conform closely with existing asymptotic approximations. However, the ratio ϕ can be chosen *optimally* on the basis of a finite sample MSE criterion. In other words, the finite sample properties of HAC-type integrated variance estimators can be optimized.

Our approach relates to the optimal MSE approach to integrated variance estimation by virtue of *realized variance* of Bandi and Russell (2004a, 2004b). Following Bandi and Russell (2004a, 2004b) we focus on finite sample performance and study an MSE-based method to optimize such a performance. Bandi and Russell (2004a, 2004b) write the conditional MSE of the classical realized variance estimator of Andersen et al. (2003) and Barndorff-Nielsen and Shephard (2002) as a function of the sampling frequency and select an optimal sampling frequency that minimizes the MSE. Here the conditional MSEs of alternative integrated variance estimators are written as a function of ϕ and selection of an optimal ϕ is conducted for a *given* number of intra-daily observations.

Interestingly, Kiefer and Vogelsang (2002) have also recently highlighted the importance of treating the ratio ϕ as fixed in deriving asymptotic approximations to the properties of HAC estimators. Differently from Kiefer and Vogelsang (2002), however, we do not aim to derive asymptotic approximations for HAC estimators (and corresponding test statistics) for any value of ϕ. Rather, we study selection of ϕ in order to optimize the estimator's finite sample performance as summarized by its conditional (on the volatility path of the underlying price process) MSE.

Using midpoints of bid-ask quotes for a sample of S&P 100 stocks, we find that the root MSEs of HAC-type integrated variance estimators at the optimal ϕ value imply fairly precise estimation of the integrated price variance over the period. However, estimation accuracy deteriorates quickly with suboptimal choices of ϕ. Furthermore, we show that the optimal finite sample MSE values of these estimators are smaller than the optimal finite sample MSE values of the classical realized variance estimator. Nonetheless, the gains that HAC-type estimators provide over the realized variance estimator can be either reduced or lost by suboptimal choices of ϕ.

The paper proceeds as follows. Section 2 discusses the model and the class of HAC-type estimators which are the focus of the present work. Section 3 presents the finite sample MSEs (as a function of ϕ) of some recently-proposed HAC estimators of the integrated price variance and discusses choice of ϕ. In Section 4 we apply the methods to three representative stocks, i.e., Goldman Sachs, SBC Communications, and EXXON Mobile Corporation. Section 5 concludes. The Appendix contains the proofs.

2 The framework

Following the notation in Bandi and Russell (2004b) and Bandorff-Nielsen et al. (2005), *inter alia*, denote a trading day by $h = [0, 1]$. The trading day is divided into m subperiods $t_i - t_{i-1}$ with $i = 1, ..., m$ so that $t_0 = 0$ and $t_m = 1$. Now define

$$\underbrace{p(t_i) - p(t_{i-1})}_{r_i} = \underbrace{p^e(t_i) - p^e(t_{i-1})}_{r_i^e} + \underbrace{\eta(t_i) - \eta(t_{i-1})}_{\varepsilon_i}, \tag{1}$$

where r_i is an observed continuously-compounded intra-daily return, r_i^e is an efficient continuously-compounded intra-daily return, and ε_i is a market microstructure contamination in the intra-daily return process. As in Bandi and Russell (2004b), Bandorff-Nielsen et al. (2005), Hansen and Lunde (2004b), and Zhang et al. (2004), among others, we make the following assumptions:

Assumption 1. *The efficient price process* p^e *is a stochastic volatility local martingale, namely,*

$$p^e(t) = \int_0^t \sigma_s dW_s, \tag{2}$$

where $\{W_t : t \geq 0\}$ *is a standard Brownian motion assumed to be independent of the càdlàg spot volatility process* σ_t *for all* t. *Furthermore,*

$$Q(t) = \int_0^t \sigma_s^4 ds < \infty \tag{3}$$

for all t.

Assumption 2. *The logarithmic price contaminations* $\eta(t)$ *are i.i.d. with a bounded fourth moment and independent of* $p^e(t)$.[1]

[1] The empirical validity of these assumptions depends on the market structure (centralized versus decentralized markets), the nature of the price measurements (transaction prices versus midpoints of bid-ask spreads, for instance), and the sampling method (calendar time sampling versus event time sampling). We refer the reader to Bandi and Russell (2005) for discussions. Hansen and Lunde (2004a, 2004b) study the

The object of econometric interest is the integrated price variance over the trading day, namely, $V = \int_0^1 \sigma_s^2 ds$. To this extent, consider the "regular" (in the terminology of Barndorff-Nielsen et al., 2005) kernel-based estimator

$$\widehat{V}_w^{HL} = w_0 \widehat{\gamma}_0 + 2 \sum_{s=1}^{q} w_s \widehat{\gamma}_s, \tag{4}$$

where $\widehat{\gamma}_s = \sum_{i=1}^{m-s} r_i r_{i+s}$ and the $w_s's$ are generic weights. This class of integrated variance estimators was suggested by Hansen and Lunde (2004a) and is in the tradition of zero frequency nonparametric spectral density estimators, or HAC estimators (Andrews, 1991, Andrews and Monahan, 1992, and Newey and West, 1987, among others). Hansen and Lunde (2004b) study the finite sample mean-squared error properties of $\widehat{V}_w^{HL}$ for the case $q = 1$, $w_0 = 1$, and $w_1 = \frac{m}{m-1}$ (see, also, Zhou, 1996).[2] Hansen and Lunde (2004a) discuss the finite sample bias properties of $\widehat{V}_w^{HL}$ for the more general case of an unrestricted q with $w_0 = 1$ and $w_s = \frac{m}{m-s}$. The limiting features of $\widehat{V}_w^{HL}$ as a "near-consistent" estimator of the integrated variance of the efficient price process V are examined in Barndorff-Nielsen et al. (2005). Under Assumptions 1 and 2, Barndorff-Nielsen et al. (2005) show that, when using Bartlett-type kernel weights (i.e., when $w_0 = \frac{m-1}{m}\frac{q-1}{q}$ and $w_s = \frac{q-s}{q}$ for $s = 1, ..., q$), the asymptotic variance of $\widehat{V}_w^{HL}$ coincides with the theoretical lower bound of the limiting variance of "regular" kernel-based estimators, namely $4\left(\mathbf{E}\left(\eta^2\right)\right)^2$.

For an average stock, $\left(\mathbf{E}\left(\eta^2\right)\right)^2$ is very small relative to $V = \int_0^1 \sigma_s^2 ds$ (see Section 4), hence the "near-consistency" of "regular" Bartlett-type kernel-based estimators. "Near-consistency" requires $q, m \to \infty$ with $\frac{q}{m} \to 0$ and $\frac{q^2}{m} \to \infty$. In practise, $q = \lfloor \phi m \rfloor$, where $\lfloor x \rfloor$ denotes the largest integer that is smaller than x or equal to x, with $0 < \phi \leq 1$.

Next, we show how to select ϕ optimally on the basis of a finite sample MSE criterion. We do so both in the context of the aforementioned "regular" Bartlett-type kernel-based estimator and in the context of the subsampling estimator proposed by Zhang et al. (2004). Interestingly, Zhang et al.'s estimator is consistent for V. As noticed by Barndorff-Nielsen et al. (2005), this estimator can be interpreted as a "modified" Bartlett-type kernel-based estimator. As they show, the "modification," which we briefly discuss below, guarantees theoretical consistency.

3 Choosing ϕ

We start with Hansen and Lunde's "regular" kernel-based estimator computed using Bartlett-type kernel weights (see Barndorff-Nielsen et al., 2005). Theorem 1 contains the conditional (on the volatility path as in Bandi and Russell, 2004a, 2004b) MSE of the

empirical features of the noise for a sample of NYSE and NASDAQ stocks. Awartani et al. (2004) propose hypothesis tests on the noise properties.

[2] The finite sample mean-squared error properties of this estimator in the context of a pure jump process of finite variation for the efficient price are studied by Oomen (2004a).

estimator expressed as a function of the ratio $\phi = \frac{q}{m}$. The optimal ϕ, ϕ^*, is defined as the *argmin* of the conditional MSE.

Theorem 1 (The estimator of Hansen and Lunde, 2004a) *Assume the η's are mean-zero normal.*[3] *Consider*

$$\widehat{V}_w^{HL} = w_0 \widehat{\gamma}_0 + 2 \sum_{s=1}^{q} w_s \widehat{\gamma}_s, \tag{5}$$

with

$$\widehat{\gamma}_s = \sum_{i=1}^{m-s} r_i r_{i+s}. \tag{6}$$

Assume $w_0 = \left(\frac{m-1}{m}\right)\left(\frac{q-1}{q}\right)$ *and* $w_s = \frac{q-s}{q}$ *for* $s = 1, ..., q$. *The optimal (in a conditional MSE sense)* ϕ *is defined as*

$$\phi^*_{HL} = \underset{0<\phi\le 1}{\arg\min} \left[(bias(\phi))^2 + var(\phi) \right], \tag{7}$$

where

$$(bias(\phi))^2 = \frac{V^2}{m^2} + \left(\frac{2}{m^2} - \frac{2}{m^3}\right)\frac{V^2}{\phi} + \left(\frac{1}{m^2} - \frac{2}{m^3} + \frac{1}{m^4}\right)\frac{V^2}{\phi^2} \tag{8}$$

and

$$\begin{aligned} var(\phi) = {} & K^{HL} - \frac{1}{3}Q\phi^2 + \left(\frac{8}{3}\sigma_\eta^2 V + \frac{4}{3}Q\right)\phi + \\ & [-\frac{4}{m^4}Q + \frac{\left(4\sigma_\eta^4 + 8\sigma_\eta^2 V\right)}{m} + \frac{\left(8\sigma_\eta^4 + 16\sigma_\eta^2 V + 8Q\right)}{m^3} \\ & + \frac{\left(-\frac{56}{3}\sigma_\eta^2 V - \frac{10}{3}Q - 24\sigma_\eta^4\right)}{m^2}]\frac{1}{\phi} + \\ & [8\frac{1}{m}\sigma_\eta^4 + \frac{2}{m^5}Q + \frac{\left(-24\sigma_\eta^4 - 8\sigma_\eta^2 V\right)}{m^2} + \frac{\left(20\sigma_\eta^4 + 16\sigma_\eta^2 V + 2Q\right)}{m^3} \\ & + \frac{\left(-4\sigma_\eta^4 - 8\sigma_\eta^2 V - 4Q\right)}{m^4}]\frac{1}{\phi^2}, \end{aligned} \tag{9}$$

with

$$K^{HL} = 4\sigma_\eta^4 + \frac{4}{m}\sigma_\eta^4 + \left(-\frac{4}{m^2}\sigma_\eta^4 - \frac{8}{m^2}\sigma_\eta^2 V - \frac{11}{3}\frac{1}{m^2}Q\right) + \frac{2}{m^3}Q, \tag{10}$$

$$V = \int_0^1 \sigma_s^2 ds, \tag{11}$$

$$Q = \int_0^1 \sigma_s^4 ds, \tag{12}$$

and

$$\sigma_\eta^2 = \mathbf{E}(\eta^2). \tag{13}$$

[3]The normality assumption is convenient. This assumption can be easily relaxed. In the more general case, the MSE would be a function of the fourth noise moment too.

Proof. *The proof follows the same lines as the proof of Theorem 2 - see the Appendix.*

When $q = \lfloor \phi m \rfloor$ for $0 < \phi \leq 1$, as is the case in practise, a traditional bias-variance trade-off arises. For a given number of intra-daily observations m, larger values of ϕ lead to larger values of q and hence a smaller bias. However, being that the variance contains terms of order $O\left(\frac{q}{m}\right)$ and $O\left(\frac{q^2}{m^2}\right)$, higher values of ϕ translate into a larger variance. Interestingly, the variance itself is a convex function of ϕ. This result mirrors a similar result in the context of integrated variance estimation by virtue of realized variance ($\widehat{\gamma}_0$). There, Bandi and Russell (2004a, 2004b) show that the variance of the classical realized variance estimator is a convex function of the number of intra-daily observations used to compute realized variance when market microstructure noise plays a role.

We now turn to the subsampling estimator suggested by Zhang et al. (2004). Given the initial sampling grid $\Phi := \{t_0 = 0, t_{2,...,} t_m = 1\}$, consider non-overlapping subgrids $\Phi_u^q := \{t_{u-1}, t_{u-1+q,...,} t_{u-1+c_u q}\}$ with $c_u = \left\lfloor \frac{m-u+1}{q} \right\rfloor$ for $u = 1, ..., q$. The sub-sampling estimator is then defined as

$$\widehat{V}^{ZMA} = \frac{1}{q}\sum_{u=1}^{q}\left(\sum_{t_i \in \Phi_u^q}(p(t_{i+q}) - p(t_i))^2\right) - \frac{m-q+1}{mq}\sum_{i=1}^{m} r_i^2. \tag{14}$$

Barndorff-Nielsen et al. (2005) show that the sub-sampling estimator can be rewritten as a "modified" Bartlett-type kernel estimator. Specifically,

$$\widehat{V}^{ZMA} = \left(1 - \frac{m-q+1}{mq}\right)\widehat{\gamma}_0 + 2\sum_{s=1}^{q}\left(\frac{q-s}{q}\right)\widehat{\gamma}_s - \frac{1}{q}\vartheta_q \tag{15}$$

with $\vartheta_1 = 0$ and $\vartheta_q = \vartheta_{q-1} + (r_1 + ... + r_{q-1})^2 + (r_{m-q+2} + ... + r_m)^2$ for $q \geq 2$. The addition of the term $\frac{1}{q}\vartheta_q$, which the subsampling approach entails by construction, is what makes the estimator $\widehat{V}^{ZMA}$ consistent (Barndorff-Nielsen et al., 2005). As noted by Barndorff-Nielsen et al. (2005), Bartlett (1950) was the first to motivate the Bartlett kernel with subsampling.

Theorem 2. (The estimator of Zhang et al., 2004) *Assume the η's are mean-zero normal and $\sigma_i^2 = \int_{t_{i-1}}^{t_i}\sigma_s^2 ds \approx \frac{V}{m}$ for all i.[4] The optimal (in a conditional MSE sense) ϕ of the subsampling estimator in Eq. (15) is defined as*

$$\phi^*_{ZMA} = \underset{0<\phi\leq 1/2}{\arg\min}\left[(bias(\phi))^2 + var(\phi)\right], \tag{16}$$

where

$$(bias(\phi))^2 \approx \left(\frac{6V^2}{m^2} + \frac{2V^2}{m}\right) + V^2\phi^2 - \frac{4V^2}{m}\phi + \left(-4\frac{V^2}{m^2} - 4\frac{V^2}{m^3}\right)\frac{1}{\phi} \tag{17}$$

[4] We make this approximation to evaluate the MSE easily. The approximation is valid if volatility does not change much within the day. The expression is exact if sampling is conducted in business time. We refer the reader to Oomen (2004a, 2004b) for a thorough approach to business time sampling.

$$+\left(\frac{V^2}{m^2}+\frac{2V^2}{m^3}+\frac{V^2}{m^4}\right)\frac{1}{\phi^2}, \tag{18}$$

and, if $\phi \le 1/2$,

$$Var\left(\phi\right)\approx K^{ZMA}-\frac{1}{3}(Q+V^2)\phi^2+\left(-\frac{1}{3}V^2\frac{1}{m}-4V^2\frac{1}{m^2}+\frac{4}{3}Q\right)\phi$$
$$+[-\frac{4}{m^4}\left(Q+V^2\right)+\left(\frac{8\sigma_\eta^4+16\sigma_\eta^2V-8Q-\frac{56}{3}V^2}{m^3}\right)+\left(\frac{24\sigma_\eta^2V-\frac{10}{3}Q+8\sigma_\eta^4}{m^2}\right)$$
$$+\left(\frac{-8\sigma_\eta^4+8\sigma_\eta^2V}{m}\right)]\frac{1}{\phi}$$
$$+[\frac{2}{m^5}Q+\left(\frac{-4\sigma_\eta^4-8\sigma_\eta^2V+4Q-8V^2}{m^4}\right)+\left(\frac{-4\sigma_\eta^4-16\sigma_\eta^2V+2Q}{m^3}\right)$$
$$+\left(\frac{8\sigma_\eta^4-8\sigma_\eta^2V}{m^2}\right)+\frac{8}{m}\sigma_\eta^4]\frac{1}{\phi^2},$$

with

$$K^{ZMA} = (-4\sigma_\eta^4-8V\sigma_\eta^2)\frac{1}{m}+\left(-4\sigma_\eta^4-8\sigma_\eta^2V+\frac{13}{3}Q+\frac{79}{3}V^2\right)\frac{1}{m^2}$$
$$+\frac{1}{m^3}\left(2Q+8V^2\right),$$

$$V=\int_0^1\sigma_s^2ds, \tag{19}$$

$$Q=\int_0^1\sigma_s^4ds, \tag{20}$$

and

$$\sigma_\eta^2=\mathbf{E}(\eta^2). \tag{21}$$

Proof. *See Appendix.*

The addition of the term $\frac{1}{q}\vartheta_q$ is what makes the quantity $4\sigma_\eta^4$, which appears in K^{HL}, not appear in K^{ZMA}. Under standard asymptotic conditions, i.e, as $q,m\to\infty$ with $\frac{q}{m}\to 0$ and $\frac{q^2}{m}\to\infty$, this modification is sufficient for the consistency of Zhang et al.'s subsampling estimator (Barndorff-Nielsen et al., 2005).

In practise, though, $q=\lfloor\phi m\rfloor$ and the overall contribution of $4\sigma_\eta^4$ to the finite sample conditional MSE of the estimator is small, as we show in the next section. As earlier in the case of $\widehat{V}^{HL}$, for a given number of intra-daily observations m, the choice of the number of subsamples or, equivalently, the choice of ϕ induces a bias-variance trade-off which can be optimized.

In the next section we apply the methods to both estimators. Specifically, we discuss choice of the optimal number of autocovariances, $q^*_{HL}=\lfloor\phi^*_{HL}m\rfloor$, and choice of the optimal number of subsamples, i.e., $q^*_{ZMA}=\lfloor\phi^*_{ZMA}m\rfloor$, based on three representative stocks.

4 Some specific stocks: GS, SBC, and XOM.

We consider Goldman Sachs (GS), SBC Communications (SBC), and EXXON Mobile Corporation (XOM). The data come from the TAQ data set. They are midpoints of bid-ask quotes posted on two exchanges, the NYSE and the MIDWEST, over the month of February 2002. The relevant parameter values for σ_η^2, V, Q, and m come from Table 1 in Bandi and Russell (2004b).[5] More generally, the interested reader is referred to Bandi and Russell (2004b) for simple techniques on evaluating σ_η^2, V, and Q. We choose GS, SBC, and XOM since they represent median and extreme features of the S&P100 stocks as summarized by the ratio between the second moment of the noise returns, σ_ε^2, and the integrated variance of the underlying efficient price process, V. Specifically, GS, SBC, and XOM correspond to the first, the fifth, and the nineth decile of the cross-sectional distribution of the ratios, respectively. The parameters are:

Table 1.

	GS	**SBC**	**XOM**
σ_η^2	$0.87e-07$	$1.89e-07$	$2.1e-07$
V	0.00042	0.00041	0.00018
Q	$2.31e-07$	$2.1e-07$	$4.1e-08$
m	2,247	2,034	2,630

We use Theorem 1 and Theorem 2 to derive the optimal ϕ and q values in the case of both estimators. Recall, given ϕ^*, q^* is defined as $\lfloor \phi^* m \rfloor$. Table 2 contains these values.

[5]The second moments of the noise σ_η^2 are obtained by dividing by two the corresponding σ_ε^2 values contained in the column labelled "Mid. Var." The daily integrated variances V are in the column labelled "V^*." The integrated quarticities Q are obtained by using the approximate optimal frequencies in the column labelled "D^a" as follows. Since,

$$\frac{6.5 \times 60}{D^a} = \left(\frac{Q}{\sigma_\varepsilon^4}\right)^{1/3},$$

(c.f., Bandi and Russell, 2004b, Proposition 4), then

$$Q = \left(\frac{6.5 \times 60}{D^a}\right)^3 \sigma_\varepsilon^4.$$

Finally, m is obtained by using the average durations d (in seconds) in the column labelled "Avg. Dur." as follows:

$$m = \frac{6.5 \times 60 \times 60}{d}.$$

Similarly, the interested reader can obtain representative values for all the S&P 100 stock by referring to Table 1 in Bandi and Russell (2004b).

Table 2.

	GS	**SBC**	**XOM**
$\widehat{V}^{HL} - \phi^*$	0.006	0.007	0.006
$\widehat{V}^{HL} - q^*$	13	14	15
$\widehat{V}^{ZMA} - \phi^*$	0.006	0.007	0.006
$\widehat{V}^{ZMA} - q^*$	13	14	15

The MSEs of the two estimators are very similar. This is, of course, to be expected given the theoretical relation between the estimators, as discussed by Barndorff-Nielsen et al. (2005), and the small magnitude of the fourth noise moment. Figs. 1 and 2 contain the MSE plots of the estimators (as a function of ϕ) in the SBC case. The MSE values at the optimum ϕ (in Table 2) are:

Table 3.

	GS	**SBC**	**XOM**
$MSE\,(\phi^*_{HL})$	$2.82e-09$	$2.82e-09$	$4.78e-10$
$MSE\,(\phi^*_{ZMA})$	$2.95e-09$	$2.95e-09$	$4.98e-10$

These values are fairly small. Consider $\widehat{V}^{HL}$ and SBC, for instance. The root MSE value at the optimum is equal to $5.3e-05$. The corresponding integrated variance over the day is $4.1e-04$.

It is useful to notice that, in light of the empirically-relevant magnitudes of the price and noise moments, the dominating bias and variance terms are $\frac{V^2}{m^2}\frac{1}{\phi^2}$ and $\frac{4}{3}Q\phi$ in the case of both estimators. Hence, the expression

$$\phi^*_{HL,ZMA} \approx \left(\frac{3}{2}\frac{\frac{V^2}{m^2}}{Q}\right)^{1/3} \tag{22}$$

provides a convenient *rule-of-thumb* to choose ϕ in practise. The approximate $\phi^*_{HL,ZMA}$ value can be interpreted as a signal-to-noise ratio.

We now compare the optimum MSE values derived above to the optimum MSE values of the classical realized variance estimator ($\widehat{\gamma}_0$). In the realized variance case, the optimum values are obtained by sampling continuously-compounded returns using the optimal sampling frequency derived in Bandi and Russell (2004a, 2004b). These values are:

Table 4.

	GS	**SBC**	**XOM**
$MSE_{\widehat{\gamma}_0}$	$5e-09$	$7.5e-09$	$2.5e-09$

Table 3 and Table 4 suggest that there are substantial MSE gains to be obtained by employing kernel-based integrated variance estimators. Hansen and Lunde (2004a) derive the same conclusion by comparing the MSE of the realized variance estimator to the MSE of $\widehat{V}_w^{HL}$ for the case $q = 1$, $w_0 = 1$, and $w_1 = \frac{m}{m-1}$.

Zhang et al. (2004) provide a complete distribution theory for their proposed estimator. It is interesting to compare our finite sample optimum q^*_{ZMA} to the optimum q ($\widetilde{q}_{ZMA}$, say) implied by the limiting results in Zhang et al. (2004), namely,

$$\widetilde{q}_{ZMA} = \left(\frac{16 \left(\mathbf{E}(\eta^2) \right)^2}{\frac{8}{3} Q} \right)^{1/3} m^{2/3}$$

(see Zhang et al., 2004, Eqs. (36) and (39)).

Table 5.

	GS	**SBC**	**XOM**
$\widetilde{q}_{ZMA}$	1	2	4

Table 2 and Table 5 indicate that the $\widetilde{q}_{ZMA}$ values are quite sub-optimal in finite sample. We can now plug the $\widetilde{q}_{ZMA}$ values into the finite sample MSE expansion in Theorem 2 to obtain the corresponding finite sample MSE values.

Table 6.

	GS	**SBC**	**XOM**
$MSE\left(\widetilde{\phi}_{ZMA}\right)$	$1.5e-07$	$4.1e-08$	$1.36e-09$

These results are interesting. Zhang et al.'s estimator is promising and can provide significant MSE gains over the classical realized variance estimator (c.f. Table 3 and Table 4). However, choosing the optimal number of subsamples using asymptotic criteria can induce biases that might reduce and/or lose these gains (c.f. Table 4 and Table 6). Our finite sample methods appear to be well suited to exploit the information potential of the subsampling estimator. Similar considerations apply to Hansen and Lunde's approach.

Finally, we examine Zhang et al.'s implied asymptotic approximation to the finite sample MSE of $\widehat{V}^{ZMA}$, namely

$$asyMSE\left(\widetilde{\phi}_{ZMA}\right) = \frac{8}{m^{1/3}} \left(\left(\frac{16 \left(\mathbf{E}(\eta^2) \right)^2}{\frac{8}{3} Q} \right)^{1/3} \right)^{-2} \left(\mathbf{E}(\eta^2) \right)^2 + \frac{1}{m^{1/3}} \left(\left(\frac{16 \left(\mathbf{E}(\eta^2) \right)^2}{\frac{8}{3} Q} \right)^{1/3} \right) \frac{8}{3} Q$$

(see Zhang et al., 2004, Eq. (37)).

Table 7.

	GS	**SBC**	**XOM**
$asyMSE\left(\widetilde{\phi}_{ZMA}\right)$	$4.1e-10$	$6.6e-10$	$2.2e-10$

Table 3 and Table 7 indicate that the finite sample MSE of the estimator at its optimal ϕ value is considerably larger than the corresponding asymptotic approximation. Hence, using asymptotic approximations relying on a vanishing ϕ might severely underestimate the true sampling error of the estimator.

In sum, much can be learned from using kernel-based integrated variance estimators. However, one should exercise care when bringing these tools to the data. We find that:

[1] The finite sample MSEs of $\widehat{V}^{HL}$ and $\widehat{V}^{ZMA}$ are similar.

[2] The root MSEs at the optimal ϕ value imply fairly precise estimation of V. However, estimation accuracy deteriorates quickly when choosing ϕ suboptimally. This is particularly true when the selected ϕ value is excessively small.

[3] The finite sample MSEs are considerably larger than those implied by asymptotic approximations.

[4] The optimal finite sample MSE values of $\widehat{V}^{HL}$ and $\widehat{V}^{ZMA}$ are smaller than the optimal finite sample MSE values of the classical realized variance estimator.

[5] The gains that $\widehat{V}^{HL}$ and $\widehat{V}^{ZMA}$ provide over the classical realized variance estimator can be either reduced or lost by suboptimally choosing the number of autocovariances (or, equivalently, the number of subsamples in the case of Zhang et al.'s "modified" kernel-based approach).

5 Conclusions

Nonparametric price variance estimation in the presence of market microstructure noise contaminations is a difficult task. While the classical realized variance estimator of Andersen et al. (2003) and Barndorff-Nielsen and Shephard (2002) has been shown to be inconsistent for integrated variance under realistic price formation mechanisms with noise (Bandi and Russell, 2004a,b, and Zhang et al., 2004), some interesting recent work has proposed estimators that are either "near-consistent" or consistent for the object of interest (Bandorff-Nielsen et al., 2005, Hansen and Lunde, 2004a, and Zhang et al., 2004).

We argue that asymptotic representations relying on limiting conditions on the number of autocovariances q and the number of observations m are unsatisfactory approximations to the finite sample properties of kernel-based integrated variance estimators. In light of this observation, we treat the ratio between q and m as fixed, as it is in practise, and choose it optimally in order to minimize the finite sample MSE of the estimators as done by Bandi and Russell (2004a, 2004b) in the context of realized variance.

Several extensions are possible. First, similar methods can be applied to the (consistent) "modified" kernel-based estimator of Barndorff-Nielsen et al. (2005). Second, in keeping with Bandi and Russell (2004a), Hansen and Lunde (2004a), and Zhang (2005), it is of interest to study the dependent noise case. Third, the optimal estimates derived here can be evaluated in the context of relevant economic metrics as suggested by Bandi and Russell (2004b) and Bandi et al. (2005) when using optimally-sampled realized variances and covariances.

6 Appendix

Proof of Theorem 2. Define $x_{i,s} = r_i r_{i+s}$,

$$\begin{aligned} z_1 &= x_{1,0}, \\ z_2 &= x_{2,0} + 2x_{1,1}, \\ z_3 &= x_{3,0} + 2x_{2,1} + 2x_{1,2} \\ &\dots \end{aligned}$$

and

$$\begin{aligned} \widetilde{z}_1 &= x_{m,0}, \\ \widetilde{z}_2 &= x_{m-1,0} + 2x_{m-1,1}, \\ \widetilde{z}_3 &= x_{m-2,0} + 2x_{m-2,1} + 2x_{m-2,2} \\ &\dots \end{aligned}$$

Following Barndorff-Nielsen et al. (2005), write

$$\widehat{V}^{ZMA} = \underbrace{\left(1 - \frac{m-q+1}{mq}\right)\widehat{\gamma}_0 + 2\sum_{s=1}^{q}\left(\frac{q-s}{q}\right)\widehat{\gamma}_s}_{\alpha} \underbrace{- \frac{1}{q}\vartheta_q}_{\beta}$$

with

$$\frac{1}{q}\vartheta_q = \frac{1}{q}\sum_{s=1}^{q-1}(q-s)z_s + \frac{1}{q}\sum_{s=1}^{q-1}(q-s)\widetilde{z}_s.$$

We start with the variance of α. We rely on Corollary 2 of Barndorff-Nielsen et al. (2005) and use the appropriate kernel weights (namely, $w_0 = 1 - \frac{m-q+1}{mq}$ and $w_s = \frac{q-s}{q}$ for $s \geq 1$) to obtain

$$Var(\alpha) = \Lambda_1^\alpha \sigma_\eta^4 m + \Lambda_0^\alpha \sigma_\eta^2 + \Lambda_{-1}^\alpha \frac{1}{m},$$

where

$$\begin{aligned} \Lambda_1^\alpha = \; & 12\left(\frac{mq-m+q-1}{mq}\right)^2 + 4\left(\frac{m-1}{m}\right)\left(\frac{q-1}{q}\right)\left(7\left(\frac{q-1}{q}\right) - 8\left(\frac{mq-m+q-1}{mq}\right)\right) \\ & + \left(\frac{m-2}{m}\right)\left(\frac{q-2}{q}\right)8\left(3\frac{q-2}{q} - 4\frac{q-1}{q} + \left(\frac{mq-m+q-1}{mq}\right)\right) \\ & + \sum_{j=3}^{q}\left(\frac{m-j}{m}\right)\left(\frac{q-j}{q}\right)8\left(3\left(\frac{q-j}{q}\right) - 4\left(\frac{q-(j-1)}{q}\right) + \left(\frac{q-(j-2)}{q}\right)\right) \\ & - \frac{4}{m}\left(\frac{mq-m+q-1}{mq}\right)^2 - \frac{8}{m}\sum_{j=1}^{q}\left(\frac{q-j}{q}\right)^2, \end{aligned}$$

$$\begin{aligned} \Lambda_0^\alpha = \; & 8V\left(\frac{mq-m+q-1}{mq}\right)^2 + 16V\left(\frac{m-1}{m}\right)\left(\frac{q-1}{q}\right)\left(\left(\frac{q-1}{q}\right) - \left(\frac{mq-m+q-1}{mq}\right)\right) \\ & + \sum_{j=2}^{q} 16V_{j/m}\left(\frac{q-j}{q}\right)\left(\left(\frac{q-j}{q}\right) - \left(\frac{q-(j-1)}{q}\right)\right), \end{aligned}$$

and

$$\Lambda_{-1}^{\alpha} = 2Q\left(\frac{mq-m+q-1}{mq}\right)^2 + \sum_{j=1}^{q} 4Q_{j/m}\left(\frac{q-j}{q}\right)^2.$$

In light of the fact that

$$\begin{aligned}\sum_{j=1}^{q} j &= \frac{q^2+q}{2},\\ \sum_{j=1}^{q} j^2 &= \frac{(q+1)(2q+1)q}{6},\\ \sum_{j=1}^{q} j^3 &= \frac{q^4+2q^3+q^2}{4},\end{aligned}$$

and since $V_{j/m} = V\left(\frac{m-j}{m}\right)$ and $Q_{j/m} = Q\left(\frac{m-j}{m}\right)$, simple algebra gives

$$\Lambda_1^{\alpha} = \frac{4}{m} + \frac{8}{q^2} + \frac{20}{m^2} + \frac{4}{m^2q^2} - \frac{12}{mq} - \frac{8}{mq^2} - \frac{24}{m^2q} - \frac{4}{m^3} - \frac{4}{m^3q^2} + \frac{8}{m^3q},$$

$$\Lambda_0^{\alpha} = \left(\frac{8}{3}\frac{q}{m} - \frac{8}{q^2} + \frac{24}{m^2} + \frac{24}{m^2q^2} + \frac{8}{q} - \frac{56}{3}\frac{1}{mq} + \frac{16}{mq^2} - \frac{48}{m^2q}\right)V,$$

and

$$\Lambda_{-1}^{\alpha} = \left(\frac{2}{q^2} + \frac{2}{m^2} - \frac{10}{3}\frac{1}{q} + \frac{13}{3}\frac{1}{m} - \frac{8}{mq} + \frac{4}{mq^2} + \frac{4}{3}q + \frac{2}{m^2q^2} - \frac{1}{3}\frac{q^2}{m} - \frac{4}{m^2q}\right)Q.$$

Hence,

$$\begin{aligned}Var(\alpha) = &\left(\frac{4}{m} + \frac{8}{q^2} + \frac{20}{m^2} + \frac{4}{m^2q^2} - \frac{12}{qm} - \frac{8}{mq^2} - \frac{24}{m^2q} - \frac{4}{m^3} - \frac{4}{m^3q^2} + \frac{8}{m^3q}\right)\sigma_\eta^4 m\\ &+\left(\frac{8}{3}\frac{q}{m} - \frac{8}{q^2} + \frac{24}{m^2} + \frac{24}{m^2q^2} + \frac{8}{q} - \frac{56}{3mq} + \frac{16}{mq^2} - \frac{48}{m^2q}\right)\sigma_\eta^2 V\\ &+\left(\frac{2}{q^2} + \frac{2}{m^2} - \frac{10}{3}\frac{1}{q} + \frac{13}{3}\frac{1}{m} - \frac{8}{mq} + \frac{4}{mq^2} + \frac{4}{3}q + \frac{2}{m^2q^2} - \frac{1}{3}\frac{q^2}{m} - \frac{4}{m^2q}\right)\frac{1}{m}Q\\ = &\left(4 + \frac{8m}{q^2} + \frac{20}{m} + \frac{4}{mq^2} - \frac{12}{q} - \frac{8}{q^2} - \frac{24}{mq} - \frac{4}{m^2} - \frac{4}{m^2q^2} + \frac{8}{m^2q}\right)\sigma_\eta^4\\ &+\left(\frac{8}{3}\frac{q}{m} - \frac{8}{q^2} + \frac{24}{m^2} + \frac{24}{m^2q^2} + \frac{8}{q} - \frac{56}{3mq} + \frac{16}{mq^2} - \frac{48}{m^2q}\right)\sigma_\eta^2 V\\ &+\left(\frac{2}{mq^2} + \frac{2}{m^3} - \frac{10}{3}\frac{1}{qm} + \frac{13}{3}\frac{1}{m^2} - \frac{8}{m^2q} + \frac{4}{m^2q^2} + \frac{4}{3}\frac{q}{m} + \frac{2}{m^3q^2} - \frac{1}{3}\frac{q^2}{m^2} - \frac{4}{m^3q}\right)Q,\end{aligned}$$

which yields

$$\begin{aligned}Var(\alpha) &= 4\sigma_\eta^4 + \frac{20}{m}\sigma_\eta^4 + \left(-\frac{4}{m^2}\sigma_\eta^4 + \frac{24}{m^2}\sigma_\eta^2 V + \frac{13}{3}\frac{1}{m^2}Q\right) + \frac{2}{m^3}Q\\ &\quad +8\frac{m}{q^2}\sigma_\eta^4 + \frac{2}{m^3q^2}Q - \frac{1}{3}\frac{q^2}{m^2}Q - \frac{4}{m^3q}Q\end{aligned}$$

$$
\begin{aligned}
&+\left(-12\sigma_\eta^4+8\sigma_\eta^2 V\right)\frac{1}{q}+\left(-8\sigma_\eta^4-8\sigma_\eta^2 V\right)\frac{1}{q^2}\\
&+\left(\frac{8}{3}\sigma_\eta^2 V+\frac{4}{3}Q\right)\frac{q}{m}+\left(4\sigma_\eta^4+16\sigma_\eta^2 V+2Q\right)\frac{1}{mq^2}\\
&+\left(8\sigma_\eta^4-48\sigma_\eta^2 V-8Q\right)\frac{1}{m^2 q}+\left(-4\sigma_\eta^4+24\sigma_\eta^2 V+4Q\right)\frac{1}{m^2q^2}\\
&+\left(-\frac{56}{3}\sigma_\eta^2 V-\frac{10}{3}Q-24\sigma_\eta^4\right)\frac{1}{mq}.
\end{aligned}
$$

Consider now $Var(\beta)$, namely,

$$
Var(\beta)=Var\left(-\frac{1}{q}\vartheta_q\right)=\frac{1}{q^2}Var\left(\sum_{s=1}^{q-1}(q-s)z_s\right)+\frac{1}{q^2}Var\left(\sum_{s=1}^{q-1}(q-s)\widetilde{z}_s\right)
$$

for $q\leq m/2$. By virtue of Lemma A.3 and Lemma 7 of Barndorff-Nielsen et al. (2005), we can write

$$
\begin{aligned}
&Var\left(\sum_{s=1}^{q-1}(q-s)z_s\right)=\sum_{s=1}^{q-1}(q-s)^2 Var(z_s)+2\sum_{s=1}^{q-2}(q-s)\left(q-s-1\right)cov(z_s,z_{s+1})\\
=\;&\sum_{s=1}^{q-1}(q-s)^2\left[12\sigma_\eta^4+8\sigma_\eta^2(\sigma_1^2+...+\sigma_s^2)+\sigma_s^2\left(4\sigma_1^2+...+4\sigma_{s-1}^2+2\sigma_s^2\right)\right]\\
&+2\sum_{s=1}^{q-2}(q-s)\left(q-s-1\right)\left[-6\sigma_\eta^4-4\sigma_\eta^2(\sigma_1^2+...+\sigma_s^2)\right]-\left[(q-1)^2 4\sigma_\eta^4\right]\\
=\;&A+B+C.
\end{aligned}
$$

We start with A. Write

$$
\begin{aligned}
A\;=\;&\sum_{s=1}^{q-1}(q-s)^2\left[12\sigma_\eta^4+8\sigma_\eta^2(\sigma_1^2+...+\sigma_s^2)+\sigma_s^2\left(4\sigma_1^2+...+4\sigma_{s-1}^2+2\sigma_s^2\right)\right]\\
=\;&\sum_{s=1}^{q-1}(q-s)^2\left[12\sigma_\eta^4+8\sigma_\eta^2 s\frac{V}{m}+4(s-1)\frac{V^2}{m^2}+2\frac{V^2}{m^2}\right]\\
=\;&2\sigma_\eta^4 q-6\sigma_\eta^4 q^2+4\sigma_\eta^4 q^3-\frac{1}{3}V^2\frac{q}{m^2}+\frac{2}{3}V^2\frac{q^2}{m^2}-\frac{2}{3}V^2\frac{q^3}{m^2}+\frac{1}{3}V^2\frac{q^4}{m^2}\\
&-\frac{2}{3}\sigma_\eta^2 V\frac{q^2}{m}+\frac{2}{3}\sigma_\eta^2 V\frac{q^4}{m}.
\end{aligned}
$$

We now turn to B. Write

$$
\begin{aligned}
B\;=\;&2\sum_{s=1}^{q-2}(q-s)\left(q-s-1\right)\left[-6\sigma_\eta^4-4\sigma_\eta^2(\sigma_1^2+...+\sigma_s^2)\right]\\
=\;&q(-8\sigma_\eta^4)+q^2\left(12\sigma_\eta^4\right)+q^3(-4\sigma_\eta^4)+\frac{q}{m}\left(-\frac{4}{3}\sigma_\eta^2 V\right)+\frac{q^2}{m}\left(\frac{2}{3}\sigma_\eta^2 V\right)\\
&+\frac{q^3}{m}\left(\frac{4}{3}\sigma_\eta^2 V\right)+\frac{q^4}{m}\left(-\frac{2}{3}\sigma_\eta^2 V\right).
\end{aligned}
$$

Finally,

$$C = -(q-1)^2 4\sigma_\eta^4 = -q^2(4\sigma_\eta^4) - q(-8\sigma_\eta^4) - 4\sigma_\eta^4.$$

Thus,

$$\begin{aligned}
& \frac{1}{q^2} Var\left(\sum_{s=1}^{q-1}(q-s)z_s\right) = \frac{1}{q^2}(A+B+C) \\
= \;& 2\sigma_\eta^4 + \frac{1}{q}\left(2\sigma_\eta^4\right) + \frac{1}{q^2}(-4\sigma_\eta^4) + \frac{1}{m^2}\left(\frac{2}{3}V^2\right) + \frac{1}{mq}\left(-\frac{4}{3}V\sigma_\eta^2\right) \\
& + \frac{q}{m}\left(\frac{4}{3}\sigma_\eta^2 V\right) + \frac{q^2}{m^2}\left(\frac{1}{3}V^2\right) + \frac{1}{qm^2}\left(-\frac{1}{3}V^2\right) - \frac{2}{3}\frac{q}{m^2}V^2.
\end{aligned}$$

Since

$$Var(\widetilde{z}_1) = 8\sigma_\eta^4 + 8\sigma_\eta^2\sigma_m^2 + 2\sigma_m^4,$$

$$Var(\widetilde{z}_j) = 12\sigma_\eta^4 + 8\sigma_\eta^2\left(\sigma_m^2 + ... + \sigma_{m-j+1}^2\right) + \sigma_{m-j+1}^2\left(4\sigma_m^2 + ... + 4\sigma_{m-j+2}^2 + 2\sigma_{m-j+1}^2\right)$$

for $j = 2, ...,$

$$Cov(\widetilde{z}_1, \widetilde{z}_2) = -6\sigma_\eta^4 - 4\sigma_\eta^2\sigma_m^2,$$

and

$$Cov(\widetilde{z}_j, \widetilde{z}_{j+1}) = -6\sigma_\eta^4 - 4\sigma_\eta^2\left[\sigma_m^2 + ... + \sigma_{m-j+1}^2\right] \qquad j = 2, ...,$$

then $\frac{1}{q^2}Var\left(\sum_{s=1}^{q-1}(q-s)\widetilde{z}_s\right)$ can be represented similarly. Hence,

$$\begin{aligned}
& \frac{1}{q^2} Var\left(\sum_{s=1}^{q-1}(q-s)z_s\right) + \frac{1}{q^2} Var\left(\sum_{s=1}^{q-1}(q-s)\widetilde{z}_s\right) \\
= \;& 4\sigma_\eta^4 + \frac{1}{q}\left(4\sigma_\eta^4\right) + \frac{1}{q^2}(-8\sigma_\eta^4) + \frac{1}{m^2}\left(\frac{4}{3}V^2\right) + \frac{1}{mq}\left(-\frac{8}{3}V\sigma_\eta^2\right) \\
& + \frac{q}{m}\left(\frac{8}{3}\sigma_\eta^2 V\right) + \frac{q^2}{m^2}\left(\frac{2}{3}V^2\right) + \frac{1}{qm^2}\left(-\frac{2}{3}V^2\right) - \frac{4}{3}\frac{q}{m^2}V^2.
\end{aligned}$$

We now turn to the covariance between α and β. Start with

$$\begin{aligned}
& 2\frac{1}{q}\sum_{s=1}^{q-1}\left(\frac{mq-m+q-1}{mq}\right)(q-s)cov\left(\gamma_0, z_s\right) \\
= \;& 2\frac{1}{q}\left(\frac{m+1}{m}\right)\left(\frac{q-1}{q}\right)(q-1)[10\sigma_\eta^4 + 8\sigma_\eta^2\sigma_1^2 + 2\sigma_1^4] \\
& + 2\frac{1}{q}\left(\frac{m+1}{m}\right)\left(\frac{q-1}{q}\right)(q-2)[-4\sigma_\eta^4 + 4\sigma_\eta^2(\sigma_2^2 - \sigma_1^2) + 2\sigma_2^4] \\
& + 2\frac{1}{q}\left(\frac{m+1}{m}\right)\left(\frac{q-1}{q}\right)\sum_{s=3}^{q-1}(q-s)\left[4\sigma_\eta^2(\sigma_s^2 - \sigma_{s-1}^2) + 2\sigma_s^4\right] \\
= \;& 12\sigma_\eta^4 - \frac{16}{q}\sigma_\eta^4 + \frac{4}{q^2}\sigma_\eta^4 + \frac{1}{m}(12\sigma_\eta^4 + 16\sigma_\eta^2 V) + \frac{1}{m^2}(16\sigma_\eta^2 V - 4V^2) \\
& + 2\frac{V^2}{m^3}q + 2\frac{V^2}{m^2}q + \frac{1}{qm}\left(-16\sigma_\eta^4 - 32\sigma_\eta^2 V\right) + \frac{1}{qm^2}\left(2V^2 - 32\sigma_\eta^2 V\right) \\
& + \frac{1}{qm^3}\left(2V^2\right) + \frac{1}{q^2m^2}\left(16\sigma_\eta^2 V\right) + \frac{1}{mq^2}\left(4\sigma_\eta^4 + 16\sigma_\eta^2 V\right) + \frac{1}{m^3}(-4V^2),
\end{aligned}$$

where the first equality derives from Lemma A.4 of Barndorff-Nielsen et al. (2005). Next, we consider the terms:

$$2\frac{1}{q^2}\sum_{s=1}^{q}\sum_{k=1}^{q-1}(q-s)(q-k)cov\left(\widehat{\gamma}_s, z_k\right) = 2\frac{1}{q^2}\sum_{s=1}^{q-1}\sum_{k=1}^{q-1}(q-s)(q-k)cov\left(\widehat{\gamma}_s, z_k\right)$$

since $cov(\widehat{\gamma}_s, z_{s-i}) = 0$ for all s and all $i \geq 1$. Write

$$\underbrace{2\frac{1}{q^2}\sum_{s=1}^{q-1}(q-s)^2 cov\left(\widehat{\gamma}_s, z_s\right)}_{\alpha'} + \underbrace{2\frac{1}{q^2}\sum_{s=1}^{q-1}\sum_{k>s}(q-s)(q-k)cov\left(\widehat{\gamma}_s, z_k\right)}_{\beta'}.$$

Start with $\alpha^{'}$. Using again Lemma A.4 of Barndorff-Nielsen et al. (2005),

$$\begin{aligned}
\alpha^{'} &= 2\frac{1}{q^2}\sum_{s=1}^{q-1}(q-s)^2 cov\left(\widehat{\gamma}_s, z_s\right) \\
&= 2\frac{1}{q^2}(q-1)^2[-4\sigma_\eta^4 - 2\sigma_\eta^2\sigma_1^2] + 2\frac{1}{q^2}\sum_{s=2}^{q-1}(q-s)^2[-2\sigma_\eta^4 - 2\sigma_\eta^2\sigma_1^2] \\
&= -2\sigma_\eta^4 - \frac{4\sigma_\eta^4}{q^2} + \frac{22}{3}\frac{\sigma_\eta^4}{q} + 2\sigma_\eta^2\frac{V}{m} - \frac{2}{3}\sigma_\eta^2 V\frac{1}{qm} - \frac{4}{3}\sigma_\eta^4 q - \frac{4}{3}\sigma_\eta^2 V\frac{q}{m}.
\end{aligned}$$

Now turn to $\beta^{'}$. Write

$$\begin{aligned}
\beta^{'} &= 2\frac{1}{q^2}\sum_{s=1}^{q-1}\sum_{k>s}(q-s)(q-k)cov\left(\widehat{\gamma}_s, z_k\right) \\
= {} & \frac{2}{q^2}\sum_{s=1}^{q-2}(q-(s+1))(q-s)\left[4\sigma_\eta^4 + 4\sigma_\eta^2\sigma_1^2 + 2\sigma_\eta^2(\sigma_{s+1}^2 - \sigma_2^2) + 2\sigma_1^2\sigma_{s+1}^2\right] \\
&+\frac{2}{q^2}\sum_{s=1}^{q-3}(q-(s+2))(q-s)\left[-2\sigma_\eta^4 + 2\sigma_\eta^2\left((\sigma_2^2 - \sigma_1^2) - (\sigma_3^2 - \sigma_2^2)\right) + 2\sigma_2^2\sigma_{s+2}^2\right] \\
&+\frac{2}{q^2}\sum_{s=1}^{q-4}\sum_{k>s+2}(q-k)(q-s)\left[2\sigma_\eta^2\left((\sigma_s^2 - \sigma_{s-1}^2) - (\sigma_{s+1}^2 - \sigma_s^2)\right) + 2\sigma_s^2\sigma_k^2\right] \\
= {} & \frac{2}{q^2}\sum_{s=1}^{q-2}(q-(s+1))(q-s)\left[4\sigma_\eta^4 + 4\sigma_\eta^2\frac{V}{m} + 2\frac{V^2}{m^2}\right] \\
&+\frac{2}{q^2}\sum_{j=1}^{q-3}(q-(s+2))(q-s)\left[-2\sigma_\eta^4 + 2\frac{V^2}{m^2}\right] + \frac{2}{q^2}\sum_{s=1}^{q-4}\sum_{k>s+2}(q-k)(q-s)\left[2\frac{V^2}{m^2}\right].
\end{aligned}$$

Simple algebra leads to

$$\begin{aligned}
\beta^{'} &= -2\sigma_\eta^4 + \frac{4}{3}\sigma_\eta^4 q + \frac{1}{q}\frac{2}{3}\sigma_\eta^4 - 4\sigma_\eta^4\frac{1}{q^2} - 8\sigma_\eta^2\frac{V}{m} - \frac{17}{4}\frac{V^2}{m^2} \\
&+\frac{16}{3}\sigma_\eta^2\frac{V}{qm} + \frac{7}{2}\frac{V^2}{qm^2} - \frac{5}{4}\frac{V^2}{m^2}q + \frac{1}{4}\frac{V^2}{m^2}q^2 + \frac{8}{3}\sigma_\eta^2\frac{V}{m}q + 2\frac{1}{q^2}\frac{V^2}{m^2}.
\end{aligned}$$

When computing the covariance between α and β we can treat the terms involving $\widetilde{z}$ similarly as the terms involving z, in that

$$
\begin{aligned}
cov(\widehat{\gamma}_0, \widetilde{z}_1) &= 10\sigma_\eta^4 + 8\sigma_\eta^2\sigma_m^2 + 2\sigma_m^4, \\
cov(\widehat{\gamma}_0, \widetilde{z}_2) &= -4\sigma_\eta^4 + 4\sigma_\eta^2\left(\sigma_{m-1}^2 - \sigma_m^2\right) + 2\sigma_{m-1}^4, \\
cov(\widehat{\gamma}_0, \widetilde{z}_j) &= 4\sigma_\eta^2\left(\sigma_{m-j+1}^2 - \sigma_{m-j+2}^2\right) + 2\sigma_{m-j+1}^4 \qquad j \geq 3,
\end{aligned}
$$

and

$$
\begin{aligned}
cov(\widehat{\gamma}_1, \widetilde{z}_1) &= -4\sigma_\eta^4 - 2\sigma_\eta^2\sigma_m^2, \\
cov(\widehat{\gamma}_j, \widetilde{z}_j) &= -2\sigma_\eta^4 - 2\sigma_\eta^2\sigma_m^2 \qquad j \geq 2, \\
cov(\widehat{\gamma}_j, \widetilde{z}_{j+1}) &= 4\sigma_\eta^4 + 2\sigma_\eta^2\sigma_m^2 + 2\sigma_\eta^2(\sigma_{m-j}^2 - \sigma_{m-1}^2) + 2\sigma_m^2\sigma_{m-j}^2, \\
cov(\widehat{\gamma}_j, \widetilde{z}_{j+2}) &= -2\sigma_\eta^4 + 2\sigma_\eta^2\left[(\sigma_{m-1}^2 - \sigma_m^2) - (\sigma_{m-2}^2 - \sigma_{m-1}^2)\right] + 2\sigma_{m-1}^2\sigma_{m-j-1}^2, \\
cov(\widehat{\gamma}_j, \widetilde{z}_{j+i}) &= 2\sigma_\eta^2\left[(\sigma_{m-i+1}^2 - \sigma_{m-i+2}^2) - (\sigma_{m-i}^2 - \sigma_{m-i+1}^2)\right] + 2\sigma_{m-i+1}^2\sigma_{m-j-i+1}^2, \qquad i \geq 3.
\end{aligned}
$$

Putting all the elements together, we obtain

$$
\begin{aligned}
&Var\left(\widehat{V}^{ZMA}\right) = 4\sigma_\eta^4 + \frac{20}{m}\sigma_\eta^4 + \left(-\frac{4}{m^2}\sigma_\eta^4 + \frac{24}{m^2}\sigma_\eta^2 V + \frac{13}{3}\frac{1}{m^2}Q\right) + \frac{2}{m^3}Q \\
&+8\frac{m}{q^2}\sigma_\eta^4 + \frac{2}{m^3q^2}Q - \frac{1}{3}\frac{q^2}{m^2}Q - \frac{4}{m^3q}Q \\
&+\left(-12\sigma_\eta^4 + 8\sigma_\eta^2 V\right)\frac{1}{q} + \left(-8\sigma_\eta^4 - 8\sigma_\eta^2 V\right)\frac{1}{q^2} \\
&+\left(\frac{8}{3}\sigma_\eta^2 V + \frac{4}{3}Q\right)\frac{q}{m} + \left(4\sigma_\eta^4 + 16\sigma_\eta^2 V + 2Q\right)\frac{1}{mq^2} \\
&+\left(8\sigma_\eta^4 - 48\sigma_\eta^2 V - 8Q\right)\frac{1}{m^2q} + \left(-4\sigma_\eta^4 + 24\sigma_\eta^2 V + 4Q\right)\frac{1}{m^2q^2} \\
&+\left(-\frac{56}{3}\sigma_\eta^2 V - \frac{10}{3}Q - 24\sigma_\eta^4\right)\frac{1}{mq} \\
&+4\sigma_\eta^4 + \frac{1}{q}\left(4\sigma_\eta^4\right) + \frac{1}{q^2}(-8\sigma_\eta^4) + \frac{1}{m^2}(\frac{4}{3}V^2) + \frac{1}{mq}\left(-\frac{8}{3}V\sigma_\eta^2\right) \\
&+\frac{q}{m}\left(\frac{8}{3}\sigma_\eta^2 V\right) + \frac{q^2}{m^2}\left(\frac{2}{3}V^2\right) + \frac{1}{qm^2}(-\frac{2}{3}V^2) - \frac{4}{3}\frac{q}{m^2}V^2 \\
&-24\sigma_\eta^4 + \frac{32}{q}\sigma_\eta^4 - \frac{8}{q^2}\sigma_\eta^4 - \frac{1}{m}(24\sigma_\eta^4 + 32\sigma_\eta^2 V) - \frac{1}{m^2}(32\sigma_\eta^2 V - 8V^2) \\
&-4\frac{V^2}{m^3}q - 4\frac{V^2}{m^2}q - \frac{1}{qm}\left(-32\sigma_\eta^4 - 64\sigma_\eta^2 V\right) - \frac{1}{qm^2}\left(4V^2 - 64\sigma_\eta^2 V\right) \\
&-\frac{1}{qm^3}\left(4V^2\right) - \frac{1}{q^2m^2}\left(32\sigma_\eta^2 V\right) - \frac{1}{mq^2}\left(8\sigma_\eta^4 + 32\sigma_\eta^2 V\right) - \frac{1}{m^3}(-8V^2) \\
&+8\sigma_\eta^4 + \frac{16\sigma_\eta^4}{q^2} - \frac{88}{3}\frac{\sigma_\eta^4}{q} - 8\sigma_\eta^2\frac{V}{m} + \frac{8}{3}\sigma_\eta^2 V\frac{1}{qm} + \frac{16}{3}\sigma_\eta^4 q + \frac{16}{3}\sigma_\eta^2 V\frac{q}{m} \\
&+8\sigma_\eta^4 - \frac{16}{3}\sigma_\eta^4 q - \frac{1}{q}\frac{8}{3}\sigma_\eta^4 + 16\sigma_\eta^4\frac{1}{q^2} + 32\sigma_\eta^2\frac{V}{m} + 17\frac{V^2}{m^2} \\
&-\frac{64}{3}\sigma_\eta^2\frac{V}{qm} - 14\frac{V^2}{qm^2} + 5\frac{V^2}{m^2}q - \frac{V^2}{m^2}q^2 - \frac{32}{3}\sigma_\eta^2\frac{V}{m}q - 8\frac{1}{q^2}\frac{V^2}{m^2}.
\end{aligned}
$$

Now rewrite as

$$
Var\left(\widehat{V}^{ZMA}\right) = (-4\sigma_\eta^4 - 8V\sigma_\eta^2)\frac{1}{m} + \left(-4\sigma_\eta^4 - 8\sigma_\eta^2 V + \frac{13}{3}Q + \frac{79}{3}V^2\right)\frac{1}{m^2}
$$

$$+\frac{1}{m^3}\left(2Q+8V^2\right)+8\frac{m}{q^2}\sigma_\eta^4-\frac{1}{3}(Q+V^2)\frac{q^2}{m^2}-\frac{1}{3}V^2\frac{q}{m^2}$$
$$-4V^2\frac{q}{m^3}+\frac{2}{m^3q^2}Q-\frac{4}{m^3q}\left(Q+V^2\right)$$
$$+\left(-8\sigma_\eta^4+8\sigma_\eta^2V\right)\frac{1}{q}+\left(8\sigma_\eta^4-8\sigma_\eta^2V\right)\frac{1}{q^2}$$
$$+\left(\frac{4}{3}Q\right)\frac{q}{m}+\left(-4\sigma_\eta^4-16\sigma_\eta^2V+2Q\right)\frac{1}{mq^2}$$
$$+\left(8\sigma_\eta^4+16\sigma_\eta^2V-8Q-\frac{56}{3}V^2\right)\frac{1}{m^2q}+$$
$$+\left(-4\sigma_\eta^4-8\sigma_\eta^2V+4Q-8V^2\right)\frac{1}{m^2q^2}$$
$$+\left(24\sigma_\eta^2V-\frac{10}{3}Q+8\sigma_\eta^4\right)\frac{1}{mq}$$

or, in terms of ϕ,

$$Var\left(\widehat{V}^{ZMA}\right)=(-4\sigma_\eta^4-8V\sigma_\eta^2)\frac{1}{m}+\left(-4\sigma_\eta^4-8\sigma_\eta^2V+\frac{13}{3}Q+\frac{79}{3}V^2\right)\frac{1}{m^2}$$
$$+\frac{1}{m^3}\left(2Q+8V^2\right)+8\frac{1}{\phi^2m}\sigma_\eta^4-\frac{1}{3}(Q+V^2)\phi^2-\frac{1}{3}V^2\frac{\phi}{m}$$
$$-4V^2\frac{\phi}{m^2}+\frac{2}{m^5\phi^2}Q-\frac{4}{m^4\phi}\left(Q+V^2\right)$$
$$+\left(-8\sigma_\eta^4+8\sigma_\eta^2V\right)\frac{1}{\phi m}+\left(8\sigma_\eta^4-8\sigma_\eta^2V\right)\frac{1}{\phi^2m^2}$$
$$+\left(\frac{4}{3}Q\right)\phi+\left(-4\sigma_\eta^4-16\sigma_\eta^2V+2Q\right)\frac{1}{m^3\phi^2}$$
$$+\left(8\sigma_\eta^4+16\sigma_\eta^2V-8Q-\frac{56}{3}V^2\right)\frac{1}{m^3\phi}+$$
$$+\left(-4\sigma_\eta^4-8\sigma_\eta^2V+4Q-8V^2\right)\frac{1}{m^4\phi^2}$$
$$+\left(24\sigma_\eta^2V-\frac{10}{3}Q+8\sigma_\eta^4\right)\frac{1}{m^2\phi}.$$

Then,

$$Var\left(\widehat{V}^{ZMA}\right)=K-\frac{1}{3}(Q+V^2)\phi^2+\left(-\frac{1}{3}V^2\frac{1}{m}-4V^2\frac{1}{m^2}+\frac{4}{3}Q\right)\phi$$
$$+[-\frac{4}{m^4}\left(Q+V^2\right)+\left(\frac{8\sigma_\eta^4+16\sigma_\eta^2V-8Q-\frac{56}{3}V^2}{m^3}\right)+\left(\frac{24\sigma_\eta^2V-\frac{10}{3}Q+8\sigma_\eta^4}{m^2}\right)$$
$$+\left(\frac{-8\sigma_\eta^4+8\sigma_\eta^2V}{m}\right)]\frac{1}{\phi}$$
$$+[\frac{2}{m^5}Q+\left(\frac{-4\sigma_\eta^4-8\sigma_\eta^2V+4Q-8V^2}{m^4}\right)+\left(\frac{-4\sigma_\eta^4-16\sigma_\eta^2V+2Q}{m^3}\right)$$
$$+\left(\frac{8\sigma_\eta^4-8\sigma_\eta^2V}{m^2}\right)+\frac{8}{m}\sigma_\eta^4]\frac{1}{\phi^2},$$

where

$$K = (-4\sigma_\eta^4 - 8V\sigma_\eta^2)\frac{1}{m} + \left(-4\sigma_\eta^4 - 8\sigma_\eta^2 V + \frac{13}{3}Q + \frac{79}{3}V^2\right)\frac{1}{m^2} + \frac{1}{m^3}\left(2Q + 8V^2\right).$$

As for the bias term, write

$$\begin{aligned}
\frac{1}{q}E(\vartheta_q) &= \frac{1}{q}\sum_{s=1}^{q-1}(q-s)E(z_s) + \frac{1}{q}\sum_{s=1}^{q-1}(q-s)E(\widetilde{z}_s) \\
&= \frac{1}{q}(q-1)2\sigma_\eta^2 + \frac{1}{q}\sum_{s=1}^{q-1}(q-s)\sigma_s^2 + \frac{1}{q}(q-1)2\sigma_\eta^2 + \frac{1}{q}\sum_{s=1}^{q-1}(q-s)\sigma_{m+1-s}^2 \\
&= \left(4 - \frac{4}{q}\right)\sigma_\eta^2 + \frac{q}{m}V - \frac{V}{m},
\end{aligned}$$

where the first equality derives from Lemma A.1 of Barndorff-Nielsen et al. (2005). Hence,

$$\begin{aligned}
bias\left(\widehat{V}^{ZMA}\right) &= \left(\frac{mq - m + q - 1}{mq}\right)\left(V + 2m\sigma_\eta^2\right) - V - 2\left(\frac{q-1}{q}\right)\sigma_\eta^2(m-1) \\
&\quad - \left(4 - \frac{4}{q}\right)\sigma_\eta^2 - \frac{q}{m}V + \frac{V}{m} \\
&= -\frac{V}{q} + \frac{2V}{m} - \frac{q}{m}V - \frac{V}{qm}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left(bias\left(\widehat{V}^{ZMA}\right)\right)^2 &= \frac{V^2}{q^2} + \frac{4V^2}{m^2} + \frac{q^2}{m^2}V^2 + \frac{V^2}{q^2m^2} - \frac{4V^2}{qm} + \frac{2V^2}{m} + \frac{2V^2}{q^2m} \\
&\quad -4\frac{q}{m^2}V^2 - 4\frac{V^2}{qm^2} + \frac{2V^2}{m^2}
\end{aligned}$$

or, in terms of ϕ,

$$\begin{aligned}
\left(bias\left(\widehat{V}^{ZMA}\right)\right)^2 &= \frac{V^2}{\phi^2m^2} + \frac{4V^2}{m^2} + \phi^2V^2 + \frac{V^2}{\phi^2m^4} - \frac{4V^2}{\phi m^2} + \frac{2V^2}{m} + \frac{2V^2}{\phi^2m^3} \\
&\quad -4\frac{\phi}{m}V^2 - 4\frac{V^2}{\phi m^3} + \frac{2V^2}{m^2} \\
&= \left(\frac{6V^2}{m^2} + \frac{2V^2}{m}\right) + V^2\phi^2 - \frac{4V^2}{m}\phi + \left(-4\frac{V^2}{m^2} - 4\frac{V^2}{m^3}\right)\frac{1}{\phi} \\
&\quad + \left(\frac{V^2}{m^2} + \frac{2V^2}{m^3} + \frac{V^2}{m^4}\right)\frac{1}{\phi^2}.
\end{aligned}$$

■

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Figure 1 (HL-SBC)

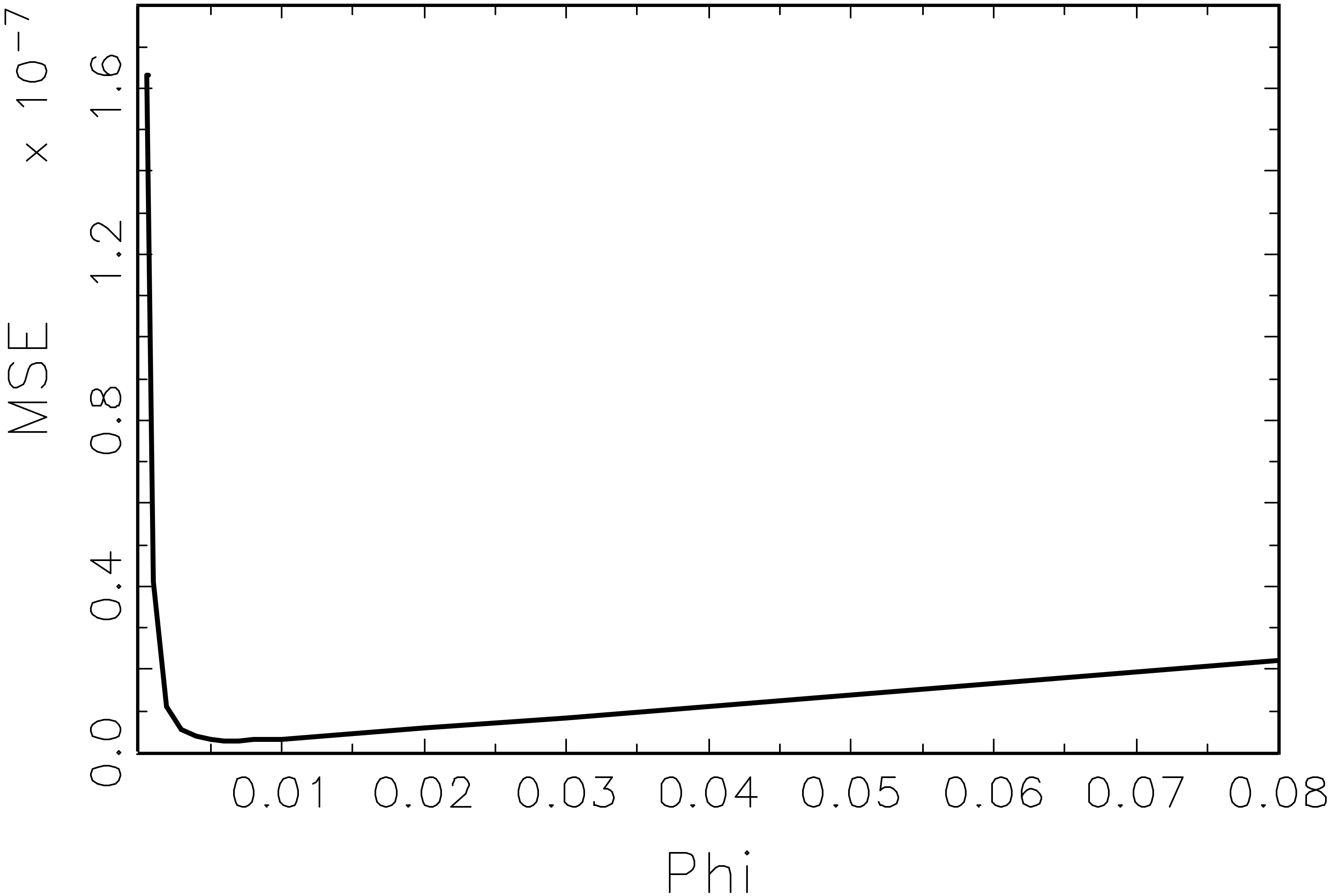

Figure 2 (ZMA-SBC)