Nonparametric Bootstrap for Quasi-Likelihood Ratio Tests*

Lorenzo Camponovo

University of Lugano visiting Yale University

First Version: October 2010. This Version: October 2011

*The author thanks participants at the 26th Annual Congress of the European Economic Association and the 65th European Meeting of the Econometric Society 2011 in Oslo, for helpful comments. This research was done when the author was visiting Yale University, financed by a fellowship of the Swiss National Science Foundation (SNSF).

Abstract

We introduce a nonparametric bootstrap approach for Quasi-Likelihood Ratio type tests of nonlinear restrictions. Our method applies to extremum estimators, such as quasi-maximum likelihood and generalized method of moments estimators. Unlike existing parametric bootstrap procedures for Quasi-Likelihood Ratio type tests, our procedure constructs bootstrap samples in a fully nonparametric way. We study the higher order properties of our nonparametric bootstrap and show the asymptotic refinements implied with respect to the standard asymptotic theory. Our approach delivers the same higher order properties of the nonparametric bootstrap methods introduced in Andrews (2002) and Kim (2003) in relation to Wald and Lagrange Multiplier tests, respectively. Monte Carlo simulations and a real data application to testing for stock return predictability confirm the accuracy of our bootstrap procedure.

1 Introduction

As pointed out in Davidson and MacKinnon (1999), the distributions of the most frequently used test statistics are known only asymptotically and, in many applications, the asymptotic distributions provide only poor approximations of the test statistic distributions. Thus, inference based on critical values derived by the standard asymptotic theory may be less accurate. To refine the approximations of the test statistic distributions, in the recent past bootstrap methods have been proposed; see e.g., Singh (1981), Beran (1988), and Hall (1992).

There are many ways to apply the bootstrap approach. In particular, much attention has been devoted to nonparametric iid and block bootstrap methods; see e.g., Efron (1979) for the definition of the nonparametric iid bootstrap, and Hall (1985), Carlstein (1986) and Künsch (1989) for the definition of nonparametric block bootstrap procedures[1]. Indeed, the straightforward implementation of the nonparametric block bootstrap makes this approach a valid alternative to more complex parametric bootstrap procedures that typically require stronger model assumptions. Important studies that establish the higher order properties of nonparametric block bootstrap methods include: Hall and Horowitz (1996), who introduce a block bootstrap procedure for over-identification tests and t-tests of generalized method of moments (GMM) estimators; Andrews (2002), who proposes block bootstrap methods and computationally attractive block bootstrap procedures for over-identification tests, t-tests, and Wald tests of nonlinear restrictions for extremum estimators, such as quasi-maximum likelihood (QML) and GMM estimators; Kim (2003), who develops nonparametric block bootstrap methods for Lagrange Multiplier (LM) tests for extremum estimators[2].

This paper completes the results of nonparametric block bootstrap procedures in Andrews (2002) and Kim (2003) by introducing a nonparametric block bootstrap approach for Quasi-Likelihood Ratio (QLR) type tests in the context of extremum estimators. In spite of the typically higher computational costs, the strong invariance properties of QLR tests make this

[1] For the sake of brevity, in this paper we will refer only to nonparametric block bootstrap procedures. Indeed, the nonparametric iid bootstrap is a special case of block bootstrap method (block length $l = 1$).

[2] Besides block bootstrap methods for LM tests, Kim (2003) develops bootstrap procedures also of Quasi-Likelihood Ratio tests. Nevertheless, his approach applies only to GMM estimators, while is not well defined for the class of QML estimators.

approach an attractive alternative in particular to Wald procedures. Therefore, accurate nonparametric block bootstrap procedures are desirable also in the QLR context.

The definition of bootstrap methods for tests, and in particular nonparametric block bootstrap procedures, requires some care. By construction, nonparametric block bootstrap samples reproduce the true data generating process, whether or not the null hypothesis is true. It turns out that when not correctly defined, nonparametric block bootstrap tests may imply empirical frequencies of rejection of the null hypothesis close to the nominal significance level, whether or not the null is true. Consequently, in these situations bootstrap tests may have no power and become useless. To avoid this problem and to introduce a valid approach for dealing with tests, the definition of bootstrap methods has to ensure that either (i) the bootstrap statistic is adjusted such that its distribution is a consistent estimator of the null distribution of the test statistic or (ii) the bootstrap distribution of the sample is adjusted such that it is a consistent estimator of the null distribution of the sample. In the QLR tests context, the latter condition is typically satisfied when considering restricted parametric bootstrap methods that impose the null hypothesis for the construction of the bootstrap samples. On the other hand, conditions (i) and (ii) are generally no met when considering nonparametric block bootstrap procedures.

The aim of this paper is to overcome this problem by introducing a nonparametric block bootstrap method that satisfies condition (i) and consequently provides a valid statistical tool for dealing with QLR type tests. To achieve this objective, we follow the setup introduced in Andrews (2002) and Kim (2003). For the sake of brevity, we focus on two-step GMM estimators, but the application of our nonparametric block bootstrap method to one-step GMM and QML estimators is straightforward[3]. After introducing the QLR statistic, we first define our nonparametric block bootstrap approach. We show that the distribution of the nonparametric block bootstrap statistic is a consistent estimator of the null distribution of the QLR statistic whether or not the null hypothesis is true. Second, we study the higher order properties of our method. Standard asymptotic theory implies an error in test rejection probability of order

[3]In Remark 1 below we discuss the one-step GMM case. In Section 5.1 we show how to implement our method for QML estimators. A rigorous proof of the higher order properties of our bootstrap procedure for QML estimators is available from the author on request.

$O(N^{-1})$, where N denotes the sample size. When data are iid, we show that our approach reduces the error from $O(N^{-1})$ to $O(N^{-2})$. In the time series case, we show instead that the nonparametric block bootstrap reduces the error by at least N^{ξ}, for some $\xi > 0$, which depends on the block length $l \propto N^{\gamma}$, $0 \leq \gamma \leq 1$. In particular, when data are independent $\xi < 1/2$, while when data are dependent and $\kappa > 0$, the improvement cannot exceed $N^{1/4}$. These asymptotic refinements match the higher order properties of the nonparametric block bootstrap methods introduced in Andrews (2002) and Kim (2003) for Wald and LM tests, respectively.

We analyze through Monte Carlo simulation the accuracy of our nonparametric block bootstrap in the context of QML and GMM estimators. The results in both settings confirm the reliability of our approach, that clearly outperforms the standard asymptotic theory. Finally, we consider also a real data application. Based on a predictive regression model, we study the predictability of stock return. To deal with the endogeneity and nonstationarity of some explanatory variables, we adopt the difference-based approach introduced in Phillips and Han (2010) and Han, Phillips and Sul (2011) for autoregressive models and Camponovo (2011) for predictive regression models. This approach eliminates the persistency of the explanatory variables and allows to apply the usual GMM technology, including QLR type tests and our nonparametric block bootstrap approach. We apply our method to US equity data from 1948-2008, and obtain results in line with the findings provided by new testing procedures recently introduced in the stock return predictability literature.

The rest of the paper is organized as follows. In Section 2, we introduce the model and the QLR statistic, while in Section 3 we define our nonparametric block bootstrap approach. The higher order properties of our procedure are presented in Section 4. Section 5 analyzes the accuracy of the nonparametric block bootstrap through Monte Carlo simulations, while in Section 6 we consider the real data application. Finally, Section 7 concludes.

2 The Model

We use the setup introduced in Andrews (2002) and Kim (2003). In particular, let $(X_1, \ldots, X_n)$ be an observation sample from a strictly stationary ergodic sequence of random vectors, where

$X_i \in \mathbb{R}^{d_x}$. Let $\theta \in \Theta \subset \mathbb{R}^{d_\theta}$ be an unknown parameter. We consider GMM estimators of θ based on the moment conditions $E[g(X_i, \theta_0)] = 0$, where $g(\cdot, \cdot)$ is a $\mathbb{R}^{d_g}$-valued function with $d_g \geq d_\theta$, and θ_0 denotes the true value of the unknown parameter θ. Finally, we focus on QLR tests of the null hypothesis $\mathcal{H}_0 : \eta(\theta_0) = 0$ versus $\mathcal{H}_1 : \eta(\theta_0) \neq 0$, where η is a (nonlinear) $\mathbb{R}^{d_\eta}$-valued function.

Furthermore, in line with Hall and Horowitz (1996), Andrews (2002) and Kim (2003), we also assume that the true moment conditions are uncorrelated beyond lags of length $0 \leq \kappa < \infty$. That is, $E[g(X_i, \theta_0) g(X_{i+j}, \theta_0)'] = 0$, for $j > \kappa$. As pointed out in Kim (2003), this condition allows us to write the covariance matrix estimator as sample averages, and then to use the Edgeworth expansion results introduced in Götze and Hipp (1983, 1994); see e.g., Kim (2003) for further details[4].

Under these assumptions the two-step GMM estimator $\hat{\theta}_N$ is defined as the solution of

$$
\begin{aligned}
\min_{\theta \in \Theta} J_N(\theta, \bar{\theta}_N) &= \left(N^{-1} \sum_{i=1}^{N} g(X_i, \theta) \right)' \Omega_N(\bar{\theta}_N) \left(N^{-1} \sum_{i=1}^{N} g(X_i, \theta) \right), & (1) \\
\Omega_N(\theta) &= W_N^{-1}(\theta), & (2) \\
W_N(\theta) &= N^{-1} \sum_{i=1}^{N} \left(g(X_i, \theta) g(X_i, \theta)' + \sum_{j=1}^{\kappa} H(X_i, X_{i+j}, \theta) \right), & (3) \\
H(X_i, X_{i+j}, \theta) &= g(X_i, \theta) g(X_{i+j}, \theta)' + g(X_{i+j}, \theta) g(X_i, \theta)', & (4)
\end{aligned}
$$

where $N = n - \kappa$, $\bar{\theta}_N$ is the solution of

$$
\min_{\theta \in \Theta} J_N(\theta) = \left(N^{-1} \sum_{i=1}^{N} g(X_i, \theta) \right)' \Omega \left(N^{-1} \sum_{i=1}^{N} g(X_i, \theta) \right), \qquad (5)
$$

and Ω is a nonrandom, positive definite, symmetric matrix.

Moreover, we define the restricted two-step GMM estimator $\tilde{\theta}_N$ as the solution of (1), subject to the restriction $\eta(\theta) = 0$, where in this case $\bar{\theta}_N$ solves (5) subject to $\eta(\theta) = 0$. Then,

[4] Note that this condition is a necessary assumption only for the higher order properties of our method.

the QLR statistic for testing $\mathcal{H}_0 : \eta(\theta_0) = 0$ versus $\mathcal{H}_1 : \eta(\theta_0) \neq 0$ is defined as

$$QLR_N = N(J_N(\tilde{\theta}_N, \bar{\theta}_N) - J_N(\hat{\theta}_N, \bar{\theta}_N)). \quad (6)$$

Remark 1 *Let* $D_0 = E\left[\frac{\partial}{\partial\theta'} g(X_i, \theta_0)\right]$ *and* $\Omega_0^{-1} = E\left[W_N(\theta_0)\right]$. *To be valid, QLR tests based only on the one-step GMM estimators defined in equation (5) require the further condition that* $D_0'\Omega\Omega_0^{-1}\Omega D_0 = cD_0'\Omega D_0$, *for some* $c > 0$. *If this condition holds, then the nonparametric block bootstrap approach introduced in the next section can be extended straightforward also to this context.*

When $\mathcal{H}_0$ is true, under some regularity conditions, the asymptotic distribution of the QLR statistic (6) is a central chi-square with d_η degrees of freedom. Therefore, the critical values of the QLR test can be computed using this distribution. Nevertheless, as pointed out for instance in Davidson and MacKinnon (1999), the asymptotic distribution may work poorly in finite sample. In Section 3 we introduce a nonparametric block bootstrap approach for dealing with QLR tests, while in Section 4 we show the asymptotic refinements provided by this method.

3 Nonparametric Block Bootstrap

To provide a valid approach for dealing with tests, the definition of bootstrap tests has to ensure that either (i) the bootstrap test statistic is adjusted such that its distribution is a consistent estimator of the null distribution of the test statistics, or (ii) the bootstrap distribution of the sample is adjusted such that it is a consistent estimator of the null distribution of the sample. In this section we introduce a nonparametric block bootstrap procedures that satisfies condition (i) in the QLR tests context.

We consider both nonoverlapping and overlapping block bootstrap, with block size length $l \propto N^\gamma$, where $0 \leq \gamma \leq 1$ (for the iid case, $l = 1$, $\gamma = 0$). For simplicity, we assume that $N/l = b \in \mathbb{N}$. Under the assumptions introduced in the previous section, the observations that have to be bootstrapped are $\{\tilde{X}_1, \ldots, \tilde{X}_N\}$, where $\tilde{X}_i = (X_i, \ldots, X_{i+\kappa})$. For the nonoverlapping block bootstrap, the nonoverlapping blocks are $(\tilde{X}_1, \ldots, \tilde{X}_l), (\tilde{X}_{l+1}, \ldots, \tilde{X}_{2l}), \ldots, (\tilde{X}_{N-l+1}, \ldots, \tilde{X}_N)$. The overlapping blocks are instead $(\tilde{X}_1, \ldots, \tilde{X}_l)$, $(\tilde{X}_2, \ldots, \tilde{X}_{l+1}), \ldots, (\tilde{X}_{N-l+1}, \ldots, \tilde{X}_N)$. Using these blocks, the nonoverlapping (overlapping) block bootstrap generates bootstrap samples by randomly selecting with replacement b blocks from the b nonoverlapping ($N-l+1$ overlapping) blocks.

Given a bootstrap sample, we define the two-step bootstrap nonrestricted GMM estimator $\hat{\theta}_N^*$ as

$$\begin{aligned}
\min_{\theta\in\Theta} \bar{J}_N^*(\theta, \bar{\theta}_N^*) &= \left(N^{-1}\sum_{i=1}^{N} g^*(X_i^*, \theta)\right)' \Omega_N^*(\bar{\theta}_N^*)\left(N^{-1}\sum_{i=1}^{N} g^*(X_i^*, \theta)\right), && (7)\\
g^*(X_i^*, \theta) &= g(X_i^*, \theta) - E^*[g(X_i^*, \hat{\theta}_N)], && (8)\\
\Omega_N^*(\theta) &= W_N^*(\theta)^{-1}, && (9)\\
W_N^*(\theta) &= N^{-1}\sum_{i=1}^{N}\left(g^*(X_i^*, \theta)g^*(X_i^*, \theta)' + \sum_{j=1}^{\kappa} H^*(X_i^*, X_{i,i+j}^*, \theta)\right), && (10)\\
H^*(X_i^*, X_{i,i+j}^*, \theta) &= g^*(X_i^*, \theta)g^*(X_{i,i+j}^*, \theta)' + g^*(X_{i,i+j}^*, \theta)g^*(X_i^*, \theta)', && (11)
\end{aligned}$$

where $X_{i,i+j}^*$ denotes the $(j+1)$st element of $\tilde{X}_i^*$, E^* denotes the expectation with respect to the induced nonoverlapping (overlapping) block bootstrap distribution, and $\bar{\theta}_N^*$ is the solution of

$$\min_{\theta\in\Theta} \bar{J}_N^*(\theta) = \left(N^{-1}\sum_{i=1}^{N} g^*(X_i^*, \theta)\right)' \Omega\left(N^{-1}\sum_{i=1}^{N} g^*(X_i^*, \theta)\right). \qquad (12)$$

Note that as in Hall and Horowitz (1996), and Andrews (2002), the bootstrap sample moment conditions in (8) are recentered by subtracting off $E^*[g(X_i^*, \hat{\theta}_N)]$. The recentering is necessary in order to ensure that the bootstrap moments $E^*[g^*(X_i^*, \theta)] = 0$, when $\theta = \hat{\theta}_N$, which mimics the population moments $E[g(X_i, \theta)] = 0$, when $\theta = \theta_0$. In particular, for the nonoverlapping and overlapping block bootstrap we have, respectively

$$E^*[g(X_i^*,\theta)] = N^{-1}\sum_{i=1}^{N} g(X_i,\theta), \text{ and} \tag{13}$$

$$E^*[g(X_i^*,\theta)] = (N-l+1)^{-1}\sum_{i=1}^{N} \omega(i,l,N) g(X_i,\theta), \text{ where} \tag{14}$$

$$\omega(i,l,N) = \begin{cases} i/l & \text{if } i \in [1, l-1] \\ 1 & \text{if } i \in [l, N-l+1] \\ (N-i+1)/l & \text{if } i \in [N-l+2, N] \end{cases} . \tag{15}$$

To define the nonparametric block bootstrap QLR statistic, we finally introduce the two-step bootstrap restricted GMM estimators. Intuitively, the natural candidate is the two-step GMM estimator solution of (7) subject to the restriction $\eta(\theta)=0$. Nevertheless, it is easy to verify that the distribution of the relating nonparametric block bootstrap QLR statistic, defined using this restricted estimator and the the nonrestricted estimator $\hat{\theta}_N^*$, is a consistent estimator of the null distribution of statistic (6) only when $\mathcal{H}_0$ is true. It turns out that this bootstrap approach does not satisfy neither condition (i) nor condition (ii).

We overcome this problem by defining the two-step bootstrap restricted GMM estimator $\tilde{\theta}_N^*$ as the solution of (7), subject to

$$\eta(\theta) = \eta(\hat{\theta}_N), \tag{16}$$

where in this case $\bar{\theta}_N^*$ solves (12) subject to (16). The bootstrap restriction (16) exactly mimics the population restriction $\eta(\theta) = \eta(\theta_0)$, which is trivially satisfied by θ_0, whether or not $\mathcal{H}_0$ is true. The restriction (16) represents the key condition for defining a valid nonparametric block bootstrap approach for QLR tests. Indeed, with this restriction the distribution of the nonparametric block bootstrap QLR statistic defined in equation (17) below is a consistent estimator of the null distribution of statistic (6) whether or not $\mathcal{H}_0$ is true. Note that the restriction (16) is also proposed in Chen and Pouzo (2009) for the definition of bootstrap methods for QLR tests in the context of penalized sieve minimum distance estimators of nonparametric conditional moment models.

Finally, we define the bootstrap QLR statistic as

$$QLR_N^* = N(J_N^*(\tilde{\theta}_N^*, \bar{\theta}_N^*) - J_N^*(\hat{\theta}_N^*, \bar{\theta}_N^*)), \tag{17}$$

where

$$J_N^*(\theta, \bar{\theta}_N^*) = \left(N^{-1}\sum_{i=1}^{N} g^*(X_i^*, \theta)\right)' \Omega_N^*(\bar{\theta}_N^*)^{1/2}\bar{\Xi}_N\Omega_N^*(\bar{\theta}_N^*)^{1/2}\left(N^{-1}\sum_{i=1}^{N} g^*(X_i^*, \theta)\right), \tag{18}$$

$$\bar{\Xi}_N = \Omega_N(\bar{\theta}_N)^{-1/2}\bar{W}_N^{-1}\Omega_N(\bar{\theta}_N)^{-1/2}, \tag{19}$$

$$\bar{W}_N = E^*\left[N^{-1}\sum_{i=1}^{N}\sum_{j=1}^{N} g^*(X_i^*, \hat{\theta}_N)g^*(X_i^*, \hat{\theta}_N)'\right], \tag{20}$$

and for the nonoverlapping and overlapping block bootstrap we have, respectively

$$\bar{W}_N = N^{-1}\sum_{i=0}^{b-1}\sum_{j=1}^{l}\sum_{m=1}^{l} g^*(X_{il+j}, \hat{\theta}_N)g^*(X_{il+m}, \hat{\theta}_N)', \text{ and} \tag{21}$$

$$\bar{W}_N = bN^{-1}(N-l+1)^{-1}\sum_{i=0}^{N-l}\sum_{j=1}^{l}\sum_{m=1}^{l} g^*(X_{i+j}, \hat{\theta}_N)g^*(X_{i+m}, \hat{\theta}_N)'. \tag{22}$$

The correction factor defined in equation (19) is introduced in order to correct the fact that the independence between the bootstrap blocks does not properly mimic the dependence in the original sample. The correction factor is necessary for implying the asymptotic refinements of the nonparametric block bootstrap method, and applies only to the time series case; see e.g., Hall and Horowitz (1996), Andrews (2002), and Kim (2003) for further details.

Let α denotes the significance level, and let $z^*_{QLR,\alpha}$ be the $1-\alpha$ quantile of QLR_N^*. The bootstrap QLR test of $\mathcal{H}_0 : \eta(\theta_0) = 0$ versus $\mathcal{H}_1 : \eta(\theta_0) \neq 0$ rejects the null hypothesis if

$$QLR_N > z^*_{QLR,\alpha}. \tag{23}$$

After the definition of our nonparametric block bootstrap approach, in the next section we analyze its higher order properties.

4 Results

We first introduce the assumptions required for establishing the higher order properties of our nonparametric block bootstrap approach. In particular, we consider the same assumptions introduced in Kim (2003) in relation to GMM estimators. As in Andrews (2002) and Kim (2003), we denote by $f(\tilde{X}_i,\theta)$ the vector containing the unique components of $g(X_i,\theta)$ and $g(X_i,\theta)g(X_{i+j})'$, for $j=0,\ldots,\kappa$, and their derivatives through order $d_1\geq 3$, with respect to θ. Moreover, let $(\partial^j/\partial\theta^j)g(X_i,\theta)$ and $(\partial^j/\partial\theta^j)f(\tilde{X}_i,\theta)$ denote the vectors of partial derivatives of $g(X_i,\theta)$ and $f(\tilde{X}_i,\theta)$, respectively, with respect to θ of order j.

Assumption 2 *There exists a sequence of iid vectors $\{\epsilon_i : i=-\infty,\ldots,\infty\}$ of dimension $d_\epsilon\geq d_x$ and $d_x\times 1$ function h such that $X_i=h(\epsilon_i,\epsilon_{i-1},\epsilon_{i-2},\ldots)$. There are constants $K<\infty$ and $\xi>0$ such that for all $m\geq 1$, $E[||h(\epsilon_i,\epsilon_{i-1},\ldots)-h(\epsilon_i,\epsilon_{i-1},\ldots,\epsilon_{i-m},0,0,\ldots)||]\leq K\exp(-\xi m)$.*

Assumption 3 *(a) Θ is compact and θ_0 is an interior point of Θ. (b) $E[||g(X_i,\theta)||]<\infty$ for all $\theta\in\Theta$. (c) The two-step GMM estimators $\hat{\theta}_n$ minimizes $J_N(\theta,\bar{\theta}_N)$ over $\theta\in\Theta$, while for the restricted parameter space $\Theta_0=\{\theta\in\Theta:\eta(\theta)=0\}$, $\tilde{\theta}_N$ minimizes $J_N(\theta,\bar{\theta}_N)$ over $\theta\in\Theta_0$; θ_0 is the unique solution in Θ to $E[g(X_1,\theta)]=0$; for some function $C_g(x)$, $||g(x,\theta_1)-g(x,\theta_2)||\leq C_g(x)||\theta_1-\theta_2||$ for all x in the support of X_1 and all θ_1, $\theta_2\in\Theta$; and $E[C_g^{q_0}(X_i)]<\infty$ for some $q_0\geq 2$.*

Assumption 4 *(a) $E[g(X_i,\theta_0)g(X_{i+j},\theta_0)']=0$ for all $j>\kappa$, for some $0\leq\kappa<\infty$. (b) Ω and Ω_0 are positive definite and D_0 is of full rank L_θ. (c) $E[||g(X_i,\theta_0)||^{q_1}]<\infty$, for some $q_1\geq 2$. (d) $g(x,\theta)$ is $d=d_1+d_2$ times differentiable with respect to θ on N_0, some neighborhood of θ_0, for all x in the support of X_1, where $d_1\geq 3$ and $d_2\geq 0$. (e) There is a function $C_{\partial f}(\tilde{X}_i)$ such that $||(\partial^j/\partial\theta^j)f(\tilde{X}_1,\theta)-(\partial^j/\partial\theta^j)f(\tilde{X}_1,\theta_0)||\leq C_{\partial f}(\tilde{X}_1)||\theta-\theta_0||$ for all $\theta\in N_0$, for all $j=0,\ldots,d_2$. (f) $E[C_{\partial f}^{q_2}(\tilde{X}_i)]<\infty$ and $E[||(\partial^j/\partial\theta^j)f(\tilde{X}_1,\theta_0)||^{q_2}]\leq C_f<\infty$, for all $j=0,\ldots,d_2$, for some constants q_2 and C_f. (g) $f(\tilde{X}_1,\theta_0)$ is once differentiable with respect to $\tilde{X}_1$ with uniformly continuous first derivative. (h) The $\mathbb{R}^{L_\eta}$-valued function $\eta(\cdot)$ is d_1 times differentiable at θ_0 and $(\partial/\partial\theta')\eta(\theta_0)$ is of full rank $L_\eta\leq L_\theta$.*

Assumption 5 *There exist constant* $K_1 < \infty$ *and* $\delta > 0$ *such that for arbitrarily large* $\zeta > 1$ *and all integers* $m \in (\delta^{-1}, N)$ *and* $t \in \mathbb{R}^{dim(f)}$ *with* $\delta < ||t|| < N^{\zeta}$, $E[|E(\exp(it' \sum_{s=1}^{2m+1} f(\tilde{X}_s, \theta_0) : \{\epsilon_j : |j - m| > K_1\})|] \leq \exp(-\delta)$*, where* $i = \sqrt{-1}$.

Assumption 2 implies that $\{X_i : i \geq 1\}$ is strong mixing, and the mixing coefficient is an exponentially decreasing function of the lag length. Assumption 3 provides conditions typically required for the consistency of GMM estimators. Assumptions 4 and 5 supply conditions that allow us to use Edgeworth expansion results in Götze and Hipp (1994).

The higher order properties of our nonparametric block bootstrap approach are presented in the next theorem.

Theorem 6 *(a) Let assumptions (2)-(5) hold with* $q_0 > 6$, $q_1 > max\{6/(1-2c), 12\}$, $q_2 > 9/(1-2\gamma-2\xi)$, $d_1 \geq 2/c$, $d_2 \geq -1 + (3/2+\gamma+\xi)/c$*, and for some* $c \in (0, 1/2)$*. Suppose* $0 \leq \xi < 1/2 - \gamma$, $0 < \gamma < 1/2$*, and either* $\xi < \gamma$ *or* $k = 0$ *(if* $\{X_i : i \geq 1\}$ *are independent, then* $\gamma = 0$*, and* $\kappa = 0$*). Then, under* $\mathcal{H}_0 : \eta(\theta_0) = 0$,

$$P(QLR_N > z^*_{QLR,\alpha}) = \alpha + o(N^{-(1+\xi)}). \tag{24}$$

(b) Suppose $\{X_i : i \geq 1\}$ *are iid, let assumptions (2)-(5) hold with* $q_0 > 8$, $q_1 > max\{8/(1-2c), 16\}$, $q_2 > 12$, $d_1 \geq 5/2c$, $d_2 \geq -1 + 2/c$*, and for some* $c \in (0, 1/2)$*. Then, under* $\mathcal{H}_0 : \eta(\theta_0) = 0$,

$$P(QLR_N > z^*_{QLR,\alpha}) = \alpha + O(N^{-2}). \tag{25}$$

Remark 7 *Besides assumptions (2)-(5), if the condition in Remark 1 holds, then the results of Theorem 6 extend straightforward also to QLR tests based on one-step GMM estimators.*

Standard asymptotic theory that uses the chi-square distribution for computing the critical values of QLR tests implies an error in test rejection probability of order $O(N^{-1})$. Part (a) of Theorem 6 shows that using the nonparametric block bootstrap we can reduce the error by at least N^{ξ}. When data are independent part (a) holds for $\xi < 1/2$. When data are dependent and $\kappa > 0$, the improvement cannot instead exceed $N^{-1/4}$.

When data are iid, part (b) of Theorem 6 shows that the refinement provided by the nonparametric block bootstrap is even larger. In particular, the error in rejection probability is of order $O(N^{-2})$ instead of $O(N^{-1})$. Finally, Theorem 6 shows that our nonparametric block bootstrap approach for QLR tests implies the same asymptotic refinements provided by the nonparametric block bootstrap methods introduced for Wald and LM tests; see e.g., Andrews (2002), and Kim (2003), respectively.

5 Monte Carlo Study

In this section we study through Monte Carlo simulation the accuracy of our nonparametric block bootstrap approach. In particular, in Section 5.1 we consider a QML setting, while in Section 5.2 we consider a GMM setting. Finally, in Section 5.3, we compare the accuracy of our procedures with the nonparametric block bootstrap methods for Wald and LM tests introduced in Andrews (2002) and Kim (2003), respectively.

5.1 Quasi-Maximum Likelihood Case

We show how to implement the nonparametric block bootstrap approach for QLR tests based on QML estimators. As benchmark setting we consider the linear regression model. After introducing the QLR statistic, we explain how to define the nonparametric block bootstrap QLR statistic. Finally, in Section 5.1.1, we study through Monte Carlo simulations the accuracy of our method in this context.

Consider the linear regression model

$$Y_t = \theta' Z_t + U_t, \qquad t = 1, \ldots, n, \tag{26}$$

where $\{Y_t\}$ are scalar random variables, θ_0 is the true value of the unknown $\mathbb{R}^d$-dimensional parameter θ, $\{Z_t\}$ are iid $\mathbb{R}^d$-valued random variables, and the error terms are iid with distribution function $F(x)$. Let $(X_1, \ldots, X_n)$, $X_i = (Y_t, Z_t')'$, be the sample generated according to

(26). Then, the QML estimator, denoted by $\hat{\theta}_n$, is defined as the solution of

$$\min_{\theta \in \Theta} \rho_n(\theta) = n^{-1} \sum_{i=1}^{n} \rho(X_i, \theta), \tag{27}$$

where $\rho(\cdot, \cdot)$ denotes the quasi-loglikelihood function. Let $\tilde{\theta}_n$ be the restricted QML estimator, defined as the solution of (27), subject to the restriction $\eta(\theta) = 0$. Then, the QLR statistic for testing $\mathcal{H}_0 : \eta(\theta_0) = 0$ versus $\mathcal{H}_1 : \eta(\theta_0) \neq 0$ is defined as

$$QLR_n = 2n(\rho_n(\tilde{\theta}_n) - \rho_n(\hat{\theta}_n)). \tag{28}$$

We introduce the nonparametric block bootstrap QLR statistic. Let $(X_1^*, \ldots, X_n^*)$ be a bootstrap sample (in the iid case $l = 1$). The nonrestricted block bootstrap QLR estimator, denoted by $\hat{\theta}_n^*$, is defined as the solution of

$$\min_{\theta \in \Theta} \rho_n^*(\theta) = n^{-1} \sum_{i=1}^{n} \rho(X_i^*, \theta). \tag{29}$$

The restricted bootstrap QML estimator $\tilde{\theta}_n^*$ is defined as the solution of (29), subject to the restriction $\eta(\theta) = \eta(\hat{\theta}_n)$. As for the GMM setting, this restriction represents the key condition for defining a valid nonparametric block bootstrap approach for QLR tests based on QML estimators. Finally, the bootstrap QLR statistic is given by

$$QLR_n^* = 2n(\rho_n^*(\tilde{\theta}_n^*) - \rho_n^*(\hat{\theta}_n^*)). \tag{30}$$

Let α denotes the significance level, and let $z_{QLR,\alpha}^*$ be the $1 - \alpha$ quantile of QLR_n^*. The bootstrap QLR test of $\mathcal{H}_0 : \eta(\theta_0) = 0$ versus $\mathcal{H}_1 : \eta(\theta_0) \neq 0$ rejects the null hypothesis if

$$QLR_n > z_{QLR,\alpha}^*. \tag{31}$$

Remark 8 *For the time series case, when considering overlapping block bootstrap, (29) is* *replaced by*

$$\min_{\theta\in\Theta}\rho_n^*(\theta)=n^{-1}\sum_{i=1}^{n}\Big(\rho(X_i^*,\theta)-E^*[g(X_i^*,\hat{\theta}_n)']\theta\Big), \tag{32}$$

where $g(X_i,\theta)=(\partial/\partial\theta)\rho(X_i,\theta)$. *The recentering is necessary in order to ensure that the bootstrap population first-order conditions equal zero at* $\hat{\theta}_n$. *For the nonoverlapping block bootstrap the recentering is not necessary. Indeed, we have* $E^*[g(X_i^*,\hat{\theta}_n)]=0$.

Remark 9 *As for the GMM setting, note that for the time series case the QLR statistic (30) requires the same correction factor defined in (19).*

Remark 10 *By replacing assumptions 3 with the typically assumptions required for the consistency of QML estimators, the results in Theorem 6 extend straightforward also to QLR tests based on QML estimators. A rigorous proof of the theorem for QML estimators is available from the author on request.*

5.1.1 Numerical Results

We consider the sample size $n=50$, $d=3$, and the true value $\theta_0=(1,0,0.5)'$. The regressors are $Z_t=(1,Z_{1t},Z_{2t})'$, where $(Z_{1t},Z_{2t})'$ are drawn from $N(0,I_2)$, and I_2 denotes the 2×2 identity matrix. Furthermore, we consider the case where the errors $\{\epsilon_t\}$ are sampled independently from t_5-distribution.

For each of the 1000 Monte Carlo samples generated according to these parameter choices, we implement QLR tests of the null hypothesis $\mathcal{H}_0:\theta_{0,2}=0$ versus $\mathcal{H}_1:\theta_{0,2}\neq 0$, where $\theta_{0,2}$ denotes the second component of θ_0. More precisely, we implement QLR tests using standard asymptotic critical values based on the chi-square distribution, and critical values computed using our nonparametric bootstrap approach based on $B=199$ bootstrap replications. Table 1 summarizes the empirical frequencies of rejection of the null hypothesis $\mathcal{H}_0$ for significance levels $\alpha=0.01,0.05,0.10$.

The results in Table 1 point out the reliability of the nonparametric bootstrap method. Indeed, the empirical frequencies of rejection of $\mathcal{H}_0$ implied by our approach are closer to the nominal levels than those implied by classic asymptotic theory. For instance, for $\alpha=0.05$, the frequency of rejection using the nonparametric bootstrap is 0.063. Using standard asymptotic theory it is 0.072.

5.2 GMM Case

We consider the example introduced in Hall and Horowitz (1996), who consider a simplified version of an asset pricing model defined by the moment conditions

$$E[g(X,\theta_0)] = E\left[\begin{pmatrix} 1 \\ X_2 \end{pmatrix} (\exp(\mu - \theta_0(X_1 + X_2) + 3X_2) - 1)\right] = 0, \tag{33}$$

where $X = (X_1, X_2)'$, $\theta_0 = 3$ is the parameter of interest, μ is a known normalization constant, and X_1, X_2 are independent random scalars. In particular, we consider the cases where (i) $(X_1, X_2)' \sim N(0, 0.2^2 \cdot I_2)$, (ii) $(X_1, X_2)' \sim N(0, 0.4^2 \cdot I_2)$, and (iii) $X_1 \sim N(0, 0.2^2)$, X_2 follows a strictly stationary AR(1) process with first-order serial correlation coefficient $r_{X_2} = 0.5$.

For each of the 1000 Monte Carlo samples of size $n = 50$ generated according to these parameter choices, we implement QLR tests of the null hypothesis $\mathcal{H}_0 : \theta_0 = 3$ versus $\mathcal{H}_1 : \theta_0 \neq 3$. As in the previous exercise, we consider standard asymptotic critical values based on the chi-square distribution, and critical values computed using our nonparametric block bootstrap approach based on $B = 199$ bootstrap replications. For the iid cases (i) and (ii) we consider the block size $l = 1$. For the time series case (iii) we consider instead nonoverlapping and overlapping blocks of size $l = 5$. Table 2 and Table 3 summarize the empirical frequencies of rejection of the null hypothesis $\mathcal{H}_0$ for significance levels $\alpha = 0.01, 0.05, 0.10$.

The Monte Carlo results confirm the accuracy of the nonparametric block bootstrap approach. In all cases under investigation, the empirical frequencies of rejection of $\mathcal{H}_0$ implied by our method are very close to the nominal levels. Standard asymptotic theory implies instead values quite far from the nominal levels. For the iid cases (Table 2), for $\alpha = 0.01$ the frequencies of rejection using the nonparametric bootstrap are less than 0.014. Using standard asymptotic theory they are instead larger than 0.045. Also for the time series case (Table 3), both nonoverlapping and overlapping block bootstrap show a desirable accuracy.

5.3 Comparison Nonparametric Block Bootstrap Procedures

In this section, we compare through Monte Carlo simulation the accuracy of the nonparametric block bootstrap procedures for Wald and LM tests introduced in Andrews (2002) and Kim (2003), respectively, and our nonparametric block bootstrap approach for QLR tests. To this end, we consider the following instrumental variable regression model

$$
\begin{aligned}
Y_t &= \theta_1 X_{1t} + \theta_2 X_{2t} + U_t, && (34)\\
X_{2t} &= \pi Z_t + V_t, && (35)
\end{aligned}
$$

where, for $t = 1, \ldots, n$, θ_1 and θ_2 are the unknown parameter of interest, $X_{1t} \sim N(0,1)$, $Z_t \sim N(1,1)$, $\begin{pmatrix} U_t \\ V_t \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0.2 \\ 0.2 & 1 \end{pmatrix} \right)$, and $\pi = 0.5$.

Let $\theta_{0,1}$ and $\theta_{0,2}$ denote the true values of parameter θ_1 and θ_2, respectively. We generate 1000 samples of size $n = 50, 100$, according to model (34)-(35) for the parameter choices $\theta_{0,1} = 1$ and $\theta_{0,2} = 1, 1.25, 1.5, 1.75, 2$. We study the accuracy of the different nonparametric block bootstrap procedures in a test of the null hypothesis $\mathcal{H}_0 : \theta_{0,2} = 1$. More precisely, we consider M-estimators on the unknown parameters based on the moment conditions

$$
\begin{aligned}
E[(Y_t - \theta_{0,1} X_{1t} - \theta_{0,2} X_{2t}) X_{1t}] &= 0, && (36)\\
E[(Y_t - \theta_{0,1} X_{1t} - \theta_{0,2} X_{2t}) Z_t] &= 0, && (37)
\end{aligned}
$$

and apply the nonparametric block bootstrap for Wald test introduced in Section 4, in Andrews (2002); the nonparametric block bootstrap for LM test introduced in Section 2.3, in Kim (2002); and our nonparametric block bootstrap for QLR test introduced in our Section 3. Figure 1 and Figure 2 plot the empirical frequencies of rejection of the null hypothesis $\mathcal{H}_0 : \theta_{0,2} = 1$, for different values $\theta_{0,2} = 1, 1.25, 1.5, 1.75, 2$, for the three different testing procedures under investigation, with sample sizes $n = 50$ and $n = 100$, respectively, based on $B = 199$ bootstrap replications.

In Figure 1 and Figure 2, we observe that when $\theta_{0,2} = 1$, the size values of all the nonparametric bootstrap procedures are quite close to the significance level $\alpha = 0.05$. In particular, when $n = 100$ (Figure 2) the empirical proportion of rejection of the bootstrap methods for Wald, LM and QLR are respectively 0.04, 0.043 and 0.046. When $\theta_{0,2} \neq 1$, as expected the proportion of rejections of the testing procedures under investigation increases. It is interesting to note that the power of our approach in both cases is slightly larger than the power of the bootstrap procedures for Wald and LM test. In particular, the difference in power between our method and Andrews's and Kim's approaches is close to 0.05 and 0.1, respectively, when $n = 100$ and $\theta_{2,0} = 1.5$.

6 A Real Data Example

6.1 Stock Return Predictability

The econometric approach to test for predictability is mostly based on a predictive regression of the form

$$Y_t = \alpha + \beta X_{t-1} + U_t, \tag{38}$$

$$X_t = \rho X_{t-1} + V_t, \tag{39}$$

where for $t = 1, \ldots, n$, Y_t denotes the stock return at time t, X_{t-1} is the explanatory variable observed at time $t-1$ assumed to predict the return Y_t, $\alpha \in \mathbb{R}$, β is the unknown parameter of interest, ρ is the unknown degree of persistence in the variable X_t, and U_t, V_t are iid zero-mean error terms; see, e.g., Stambaugh (1999).

Let β_0 and ρ_0 denote the true values of the unknown parameters β and ρ, respectively. To test the null hypothesis of no predictability $\mathcal{H}_0 : \beta_0 = 0$, the usual approach relies on standard asymptotic theory for the OLS estimator of β. However, standard asymptotic theory may be not appropriate in this setting. Indeed, as pointed out in Torous, Valkanov and Yan (2004) various state variables considered as predictors are well approximated by a nearly integrated process. Consequently, this suggests a local-to-unit framework $\rho = 1 + c/n$, for the autoregressive model (39), which may imply a nonstandard asymptotic distribution for the OLS estimators of β. To overcome this problem recent research has proposed new approaches for testing $\mathcal{H}_0$ in presence of integrated or nearly integrated regressors; see, e.g., Campbell and Yogo (2006), Jansson and Moreira (2006), Camponovo, Scaillet and Trojani (2010), Kostakis, Magdalinos, Stamatogiannis (2011), and Camponovo (2011). In this exercise we use the approach introduced in Camponovo (2011) and develop a QLR test for the null hypothesis $\mathcal{H}_0$.

Camponovo (2011) extends the difference-based approach introduced in Phillips and Han (2010) and Han, Phillips and Sul (2011) for autoregressive models to the predictive regression model (38)-(39). More precisely, let $\Delta X_s^t := X_t - X_s$ and $\Delta Y_s^t := Y_t - Y_s$, where $s < t$. Camponovo (2011) proves the validity of the following moment conditions[5]

$$E[(\Delta Y_s^t - \beta_0 \Delta X_{s-1}^{t-1})(\Delta X_s^t + (2 - \rho_0^{t-s})\Delta X_{s-1}^s] = 0, \quad (40)$$

$$E[\Delta X_s^t \Delta X_{s+1}^{t-1} - \rho_0 (\Delta X_{s+1}^{t-1})^2] = 0, \quad (41)$$

where in these cases $s \leq t-3$. Note that the differencing transformations eliminate the nonstationarity of the explanatory variables. It turns out that this approach allows to apply the usual GMM technology to the moment conditions (40)-(41), including QLR type tests; see e.g., Camponovo (2011) for more details. In particular, in this exercise we consider the moment conditions (40)-(41) with $s = t - j$, $j = 3, \ldots, 12$, and define a QLR test as in Section 2. Finally, to test the null hypothesis $\mathcal{H}_0$ we apply our nonparametric block bootstrap approach defined in Section 3.[6]

6.2 Empirical Predictability Findings

We analyze the predictive power of dividend price ratio (d/p), earning price ratio (e/p) and dividend payout ratio (d/e) for excess returns. We consider monthly S&P 500 index data from

[5]The moment conditions (41) are first proved in Han, Phillips and Sul (2011).

[6]Note that in this exercise the blocks to be bootstrapped are subsamples constructed using the difference-based observations Δx_s^t and Δ_s^t. Since for $\rho_0 < 1$ the true moment conditions are correlated, we use the Newey-West covariance estimator instead of the covariance estimator defined in (2)-(4). The higher order results in Theorem 6 apply only under the further assumption that $\rho_0 = 1$.

Shiller (2000). The dividend price ratio are computed as the difference between the log of dividends and the log of stock prices, where dividends are calculated using a 12-month rolling sum. Similarly, the earning price ratio are computed as the difference between the log of earnings and the log of stock prices, where earnings are calculated using a 12-month rolling sum. Finally, the dividend payout ratio are computed as the difference between the log of dividends and the log of earnings, as previously defined. We compute the excess returns as the returns on the S&P 500 index, including dividends, minus short-term interest rate. In Figure 3 and Figure 4 we plot the excess returns and the earning price ratio of the S&P 500 for the period from January 1948 to December 2007. The nearly integrated features of the earning price ratio are well-known and apparent in Figure 4.

We apply our nonparametric block bootstrap QLR test introduced in Section 3 to the period from January 1948 to December 2007. Table 4 reports the numerical results. The null hypothesis of no predictability is rejected at the 5% significance level only for dividend price ratio and earning price ratio. For the dividend payout ratio the hypothesis of no predictability is instead not rejected. These results are exactly in line with the findings in Campbell and Yogo (2006) and Kostakis, Magdalinos, Stamatogiannis (2011). This suggests that our approach correctly applies to nearly integrated settings. A more detailed analysis based on different explanatory variables and also different subperiods is currently under investigation by the author.

7 Conclusions

Bootstrap methods provide useful approaches for reducing the error in test rejection probability. To achieve this objective, the definition of bootstrap methods for tests requires some care. In the QLR tests context, recent research propose restricted parametric bootstrap procedures that impose the null hypothesis for the construction of the bootstrap samples. Nevertheless, a nonparametric block bootstrap approach is not yet available. In this paper we fill this gap by introducing a nonparametric block bootstrap method for QLR type tests of nonlinear restrictions. The bootstrap method applies to extremum estimators, such as QML and GMM estimators. We study the higher order properties of this method and we show the asymptotic refinements implied with respect to standard asymptotic theory. In particular, when data are iid our method reduces the error in test rejection probability from $O(N^{-1})$ to $O(N^{-2})$, where N is the sample size. In the time series case, the improvement is of at least N^{ξ}, for some $\xi > 0$ which depends on the block length $l \propto N^{\gamma}$. When data are independent $\xi < 1/2$, while when data are dependent and $\kappa > 0$, the improvement cannot exceed $N^{1/4}$. Our method implies the same asymptotic refinements provided by the nonparametric block bootstrap methods introduced for Wald and LM tests in Andrews (2002) and Kim (2003), respectively. Monte Carlo simulations for QLR tests of QML and GMM estimators confirm the better performance of our procedure over the standard asymptotic theory. Finally, a real data application to testing for stock return predictability confirm the accuracy and reliability of our bootstrap procedure.

Appendix: Proofs

The proof of Theorem 6 relies heavily on the proof of Theorem 2.1 in Kim (2003). To prove Theorem 6, we first introduce and prove some auxiliary Lemmas. We also borrow some results established in Andrews (2002) and Kim (2003). In particular, we make us of Lemma 2, 3, 6, 7, 8, 10, in Andrews (2002), and Lemma 2.3, 2.10, 2.11, 2.12, 2.13 in Kim (2003).

In the next Lemmas, a and c denote constants that satisfy $a \geq 0$, $2a$ is an integer, and $c \in [0, 1/2)$. Moreover, C denotes a generic constant that may change from one equality or inequality to another.

Lemma 11 *Suppose assumptions (2)-(4) hold with $q_0 > 4a$, $q_1 > \max(4a/(1-2c), 8a)$, and $q_2 > 4a$. Let $\hat{\theta}_N$ denotes the two-step GMM estimator. Let $\tilde{\theta}_N^*$ be the restricted two-step GMM estimator. Then,*

$$\lim_{N\to\infty} N^a P\left(N^a P^*\left(||\tilde{\theta}_N^* - \hat{\theta}_N|| > N^{-c}\epsilon\right) > \epsilon\right) = 0. \tag{42}$$

Proof. We first prove the Lemma for one-step restricted bootstrap GMM estimators, also denoted by $\tilde{\theta}_N^*$, based on a nonrandom, positive definite, symmetric matrix Ω, under the further assumption that the one-step GMM estimator, also denoted by $\hat{\theta}_N$, minimizes $J_N(\theta) = \big(N^{-1}\sum_{i=1}^N g(X_i,\theta)\big)'\Omega\big(N^{-1}\sum_{i=1}^N g(X_i,\theta)\big)$ over $\theta \in \Theta$.

Consider the case $c = 0$, and nonoverlapping block bootstrap. Let $J_N^*(\theta) = \big(g^*(\theta)\big)'\Omega\big(g^*(\theta)\big)$, and let $J^*(\theta) = E^*\left[g^*(\theta)\right]'\Omega E^*\left[g^*(\theta)\right]$, where $g^*(\theta) = N^{-1}\sum_{i=1}^N (g(X_i^*,\theta) - E^*[g(X_i^*,\hat{\theta}_N)])$. Given $\epsilon > 0$, there exists a $\delta > 0$ independent of N such that $||\theta - \hat{\theta}_N|| > \epsilon$ implies $J^*(\theta) - J^*(\hat{\theta}_N) \geq \delta > 0$, with probability $1 - o(N^{-a})$. This is valid because,

(i) $E^*[N^{-1}\sum_{i=1}^N g(X_i^*,\theta)] = N^{-1}\sum_{i=1}^N g(X_i,\theta)$, with probability $1 - o(N^{-a})$.

(ii) $E^*[g^*(\hat{\theta}_N)] = 0$, with probability $1 - o(N^{-a})$.

(iii) $\lim_{N\to\infty} N^a P(\sup_{\theta\in\Theta} |N^{-1}\sum_{i=1}^N g(X_i,\theta) - N^{-1}\sum_{i=1}^N g(X_i,\hat{\theta}_n) + E[g(X_1,\theta)] - E[g(X_1,\hat{\theta}_N)]| > \lambda) = 0$, for all $\lambda > 0$, by Lemma 2 in Andrews (2002).

(iv) $\lim_{N\to\infty} N^a P(|E[g(X_i, \hat{\theta}_N)] - E[g(X_i, \theta_0)] > \lambda) = 0$, for all $\lambda > 0$, by Lemma 3 in Andrews (2002).

(v) θ_0 is the unique solution to $E[g(X_1, \theta)] = 0$.

Therefore, we have

$$
\begin{aligned}
& N^a P(N^a P^*(||\tilde{\theta}_N^* - \hat{\theta}_N|| > \epsilon) > \epsilon), \\
\leq\ & N^a P(N^a P^*(J^*(\tilde{\theta}_N^*) - J^*(\hat{\theta}_N) > \delta) > \epsilon), \\
\leq\ & N^a P(N^a P^*(J^*(\tilde{\theta}_N^*) - J_N^*(\tilde{\theta}_N^*) + J_N^*(\tilde{\theta}_N^*) - J^*(\hat{\theta}_N) > \delta) > \epsilon), \\
\leq\ & N^a P(N^a P^*(J^*(\tilde{\theta}_N^*) - J_N^*(\tilde{\theta}_N^*) + J_N^*(\hat{\theta}_N) - J^*(\hat{\theta}_N) > \delta) > \epsilon), \\
\leq\ & N^a P(N^a P^*(2 \sup_{\theta\in\Theta} |J_N^*(\theta) - J^*(\theta)| > \delta) > \epsilon) \to 0,
\end{aligned}
$$

where in the last inequality we use Lemma 7, with $j = 1$, in Andrews (2002).

For the case of overlapping blocks, condition (i) is no longer valid. We have instead $E^*[g(X_i^*, \theta)] = (N - l + 1)^{-1} \sum_{i=1}^N \omega(i, l, N) g(X_i, \theta)$, with probability $1 - o(N^{-a})$. Nevertheless, using the arguments of the proof of Lemma 2 in Andrews (2002), condition (iii) is still valid with $N^{-1} \sum_{i=1}^N g(X_i, \theta)$ replaced by $(N - l + 1)^{-1} \sum_{i=1}^N \omega(i, l, N) g(X_i, \theta)$. Consequently, the remainder of the proof remains the same for the overlapping block bootstrap as well.

For the case $c > 0$, note that $\tilde{\theta}_N^*$ is in the interior of Θ and $(\partial/\partial\theta) J_N^*(\tilde{\theta}_N^*) + ((\partial/\partial\theta')\eta(\tilde{\theta}_N^*))'\tilde{\lambda}_N^* = 0$, with probability $1 - o(N^{-a})$, where $\tilde{\lambda}_N^*$ is the corresponding vector of Lagrange multipliers. Using mean value expression line by line of $(\partial/\partial\theta) J_N^*(\tilde{\theta}_N^*)$ and $\eta(\tilde{\theta}_N^*) - \eta(\hat{\theta}_N) = 0$ about $\hat{\theta}_N$ we have

$$
\begin{aligned}
\frac{\partial}{\partial\theta} J_N^*(\tilde{\theta}_N^*) &= \frac{\partial}{\partial\theta} J_N^*(\hat{\theta}_N) + \frac{\partial^2}{\partial\theta\partial\theta'} J_N^*(\theta_N^{+*})(\tilde{\theta}_N^* - \hat{\theta}_N), && (43) \\
0 &= \frac{\partial}{\partial\theta} \eta(\theta_N^{++*})(\tilde{\theta}_N^* - \hat{\theta}_N), && (44)
\end{aligned}
$$

with probability $1 - o(N^{-a})$, where θ_N^{+*} and θ_N^{++*} lie between $\hat{\theta}_N$ and $\tilde{\theta}_N^*$ and may differ across row respectively. Furthermore, we introduce the matrix M_N^* as

$$M_N^* = \left(\frac{\partial}{\partial\theta}\eta(\tilde{\theta}_N^*)\right)' \left(\frac{\partial}{\partial\theta}\eta(\theta_N^{++*})\frac{\partial^2}{\partial\theta\partial\theta'}J_N^*(\theta_N^{+*})^{-1}\frac{\partial}{\partial\theta}\eta(\tilde{\theta}_N^*)'\right)^{-1} \left(\frac{\partial}{\partial\theta}\eta(\theta_N^{++*})\right)\left(\frac{\partial^2}{\partial\theta\partial\theta'}J_N^*(\theta_N^{+*})\right)^{-1}. \tag{45}$$

Using the same arguments of the proof of Lemma 2.3 in Kim (2003) we get

$$\tilde{\theta}_N^* - \hat{\theta}_N = -\left(\frac{\partial^2}{\partial\theta\partial\theta'}J_N^*(\theta_N^{+*})\right)^{-1}(I_{L_\theta} - M_n^*)\left(\frac{\partial}{\partial\theta}J_N^*(\hat{\theta}_N)\right), \tag{46}$$

with probability $1 - o(N^{-a})$. Consequently, the result of the Lemma for $c > 0$ follows from

$$\lim_{N\to\infty} N^a P(N^a P^*(||\frac{\partial^2}{\partial\theta\partial\theta'}J_N^*(\theta_N^{+*}) - D_0\Omega D_0|| > \epsilon) > \epsilon) = 0, \tag{47}$$

$$\lim_{N\to\infty} N^a P(N^a P^*(||D_N^*(\hat{\theta}_N^*) - D_0|| > \epsilon) > \epsilon) = 0, \tag{48}$$

$$\lim_{N\to\infty} N^a P(N^a P^*(||\frac{\partial}{\partial\theta}\eta(\tilde{\theta}_N^*) - \frac{\partial}{\partial\theta}\eta(\theta_0)|| > \epsilon) > \epsilon) = 0, \tag{49}$$

$$\lim_{N\to\infty} N^a P(N^a P^*(||N^{-1}\sum_{i=1}^N g^*(X_i^*, \hat{\theta}_N)|| > N^{-c}\epsilon) > \epsilon) = 0, \tag{50}$$

where to prove (47), (48), and (49) we first use the results for the case with $c = 0$, we apply Taylor expansion about $\hat{\theta}_N$, and use Lemma 7 in Andrews (2002). Equation (50) holds instead by Lemma 6 (c) in Andrews (2002). Finally, the results of Lemma 8 in Andrews (2002) conclude the proof. ■

Our Lemma 11 combined with Lemma 10 in Andrews (2002) implies that

$$\lim_{N\to\infty} N^a P\left(N^a P^*\left(||\tilde{\theta}_N^* - \hat{\theta}_N^*|| > N^{-c}\epsilon\right) > \epsilon\right) = 0. \tag{51}$$

The result in equation (51) allows us to approximate the bootstrap QLR_N^* by a quadratic form and to apply the results of Chandra and Ghosh (1979); see Lemma 12 below.

As in Andrews (2002) and Kim (2003), we introduce some additional notation. In particular, let $f^*(\tilde{X}_i, \theta)$ denotes the vector containing the unique components of $g^*(X_i^*, \theta)$ and $g^*(X_i^*, \theta)g^*(X_{i,i+j}^*)'$, for $j = 0, \ldots, \kappa$, and their derivatives through order d_1 with respect to

θ. Moreover, let $S_N = N^{-1}\sum_{i=1}^N f(X_i, \theta_0)$, $S = E[S_N]$, $S_N^* = N^{-1}\sum_{i=1}^N f^*(\tilde{X}_i^*, \hat{\theta}_N)$ and $S^* = E^*[S^*]$.

We now define the components of the Edgeworth expansion of the test statistic QLR_N and the bootstrap analogue QLR_N^*. Let $\Psi_N = N^{1/2}(S_N - S)$ and $\Psi_N^* = N^{1/2}(S_N^* - S^*)$. Let $\Psi_{N,j}$ and $\Psi_{N,j}^*$ denote the j-th element of Ψ_N and Ψ_N^*, respectively. Let $\nu_{N,a}$ and $\tilde{\nu}_{N,a}$ denote vectors of moment of the form, $N^{\alpha(m)}E[\Pi_{r=1}^m \Psi_{N,j_r}]$ and $N^{\alpha(m)}E^*[\Pi_{r=1}^m \Psi_{N,j_r}^*]$ respectively, where $2 \leq m \leq 2a+2$, $\alpha(m) = 0$, if m is even, and $\alpha(m) = 1/2$, if m is odd. Let $\nu_a = \lim_{N\to\infty} \nu_{N,a}$.

The Edgeworth expansions of QLR_N depend on $\pi_{QLR}(y, \nu_a)$, where $\pi_{QLR}(y, \nu_a)$ denote polynomial functions of y whose coefficients are continuous function of ν_a, for $i = 1, \ldots, [a]$. Here $[a]$ denotes the largest integer smaller than or equal a.

The Edgeworth expansions of QLR_N^* depend on $\pi_{QLR}(y, \nu_{QLR,N,a}^*)$, where the term $\nu_{QLR,N,a}^* = \lambda_{QLR}(\bar{\Xi}_N, \tilde{\nu}_{N,a})$, $\lambda_{QLR}(\cdot,\cdot)$ is a function continuously and differentiable at (I, ν_a) and $\lambda_{QLR}(I, \nu_a) = \nu_a$. The function $\lambda_{QLR}(\cdot,\cdot)$ is determined by the effect of the correction factor $\bar{\Xi}_N$ on the Edgeworth expansion of the bootstrap QLR statistic. Let χ_λ^2 denote a chi-square random variable with λ degrees of freedom.

Lemma 12 *(a) Suppose assumptions (2)-(5) hold with* $q_0 > 2a$, $q_1 > max(2a/(1-2c), 4a)$, *and* $q_2 > 2a+3$, $d_1 \geq (2a+1)/(2c)$, *and* $2a$ *is an integer number. Then, under* $\mathcal{H}_0 : \eta(\theta_0) = 0$,

$$\lim_{N\to\infty} N^a \sup_z |P(QLR_N \leq z) - \int_{-\infty}^z d[1 + \sum_{i=1}^a N^{-i}\pi_{QLRi}(y, \nu_a)]P(\chi_{L_\eta}^2 \leq y)| = 0. \tag{52}$$

(b) Suppose assumptions (2)-(5) hold with $q_0 > 4a$, $q_1 > max(4a/(1-2c), 8a)$, *and* $q_2 > 2a+3$, $q_2 > 6a/(1-2\gamma)$, $d_1 \geq (2a+1)/(2c)$, $d_2 \geq -1 + (a+\gamma)/c$, $2a$ *is an integer number, and* $0 < \gamma < 1/2$ *(where* $\gamma = 0$ *is permitted if* $\{X_i : i \geq 1\}$ *are independent). Then, for all* $\epsilon > 0$,

$$\lim_{N\to\infty} N^a P\left(\sup_z |P^*(QLR_N^* \leq z) - \int_{-\infty}^z d[1 + \sum_{i=1}^a N^{-i}\pi_{QLRi}(y, \nu_{QLR,N,a}^*)]P(\chi_{L_\eta}^2 \leq y)| > \epsilon\right) = 0. \tag{53}$$

Proof. of Lemma 12 The proof of part (a) is analogous to the proof of Lemma 2.13 part (a) in Kim (2003).

For part (b), using the results of Lemma 11, note that QLR^*_N can be approximated by a quadratic form $K^*_N(\tilde{\theta}^*_N, \bar{\Xi}_N)' K^*_N(\tilde{\theta}^*_N, \bar{\Xi}_N)$ by using its second order Taylor expansion around $\hat{\theta}^*_N$ and by further approximating the second order term with the quadratic form of the $L_\eta \times 1$ vector

$$K^*_N(\tilde{\theta}^*_N, \bar{\Xi}_N) = (1/\sqrt{2}) A^*_{1N} A^{*-1}_{2N} ((\partial/\partial\theta')\eta(\tilde{\theta}^*_N))((\partial^2/\partial\theta\partial\theta') J_N(\theta^{+++*}_N, \bar{\theta}^*_N))^{-1} N^{1/2} (\partial/\partial\theta) J_N(\tilde{\theta}^*_N, \bar{\theta}^*_N), \tag{54}$$

where

$$\begin{aligned} A^*_{1N} &= (((\partial/\partial\theta')\eta(\tilde{\theta}^*_N))((\partial^2/\partial\theta\partial\theta') J_N(\theta^{+++*}_N, \bar{\theta}^*_N))^{-1} (\partial^2/\partial\theta\partial\theta') J_N(\theta^{++*}_N, \bar{\theta}^*_N) \times \\ &\quad ((\partial^2/\partial\theta\partial\theta') J_N(\theta^{+++*}_N, \bar{\theta}^*_N))^{-1} ((\partial/\partial\theta')\eta(\tilde{\theta}^*_N))')^{1/2}, \\ A^*_{2N} &= ((\partial/\partial\theta')\eta(\tilde{\theta}^*_N))((\partial^2/\partial\theta\partial\theta') J_N(\theta^{+++*}_N, \bar{\theta}^*_N))^{-1} ((\partial/\partial\theta')\eta(\tilde{\theta}^*_N))'. \end{aligned}$$

The vectors θ^{++*}_N and θ^{+++*}_N lie between $\hat{\theta}^*_N$ and $\tilde{\theta}^*_N$ and are respectively defined by the Taylor expansion of QLR^*_N and the mean value expansion of $(\partial/\partial\theta) J_N(\tilde{\theta}^*_N, \bar{\theta}^*_N)$.

Therefore, using the results of Lemma 2.10 part (b) in Kim (2003), there exists an infinitely differentiable function $G(\cdot, \cdot)$ with $G(S^*, \bar{\Xi}_N) = 0$ such that,

$$\lim_{N\to\infty} N^a P\left(\sup_{z \in \mathbb{R}^{L_\eta}} N^a |P^*(K^*_N(\tilde{\theta}^*_N, \bar{\Xi}_N) \leq z) - P^*(N^{1/2} G^*(S^*_N, \bar{\Xi}_N) \leq z)| > \epsilon \right) = 0. \tag{55}$$

Lemma 2.12 in Kim (2003) ensures that the coefficients $\nu^*_{QLR,N,a}$ are well behaved, while considering Theorem 3.3 of Battacharya (1987), it turns out that $G^*(S^*_N, \bar{\Xi}_N)$ possess multivariate Edgeworth expansions with remainder $o(N^{-a})$. Finally, part (b) follows by applying Theorem 1 and Remark 2.2 of Chandra and Ghosh (1979). ■

Proof. of Theorem 6. We first prove part (a). Lemma 2.11 in Kim (2003), for $a = 3/2$, implies that

$$\lim_{N\to\infty} N^{3/2} P\left(N^{\xi} \sup_z \left| \int_{-\infty}^{z} d[\pi_{QLR1}(y, \nu^*_{LR,N,3/2}) - \pi_{QLR1}(y, \nu_{3/2})] P(\chi^2_{L_\eta} \leq y) \right| > \epsilon \right) = 0. \quad (56)$$

Combining (56) with the results of the previous Lemma we get

$$\lim_{N\to\infty} N^{3/2} P\left(N^{1+\xi} \sup_z |P^*(QLR^*_N \leq z) - P^*(QLR_N \leq z)| > \epsilon \right) = 0. \quad (57)$$

$$\lim_{N\to\infty} N^{3/2} P\left(N^{1+\xi} |1 - \alpha - F_{QLR}(z^*_{QLR,\alpha})| > \epsilon \right) = 0. \quad (58)$$

where $F_{QLR}(\cdot)$ denotes the distribution function of QLR_N. Consequently,

$$\begin{aligned} P(QLR_N \leq z^*_{QLR,\alpha}) &= P(F_{QLR}(QLR_N) > F_{QLR}(z^*_{QLR,\alpha}), N^{1+\xi}|1 - \alpha - F_{QLR}(z^*_{QLR,\alpha})| \leq \epsilon) \\ &+ P(F_{QLR}(QLR_N) > F_{QLR}(z^*_{QLR,\alpha}), N^{1+\xi}|1 - \alpha - F_{QLR}(z^*_{QLR,\alpha})| > \epsilon) \\ &\leq P(F_{QLR}(QLR_N) > 1 - \alpha - \epsilon/N^{1+\epsilon}) + o(N^{-3/2}) \\ &\leq \alpha + \epsilon/N^{1+\epsilon} + o(N^{-3/2}). \end{aligned}$$

Since this inequality holds also reversed, and $-\epsilon/N^{1+\xi}$ and $\epsilon/N^{1+\xi}$ interchanged, part (a) is established.

For part (b) as in the proof of Lemma 12, we first approximate QLR_N and QLR^*_N by the quadratic form $K_N(\tilde{\theta}_N)'K_N(\tilde{\theta}_N)'$ and $K^*_N(\tilde{\theta}^*_N)'K^*_N(\tilde{\theta}^*_N)'$, which in this case does not depend on the correction factor. From Lemma 2.10 in Kim (2003), $K_N(\tilde{\theta}_N)$ and $K^*_N(\tilde{\theta}^*_N)$ can be replaced by $N^{-1/2}G(S_N)$ and $N^{-1/2}G(S^*_N)$, respectively, where $G(\cdot)$ denotes an infinite differentiable function.

Let $\mathcal{S}$ and $\mathcal{S}^*$ denote the L_η-dimensional sphere centered at the origin with radius $(z_{QLR,\alpha})^{1/2}$ and $(z^*_{QLR,\alpha})^{1/2}$, respectively. Using Theorem 3.1 and Theorem 3.3 of Bhattacharya (1987) we get

$$
\begin{aligned}
P(N^{1/2}G(S_N)\in\mathcal{S}) &= \int_{\mathcal{S}}\{1+N^{-1}s_2(x)\}\phi(x)dx+O(N^{-2}), &(59)\\
P^*(N^{1/2}G(S_N^*)\in\mathcal{S}) &= \int_{\mathcal{S}}\{1+N^{-1}\hat{s}_2(x)\}\phi(x)dx+O_p(N^{-2}), &(60)
\end{aligned}
$$

where $s_2(x)$ is an even polynomial of degree 6, $\hat{s}_2(x)$ is the same as s_2 with population moments replaced by sample moments, and $\phi(x)$ is the L_η-dimensional standard normal density. Furthermore, consider the Cornish-Fisher expansions

$$
\begin{aligned}
(z_{QLR,\alpha})^{1/2} &= c_0+N^{-1}c_1+O(N^{-2}), &(61)\\
(z^*_{QLR,\alpha})^{1/2} &= c_0+N^{-1}\hat{c}_1+O_p(N^{-2}), &(62)
\end{aligned}
$$

where $\epsilon<\alpha<1-\epsilon$, with $\epsilon\in(0,1/2)$, c_0 is the radius of sphere S_0 centered at the origin such that $P(Z\in S_0)=1-\alpha$, for $Z\sim N(0,I_{L_\eta})$. The term c_1 is constant and depends on the population moments, and $\hat{c}_1$ is the same as c_1 but based on the sample moments.

Then, we get

$$
\begin{aligned}
P(QLR_N<z^*_{QLR\alpha}) &= P(N^{1/2}G(S_N)\in S^*)+O(N^{-2}), &(63)\\
&= P(N^{1/2}G(S_N)\in(z^*_{QLR,\alpha}/z_{QLR,\alpha})^{-1/2}S)+O(N^{-2}), &(64)\\
&= P(N^{1/2}G(S_N)\in\{1+(Nc_0)^{-1}(\hat{c}_1-c_1)+O_p(N^{-2})\}S)+O(N^{-2}), &(65)\\
&= P(\{1-(Nc_0)^{-1}(\hat{c}_1-c_1)+O_p(N^{-2})\}N^{1/2}G(S_N)\in S)+O(N^{-2}), &(66)\\
&= \int_S\{1+N^{-1}s_2(x)\}\phi(x)dx+O(N^{-2}), &(67)\\
&= 1-\alpha+O(N^{-2}), &(68)
\end{aligned}
$$

where (65) and (66) hold by Taylor expansion of $(z_{QLR,\alpha})^{1/2}$ and delta-method, respectively, while

(67) hold by the Edgeworth expansion

$$
\begin{aligned}
& P(\{1-(Nc_0)^{-1}(\hat{c}_1-c_1)\}N^{1/2}G(S_N)\in S), && (69)\\
= & \int_{\mathcal{S}}\{1+N^{-1/2}t_1(x)+N^{-1}t_2(x)+N^{-3/2}t_3(x)\}\phi(x)dx+O(N^{-2}), && (70)\\
= & \int_{\mathcal{S}}\{1+N^{-1}t_2(x)\}\phi(x)dx+O(N^{-2}), && (71)
\end{aligned}
$$

since the integral of $t_j(x)\phi(x)$ over $\mathcal{S}$ is zero for j odd, and $t_j = s_j$ for $j = 1, 2$, because $N^{1/2}G(S_N)$ and $\{1-(Nc_0)^{-1}(\hat{c}_1-c_1)\}N^{1/2}G(S_N)$ differ only in terms of order $O(N^{-3/2})$. ■

References

[1] Andrews, D.W.K., 2002. Higher-order improvements of a computationally attractive k-step bootstrap for extremum estimators. Econometrica, 70, 119–162.

[2] Beran, R., 1988. Prepivoting test statistic: A bootstrap view of asymptotic refinements. Journal of the American Statistical Association, 83, 687–697.

[3] Bhattacharya, R.N., 1987. Some aspect of Edgeworth expansions in statistics and probability, in New Perspectives in Theoretical and Applied Statistics, ed. by M.L. Puri, J.P. Vilaploma, and W. Wertz. Wiley, New York.

[4] Campbell, J.Y., and M. Yogo 2006. Efficient tests of stock return predictability. Journal of Financial Economics, 81, 27–60.

[5] Camponovo, L., 2011. Predictive regression with nearly integrated explanatory variables. Working paper.

[6] Camponovo, L., O. Scaillet, and F. Trojani, 2010. Robust predictive regression. Working paper.

[7] Carlstein, E., 1986. The use of subseries methods for estimating the variance of a general statistic from a stationary time series. Annals of Statistics, 14, 1171–1179.

[8] Chandra, T.K., and J.K. Ghosh, 1979. Valid asymptotic expansions for the likelihood ratio statistic and other perturbed chi-square variables. Sankhya, 41, Series A, 22-47.

[9] Chen, X, and D. Pouzo, 2009. On inference of penalized sieve minimum distance estimators estimators of nonparametric conditional moment models. Working paper.

[10] Davidson, R., and J.G. MacKinnon, 1999. The size distortion of bootstrap tests. Econometric Theory, 15, 361–376.

[11] Efron, B., 1979. Bootstrap methods: Another look at the jackknife. Annals of Statistics, 7, 1–26.

[12] Götze, F., and C. Hipp, 1983. Asymptotic expansions for sums of weakly dependent random vectors. Z. Wahrscheinlichkeitstheorie verw. Gebiete, 64, 211–239.

[13] Götze, F., and C. Hipp, 1994. Asymptotic distribution of statistic in time series. Annals of Statistics, 22, 2068–2088.

[14] Götze, F., and H. R. Künsch, 1996. Second-order correctness of the blockwise bootstrap for stationary observations. Annals of Statistics, 24, 19141933.

[15] Hall, P., 1985. Resampling a coverage process. Stochastic Processes and their Applications, 19, 259–269.

[16] Hall, P., 1992. The bootstrap and Edgeworth expansion. New York: Springer-Verlag.

[17] Hall, P., and J. Horowitz, 1996. Bootstrap critical values for tests based on Generalized-Method-of-Moment estimators. Econometrica, 64, 891–916.

[18] Han, C., Phillips, P.C.B., and D. Sul, 2011. Uniform asymptotic normality in stationary and unit root autoregression. Econometric Theory, forthcoming.

[19] Jansson, M., and M.J. Moreira, 2006. Optimal inference in regression models with nearly integrated regessors. Econometrica, 74, 681–714.

[20] Kim, J. H., 2003. Econometric analysis of bootstrap performances. Ph.D. Thesis dissertation, Yale University.

[21] Kostakis, A., A. Magdalinos, and M. Stamatogiannis, 2010. Robust econometric inference for stock return predictability. Working paper.

[22] Künsch, H., 1989. The jacknife and the bootstrap for general stationary observations. Annals of Statistics, 17, 1217–1241.

[23] Lahiri, S.N., 1996. On Edgeworth expansion and moving block bootstrap for studentized M-estimators in multiple linear regression models. Journal of Multivariate Analysis, 56, 4259.

[24] Phillips, P.C.B., and C. Han, 2008. Gaussian inference in AR(1) time series with or without a unit root. Econometric Theory, 24, 631-650.

[25] Shiller, R.J., 2000. Irrational Exuberance. Princeton University Press, Princeton, New York.

[26] Singh, K., 1981. On the asymptotic accuracy of Efron's bootstrap. Annals of Statistics, 9, 1187–1195.

[27] Stambaugh, R.F. 1999. Predictive regressions. Jornal of Financial Economics, 54, 375-421.

[28] Torous, W., R. Valkanov, and S. Yan, 2004. On predicting stock returns with nearly integrated explanatory variables. Journal of Business, 77, 937-966.

α	Asymptotic critical values	Bootstrap critical values
0.10	0.128	0.114
0.05	0.072	0.063
0.01	0.015	0.011

Table 1: **Empirical Frequencies of Rejection of $\mathcal{H}_0$.** We compute the empirical frequencies of rejection of QLR tests of $\mathcal{H}_0 : \theta_{2,0} = 0$ versus $\mathcal{H}_1 : \theta_{2,0} \neq 0$, in the linear regression model (26). The significance level of the tests is $\alpha = 0.01, 0.05, 0.10$. We implement the QLR tests using standard asymptotic critical values based on the chi-square distribution (second column), and critical values computed using our nonparametric bootstrap approach (third column). The sample size is $n = 50$.

α	Asymptotic critical values	Bootstrap critical values
0.10	0.154	0.102
0.05	0.109	0.059
0.01	0.056	0.014

α	Asymptotic critical values	Bootstrap critical values
0.10	0.136	0.098
0.05	0.085	0.049
0.01	0.045	0.012

Table 2: **Empirical Frequencies of Rejection of $\mathcal{H}_0$.** We compute the empirical frequencies of rejection of QLR tests of $\mathcal{H}_0 : \theta_0 = 3$ versus $\mathcal{H}_1 : \theta_0 \neq 3$, in the GMM setting introduced in Section 5.2. The significance level of the tests is $\alpha = 0.01, 0.05, 0.10$. We implement the QLR tests using standard asymptotic critical values based on the chi-square distribution (second column), and critical values computed using our nonparametric bootstrap approach (third column). The sample size is $n = 50$. In the top table, $(X_1, X_2)' \sim N(0, 0.2^2 \cdot I_2)$. In the bottom table we have instead $(X_1, X_2)' \sim N(0, 0.4^2 \cdot I_2)$.

α	Asymptotic critical values	Bootstrap critical values
0.10	0.164	0.119
0.05	0.109	0.064
0.01	0.038	0.014

α	Asymptotic critical values	Bootstrap critical values
0.10	0.164	0.119
0.05	0.109	0.068
0.01	0.038	0.016

Table 3: **Empirical Frequencies of Rejection of $\mathcal{H}_0$.** We compute the empirical frequencies of rejection of QLR tests of $\mathcal{H}_0 : \theta_0 = 3$ versus $\mathcal{H}_1 : \theta_0 \neq 3$, in the GMM setting introduced in Section 5.2. The significance level of the tests is $\alpha = 0.01, 0.05, 0.10$. We implement the QLR tests using standard asymptotic critical values based on the chi-square distribution (second column), and critical values computed using our nonparametric block bootstrap approach (third column). The sample size is $n = 50$. For both tables, $X_1 \sim N(0, 0.4^2)$, while X_2 follows an AR(1) process with first-order serial correlation coefficient $r_{X_2} = 0.5$. In the top table, the third column considers the nonoverlapping block bootstrap, while in the bottom table we consider the overlapping block bootstrap.

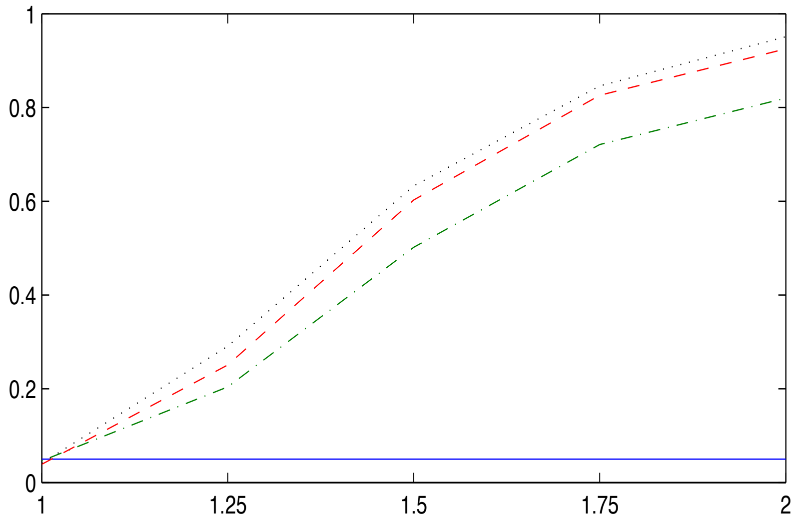

Figure 1: **Power curves in the instrumental variable regression model,** $n = 50$**.** We plot the proportion of rejections of the null hypothesis $\mathcal{H}_0 : \theta_{0,2} = 1$, when the true parameter value is $\theta_{0,2} \in \{1, 1.25, 1.5, 1.75, 2\}$. We consider our nonparametric bootstrap for QLR (black dotted line), the bootstrap approach for Wald test (red dashed line), and the bootstrap method for LM test (green dash-dotted line).

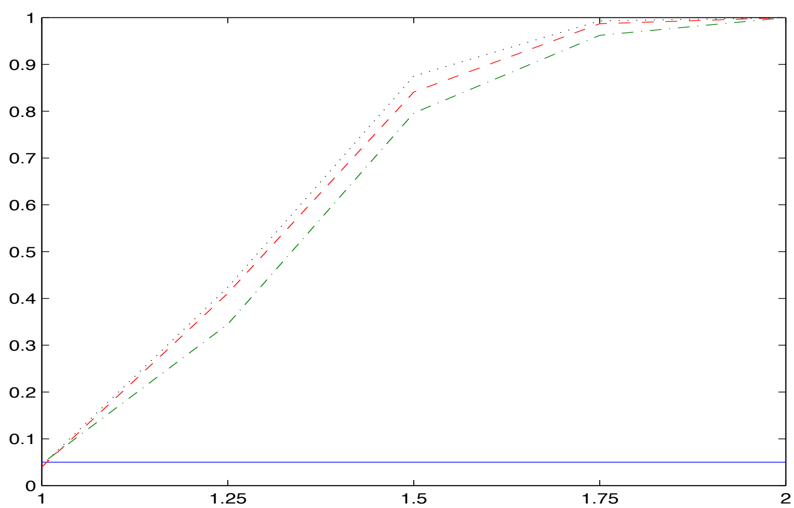

Figure 2: **Power curves in the instrumental variable regression model,** $n = 100$**.** We plot the proportion of rejections of the null hypothesis $\mathcal{H}_0 : \theta_{0,2} = 1$, when the true parameter value is $\theta_{0,2} \in \{1, 1.25, 1.5, 1.75, 2\}$. We consider our nonparametric bootstrap for QLR (black dotted line), the bootstrap approach for Wald test (red dashed line), and the bootstrap method for LM test (green dash-dotted line).

	d/p	e/p	d/e
QLR_N	5.6906^{**}	6.0483^{**}	1.0620

Table 4: **QLR Test.** We compute the QLR statistic defined in Section 6.1 for dividend price ratio (d/p), earning price ratio (e/p) and dividend payout ratio (d/e).
($*$) means rejection at 10% significance level using our bootstrap approach.
($**$) means rejection at 5% significance level using our bootstrap approach.
($***$) means rejection at 1% significance level using our bootstrap approach.

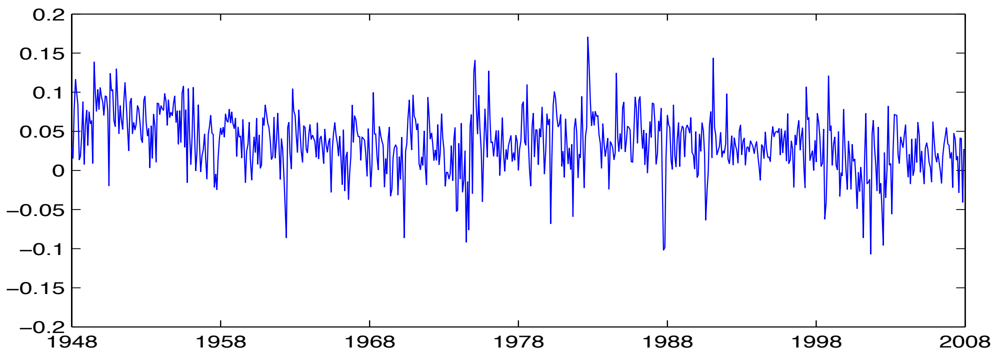

Figure 3: **Excess Return.** We plot the excess return of the S&P 500 for the period 1948-2008. We consider monthly S&P 500 index data from Shiller (2000).

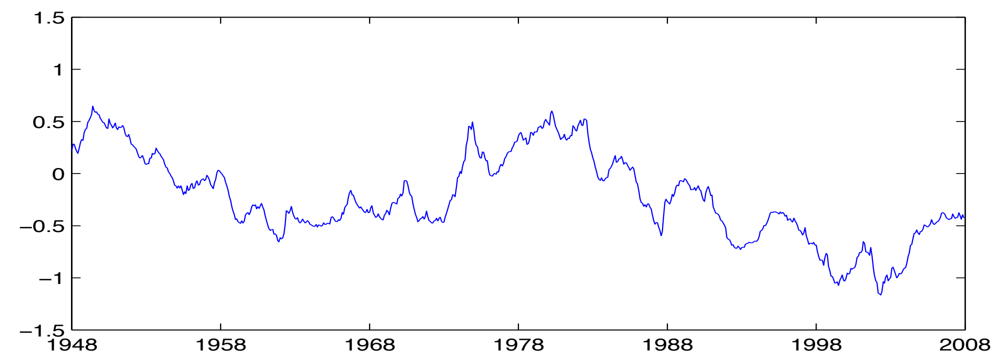

Figure 4: **Earning Price Ratio.** We plot the earning price ratio of the S&P 500 for the period 1948-2008. We consider monthly S&P 500 index data from Shiller (2000).