Independence and Conditional Independence in Causal Systems

Karim Chalak[1]
Department of Economics
Boston College

Halbert White
Department of Economics
University of California, San Diego

May 5, 2007

Abstract: This paper studies the interrelations between independence or conditional independence and causal relations, defined in terms of functional dependence, that hold among variables of interest within the settable system framework of White and Chalak. We provide formal conditions ensuring the validity of Reichenbach's principle of common cause and introduce a new conditional counterpart, the conditional Reichenbach principle of common cause. We then provide necessary and sufficient causal conditions for probabilistic dependence and conditional dependence among certain random vectors in settable systems. Immediate corollaries of these results constitute causal conditions sufficient to ensure independence or conditional independence among random vectors in settable systems. We demonstrate how our results relate to and generalize results in the artificial intelligence literature, particularly results concerning *d*-separation.

PRELIMINARY AND INCOMPLETE

Acknowledgment: The authors thank Julian Betts, Graham Elliott, Clive Granger, Mark Machina, Dimitris Politis, and especially Ruth Williams for their helpful comments and suggestions. All errors and omissions are the authors' responsibility.

[1] Karim Chalak, Department of Economics, Boston College, 140 Commonwealth Avenue, Chestnut Hill, MA 02467 (email: chalak@bc.edu).
Halbert White, Department of Economics 0508, University of California, San Diego, 9500 Gilman Drive, La Jolla, CA 92093-0508 (email: hwhite@ucsd.edu).

1. Introduction

The concepts of independence and conditional independence are central to many disciplines, including probability theory, statistics, econometrics, and artificial intelligence (see e.g Granger, 1969; Dawid, 1979, 1980; Florens, Mouchart, and Rolin 1990; Studeny 1993; Chalak and White, 2007a). In particular, they are concepts central to the study of causal inference (see, e.g., Rubin 1974; Holland, 1986; Spirtes, Glymour, and Scheines, 1993 (SGS); Pearl 1988, 1993, 1995, 2000; Dawid, 2000, 2002; Rosenbaum, 2002). In this paper, we study the interrelations between independence or conditional independence and causal relations, defined in terms of functional dependence, that may hold among variables of interest within the settable system framework of White and Chalak (2006) (WC), a formal mathematical framework for defining, identifying, modeling, and estimating causal effects.

The interrelations between conditional independence and causal relations have been extensively studied in the artificial intelligence literature. This literature introduced graphical algorithms applicable to directed acyclic graphs (DAG) to characterize independence and conditional independence among variables in "Bayesian Networks" or more specifically "directed Markov fields." In particular, our results relate to notions of "*d*-separation" (Verma and Pearl, 1988; Geiger, Verma, and Pearl, 1990; Geiger and Pearl, 1993; Pearl, 2000), "*D*-separation" (Geiger, Verma, and Pearl, 1990), and alternative graphical criteria (Lauritzen and Spiegelhalter, 1988; Lauritzen, Dawid, Larsen, and Leimer, 1990).

The topic of this paper is also central to certain strands of the philosophy literature (Spohn, 1980; Hausman and Woodward, 1999; Cartwright, 2000). In particular, Reichenbach (1956) introduced his "principle of common cause," which states that if two variables are associated (e.g., correlated) then either one causes the other or they both share a third common cause. Although this principle has intuitive appeal and despite its venerated status, its formal standing is nevertheless ambiguous. Is it an axiom or a postulate, or is it a logical consequence of assumptions as yet unformulated? Here we provide a formal proof of the Reichenbach principle of common cause. The framework supporting this proof is that of WC's canonical recursive partitioned settable systems. We also introduce a new conditional counterpart to this principle that we term the *conditional Reichenbach principle of common cause*. We then build on these results to provide necessary and sufficient causal conditions for dependence and conditional dependence among certain random vectors in settable systems. Immediate corollaries of these results constitute causal conditions sufficient to ensure independence or conditional independence among random vectors in settable systems. This has direct connections to notions of *d*-separation and *D*-separation introduced in the artificial intelligence literature.

Thus, our results contribute to answering two questions of interest for the study of empirical relationships. First, what restrictions (if any) on the possible functionally defined causal relationships holding between variables of interest follow from knowledge of the probability distribution governing these variables? Conversely, what implications for their probability distribution derive from knowledge of functionally defined causal relationships between variables of interest?

This paper is organized as follows. In Section 2, we introduce a version of WC's settable system framework. Using this, we provide rigorous definitions of *direct* and *indirect causality* based on functional dependence, relevant to the study of direct and indirect causality in experimental and observational studies. Although graphical representation of our definitions and related concepts is helpful to heuristic understanding, our analysis does not rely on properties of graphs. Instead, the analysis is entirely standard, driven by functional relationships holding among the various components of a given settable system. We refine previous definitions of indirect causality to accommodate notions of causality *via* a set of variables and *exclusive of* a set of variables in recursive systems, extending notions introduced by Pearl (2001). Section 3 provides a proof of the Reichenbach principle of common cause for recursive systems. We then provide necessary and sufficient conditions for probabilistic dependence among random variables in settable systems. In Section 4, we build on the results in Section 3 to introduce and prove the conditional Reichenbach principle of common cause and provide necessary and sufficient conditions for probabilistic conditional dependence of certain vectors of random variables in settable systems. In Section 5, we discuss how our results relate to and generalize results of the artificial intelligence literature related to *d*-separation and *D*-separation. Section 6 concludes and discusses directions for future research. Formal mathematical proofs are collected in the Mathematical Appendix.

2. Direct and Indirect Causality in Settable Systems

We work with the following version of WC's definition of *settable systems*. We let $\mathbb{R}^\infty \equiv \mathbb{R} \times \mathbb{R} \times \ldots$.

Definition 2.1: Settable System Let $(\Omega, \mathcal{F})$ be a measurable space. For $h = 1, 2, \ldots$ and $j = 1, 2, \ldots$, let *settings* $Z_{h,j} : \Omega \to \mathbb{R}$ be measurable functions. For $h = 0$ let setting $Z_0 : \Omega \to \mathbb{R}$ be a measurable surjective mapping. For $h = 1, 2, \ldots$, and $j = 1, 2, \ldots$, let *response functions* $r_{h,j} : \mathbb{R}^\infty \to \mathbb{R}$ be measurable functions.

For $h = 1, 2, \ldots$, and $j = 1, 2, \ldots$, let $Z_{(h,j)}$ be the vector including every setting except $Z_{h,j}$, and define the *settable variables* $\mathcal{X}_{h,j} : \{0,1\} \times \Omega \to \mathbb{R}$

$$\mathcal{X}_{h,j}(1, \cdot) = Z_{h,j}$$

$$\mathcal{X}_{h,j}(0, \cdot) = r_{h,j}(Z_{(h,j)}).$$

For $h = 0$ let $\mathcal{X}_h(0, \cdot) = \mathcal{X}_h(1, \cdot) = Z_h$.

Put $Z_h \equiv \{ Z_{h,j} : j = 1, 2, \ldots \}$, $Z \equiv \{ Z_0, Z_1, \ldots\}$, $r_h \equiv \{ r_{h,j} ,\ j = 1, 2, \ldots \}$, $r \equiv \{ r_h \}$, $\mathcal{X}_h \equiv \{ \mathcal{X}_{h,j} : j = 1, 2, \ldots \}$, and $\mathcal{X} \equiv \{ \mathcal{X}_0, \mathcal{X}_1, \ldots\}$.

The pair $\mathcal{S} \equiv \{(\Omega, \mathcal{F}), (Z, r, \mathcal{X})\}$ is a *settable system*. ■

In Definition 2.1, we have a number of *agents* indexed by h responding to other agents in the system. Each agent governs responses indexed by j. We denote the jth response of agent h by $Y_{h,j} = \mathcal{X}_{h,j}(0, \cdot)$. Thus, agent h's jth response depends on the settings of all other variables in the system, including not only the settings corresponding to other agents but also those corresponding to agent h other than j. We refer to $\mathcal{X}_{h,j}$ as a "settable" or "causal" variable. A settable variable has a dual aspect: $\mathcal{X}_{h,j}(1, \cdot)$ is a setting, whereas $\mathcal{X}_{h,j}(0, \cdot)$ is a response. As we discuss shortly, we define causality in terms of settable variables rather than random variables or events.

The setting Z_0 plays a special role as a *fundamental variable*, that is, a variable that does not respond to any other variable in the system, apart from its dependence on $\omega \in \Omega$. This motivates the convention that $\mathcal{X}_0(0, \cdot) = \mathcal{X}_0(1, \cdot) = Z_0$. Indeed, we can identify $Z_0(\omega)$ with ω, as there is no loss of generality in choosing $\Omega = \mathbb{R}$ and letting Z_0 be the (measurable and surjective) identity map, $Z_0(\omega) = \omega$. WC's definition explicitly accommodates attributes; for conciseness and without essential loss of generality, we suppress these here. Thus a settable system $\mathcal{S} \equiv \{(\Omega, \mathcal{F}), (Z, r, \mathcal{X})\}$ is composed of a stochastic component: the measurable space $(\Omega, \mathcal{F})$ and a causal structure: the settings, responses, and settable variables $(Z, r, \mathcal{X})$. WC and White and Chalak (2007) discuss in detail the relationship of the settable system framework to Rubin's treatment effects framework as formalized in Holland (1986) and to Pearl's causal model (Pearl, 2000).

It is useful to partition the system under study to group certain variables into specific blocks. WC and White and Chalak (2007) discuss a number of examples where it is appropriate to treat the system under study at the level of a certain partition but not another.

Definition 2.2: Partitioned Settable System Let $(\Omega, \mathcal{F})$ and Z be as in Definition 2.1. Suppose that settings of $\mathcal{X}$ are given by $\mathcal{X}_{h,j}(1, \cdot) = Z_{h,j}$, $h = 1, 2, \ldots;\ j = 1, 2, \ldots$, and by $\mathcal{X}_h(1, \cdot) = Z_h$ for $h = 0$. Let $\Pi = \{\Pi_b\}$ be a partition of the ordered pairs $\{(h, j): h = 1, 2, \ldots ; j = 1, 2, \ldots\}$. Suppose there exists a countable sequence of measurable functions $r^{\Pi} \equiv \{r^{\Pi}_{h,j} : \mathbb{R}^{\infty} \to \mathbb{R}\}$ such that for all (h, j) in Π_b the responses $Y_{h,j} = \mathcal{X}_{h,j}(0, \cdot)$ are jointly determined as

$$Y_{h,j} = r^{\Pi}_{h,j}(Z_{(b)}), \quad b = 1, 2, \ldots,$$

where $Z_{(b)}$ is the countable vector containing Z_0 and $Z_{i,k}$, $(i, k) \notin \Pi_b$. Then we say that r^{Π} is *adapted to* Π. The pair $\mathcal{S} \equiv \{(\Omega, \mathcal{F}), (Z, \Pi, r^{\Pi}, \mathcal{X})\}$ is a *partitioned settable system*. ■

Settable systems provide a suitable framework for the study of causality. We next give a definition of direct causality within this framework, based on functional dependence. For this, we let $z_{(b)(h,j)}$ be the vector containing all elements of $z_{(b)}$ except $z_{h,j}$.

Definition 2.3: Direct Causality Let $\mathcal{S} \equiv \{(\Omega, \mathcal{F}), (Z, \Pi, r^{\Pi}, \mathcal{X})\}$ be a partitioned settable system. For given positive integer b, let $(i, k) \in \Pi_b$. (i) If for given $(h, j) \notin \Pi_b$ the function $z_{h,j} \to r^{\Pi}_{i,k}(z_{(b)})$ is constant in $z_{h,j}$ for every $z_{(b)(h,j)}$, then we say $\mathcal{X}_{h,j}$ *does not directly cause* $\mathcal{X}_{i,k}$ *in* $\mathcal{S}$ and write $\mathcal{X}_{h,j} \overset{d}{\not\Rightarrow}_{\mathcal{S}} \mathcal{X}_{i,k}$. Otherwise, we say $\mathcal{X}_{h,j}$ *directly causes* $\mathcal{X}_{i,k}$ *in* $\mathcal{S}$ and write $\mathcal{X}_{h,j} \overset{d}{\Rightarrow}_{\mathcal{S}} \mathcal{X}_{i,k}$. (ii) For $(h, j), (i, k) \in \Pi_b$, $\mathcal{X}_{h,j} \overset{d}{\not\Rightarrow}_{\mathcal{S}} \mathcal{X}_{i,k}$. ■

In Definition 2.3, a settable variable $\mathcal{X}_{h,j}$ is said to directly cause $\mathcal{X}_{i,k}$ relative to the system $\mathcal{S}$ if there exist some settings of all other variables in the system such that $\mathcal{X}_{i,k}$ systematically responds to settings in $\mathcal{X}_{h,j}$. Note that, by definition, variables within the same block do not cause each other. In particular $\mathcal{X}_{h,j} \overset{d}{\not\Rightarrow}_{\mathcal{S}} \mathcal{X}_{h,j}$. Also, Definition 2.3 permits mutual causality, so that $\mathcal{X}_{h,j} \overset{d}{\Rightarrow}_{\mathcal{S}} \mathcal{X}_{i,k}$ and $\mathcal{X}_{i,k} \overset{d}{\Rightarrow}_{\mathcal{S}} \mathcal{X}_{h,j}$ without contradiction. Mutual causality is ruled out in SGS (p. 42), for example, where it is an axiom that if A causes B then B does not cause A. What we define here as "direct causality" is the notion introduced in WC as simply "causality." Here we require more refined definitions, distinguishing in particular between direct and indirect causality.

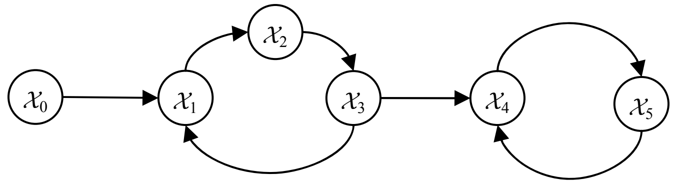
We now introduce notions of paths, successors, predecessors, and intercessors, adapting graph theoretic concepts discussed, for example, by Bang-Jensen and Gutin (2001) (BG).

Definition 2.4: Paths, Successors, Predecessors, and Intercessors Let $\mathcal{S} \equiv \{(\Omega, \mathcal{F}), (Z, \Pi, r^{\Pi}, \mathcal{X})\}$ be a partitioned settable system. For given positive integer b let $(i, k) \in \Pi_b$ and $(h, j) \notin \Pi_b$. We call the collection of settable variables $\{\mathcal{X}_{h,j}, \mathcal{X}_{h_1,j_1}, \ldots, \mathcal{X}_{h_n,j_n}, \mathcal{X}_{i,k}\}$ an $(\mathcal{X}_{h,j}, \mathcal{X}_{i,k})$-*walk* of *length* n+1 if $\mathcal{X}_{h,j} \overset{d}{\Rightarrow}_{\mathcal{S}} \mathcal{X}_{h_1,j_1} \overset{d}{\Rightarrow}_{\mathcal{S}} \ldots \overset{d}{\Rightarrow}_{\mathcal{S}} \mathcal{X}_{h_n,j_n} \overset{d}{\Rightarrow}_{\mathcal{S}} \mathcal{X}_{i,k}$. When the elements of an $(\mathcal{X}_{h,j}, \mathcal{X}_{i,k})$-*walk* are distinct, we call it an $(\mathcal{X}_{h,j}, \mathcal{X}_{i,k})$-*path*. We say $\mathcal{X}_{h,j}$ $d(\mathcal{S})$-*precedes* $\mathcal{X}_{i,k}$ or $\mathcal{X}_{i,k}$ $d(\mathcal{S})$-*succeeds* $\mathcal{X}_{h,j}$ if there exists at least one $(\mathcal{X}_{h,j}, \mathcal{X}_{i,k})$-path of positive length. If $\mathcal{X}_{h,j}$ $d(\mathcal{S})$-precedes $\mathcal{X}_{i,k}$, we call $\mathcal{X}_{h,j}$ a $d(\mathcal{S})$-*predecessor* of $\mathcal{X}_{i,k}$, and we call $\mathcal{X}_{i,k}$ a $d(\mathcal{S})$-*successor* of $\mathcal{X}_{h,j}$. If $\mathcal{X}_{h,j}$ $d(\mathcal{S})$-precedes $\mathcal{X}_{i,k}$ and $\mathcal{X}_{h,j}$ $d(\mathcal{S})$-succeeds $\mathcal{X}_{i,k}$, we say $\mathcal{X}_{h,j}$ and $\mathcal{X}_{i,k}$ belong to an $\mathcal{S}$-*cycle*. If $\mathcal{X}_{h,j}$ and $\mathcal{X}_{i,k}$ do not belong to an $\mathcal{S}$-cycle, $\mathcal{X}_{g,l}$

$d(\mathcal{S})$-succeeds $\mathcal{X}_{h,j}$, and $\mathcal{X}_{g,l}$ $d(\mathcal{S})$-precedes $\mathcal{X}_{i,k}$, we say $\mathcal{X}_{g,l}$ $d(\mathcal{S})$-*intercedes* $\mathcal{X}_{h,j}$ and $\mathcal{X}_{i,k}$. If $\mathcal{X}_{g,l}$ $d(\mathcal{S})$-intercedes $\mathcal{X}_{h,j}$ and $\mathcal{X}_{i,k}$, we call $\mathcal{X}_{g,l}$ an $(\mathcal{X}_{h,j}, \mathcal{X}_{i,k})$ $d(\mathcal{S})$-*intercessor*. We denote by $\mathcal{I}^{d(\mathcal{S})}_{(h,j):(i,k)}$ the set of $(\mathcal{X}_{h,j}, \mathcal{X}_{i,k})$ $d(\mathcal{S})$-intercessors. ■

Directed graphs can be used to illustrate a variety of aspects of settable systems. In particular, they are helpful in visualizing the notions just introduced. For this, let $G = (V, E)$ be a directed graph with a non-empty finite set of vertices $V = \{\mathcal{X}_{h,j} : j = 1, \ldots, J_h;\ h = 0, 1, \ldots, H\}$ and a set of arcs $E \subset V \times V$ of ordered pairs of distinct vertices such that an arc $(\mathcal{X}_{h,j}, \mathcal{X}_{i,k})$ belongs to E if $\mathcal{X}_{h,j} \overset{d}{\Rightarrow}_{\mathcal{S}} \mathcal{X}_{i,k}$. From Definition 2.3, there exists at most one $(\mathcal{X}_{h,j}, \mathcal{X}_{i,k})$ arc, and G thus does not contain "parallel arcs." Since $\mathcal{X}_{h,j} \overset{d}{\not\Rightarrow}_{\mathcal{S}} \mathcal{X}_{h,j}$, there can be no arc $(\mathcal{X}_{h,j}, \mathcal{X}_{h,j})$ in E, so G does not contain self directed arcs or "loops." (Loops and parallel arcs can nevertheless be useful in other contexts; see for example Golubitsky and Stewart, 2006.) With loops and parallel arcs ruled out, a graph G associated with a settable system $\mathcal{S}$ can be neither a "directed pseudograph" nor a "directed multigraph" (see BG, p.4).

Graph G_1 illustrates the concepts of Definition 2.4. (Whenever convenient, we may work with singly indexed settable variables or nodes. These can be viewed either as associated with a single agent or with multiple agents, each of whom govern a single response.) We have that $\{\mathcal{X}_0, \mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{X}_4\}$ and $\{\mathcal{X}_0, \mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \mathcal{X}_4\}$ are an $(\mathcal{X}_0, \mathcal{X}_4)$-walk of length 7 and an $(\mathcal{X}_0, \mathcal{X}_4)$-path of length 4, respectively. We also have that $\mathcal{X}_0$ $d(\mathcal{S})$-precedes $\mathcal{X}_4$, $\mathcal{X}_3$ $d(\mathcal{S})$-succeeds $\mathcal{X}_1$, and that $\mathcal{X}_1$ and $\mathcal{X}_3$ belong to an $\mathcal{S}$-cycle, as do $\mathcal{X}_4$ and $\mathcal{X}_5$. The set of $(\mathcal{X}_1, \mathcal{X}_4)$ $d(\mathcal{S})$-intercessors is given by $\mathcal{I}^{d(\mathcal{S})}_{1:4} = \{\mathcal{X}_2, \mathcal{X}_3\}$.



Graph 1 (G_1)

This definition of predecessors and successors extends that of White and Chalak (2006), who define these concepts only for recursive systems, defined next. In recursive systems, the current definition is stronger, in that if $\mathcal{X}_{h,j}$ $d(\mathcal{S})$-*precedes* $\mathcal{X}_{i,k}$ according to our definition here, then $\mathcal{X}_{h,j}$ $\mathcal{S}$-*precedes* $\mathcal{X}_{i,k}$ (denoted by $\mathcal{X}_{i,k} \Leftarrow_{\mathcal{S}} \mathcal{X}_{h,j}$) in White and Chalak's (2006) terminology. The converse does not hold, however. We use the term "intercessor" instead of the possible descriptor "mediator," as the latter may connote transmission; we want to avoid this, because intercessors need not transmit effects, as we explain further below.

In what follows, we focus on recursive partitioned settable systems, defined next. We leave the study of the interrelations between (conditional) independence and causal relationships in non-recursive systems for other work.

Definition 2.5: Recursive Partitioned Settable System Let $(\Omega, \mathcal{F})$, Z, $\Pi = \{\Pi_b\}$, and $\mathcal{X}(1, \cdot)$ be as in Definition 2.2. For b = 1, 2, …, let $Z_{[b]}$ denote the vector containing the settings $Z_{h,j}$ for $(h, j) \in \Pi_b$, and let $Z_{[0]} = Z_0$. Suppose there exists $r^{\Pi} \equiv \{r^{\Pi}_{h,j}\}$ adapted to Π such that for all (h, j) in Π_b the responses $Y_{h,j} = \mathcal{X}_{h,j}(0, \cdot)$ are jointly determined as

$$Y_{h,j} = r^{\Pi}_{h,j}(Z_{[0]}, \ldots, Z_{[b-1]}), \quad b = 1, 2, \ldots .$$

Then we say that r^{Π} is *recursive* and that Π is a *recursive partition*. If r^{Π} is recursive, the pair $\mathcal{S} \equiv \{(\Omega, \mathcal{F}), (Z, \Pi, r^{\Pi}, \mathcal{X})\}$ is a *recursive partitioned settable system*. We also say simply that $\mathcal{S}$ is *recursive*. ■

Parallel to the notation above, let $\mathcal{X}_{[b]}$ denote the vector containing $\mathcal{X}_{h,j}$ for $(h, j) \in \Pi_b$. It follows that if $\mathcal{S}$ is recursive, then $\mathcal{X}_{[b]} \not\overset{d}{\Rightarrow}_{S} \mathcal{X}_{[0]}, \ldots, \mathcal{X}_{[b-1]}$, b = 1, 2, … . In particular, it follows from Definition 2.4 that in recursive systems if $\mathcal{X}_{h,j} \overset{d}{\Rightarrow}_{S} \mathcal{X}_{i,k}$ then $\mathcal{X}_{i,k} \not\overset{d}{\Rightarrow}_{S} \mathcal{X}_{h,j}$. Recursive systems thus do not admit mutual causality. For the graphical depiction, this means that we cannot have both arcs $(\mathcal{X}_{h,j}, \mathcal{X}_{i,k})$ and $(\mathcal{X}_{i,k}, \mathcal{X}_{h,j})$ belonging to E. In addition, we have from Definition 2.4 that a recursive system $\mathcal{S}$ does not admit cycles of the form $\mathcal{X}_{h,j} \overset{d}{\Rightarrow}_{S} \mathcal{X}_{h_1,j_1} \overset{d}{\Rightarrow}_{S} \ldots \overset{d}{\Rightarrow}_{S} \mathcal{X}_{h_n,j_n} \overset{d}{\Rightarrow}_{S} \mathcal{X}_{h,j}$. Thus, when $\mathcal{S}$ is recursive, its corresponding graph G is termed *acyclic*, i.e., it admits an acyclic ordering of its vertices (see proposition 1.4.3 in BG). BG's (p. 175) DFSA algorithm outputs an acyclic ordering of the vertices of a directed acyclic graph (DAG).

We focus specifically on the canonical systems introduced next.

Definition 2.6: Canonical Recursive Partitioned Settable Systems Let $\mathcal{S} \equiv \{(\Omega, \mathcal{F}), (Z, \Pi, r^{\Pi}, \mathcal{X})\}$ be a recursive settable system. Suppose the settings are the *canonical settings* such that

$$\mathcal{X}_{[b]}(1, \cdot) = r^{\Pi}_{[b]}(\mathcal{X}_{[0]}(1, \cdot), \ldots, \mathcal{X}_{[b-1]}(1, \cdot)), \quad b = 1, 2, \ldots .$$

Then $\mathcal{S}$ is a *canonical* recursive partitioned settable system, or simply a *canonical* system. ■

In canonical systems, each setting $Z_{h,j}$ is determined as a response to its predecessors' settings. These systems are particularly relevant in observational studies where control is not feasible. When $\mathcal{S}$ is canonical, the setting and response values are determined entirely by Z_0 and r^{Π}. Thus in this case we also write $\mathcal{S} = \{(\Omega, \mathcal{F}), (Z_0, \Pi, r^{\Pi}, \mathcal{X})\}$.

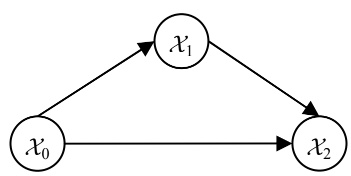
We next define notions of indirect causality in canonical systems. Pearl (2000, p. 165) states that, "the notion of indirect causality has no intrinsic operational meaning apart from providing a comparison between the direct and total effects." Nevertheless, our interest in the relations between causality and conditional dependence turns out to make it natural to define several related notions of indirect causality as primitives, thus endowing them with "intrinsic operational meaning." We then relate notions of total causality to those of direct and indirect causality.

The basic idea of indirect causality adopted here is straightforward. Consider a canonical system of three settable variables (see Graph G_2), where $\mathcal{X}_1(0, \cdot) = r_1(\mathcal{X}_0(1, \cdot))$ and

$$\mathcal{X}_2(0, \cdot) = r_2(\mathcal{X}_0(1, \cdot), r_1(\mathcal{X}_0(1, \cdot))).$$

Then $\mathcal{X}_0$ indirectly causes $\mathcal{X}_2$ via $\mathcal{X}_1$ if there exist z_0 and z_0^*, $z_0 \neq z_0^*$, such that

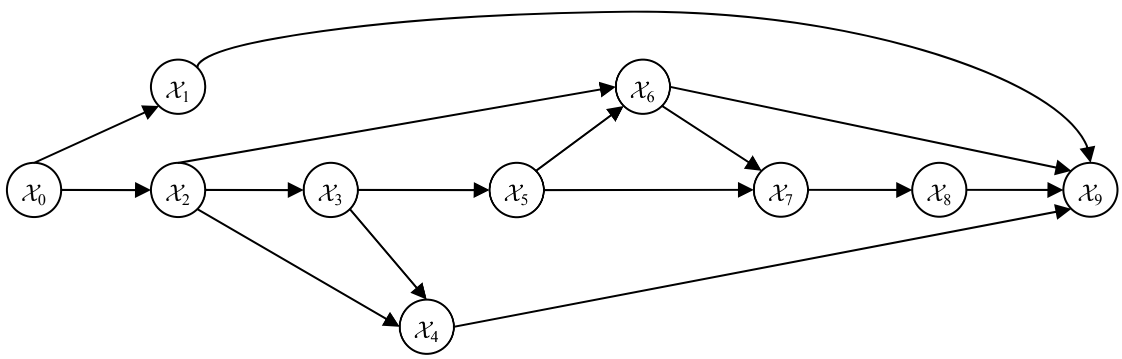
$$r_2(z_0^*, r_1(z_0^*)) - r_2(z_0^*, r_1(z_0)) \neq 0.$$



Graph 2 (G_2)

It is necessary but not sufficient for this that $\mathcal{X}_0$ directly cause $\mathcal{X}_1$ and that $\mathcal{X}_1$ directly cause $\mathcal{X}_2$. We emphasize that transitivity of causation is *not* guaranteed here, unlike classical treatments such as SGS (p. 42), where transitivity of causation is axiomatic. Instead, transitivity depends on the response functions. For example, if $r_1(z_0) = \max(z_0, 0)$ and $r_2(z_0, z_1) = \min(z_1, 0)$, then $\mathcal{X}_0 \stackrel{d}{\Rightarrow}_{\mathcal{S}} \mathcal{X}_1$ and $\mathcal{X}_1 \stackrel{d}{\Rightarrow}_{\mathcal{S}} \mathcal{X}_2$, but $\mathcal{X}_0$ does not indirectly cause $\mathcal{X}_2$, as $r_2(z_0^*, r_1(z_0)) = \min(\max(z_0, 0), 0) = 0$ for all z_0. With transitivity, $\mathcal{X}_{h,j}$ is an indirect cause of $\mathcal{X}_{i,k}$ if there exists an $(\mathcal{X}_{h,j}, \mathcal{X}_{i,k})$-path of length greater than 2 (SGS, p. 44-45). We work with somewhat more refined notions of indirect causality, elaborated below.

In the general case, the idea underlying indirect causality is essentially the same as in G_2, but to express this rigorously requires particular care and non-trivial further notation. For illustration, we employ a canonical system $\mathcal{S}_3$ illustrated in graph G_3, where $\Pi_1 = \{1, 2\}$ and $\Pi_b = \{b+1\}$ for $b = 2, \ldots, 8$. The complexity of this example is not capricious. This is the simplest system permitting a full illustration of the relationships that must be considered in a general definition of indirect causality. For brevity, we may drop explicit reference to $d(\mathcal{S})$ below when referring to $d(\mathcal{S})$-predecessors, intercessors, or successors. In particular, we write $\mathcal{I}_{(h,j):(i,k)}$ instead of $\mathcal{I}^{d(\mathcal{S})}_{(h,j):(i,k)}$.



Graph 3 (G_3)

To begin the illustration, take $b_1 < b_2$, $(h, j) \in \Pi_{b_1}$, $(i, k) \in \Pi_{b_2}$, and let A be a subset of the indexes of the elements of the intercessors $\mathcal{I}_{(h,j):(i,k)}$, denoted $ind(\mathcal{I}_{(h,j):(i,k)})$. For example, in $\mathcal{S}_3$, let $b_1 = 1$ and $b_2 = 8$, let (h, j) correspond to the index 2 (the second element of $\Pi_1 = \{1, 2\}$), and let (i, k) correspond to the index 9 (the sole element of Π_8). Then we have $ind(\mathcal{I}_{2:9}) = \{3, 4, 5, 6, 7, 8\}$, and we can let $A = \{5, 7\}$, say.

For given $(g, l) \in A$, let $\mathcal{I}^{(g,l)}_{(h,j):(i,k)} \equiv \mathcal{I}_{(h,j):(g,l)} \cup \{\mathcal{X}_{g,l}\} \cup \mathcal{I}_{(g,l):(i,k)}$ denote the $(\mathcal{X}_{h,j}, \mathcal{X}_{i,k})$ *intercessors for paths through* $\mathcal{X}_{g,l}$, and for $\mathcal{X}_A \equiv \bigcup_{(g,l)\in A} \{\mathcal{X}_{g,l}\}$, let $\mathcal{I}^{A}_{(h,j):(i,k)} \equiv \bigcup_{(g,l)\in A} \mathcal{I}^{(g,l)}_{(h,j):(i,k)}$ denote the $(\mathcal{X}_{h,j}, \mathcal{X}_{i,k})$ *intercessors for paths through* $\mathcal{X}_A$. (For $A = \varnothing$ we let $\mathcal{I}^{A}_{(h,j):(i,k)} = \varnothing$.) Thus, in $\mathcal{S}_3$ we have $ind(\mathcal{I}^5_{2:9}) = ind(\mathcal{I}^7_{2:9}) = \{3, 5, 6, 7, 8\}$ and it follows that $ind(\mathcal{I}^A_{2:9}) = \{3, 5, 6, 7, 8\}$ as well.

Let $\mathcal{X}_{\underline{A}} \equiv \mathcal{I}_{(h,j):(i,k)} \setminus \mathcal{I}^{A}_{(h,j):(i,k)}$ denote the $(\mathcal{X}_{h,j}, \mathcal{X}_{i,k})$ *intercessors not belonging to paths through* $\mathcal{X}_A$ and let $\underline{A}$ denote the set of indexes of the elements of $\mathcal{X}_{\underline{A}}$. In system $\mathcal{S}_3$, $\underline{A} = ind(\mathcal{I}_{2:9}) \setminus ind(\mathcal{I}^A_{2:9}) = \{3, 4, 5, 6, 7, 8\} \setminus \{3, 5, 6, 7, 8\} = \{4\}$.

Let $\mathcal{X}_{\bar{A}} \equiv \bigcup_{(g,l),(f,m)\in A} \mathcal{I}_{(g,l):(f,m)} \setminus \mathcal{X}_A$ denote the *inter-*$\mathcal{X}_A$ *intercessors excluded from* $\mathcal{X}_A$, and let $\bar{A}$ denote the set of indexes of the elements of $\mathcal{X}_{\bar{A}}$. In $\mathcal{S}_3$, we have $\bar{A} = \{6\}$.

Next, we distinguish between the ($\mathcal{X}_{h,j}$, $\mathcal{X}_{i,k}$) intercessors for paths through $\mathcal{X}_A$ that *strictly* precede or succeed $\mathcal{X}_A$.

We define the $\mathcal{X}_A$ *predecessors excluded from* $\mathcal{X}_A \cup \mathcal{X}_{\bar{A}}$:

$$\mathcal{P}^A_{(h,j):(i,k)} \equiv \bigcup_{(g,l)\in A} \{\mathcal{X}_{f,m} \in \mathcal{I}^A_{(h,j):(i,k)} \text{ and } \mathcal{X}_{f,m} \notin (\mathcal{X}_A \cup \mathcal{X}_{\bar{A}}) \text{: } \mathcal{X}_{f,m}\ d(\mathcal{S})\text{-}precedes\ \mathcal{X}_{g,l}\},$$

and the $\mathcal{X}_A$ *successors excluded from* $\mathcal{X}_A \cup \mathcal{X}_{\bar{A}}$:

$$\mathcal{S}^A_{(h,j):(i,k)} \equiv \bigcup_{(g,l)\in A} \{\mathcal{X}_{f,m} \in \mathcal{I}^A_{(h,j):(i,k)} \text{ and } \mathcal{X}_{f,m} \notin (\mathcal{X}_A \cup \mathcal{X}_{\bar{A}}) \text{: } \mathcal{X}_{f,m}\ d(\mathcal{S})\text{-}succeeds\ \mathcal{X}_{g,l}\}.$$

In the example illustrated in G_3, we have $ind(\mathcal{P}^A_{2:9}) = \{3\}$ and $ind(\mathcal{S}^A_{2:9}) = \{8\}$.

By construction, $\mathcal{I}^A_{(h,j):(i,k)} = \mathcal{P}^A_{(h,j):(i,k)} \cup \mathcal{X}_A \cup \mathcal{X}_{\bar{A}} \cup \mathcal{S}^A_{(h,j):(i,k)}$, and $ind(\mathcal{I}^A_{(h,j):(i,k)})$ is a subset of $\Pi_{[b_1+1:b_2-1]}$, where for any blocks a, b, $0 \le a \le b$, $\Pi_{[a:b]} \equiv \Pi_a \cup ... \cup \Pi_{b-1} \cup \Pi_b$. We verify this equality in our example, where $\mathcal{I}^A_{2:9} = \{\mathcal{X}_3, \mathcal{X}_5, \mathcal{X}_6, \mathcal{X}_7, \mathcal{X}_8\}$ and $\mathcal{P}^A_{2:9} \cup \mathcal{X}_A \cup \mathcal{X}_{\bar{A}} \cup \mathcal{S}^A_{2:9} = \{\mathcal{X}_3\} \cup \{\mathcal{X}_5, \mathcal{X}_7\} \cup \{\mathcal{X}_6\} \cup \{\mathcal{X}_8\}$. We write values of settings corresponding to $\Pi_{[a:b]}$ as $z_{[a:b]}$. In particular, when $(i, k) \in \Pi_{b_2}$, we can express response values for $\mathcal{X}_{i,k}$ as $r^{\Pi}_{i,k}(z_{[0:b_2-1]})$.

Next, let $z_{[0:b_1](h,j)}$ denote a vector of values for settings for all elements corresponding to $\Pi_{[0:b_1]}$ except $\mathcal{X}_{h,j}$. Thus, in $\mathcal{S}_3$, $z_{[0:1](2)}$ denotes values of settings for $\mathcal{X}_0$ and $\mathcal{X}_1$. Similarly, let $z_{(h,j):A}$, $z_{\underline{A}}$, z_A, $z_{\bar{A}}$, and $z_{A:(i,k)}$ denote vectors of values of settings for elements of $\mathcal{P}^A_{(h,j):(i,k)}$, $\mathcal{X}_{\underline{A}}$, $\mathcal{X}_A$, $\mathcal{X}_{\bar{A}}$, and $\mathcal{S}^A_{(h,j):(i,k)}$ respectively. We treat elements of $\Pi_{[0:b_2-1]}$ that do not correspond to ($\mathcal{X}_{h,j}$, $\mathcal{X}_{i,k}$) intercessors as elements of $\Pi_{[0:b_1]}$ without loss of generality. We now represent response values for $\mathcal{X}_{i,k}$ (recall (i,k) belongs to Π_{b_2}) as

$$r^{A,(h,j)}_{i,k}(z_{[0:b_1](h,j)}, z_{h,j}, z_{(h,j):A}, z_{\underline{A}}, z_A, z_{\bar{A}}, z_{A:(i,k)}) \equiv r^{\Pi}_{i,k}(z_{[0:b_2-1]}).$$

The superscript notation $A,(h,j)$ just introduced does not designate a partition but instead acts to specify that the arguments of $r^{\Pi}_{i,k}$ have been permuted in a particular way, so as to focus attention on settings of $\mathcal{X}_A$ and $\mathcal{X}_{h,j}$.

Observe that when $A = ind(\mathcal{I}_{(h,j):(i,k)})$, the sets $ind(\mathcal{P}^A_{(h,j):(i,k)})$, $\underline{A}$, $\bar{A}$, and $ind(\mathcal{S}^A_{(h,j):(i,k)})$ are empty and we write $r^{A,(h,j)}_{i,k}(z_{[0:b_1](h,j)}, z_{h,j}, z_A) \equiv r^{\Pi}_{i,k}(z_{[0:b_2-1]})$. Alternatively, when $A = \varnothing$, the sets $ind(\mathcal{P}^A_{(h,j):(i,k)})$, $\bar{A}$, and $ind(\mathcal{S}^A_{(h,j):(i,k)})$ are empty, whereas $\underline{A} = ind(\mathcal{I}_{(h,j):(i,k)})$, and we write $r^{A,(h,j)}_{i,k}(z_{[0:b_1](h,j)}, z_{h,j}, z_{\underline{A}}) \equiv r^{\Pi}_{i,k}(z_{[0:b_2-1]})$.

Similar to response values for $\mathcal{X}_{i,k}$, we represent vectors of response values for elements of $\mathcal{P}^A_{(h,j):(i,k)}$, $\mathcal{X}_{\underline{A}}$, $\mathcal{X}_A$, $\mathcal{X}_{\bar{A}}$, and $\mathcal{S}^A_{(h,j):(i,k)}$ respectively as

$$y_{(h,j):A} = r^{A,(h,j)}_{(h,j):A}(z_{[0:b_1](h,j)},\ z_{h,j}),$$

$$y_{\underline{A}} = r^{A,(h,j)}_{\underline{A}}(z_{[0:b_1](h,j)},\ z_{h,j},\ y_{(h,j):A}),$$

$$y_{A,\bar{A}} = r^{A,(h,j)}_{A,\bar{A}}(z_{[0:b_1](h,j)},\ z_{h,j},\ y_{(h,j):A},\ y_{\underline{A}}),$$

$$y_{A:(i,k)} = r^{A,(h,j)}_{A:(i,k)}(z_{[0:b_1](h,j)},\ z_{h,j},\ y_{(h,j):A},\ y_{\underline{A}},\ y_{\bar{A}},\ y_A).$$

For convenience, we write $(y_A, y_{\bar{A}}) = y_{A,\bar{A}}$. The elements of these response vectors obtain by recursive substitution. Any given element of one of these vectors will depend only on its corresponding predecessors. Thus, although $y_{\underline{A}}$ appears as an argument in $r^{A,(h,j)}_{A,\bar{A}}$, only the predecessor elements of $y_{\underline{A}}$ for a given response determine that response. In particular, $y_{\underline{A}}$ does not determine y_A by definition.

To account for interventions to settings of $\mathcal{X}_{h,j}$ (denoted $z_{h,j} \to z^*_{h,j}$) and their consequences relevant for indirect causality, we introduce one last notation:

$$y^*_{(h,j):A} = r^{A,(h,j)}_{(h,j):A}(z_{[0:b_1](h,j)},\ z^*_{h,j}),$$

$$y^*_{\underline{A}} = r^{A,(h,j)}_{\underline{A}}(z_{[0:b_1](h,j)},\ z^*_{h,j},\ y^*_{(h,j):A}),$$

$$y^*_{A,\bar{A}} = r^{A,(h,j)}_{A,\bar{A}}(z_{[0:b_1](h,j)},\ z^*_{h,j},\ y^*_{(h,j):A},\ y^*_{\underline{A}}),$$

$$y^*_{A:(i,k)} = r^{A,(h,j)}_{A:(i,k)}(z_{[0:b_1](h,j)},\ z^*_{h,j},\ y^*_{(h,j):A},\ y^*_{\underline{A}},\ y^*_A,\ y^*_{\bar{A}}).$$

We also write $(y^*_A, y^*_{\bar{A}}) = y^*_{A,\bar{A}}$. We can now define our first notion of indirect causality.

Definition 2.7(I): Indirect Causality via $\mathcal{X}_A$ Let $\mathcal{S} = \{(\Omega, \mathcal{F}), (Z_0, \Pi, r^{\Pi}, \mathcal{X})\}$ be canonical. For given non-negative integers b_1 and b_2 with $b_1 < b_2$, let $(h, j) \in \Pi_{b_1}$, let $(i, k) \in \Pi_{b_2}$, and let A be a subset of $ind(\mathcal{I}_{(h,j):(i,k)})$. Then $\mathcal{X}_{h,j}$ *indirectly causes* $\mathcal{X}_{i,k}$ *via* $\mathcal{X}_A$ *with respect to* $\mathcal{S}$ if there exist: (a) $z_{[0:b_1](h,j)}$; and (b) $z_{h,j}$ and $z^*_{h,j}$ with $z_{h,j} \neq z^*_{h,j}$ such that

$r_{i,k}^{A,(h,j)}(z_{[0:b_1](h,j)},\, z^*_{h,j},\, y^*_{(h,j):A},\, y^*_{\underline{A}},\, y^*_A,\, y^*_{\bar{A}},\, y^*_{A:(i,k)})$
$- r_{i,k}^{A,(h,j)}(z_{[0:b_1](h,j)},\, z^*_{h,j},\, y^*_{(h,j):A},\, y^*_{\underline{A}},\, y_A,\, y^*_{\bar{A}},\, r_{A:(i,k)}^{A,(h,j)}(z_{[0:b_1](h,j)},\, z^*_{h,j},\, y^*_{(h,j):A},\, y^*_{\underline{A}},\, y_A,\, y^*_{\bar{A}}))$
$\neq 0;$

and we write $\mathcal{X}_{h,j} \overset{i[A]}{\Rightarrow}_S \mathcal{X}_{i,k}$. Otherwise, we say that $\mathcal{X}_{h,j}$ *does not indirectly cause* $\mathcal{X}_{i,k}$ *via* $\mathcal{X}_A$ *with respect to* $\mathcal{S}$ and we write $\mathcal{X}_{h,j} \overset{i[A]}{\not\Rightarrow}_S \mathcal{X}_{i,k}$. When $A = ind(\mathcal{I}^{d(\mathcal{S})}_{(h,j):(i,k)})$ and $\mathcal{X}_{h,j} \overset{i[A]}{\Rightarrow}_S \mathcal{X}_{i,k}$, we say $\mathcal{X}_{h,j}$ *indirectly causes* $\mathcal{X}_{i,k}$ *with respect to* $\mathcal{S}$ and we write $\mathcal{X}_{h,j} \overset{i}{\Rightarrow}_S \mathcal{X}_{i,k}$; when $A = ind(\mathcal{I}_{(h,j):(i,k)})$ and $\mathcal{X}_{h,j} \overset{i[A]}{\not\Rightarrow}_S \mathcal{X}_{i,k}$, we say that $\mathcal{X}_{h,j}$ *does not indirectly cause* $\mathcal{X}_{i,k}$ *with respect to* $\mathcal{S}$ and we write $\mathcal{X}_{h,j} \overset{i}{\not\Rightarrow}_S \mathcal{X}_{i,k}$. ■

Consider system $\mathcal{S}_3$ illustrated in G_3, for example. With $A = \{5, 7\}$, Definition 2.7(I) states that $\mathcal{X}_2 \overset{i[A]}{\Rightarrow}_{S_3} \mathcal{X}_9$ if there exists (a) $z_{[0:1](2)}$; and (b) z_2 and z_2^* with $z_2 \neq z_2^*$ such that

$$r_9^{A,2}(z_{[0:1](2)},\, z_2^*,\, y^*_{2:A},\, y^*_{\underline{A}},\, y^*_A,\, y^*_{\bar{A}},\, y^*_{A:9})$$
$$- r_9^{A,2}(z_{[0:1](2)},\, z_2^*,\, y^*_{2:A},\, y^*_{\underline{A}},\, y_A,\, y^*_{\bar{A}},\, r_{A:9}^{A,2}(z_{[0:1](2)},\, z_2^*,\, y^*_{2:A},\, y^*_{\underline{A}},\, y_A,\, y^*_{\bar{A}})) \neq 0;$$

Observe that among the arguments of $r_9^{A,2}$ and $r_{A:9}^{A,2}$ in both responses of $\mathcal{X}_9$ above, only settings y_A and y_A^* differ. Intuitively, Definition 2.7(I) concludes that $\mathcal{X}_2 \overset{i[A]}{\Rightarrow}_{S_3} \mathcal{X}_9$ if there exist an intervention $z_2 \rightarrow z_2^*$ to $\mathcal{X}_2$ and corresponding responses of $\mathcal{X}_9$ that differ when the effects of setting z_2^* fully propagate in the system, as opposed to when they propagate through all variables in the system except elements of $\mathcal{X}_A$. Thus, setting values for $\mathcal{X}_0$, $\mathcal{X}_1$, $\mathcal{X}_2$, $\mathcal{X}_3$, $\mathcal{X}_4$, and $\mathcal{X}_6$ are $z_{[0:1](2)}$, z_2^*, $y^*_{2:A}$, $y^*_{\underline{A}}$, $y^*_{\bar{A}}$ in determining both responses of $\mathcal{X}_9$. On the other hand, setting values for ($\mathcal{X}_5$, $\mathcal{X}_7$) are y_A^* in the first response but are y_A in the second. Also, settings of $\mathcal{X}_8$ in both responses of $\mathcal{X}_9$ are allowed to differ only in response to different settings of ($\mathcal{X}_5$, $\mathcal{X}_7$).

When $\mathcal{X}_{h,j} \overset{i}{\Rightarrow}_S \mathcal{X}_{i,k}$, it follows that for some non-empty $A \subset \mathcal{I}_{(h,j):(i,k)}$ we have $\mathcal{X}_{h,j} \overset{i[A]}{\Rightarrow}_S \mathcal{X}_{i,k}$. The converse need not hold, because $\mathcal{X}_{h,j}$ can indirectly cause $\mathcal{X}_{i,k}$ through each of two distinct intercessors whose associated effects may cancel each other. For example, it may be that $\mathcal{X}_2$ indirectly causes $\mathcal{X}_9$ via $\mathcal{X}_4$ as well as via $\mathcal{X}_6$ with respect to $\mathcal{S}_3$ but that $\mathcal{X}_2$ does not indirectly cause $\mathcal{X}_9$ via $\{\mathcal{X}_4, \mathcal{X}_6\}$ with respect to $\mathcal{S}_3$.

We next introduce an indirect causality concept complementary to that above.

Definition 2.7(II): Indirect Causality exclusive of $\mathcal{X}_A$ Let $\mathcal{S}$ and A be as Definition 2.7(I). Then $\mathcal{X}_{h,j}$ *indirectly causes* $\mathcal{X}_{i,k}$ *exclusive of* $\mathcal{X}_A$ *with respect to* $\mathcal{S}$, if there exist: (a) $z_{[0:b_1](h,j)}$; and (b) $z_{h,j}$ and $z^*_{h,j}$ with $z_{h,j} \neq z^*_{h,j}$ such that

$$r^{A,(h,j)}_{i,k}(z_{[0:b_1](h,j)}, z^*_{h,j}, y^*_{(h,j):A}, y^*_{\underline{A}}, y^*_{A}, y^*_{\bar{A}}, y^*_{A:(i,k)})$$
$$- r^{A,(h,j)}_{i,k}(z_{[0:b_1](h,j)}, z^*_{h,j}, y_{(h,j):A}, y_{\underline{A}}, y^*_{A}, y_{\bar{A}}, r^{A,(h,j)}_{A:(i,k)}(z_{[0:b_1](h,j)}, z_{h,j}, y_{(h,j):A}, y_{\underline{A}}, y^*_{A}, y_{\bar{A}}))$$
$$\neq 0;$$

and we write $\mathcal{X}_{h,j} \overset{i[\sim A]}{\Rightarrow}_{S} \mathcal{X}_{i,k}$. Otherwise, we say that $\mathcal{X}_{h,j}$ *does not indirectly cause* $\mathcal{X}_{i,k}$ *exclusive of* $\mathcal{X}_A$ *with respect to* $\mathcal{S}$ and we write $\mathcal{X}_{h,j} \overset{i[\sim A]}{\not\Rightarrow}_{S} \mathcal{X}_{i,k}$. ■

In system $\mathcal{S}_3$, Definition 2.7(II) says that $\mathcal{X}_2 \overset{i[\sim A]}{\Rightarrow}_{S_3} \mathcal{X}_9$ if there exists (a) $z_{[0:1](2)}$; and (b) z_2 and z^*_2 with $z_2 \neq z^*_2$ such that

$$r^{A,2}_{9}(z_{[0:1](2)}, z^*_2, y^*_{2:A}, y^*_{\underline{A}}, y^*_{A}, y^*_{\bar{A}}, y^*_{A:9})$$
$$- r^{A,2}_{9}(z_{[0:1](2)}, z^*_2, y_{2:A}, y_{\underline{A}}, y^*_{A}, y_{\bar{A}}, r^{A,2}_{A:9}(z_{[0:1](2)}, z_2, y_{2:A}, y_{\underline{A}}, y^*_{A}, y_{\bar{A}})) \neq 0.$$

In contrast to Definition 2.7(I), here all arguments of $r^{A,2}_{9}$ can differ across the above two responses of $\mathcal{X}_9$ except for settings z^*_2, and y^*_A – these are identical in determining both responses. Similarly, all arguments of $r^{A,2}_{A:9}$ except for setting y^*_A can differ across the two responses of $\mathcal{X}_9$. Intuitively, Definition 2.7(II) concludes that $\mathcal{X}_2 \overset{i[\sim A]}{\Rightarrow}_{S_3} \mathcal{X}_9$ if there exist an intervention $z_2 \rightarrow z^*_2$ to $\mathcal{X}_2$ and corresponding responses of $\mathcal{X}_9$ that differ when the effects of setting z^*_2 in $\mathcal{X}_2$ fully propagate in the system as opposed to when they propagate only through the direct effect of $\mathcal{X}_2$ on $\mathcal{X}_9$ and through elements of $\mathcal{X}_A$.

Definitions 2.7(I,II) provide rigorous, operational definitions of indirect causality, extending definitions provided in Pearl (2001) by introducing the notions of indirect causality *via* a subset $\mathcal{X}_A$ of the ($\mathcal{X}_{h,j}$, $\mathcal{X}_{i,k}$) intercessors and of indirect causality *exclusive of* $\mathcal{X}_A$.

In analyzing relations between causality and conditional independence, it is important to keep track of channels of indirect causality as well as direct causality. Our next definition, a form of total causality, facilitates this.

Definition 2.8(I): *A*-Causality Let $\mathcal{S} = \{(\Omega, \mathcal{F}), (Z_0, \Pi, r^{\Pi}, \mathcal{X})\}$ be canonical. For given non-negative integers b_1 and b_2 with $b_1 < b_2$, let $(h, j) \in \Pi_{b_1}$, let $(i, k) \in \Pi_{b_2}$, and let A

$\subset ind(\mathcal{I}_{(h,j):(i,k)})$. Then $\mathcal{X}_{h,j}$ *causes* $\mathcal{X}_{i,k}$ *via* $\mathcal{X}_A$ (or $\mathcal{X}_{h,j}$ *A-causes* $\mathcal{X}_{i,k}$) *with respect to* $\mathcal{S}$ if there exist: (a) $z_{[0:b_1](h,j)}$; and (b) $z_{h,j}$ and $z^*_{h,j}$ with $z_{h,j} \neq z^*_{h,j}$ such that

$$r^{A,(h,j)}_{i,k}(z_{[0:b_1](h,j)}, z^*_{h,j}, y^*_{(h,j):A}, y^*_{\underline{A}}, y^*_A, y^*_{\bar{A}}, y^*_{A:(i,k)})$$
$$-r^{A,(h,j)}_{i,k}(z_{[0:b_1](h,j)}, z_{h,j}, y^*_{(h,j):A}, y^*_{\underline{A}}, y_A, y^*_{\bar{A}}, r^{A,(h,j)}_{A:(i,k)}(z_{[0:b_1](h,j)}, z^*_{h,j}, y^*_{(h,j):A}, y^*_{\underline{A}}, y_A, y^*_{\bar{A}}))$$
$$\neq 0;$$

and we write $\mathcal{X}_{h,j} \overset{A}{\Rightarrow}_S \mathcal{X}_{i,k}$. Otherwise, we say that $\mathcal{X}_{h,j}$ *does not A-cause with respect to* $\mathcal{S}$, and we write $\mathcal{X}_{h,j} \overset{A}{\not\Rightarrow}_S \mathcal{X}_{i,k}$. When $A = ind(\mathcal{I}_{(h,j):(i,k)})$ and $\mathcal{X}_{h,j} \overset{A}{\Rightarrow}_S \mathcal{X}_{i,k}$, we say $\mathcal{X}_{h,j}$ *causes* $\mathcal{X}_{i,k}$ *with respect to* $\mathcal{S}$ and we write $\mathcal{X}_{h,j} \Rightarrow_S \mathcal{X}_{i,k}$; when $A = ind(\mathcal{I}_{(h,j):(i,k)})$ and $\mathcal{X}_{h,j} \overset{A}{\not\Rightarrow}_S \mathcal{X}_{i,k}$, we say that $\mathcal{X}_{h,j}$ *does not cause* $\mathcal{X}_{i,k}$ *with respect to* $\mathcal{S}$, and we write $\mathcal{X}_{h,j} \not\Rightarrow_S \mathcal{X}_{i,k}$. ■

This definition yields direct causality in the special case where A is the empty set. That is, $\varnothing$-causality is direct causality restricted to canonical systems. To see this, observe that when $A = \varnothing$ and $\mathcal{X}_{h,j} \overset{A}{\Rightarrow}_S \mathcal{X}_{i,k}$ we have from Definition 2.8(I) that there exit (a) $z_{[0:b_1](h,j)}$; and (b) $z_{h,j}$ and $z^*_{h,j}$ with $z_{h,j} \neq z^*_{h,j}$ such that

$$r^{A,(h,j)}_{i,k}(z_{[0:b_1](h,j)}, z^*_{h,j}, y^*_{\underline{A}}) - r^{A,(h,j)}_{i,k}(z_{[0:b_1](h,j)}, z_{h,j}, y^*_{\underline{A}}) \neq 0,$$

and thus $\mathcal{X}_{h,j} \overset{d}{\Rightarrow}_S \mathcal{X}_{i,k}$.

Definition 2.8(II): ~A-Causality Let $\mathcal{S}$ and A be as Definition 2.8(I). Then $\mathcal{X}_{h,j}$ *causes* $\mathcal{X}_{i,k}$ *exclusive of* $\mathcal{X}_A$ (or $\mathcal{X}_{h,j}$ *~A-causes* $\mathcal{X}_{i,k}$) *with respect to* $\mathcal{S}$ if there exist: (a) $z_{[0:b_1](h,j)}$; and (b) $z_{h,j}$ and $z^*_{h,j}$ with $z_{h,j} \neq z^*_{h,j}$ such that

$$r^{A,(h,j)}_{i,k}(z_{[0:b_1](h,j)}, z^*_{h,j}, y^*_{(h,j):A}, y^*_{\underline{A}}, y^*_A, y^*_{\bar{A}}, y^*_{A:(i,k)})$$
$$-r^{A,(h,j)}_{i,k}(z_{[0:b_1](h,j)}, z_{h,j}, y_{(h,j):A}, y_{\underline{A}}, y^*_A, y_{\bar{A}}, r^{A,(h,j)}_{A:(i,k)}(z_{[0:b_1](h,j)}, z_{h,j}, y_{(h,j):A}, y_{\underline{A}}, y^*_A, y_{\bar{A}}))$$
$$\neq 0;$$

and we write $\mathcal{X}_{h,j} \overset{\sim A}{\Rightarrow}_S \mathcal{X}_{i,k}$. Otherwise, we say that $\mathcal{X}_{h,j}$ *does not cause* $\mathcal{X}_{i,k}$ *exclusive of* $\mathcal{X}_A$ (or $\mathcal{X}_{h,j}$ *does not ~A-cause* $\mathcal{X}_{i,k}$) *with respect to* $\mathcal{S}$, and we write $\mathcal{X}_{h,j} \overset{\sim A}{\not\Rightarrow}_S \mathcal{X}_{i,k}$. ■

Thus, Definitions 2.8(I,II) are analogous to Definitions 2.7(I,II) with the difference that the direct effect of $\mathcal{X}_{h,j}$ on $\mathcal{X}_{i,k}$ is now further taken into account.

Observe that $\sim\varnothing$-Causality is equivalent to A-Causality with $A = ind(\mathcal{I}_{(h,j):(i,k)})$. To see this, let $A = ind(\mathcal{I}_{(h,j):(i,k)})$ and suppose that $\mathcal{X}_{h,j} \overset{A}{\Rightarrow}_S \mathcal{X}_{i,k}$; then $\mathcal{X}_{h,j} \Rightarrow_S \mathcal{X}_{i,k}$. We have from Definition 2.8(I) that there exist: (a) $z_{[0:b_1](h,j)}$; and (b) $z_{h,j}$ and $z^*_{h,j}$ with $z_{h,j} \neq z^*_{h,j}$ such that

$$r^{A,(h,j)}_{i,k}(z_{[0:b_1](h,j)},\ z^*_{h,j},\ y^*_A) - r^{A,(h,j)}_{i,k}(z_{[0:b_1](h,j)},\ z_{h,j},\ y_A) \neq 0.$$

Also, let $B = \varnothing$ and suppose that $\mathcal{X}_{h,j} \overset{\sim B}{\Rightarrow}_S \mathcal{X}_{i,k}$. We have from Definition 2.8(II) that there exist: (a) $z_{[0:b_1](h,j)}$; and (b) $z_{h,j}$ and $z^*_{h,j}$ with $z_{h,j} \neq z^*_{h,j}$ such that

$$r^{B,(h,j)}_{i,k}(z_{[0:b_1](h,j)},\ z^*_{h,j},\ y^*_{\underline{B}}) - r^{B,(h,j)}_{i,k}(z_{[0:b_1](h,j)},\ z_{h,j},\ y_{\underline{B}}) \neq 0.$$

But when $B = \varnothing$, we have $\underline{B} = ind(\mathcal{I}_{(h,j):(i,k)}) = A$. The claim is verified, as the two definitions coincide.

It can be analogously shown that $\varnothing$-Causality is equivalent to $\sim A$-Causality when $A = ind(\mathcal{I}_{(h,j):(i,k)})$. Thus, $\varnothing$-causality and $\sim A$-Causality with $A = ind(\mathcal{I}_{(h,j):(i,k)})$ both reduce to direct causality in canonical systems.

Our first formal result links A-causality, direct causality, and indirect causality via $\mathcal{X}_A$.

Proposition 2.9: Let $\mathcal{S} = \{(\Omega, \mathcal{F}), (Z_0, \Pi, r^\Pi, \mathcal{X})\}$ be canonical. For given non-negative integers b_1 and b_2 with $b_1 < b_2$, let $(h, j) \in \Pi_{b_1}$, let $(i, k) \in \Pi_{b_2}$, let $A \subset ind(\mathcal{I}_{(h,j):(i,k)})$, and suppose that $\mathcal{X}_{h,j} \overset{A}{\Rightarrow}_S \mathcal{X}_{i,k}$. Then $\mathcal{X}_{h,j} \overset{d}{\Rightarrow}_S \mathcal{X}_{i,k}$ or $\mathcal{X}_{h,j} \overset{i[A]}{\Rightarrow}_S \mathcal{X}_{i,k}$ or both. ■

An important special case of Proposition 2.9 occurs when $A = ind(\mathcal{I}_{(h,j):(i,k)})$.

Corollary 2.10: Let $\mathcal{S}$, (h, j), and (i, k) be as in Proposition 2.9, and suppose that $\mathcal{X}_{h,j} \Rightarrow_S \mathcal{X}_{i,k}$. Then $\mathcal{X}_{h,j} \overset{d}{\Rightarrow}_S \mathcal{X}_{i,k}$ or $\mathcal{X}_{h,j} \overset{i}{\Rightarrow}_S \mathcal{X}_{i,k}$ or both. ■

Corollary 2.10 verifies the plausible claim that if $\mathcal{X}_{h,j}$ causes $\mathcal{X}_{i,k}$, it does so directly, indirectly, or both. The converse need not hold, however, as direct and indirect causal channels can cancel one another. Proposition 2.9 extends this proposition to A-causality. A similar result holds for $\sim A$-causality:

Proposition 2.11: Let $\mathcal{S}$, (h, j), (i, k), and A be as in Proposition 2.9, and suppose that $\mathcal{X}_{h,j}$ $\overset{\sim A}{\Rightarrow}_S$ $\mathcal{X}_{i,k}$. Then $\mathcal{X}_{h,j} \overset{d}{\Rightarrow}_S \mathcal{X}_{i,k}$ or $\mathcal{X}_{h,j} \overset{i[\sim A]}{\Rightarrow}_S \mathcal{X}_{i,k}$ or both. ■

3. Independence in Settable Systems

Our next result plays a key role in formalizing Reichenbach's principle of common cause.

Lemma 3.1: Let $\mathcal{S} = \{(\Omega, \mathcal{F}), (Z_0, \Pi, r^{\Pi}, \mathcal{X})\}$ be canonical. Then a settable variable $\mathcal{X}_{h,j}$ is constant if and only if $\mathcal{X}_0 \not\Rightarrow_S \mathcal{X}_{h,j}$. ■

So far, none of our definitions or results have required any probabilistic elements. Our next result, formalizing Reichenbach's principle of common cause, relates the functionally defined notion of causality adopted here to the stochastic concept of dependence. This requires us to explicitly introduce probability measures P on $(\Omega, \mathcal{F})$. For canonical systems $\mathcal{S}$, the stochastic behavior of settings and responses is determined entirely by P and r^{Π}. In stating our results, we follow Dawid (1979) and write $X \perp Y$ when random variables X and Y are independent under P, whereas $X \not\perp Y$ means that X and Y are not independent under P.

Proposition 3.2: The Reichenbach Principle of Common Cause (I) Let $\mathcal{S} = \{(\Omega, \mathcal{F}), (Z_0, \Pi, r^{\Pi}, \mathcal{X})\}$ be canonical. For given non-negative integers b_1 and b_2, let $(h, j) \in \Pi_{b_1}$ and $(i, k) \in \Pi_{b_2}$ $((h, j) \neq (i, k))$. Let $\mathcal{X}_{h,j}$ and $\mathcal{X}_{i,k}$ be settable variables, and let $Y_{h,j} \equiv \mathcal{X}_{h,j}(0, \cdot)$, and $Y_{i,k} \equiv \mathcal{X}_{i,k}(0, \cdot)$. For every probability measure P on $(\Omega, \mathcal{F})$, if $Y_{h,j} \not\perp Y_{i,k}$, then either:

(i) $\mathcal{X}_{h,j} \not\Rightarrow_S \mathcal{X}_{i,k}$, $\mathcal{X}_{i,k} \not\Rightarrow_S \mathcal{X}_{h,j}$, and $\mathcal{X}_0 \Rightarrow_S \mathcal{X}_{h,j}$ and $\mathcal{X}_0 \Rightarrow_S \mathcal{X}_{i,k}$; or

(ii) $\mathcal{X}_{i,k} \not\Rightarrow_S \mathcal{X}_{h,j}$ and either (a) $(h, j) = 0$ and $\mathcal{X}_{h,j} \Rightarrow_S \mathcal{X}_{i,k}$ or (b) $(h, j) \neq 0$ and $\mathcal{X}_0 \Rightarrow_S \mathcal{X}_{h,j}$ and $\mathcal{X}_0 \Rightarrow_S \mathcal{X}_{i,k}$; or

(iii) $\mathcal{X}_{h,j} \not\Rightarrow_S \mathcal{X}_{i,k}$ and either (a) $(i, k) = 0$ and $\mathcal{X}_{i,k} \Rightarrow_S \mathcal{X}_{h,j}$ or (b) $(i, k) \neq 0$ and $\mathcal{X}_0 \Rightarrow_S \mathcal{X}_{i,k}$ and $\mathcal{X}_0 \Rightarrow_S \mathcal{X}_{h,j}$. ■

This result provides a fully explicit statement of conditions, both causal and stochastic, under which the Reichenbach principle of common cause holds -- that is, under which it is true that when two random variables in a causal system are stochastically dependent, either one causes the other or there exists an underlying common cause. Note that the possibility that one variable causes the other is not explicit in (ii), except when $(h, j) = 0$. The possibility that $\mathcal{X}_{h,j} \Rightarrow_S \mathcal{X}_{i,k}$ when $(h, j) \neq 0$ is nevertheless implicit, as one way in which we may have $\mathcal{X}_0 \Rightarrow_S \mathcal{X}_{i,k}$ is via the indirect channel $\mathcal{X}_0 \Rightarrow_S \mathcal{X}_{h,j} \Rightarrow_S \mathcal{X}_{i,k}$. If this fails in (ii), then there nevertheless must be a common cause, $\mathcal{X}_0$. (The analogous statement also holds in (iii).)

Although Reichenbach's principle holds generally for canonical systems, the proof reveals that this is not a particularly deep fact. The reason is that the fundamental settable variable $\mathcal{X}_0$ can always serve as a common cause. Moreover, because the fundamental settings Z_0 can, without loss of generality, be identified with the underlying elements ω of the universe Ω, *one cannot dispense with this universal common cause without dispensing with the underlying structure supporting probability statements*.

Another way of appreciating the content of the common cause principle is to examine what it tells us about *lack* of dependence, via the contrapositive of Proposition 3.2. Specifically, the contrapositive says that, for all probability measures, if two random variables neither cause each other nor share the common cause $\mathcal{X}_0$, then they are independent. But Lemma 3.1 implies that when two random variables do not share the common cause $\mathcal{X}_0$, then at least one of them must be constant. Knowing that if at least one of two random variables is constant, then the two are independent for all probability measures does not afford deep insight into independence. Nevertheless, deeper insights emerge when we relax Reichenbach's principle to its conditional counterpart in Section 4 and study its relationship to notions of *d* /*D*-separation in Section 5.

The principle of common cause gives necessary but not sufficient causal conditions for dependence. Thus, its contrapositive gives sufficient but not necessary conditions for independence. Specifically, independence between responses can arise in the presence of a common cause other than the universal common cause. Independence can also arise between two responses in the presence of an underlying causal relation.

Examples in which two independent non-constant responses share one or more common causes are ubiquitous. Examples involving just one common cause typically require seemingly strange or artificial (but nevertheless well defined and measurable) response functions. On the other hand, examples of independent responses determined by two or more common causes can arise from familiar and well-behaved functions. For example, Cochran's theorem (Cochran, 1934), which underlies the behavior of the standard *t*- and *F*- statistics for hypothesis testing, ensures the independence of two distinct functions of the squares of two or more independent identically distributed normal random variables.

It is also easy to construct examples in which independence holds between directly causally related variables. Specifically, suppose $\mathcal{X}_1 \Rightarrow_S \mathcal{X}_3$ and $\mathcal{X}_2 \Rightarrow_S \mathcal{X}_3$ such that

$$Y_3 \overset{c}{=} a\, Z_2 + Z_1,$$

where Z_1 and Z_2 are jointly normally distributed with mean zero and variance one. (Following WC, we use the $\overset{c}{=}$ symbol instead of the usual equality sign to emphasize that this is a causal relationship.) Then Y_3 and Z_2 are also jointly normally distributed with mean zero. Thus, if Y_3 and Z_2 have zero correlation, then they are independent. But this can be ensured by taking $a = -\rho$, where ρ is the correlation between Z_1 and Z_2. (Note that Y_3 has non-zero variance as long as $|\, a \,| < 1$.)

It is thus useful to refine the possibilities for independence to distinguish situations in which causal restrictions among settable variables ensure that their responses are independent for any probability measure and those where independence among random variables is due to a particular choice of P or a particular configuration of the response functions. The following definitions are useful for this.

Definition 3.3: Causal Isolation and Stochastic Isolation Let $\mathcal{S}$, $\mathcal{X}_{h,j}$, $\mathcal{X}_{i,k}$ be as in Proposition 3.2. Suppose that (a) $(h, j) = 0$ and $\mathcal{X}_{h,j} \not\rightrightarrows_S \mathcal{X}_{i,k}$; or (b) $(i, k) = 0$ and $\mathcal{X}_{i,k} \not\rightrightarrows_S \mathcal{X}_{h,j}$; or (c) $(h, j) \neq 0$, $(i, k) \neq 0$, and $\mathcal{X}_0 \not\rightrightarrows_S \mathcal{X}_{h,j}$ or $\mathcal{X}_0 \not\rightrightarrows_S \mathcal{X}_{i,k}$. Then $\mathcal{X}_{h,j}$ and $\mathcal{X}_{i,k}$ are *causally isolated*. Put $Y_{h,j} \equiv \mathcal{X}_{h,j}(0, \cdot)$ and $Y_{i,k} \equiv \mathcal{X}_{i,k}(0, \cdot)$. If P is a probability measure on $(\Omega, \mathcal{F})$ such that $Y_{h,j} \perp Y_{i,k}$ when $\mathcal{X}_{h,j}$ and $\mathcal{X}_{i,k}$ are not causally isolated, then we say that *P stochastically isolates* $\mathcal{X}_{h,j}$ and $\mathcal{X}_{i,k}$. ■

Thus, $\mathcal{X}_{h,j}$ and $\mathcal{X}_{i,k}$ are causally isolated when the conclusion of Proposition 3.2 does not hold, that is, when $\mathcal{X}_0 \not\rightrightarrows_S \mathcal{X}_{h,j}$ or $\mathcal{X}_0 \not\rightrightarrows_S \mathcal{X}_{i,k}$. Causal isolation arises when, for one or the other of $\mathcal{X}_{h,j}$ and $\mathcal{X}_{i,k}$, the effects of the fundamental cause $\mathcal{X}_0$ operate through multiple indirect channels in just the right way as to cancel out. For variables that are not causally isolated, independence (stochastic isolation) can arise either directly from P (e.g., when $\mathcal{X}_{h,j}$ and $\mathcal{X}_{i,k}$ are both caused solely and directly by $\mathcal{X}_0$) or from just the right interaction between multiple causes (common or direct) and the probability measure P, as in the example preceding Definition 3.3.

Stochastic isolation is a very strong restriction on P. Nevertheless, there always exists a stochastically isolating probability measure. In fact, we can always construct a probability measure P on $(\Omega, \mathcal{F})$ that is capable of ensuring the stochastic independence of any two random variables in $\mathcal{S}$ regardless of the causal relations that hold among them. Proposition 3.4 formalizes this.

Proposition 3.4: Let $\mathcal{S} = \{(\Omega, \mathcal{F}), (Z_0, \Pi, r^{\Pi}, \mathcal{X})\}$ be canonical. Let P be a probability measure on $(\Omega, \mathcal{F})$ such that $Y_{h,j} \equiv \mathcal{X}_{h,j}(0, \cdot)$ and $Y_{i,k} \equiv \mathcal{X}_{i,k}(0, \cdot)$ have probability distributions $P_{h,j}$ and $P_{i,k}$, respectively. Then there exists a probability measure P^* on $(\Omega, \mathcal{F})$ such that $P^*_{h,j} = P_{h,j}$, $P^*_{i,k} = P_{i,k}$, and the joint probability distribution of $Y_{h,j}$ and $Y_{i,k}$ is $P^*_{h,j}\, P^*_{i,k}$. In particular, if $\mathcal{X}_{h,j}$ and $\mathcal{X}_{i,k}$ are not causally isolated, then P^* stochastically isolates $\mathcal{X}_{h,j}$ and $\mathcal{X}_{i,k}$. ■

It follows from Proposition 3.4 that the converse of Proposition 3.2 does not hold. That is, for any canonical system $\mathcal{S}$, we can always find a probability measure P^* such that $\mathcal{X}_{h,j}$ and $\mathcal{X}_{i,k}$ are *not* causally isolated and $Y_{h,j} \perp Y_{i,k}$.

We are now ready to state necessary and sufficient conditions for probabilistic dependence among random variables in settable systems.

Corollary 3.5: The Reichenbach Principle of Common Cause (II) Suppose the conditions of Proposition 3.2 hold. For every probability measure P on $(\Omega, \mathcal{F})$, $Y_{h,j} \not\perp Y_{i,k}$ if and only if (a) either:

(i) $(h, j) = 0$ and $\mathcal{X}_{h,j} \Rightarrow_S \mathcal{X}_{i,k}$; or

(ii) $(i, k) = 0$ and $\mathcal{X}_{i,k} \Rightarrow_S \mathcal{X}_{h,j}$; or

(iii) $(h, j) \neq 0$, $(i, k) \neq 0$, and $\mathcal{X}_0 \Rightarrow_S \mathcal{X}_{i,k}$ and $\mathcal{X}_0 \Rightarrow_S \mathcal{X}_{h,j}$;

and (b) P does not stochastically isolate $\mathcal{X}_{h,j}$ and $\mathcal{X}_{i,k}$. ■

Stated in the contrapositive, the key idea is deceptively straightforward: $Y_{h,j}$ and $Y_{i,k}$ are independent if and only if (a) $\mathcal{X}_{h,j}$ and $\mathcal{X}_{i,k}$ are causally isolated or (b) they are not causally isolated (so that $Y_{h,j}$ and $Y_{i,k}$ are nondegenerate) and P stochastically isolates $\mathcal{X}_{h,j}$ and $\mathcal{X}_{i,k}$.

This result serves several purposes. First, it characterizes stochastic dependence (and thus independence) in terms of functionally defined causal relations, rigorously establishing and refining the Reichenbach principle. Second, it demonstrates the crucial and dramatically simplifying role played by the fundamental variable $\mathcal{X}_0$ as a universal common cause. Once this role is accepted and understood in the context of canonical settable systems, the content of the Reichenbach principle is no longer mysterious, nor is it apparently deep. Third, and of particular significance, this result provides an accessible template for understanding the more involved relations between causal structure and conditional independence. In turn, these relations are central to the identification of causal effects in observational studies. We thus now direct our attention to the relations between causality and conditional independence.

4. Conditional Independence in Settable Systems

First, we state a result analogous to Lemma 3.1 that plays a key role in formalizing our conditional Reichenbach principle of common cause.

Lemma 4.1 Let $\mathcal{S}$, (h, j), (i, k), and $A \subset ind(\mathcal{I}_{(h,j):(i,k)})$ be as in Definition 2.8 (I, II). Then there exists a measurable function $\tilde{r}_{i,k}^{A,(h,j)}$ such that

$$\tilde{r}_{i,k}^{A,(h,j)}\ (z_{[0:b_1](h,j)},\ y_A) = r_{i,k}^{A,(h,j)}\ (z_{[0:b_1](h,j)}, z_{h,j},\ y_{(h,j):A},\ y_{\underline{A}},\ y_A,\ y_{\bar{A}},\ y_{A:(i,k)})$$

if and only if $\mathcal{X}_{h,j} \overset{\sim A}{\not\Rightarrow}_S \mathcal{X}_{i,k}$.

In particular, there exists a measurable function $\tilde{r}_{i,k}^{A,0}$ such that

$$\tilde{r}_{i,k}^{A,0}\ (y_A) = r_{i,k}^{A,0}\ (z_0,\ y_{0:A},\ y_{\underline{A}},\ y_A,\ y_{\bar{A}},\ y_{A:(i,k)})$$

if and only if $\mathcal{X}_0 \overset{\sim A}{\not\Rightarrow}_S \mathcal{X}_{i,k}$. ■

When $A = \varnothing$, Lemma 4.1 contains the result of Lemma 3.1 as a special case. In fact, when $A = \varnothing$ we have from Lemma 4.1 that there exists a measurable function $\tilde{r}_{i,k}^{A,0}$ such that

$$\tilde{r}_{i,k}^{A,0}\ (y_A) = r_{i,k}^{A,0}\ (z_0,\ y_{\underline{A}})$$

if and only if $\mathcal{X}_0 \overset{\sim\varnothing}{\not\Rightarrow}_S \mathcal{X}_{i,k}$. It follows by canonicality that the response of $\mathcal{X}_{i,k}$ is equal to $\tilde{r}_{i,k}^{A,0}\ (y_A)$, which must be constant. Recalling that $\mathcal{X}_0 \overset{\sim\varnothing}{\not\Rightarrow}_S \mathcal{X}_{i,k}$ is equivalent to $\mathcal{X}_0 \not\Rightarrow_S \mathcal{X}_{i,k}$ establishes that this is precisely the content of Lemma 3.1.

We are now ready to formalize the conditional Reichenbach principle of common cause.

Proposition 4.2: Conditional Reichenbach Principle of Common Cause (I) Let $\mathcal{S}$ = $\{(\Omega, \mathcal{F}), (Z_0, \Pi, r^{\Pi}, \mathcal{X})\}$ be canonical. For given non-negative integers b_1 and b_2, let $(h, j) \in \Pi_{b_1}$ and $(i, k) \in \Pi_{b_2}$ ($(h, j) \neq (i, k)$). Let $\mathcal{X}_{h,j}$ and $\mathcal{X}_{i,k}$ be settable variables, and let $Y_{h,j} \equiv \mathcal{X}_{h,j}\ (0, \cdot)$ and $Y_{i,k} \equiv \mathcal{X}_{i,k}\ (0, \cdot)$. Let $A \subset \Pi_{1:[\max(b_1,b_2)-1]} \setminus \{(h, j), (i, k)\}$, let $\mathcal{X}_A$ be the corresponding vector of settable variables, and let $Y_A = \mathcal{X}_A\ (0, \cdot)$. For every probability measure P on $(\Omega, \mathcal{F})$, if $Y_{h,j} \perp Y_{i,k} \mid Y_A$ then either:

(i) $\mathcal{X}_{h,j} \not\Rightarrow_S \mathcal{X}_{i,k}$, $\mathcal{X}_{i,k} \not\Rightarrow_S \mathcal{X}_{h,j}$, and with $A_1 \equiv A \cap ind(\mathcal{I}_{0:(h,j)})$ and $A_2 \equiv A \cap ind(\mathcal{I}_{0:(i,k)})$, $\mathcal{X}_0 \overset{\sim A_1}{\Rightarrow}_S \mathcal{X}_{h,j}$ and $\mathcal{X}_0 \overset{\sim A_2}{\Rightarrow}_S \mathcal{X}_{i,k}$; or

(ii) $\mathcal{X}_{i,k} \not\Rightarrow_S \mathcal{X}_{h,j}$ and either (a) $(h, j) = 0$ and $\mathcal{X}_{h,j} \overset{\sim A_2}{\Rightarrow}_S \mathcal{X}_{i,k}$ or (b) $(h, j) \neq 0$ and $\mathcal{X}_0 \overset{\sim A_1}{\Rightarrow}_S \mathcal{X}_{h,j}$ and $\mathcal{X}_0 \overset{\sim A_2}{\Rightarrow}_S \mathcal{X}_{i,k}$; or

(iii) $\mathcal{X}_{h,j} \not\Rightarrow_S \mathcal{X}_{i,k}$ and either (a) $(i, k) = 0$ and $\mathcal{X}_{i,k} \overset{\sim A_1}{\Rightarrow}_S \mathcal{X}_{h,j}$ or (b) $(i, k) \neq 0$ and $\mathcal{X}_0 \overset{\sim A_2}{\Rightarrow}_S \mathcal{X}_{i,k}$ and $\mathcal{X}_0 \overset{\sim A_1}{\Rightarrow}_S \mathcal{X}_{h,j}$. ■

Proposition 4.2 is an explicit statement of causal and stochastic conditions that extend Reichenbach's principle to its conditional counterpart. This is significant, as it implies that in recursive causal systems, knowledge of conditional dependence relations such as $Y_{h,j} \not\perp Y_{i,k} \mid Y_A$ is informative about the possible causal relations that involve settable variables $\mathcal{X}_{h,j}$ and $\mathcal{X}_{i,k}$. Proposition 4.2 implies that in recursive systems, in order for two random variables $Y_{h,j}$ and $Y_{i,k}$ to be conditionally dependent given a vector of random variables Y_A, it must be that the fundamental variable $\mathcal{X}_0$ causes at least $\mathcal{X}_{h,j}$ or $\mathcal{X}_{i,k}$ exclusive of the relevant subsets of $\mathcal{X}_A$. Otherwise, we can express $Y_{h,j}$ or $Y_{i,k}$ (or both) as a function of the relevant sub-vector of Y_A. As Proposition 4.2 has $A \subset \prod_{1:[\max(b_1,b_2)-1]}$, it is necessary that $0 \notin A$. In fact, let $\mathcal{X}_{h,j}$ and $\mathcal{X}_{i,k}$ be as in Proposition 4.2, let $B \subset \prod_{0:[\max(b_1,b_2)-1]}$ with $0 \in B$, let $\mathcal{X}_B$ be the corresponding vector of settable variables, and let $Y_B = \mathcal{X}_B(0, \cdot)$. Then it is straightforward to show that for every probability measure P on $(\Omega, \mathcal{F})$, $Y_{h,j} \perp Y_{i,k} \mid Y_B$.

The contrapositive of Proposition 4.2 gives conditions sufficient for $Y_{h,j} \perp Y_{i,k} \mid Y_A$. In Section 5, we describe how the notions of *d*-separation and *D*-separation provide sufficient conditions for conditional independence, and we discuss how our present results relate to these. For now, we illustrate by applying Proposition 4.2 to system $\mathcal{S}_3$. We have $Y_0 \perp Y_n \mid Y_2$ for $n = 3, \ldots, 8$, as $\mathcal{X}_0 \overset{\sim\{2\}}{\not\Rightarrow}_S \mathcal{X}_n$ for $n = 3, \ldots, 8$. Similarly, $Y_0 \perp Y_9 \mid (Y_1, Y_2)$, as $\mathcal{X}_0 \overset{\sim\{1,2\}}{\not\Rightarrow}_S \mathcal{X}_9$. Also, we have that $Y_2 \perp Y_5 \mid Y_3$, since $\mathcal{X}_0 \overset{\sim\{3\}}{\not\Rightarrow}_S \mathcal{X}_5$. Similarly, we have $Y_2 \perp Y_7 \mid (Y_3, Y_6)$, as $\mathcal{X}_0 \overset{\sim\{3,6\}}{\not\Rightarrow}_S \mathcal{X}_7$. In addition, we have that $Y_3 \perp Y_5 \mid Y_2$, as $\mathcal{X}_0 \overset{\sim\{2\}}{\not\Rightarrow}_S \mathcal{X}_3$ and $\mathcal{X}_0 \overset{\sim\{2\}}{\not\Rightarrow}_S \mathcal{X}_5$. Observe, however, that in G_3 we have that $\mathcal{X}_3$ and $\mathcal{X}_5$ are not *d*-separated given $\mathcal{X}_2$, although they are *D*-separated given $\mathcal{X}_2$. Similar arguments establish that $Y_3 \perp Y_4 \mid Y_2$, even though in G_3 we have that $\mathcal{X}_3$ and $\mathcal{X}_4$ are not *d*-separated given $\mathcal{X}_2$.

None of the cases we have just considered require knowledge of the response functions. This knowledge may, however, be important. To illustrate, consider determining whether $Y_2 \perp Y_7 \mid Y_3$. From the contrapositive of Proposition 4.2 we know that this will hold if $\mathcal{X}_0 \overset{\sim\{3\}}{\not\Rightarrow}_S \mathcal{X}_7$. However, determining whether $\mathcal{X}_0 \overset{\sim\{3\}}{\not\Rightarrow}_S \mathcal{X}_7$ unavoidably requires knowledge of the functional forms of the response functions involved. Because the path $\{\mathcal{X}_2, \mathcal{X}_6, \mathcal{X}_7\}$ does not contain $\mathcal{X}_3$, we have that $\mathcal{X}_2$ and $\mathcal{X}_7$ are neither *d*-separated nor *D*-separated given $\mathcal{X}_3$

in G_3. To conclude that $Y_2 \perp Y_7 \mid Y_3$ in such situations, Pearl (2000, p. 48-49) and Spirtes et al (1993, p. 35, 56) introduce the assumptions of "stability" or "faithfulness" of P. Proposition 4.2 does not impose such restrictions on P; instead the properties of the response functions play the key role.

Parallel to Definition 3.3, we now provide definitions useful in distinguishing between situations in which causal restrictions among settable variables ensure that their responses are conditionally independent for any probability measure and those where conditional independence among random variables is due to a particular choice of P or a particular configuration of the response functions.

Definition 4.3: Conditional Causal Isolation and Conditional Stochastic Isolation Let $\mathcal{S}$, $\mathcal{X}_{h,j}$, $\mathcal{X}_{i,k}$, $\mathcal{X}_A$, A_1, and A_2 be as in Proposition 4.2. Suppose that (a) $(h, j) = 0$ and $\mathcal{X}_{h,j} \overset{\sim A_2}{\not\Rightarrow}_S \mathcal{X}_{i,k}$; or (b) $(i, k) = 0$ and $\mathcal{X}_{i,k} \overset{\sim A_1}{\not\Rightarrow}_S \mathcal{X}_{h,j}$; or (c) $(h, j) \neq 0$, $(i, k) \neq 0$, and $\mathcal{X}_0 \overset{\sim A_1}{\not\Rightarrow}_S \mathcal{X}_{h,j}$ or $\mathcal{X}_0 \overset{\sim A_2}{\not\Rightarrow}_S \mathcal{X}_{i,k}$, then $\mathcal{X}_{h,j}$ and $\mathcal{X}_{i,k}$ are *causally isolated given* $\mathcal{X}_A$ (or $\mathcal{X}_A$*-conditionally causally isolated*). Put $Y_{h,j} \equiv \mathcal{X}_{h,j}(0, \cdot)$, $Y_{i,k} \equiv \mathcal{X}_{i,k}(0, \cdot)$, and $Y_A \equiv \mathcal{X}_A(0, \cdot)$. If P is a probability measure on $(\Omega, \mathcal{F})$ such that $Y_{h,j} \perp Y_{i,k} \mid Y_A$ when $\mathcal{X}_{h,j}$ and $\mathcal{X}_{i,k}$ are not causally isolated given $\mathcal{X}_A$, then we say that P *stochastically isolates* $\mathcal{X}_{h,j}$ *and* $\mathcal{X}_{i,k}$ *given* $\mathcal{X}_A$ (or P *stochastically* $\mathcal{X}_A$*-conditionally isolates* $\mathcal{X}_{h,j}$ *and* $\mathcal{X}_{i,k}$). ■

From Definition 4.3, we have that $\mathcal{X}_{h,j}$ and $\mathcal{X}_{i,k}$ are causally isolated given $\mathcal{X}_A$ when the conclusion of Proposition 4.2 does not hold, that is when $\mathcal{X}_0 \overset{\sim A_1}{\not\Rightarrow}_S \mathcal{X}_{h,j}$ or $\mathcal{X}_0 \overset{\sim A_2}{\not\Rightarrow}_S \mathcal{X}_{i,k}$. Just as in the unconditional case, stochastic isolation given $\mathcal{X}_A$ is a nontrivial restriction on P. However, for sufficiently well behaved probability measures, there always exists such a stochastically isolating P. In particular, if given Y_A, the conditional probabilities of $Y_{h,j}$ and $Y_{i,k}$ are *regular* (see e.g. Dudley, 2002, p. 268-270), then one can always construct a probability measure ensuring that $Y_{h,j}$ and $Y_{i,k}$ are conditionally independent given Y_A. The next result formally establishes this.

Proposition 4.4: Let $\mathcal{S} = \{(\Omega, \mathcal{F}), (Z_0, \Pi, r^\Pi, \mathcal{X})\}$ be canonical. Let P be a probability measure on $(\Omega, \mathcal{F})$ such that given $Y_A \equiv \mathcal{X}_A(0, \cdot)$, the responses $Y_{h,j} \equiv \mathcal{X}_{h,j}(0, \cdot)$ and $Y_{i,k} \equiv \mathcal{X}_{i,k}(0, \cdot)$ have regular conditional probability distributions $P_{h,j|A}$ and $P_{i,k|A}$, respectively. Then there exists a probability measure P^* on $(\Omega, \mathcal{F})$ such that given Y_A, the conditional probability distributions of $Y_{h,j}$ and $Y_{i,k}$, $P^*_{h,j|A}$ and $P^*_{i,k|A}$, respectively, satisfy $P^*_{h,j|A} = P_{h,j|A}$, $P^*_{i,k|A} = P_{i,k|A}$, and the conditional joint probability distribution of $Y_{h,j}$ and $Y_{i,k}$ is $P^*_{h,j|A}\, P^*_{i,k|A}$. In particular, if $\mathcal{X}_{h,j}$ and $\mathcal{X}_{i,k}$ are not causally isolated given $\mathcal{X}_A$, then P^* stochastically isolates $\mathcal{X}_{h,j}$ and $\mathcal{X}_{i,k}$ given $\mathcal{X}_A$. ■

Analogous to Corollary 3.6, our next result provides necessary and sufficient conditions for conditional dependence among random variables $Y_{h,j}$ and $Y_{i,k}$ in canonical systems given a vector of random variables Y_A corresponding to a subset $\mathcal{X}_A$ of the ($\mathcal{X}_{h,j}$, $\mathcal{X}_{i,k}$) intercessors.

Corollary 4.5: Conditional Reichenbach Principle of Common Cause (II) Suppose the conditions of Proposition 4.2 hold. For every probability measure P on $(\Omega, \mathcal{F})$, $Y_{h,j} \perp Y_{i,k} \mid Y_A$ if and only if (a) either

(i) $(h, j) = 0$ and with $A_2 \equiv A \cap ind(\mathcal{I}^{d(\mathcal{S})}_{0:(i,k)})$, $\mathcal{X}_{h,j} \overset{\sim A_2}{\Rightarrow}_S \mathcal{X}_{i,k}$; or

(ii) $(i, k) = 0$ and with $A_1 \equiv A \cap ind(\mathcal{I}^{d(\mathcal{S})}_{0:(h,j)})$, $\mathcal{X}_{i,k} \overset{\sim A_1}{\Rightarrow}_S \mathcal{X}_{h,j}$; or

(iii) $(h, j) \neq 0$, $(i, k) \neq 0$, and $\mathcal{X}_0 \overset{\sim A_2}{\Rightarrow}_S \mathcal{X}_{i,k}$ and $\mathcal{X}_0 \overset{\sim A_1}{\Rightarrow}_S \mathcal{X}_{h,j}$;

and (b) P does not stochastically isolate $\mathcal{X}_{h,j}$ and $\mathcal{X}_{i,k}$ given $\mathcal{X}_A$. ■

Corollary 4.5 tells us that, for all probability measures, if two settable variables neither cause each other exclusive of the relevant elements of $\mathcal{X}_A$ nor have a common cause exclusive of the relevant elements of $\mathcal{X}_A$, then their responses are conditionally independent given Y_A. In fact, Lemma 4.1 implies that when two non-degenerate random variables do not share a common cause exclusive of the relevant subsets of $\mathcal{X}_A$, then at least one of them must be a function only of the relevant subs-vector of Y_A. In addition, Corollary 4.5 tells us that conditional independence can also occur when $\mathcal{X}_{h,j}$ and $\mathcal{X}_{i,k}$ are not causally isolated given $\mathcal{X}_A$, provided P stochastically isolates $\mathcal{X}_{h,j}$ and $\mathcal{X}_{i,k}$ given $\mathcal{X}_A$. For example, in system $\mathcal{S}_3$ suppose that P stochastically isolates $\mathcal{X}_1$ and $\mathcal{X}_4$ given $\mathcal{X}_3$; then we have that $Y_1 \perp Y_4 \mid Y_3$ even though $\mathcal{X}_1$ and $\mathcal{X}_4$ are neither *d*-separated nor *D*-separated given $\mathcal{X}_3$ in G_3.

In applications, we are usually interested in conditional independence relations between vectors of variables. Accordingly, we now generalize the results of this section to accommodate vectors of variables.

First, we note that the meaning of the notations $\overset{d}{\Rightarrow}_S$ and $\overset{d}{\not\Rightarrow}_S$ extends to accommodate disjoint sets of multiple settable variables appearing on the right and left hand sides. For example, if A and B are non-empty collections of indexes, we let $\mathcal{X}_A$ be a vector of settable variables whose indexes belong to A and similarly for $\mathcal{X}_B$, and we write $\mathcal{X}_A \overset{d}{\Rightarrow}_S$

$\mathcal{X}_B$ if $\mathcal{X}_{h,j} \overset{d}{\Rightarrow}_S \mathcal{X}_{i,k}$ for some $(h, j) \in A$ and $(i, k) \in B$. Otherwise, we write $\mathcal{X}_A \overset{d}{\not\Rightarrow}_S \mathcal{X}_B$ indicating that $\mathcal{X}_{h,j} \overset{d}{\not\Rightarrow}_S \mathcal{X}_{i,k}$ for all $(h, j) \in A$ and $(i, k) \in B$. Observe that even though $\mathcal{S}$ is a recursive system, it is possible to have $\mathcal{X}_A \overset{d}{\Rightarrow}_S \mathcal{X}_B$ and $\mathcal{X}_B \overset{d}{\Rightarrow}_S \mathcal{X}_A$.

Similarly, we can extend the meaning of the notations $\overset{i[A]}{\Rightarrow}_S$, $\overset{i[A]}{\not\Rightarrow}_S$, $\overset{i[\sim A]}{\Rightarrow}_S$, $\overset{i[\sim A]}{\not\Rightarrow}_S$, $\overset{A}{\Rightarrow}_S$, $\overset{A}{\not\Rightarrow}_S$, $\overset{\sim A}{\Rightarrow}_S$, and $\overset{\sim A}{\not\Rightarrow}_S$ to accommodate disjoint sets of multiple settable variables appearing on the right and left hand sides. To do this requires some further notation. Let $\mathcal{I}_{A:B} = \bigcup_{(h,j)\in A} \bigcup_{(i,k)\in B} \mathcal{I}_{(h,j):(i,k)} \setminus (\mathcal{X}_A \cup \mathcal{X}_B)$ denote the set of $(\mathcal{X}_A, \mathcal{X}_B)$ $d(\mathcal{S})$-intercessors and let $C \subset ind(\mathcal{I}_{A:B})$. For given $(h, j) \in A$ and $(i, k) \in B$, let $C_{(h,j):(i,k)} = C \cap ind(\mathcal{I}_{(h,j):(i,k)})$. Then, we say that $\mathcal{X}_A \overset{C}{\Rightarrow}_S \mathcal{X}_B$ if there exists $(h, j) \in A \cap \Pi_{b_1}$ and $(i, k) \in B \cap \Pi_{b_2}$ with $b_1 < b_2$, such that $\mathcal{X}_{h,j} \overset{C_{(h,j)(i,k)}}{\Rightarrow}_S \mathcal{X}_{i,k}$. Otherwise, we write $\mathcal{X}_A \overset{C}{\not\Rightarrow}_S \mathcal{X}_B$, indicating that $\mathcal{X}_{h,j} \overset{C_{(h,j)(i,k)}}{\not\Rightarrow}_S \mathcal{X}_{i,k}$ for all $(h, j) \in A \cap \Pi_{b_1}$ and $(i, k) \in B \cap \Pi_{b_2}$ with $b_1 < b_2$. The notations other than $\overset{A}{\Rightarrow}_S$ and $\overset{A}{\not\Rightarrow}_S$ in the list above are defined analogously for vectors of variables.

The definitions of conditional causal isolation and conditional stochastic isolation directly generalize to the vector case in the obvious way. Thus, if $Y_A \perp Y_B \mid Y_C$ when $\mathcal{X}_A$ and $\mathcal{X}_B$ are not causally isolated given $\mathcal{X}_C$ (that is conditions (i), (ii), or (iii) of (a) in Theorem 4.6 below hold) then we say that *P stochastically isolates $\mathcal{X}_A$ and $\mathcal{X}_B$ given $\mathcal{X}_C$.*

We introduce one last notation. Let $\underline{a} = \min\{b$: there exists $(h, j) \in \Pi_b \cap A\}$ and $\underline{b} = \min\{b$: there exists $(i, k) \in \Pi_b \cap B\}$. Similarly, Let $\bar{a} = \max\{b$: there exists $(h, j) \in \Pi_b \cap A\}$ and $\bar{b} = \max\{b$: there exists $(i, k) \in \Pi_b \cap B\}$.

Theorem 4.6: Conditional Reichenbach Principle of Common Cause (III) Let $\mathcal{S} \equiv \{(\Omega, \mathcal{F}), (Z, \Pi, r^{\Pi}, \mathcal{X})\}$ be canonical. Let A and B be non-empty disjoint subsets of $\Pi \cup \Pi_0$, let $\mathcal{X}_A$ and $\mathcal{X}_B$ be the corresponding vectors of settable variables, and let $Y_A = \mathcal{X}_A(0, \cdot)$ and $Y_B = \mathcal{X}_B(0, \cdot)$. Let $C \subset \prod_{1:[\max(\bar{a},\bar{b})-1]} \setminus (A \cup B)$, let $\mathcal{X}_C$ be the corresponding vector of settable variables, and let $Y_C = \mathcal{X}_C(0, \cdot)$. For every probability measure P on $(\Omega, \mathcal{F})$, $Y_A \not\perp Y_B \mid Y_C$ if and only if (a) either:

(i) $\underline{a} = 0$, $\underline{b} \neq 0$, and with $C_2 \equiv C \cap ind(\mathcal{I}_{\{0\}:B})$, $\mathcal{X}_0 \overset{\sim C_2}{\Rightarrow}_S \mathcal{X}_B$; or

(ii) $\underline{a} \neq 0$, $\underline{b} = 0$, and with $C_1 \equiv C \cap ind(\mathcal{I}_{\{0\}:A})$, $\mathcal{X}_0 \overset{\sim C_1}{\Rightarrow}_S \mathcal{X}_A$; or

(iii) $\underline{a} \neq 0$, $\underline{b} \neq 0$, and with $C_1 \equiv C \cap ind(\mathcal{I}_{\{0\}:A})$ and $C_2 \equiv C \cap ind(\mathcal{I}_{\{0\}:B})$, $\mathcal{X}_0 \overset{\sim C_1}{\Rightarrow}_S \mathcal{X}_A$ and $\mathcal{X}_0 \overset{\sim C_2}{\Rightarrow}_S \mathcal{X}_B$;

and (b) P does not stochastically isolate $\mathcal{X}_A$ and $\mathcal{X}_B$ given $\mathcal{X}_C$. ■

Corollary 4.5 obtains as a special case of Theorem 4.6 by taking $A = \{(h, j)\}$ and $B = \{(i, k)\}$. Corollary 3.5 follows from Theorem 4.6 by further taking $C = \varnothing$.

5. Settable Systems and Graphical Separation

The results of the previous section pertain directly to the artificial intelligence literature that aims at encoding conditional independence relationships by graphical means. In particular, as we now discuss, Proposition 4.2 and Corollary 4.5 and Theorem 4.6 relate directly to notions of *d*-separation (Verma and Pearl, 1988; Geiger and Pearl, 1993; Pearl, 2000, p. 16-17), *D*-separation (Geiger, Verma, and Pearl, 1990), and of "separation" in "moral graphs" (Lauritzen and Spiegelhalter, 1988; Lauritzen, Dawid, Larsen, and Leimer, 1990).

5.1 Graphical Separation

For finite systems, Geiger, et. al. (1990) provide a set of graphical criteria for DAGs that they refer to as *d*-separation that can identify exactly the conditional independence relations that can be derived from a "recursive basis[2]" under a set of axioms they refer to as the "graphoid axioms[3]." Lauritzen, et. al. (1990, proposition 3) provide an equivalent graphical algorithm applicable to the "moral" graph of the smallest relevant "ancestral" sets in the original DAG.

---

[2] To express the Geiger, et. al. (1990) notion of a recursive basis in our framework, let H and J_h be finite integers, and let $\mathcal{P}_{(h,j)}$, $h = 1, \ldots, H$; $j = 1, \ldots, J_h$, be the set of $\mathcal{X}_{h,j}$ $d(\mathcal{S})$-predecessors. A recursive basis is then a list of conditional independence statements, one for every (h, j), of the form $Y_{h,j} \perp Y_{\mathcal{P}^{d(\mathcal{S})}_{(h,j)} \backslash B_{(h,j)}} \mid Y_{B_{h,j}}$ for some $B_{(h,j)} \subset \mathcal{P}_{(h,j)}$.

[3] In Geiger, et. al. (1990), the four graphoid axioms are properties of conditional independence relations discussed, for example, in Dawid (1979). Geiger, et. al. (1990) list, but do not use, a fifth axiom they refer to as the "intersection" axiom, which does not hold for all probability measures. A sufficient but not necessary condition under which the intersection axiom holds is that the probability measure P is strictly positive in the sense that for any event A in $\mathcal{F}$, we have $P(A) = 0$ only if $A = \varnothing$ (Verma and Pearl 1988; Geiger and Pearl 1993; Pearl 2000). San Martin, Mouchart, and Rolin (2005, theorem 2.2) employ the concept of "no common information" or "measurable separability" to provide a weaker condition under which the intersection axiom holds.

A key assumption in the *d*-separation literature is that the governing probability measure *P* obeys a "directed local Markov property" (Lauritzen, et. al., 1990). This is the assumption that conditioning on the direct causes of a variable (its "parents") renders it conditionally independent from its remaining predecessors. Pearl (2000, theorem 1.2.7) calls this the "parental Markov condition" and Spirtes, Glymour, and Scheines (1993, p. 54) refer to it as the "Causal Markov Property."

Lauritzen, et. al. (1990) further show that this property is equivalent to two other properties that they refer to as the "directed global Markov property" and the "well-numbering Markov property" (Lauritzen, et. al. 1990, corollary 2). When *P* is absolutely continuous with respect to a product measure, the directed local Markov property is also equivalent to *P* admitting a particular recursive factorization (Lauritzen, et. al., 1990, theorem 1). Under these equivalent Markov assumptions, their graphical algorithm or equivalently *d*-separation permits verification of valid conditional independence relations from the graph (Lauritzen, et. al., 1990, proposition 2).

The literature on causality and conditional independence (e.g., Spohn, 1980; Lauritzen and Spiegelhalter, 1988), particularly as expressed in terms of the *d*-separation criteria (e.g., Pearl, 2000, p. 16-17), embodies three particular relations between causality and conditional independence. Absent other causal relationships and expressed in the present notation and nomenclature, these can be stated as:

d.1 Conditioning on variables that fully mediate the effect of $\mathcal{X}_{h,j}$ on $\mathcal{X}_{i,k}$ renders $Y_{h,j}$ and $Y_{i,k}$ conditionally independent;

d.2 Conditioning on common causes of $\mathcal{X}_{h,j}$ and $\mathcal{X}_{i,k}$, or conditioning on variables that fully mediate the effects of these common causes on either (or both) $\mathcal{X}_{h,j}$ or $\mathcal{X}_{i,k}$, renders $Y_{h,j}$ and $Y_{i,k}$ conditionally independent;

d.3 Conditioning on the response of a settable variable caused by both $\mathcal{X}_{h,j}$ and $\mathcal{X}_{i,k}$ (a "common response") renders $Y_{h,j}$ and $Y_{i,k}$ as well as other respective successors of $\mathcal{X}_{h,j}$ and $\mathcal{X}_{i,k}$ conditionally dependent.

D-separation adds to these the following property:

D.4 Conditioning on causes that entirely determine $\mathcal{X}_{h,j}$ renders $Y_{h,j}$ conditionally independent of all other variables in the system.

D-separation accommodates the presence of "deterministic nodes" or variables that are functionally dependent on their predecessors. Because the fundamental variable is not represented in deterministic DAGs, random behavior is at best not explicitly treated. In the absence of random behavior, a formal role for probability and conditional probability is lacking. To avoid this difficulty, Pearl (2000, p. 68) analyzes "chance-node" DAGs, in which every node is a function of its parents and a non-degenerate unobserved random variable. These "arbitrary distributed random disturbances" are assumed to be "jointly independent" and are not explicitly represented in the DAG (see Pearl, 2000, pp. 68 - 69). Pearl (2000, p. 68) states that, "these disturbances represent independent background factors that the investigator chooses not to include in the analysis." In our terminology, this amounts to assuming that the probability measure P jointly stochastically isolates these random disturbances. This is indeed a strong assumption. Further, it is one that does not emerge naturally from the system of interest; rather, it seems artificially annexed.

Geiger, et. al. (1990, p. 526) state explicitly that "no functional dependencies exist in this [chance-node] interpretation of DAGs." It follows that d-separation and D-separation coincide in chance-node DAGs, since none of the graph's nodes are fully determined by its parents. This also creates a serious conceptual obstacle: applying a definition of causality based on functional dependence to chance-node DAGs is inherently contradictory (see Pearl, 2000, pp. 26, 68). Although Lauritzen, et. al. (1988) refer to "causal networks" throughout, they avoid a causal interpretation based on functional dependence, saying that "causality has a broad interpretation as any natural ordering in which knowledge of a parent influences opinion concerning a child – this influence could be logical, physical, temporal, or simply conceptual in that it may be most appropriate to think of the probability of children given parents" (Lauritzen, et. al., 1988, p. 160). Indeed, there is no reference whatsoever to causality in Lauritzen, et. al. (1990); the discussion there is solely concerned with properties of conditional dependence and its graphical representation. Significantly, Dawid (2002, p. 165) states that, "if we were to base causal understanding on the structure of graphical representations, these must be supplied with a richer semantics, already embodying causal properties." Thus, chance-node DAGs do not support an analysis of causality based on functional dependence of the sort considered here. Nor does there appear to be a solid basis for using the notions of d-separation (or D-separation) to obtain insight into the relations between conditional probability and causal concepts based on functional relationships.

How, then, is one to interpret the statements of properties $d.1$ – $d.3$ and $D.4$ above, where concepts of causality and conditional probability are explicitly intermixed? The answer is that these statements are fully meaningful within the context of canonical settable systems. The compatibility of the settable system framework with classical probability theory and the above function-based definitions of direct and indirect causality enable us to avoid the difficulties encountered by chance-node DAGs. As discussed extensively in WC, causality is properly defined between settable variables, rather than in terms of random variables or events. Consequently, distinctions between "deterministic" and "chance" nodes are no longer relevant, eliminating the difficulties for causal discourse that these distinctions create. The canonical settable system framework supplies the "richer semantics, already embodying causal properties" for graphical representation required by Dawid (2002).

Given that $d.1$– $d.3$ and $D.4$ are meaningful statements within the framework of canonical settable systems, it is possible to investigate their validity. For this, we first note that our results in Section 4 do not impose any "Markov" assumptions on the probability measure P. Instead, the directed local Markov property is a *consequence* of our framework.

Specifically, the inclusion of the fundamental variable $\mathcal{X}_0$ and Proposition 4.2 implies that this property always holds for canonical systems. For example, in system $\mathcal{S}_3$ we have that $Y_3 \perp Y_0 \mid Y_2$, $Y_4 \perp Y_0 \mid (Y_2, Y_3)$, etc.

Further, and despite the fact that we operate only with the usual properties of conditional probability (e.g., Dawid, 1979), Proposition 4.2 (and Corollary 4.5 and Theorem 4.6) deliver *d*.1, *d*.2, and *D*.4 as necessary consequences of settable systems. That is, for canonical systems, the conditional Reichenbach principle of common cause holds, *ensuring* these three properties. Thus, not only are *d*.1, *d*.2, and *D*.4 fully meaningful statements in the settable system framework, but Proposition 4.2 also demonstrates that recursive bases, the intersection axiom, and the Markov properties are not fundamental to establishing these interrelations between causal relations and conditional independence. This by no means suggests that the notions of *d*-separation are inappropriate for the study of conditional probability relationships; indeed they are. Rather, they are not a natural starting point or context for a study of the interrelations between functionally defined causality and conditional probability.

Thus, the *d*-separation literature provides sufficient conditions for inferring conditional independence from graphical structures, but does not relate this to causal notions based on functional dependence. Among other things, the *d*-separation conditions apply to chance-node DAGs with a recursive basis, satisfying certain Markov properties. In contrast, our conditional Reichenbach principle provides necessary and sufficient conditions for inferring conditional independence from causal structures where causality is defined in terms of functional dependence. We do not require chance nodes or recursive bases. In addition, the Markov properties emerge as a consequence of our structure, rather than having to be imposed. Other conditions imposed in the *d*-separation framework are either consequences of our set-up or are held in common, such as the absence of conditioning on successors.

5.2 Conditioning on Successors

Above, we have explicitly discussed *d*.1, *d*.2, and *D*.4, but what about property *d*.3? It, too, is formally meaningful in the canonical settable system framework; but is it a consequence of the properties of a canonical recursive settable system in the same way that *d*.1, *d*.2, and *D*.4 are? We begin to address this question by noting that unlike the latter properties, which all involve the conditional independence of *successors* conditioning on predecessors, *d*.3 is a statement about the (lack of) conditional independence of *predecessors*, conditioning on a successor. Thus, although *d*.3 may indeed be a valid property of canonical systems under given conditions, it will not follow from the conditional Reichenbach principle of common cause.

A general analysis of *d*.3 is sufficiently involved that a full treatment is beyond the scope of the present treatment. First, we demonstrate in Proposition 5.1 by counterexample that *d*.3 does not hold for all probability measures.

Proposition 5.1: Let $(Z_1, Z_2)' \sim N(0, \Sigma_0)$, where

$$\Sigma_0 \equiv \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix},$$

and for some $b \neq 0$, let $Y_3 \overset{c}{=} Z_1 + b\, Z_2$. Then $(Z_1, Z_2, Y_3)' \sim N(0, \Sigma)$, where

$$\Sigma \equiv \begin{bmatrix} 1 & \rho & 1+b\rho \\ \rho & 1 & \rho+b \\ 1+b\rho & \rho+b & 1+2b\rho+b^2 \end{bmatrix} = \begin{bmatrix} \Sigma_0 & \Sigma_{0,1} \\ \Sigma_{1,0} & \Sigma_1 \end{bmatrix},$$

where $\Sigma_{0,1}' = \Sigma_{1,0} \equiv (1 + b\rho, \rho + b)$ and $\Sigma_1 \equiv (1 + 2b\rho + b^2)$. Further, $Z_1 \perp Z_2 \mid Y_3$ if and only if $|\rho| = 1$. ■

In Proposition 5.1, when $|\rho| = 1$ it follows that $Z_1 \perp Z_2 \mid Y_3$ even though Y_3 is a common response to settings Z_1 and Z_2. In fact, Pearl (2000, p. 48-49) and Spirtes et al (1993, p. 35, 56) introduce the assumptions of "stability" or "faithfulness" of P to rule out such situations, which they view as unlikely. Without doubt, the case $|\rho| = 1$ is a very special one.

Indeed, we can sketch arguments suggesting that *d*.3 may well hold under plausible conditions, using only standard methods of analysis. For this, consider a simple system in which $\mathcal{X}_1$ and $\mathcal{X}_2$ are the sole causes of $\mathcal{X}_3$, so that

$$Y_3 = r_3(Y_1, Y_2).$$

We seek conditions ensuring that $Y_1 \not\perp Y_2 \mid Y_3$. First, we rule out the possibility that one or both of Y_1 and Y_2 are constant under P, as in this case we have that Y_1 and Y_2 are independent, both conditionally and unconditionally. Further, if Y_1 and Y_2 are properly behaved, we can take their marginal distributions to be standard uniform, $U[0,1]$, as we can write

$$Y_3 = r_3(Y_1, Y_2) = r_3(F_1^{-1}[F_1(Y_1)], F_2^{-1}[F_2(Y_2)])$$

$$\equiv \tilde{r}_3(F_1(Y_1), F_2(Y_2)) = \tilde{r}_3(\tilde{Y}_1, \tilde{Y}_2),$$

where F_1 and F_2 are the marginal cumulative distribution functions (CDFs) of Y_1 and Y_2, respectively. By construction, $\tilde{Y}_1 \equiv F_1(Y_1)$, and $\tilde{Y}_2 \equiv F_2(Y_2)$ each have the distribution $U[0,1]$. In what follows, we revert to writing Y_1 and Y_2.

It will suffice for *d*.3 to show that there exist y_1, y_2, and a set A_3 (possibly depending on y_1, y_2) with $P[Y_3 \in A_3] > 0$ such that

$$P[Y_1 < y_1, Y_2 < y_2 \mid Y_3 \in A_3] \neq P[Y_1 < y_1 \mid Y_3 \in A_3]\, P[Y_2 < y_2 \mid Y_3 \in A_3].$$

It follows easily that it will suffice for this to find y_1, y_2, and A_3 with $P[Y_3 \in A_3] > 0$ such that

$$P[Y_1 < y_1, Y_2 < y_2, Y_3 \in A_3] = 0$$

$$P[Y_1 < y_1, Y_3 \in A_3] > 0$$

$$P[Y_2 < y_2, Y_3 \in A_3] > 0.$$

Consider the first quantity. If P is suitably regular (see Dudley, 2002, ch.10), we have

$$P[Y_1 < y_1, Y_2 < y_2, Y_3 \in A_3] = \int P[Y_1 < y_1, Y_2 < y_2 | Y_3 = y_3]\, 1\{ y_3 \in A_3\}\, dF_3(y_3),$$

where dF_3 is the density of Y_3. For any y_3 we have

$$P[Y_1 < y_1, Y_2 < y_2 \mid Y_3 = y_3] = P[Y_1 < y_1, Y_2 < y_2 \mid r_3(Y_1, Y_2) = y_3]$$

$$= P[Y_1 < y_1, Y_2 < y_2 | Y_2 = r_{3,2}^{-1}(Y_1,y_3)],$$

given regularity of r_3 sufficient to invoke the implicit function theorem to ensure the existence of $r_{3,2}^{-1}$. It follows that

$$P[Y_1 < y_1, Y_2 < y_2 | Y_2 = r_{3,2}^{-1}(Y_1,y_3)] = P[Y_1 < y_1, r_{3,2}^{-1}(Y_1,y_3) < y_2],$$

so that

$$P[Y_1 < y_1, Y_2 < y_2, Y_3 \in A_3] = \int P[Y_1 < y_1, r_{3,2}^{-1}(Y_1,y_3) < y_2]\, 1\{ y_3 \in A_3\}\, dF_3(y_3).$$

Similarly, with sufficient regularity of P and r_3, we have for the other quantities that

$$P[Y_1 < y_1, Y_3 \in A_3] = \int P[Y_1 < y_1 \mid Y_3 = y_3]\, 1\{ y_3 \in A_3\}\, dF_3(y_3)$$

$$= \int P[r_{3,1}^{-1}(Y_2, y_3) < y_1]\, 1\{ y_3 \in A_3\}\, dF_3(y_3),$$

and

$$P[Y_2 < y_2, Y_3 \in A_3] = \int P[Y_2 < y_2 | Y_3 = y_3]\, 1\{ y_3 \in A_3\}\, dF_3(y_3)$$

$$= \int P[r_{3,2}^{-1}(Y_1,y_3) < y_2]\, 1\{ y_3 \in A_3\}\, dF_3(y_3).$$

Our result thus holds if for some y_1, y_2, and all y_3 in some A_3 with positive probability we have

$$v_3(y_1, y_2, y_3) \equiv P[Y_1 < y_1, r_{3,2}^{-1}(Y_1, y_3) < y_2] = 0$$

$$v_2(y_1, y_3) \equiv P[\, r_{3,1}^{-1}(y_3, Y_2) < y_1] > 0$$

$$v_1(y_2, y_3) \equiv P[\, r_{3,2}^{-1}(Y_1, y_3) < y_2] > 0.$$

Given the uniform marginal distributions, each of these is the volume (Lebesgue measure) of a real Borel set.

To see how these relations can be ensured, consider the linear case, say $r_3(y_1, y_2) = y_1 + b\, y_2$ for $b > 0$. (Although of course not true generally, linearity may hold approximately locally). Then $r_{3,2}^{-1}(y_1, y_3) = (y_3 - y_1)/b$, and $r_{3,1}^{-1}(y_2, y_3) = y_3 - b\, y_2$, so that

$$v_1(y_2, y_3) = P[(y_3 - Y_1)/b < y_2] = 1 - (y_3 - b\, y_2)$$

$$v_2(y_1, y_3) = P[(y_3 - bY_2) < y_1] = 1 - (y_3 - y_1)/\, b$$

$$v_3(y_1, y_2, y_3) = P[\, Y_1 < y_1, (y_3 - Y_1)/b < y_2] = P[\, Y_1 < y_1, Y_1 > (y_3 - b\, y_2)].$$

It is now straightforward to see how to pick y_1, y_2, and y_3 so that $v_3(y_1, y_2, y_3) = 0$, but $v_1(y_2, y_3) = 1 - (y_3 - b\, y_2) > 0$ and $v_2(y_1, y_3) = 1 - (y_3 - y_1) / b > 0$. For $v_3(y_1, y_2, y_3) = 0$, it suffices that $y_3 > y_1 + b\, y_2$, while $v_1(y_2, y_3) > 0$ if $y_3 < 1 + b\, y_2$, and $v_2(y_1, y_3) > 0$ if $y_3 < y_1 + b$. Taking $y_1 = y_2$ for convenience, we see that any choice of y_1 will deliver A_3, as for all $0 < y_1 < 1$ we have

$$y_1 + b\, y_1 < \min(1 + by_1, y_1 + b).$$

For example, pick $y_1 = y_2 = .5$. Then it suffices to pick $A_3 = (.5 + .5b, \min(.5 + b, 1 + .5b))$. For all y_3 in this set, $v_3(y_1, y_2, y_3) = 0$, $v_1(y_2, y_3) > 0$, and $v_2(y_1, y_3) > 0$. Further, for any $b > 0$, this set has positive Lebesgue measure: $P[Y_3 \in A_3] = \min(.5 + b, 1 + .5b) - (.5 + .5b) > 0$. This establishes the desired result.

This is merely a sketch of how a proof might work. For a general result, we cannot rely on linearity. Nevertheless, for plausible choices of r_3, this may be a sufficiently good approximation locally that a similar argument can work. Also, the implicit function theorem may be stronger than strictly necessary. Nevertheless, this sketch illustrates the basic ideas and provides some reason for thinking that a property such as *d*.3 may hold for the three-variable case under plausible general conditions. We leave a rigorous treatment of a general version of *d*.3 with three or more variables to other work.

6. Conclusion

We study the interrelations between independence or conditional independence and causal relations that hold among variables of interest within the *settable system* framework of White and Chalak (2006). We provide rigorous functional definitions of *direct* and *indirect causality* as well as notions of causality *via* a set of variables and *exclusive of* a set of variables. These definitions add to and extend the definitions provided in SGS (1993) and Pearl (2000, 2001). We provide a proof for the *Reichenbach principle of common cause*, and we introduce and prove its conditional counterpart, the *conditional Reichenbach principle of common cause*. We distinguish between situations in which causal restrictions among settable variables ensure that their responses are (conditionally) independent for any probability measure and those where conditional independence among random variables is due to a particular choice of the probability measure or a particular configuration of the response functions. We introduce concepts of (*conditional*) *causal* and *stochastic isolation* to support these results. We then state necessary and sufficient conditions for (conditional) dependence among certain random vectors in settable systems.

We relate our results to notions of *d*-separation, *D*-separation, and alternative graphical criteria in the artificial intelligence literature. We show thereby that canonical settable systems constitute an appropriate framework for studying the interrelations between functionally defined causal relations and conditional independence, and that recursive bases, the intersection axiom, and the Markov properties are not fundamental to establishing these interrelations. In particular, we see that the "directed local Markov property" always holds in canonical settable systems. Also, we do not impose the assumption of "faithfulness" (SGS, 1993) or "stability" (Pearl, 2000); rather, we permit our conclusions about causality and conditional independence to emerge from the properties of the response functions involved. In this work, we restrict our attention to studying the implications for conditional independence of conditioning on predecessors of the variables of interest. We leave to other work studying the implications for conditional independence of conditioning on successors of variables of interest, as also considered in the *d*-separation literature.

We restrict attention here to recursive systems. An interesting direction for further research is to extend our concepts and results to non-recursive systems (see, e.g., Lauritzen and Richardson, 2002; White and Chalak, 2007). The results of the present work also have direct implications for the structural identification of causal effects in observational studies with use of conditioning instruments and predictive proxies discussed in WC, Chalak and White (2007a, 2007b). Also, our results have direct implications for suggesting and testing for causal models. We leave investigating these relationships to other work.

Our framework also constitutes an appropriate foundation for studying the identification of "path-specific" causal effects (See Pearl, 2001; Avin, Shpitser, and Pearl, 2005). It follows that this framework also provides a solid foundation for studying policy evaluation (see, e.g., Heckman, 2005), where distinctions between direct and indirect effects are focal. It is also of interest to investigate the relationship between our results and notions of Granger and Sims causality (Granger, 1969; Sims, 1972). We leave these studies to other work.

7. Mathematical Appendix

Proof of Proposition 2.9: We prove the contrapositive. Let *A*, $z_{[0:b_1](h,j)}$, $z_{h,j}$, and $z^*_{h,j}$ be as in Definitions 2.7 and 2.8. We have:

$$r^{A,(h,j)}_{i,k}(z_{[0:b_1](h,j)},\ z^*_{h,j},\ y^*_{(h,j):A},\ y^*_{\underline{A}},\ y^*_A,\ y^*_{\bar{A}},\ y^*_{A:(i,k)})$$
$$-r^{A,(h,j)}_{i,k}(z_{[0:b_1](h,j)},\ z_{h,j},\ y^*_{(h,j):A},\ y^*_{\underline{A}},\ y_A,\ y^*_{\bar{A}},\ r^{A,(h,j)}_{A:(i,k)}(z_{[0:b_1](h,j)},\ z^*_{h,j},\ y^*_{(h,j):A},\ y^*_{\underline{A}},\ y_A,\ y^*_{\bar{A}}))$$

$$= r^{A,(h,j)}_{i,k}(z_{[0:b_1](h,j)},\ z^*_{h,j},\ y^*_{(h,j):A},\ y^*_{\underline{A}},\ y^*_A,\ y^*_{\bar{A}},\ y^*_{A:(i,k)})$$
$$-r^{A,(h,j)}_{i,k}(z_{[0:b_1](h,j)},\ z^*_{h,j},\ y^*_{(h,j):A},\ y^*_{\underline{A}},\ y_A,\ y^*_{\bar{A}},\ r^{A,(h,j)}_{A:(i,k)}(z_{[0:b_1](h,j)},\ z^*_{h,j},\ y^*_{(h,j):A},\ y^*_{\underline{A}},\ y_A,\ y^*_{\bar{A}}))$$

$$+ r^{A,(h,j)}_{i,k}(z_{[0:b_1](h,j)},\ z^*_{h,j},\ y^*_{(h,j):A},\ y^*_{\underline{A}},\ y_A,\ y^*_{\bar{A}},\ r^{A,(h,j)}_{A:(i,k)}(z_{[0:b_1](h,j)},\ z^*_{h,j},\ y^*_{(h,j):A},\ y^*_{\underline{A}},\ y_A,\ y^*_{\bar{A}}))$$
$$-r^{A,(h,j)}_{i,k}(z_{[0:b_1](h,j)},\ z_{h,j},\ y^*_{(h,j):A},\ y^*_{\underline{A}},\ y_A,\ y^*_{\bar{A}},\ r^{A,(h,j)}_{A:(i,k)}(z_{[0:b_1](h,j)},\ z^*_{h,j},\ y^*_{(h,j):A},\ y^*_{\underline{A}},\ y_A,\ y^*_{\bar{A}})).$$

Suppose $\mathcal{X}_{h,j} \overset{d}{\not\Rightarrow}_S \mathcal{X}_{i,k}$. Then by Definition 2.3, for all function argument values

$$r^{A,(h,j)}_{i,k}(z_{[0:b_1](h,j)},\ z^*_{h,j},\ y^*_{(h,j):A},\ y^*_{\underline{A}},\ y_A,\ y^*_{\bar{A}},\ r^{A,(h,j)}_{A:(i,k)}(z_{[0:b_1](h,j)},\ z^*_{h,j},\ y^*_{(h,j):A},\ y^*_{\underline{A}},\ y_A,\ y^*_{\bar{A}}))$$
$$-r^{A,(h,j)}_{i,k}(z_{[0:b_1](h,j)},\ z_{h,j},\ y^*_{(h,j):A},\ y^*_{\underline{A}},\ y_A,\ y^*_{\bar{A}},\ r^{A,(h,j)}_{A:(i,k)}(z_{[0:b_1](h,j)},\ z^*_{h,j},\ y^*_{(h,j):A},\ y^*_{\underline{A}},\ y_A,\ y^*_{\bar{A}}))$$
$$= 0.$$

Also suppose $\mathcal{X}_{h,j} \overset{i[A]}{\not\Rightarrow}_S \mathcal{X}_{i,k}$. Then by Definition 2.7(I), for all function argument values

$$r^{A,(h,j)}_{i,k}(z_{[0:b_1](h,j)},\ z^*_{h,j},\ y^*_{(h,j):A},\ y^*_{\underline{A}},\ y^*_A,\ y^*_{\bar{A}},\ y^*_{A:(i,k)})$$
$$-r^{A,(h,j)}_{i,k}(z_{[0:b_1](h,j)},\ z^*_{h,j},\ y^*_{(h,j):A},\ y^*_{\underline{A}},\ y_A,\ y^*_{\bar{A}},\ r^{A,(h,j)}_{A:(i,k)}(z_{[0:b_1](h,j)},\ z^*_{h,j},\ y^*_{(h,j):A},\ y^*_{\underline{A}},\ y_A,\ y^*_{\bar{A}}))$$
$$= 0.$$

It follows that for all function argument values

$$r^{A,(h,j)}_{i,k}(z_{[0:b_1](h,j)},\ z^*_{h,j},\ y^*_{(h,j):A},\ y^*_{\underline{A}},\ y^*_A,\ y^*_{\bar{A}},\ y^*_{A:(i,k)})$$
$$-r^{A,(h,j)}_{i,k}(z_{[0:b_1](h,j)},\ z_{h,j},\ y^*_{(h,j):A},\ y^*_{\underline{A}},\ y_A,\ y^*_{\bar{A}},\ r^{A,(h,j)}_{A:(i,k)}(z_{[0:b_1](h,j)},\ z^*_{h,j},\ y^*_{(h,j):A},\ y^*_{\underline{A}},\ y_A,\ y^*_{\bar{A}}))$$
$$= 0,$$

that is, $\mathcal{X}_{h,j} \overset{A}{\not\Rightarrow}_S \mathcal{X}_{i,k}$. This verifies the contrapositive, so the claimed result follows. ■

Proof of Corollary 2.10: Apply Lemma 2.9 with $A = ind(\mathcal{I}_{(h,j):(i,k)})$. ■

Proof of Proposition 2.11: We prove the contrapositive. Let *A*, $z_{[0:b_1](h,j)}$, $z_{h,j}$, and $z^*_{h,j}$ be as in Definitions 2.7 and 2.8. We have:

$$r_{i,k}^{A,(h,j)}(z_{[0:b_1](h,j)},\, z^*_{h,j},\, y^*_{(h,j):A},\, y^*_{\underline{A}},\, y^*_A,\, y^*_{\bar{A}},\, y^*_{A:(i,k)})$$
$$-r_{i,k}^{A,(h,j)}(z_{[0:b_1](h,j)},\, z_{h,j},\, y_{(h,j):A},\, y_{\underline{A}},\, y^*_A,\, y_{\bar{A}},\, r_{A:(i,k)}^{A,(h,j)}(z_{[0:b_1](h,j)},\, z_{h,j},\, y_{(h,j):A},\, y_{\underline{A}},\, y^*_A,\, y_{\bar{A}}))$$

$$= r_{i,k}^{A,(h,j)}(z_{[0:b_1](h,j)},\, z^*_{h,j},\, y^*_{(h,j):A},\, y^*_{\underline{A}},\, y^*_A,\, y^*_{\bar{A}},\, y^*_{A:(i,k)})$$
$$-r_{i,k}^{A,(h,j)}(z_{[0:b_1](h,j)},\, z^*_{h,j},\, y_{(h,j):A},\, y_{\underline{A}},\, y^*_A,\, y_{\bar{A}},\, r_{A:(i,k)}^{A,(h,j)}(z_{[0:b_1](h,j)},\, z_{h,j},\, y_{(h,j):A},\, y_{\underline{A}},\, y^*_A,\, y_{\bar{A}}))$$

$$+ r_{i,k}^{A,(h,j)}(z_{[0:b_1](h,j)},\, z^*_{h,j},\, y_{(h,j):A},\, y_{\underline{A}},\, y^*_A,\, y_{\bar{A}},\, r_{A:(i,k)}^{A,(h,j)}(z_{[0:b_1](h,j)},\, z_{h,j},\, y_{(h,j):A},\, y_{\underline{A}},\, y^*_A,\, y_{\bar{A}}))$$
$$-r_{i,k}^{A,(h,j)}(z_{[0:b_1](h,j)},\, z_{h,j},\, y_{(h,j):A},\, y_{\underline{A}},\, y^*_A,\, y_{\bar{A}},\, r_{A:(i,k)}^{A,(h,j)}(z_{[0:b_1](h,j)},\, z_{h,j},\, y_{(h,j):A},\, y_{\underline{A}},\, y^*_A,\, y_{\bar{A}}))$$

Suppose $\mathcal{X}_{h,j} \overset{d}{\not\Rightarrow}_S \mathcal{X}_{i,k}$. Then by Definition 2.3, for all function argument values

$$r_{i,k}^{A,(h,j)}(z_{[0:b_1](h,j)},\, z^*_{h,j},\, y_{(h,j):A},\, y_{\underline{A}},\, y^*_A,\, y_{\bar{A}},\, r_{A:(i,k)}^{A,(h,j)}(z_{[0:b_1](h,j)},\, z_{h,j},\, y_{(h,j):A},\, y_{\underline{A}},\, y^*_A,\, y_{\bar{A}}))$$
$$-r_{i,k}^{A,(h,j)}(z_{[0:b_1](h,j)},\, z_{h,j},\, y_{(h,j):A},\, y_{\underline{A}},\, y^*_A,\, y_{\bar{A}},\, r_{A:(i,k)}^{A,(h,j)}(z_{[0:b_1](h,j)},\, z_{h,j},\, y_{(h,j):A},\, y_{\underline{A}},\, y^*_A,\, y_{\bar{A}}))$$
$$= 0.$$

Also suppose $\mathcal{X}_{h,j} \overset{i[\sim A]}{\Rightarrow}_S \mathcal{X}_{i,k}$. Then by Definition 2.7(II), for all function argument values

$$r_{i,k}^{A,(h,j)}(z_{[0:b_1](h,j)},\, z^*_{h,j},\, y^*_{(h,j):A},\, y^*_{\underline{A}},\, y^*_A,\, y^*_{\bar{A}},\, y^*_{A:(i,k)})$$
$$-r_{i,k}^{A,(h,j)}(z_{[0:b_1](h,j)},\, z^*_{h,j},\, y_{(h,j):A},\, y_{\underline{A}},\, y^*_A,\, y_{\bar{A}},\, r_{A:(i,k)}^{A,(h,j)}(z_{[0:b_1](h,j)},\, z_{h,j},\, y_{(h,j):A},\, y_{\underline{A}},\, y^*_A,\, y_{\bar{A}}))$$
$$= 0;$$

that is, $\mathcal{X}_{h,j} \overset{\sim A}{\not\Rightarrow}_S \mathcal{X}_{i,k}$. This verifies the contrapositive, so the claimed result follows. ■

Proof of Lemma 3.1: First, suppose $\mathcal{X}_0 \not\Rightarrow_S \mathcal{X}_{h,j}$. By Definition 2.8(I), for all z_0 and z_0^* it must hold that

$$r_{h,j}^{A,0}(z_0^*,\, r_A^{A,0}(z_0^*)) - r_{h,j}^{A,0}(z_0,\, r_A^{A,0}(z_0)) = 0,$$

where $A = \text{ind}(\mathcal{I}_{0:(h,j)})$. If so, the response $\mathcal{X}_{h,j}(0, \cdot)$ must be constant. By canonicality, $\mathcal{X}_{h,j}(1, \cdot) = \mathcal{X}_{h,j}(0, \cdot)$ must also be constant.

Next, suppose that $\mathcal{X}_{h,j}$ is constant, but that $\mathcal{X}_0 \not\Rightarrow_S \mathcal{X}_{h,j}$ does not hold, that is $\mathcal{X}_0 \Rightarrow_S \mathcal{X}_{h,j}$. By Definition 2.8(I), there exist z_0 and z_0^*, $z_0 \neq z_0^*$ such that

$$r_{h,j}^{A,0}(z_0^*,\, r_A^{A,0}(z_0^*)) - r_{h,j}^{A,0}(z_0,\, r_A^{A,0}(z_0)) \neq 0.$$

But then $\mathcal{X}_{h,j}(0, \cdot)$ cannot be constant, so the result follows. ■

Proof of Proposition 3.2: Let P be any probability measure. Without loss of generality, for non-negative integers b_1 and b_2, let $(h, j) \in \Pi_{b_1}$ and $(i, k) \in \Pi_{b_2}$. (i) Suppose $b_1 = b_2$. Then necessarily $\mathcal{X}_{h,j} \not\Rightarrow_S \mathcal{X}_{i,k}$ and $\mathcal{X}_{i,k} \not\Rightarrow_S \mathcal{X}_{h,j}$. Suppose that $\mathcal{X}_0 \not\Rightarrow_S \mathcal{X}_{h,j}$ or $\mathcal{X}_0 \not\Rightarrow_S \mathcal{X}_{i,k}$. By Lemma 3.1, it follows that either $Y_{h,j}$ or $Y_{i,k}$ is constant. If so, then $Y_{h,j} \perp Y_{i,k}$, a contradiction. (ii) Suppose $b_1 < b_2$. Then necessarily $\mathcal{X}_{i,k} \not\Rightarrow_S \mathcal{X}_{h,j}$. Next, suppose that neither (a) nor (b) hold. First, if $(h, j) = 0$ then $\mathcal{X}_{h,j} \not\Rightarrow_S \mathcal{X}_{i,k}$. By Lemma 3.1, it follows that $Y_{i,k}$ is constant. If so, then $Y_{h,j} \perp Y_{i,k}$, a contradiction. Second, if $(h, j) \neq 0$, then $\mathcal{X}_0 \not\Rightarrow_S \mathcal{X}_{h,j}$ or $\mathcal{X}_0 \not\Rightarrow_S \mathcal{X}_{i,k}$. Lemma 3.1 entails that either $Y_{h,j}$ or $Y_{i,k}$ is constant. If so, then $Y_{h,j} \perp Y_{i,k}$, a contradiction. (iii) Suppose $b_2 < b_1$. The proof is symmetric to (ii). As P is arbitrary, the result holds for all P. ■

Proof of Proposition 3.4: Immediate from theorem 18.2 of Billingsley (1979). ■

Proof of Corollary 3.5: The result is immediate from Proposition 3.2 and the contrapositive of the definition of stochastic isolation. ■

Proof of Lemma 4.1 First, suppose that $\mathcal{X}_{h,j} \overset{\sim A}{\not\Rightarrow_S} \mathcal{X}_{i,k}$. By definition 2.8 (II) we have that for all function argument values that

$$\begin{aligned}&r_{i,k}^{A,(h,j)}(z_{[0:b_1](h,j)},\ z_{h,j}^*,\ y_{(h,j):A}^*,\ y_{\underline{A}}^*,\ y_A^*,\ y_{\bar{A}}^*,\ y_{A:(i,k)}^*)\\&-r_{i,k}^{A,(h,j)}(z_{[0:b_1](h,j)},\ z_{h,j},\ y_{(h,j):A},\ y_{\underline{A}},\ y_A^*,\ y_{\bar{A}},\ r_{A:(i,k)}^{A,(h,j)}(z_{[0:b_1](h,j)},\ z_{h,j},\ y_{(h,j):A},\ y_{\underline{A}},\ y_A^*,\ y_{\bar{A}}))\\&=0;\end{aligned}$$

Thus the function

$$\begin{aligned}&(z_{[0:b_1](h,j)},\ z_{h,j},\ y_{(h,j):A},\ y_{\underline{A}},\ y_A,\ y_{\bar{A}}) \to\\&\quad r_{i,k}^{A,(h,j)}(z_{[0:b_1](h,j)},\ z_{h,j},\ y_{(h,j):A},\ y_{\underline{A}},\ y_A,\ y_{\bar{A}},\ r_{A:(i,k)}^{A,(h,j)}(z_{[0:b_1](h,j)},\ z_{h,j},\ y_{(h,j):A},\ y_{\underline{A}},\ y_A,\ y_{\bar{A}}))\end{aligned}$$

is constant in $(z_{h,j},\ y_{(h,j):A},\ y_{\underline{A}},\ y_{\bar{A}})$ for every $(z_{[0:b_1](h,j)},\ y_A)$.

Thus, there exits a measurable function $(z_{[0:b_1](h,j)},\ y_A) \to \tilde{r}_{i,k}^{A,(h,j)}(z_{[0:b_1](h,j)},\ y_A)$ such that for every $(z_{[0:b_1](h,j)},\ y_A)$ and all $(z_{h,j},\ y_{(h,j):A},\ y_{\underline{A}},\ y_{\bar{A}})$ we have

$$\tilde{r}_{i,k}^{A,(h,j)}(z_{[0:b_1](h,j)},\ y_A) = r_{i,k}^{A,(h,j)}(z_{[0:b_1](h,j)}, z_{h,j},\ y_{(h,j):A},\ y_{\underline{A}},\ y_A,\ y_{\bar{A}},\ y_{A:(i,k)}).$$

In particular, when $\mathcal{X}_0 \overset{\sim A}{\not\Rightarrow_S} \mathcal{X}_{i,k}$ we have $\tilde{r}_{i,k}^{A,0}(y_A) = r_{i,k}^{A,0}(z_0,\ y_{0:A},\ y_{\underline{A}},\ y_A,\ y_{\bar{A}},\ y_{A:(i,k)})$.

Next, suppose that there exists a measurable function $\tilde{r}_{i,k}^{A,(h,j)}$ such that

$$\tilde{r}_{i,k}^{A,(h,j)}\ (z_{[0:b_1](h,j)},\ y_A) = r_{i,k}^{A,(h,j)}\ (z_{[0:b_1](h,j)}, z_{h,j},\ y_{(h,j):A},\ y_{\underline{A}},\ y_A,\ y_{\bar{A}},\ y_{A:(i,k)}),$$

but that $\mathcal{X}_{h,j} \overset{\sim A}{\Rightarrow}_S \mathcal{X}_{i,k}$ does not hold, that is $\mathcal{X}_{h,j} \overset{\sim A}{\not\Rightarrow}_S \mathcal{X}_{i,k}$. By Definition 2.8(II) there exist (a) $z_{[0:b_1](h,j)}$; and (b) $z_{h,j}$ and $z_{h,j}^*$ with $z_{h,j} \neq z_{h,j}^*$ such that

$$\begin{aligned}&r_{i,k}^{A,(h,j)}(z_{[0:b_1](h,j)},\ z_{h,j}^*,\ y_{(h,j):A}^*,\ y_{\underline{A}}^*,\ y_A^*,\ y_{\bar{A}}^*,\ y_{A:(i,k)}^*)\\ &-\ r_{i,k}^{A,(h,j)}(z_{[0:b_1](h,j)},\ z_{h,j},\ y_{(h,j):A},\ y_{\underline{A}},\ y_A^*,\ y_{\bar{A}},\ r_{A:(i,k)}^{A,(h,j)}(z_{[0:b_1](h,j)},\ z_{h,j},\ y_{(h,j):A},\ y_{\underline{A}},\ y_A^*,\ y_{\bar{A}}))\\ &\neq 0.\end{aligned}$$

But then the function

$$\begin{aligned}&(z_{[0:b_1](h,j)},\ z_{h,j},\ y_{(h,j):A},\ y_{\underline{A}},\ y_A,\ y_{\bar{A}}) \rightarrow\\ &\quad r_{i,k}^{A,(h,j)}(z_{[0:b_1](h,j)},\ z_{h,j},\ y_{(h,j):A},\ y_{\underline{A}},\ y_A,\ y_{\bar{A}},\ r_{A:(i,k)}^{A,(h,j)}(z_{[0:b_1](h,j)},\ z_{h,j},\ y_{(h,j):A},\ y_{\underline{A}},\ y_A,\ y_{\bar{A}}))\end{aligned}$$

is not constant in $(z_{h,j},\ y_{(h,j):A},\ y_{\underline{A}},\ y_{\bar{A}})$ for every $(z_{[0:b_1](h,j)},\ y_A)$, a contradiction.

In particular, when $\mathcal{X}_0 \overset{\sim A}{\not\Rightarrow}_S \mathcal{X}_{i,k}$, the function

$$\begin{aligned}&r_{i,k}^{A,0}\ (z_0,\ y_{0:A},\ y_{\underline{A}},\ y_A,\ y_{\bar{A}},\ y_{A:(i,k)}) \rightarrow\\ &\qquad\qquad r_{i,k}^{A,0}\ (z_0,\ y_{0:A},\ y_{\underline{A}},\ y_A,\ y_{\bar{A}},\ r_{A:(i,k)}^{A,(h,j)}(z_0,\ y_{0:A},\ y_{\underline{A}},\ y_A,\ y_{\bar{A}}))\end{aligned}$$

is not constant in $(z_0,\ y_{0:A},\ y_{\underline{A}},\ y_{\bar{A}})$ for every y_A. This completes the proof. ■

Proof of Proposition 4.2: The proof parallels that of Proposition 3.2. Let P be any probability measure. (i) Suppose $b_1 = b_2$. Then necessarily $\mathcal{X}_{h,j} \not\Rightarrow_S \mathcal{X}_{i,k}$ and $\mathcal{X}_{i,k} \not\Rightarrow_S \mathcal{X}_{h,j}$. Suppose that $\mathcal{X}_0 \overset{\sim A_1}{\not\Rightarrow}_S \mathcal{X}_{h,j}$ or $\mathcal{X}_0 \overset{\sim A_2}{\not\Rightarrow}_S \mathcal{X}_{i,k}$. Let $\mathcal{X}_{A_1}$ be a vector of settable variables and let $Y_{A_1} = \mathcal{X}_{A_1}(0,\cdot)$. If $\mathcal{X}_0 \overset{\sim A_1}{\not\Rightarrow}_S \mathcal{X}_{h,j}$, Lemma 4.1 gives that $Y_{h,j} = \tilde{r}_{h,j}^{A_1,0}(Y_{A_1})$. Let $A_1^c = A \setminus A_1$, let $\mathcal{X}_{A_1^c}$ be the corresponding vector of settable variables, and let $Y_{A_1^c} = \mathcal{X}_{A_1^c}(0,\cdot)$; then $Y_A = (Y_{A_1}, Y_{A_1^c})$. Since $Y_{h,j} = \tilde{r}_{h,j}^{A_1,0}(Y_{A_1})$, we have that $Y_{h,j} \perp (Y_{i,k}, Y_{A_1^c}) \mid Y_{A_1}$. We then have that $Y_{h,j} \perp Y_{i,k} \mid (Y_{A_1}, Y_{A_1^c})$ (see, for example, Dawid, 1979, section 4; Dohler, 1980, lemma 3; Smith 1989, property 3; and Florens, Mouchart, and Rolin 1990, theorem 2.2.10), that is $Y_{h,j} \perp Y_{i,k} \mid Y_A$. (Note that when $A_1 = A$ the result is immediate. Also, when $A_1 = \varnothing$, $\mathcal{X}_{h,j}$ is constant and the result is trivial.) Alternatively, suppose that $\mathcal{X}_0 \overset{\sim A_2}{\not\Rightarrow}_S \mathcal{X}_{i,k}$. Then by a parallel argument, we obtain that $Y_{h,j} \perp Y_{i,k} \mid Y_A$, a contradiction. (ii) Suppose $b_1 < b_2$. Then necessarily $\mathcal{X}_{i,k} \not\Rightarrow_S \mathcal{X}_{h,j}$. Next, suppose that neither (a) nor (b) hold. First, if $(h, j) =$

0 then $\mathcal{X}_{h,j} \overset{\sim A_2}{\not\Rightarrow}_S \mathcal{X}_{i,k}$. By Lemma 4.1, it follows that $Y_{i,k} = \tilde{r}_{i,k}^{A_2,0}(Y_{A_2})$ with $Y_{A_2} = \mathcal{X}_{A_2}(0,\cdot)$. If so, an argument parallel to (i) establishes that $Y_{h,j} \perp Y_{i,k} \mid Y_A$, a contradiction. Second, if $(h, j) \neq 0$ then $\mathcal{X}_0 \overset{\sim A_1}{\not\Rightarrow}_S \mathcal{X}_{h,j}$ or $\mathcal{X}_0 \overset{\sim A_2}{\not\Rightarrow}_S \mathcal{X}_{i,k}$. The proof is symmetric to (i), leading to $Y_{h,j} \perp Y_{i,k} \mid Y_A$, a contradiction. (iii) Suppose $b_2 < b_1$. The proof is symmetric to (ii). As P is arbitrary, the result holds for all P. ■

Proof of Proposition 4.4: The result follows from proposition III.2.1 of Neveu (1965, p. 74-75). ■

Proof of Corollary 4.5: The result is immediate from Proposition 4.2 and the contrapositive of the definition of conditional stochastic isolation. ■

Proof of Theorem 4.6: Let P be any probability measure. First, we prove that if $Y_A \not\perp Y_B \mid Y_C$ then $\mathcal{X}_A$ and $\mathcal{X}_B$ are not causally isolated given $\mathcal{X}_C$. (i) Suppose $\underline{a} = 0$, $\underline{b} \neq 0$, and that $\mathcal{X}_0 \overset{\sim C_2}{\not\Rightarrow}_S \mathcal{X}_B$. Then $\mathcal{X}_0 \overset{\sim C_{0:(i,k)}}{\not\Rightarrow}_S \mathcal{X}_{i,k}$ for all $(i, k) \in B$. For given $(i, k) \in B$, let $\mathcal{X}_{C_{0:(i,k)}}$ be a vector of settable variables and let $Y_{C_{0:(i,k)}} = \mathcal{X}_{C_{0:(i,k)}}(0,\cdot)$. By Lemma 4.1, it follows that $Y_{i,k} = \tilde{r}_{i,k}^{C_{0:(i,k)},0}(Y_{C_{0:(i,k)}})$ for all $(i, k) \in B$. Let $\mathcal{X}_{C_2}$ be a vector of settable variables and let $Y_{C_2} = \mathcal{X}_{C_2}(0,\cdot)$; then we have $Y_B = \tilde{r}_B^{C_2,0}(Y_{C_2})$. Let $C_2^c = C \setminus C_2$, let $\mathcal{X}_{C_2^c}$ be the corresponding vector of settable variables, and let $Y_{C_2^c} = \mathcal{X}_{C_2^c}(0,\cdot)$; then $Y_C = (Y_{C_2}, Y_{C_2^c})$. Since $Y_B = \tilde{r}_B^{C_2,0}(Y_{C_2})$, we have that $(Y_A, Y_{C_2^c}) \perp Y_B \mid Y_{C_2}$. We then have that $Y_A \perp Y_B \mid (Y_{C_2}, Y_{C_2^c})$ (see, for example, Dawid, 1979, section 4; Dohler, 1980, lemma 3; Smith 1989, property 3; and Florens, Mouchart, and Rolin 1990, theorem 2.2.10), that is, $Y_A \perp Y_B \mid Y_C$, a contradiction. (Note that when $C_2 = C$ the result is immediate. Also, when $C_2 = \varnothing$, $\mathcal{X}_B$ is constant and the result is trivial.) (ii) Suppose $\underline{a} \neq 0$, $\underline{b} = 0$, and that $\mathcal{X}_0 \overset{\sim C_1}{\not\Rightarrow}_S \mathcal{X}_A$. The result is symmetric to (i) yielding that $Y_A \perp Y_B \mid Y_C$, a contradiction. (iii) Suppose that $\underline{a} \neq 0$, $\underline{b} \neq 0$, and that $\mathcal{X}_0 \overset{\sim C_1}{\not\Rightarrow}_S \mathcal{X}_A$ or $\mathcal{X}_0 \overset{\sim C_2}{\not\Rightarrow}_S \mathcal{X}_B$. Suppose that $\mathcal{X}_0 \overset{\sim C_1}{\not\Rightarrow}_S \mathcal{X}_A$; then $\mathcal{X}_0 \overset{\sim C_{0:(h,j)}}{\not\Rightarrow}_S \mathcal{X}_{h,j}$ for all $(h, j) \in A$. For given $(h, j) \in A$, let $\mathcal{X}_{C_{0:(h,j)}}$ be a vector of settable variables and let $Y_{C_{0:(h,j)}} = \mathcal{X}_{C_{0:(h,j)}}(0,\cdot)$. From Lemma 4.1 it follows that $Y_{h,j} = \tilde{r}_{h,j}^{C_{0:(h,j)},0}(Y_{C_{0:(h,j)}})$ for all $(h, j) \in A$. Let $\mathcal{X}_{C_1}$ be a vector of settable variables and let $Y_{C_1} = \mathcal{X}_{C_1}(0,\cdot)$, then we have that $Y_A = \tilde{r}_A^{C_1,0}(Y_{C_1})$. Let $C_1^c = C \setminus C_1$, let $\mathcal{X}_{C_1^c}$ be the corresponding vector of settable variables, and let $Y_{C_1^c} = \mathcal{X}_{C_1^c}(0,\cdot)$; then $Y_C = (Y_{C_1}, Y_{C_1^c})$. Since $Y_A = \tilde{r}_A^{C_1,0}(Y_{C_1})$, we have that $Y_A \perp (Y_B, Y_{C_1^c}) \mid Y_{C_1}$. We then have that $Y_A \perp Y_B \mid (Y_{C_1}, Y_{C_1^c})$ (see, e.g., Dawid, 1979, section 4), that is, $Y_A \perp Y_B \mid Y_C$, a contradiction. Alternatively, suppose that $\mathcal{X}_0 \overset{\sim C_2}{\not\Rightarrow}_S \mathcal{X}_B$. Then by a parallel argument, we obtain that $Y_A \perp Y_B \mid Y_C$, a contradiction. That P does not stochastically isolate $\mathcal{X}_A$ and $\mathcal{X}_B$ given $\mathcal{X}_C$ follows by the definition of conditional stochastic isolation. As P is arbitrary, the result holds for all P. The rest of the proof follows from the contrapositive of the definition of conditional stochastic isolation. ■

Proof of Proposition 5.1: The joint normality of $(Z_1, Z_2, Y_3)'$ holds as a standard property of the normal distribution. The elements of the covariance matrix Σ follow by elementary computations. It is also a standard property of the normal distribution (e.g., Hamilton, 1994, p.100) that

$$(Z_1, Z_2)' \mid Y_3 \sim N(0, \Sigma_0 - \Sigma_{1,0} \Sigma_1^{-1} \Sigma_{0,1}).$$

For the normal distribution, $Z_1 \perp Z_2 \mid Y_3$ if and only if $E(Z_1 Z_2 \mid Y_3) = 0$. Computing the off-diagonal element of $\Sigma_0 - \Sigma_{1,0} \Sigma_1^{-1} \Sigma_{0,1}$ corresponding to $E(Z_1 Z_2 \mid Y_3)$, we obtain

$$E(Z_1 Z_2 \mid Y_3) = b(\rho^2 - 1).$$

As $b \neq 0$, this equals zero if and only if $|\rho| = 1$. ■

References

Avin, C., I. Shpitser, and J. Pearl (2005), "Identifiability of Path-Specific Effects," In *Proceedings of International Joint Conference on Artificial Intelligence*, Edinburgh, Scotland, 357-363.

Bang-Jensen, J. and G. Gutin (2001). *Digraphs: Theory, Algorithms and Applications*. London: Springer-Verlag.

Billingsley, P. (1979). *Probability and Measure*. New York: Wiley.

Cartwright, N. (2000). *Measuring Causes: Invariance, Modularity and the Causal Markov Condition*. Monograph, London: Centre for Philosophy of Natural and Social Science.

Chalak, K., and H. White (2007a), "An Extended Class of Instrumental Variables for the Estimation of Causal Effects," UCSD Department of Economics Discussion Paper.

Chalak, K., and H. White (2007b), "Identification with Conditioning Instruments in Causal Systems," UCSD Department of Economics Discussion Paper.

Cochran, W. G. (1934), "The Distribution of Quadratic Forms in a Normal System with Application to the Analysis of Variance," *Proceedings of the Cambridge Philosophical Society*, 30, 178-191.

Dawid, A.P. (1979), "Conditional Independence in Statistical Theory," *Journal of the Royal Statistical Society*, Series B, 41, 1-31 (with discussion).

Dawid, A.P. (1980), "Conditional Independence for Statistical Operation," *The Annals of Statistics*, 8, 598-617.

Dawid, A.P. (2000), "Causal Inference without Counterfactuals," *Journal of the American Statistical Association*, 95, 407-448 (with discussion).

Dawid, A.P. (2002), "Influence Diagrams for Causal Modeling and Inference," *International Statistical Review*, 70, 161-189.

Dudley, R.M. (2002). *Real Analysis and Probability*. New York: Cambridge University Press.

Florens, J.-P., M. Mouchart, and J.-M. Rolin (1990). *Elements of Bayesian Statistics*. New York: Marcel Dekker.

Geiger, D., T. S. Verma, and J. Pearl (1990), "Identifying Independence in Bayesian Networks," *Networks*, 20, 507-534.

Geiger, D. and J. Pearl (1993), "Logical and Algorithmic Properties of Conditional Independence and Graphical Models," *The Annals of Statistics*, 21, 2001-2021.

Golubitsky, M. and I. Stewart (2006), "Nonlinear Dynamics of Networks: the Groupoid Formalism," *Bulletin of the American Mathematical Society*, 43, 305-364.

Granger, C. W. J. (1969), "Investigating Causal Relations by Econometric Models and Cross-Spectral Methods," *Econometrica*, 37, 424-438.

Hamilton, J.D. (1994). *Time Series Analysis*. Princeton: Princeton University Press.

Hausman, D. and J. Woodward (1999), "Independence, Invariance and the Causal Markov Condition," *British Journal for the Philosophy of Science*, 50, 521-583.

Heckman, J. (2005), "The Scientific Model of Causality," *Sociological Methodology*, 35, 1-97.

Holland, P.W. (1986), "Statistics and Causal inference," *Journal of the American Statistical Association*, 81, 945-970 (with discussion).

Lauritzen, S. L., and D. J. Spiegelhalter (1988), "Local Computations with Probabilities on Graphical Structures and their Application to Expert Systems," *Journal of the Royal Statistical Society,* Series B, 50, 157-224 (with discussion).

Lauritzen, S. L., A. P. Dawid, B. N. Larsen, and H.-G. Leimer (1990), "Independence Properties of Directed Markov Fields," *Networks*, 20, 491-505.

Lauritzen, S. L., and T. S. Richardson (2002), "Chain Graph Models and their Causal Interpretation," *Journal of the Royal Statistical Society*, Series B, 64, 321-361 (with discussion).

Neveu, J. (1965). *Mathematical Foundations of the Calculus of Probability*. Translated by Amiel Feinstein. San Francisco: Holden-Day.

Pearl, J. (1988). *Probabilistic Reasoning in Intelligent Systems: Networks of Plausible Inference*. San Mateo, CA: Morgan Kaufman.

Pearl, J. (1993), "Aspects of Graphical Methods Connected with Causality," in *Proceedings of the 49th Session of the International Statistical Institute*, pp. 391-401.

Pearl, J. (1995), "Causal Diagrams for Empirical Research," *Biometrika*, 82, 669-710 (with Discussion).

Pearl, J. (2000). *Causality: Models, Reasoning, and Inference*. New York: Cambridge University Press.

Pearl, J. (2001), "Direct and Indirect Effects," In *Proceedings of the Seventeenth Conference on Uncertainty in Artificial Intelligence*, San Francisco, CA: Morgan Kaufmann, 411-420.

Reichenbach, H. (1956). *The Direction of Time*. Berkeley: University of California Press.

Rosenbaum, P. R. (2002). *Observational Studies*. 2nd ed., Berlin: Springer-Verlag.

Rubin, D. (1974), "Estimating Causal Effects of Treatments in Randomized and Non-randomized Studies," *Journal of Educational Psychology*, 66, 688-701.

San Martin, E., M. Mouchart, and J.-M. Rolin (2005), "Ignorable Common Information, Null Sets and Basu's First Theorem," Discussion Paper, Louvain-la-Neuve: Institut de Statistique.

Sims, C. (1972), "Money, Income, and Causality," *American Economic Review*, 62, 540-52.

Smith, J. Q. (1989), “Influence Diagrams for Statistical Modeling,” *The Annals of Statistics*, 17, 654-672.

Spirtes, P., C. Glymour, and R. Scheines (1993). *Causation, Prediction and Search*. Berlin: Springer-Verlag.

Studeny, M. (1993), “Formal Properties of Conditional Independence in Different Calculi of AI.” In M. Clarke, R. Kruse, S. Moral (eds.), *Symbolic and Quantitative Approaches to Reasoning and Uncertainty*. Berlin: Springer-Verlag, pp. 341-348.

Spohn, W. (1980), “Stochastic Independence, Causal Independence, and Shieldability,” *Journal of Philosophical Logic*, 9, 73-99.

Verma, T. and J. Pearl (1988), “Causal Networks: Semantics and Expressiveness,” in *Proceedings, 4th Workshop on Uncertainty in Artificial Intelligence*, Minneapolis, MN, Mountain View, CA, pp. 352-359.

White, H. and K. Chalak (2007), “Settable Systems: An Extension of Pearl's Causal Model with Optimization, Equilibrium, and Learning,” UCSD Department of Economics Discussion Paper.

White, H. and K. Chalak (2006), “A Unified Framework for Defining and Identifying Causal Effects,” UCSD Department of Economics Discussion Paper.