Convolution Based Copulas with Applications to Econometrics and Finance

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MatemateS
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Reference papers


Motivation
A review of copula functions
Convolution Based Copulas
Building Markov processes by increments aggregation
Application: efficient market dynamics
Application: Managed fund analysis
Copula functions are flexible representations of joint distributions and provide separate specifications of marginal distributions and dependence.

In many applications we are interested in the dependence structure of variables that are linked by specific relationships: assume $X$ and $Y = f(X, \epsilon)$ (a non-linear function of $X$ and a random variable $\epsilon$), and $Z = X + Y$. Which is the distribution of $Z$, and what is the dependence between $X$ and $Z$?
Application I: take $X_t$ the value of a stochastic process in discrete time, and $Y_t = X_{t+1} - X_t$ the increment. Determine the dependence structure of $X_{t+1}$ and $X_t$. Impose efficient market restrictions to $X_t$.

Application II: take $X_t$ the value of the market portfolio (or a benchmark) and $Y_t$ the return on a management strategy of a fund, as a stochastic function of $X_t$. Define $Z_t = X_t + Y_t$ the return on the managed fund. Determine the distribution of the fund and the dependence structure with the market.
COPULA FUNCTIONS: FINANCE

Copula functions in finance are mostly used in a spatial dependence sense


Copula functions used in a temporal dependence sense

Equity: Cherubini - Romagnoli (2010) (DNO Approach)

Credit: Cherubini - Mulinacci - Romagnoli (2008) (Convolution based approach)
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COPULA FUNCTIONS: ECONOMETRICS

Copula functions in econometrics are mostly used in a temporal dependence sense

Definition

A copula is a function $C : [0, 1]^2 \rightarrow [0, 1]$ satisfying the following requirements

1. **Grounded:** $C(0, v) = C(u, 0) = 0$,
2. **Uniform marginals:** $C(u, 1) = u$, $C(1, v) = v$,
3. **2-Increasing:**
   
   $C(u_1, v_1) - C(u_1, v_2) - C(u_2, v_1) + C(u_2, v_2) \geq 0$
   
   for $u_1 > u_2$ and $v_1 > v_2$. 

S. Mulinacci
Convolution Based Copulas with Applications
Darsow et al. (1992) showed that first order Markov processes $X_t$ are characterized by the following relation

$$C_{X_j, \ldots, X_n} = C_{X_{j_1}, X_{j_2}} \ast C_{X_{j_2}, X_{j_3}} \ast \cdots \ast C_{X_{j_{n-1}}, X_j}$$

where $C_{X_{j_1}, X_{j_2}, \ldots, X_n}$ is the copula associated to the vector $(X_{j_1}, X_{j_2}, \ldots, X_n)$ and $C_{X_{j_k}, X_{j_{k+1}}}$ stands for the copula function linking $X_{j_k}$ and $X_{j_{k+1}}$ and the $\ast$-product operator is defined as

$$A \ast B(u, w, v) \equiv \int_0^w \frac{\partial A(u, t)}{\partial t} \frac{\partial B(t, v)}{\partial t} \, dt \quad (1)$$

for arbitrary bivariate copula functions $A$ and $B$. This operator allows to express the Chapman-Kolmogorov equation in the language of copulas. Ibragimov (2005, 2009) extended the representation to the case of Markov processes of order $k$. 
C-CONVOLUTION

Let $F$, $H$ be two continuous c.d.f’s and $C$ be a copula function. We define the **C-convolution** of $H$ and $F$ the c.d.f.

$$H^C F(t) = \int_0^1 D_1 C \left( w, F(t - H^{-1}(w)) \right) dw$$

Alternatively, the **C-convolution** can be expressed in terms of densities as

$$h^C f(t) = \int_0^1 c \left( w, F(t - H^{-1}(w)) \right) f(t - H^{-1}(w)) dw$$
CONVOLUTION BASED COPULAS

Proposition

Let $X$ and $Y$ be two real-valued random variables on the same probability space $(\Omega, \mathcal{F}, P)$ with a dependence structure represented by the copula function $C_{X,Y}$ and continuous marginal distributions $F_X$ and $F_Y$. Then,

$$C_{X,X+Y}(u, v) = \int_0^u D_1 C_{X,Y}(w, F_Y(F_{X+Y}^{-1}(v) - F_X^{-1}(w))) \, dw$$

$$F_{X+Y}(t) = \int_0^1 D_1 C_{X,Y}(w, F_Y(t - F_X^{-1}(w))) \, dw = F_X \ast F_Y(t).$$
Proof.

Using the substitution \( w = F_X(x) \in (0, 1) \)

\[
F_{X, X+Y}(s, t) = \mathbb{P}(X \leq s, X + Y \leq t) = \int_{-\infty}^{s} \mathbb{P}(Y \leq t - x | X = x) \, dF_X(x) = \\
= \int_{-\infty}^{s} \mathbb{P}(Y \leq t - x | X = x) \, dF_X(x) = \int_{-\infty}^{s} D_1 C_{X, Y}(F_X(x), F_Y(t - x)) \, dF_X(x) = \int_{0}^{F_X(s)} D_1 C_{X, Y}(w, F_Y(t - F_X^{-1}(w))) \, dw.
\]

Then, the copula function linking \( X \) and \( X + Y \) is

\[
C_{X, X+Y}(u, v) = \int_{0}^{u} D_1 C_{X, Y}(w, F_Y(F_X^{-1}(v) - F_X^{-1}(w))) \, dw.
\]

Moreover

\[
F_{X+Y}(t) = \lim_{s \to +\infty} F_{X, X+Y}(s, t) = \int_{0}^{1} D_1 C_{X, Y}(w, F_Y(t - F_X^{-1}(w))) \, dw.
\]
CONVOLUTION BASED COPULAS

Proposition

Let $F$, $G$, $H$ be three continuous c.d.f’s, $C(w, v)$ a copula function and

$$
\hat{C}(u, v) = \int_0^u D_1 C \left( w, F(G^{-1}(v) - H^{-1}(w)) \right) \, dw.
$$

$\hat{C}(u, v)$ is a copula function iff

$$
G = H^C F.
$$
Proof.

Let \( \hat{C} \) be a copula function. Necessarily \( \hat{C}(1, v) = v \) holds. But

\[
\hat{C}(1, v) = \int_0^1 D_1 C \left( w, F \left( G^{-1}(v) - H^{-1}(w) \right) \right) \, dw = \\
= H^C * F \left( G^{-1}(v) \right) = v
\]

for all \( v \in (0, 1) \) if and only if \( G = H^C * F \).

The converse is the content of the Proposition above. \( \square \)
The C-convolution operator is closed with respect to mixtures of copula functions.

Let $A$ and $B$ be bivariate copula functions.

\[ C(u, v) = \lambda A(u, v) + (1 - \lambda) B(u, v) \]

for $\lambda \in [0, 1]$. For all c.d.f’s $H$ and $F$,

\[ H^C \ast F = H^{\lambda A + (1 - \lambda) B} \ast F = \lambda H^A \ast F + (1 - \lambda) H^B \ast F. \]
It is likewise trivial to observe that this is not true for the corresponding convolution based copula function \( \hat{C}(u, v) \)

However, we have

\[
\hat{C}(u, v) = \lambda \int_0^u D_1 A \left( w, F((H \ast F)^{-1}(v) - H^{-1}(w)) \right) dw + \\
(1 - \lambda) \int_0^u D_1 B \left( w, F((H \ast F)^{-1}(v) - H^{-1}(w)) \right) dw
\]
CONVOLUTION BASED COPULAS: EXAMPLES

The co-monotonic case

In the case $C(w, v) = w \wedge v = \min(w, v)$

$$F_{X_{i-1}} * F_Y(t) = \sup \left\{ w \in (0, 1) : F_Y^{-1}(w) + F_{X_{i-1}}^{-1}(w) < t \right\}$$

that implies the well known result (Prop. 6.15 in McNeil et al. (2005))

$$F_Y^{-1}(F_{X_{i-1}} * F_Y(t)) + F_{X_{i-1}}^{-1}(F_{X_{i-1}} * F_Y(t)) = t.$$ 

In this case the time series is deterministic (Chen and Fan (2006))

$$C_{X_{i-1}, X_i}(u, v) =$$

$$= u \wedge \sup \left\{ w \in (0, 1) : F_Y^{-1}(w) + F_{X_{i-1}}^{-1}(w) < (F_{X_{i-1}} * F_Y)^{-1}(v) \right\} =$$

$$= u \wedge v = \min(u, v).$$
CONVOLUTION BASED COPULAS: EXAMPLES

The independence case

If \( C \) is the product copula, the \( C \)-convolution of \( F_{X_{i-1}} \) and \( F_{Y_i} \) coincides with the convolution \( F_{X_{i-1}} \ast F_{Y_i} \) of \( F_{X_{i-1}} \) and \( F_{Y_i} \), while the convolution based copula takes the form

\[
C_{X_{i-1},X_i}(u, v) = \int_0^u F_{Y_i}((F_{X_{i-1}} \ast F_{Y_i})^{-1}(v) - F_{X_{i-1}}^{-1}(w)) \, dw.
\]

In this case, through our construction, we recover the law of all random walks.
CONVOLUTION BASED COPULAS: ELLIPTICAL COPULAS

By Sklar’s Theorem we can define the copula function with associated matrix \( \begin{pmatrix} a & b \\ b & c \end{pmatrix} \).

\[
C_{X_{i-1},Y_i}(u, v) = \int_{-\infty}^{F_{X_{i-1}}^{-1}(u)} \int_{-\infty}^{F_{Y_i}^{-1}(v)} \sqrt{ac - b^2} g(as^2 + 2bst + ct^2) \, ds \, dt.
\]

The C-convolution is

\[
F_{X_{i-1}} \overset{C}{*} F_{Y_i}(z) = \sqrt{ac - b^2} \int_0^1 \frac{1}{f_{X_{i-1}}(F_{X_{i-1}}^{-1}(w))} \\
\cdot \left[ \int_{-\infty}^{F_{Y_i}^{-1}(F_{Y_i}(z - F_{X_{i-1}}^{-1}(w)))} g(aF_{X_{i-1}}^{-1}(w)^2 + 2bF_{X_{i-1}}^{-1}(w)t + ct^2) \, dt \right] \, dw
\]
Since $F_{X_{i-1}} \ast F_{Y_i} = F_{X_i}$, we have

$$C_{X_{i-1},X_i}(u,v) = \sqrt{ac-b^2} \times$$

$$\int_0^u \frac{1}{f_{X_{i-1}}(F_{X_{i-1}}^{-1}(w))} \int_{-\infty}^{F_{X_i}^{-1}(v)-F_{X_{i-1}}^{-1}(w)} g(aF_{X_{i-1}}^{-1}(w)^2 + 2bF_{X_{i-1}}^{-1}(w)t + ct^2) \, dt \, dw$$

$$= \sqrt{ac-b^2} \int_{-\infty}^{F_{X_{i-1}}^{-1}(u)} \int_{-\infty}^{F_{X_i}^{-1}(v)-s} g(as^2 + 2bst + ct^2) \, dt \, ds =$$

$$= \sqrt{ac-b^2} \int_{-\infty}^{F_{X_{i-1}}^{-1}(u)} \int_{-\infty}^{F_{X_i}^{-1}(v)} g((a+c-2b)s^2 + 2\hat{t}s(b-c) + c\hat{t}^2) \, d\hat{t} \, ds$$

and this is again of elliptical type with associated matrix

$$\begin{pmatrix} a + c - 2b & b - c \\ b - c & c \end{pmatrix}.$$
Let $F_{Y_i}$, $F_{X_{i-1}}$ and $F_{X_i}$ three cumulative distribution functions, with corresponding densities $f_{Y_i}$, $f_{X_{i-1}}$ and $f_{X_i}$. Let us consider a copula $C_{X_{i-1}, Y_i}$ whose density is

$$c_{X_{i-1}, Y_i}(u, u') = \left[ 1 + \theta(1 - 2u)(1 - 2F_{X_i}(F_{Y_i}^{-1}(u') + F_{X_{i-1}}^{-1}(u))) \right] \cdot \frac{f_{X_i}(F_{Y_i}^{-1}(u') + F_{X_{i-1}}^{-1}(u))}{f_{Y_i}(F_{Y_i}^{-1}(u'))}.$$ 

Then

$$C_{X_{i-1}, X_i}(u, v) = uv(1 + \theta(1 - u)(1 - v)), \quad F_{X_i} = F_{X_{i-1}} \ast F_{Y_i}.$$
APPLICATION: DYNAMIC MODELS

- Symmetric processes
- Stationary processes
- Independent increment processes
- Martingale processes
SYMMETRIC PROCESSES

Proposition

Let $\overline{C}$ be the survival copula that is

$$\overline{C}(u, v) = u + v - 1 + C(1 - u, 1 - v)$$

and $\overline{F}(t) = 1 - F(t)$. If

$$\overline{C}_{X_{i-1}, Y_i}(u, v) = C_{X_{i-1}, Y_i}(u, v)$$

and

$$\overline{F}_{X_{i-1}}(t) = F_{X_{i-1}}(-t), \overline{F}_{Y_i}(t) = F_{Y_i}(-t),$$

then $\overline{F}_{X_i}(t) = F_{X_i}(-t)$. 
Proof.

Since \( \bar{F}^{-1}_{X_{i-1}}(w) = F^{-1}_{X_{i-1}}(1 - w) = -F^{-1}_{X_{i-1}}(w) \),

\[
\begin{align*}
\bar{F}_{X_i}(t) &= \int_0^1 D_1 \bar{C}_{X_{i-1}, Y_i} \left( w, \bar{F}_{Y_i}(t - \bar{F}^{-1}_{X_{i-1}}(w)) \right) dw \\
&= \int_0^1 D_1 C_{X_{i-1}, Y_i} \left( w, \bar{F}_{Y_i}(-t + \bar{F}^{-1}_{X_{i-1}}(w)) \right) dw \\
&= \int_0^1 D_1 C_{X_{i-1}, Y_i} \left( w, \bar{F}_{Y_i}(-t - \bar{F}^{-1}_{X_{i-1}}(w)) \right) dw \\
&= F_{X_i}(-t).
\end{align*}
\]
A strictly stationary Markov process is characterized by assuming \( C_{X_{i-1},X_i} \equiv C, \forall i \) and \( F_{X_i} \equiv F, \forall i \).

In our setting we can recover \( C_{X_{i-1},X_i} \) through

\[
C_{X_{i-1},Y_i}(u, v) = \int_0^u D_1 C_{X_{i-1},X_i} \left( w, F_{X_i}(F_{Y_i}^{-1}(v) + F_{X_{i-1}}^{-1}(w)) \right) \, dw
\]

where \( F_{X_{i-1}}, F_{X_i} \) and \( F_{Y_i} \) must satisfy

\[
F_{Y_i}(t) = \int_0^1 D_1 C_{X_{i-1},X_i} \left( w, F_{X_i}(t + F_{X_{i-1}}^{-1}(w)) \right) \, dw.
\]
Stationarity can be recovered in our framework, for any given $C$ and $F$, by setting

$$F_{Y_i}(t) = \int_0^1 D_1 C \left( w, F(t + F^{-1}(w)) \right) \, dw \equiv G(t)$$

and

$$C_{X_{i-1},Y_i}(u,v) = \int_0^u D_1 C \left( w, F(G^{-1}(t) + F^{-1}(w)) \right) \, dw \equiv A(u,v).$$

Notice that both the distribution of the increments and the copula between the level and the increments are stationary.
INDEPENDENT INCREMENT $\alpha$-STABLE PROCESSES

A symmetric (around the origin) $\alpha$-stable distribution is characterized by a characteristic function of type

$$
\phi(\lambda) = e^{-\gamma \lambda^\alpha}
$$

with $0 < \alpha \leq 2$, $\gamma \geq 0$. Note that $\phi(\lambda) = e^{-(\gamma \lambda)^\alpha} = \phi_Z(\gamma \lambda)$ where $Z$ is $\alpha$-stable with $\gamma = 1$.

For $\alpha = 2$ we get the Normal distribution.
If \((X_{i-1}, Y_i)\) is a symmetric \(\alpha\)-stable vector with independent components, then by simply applying the standard convolution formula it is easy to find that \(X_i\) is symmetrically \(\alpha\)-stable distributed with \(\gamma_{X_i} = \gamma_{X_{i-1}} + \gamma_Y\) and, if \(\rho = \frac{\gamma_{X_{i-1}}}{\gamma_{X_i}}\)

\[
C_{X_{i-1},X_i}(u, v) = \int_0^u \Phi_Z \left( \frac{\Phi_Z^{-1}(v) - \rho \Phi_Z^{-1}(w)}{(1 - \rho^\alpha)^{\frac{1}{\alpha}}} \right) dw.
\] (2)

For \(\alpha = 2\) we get the Gaussian copula generating all Gaussian processes
GENERAL PROCESSES: SIMULATION ALGORITHM

1. For $i = 1$ to $n$
2. Generate $u$ from the uniform distribution
3. Compute $X_i = F_Y^{-1}(u)$
4. Use conditional sampling to generate $v$ from $D_1 C(u, v)$
5. Compute $Y_{i+1} = F_Y^{-1}(v)$
6. $X_{i+1} = X_i + Y_{i+1}$
7. Compute the distribution $F_{X_{i+1}}(t)$ by $C$-convolution
8. Compute $u = F_{X_{i+1}}(X_{i+1})$
It is well known that

Any process whose increments $Y_i \equiv X_i - X_{i-1}$, are independent of $X_{i-1}$ ($C_{X_{i-1}, Y_i}(u, v) \equiv uv$) and whose distributions $F_{Y_i}$ have zero mean is a martingale.
Definition

A copula function $C(u, v)$ is said to be “symmetric around the first coordinate”, if

$$\tilde{C}(u, v) \equiv u - C(u, 1 - v) = C(u, v).$$

so that the pairs $(U, V)$ and $(U, 1 - V)$ have the same joint distribution $C$.

Proposition

*We recover a discrete times martingale Markov process, for any choice of symmetric distribution of increments $F_{Y_i}$ if and only if the copula between the increments and the levels is symmetric (around the first coordinate).*
Proposition

Take any bivariate copula \( A(u, v) \) and its symmetric part \( \tilde{A}(u, v) \equiv u - A(u, 1 - v) \). Define: \( C(u, v) \equiv 0.5A(u, v) + 0.5\tilde{A}(u, v) \). Then, \( C(u, v) \) is a copula and it is symmetric in the sense that \( C(u, v) = \tilde{C}(u, v) \).

Proof.

First, notice that \( \tilde{A}(u, v) \) is a copula and \( C(u, v) \) is a copula because it is a mixture of copulas. As for the symmetry property of \( C(u, v) \)

\[
\tilde{C}(u, v) = u - C(u, 1 - v) = \\
= u - (0.5A(u, 1 - v) + 0.5u - 0.5A(u, v)) \\
= 0.5A(u, v) + 0.5u - 0.5A(u, 1 - v) = C(u, v)
\]
SYMMETRIC MARTINGALES: SIMULATION ALGORITHM

1. For $i = 1$ to $n$
2. Generate $u$ from the uniform distribution
3. Compute $X_i = F_Y^{-1}(u)$
4. Generate $\xi$ from the uniform distribution
5. Use conditional sampling to generate $v$ from $D_1 C(u, v)$
6. Compute $Y_{i+1} = F_Y^{-1}(v)$
7. If $\xi \leq 0.5$, $Y_{i+1} = -Y_{i+1}$
8. $X_{i+1} = X_i + Y_{i+1}$
9. Compute the distribution $F_{X_{i+1}}(t)$ by $C$-convolution
10. Compute $u = F_{X_{i+1}}(X_{i+1})$
Typical performance analysis of managed funds rely on the moments of the distribution (expected excess return, or *alpha*, its standard deviation, or *tracking error*). We are interested in all the distribution.

Analysis motivated by the requirement from the Italian securities exchange commission to disclose the probability distribution of managed funds (compared to the market or the benchmark).

The analysis is assumed to be carried out in an efficient market so that the return on managed fund cannot be predicted either from past returns on the market or on the fund itself.
The return on a managed fund is represented as $Z = X + Y$ where

- $X$ is the return on the benchmark, with marginal cdf $F_X$
- $Y$ is the return on the management strategy, with marginal cdf $F_Y$

The dependence of $X$ and $Y$ is represented by copula function $C(u, v)$. This yields a C-convolution problem that allows to recover

- the distribution of the return on the managed fund $F_Z$
- the dependence structure between the return on the managed fund $Z$ and that of the market $X$
EXAMPLE: HENRIKSSON-MERTON COPULA

Assume the portfolio insurance management strategy
\[ Y = \alpha + \gamma \max(0, -X) + \epsilon \]
in Henriksson-Merton. The dependence structure between \( X \) and \( Y \) is represented by the HM-copula

\[
C_X, \gamma(u, v) = G_{X, \gamma}(F^{-1}_X(u), F^{-1}_Y(v))1_{\{u > F_X(0)\}} + H_{X, \gamma}(F^{-1}_X(u), F^{-1}_Y(v))1_{\{u \leq F_X(0)\}}
\]

where

\[
G_{X, \gamma}(x, y) = F_X(x)F_\epsilon(y - \alpha) - \int_{-\infty}^{y-\alpha} F_X\left(\frac{e - y + \alpha}{\gamma}\right)F_\epsilon(de)
\]

and

\[
H_{X, \gamma}(x, y) = F_X(x)F_\epsilon(\gamma x + y - \alpha) - \int_{-\infty}^{\gamma x + y - \alpha} F_X\left(\frac{e - y + \alpha}{\gamma}\right)F_\epsilon(de).
\]
The distribution of the fund is given by

\[ F_Z(z) = \int_0^{F_X(0)} F_\epsilon(z + (\gamma - 1)F_X^{-1}(w) - \alpha) \, dw + \]
\[ + \int_{F_X(0)}^1 F_\epsilon(z - F_X^{-1}(w) - \alpha) \, dw \]

If the noise term gets to zero, \( F_Y(y) = \left(1 - F_X\left(\frac{\alpha \cdot y}{\gamma}\right)\right) \cdot 1_{y \geq v} \)

and on \( \{(u, v) : v \geq F_Y(\alpha) = 1 - F_X(0)\} \) the copula tends to the perfectly negative dependence copula \( W \).
EXAMPLE: HENRIKSSON-MERTON COPULA

Notice that the parameter $\gamma$, which is a concordance measure between the asset manager forecast and actual market movements, determines the skewness of the managed fund return, with standard convolution for $\gamma = 0$. 
Core-satellite strategies consist of adding to a fund $Z$ a small investment in a satellite asset, typically uncorrelated with $Y$. In this case, the return is given by

$$F_{Z}(t) = \int_{0}^{1} F_{H}(t - F_{Z}^{-1}(w)))dw.$$  \hspace{1cm} (3)

where $Z$ denotes the core-satellite portfolio and $H$ is the c.d.f. of the investment in the satellite asset. The gain in diversification can be gauged measuring

$$C_{Z,H}(u, v) = \int_{0}^{u} F_{H}(F_{Z}^{-1}(v) - F_{Z}^{-1}(w)))dw.$$  \hspace{1cm} (4)
THE INVERSE PROBLEM: RECOVERING THE MANAGEMENT STRATEGY

Assume you observe: the return on the managed fund $Z$ and that on the benchmark $X$ and their dependence represented by the copula $C_{ZX}$. This yields a C-convolution problem that allows to recover

- the distribution of the return on the management strategy $F_Y$
- the dependence structure between the return on the management strategy $Y$ and that of the market $X$
Hedge funds and other alternative investment strategies are characterized by the property of being market neutral, that is $C_{ZX} = F_Z F_X$. Then, the return on the strategy is

$$F_Y(t) = 1 - \int_0^1 F_X(F_Z^{-1}(w) - t)dw.$$ 

and the dependence with the market is

$$C_{X,Y}(u, v) = u - \int_0^u F_X(F_Z^{-1}(w) - t)dw.$$ 

Warning

Applications to hedge funds data may suffer from liquidity problems and then violate the assumption at the foundation of the analysis.
EXAMPLE: MARKET NEUTRAL INVESTMENT

Hedge funds and other alternative investment strategies are characterized by the property of being market neutral, that is $C_{ZX} = F_Z F_X$. Then, the return on the strategy is

$$F_Y(t) = 1 - \int_0^1 F_X(F_Z^{-1}(w) - t)dw.$$ 

and the dependence with the market is

$$C_{X,Y}(u, v) = u - \int_0^u F_X(F_Z^{-1}(w) - t)dw.$$ 

Warning

Applications to hedge funds data may suffer from liquidity problems and then violate the assumption at the foundation of the analysis.
Three Italian mutual funds and their benchmark

- "Bilanciato" (BIL), referring to investment equally split between equity and bonds;
- "Azionario Multidivisa" (AzMul), representing investment in multi-currency equity;
- "Corporate IG" (CIG), representing investment in corporate bonds.
Estimation technique

- Extension of the $C$-convolution concept to the conditional copula framework as in Patton (2006)
- Two-Stages Maximum Likelihood estimation: likelihood maximization with respect to of marginal distributions in a first step and then with respect to the dependence parameter as a second step
- Several distributions fitted and tested on the marginals: finally, the $t$-Garch(1,1) model was selected in all cases
- Several copula functions estimated: Clayton, Frank, Student-$t$
For both $X_t$ and $Y_t$ $t$-GARCH(1,1)

$$X_t = \alpha_x + \epsilon_t$$

$$\sigma_{x,t}^2 = \gamma_{x,0} + \gamma_{x,1}\epsilon_{t-1}^2 + \gamma_{x,2}\sigma_{x,t-1}^2$$

$$h_t \epsilon_t | F_{t-1} \overset{iid}{\sim} t(\nu_x),$$

where $h_t = \frac{1}{\sigma_{x,t}} \sqrt{\frac{\nu_x}{\nu_x - 2}}$
Table: Estimated marginal distribution parameters in the case where $X$ and $Y$ are $t$-Garch$(1,1)$. The asterisk denotes the parameters which are significantly different from zero at the 5% level.
Table: Estimated copula parameter and relative standard error when $X$ and $Y$ are $t$-Garch(1,1). The asterisk denotes the parameters which are significantly different from zero at the 5% level.
Balanced fund (BIL). Either Student-t copula or Clayton. Fat-tails on both sides? From the test, Clayton is preferred.

Multi-currency equity. Student-t copula. Symmetric tail dependence.

Corporate fund (CIG): Gaussian copula. But the goodness of fit test failed.
Given the marginal distributions

\[
F_{X_t|\mathcal{F}_{t-1}}(x_t) = t_{(\nu_x)}((x_t - \alpha_x)h_t),
\]

\[
F_{Y_t|\mathcal{F}_{t-1}}(y_t) = t_{(\nu_y)}((y_t - \alpha_y)k_t).
\]

and the copula function we have the \( C \)-convolution

\[
F_{Z_t|\mathcal{F}_{t-1}} = F_{X_t|\mathcal{F}_{t-1}} * C F_{Y_t|\mathcal{F}_{t-1}} =
\]

\[
= \int_0^1 D_1 C \left( w, F_{Y_t|\mathcal{F}_{t-1}} \left( z_t - F_{X_t|\mathcal{F}_{t-1}}^{-1}(w) \right) \right) dw.
\]
Since \( F_{X_t|\mathcal{F}_{t-1}}^{-1}(w) = h_t^{-1} t_{(\nu_x)}^{-1}(w) + \alpha_x \) we get

\[
F_{Z_t|\mathcal{F}_{t-1}}(z_t) = \int_0^1 D_1 C \left( w, t_{(\nu_y)} \left( \left( z_t - \alpha_x - \alpha_y - h_t^{-1} t_{(\nu_x)}^{-1}(w) \right) k_t \right) \right) dw.
\]
Motivation
A review of copula functions
Convolution Based Copulas
Building Markov processes by increments aggregation
Application: efficient market dynamics
Application: Managed fund analysis

MANAGED FUND RETURNS (BIL)

<table>
<thead>
<tr>
<th>Max market volatility</th>
<th>Min market volatility</th>
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<tr>
<td>%</td>
<td>Excess return</td>
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<tr>
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<td>--------------</td>
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**Table:** Quantiles of the estimated distributions of $Y_t|\mathcal{F}_{t-1}$ and $Z_t|\mathcal{F}_{t-1}$ (percentage values). Data source: BIL.
Motivation
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MANAGED FUND RETURNS (AZ.MULT)

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<tr>
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<tr>
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Table: Quantiles of the estimated distributions of $Y_t | \mathcal{F}_{t-1}$ and $Z_t | \mathcal{F}_{t-1}$ (percentage values). Data source: AzMul.
### MANAGED FUND RETURNS (CIG)

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</table>

**Table:** Quantiles of the estimated distributions of $Y_t|\mathcal{F}_{t-1}$ and $Z_t|\mathcal{F}_{t-1}$ (percentage values). Data source: CIG.
CONCLUSIONS AND FURTHER RESEARCH

We propose a family of copulas based on convolution, with applications to

- Dynamics of speculative prices
- Performance analysis of managed funds

Other applications

- Basket credit derivatives (CMR, 2008)
- Basket equity derivatives (CGMR, work in progress)
- Reinsurance of portfolios of losses (CMR, 2011)
- Longevity analysis (colleagues in Milan)
- Factor model estimation ($X_t = b' R_t$)
Open issues

- Behavior of the model in the long run
- Behavior of the model in continuous time
- Estimation and simulation issues (speed)
Dynamic copula methods in finance

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SILVIA ROMAGNOLI